ENDOMORPHISMS OF SMOOTH PROJECTIVE 3-FOLDS WITH NONNEGATIVE KODAIRA DIMENSION, II

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Dedicated to Professor Kenji Ueno on the occasion of his sixtieth birthday

ABSTRACT. This article is a continuation of the paper [2]. Smooth complex projective 3-folds with nonnegative Kodaira dimension admitting nontrivial surjective endomorphisms are completely determined. Especially, it is proved that, for such a 3-fold X, there exist a finite étale Galois covering $\widetilde{X} \to X$ and an abelian scheme structure $\widetilde{X} \to T$ over a smooth variety T of dimension ≤ 2.

1. INTRODUCTION

A surjective *endomorphism* $f\colon X \to X$ of a variety X is called *nontrivial* if it is not an automorphism. Our purpose is to determine the structure of smooth complex projective 3-folds X with nonnegative Kodaira dimension admitting nontrivial surjective endomorphisms. Since the objects of our interest are not the endomorphisms f but the varieties X, we replace freely f with a power $f^k = f \circ \cdots \circ f$ in the discussion below. Abelian varieties and toric varieties are typical examples of varieties admitting nontrivial surjective endomorphisms. Moreover, the direct product $X \times Y$ admits a nontrivial surjective endomorphism if so does X. On the other hand, the existence of nontrivial surjective endomorphisms f induces strong restrictions on the varieties X, as follows:

- X is not of general type, i.e., the Kodaira dimension $\kappa(X)$ is less than $\dim X$.
- If $\kappa(X) \geq 0$, then f is étale. In particular, the Euler–Poincaré characteristic $\chi(X, \mathcal{O}_X)$ and the Euler number $\chi_{\mathrm{top}}(X)$ are zero.

A smooth projective curve C admits a nontrivial surjective endomorphism if and only if C is isomorphic to the projective line $\mathbb{P}^1$ or an elliptic curve. The classification of the compact complex varieties X of $\dim X > 1$ admitting nontrivial surjective endomorphisms has been done in the following cases: smooth projective surfaces (cf. [2], [20]); smooth compact complex surfaces (cf. [3]); projective bundles (cf. [1]); smooth projective 3-folds

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with $\kappa \geq 0$ except for the case where a general fiber of the Iitaka fibration is an abelian surface (cf. [2]).

The purpose of this paper is to complete the classification of smooth complex projective 3-folds with nonnegative Kodaira dimension admitting nontrivial surjective endomorphisms by showing the following:

Main Theorem. *Let X be a smooth complex projective 3-fold with $\kappa(X) \geq 0$. Then the following conditions are equivalent to each other*:

(A) *X admits a nontrivial surjective endomorphism.*

(B) *There exist a finite étale Galois covering $\tau\colon \widetilde{X} \to X$ and an abelian scheme structure $\varphi\colon \widetilde{X} \to T$ over a variety T of dimension ≤ 2 such that the Galois group $\mathrm{Gal}(\tau)$ acts on T and φ is $\mathrm{Gal}(\tau)$-equivariant.*

The implication (B) $\Rightarrow$ (A) holds in any dimension by Theorem 2.26 below. The implication (A) $\Rightarrow$ (B) can be checked easily for smooth projective 3-folds classified in our previous paper [2] (cf. Section 3.3). Therefore, we shall focus our attention to the remaining case, i.e., the case where the Iitaka fibration is an abelian fibration over a curve.

Note that Main Theorem solves the question $E_{n,a}$ for $n = 3$ in [2], and gives a refinement of $E_{n,a}$. Our method in [2] and in this article is not enough for solving the question $E_{n,a}$ for $n > 3$. Indeed, our proof in dimension three uses special properties of threefolds and elliptic curves; especially, the existence of flips, flops, and the abundance theorem in the minimal model theory and the finiteness of the order of automorphism group of an elliptic curve preserving the origin.

In order to study compact complex manifolds X admitting nontrivial surjective endomorphisms, it is important to analyze data of X preserved by the endomorphisms, since they reveal much of the deeper structure of the variety X. We have considered the following data in our previous papers [2], [3], [20]:

(1) *Iitaka fibration*: Let $\varphi\colon X \cdots\to Z$ be the Iitaka fibration of X. Then for a surjective endomorphism f of X, there exists an automorphism h of Z with $\varphi \circ f = h \circ \varphi$.

(2) *Extremal rays*: A surjective endomorphism of X with $\kappa(X) \geq 0$ induces a permutation of the set of extremal rays of X (cf. [2]).

(3) *Curves with negative self-intersection number*: If $\dim X = 2$, then X has only finitely many irreducible curves with negative self-intersection number, and any surjective endomorphism of X induces a permutation of the set of such curves (cf. [3], [20]).

The automorphism h of (1) is expected to be of finite order. If Z is a curve and a general fiber of φ is an abelian variety, then this is true by a similar argument to Lemma 3.7

below using Corollary 2.12. In particular, combined with results in [2], the finiteness of order of h is established in case $\dim X \leq 3$.

Our proof of Main Theorem is based on an argument used in [2]. The outline is as follows: Let $f\colon X \to X$ be a nontrivial surjective endomorphism of a smooth projective 3-fold X with $\kappa(X) \geq 0$. In the first step, we assume that X is not minimal, i.e., the canonical bundle K_X is not nef. We apply the minimal model program. For any extremal ray R of $\overline{\mathrm{NE}}(X)$, the contraction morphism $\mathrm{Cont}_R\colon X \to X'$ associated to R is just the blowing up of a smooth projective 3-fold X' along an *elliptic* curve E. This is shown by Mori's cone theorem and the classification of extremal rays on smooth projective 3-folds in [13]. Since the exceptional divisor of Cont_R is contained in the fixed part of the linear systems $|mK_X|$ for $m > 0$, X has only finitely many extremal rays. Thus f induces a permutation of the finite set of extremal rays. By replacing f with a suitable power f^k, we may assume that $f_* R = R$ for any extremal ray R. Then the contraction morphism Cont_R induces a nontrivial surjective endomorphism f' of X' such that $f'^{-1}E = E$ and $\mathrm{Cont}_R \circ f = f' \circ \mathrm{Cont}_R$. Taking contractions of extremal rays successively, we eventually obtain a nontrivial surjective endomorphism $f_{\min}$ of a smooth minimal model $X_{\min}$ of X.

In the second step, we assume that X is minimal. Then K_X is semi-ample by the abundance theorem (cf. [8], [11], [12]). Let $\varphi\colon X \to Z$ be the Iitaka fibration. Then $\varphi \circ f = h \circ \varphi$ for an automorphism h of Z of finite order (cf. [2], Proposition 3.7). We can prove that a suitable finite étale Galois covering $\widetilde{X}$ of X has a structure of an abelian scheme over a variety of dimension at most two. In fact, this is shown as follows:

(i) If $\kappa(X) = 0$, then this is a consequence of Bogomolov's decomposition theorem.

(ii) If $\kappa(X) = 2$, then φ is an elliptic fibration. By considering the equi-dimensional models of φ in the sense of [18, Appendix A] and by an argument in [15], [16], we can find a finite étale Galois covering $\widetilde{X}$ isomorphic to $E \times S$ for an elliptic curve E and a smooth surface S of general type.

(iii) If $\kappa(X) = 1$ and a general fiber of φ is a hyperelliptic surface, then we can find a finite étale covering $\widetilde{X}$ isomorphic to $E \times S$ for an elliptic curve E and a surface S by applying Fujiki's generic quotient theorem [5], [6], and by a similar argument to (ii).

(iv) In the remaining case, a general fiber of φ is an abelian surface. The existence of $\widetilde{X}$ is proved in Sections 4 and 5 below. For the proof, we need some results related to abelian fibrations prepared in Section 2 and the theory of global structure of elliptic fibrations in [19].

In the final step, we go back to the situation where X is not minimal. Then a finite étale Galois covering $\widetilde{X}_{\min}$ of the smooth minimal model $X_{\min}$ is isomorphic to $E \times S'$ for an elliptic curve E and a smooth projective surface S' by the second step (cf. [2, MAIN THEOREM (A)] for (i)–(iii), Sections 4 and 5 for (iv)). Let $\widetilde{X} \to X$ be the étale covering obtained as the pullback of $\widetilde{X}_{\min} \to X_{\min}$ by the birational morphism $X \to X_{\min}$. Then, by analyzing the centers of blowups connecting X to $X_{\min}$, we can show that $\widetilde{X} \simeq E \times S$ for another smooth projective surface S and that $f_{\min}$ can be lifted to recover the original endomorphism f or a suitable power f^k.

We shall explain more on the situation (iv). Let $\varphi\colon X \to C$ be an abelian fibration from a smooth projective 3-fold X to a smooth curve C and let $f\colon X \to X$ be a nontrivial surjective endomorphism satisfying $\varphi \circ f = f$. Then the natural homomorphism $\pi_1(X_t) \to \pi_1(X)$ of fundamental groups is not a zero map for a general fiber $X_t = \varphi^{-1}(t)$. If $\pi_1(X_t) \to \pi_1(X)$ is injective, then φ is called *primitive*; if not, called *imprimitive*. In the imprimitive case, the kernel of $\pi_1(X_t) \to \pi_1(X)$ contains a nonzero proper Hodge substructure of $\pi_1(X_t) \simeq \mathrm{H}_1(X_t, \mathbb{Z})$ by Corollary 2.15 below.

Suppose that φ is primitive. The proof of Main Theorem in this case is treated in Section 4. If X is minimal, then φ is a Seifert abelian fibration by Corollary 2.11; Thus there exists a finite étale Galois covering $\widetilde{X} \to X$ such that the Stein factorization of $\widetilde{X} \to X \to C$ induces an abelian scheme $\widetilde{X} \to \widetilde{C}$ over a smooth curve $\widetilde{C}$ (cf. Lemma 2.4). In particular, Main Theorem holds in this case. If a fiber of φ is a simple abelian surface, then X is minimal and Main Theorem holds in this case, by Theorem 4.1. If any smooth fiber of φ is not simple and if X is not minimal, then φ is factored by elliptic fibrations $X \to S$ and $S \to C$ in which $X \to S$ is an elliptic bundle (cf. Proposition 4.2). This essentially follows from the argument on H-factorization in Section 2.3 based on an idea of Ueno in [22]. From the elliptic bundle $X \to S$, we can find an expected finite étale Galois covering of X.

Suppose that φ is imprimitive. The proof of Main Theorem in this case is treated in Section 5. We apply the argument on H-factorization to a nonzero proper Hodge substructure of $\mathrm{H}_1(X_t, \mathbb{Z})$ contained in the kernel of $\mathrm{H}_1(X_t, \mathbb{Z}) \simeq \pi_1(X_t) \to \pi_1(X)$ and perform a finite succession of flops to X as in [2] (cf. [18, Appendix]). Then we infer that the Iitaka fibration of the minimal model $Y = X_{\min}$ is factored as $Y \to T \to C$, where $Y \to T$ is an equi-dimensional elliptic fibration over a normal projective surface T with only quotient singularities. Moreover, the endomorphism f of X induces a nontrivial surjective endomorphism $T \to T$, and the fibers of $Y \to T$ over a certain prime divisor of T consist of rational curves. We can find a suitable finite ramified covering $\widetilde{C} \to C$ such that the normalization $\widetilde{T}$ of $T \times_C \widetilde{C}$ is étale in codimension one over T and $\widetilde{T} \simeq E \times \widetilde{C}$ for an elliptic curve E. Here, the normalization $\widetilde{Y}$ of $Y \times_C \widetilde{C}$ is also étale over Y. By the ∂-étale cohomological description [19] of the elliptic fibration $\widetilde{Y} \to \widetilde{T}$, we have a finite étale covering $E' \to E$ such that $E' \times_E \widetilde{Y} \simeq E' \times S$ for an elliptic surface $S \to \widetilde{C}$ (cf. Theorem 5.10). This part is a core of our proof in the imprimitive case.

This article is organized as follows: In Section 2, we study abelian fibrations in a general setting from the viewpoint of variation of Hodge structures. Especially, we analyze Seifert abelian fibrations, primitive and imprimitive abelian fibrations, simple and non-simple abelian fibrations, the construction of H-factorization, and abelian fibrations admitting endomorphisms. In Section 3, we summarize known results on smooth projective 3-folds X of $\kappa(X) \geq 0$ admitting nontrivial surjective endomorphisms f, recall the construction of the minimal reduction of f, and note special properties in the case where the Iitaka fibration of X is an abelian fibration over a curve. Sections 4, 5 are devoted to the proof of Main Theorem for the case not treated in our previous paper [2].

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Notation and Terminology. In this article, we work over the complex number field $\mathbb{C}$.

Varieties: A *variety* means a reduced and irreducible complex algebraic scheme, or a reduced and irreducible complex analytic space. A *projective variety* is a complex variety embedded in a projective space $\mathbb{P}^n$, and a *quasi-projective* variety is a Zariski open subset of a projective variety. A *smooth projective n-fold* means a nonsingular projective variety of dimension n. The following symbols are used for a variety X as usual:

K_X	:	the canonical divisor of X (when X is normal).
$\kappa(X)$	:	the Kodaira dimension of X.
$p_g(X)$	:	the (geometric) genus of X.
$\chi(\mathcal{O}_X)$	:	the Euler–Poincaré characteristic of the structure sheaf $\mathcal{O}_X$.
$\chi_{\text{top}}(X)$	:	the topological Euler characteristic of X.
$b_i(X)$	:	the i-th Betti number of X.
$\text{Sing}(X)$	:	the singular locus of X.
$\text{Aut}(X)$	:	the space of holomorphic automorphisms of X.
$\text{Aut}^0(X)$	:	the identity component of $\text{Aut}(X)$.

For a scheme Y, the reduced part is denoted by Y_{red}, which is a reduced scheme with the same support as Y.

Minimal models: A normal projective variety X is called a minimal model if X has only terminal singularities and K_X is nef. Let $\pi\colon Y \to Z$ be a projective morphism from a normal quasi-projective variety. A Cartier divisor D on Y is called π-nef (or relatively nef over Z) if $D\Gamma \geq 0$ for any irreducible curve Γ with $\pi(\Gamma)$ being a point. If Y has only terminal singularities and K_Y is relatively nef over Z, then Y is called a relative minimal model over Z.

Fibrations: A proper surjective morphism $\pi\colon V \to S$ is called a *fibration* or a *fiber space* if V and S are normal complex varieties and f has a connected fiber. Then all the fibers of a fibration are connected. The closed subset

$$\Delta_\pi = \{s \in S \mid \pi \text{ is not smooth at some point of } \pi^{-1}(s)\}$$

is called the *discriminant locus* of π. The restriction $V^\star \to S^\star$ of π to $S^\star = S \setminus \Delta_\pi$ and $V^\star = \pi^{-1}(S^\star)$ is called the smooth part of π. The smooth part is a topological fiber bundle.

Abelian fiber spaces: If a general fiber of a fiber space $\pi\colon V \to S$ is an abelian variety, then π is called an *abelian fibration* or an *abelian fiber space*. If a general fiber of π is an elliptic curve, then π is called an *elliptic fibration* or an *elliptic fiber space*. If π is a holomorphic fiber bundle of an elliptic curve, then it is called an *elliptic bundle*. If π is a smooth abelian fibration, then the local constant system $R_1\pi_*\mathbb{Z}_V$ forms a variation of Hodge structure $H(\pi)$ of weight -1 on S. Conversely, if H is a polarized variation of Hodge structure of weight -1 on S, then there exists uniquely up to isomorphism a smooth abelian fiber space $p\colon B(H) \to S$ such that p admits a global section and $H(p) \simeq H$. This is called the smooth basic abelian fibration associated with H. An *abelian scheme* is a proper smooth morphism $\pi\colon X \to S$ of schemes such that π has a structure of S-group scheme. In this case, any fiber of π is an abelian variety.

Relative settings: Let $u\colon X \to S$ and $v\colon Y \to S$ be two morphisms into the same variety S. A morphism $h\colon X \to Y$ is called a morphism *over S* if $u = v \circ h$. If there is an isomorphism $X \xrightarrow{\sim} Y$ over S, then X and Y are called isomorphic to each other over S. Similarly, if X and Y are algebraic varieties and there is a birational map $X \cdots\to Y$ over S, then X and Y are called birational to each other over S.

Endomorphisms: A *nontrivial endomorphism* $f\colon X \to X$ of a complex variety X is, by definition, a nonconstant non-isomorphic morphism from X to itself. The subset

$$\mathrm{Fix}(f) := \{x \in X \mid f(x) = x\}$$

is called the *fixed point set* by f. For a positive integer k, the power f^k stands for the k-times composite $f \circ \cdots \circ f$ of f.

Hilbert schemes: Let V be a quasi-projective variety and let $V \to T$ be a projective morphism into another variety. We set:

$$\begin{aligned}\mathrm{Hilb}(V) &: \text{ the Hilbert scheme of } V.\\ \mathcal{Z}(V) &: \text{ the universal family} \subset V \times \mathrm{Hilb}(V).\\ \mathrm{Hilb}(V/T) &: \text{ the relative Hilbert scheme of } V/T.\\ \mathcal{Z}(V/T) &: \text{ the universal family} \subset V \times_T \mathrm{Hilb}(V/T).\end{aligned}$$

For a scheme S over T and for a proper flat morphism $\varphi\colon U \to S$ over T from a subscheme U of V, the graph Γ_φ of φ is a subscheme of $V \times_T S$ which is proper and flat over S. Hence Γ_φ coincides with the pullback of $\mathcal{Z}(V/T)$ by a morphism $u\colon S \to \mathrm{Hilb}(V/T)$, which is called the universal morphism associated with φ.

Rigidity Lemma: The following is referred as the rigidity lemma (cf. [14, Proposition 6.1]): Let $f\colon X \to Y$ and $q\colon Y \to S$ be morphisms of varieties such that the composite $p = q \circ f\colon X \to S$ is a proper smooth morphism with connected fibers. Suppose that $f(p^{-1}(s))$ is set-theoretically a single point for one point $s \in S$, then there exists a section $\eta\colon S \to Y$ of q such that $f = \eta \circ p$.

2. Abelian fiber spaces

2.1. **Seifert abelian fibrations.** We recall some facts on abelian fibrations which are almost smooth in a certain sense. To begin with, we recall the following result on smooth abelian fibration by Kollár [10, Proposition 5.9], which is generalized to the Kähler situation in [17, Lemma 2.20]:

Lemma 2.1. *Let $\pi\colon M \to S$ be a smooth abelian fibration over a smooth projective variety S. Then $\kappa(M) = \kappa(S)$.*

In the statement for the Kähler situation, we need to assume that the variation of Hodge structure $R_1\pi_*\mathbb{Z}_M$ admits an $\mathbb{R}$-polarization.

Lemma 2.2. *Let $\varphi\colon M \to S$ be a smooth abelian fibration over a smooth quasi-projective variety S. Then there is a finite étale morphism $\widetilde{S} \to S$ such that $M \times_S \widetilde{S} \to \widetilde{S}$ is an abelian scheme.*

Proof. Let $H = H(\varphi)$ be the variation of Hodge structure $R_1\varphi_*\mathbb{Z}_M$. Let $p\colon B = B(H) \to S$ be the associated basic smooth abelian fibration. Then p is an abelian scheme and φ is regarded as a torsor associated with an element η of $\mathrm{H}^1(S, \mathfrak{S}_H)$, where $\mathfrak{S}_H$ is the sheaf of germs of sections of p (cf. [17, Section 2]). Since φ is projective, there is a subvariety $\widetilde{S} \subset M$ such that $\widetilde{S} \to S$ is finite étale by [17, Corollary 2.13]. Then $\varphi \times_S \mathrm{id}_{\widetilde{S}}\colon M \times_S \widetilde{S} \to \widetilde{S}$ has a section, thus $M \times_S \widetilde{S}$ is isomorphic to the abelian scheme $B \times_S \widetilde{S}$ over $\widetilde{S}$. □

Definition 2.3 (cf. [17])**.** Let $V \to S$ be a projective fiber space from a smooth variety V onto a normal variety S whose general fibers are abelian varieties. It is called a *Seifert* *abelian fiber space* if there exist finite surjective morphisms $W \to V$ and $T \to S$ satisfying the following conditions:

(1) W and T are smooth varieties;
(2) W is isomorphic to the normalization of $V \times_S T$ over V;
(3) $W \to V$ is étale;
(4) $W \to T$ is smooth.

If $V \to S$ is a Seifert abelian fiber space, then V is a unique relative minimal model over S, since K_V is relatively numerically trivial and there are no rational curves contained in fibers. If S is compact and $\dim V = \dim S + 1$, then we may replace the condition (4) with that $W \simeq E \times T$ over T for an elliptic curve E. The notion of Seifert abelian fiber space is introduced in [17] as the name of *Q-smooth* abelian fibration.

Lemma 2.4. *Let $V \to S$ be a Seifert abelian fiber space. Then there exists a finite Galois covering $T \to S$ such that the normalization W of $V \times_S T$ is étale over V and W is an abelian scheme over T.*

Proof. By Definition 2.3 and by Lemma 2.2, we have a finite surjective morphism $T \to S$ satisfying the required properties except for that $T \to S$ is Galois. Taking the Galois closure $\widehat{T} \to S$ of $T \to S$ is equivalent to taking the Galois closure $\widehat{W} \to V$ of the finite étale covering $W \to V$. Hence, $\widehat{W}$ is isomorphic to the normalization of $V \times_S \widehat{T}$ and also to the fiber product $W \times_T \widehat{T}$. Therefore, $\widehat{T} \to S$ satisfies the required condition. □

The following gives a sufficient condition on elliptic fibrations to be Seifert:

Theorem 2.5 (cf. [15], [16])**.** *Let $\pi\colon V \to S$ be an elliptic fibration from a smooth projective n-fold V into a normal projective variety S. Suppose that*

(a) *no prime divisor Θ of V with $\operatorname{codim} \pi(\Theta) \geq 2$ is uniruled,*
(b) *no prime divisor Θ of V with $\operatorname{codim} \pi(\Theta) = 1$ is covered by a family of rational curves contained in fibers of π,*
(c) *K_V is π-numerically trivial.*

Then there exists a generically finite surjective morphism $T \to S$ satisfying the following conditions:

(1) *T is a smooth projective variety.*
(2) *For the normalization W of the main component $V \times_S T$, the induced morphism $W \to V$ is a finite étale covering.*
(3) *W is isomorphic to the product $E \times T$ over T for an elliptic curve E.*

Corollary 2.6. *Let $\pi\colon V \to S$ be an elliptic fibration from a smooth projective n-fold V onto a normal projective variety S. If the following conditions are satisfied, then π is a Seifert elliptic fibration*:

(1) *π is equi-dimensional*;
(2) *K_V is π-numerically trivial*;
(3) *For an irreducible divisor Γ contained in the discriminant locus Δ of π, the singular fiber type of π along Γ is ${}_m\mathrm{I}_0$ for some $m \geq 1$.*

Here, the singular fiber type is defined as follows (cf. [18], [19]): For a generic analytic arc C in S intersecting Γ transversally at a general point $x \in \Gamma$, $\pi^{-1}(C) \to C$ is a nonsingular minimal elliptic surface over C. The singular fiber type of f along Γ is defined to be the type of singular fiber $\pi^{-1}(x)$ in the sense of Kodaira. A singular fiber of type ${}_m\mathrm{I}_0$ of an elliptic surface is expressed as a divisor mE for an elliptic curve E.

Proposition 2.7. *Let $\pi\colon V \to C$ be an abelian fiber space over a smooth curve C. If the normalization of any fiber is an abelian variety, then π is a Seifert abelian fibration.*

The following proof contains an argument used in the proof of [17, Theorem 4.3].

Proof. By localizing C, we may assume C to be the unit disc $\{t \in \mathbb{C}\,;\,|t| < 1\}$. Moreover, the scheme-theoretic fiber $V_t = \pi^{-1}(t)$ is abelian for $t \neq 0$. The reduced part $V_{0,\mathrm{red}}$ of the central fiber V_0 is irreducible and the normalization $V^{\nu}_{0,\mathrm{red}}$ of $V_{0,\mathrm{red}}$ is abelian by assumption. Let m be the multiplicity of V_0, i.e., $V_0 = mV_{0,\mathrm{red}}$ and let $C' = \{t' \in \mathbb{C}\,;\,|t'| < 1\} \to C$ be the cyclic covering given by $t' \mapsto t = t'^m$. Let V' be the normalization of $V \times_C C'$ and let $\widetilde{V} \to V'$ be the resolution of singularities. Then $V' \to V$ is étale outside $\operatorname{Sing} V_{0,\mathrm{red}}$, and the scheme-theoretic fiber V'_0 of $\pi'\colon V' \to C'$ over $0 \in C'$ is reduced. The scheme-theoretic fiber $\widetilde{V}_{t'}$ of $\widetilde{V} \to C'$ over t' is isomorphic to V_t for $t = t'^m$ if $t' \neq 0$. Let $\widetilde{V}_0 = \bigcup \Gamma_j$ be the irreducible decomposition. We have the lower semicontinuity $1 = p_g(V_t) \geq \sum p_g(\Gamma_j)$ of the geometric genus p_g for the degeneration $\widetilde{V} \to C'$ of abelian varieties. On the other hand, $\Gamma_j \to V_{0,\mathrm{red}}$ is a finite surjective morphism if Γ_j is not exceptional for $\widetilde{V} \to V'$. Therefore, by the characterization [9] (cf. [21, Theorem 10.3]) of abelian varieties for varieties finite over an abelian variety, V'_0 is irreducible and its normalization is an abelian variety. Since $\pi'_*\omega_{V'} \to \pi'_*\omega_{V'}(V'_0)$ is not isomorphic, $\pi'^*\pi'_*\omega_{V'} \to \omega_{V'}$ is an isomorphism. In particular, V' and V'_0 are Gorenstein, and $\omega_{V'_0} \simeq \mathcal{O}_{V'_0}$. Hence, the conductor of the normalization of V'_0 is zero. Thus V'_0 is an abelian variety and $\pi'\colon V' \to C'$ is a smooth abelian fibration.

The variety V is regarded as the quotient space of V' by an action of the Galois group $\mathrm{Gal}(C'/C) \simeq \mathbb{Z}/m\mathbb{Z}$. Similarly, the normalization $V^{\nu}_{0,\mathrm{red}}$ of $V_{0,\mathrm{red}}$ is regarded as the quotient space of V'_0. Here, $V'_0 \to V^{\nu}_{0,\mathrm{red}}$ is étale since both are abelian varieties. Thus the action of

$\mathrm{Gal}(C'/C)$ on V_0' is free and that on V' is also free. Hence, V is nonsingular, $V' \to V$ is étale, and $V_{0,\mathrm{red}}$ is abelian. Therefore, $V \to C$ is a Seifert abelian fibration. □

Proposition 2.8. *Let $\pi\colon M \to C$ be a smooth abelian fibration over a smooth curve C. Suppose that a subvariety $Y \subset M$ defines a proper surjective morphism $Y \to C$ whose general fiber is an abelian variety. Then $Y \to C$ is a smooth abelian fibration.*

Proof. By localizing C, we may assume C to be a unit disc $\{t \in \mathbb{C}\,;\, |t| < 1\}$ and $Y \to C$ to be smooth outside $0 \in C$. Let M_t be the scheme-theoretic fiber of π over $t \in C$ and let Y_t be the scheme-theoretic intersection $M_t \cap Y$. Let $\nu\colon V \to Y$ be the normalization and let V_t be the scheme-theoretic fiber of $\pi|_Y \circ \nu\colon V \to C$ over t. For the irreducible decomposition $V_0 = \bigcup \Gamma_j$, we have $p_g(\Gamma_j) \leq 1$ by the lower semi-continuity $p_g(V_t) \geq \sum p_g(\Gamma_j)$ for $t \neq 0$. By [9], we infer that V_0 is irreducible and that $Y_{0,\mathrm{red}}$ and the normalization of $V_{0,\mathrm{red}}$ are abelian varieties. By Proposition 2.7, $V \to C$ is a Seifert abelian fibration and $V_{0,\mathrm{red}}$ is also abelian.

Let m be the multiplicity of V_0 and let $C' = \{t' \in \mathbb{C}\,;\, |t'| < 1\} \to C$ be the cyclic covering given by $t' \mapsto t'^m$. Then the normalization V' of $V \times_C C'$ is smooth over C' by the proof of Proposition 2.7. For the morphism $V' \to M' = M \times_C C'$, the induced homomorphism $\mathrm{H}_1(V_0', \mathbb{Z}) \to \mathrm{H}_1(M_0, \mathbb{Z})$ between the first homology groups of central fibers are isomorphic to the homomorphism $\mathrm{H}_1(Y_t, \mathbb{Z}) \to \mathrm{H}_1(M_t, \mathbb{Z})$ for $t \neq 0$ induced from $Y_t \subset M_t$. In particular, $\mathrm{H}_1(V_0', \mathbb{Z}) \to \mathrm{H}_1(M_0, \mathbb{Z})$ is injective and its cokernel is torsion free. Hence, the composite $\mathrm{H}_1(V_0', \mathbb{Z}) \to \mathrm{H}_1(V_{0,\mathrm{red}}, \mathbb{Z}) \to \mathrm{H}_1(Y_{0,\mathrm{red}}, \mathbb{Z})$ is an isomorphism. Therefore, $m = 1$, V_0 is reduced, and $V_0 \simeq Y_0$. Hence, $V \simeq Y$, and $Y \to C$ is smooth. □

2.2. **Primitive and imprimitive abelian fibrations.** The following result of Kollár [10] plays a key role in our argument below (the result itself is generalized to the compact Kähler situation in [17]):

Theorem 2.9 (cf. [10, 6.5–6.8], [17, Proposition 8.5])**.** *Let $\varphi\colon M \to S$ be a proper surjective morphism between smooth projective varieties such that which is smooth outside a normal crossing divisor $D \subset S$. Suppose that*

- *M_s is birationally equivalent to an abelian variety,*
- *the kernel of $\pi_1(M_s) \to \pi_1(M)$ contains no nonzero proper Hodge substructure of $\mathrm{H}_1(M_s, \mathbb{Z}) \simeq \pi_1(M_s)$,*

for a general smooth fiber $M_s = \varphi^{-1}(s)$. Then the following properties hold:

(1) *The local monodromies of $(R_1\varphi_*\mathbb{Z}_M)|_{S^\star}$ around D are finite, where $S^\star := S \setminus D$.*

(2) *There is a finite étale morphism $M' \to M$ such that, for the Stein factorization $M' \to S' \to S$, the local monodromies of the associated variation of Hodge structure on $S'^{\star} = S' \times_S S^{\star}$ around $S' \setminus S'^{\star}$ are trivial.*
(3) *φ is birationally equivalent over S to a Seifert abelian fibration.*

The assertion (3) above is derived from an idea used in the proof of the following:

Lemma 2.10. *Let $\varphi\colon M \to S$ be an abelian fiber space between smooth quasi-projective varieties. Let $S^{\star}$ be the complement of the discriminant locus Δ_{φ}. Suppose that*
(1) *the variation of Hodge structure $H(\varphi) = R_1\varphi_*\mathbb{Z}_M|_{S^{\star}}$ of weight -1 extends to S,*
(2) *for any point $s \in \Delta_{\varphi}$, there exists a holomorphic section of φ over an open neighborhood of s.*

Then φ is birational to a smooth abelian fibration over S.

Proof. Let H be the variation of Hodge structure extended to S and let $p\colon B = B(H) \to S$ be the smooth basic abelian fibration associated with H, i.e., an abelian scheme with an isomorphism $R_1p_*\mathbb{Z}_B \simeq H$. There exist an analytic open covering $\{\mathcal{U}_{\lambda}\}$ and analytic sections $\sigma_{\lambda}\colon \mathcal{U}_{\lambda} \to M$ of φ by assumption. If φ is smooth, then σ_{λ} induces a bimeromorphic morphism $\phi_{\lambda}\colon \varphi^{-1}(\mathcal{U}_{\lambda}) \to p^{-1}(\mathcal{U}_{\lambda})$ as the relative Albanese map over $\mathcal{U}_{\lambda}$ (cf. [17]). Even if φ is not smooth, we have the bimeromorphic morphism ϕ_{λ} by [17, Proposition 1.6]. The difference $\phi_{\lambda} \circ \phi_{\nu}^{-1}$ is described as the translation map of B/S by a section $\eta_{\lambda,\nu}\colon \mathcal{U}_{\lambda} \cap \mathcal{U}_{\nu} \to M$. Gluing $\{p^{-1}(\mathcal{U}_{\lambda})\}$ by the translation maps, we have a new smooth torus fibration $B^{\eta} \to S$, which depends on the cohomology class $\eta \in \mathrm{H}^1(S, \mathfrak{S}_H)$ of the collection $\{\eta_{\lambda,\mu}\}$, where $\mathfrak{S}_H$ denotes the sheaf of germs of holomorphic sections of $B \to S$. In other words, $B^{\eta} \to S$ is an analytic torsor of $B \to S$ associated with η. It is known that η is of finite order if and only if φ is a projective morphism. For the bimeromorphic morphism $M \to B^{\eta}$ over S, the image of an intersection of general ample divisors of M in B^{η} dominates S and has the same dimension as $\dim S$. Thus η is of finite order and $B^{\eta} \to S$ is a projective morphism by [17, Corollary 2.13]. □

Corollary 2.11. *Let $\pi\colon X \to C$ be an abelian fiber space from a smooth projective variety X onto a smooth projective curve C such that K_X is π-nef. Suppose that the kernel of $\pi_1(X_t) \to \pi_1(X)$ contains no nonzero proper Hodge substructure of $\mathrm{H}_1(X_t, \mathbb{Z}) \simeq \pi_1(X_t)$. Then π is a Seifert abelian fiber space.*

Proof. By Theorem 2.9, there exists a finite covering $\widehat{C} \to C$ such that the normalization $\widehat{X}$ of $X \times_C \widehat{C}$ is étale over X and $\widehat{X} \to \widehat{C}$ is birationally equivalent to a smooth abelian fibration $\widehat{Y} \to \widehat{C}$ over $\widehat{C}$. Since K_X is nef, $\widehat{X} \to \widehat{C}$ is also a relative minimal model. Thus $\widehat{Y}$ and $\widehat{X}$ are isomorphic in codimension one. Since any fiber of $\widehat{Y} \to \widehat{C}$ contains no rational curves, the rational map $\widehat{X} \to \widehat{Y}$ is holomorphic, and hence isomorphic. Thus $X \to C$ is a Seifert abelian fibration. □

Corollary 2.12. *Let $\varphi\colon M \to C$ be an abelian fibration over a smooth rational curve C. If φ is smooth outside two points of C, then $\kappa(M) = -\infty$.*

Proof. Let $U \subset C$ be the complement of the two points. Then $U \simeq \mathbb{C}^\star$. Hence the period map of the variation of Hodge structure $H := R^1\varphi_*\mathbb{Z}_M|_U$ is constant by the hyperbolicity of the Siegel upper half spaces. In particular, the image of the monodromy representation $\mathbb{Z} \simeq \pi_1(U, u) \to \mathrm{Aut}(H_u)$ is finite. Let $C' \simeq \mathbb{P}^1 \to C$ be the finite cyclic covering extending $\mathbb{C} \setminus \{0\} \ni z \to z^m \in \mathbb{C} \setminus \{0\} \simeq U$ for suitable m. Let M' be a nonsingular model of $M \times_C C'$. Then we may assume that the following conditions are satisfied:

(1) the pullback of H to C' is a trivial variation of Hodge structure;
(2) $M' \to C'$ admits a local section over any point of C.

Then there exist a smooth abelian fibration $Y \to C'$ and a birational morphism $M' \to Y$ over C' by Lemma 2.10. In particular,

$$\kappa(M) \le \kappa(M') = \kappa(Y) = \kappa(C') = -\infty.$$

□

Definition 2.13. Let $\varphi\colon M \to S$ be an abelian fiber space between smooth varieties and let M_s denote the fiber $\varphi^{-1}(s)$ for a point $s \in S$. Let $M^\star \to S^\star = S \setminus \Delta_\varphi$ be the smooth part of φ. If $\pi_1(M_s) \to \pi_1(M)$ is injective for a point $s \in S^\star$, then it is so for any other point of $S^\star$. In this case, φ is called a *primitive* abelian fiber space. If φ is not primitive, then it is called *imprimitive*.

Remark 2.14. A primitive abelian fibration is called a *homotopically Q-smooth* abelian fibration in [17, Section 7]. A smooth abelian fiber space is primitive if it is a projective morphism. In fact, the homomorphism $\pi_2(S) \to \pi_1(M_s)$ appearing at the homotopy exact sequence

$$\pi_2(S) \to \pi_1(M_s) \to \pi_1(M) \to \pi_1(S) \to 1$$

is zero by [17, Corollary 2.18]. If S is a smooth curve, then this is shown as follows: If S is not isomorphic to $\mathbb{P}^1$, then it follows from the vanishing $\pi_2(S) = 0$. Suppose that $S \simeq \mathbb{P}^1$. Then a smooth projective abelian fibration $\varphi\colon M \to S$ has a constant variation of Hodge structure $H = R_1\varphi_*\mathbb{Z}_M$ and $B \simeq A \times S$ for the basic abelian fibration $B = B(H) \to S$ associated with H and for an abelian variety A. The sheaf $\mathfrak{S}_H$ of germs of holomorphic sections of $B \to S$ is represented by an exact sequence $0 \to H = \mathbb{Z}_S^{\oplus 2g} \to \mathcal{O}_S^{\oplus g} \to \mathfrak{S}_H \to 0$ for $g = \dim M - \dim S$. Thus $\mathrm{H}^1(S, \mathfrak{S}_H) \simeq \mathrm{H}^2(S, \mathbb{Z}^{\oplus 2g}) \simeq \mathbb{Z}^{\oplus 2g}$ is torsion free. Thus $M \simeq B \simeq A \times S$, since M is isomorphic to the torsor of B associated with a torsion element of $\mathrm{H}^1(S, \mathfrak{S}_H)$. Therefore, $\pi_2(S) \to \pi_1(M_s)$ is zero. We infer also that a Seifert abelian fibration is primitive, since it has an étale covering from an abelian scheme.

As a corollary of Theorem 2.9, we have:

Corollary 2.15. *If $\varphi\colon M \to S$ is an imprimitive abelian fiber space, then the kernel of $\pi_1(M_s) \to \pi_1(M)$ contains a nonzero proper Hodge substructure of $\mathrm{H}_1(M_s, \mathbb{Z}) \simeq \pi_1(M_s)$ for any $s \in S^\star$.*

Proof. Assume the contrary. Then, by Theorem 2.9, there exist a finite covering $S' \to S$, a finite étale covering $M' \to M$, a smooth abelian fiber space $Y \to S'$, a birational morphism $M' \to M \times_S S'$ over M, and a birational morphism $M' \to Y$ over S'. Let s be a point of $S^\star$ over which $S' \to S$ is étale and let $s' \in S'$ be a point lying over s. Then we have a contradiction by

$$\pi_1(M_s) \simeq \pi_1(M'_{s'}) \simeq \pi_1(Y_{s'}) \subset \pi_1(Y) \simeq \pi_1(M') \subset \pi_1(M). \qquad \square$$

2.3. **Non-simple abelian fibrations.** We shall study abelian fibrations whose very general fiber is a non-simple abelian variety. We follow several arguments by Ueno in [22] which deal with Hilbert schemes.

Lemma 2.16. *Let $\psi\colon M \to T$ be a proper flat surjective morphism of smooth projective varieties. Suppose that $\dim \mathrm{H}^0(M_t, \mathcal{O}_{M_t}) = 1$ for the scheme-theoretic fiber $M_t = \psi^{-1}(t)$ over a point $t \in T$. Then the universal morphism $u\colon T \to \mathrm{Hilb}(M)$ associated with ψ is a local isomorphism at t and $u(T)$ is an irreducible component of $\mathrm{Hilb}(M)$.*

Proof. We have $\dim \mathrm{H}^0(M_t, N_{M_t/M}) = n$ for $n = \dim T$ since the normal sheaf $N_{M_t/M}$ is a free sheaf of rank n. In particular, the Zariski tangent space of $\mathrm{Hilb}(M)$ at the point $[M_t]$ corresponding to M_t is n-dimensional. Hence, $\mathrm{Hilb}(M)$ is nonsingular of dimension n at $[M_t]$, since the morphism u is injective by construction. Thus the assertion holds. $\square$

Let $\varphi\colon M \to T$ be a projective flat surjective morphism of smooth quasi-projective varieties and let $M^\star \to T^\star$ be the smooth part of φ. Suppose that φ is an abelian fibration and that there is a proper positive-dimensional abelian subvariety A_t of the fiber $M_t = \varphi^{-1}(t)$ over a fixed point $t \in T^\star$. Let $[A_t]$ denote the point of $\mathrm{Hilb}(M/T)$ corresponding to the subscheme $A_t \subset M_t$. Then, by Lemma 2.16, $[A_t]$ is a nonsingular point of $\mathrm{Hilb}(M/T) \times_T \{t\} = \mathrm{Hilb}(M_t)$ and the connected component L_t of $\mathrm{Hilb}(M_t)$ containing $[A_t]$ is isomorphic to an abelian variety. In fact, A_t is a fiber of a surjective morphism $M_t \to L_t$. Let S be an irreducible component of $\mathrm{Hilb}(M/T)$ containing L_t and let $q\colon S \to T$ be the induced morphism. If s is a point of an open neighborhood of $[A_t]$ in S, then s defines an abelian subvariety $A(s)$ of the fiber $M_{q(s)}$ of $M \to T$ over $q(s)$ with

$\dim A_t = \dim A(s)$. Moreover, any point s of $S \times_T T^\star$ defines also an abelian subvariety $A(s) \subset M_{q(s)}$ with $\dim A_t = \dim A(s)$ by Proposition 2.8.

Lemma 2.17. *Suppose that $q\colon S \to T$ is surjective. Then q is smooth over $T^\star$.*

Proof. The scheme-theoretic fiber of $S \to T$ over t is smooth at the point $s_0 = [A_t]$. Hence, the dimension of the Zariski tangent space of S at s_0 is at most $\dim T + \dim L_t$.

For a point s of an open neighborhood of $s_0 \in S$ with $q(s) \neq t$, the connected component $L(s)$ of $\mathrm{Hilb}(M/T) \times_T \{q(s)\} = \mathrm{Hilb}(M_{q(s)})$ containing s is an abelian variety and $A(s)$ is a fiber of a surjective morphism $M_{q(s)} \to L(s)$. In particular, $L(s)$ contains any irreducible component of the fiber $S \times_T \{q(s)\}$ containing s. Since $\mathrm{Hilb}(M/T)$ has at most countably many irreducible components, there exist an irreducible component S' of $\mathrm{Hilb}(M/T)$ and a dense subset $\mathcal{U} \subset S \times_T T^\star$ such that $L(s) \subset S'$ for $s \in \mathcal{U}$. Hence, $S = S'$ by $\mathcal{U} \subset S \cap S'$. Therefore, $\dim S = \dim T + \dim L_t$. Consequently, $q\colon S \to T$ is smooth at s_0. For any other point $s \in S \times_T T^\star$, we have

$$\dim L(s) = \dim_s \mathrm{Hilb}(M_{q(s)}) \geq \dim_s S \times_T \{q(s)\} \geq \dim L(s_0) = \dim L(s).$$

Hence S is an irreducible component of $\mathrm{Hilb}(M/T)$ containing $L(s)$. Therefore, $q\colon S \to T$ is smooth over $T^\star$. □

Since $\mathrm{Hilb}(M/T)$ has only countably many irreducible components, the following conditions are equivalent to each other:

- One smooth fiber of $M \to T$ is a simple abelian variety;
- A very general fiber of $M \to T$ is a simple abelian variety;
- If $A_t \subset M_t$ is a positive-dimensional proper abelian subvariety of a smooth fiber M_t, then an irreducible component S of $\mathrm{Hilb}(M/T)$ containing $[A_t]$ does not dominate T.

Definition 2.18. If one of these conditions above is satisfied, then $M \to T$ is called a *simple* abelian fibration; If not, it is called a *non-simple* abelian fibration.

Theorem 2.19. *Let $\varphi\colon M \to T$ be a non-simple abelian fibration between smooth quasi-projective varieties and let $T^\star$ be the complement of the discriminant locus Δ_φ of φ. Then there exist a finite morphism $\widehat{T} \to T$ étale over $T^\star$ and a birational morphism $\widehat{M} \to M \times_T \widehat{T}$ from a smooth quasi-projective variety $\widehat{M}$ such that the induced morphism $\hat{\varphi}\colon \widehat{M} \to \widehat{T}$ is the composite $\beta \circ \alpha$ for abelian fibrations $\alpha\colon \widehat{M} \to \widehat{S}$ and $\beta\colon \widehat{S} \to \widehat{T}$, where α and β are smooth over the inverse image of $T^\star$, and* $\dim M > \dim \widehat{S} > \dim T$.

Proof. Let S be an irreducible component of $\mathrm{Hilb}(M/T)$ discussed in Lemma 2.17. Here, $q\colon S \to T$ is proper surjective, the restriction $S^\star = q^{-1}(T^\star) \to T^\star$ is smooth, and any irreducible component of the fiber over a point of $T^\star$ is an abelian variety. Let $\widehat{S}$ be the normalization of S and let $\widehat{S} \to \widehat{T} \to T$ be the Stein factorization. Then the induced morphism $\beta\colon \widehat{S} \to \widehat{T}$ is an abelian fibration, and $\widehat{T} \to T$ is a finite morphism étale over $T^\star$. We set $Z = \mathcal{Z}(M/T) \cap (M \times_T S)$ for the universal family $\mathcal{Z}(M/T) \subset M \times_T \mathrm{Hilb}(M/T)$. Then the second projection $Z \to S$ is an abelian fibration smooth over $S^\star$ and any connected component of $Z \times_T \{t\}$ is isomorphic to M_t for $t \in T^\star$. Thus the composite

$$Z \hookrightarrow M \times_T S \dashrightarrow M \times_T \widehat{S} \to M' := M \times_T \widehat{T}$$

is an isomorphism over $T^\star$. Hence we have the factorization $M' \dashrightarrow Z \dashrightarrow \widehat{S} \to \widehat{T}$ of $\varphi \times_T \mathrm{id}_{\widehat{T}}\colon M' \to \widehat{T}$. By taking a suitable birational morphism $\widehat{M} \to M'$, we have a desired factorization. □

Proposition 2.20. *Let $\varphi\colon M \to T$ be an abelian fibration between smooth quasi-projective varieties. Let $M^\star \to T^\star$ be the smooth part of φ and let $\widetilde{H}$ be the induced variation of Hodge structure $R_1\varphi_*\mathbb{Z}_M|_{T^\star}$ of weight -1 defined over $T^\star$. For a variation of Hodge substructure $H \subset \widetilde{H}$, there exist a rational abelian fibration $\alpha\colon M \dashrightarrow S$ and an abelian fibration $\beta\colon S \to T$ with $\varphi = \beta \circ \alpha$ such that*

(1) *$\alpha\colon M \dashrightarrow S$ is holomorphic and smooth over $T^\star$,*
(2) *$\beta\colon S \to T$ is smooth over $T^\star$,*
(3) *$\mathrm{H}_1(\alpha^{-1}(s), \mathbb{Z}) = H_{\beta(s)} \subset \widetilde{H}_{\beta(s)} = \mathrm{H}_1(\varphi^{-1}(\beta(s)), \mathbb{Z})$ for any point $s \in \beta^{-1}(T^\star)$.*

Proof. Let $B(\widetilde{H}) \to T^\star$ and $B(H) \to T^\star$ be the basic abelian fibrations associated with $\widetilde{H}$ and with H, respectively. Then $B(\widetilde{H}) \to T^\star$ is an abelian scheme and $B(H) \to T^\star$ is an abelian subscheme. The smooth abelian fibration $M^\star \to T^\star$ is regarded as a torsor of $B(\widetilde{H}) \to T^\star$. Thus we have the quotient torsor $\beta\colon S^\star \to T^\star$ of $M^\star \to T^\star$ by the relative action of $B(H) \to T^\star$. Let $\alpha\colon M^\star \to S^\star$ be the induced morphism. Then the condition (3) is satisfied for any $s \in S^\star$, i.e., $\mathrm{H}_1(\alpha^{-1}(s), \mathbb{Z}) = H_{\beta(s)} \subset \widetilde{H}_{\beta(s)} = \mathrm{H}_1(\varphi^{-1}(\beta(s)), \mathbb{Z})$. Therefore, it suffices to extend α and β to a rational map and a morphism defined over T, respectively.

Let $u\colon S^\star \to \mathrm{Hilb}(M/T)$ be the universal morphism associated with $M^\star \to S^\star$. Then the graph of $M^\star \to S^\star$ isomorphic to the pullback of the universal family $\mathcal{Z}(M/T) \subset M \times_T \mathrm{Hilb}(M/T)$ by u. By Lemma 2.16, $u(S^\star)$ is a connected component of $\mathrm{Hilb}(M^\star/T^\star) = \mathrm{Hilb}(M/T)|_{T^\star}$ and $S^\star \to u(S^\star)$ is an isomorphism. Thus there is an irreducible component $S \subset \mathrm{Hilb}(M/T)$ containing $u(S^\star)$. For the scheme-theoretic intersection $Z = \mathcal{Z}(M/T) \cap (M \times_T S)$, the first projection $Z \to M$ is an isomorphism over $T^\star$. Thus the morphism α extends to the rational map $M \dashrightarrow Z \to S$ and the other morphism β extends to the natural morphism $S \to T$. □

The factorization $M \overset{\alpha}{\dashrightarrow} S \overset{\beta}{\to} T$ is called an *H-factorization* of $\varphi\colon M \to T$.

Lemma 2.21. *Let $\varphi\colon M \to T$ be a smooth non-simple abelian fiber space between smooth compact varieties with $\dim M = \dim T + 2$. Then there exist a finite étale covering $\widetilde{T} \to T$ and a non-simple abelian surface A such that $M \times_T \widetilde{T} \simeq A \times \widetilde{T}$ over $\widetilde{T}$.*

Proof. We may assume that φ is factorized as $M \to S \to T$ for two smooth elliptic fibrations $M \to S$ and $S \to T$, by Theorem 2.19. Since T is compact, any fiber of $S \to T$ is isomorphic to a constant elliptic curve F. By replacing T with a suitable étale covering of T, we may assume that $S \simeq F \times T$ over T. The fibers of $M \to S$ are also constant. Let F' be the fiber. Then the fiber M_t over a point $t \in T$ is an abelian surface which gives an extension of F by F'. In particular, M_t is isogenous to $F \times F'$. Therefore, the period map associated with the abelian fibration $M \to T$ is also constant. Hence, $M \times_T \widetilde{T} \simeq A \times \widetilde{T}$ over a finite étale covering $\widetilde{T}$ of T. $\square$

2.4. **Abelian fibration with endomorphisms.**

Lemma 2.22. *Let $f\colon A \to A$ be a nontrivial surjective endomorphism of an abelian variety A.*

(1) *If A is simple, then the fixed point locus $\operatorname{Fix}(f)$ is a non-empty finite set.*
(2) *Suppose that there is a simple abelian subvariety $B \subset A$ of codimension one satisfying $f^{-1}(B) = B$. Then there is a positive integer k such that $\dim \operatorname{Fix}(f^k) = 1$ and the subgroup $\mathrm{H}_1(B, \mathbb{Z}) \subset \mathrm{H}_1(A, \mathbb{Z})$ is just the primitive hull of the image of*

$$f_*^k - \mathrm{id}\colon \mathrm{H}_1(A, \mathbb{Z}) \to \mathrm{H}_1(A, \mathbb{Z}).$$

Proof. Let us consider A to be a commutative group scheme and let 0 be the zero element. For the point $a = f(0) \in A$ and for the translation map $T_{-a}\colon A \to A$, the composite $g := T_{-a} \circ f\colon A \to A$ is a group homomorphism of A. Moreover $h := g - \mathrm{id}_A\colon A \to A$ is a non-zero group homomorphism of A, since $g\colon A \to A$ is a nontrivial surjective endomorphism. Here, $\operatorname{Fix}(f) \neq \emptyset$ if and only if $-a$ is contained in the image of h. Furthermore, in case $\operatorname{Fix}(f) \neq \emptyset$, $\operatorname{Fix}(f)$ is a translate of $\operatorname{Ker}(h)$ since, for a closed point $x \in \operatorname{Fix}(f)$ and for a closed point $x' \in A$, $x' \in \operatorname{Fix}(f)$ if and only if $x - x' \in \operatorname{Ker}(h)$.

If A is simple, then h is surjective and $\operatorname{Ker}(h)$ is finite; thus the assertion (1) follows.

For the abelian subvariety B in (2), the restriction $f|_B\colon B \to B$ is a nontrivial surjective endomorphism since $\deg(f|_B) = \deg(f) > 1$. In particular, f has a fixed point in B by (1). Hence, we may assume $0 \in \operatorname{Fix}(f)$, i.e., $a = 0$. Then $f = g$ and $\operatorname{Fix}(f) = \operatorname{Ker}(h)$. Let $p\colon A \to E$ be the projection to the quotient space $E = A/B$, which is an elliptic curve. There is a group automorphism $u\colon E \to E$ with $p \circ f = u \circ p$. Here $u^k = \mathrm{id}_E$ for some

$k \geq 1$, since E is an elliptic curve. The fiber $p^{-1}(b)$ over any point $b \in B$ is a translate of B. Thus the restriction of f^k to $p^{-1}(b)$ has a fixed point by (1). Therefore, $\mathrm{Fix}(f^k)$ dominates E and $\dim \mathrm{Fix}(f^k) = 1$. The homomorphism $f^k_* - \mathrm{id}\colon \mathrm{H}_1(B, \mathbb{Z}) \to \mathrm{H}_1(B, \mathbb{Z})$ is not zero since $\deg(f|_B) > 1$. The kernel of $f^k_* - \mathrm{id}$ defines a proper Hodge substructure of $\mathrm{H}_1(B, \mathbb{Z})$, which is zero since B is simple. Hence, $f^k_* - \mathrm{id}\colon \mathrm{H}_1(B, \mathbb{Z}) \to \mathrm{H}_1(B, \mathbb{Z})$ is injective. On the other hand, $u^k_* - \mathrm{id}\colon \mathrm{H}_1(E, \mathbb{Z}) \to \mathrm{H}_1(E, \mathbb{Z})$ is zero. Hence, the primitive hull of the image of $f^k_* - \mathrm{id}\colon \mathrm{H}_1(A, \mathbb{Z}) \to \mathrm{H}_1(A, \mathbb{Z})$ is just the subgroup $\mathrm{H}_1(B, \mathbb{Z}) \subset \mathrm{H}_1(A, \mathbb{Z})$. □

Theorem 2.23. *Let $\varphi\colon M \to T$ be a smooth abelian fibration over a quasi-projective variety T and let $f\colon M \to M$ be a nontrivial surjective endomorphism with $\varphi \circ f = \varphi$. Suppose that there is a simple abelian subvariety A of codimension one in a fiber $M_o = \varphi^{-1}(o)$ satisfying $f^{-1}A = A$. Then there exist a smooth abelian fibration $\alpha\colon M \to S$ and a smooth elliptic fibration $\beta\colon S \to T$ such that $\varphi = \beta \circ \alpha$, A is a fiber of β, and that $\alpha \circ f = v \circ \alpha$ for an automorphism $v \in \mathrm{Aut}(S)$. In particular, φ is a non-simple abelian fibration.*

Proof. Let $\widetilde{H}$ be the variation of Hodge structure $R_1\varphi_*\mathbb{Z}_M$ and let $f_*\colon \widetilde{H} \to \widetilde{H}$ be the homomorphism induced from f. Let $H \subset \widetilde{H}$ be the primitive hull of the image of $f^k_* - \mathrm{id}\colon \widetilde{H} \to \widetilde{H}$. Then $H_o = \mathrm{H}_1(A, \mathbb{Z}) \subset \widetilde{H}_o = \mathrm{H}_1(M_o, \mathbb{Z})$ for some k by Lemma 2.22. Applying Proposition 2.20, we have an H-factorization $M \to S \to T$. Then A and $f^{-1}(A)$ are fibers of $\alpha\colon M \to S$. We set $P = \alpha(A)$ and $Q = \alpha(f^{-1}(A))$. Since $\alpha \circ f(\alpha^{-1}(Q)) = P$, we have a morphism $v\colon S \to S$ satisfying $\alpha \circ f = v \circ \alpha$ by the rigidity lemma. Here, v is a finite étale morphism with $\beta \circ v = \beta$, since φ is smooth and α is surjective. Since $v^{-1}(P) = Q$, we have $\deg v = 1$, and hence, $v \in \mathrm{Aut}(S)$. □

Theorem 2.24. *Let $\varphi\colon M \to T$ be a smooth simple abelian fibration over a quasi-projective variety T. Suppose that there is a nontrivial surjective endomorphism $f\colon M \to M$ with $\varphi \circ f = \varphi$. Then $\mathrm{Fix}(f) \to T$ is a finite étale surjective morphism. In particular, for any point $t \in T$, the fiber $M_t = \varphi^{-1}(t)$ does not contain any simple abelian subvariety A of codimension one with $f^{-1}(A) = A$.*

Proof. By Lemma 2.22, $\mathrm{Fix}(f) \cap M_o$ is a non-empty finite set for a very general point $o \in T$. Hence, $\varphi\colon \widetilde{T} \to T$ is generically finite and surjective for an irreducible component $\widetilde{T}$ of $\mathrm{Fix}(f)$. The pullback $\widetilde{\varphi}\colon \widetilde{M} := M \times_T \widetilde{T} \to \widetilde{T}$ of φ is a smooth abelian fibration with a section. The pullback $\widetilde{f} := f \times_T \mathrm{id}_{\widetilde{T}}\colon \widetilde{M} \to \widetilde{M}$ of f is also a nontrivial surjective endomorphism defined over $\widetilde{T}$. Therefore, for the proof, we may assume $\varphi\colon M \to T$ to admit a section $\sigma\colon T \to M$ satisfying $f \circ \sigma = \sigma$. Thus $\varphi\colon M \to T$ has an abelian scheme structure whose zero section is σ, and $f\colon M \to M$ is a relative group homomorphism.

Since f is not an isomorphism, $h := f - \mathrm{id}_M : M \to M$ is a non-zero relative group homomorphism over T. Since the fiber M_o over a very general point $o \in T$ is a simple abelian variety, the restriction of h to M_o is surjective. Hence $h : M \to M$ is surjective. In particular, the restriction of h to the fiber M_t over any point $t \in T$ is a finite étale surjective morphism. Therefore, $\mathrm{Fix}(f) \cap M_t$ is a finite set isomorphic to $\mathrm{Ker}(h) \cap M_t$ by the proof of Lemma 2.22. Hence, $\mathrm{Fix}(f) \to T$ is finite, étale, and surjective.

If $f^{-1}(A) = A$ for a simple abelian subvariety A of codimension one of a fiber M_t, then $\dim \mathrm{Fix}(f^k) \cap M_t = 1$ for some $k \geq 1$ by Lemma 2.22. This is a contradiction. □

Lemma 2.25. *Let f be a nontrivial surjective endomorphism of the product $A \times T$ for a simple abelian variety A and a smooth projective variety T such that $p_2 \circ f = p_2$ for the second projection p_2. Then there is a finite étale Galois covering $\widetilde{T} \to T$ such that the lift $\tilde{f}$ of f to $A \times \widetilde{T}$ is written as $\phi \times \mathrm{id}_{\widetilde{T}}$ for an endomorphism ϕ of A with respect to a given group structure.*

Proof. An irreducible component $\widetilde{T}$ of the fixed point locus $\mathrm{Fix}(f)$ is finite and étale over T by Theorem 2.24. By replacing T with $\widetilde{T}$ and by considering a suitable automorphism of $A \times T$, we may assume that f preserves $\{0\} \times T$ for the zero element $0 \in A$. Then $f(a, t) = (\phi_t(a), t)$ for holomorphic maps $\phi_t : A \to A$ for $t \in T$, $a \in A$. Here, the induced homomorphism $\phi_{t*} : \mathrm{H}_1(A, \mathbb{Z}) \to \mathrm{H}_1(A, \mathbb{Z})$ is independent of the choice of $t \in T$. Since $\phi_t(0) = 0$, there exists an endomorphism $\phi : A \to A$ with $\phi(0) = 0$ and $\phi_t = \phi$ for any $t \in T$. □

2.5. **A part of the proof of Main Theorem.** The implication (B) ⇒ (A) of Main Theorem follows from:

Theorem 2.26. *Let X be a smooth projective n-fold. Suppose that there exist a finite Galois étale covering $\tau : M \to X$ and an abelian scheme structure $\varphi : M \to T$ such that the Galois group G of τ acts also on T with $\varphi \circ \sigma = \sigma \circ \varphi$ for $\sigma \in G$. Then there is a nontrivial surjective endomorphism Φ of M such that $\varphi \circ \Phi = \varphi$ and $\sigma \circ \Phi = \Phi \circ \sigma$ for any $\sigma \in G$. In particular, X admits a nontrivial surjective endomorphism.*

Proof. The action of $\sigma \in G$ on M is written as the composite $\mathrm{Tr}(h_\sigma) \circ \psi_\sigma$ for an automorphism ψ_σ of M over T preserving the zero section and for the translation map $\mathrm{Tr}(h_\sigma)$ by a section $h_\sigma : T \to M$. Then ψ_σ is a homomorphism between two abelian schemes $\sigma \circ \varphi : M \to T$ and $\varphi : M \to T$. The set F of sections of φ over T has a natural structure of abelian group, and furthermore, a structure of left G-module by

$$h \mapsto \sigma \cdot h = \psi_\sigma \circ h \circ \sigma^{-1}$$

for $h \in F$ and $\sigma \in G$. Then $\sigma \mapsto h_\sigma$ gives a 1-cocycle and defines an element $\eta \in \mathrm{H}^1(G, F)$. Since the order m of η is finite, we have a section $a \in F$ such that $mh_\sigma = \sigma \cdot a - a$.

Let $\mu_{m+1}\colon M \to M$ be the multiplication map by $m+1$ with respect to the group structure of M over T and let $\Phi\colon M \to M$ be the composite $\mathrm{Tr}(a) \circ \mu_{m+1}$. Then $\sigma \circ \Phi = \Phi \circ \sigma$ for any $\sigma \in G$. □

Combining with Lemma 2.4, we have:

Corollary 2.27. *Let $X \to S$ be a Seifert abelian fiber space from a smooth projective n-fold X onto a normal projective variety S. Then X admits a nontrivial surjective endomorphism.*

Lemma 2.28. *Let $\varphi\colon M \to T$ be a smooth abelian fiber space from smooth projective n-fold M to a smooth projective variety T, and let $f\colon M \to M$ be a nontrivial surjective endomorphism with $\varphi \circ f = v \circ \varphi$ for an automorphism $v \in \mathrm{Aut}(T)$. Suppose that a finite group G acts on M and that $\sigma \circ f = f \circ \sigma$ for any $\sigma \in G$. If $\dim T = n-1$, then the condition* (1) *below is satisfied*; *If $\dim T = n-2$ and $v = \mathrm{id}_T$, then one of the conditions* (1), (2) *below is satisfied*:

(1) *G acts on T and φ is G-equivariant.*
(2) *There exists a smooth elliptic fibration $\alpha\colon M \to S$ over T such that $\alpha \circ f^k = \alpha$ for a power f^k, G acts on S, and that α is G-equivariant.*

Proof. We set $M_t = \varphi^{-1}(t)$ for $t \in T$. Then we have $f^{-1}M_{v(t)} = M_t$. Hence,

$$f^{-1}(\sigma(M_{v(t)})) = \sigma(f^{-1}(M_{v(t)})) = \sigma(M_t)$$

for any $\sigma \in G$. In particular,

$$f|_{\sigma(M_t)}\colon \sigma(M_t) \to \sigma(M_{v(t)})$$

an étale surjective morphism of degree $\deg(f) > 1$. If $\varphi(\sigma(M_t))$ is a point for a point $t \in T$, then there is an automorphism $\sigma_T \in \mathrm{Aut}(T)$ with $\varphi \circ \sigma = \sigma_T \circ \varphi$, by the rigidity lemma. In particular, $\varphi(\sigma(M_t))$ is a point for any $t \in T$. Hence, if, for a point $t \in T$, $\varphi(\sigma(M_t))$ is a point for any $\sigma \in G$, then (1) is satisfied. By the commutative diagram

$$\begin{array}{ccc} \sigma(M_t) & \xrightarrow{f} & \sigma(M_{v(t)}) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \varphi(\sigma(M_t)) & \xrightarrow{v} & \varphi(\sigma(M_{v(t)})), \end{array}$$

we have $\dim \sigma(M_t) > \dim \varphi(\sigma(M_t))$ by considering the mapping degree. Hence, if $\dim T = n-1$, then $\dim \varphi(\sigma(M_t)) = 0$ and thus (1) is satisfied. We may assume $\dim T = n-2$,

$v = \mathrm{id}_T$, and $\dim \varphi(\sigma(M_t)) = 1$ for any $t \in T$. Let $\sigma(M_t) \to C \to \varphi(\sigma(M_t))$ be the Stein factorization. Since M_t is an abelian surface, C is an elliptic curve and M_t is not simple. For a fiber E of $\sigma(M_t) \to C$, we have $f^{-1}E = E$. Hence, the elliptic curve $E' = \sigma^{-1}(E) \subset M_t$ also satisfies $f^{-1}E' = E'$. By Theorem 2.23, we have a factorization $M \to S \to T$ of φ into smooth elliptic fibrations $\alpha \colon M \to S$ and $\beta \colon S \to T$, and $\alpha \circ f = u \circ \alpha$ for an automorphism u of S over T. Here, u fixes the point $\alpha(E') \in \beta^{-1}(t)$. Since this property holds for any point $t \in T$ and since β is an elliptic fibration, we infer that the order of u is finite. Thus $\alpha \circ f^k = \alpha$ for suitable $k > 0$. Since $\dim S = n - 1$, $M \to S$ is G-equivariant by the argument above. □

The following is useful in order to show the other implication (A) ⇒ (B) in Main Theorem:

Proposition 2.29. *Let X be a smooth projective 3-fold of $\kappa(X) \geq 0$. If one of the following conditions is satisfied, then the condition* (B) *of* Main Theorem *is satisfied*:

(1) *There is a finite étale covering $\widetilde{X} \to X$ from an abelian 3-fold $\widetilde{X}$.*
(2) *There exist a finite étale Galois covering $\widetilde{X} \to X$, a smooth abelian fibration $\varphi \colon \widetilde{X} \to T$ over a variety T of dimension ≤ 2, and a nontrivial surjective endomorphism $\tilde{f}$ of $\widetilde{X}$ such that*
   (a) *$\sigma \circ \tilde{f} = \tilde{f} \circ \sigma$ for any element σ of the Galois group of $\widetilde{X} \to X$,*
   (b) *$\varphi \circ \tilde{f} = v \circ \varphi$ for an automorphism $v \in \mathrm{Aut}(T)$ if $\dim T = 2$,*
   (c) *$\varphi \circ \tilde{f} = \varphi$ if $\dim T = 1$.*

Proof. (1) ⇒ (B): By Bogomolov's decomposition theorem, we may assume $\widetilde{X} \to X$ to be Galois. Thus (B) is satisfied.

(2) ⇒ (B): Let G be the Galois group of $\widetilde{X} \to X$. By Lemma 2.28, we may assume that φ is G-equivariant. Let G_0 be the kernel of $G \to \mathrm{Aut}(T)$ and let $\overline{X}$ be the quotient space of $\widetilde{X}$ by G_0. Then $\overline{X} \to T$ is a G/G_0-equivariant smooth abelian fibration and the induced nontrivial surjective endomorphism $\bar{f}$ of $\overline{X}$ from $\tilde{f}$ commutes with any element of G/G_0. By replacing $\widetilde{X}$ with $\overline{X}$, we may assume that $G \to \mathrm{Aut}(T)$ is injective. Thus, we have a Seifert abelian fibration $X \to G\backslash T$. Hence, the condition (B) is satisfied by Lemma 2.4. □

3. Threefolds admitting nontrivial surjective endomorphisms

3.1. **Basic properties on varieties with nontrivial surjective endomorphisms.**

We recall some basic properties of nontrivial surjective endomorphisms from [2].

Proposition 3.1. *Let $f \colon X \to X$ be a surjective endomorphism of a smooth projective n-fold X. Then f is a finite morphism. Moreover, the following properties hold*:

(1) *If X is not uniruled or K_X is pseudo-effective, then f is étale*;
(2) *Suppose that $\kappa(X) \geq 0$ and let $\phi\colon X \dashrightarrow Z$ be the Iitaka fibration of X. Then there exists a biregular automorphism h of Z with $\phi \circ f = h \circ \phi$*;
(3) *If X is of general type, then f is an automorphism*;
(4) *If f is not an automorphism and $\kappa(X) \geq 0$, then $\chi(\mathcal{O}_X) = \chi_{\mathrm{top}}(X) = 0$.*

For a smooth projective n-fold X, let $\mathrm{NS}(X)$ be the Néron–Severi group. The Picard number $\rho(X)$ is the rank of $\mathrm{NS}(X)$. We set

$$\mathrm{N}^1(X) := \mathrm{NS}(X) \otimes \mathbb{R}, \qquad \mathrm{N}_1(X) := \mathrm{Hom}(\mathrm{NS}(X), \mathbb{R}).$$

For an algebraic 1-cycle $Z = \sum n_i Z_i$, the numerical equivalence class $\mathrm{cl}(Z) \in \mathrm{N}_1(X)$ is defined by $D \mapsto DZ = \sum n_i DZ_i$ for divisors D. Let $\mathrm{NE}(X) \subset \mathrm{N}_1(X)$ be the cone generated by $\mathrm{cl}(Z)$ for all the effective 1-cycles Z, and let $\overline{\mathrm{NE}}(X)$ denote the closure of $\mathrm{NE}(X)$. The cone $\overline{\mathrm{NE}}(X)$ is often called the Kleiman–Mori cone. An *extremal ray* (more precisely, a K_X-negative extremal ray) is a 1-dimensional face R of $\overline{\mathrm{NE}}(X)$ with $K_X R < 0$. An extremal ray R defines a nontrivial proper surjective morphism $\mathrm{Cont}_R\colon X \to Y$ with connected fibers into a normal variety such that, for an irreducible curve $C \subset X$, $\mathrm{Cont}_R(C)$ is a point if and only if $\mathrm{cl}(C) \in R$. This is called the *contraction morphism* of R. We have proved the following results related to the extremal rays in [2]:

Proposition 3.2 (cf. [2, Propositions 4.2 and 4.12])**.** *Let $f\colon Y \to X$ be a finite surjective morphism between smooth projective n-folds with $\rho(X) = \rho(Y)$. Then, the following assertions hold*:

(1) *The push-forward map $f_*\colon \mathrm{N}_1(Y) \to \mathrm{N}_1(X)$ is an isomorphism and $f_* \overline{\mathrm{NE}}(Y) = \overline{\mathrm{NE}}(X)$.*
(2) *Let $f_*\colon \mathrm{N}^1(Y) \to \mathrm{N}^1(X)$ be the map induced from the push-forward map $D \mapsto f_*D$ of divisors D. Then the dual $f^*\colon \mathrm{N}_1(X) \to \mathrm{N}_1(Y)$* (called the pullback map) *is an isomorphism and $f^* \overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(Y)$.*
(3) *If f is étale and the canonical divisor K_X is not nef, then there is a one-to-one correspondence between the set of extremal rays of X and the set of extremal rays of Y.*
(4) *Under the same assumption as in* (3), *let $\phi\colon X \to X'$ be the contraction morphism Cont_R associated to an extremal ray $R \subset \overline{\mathrm{NE}}(X)$ and let $\psi\colon Y \to Y'$ be the contraction morphism associated to the extremal ray f^*R. Then there exists a finite surjective morphism $f'\colon Y' \to X'$ such that $\phi \circ f = f' \circ \psi$.*

Theorem 3.3 (cf. [2, Theorem 4.8])**.** *Let $f\colon X \to X$ be a nontrivial surjective endomorphism of a smooth projective 3-fold X with $\kappa(X) \geq 0$. If K_X is not nef, then the* *extremal contraction* $\mathrm{Cont}_R\colon X \to X'$ *associated to any extremal ray* R *of* $\overline{\mathrm{NE}}(X)$ *is a divisorial contraction which is* (*the inverse of*) *the blowing up along an elliptic curve on* X'.

3.2. **Construction of minimal reduction of an endomorphism.** Let us recall a construction of the *minimal reduction* of a nontrivial surjective endomorphism $f\colon X \to X$ of a smooth projective 3-fold X with $\kappa(X) \geq 0$. We apply the minimal model program to X. Assume that K_X is not nef. Then there exist only finitely many extremal rays of $\overline{\mathrm{NE}}(X)$ (cf. [2, Proposition 4.6]). Hence, by replacing f with a suitable power f^k $(k > 0)$, we may assume from the beginning that $f_*R = R$ for any extremal ray $R \subset \overline{\mathrm{NE}}(X)$. Theorem 3.3 and Proposition 3.2 imply that the contraction morphism $\mu := \mathrm{Cont}_R\colon X \to X_1$ associated with any extremal ray R is the blowing up along an elliptic curve of X_1, where a nontrivial surjective endomorphism $f_1\colon X_1 \to X_1$ with $f_1 \circ \mu = \mu \circ f$ is induced. If K_{X_1} is not nef, then, by the same way as above, we replace f_1 with a suitable power of f_1 so that $(f_1)_*R_1 = R_1$ for any extremal ray R_1 of X_1, and we take the contraction morphism Cont_{R_1} associated with an extremal ray R_1. In this way, we have successive contractions of extremal rays $X \to X_1 \to X_2 \to \cdots$ with a strictly decreasing sequence $\rho(X) > \rho(X_1) > \cdots$ of Picard numbers. Thus, after a finite number of steps, we obtain a smooth minimal model X_n of X and a nontrivial surjective endomorphism f_n of X_n. To sum up, after replacing f by a suitable power f^k, we have a sequence of extremal contractions

$$X = X_0 \xrightarrow{\mu_0} X_1 \xrightarrow{\mu_1} \cdots \xrightarrow{\mu_{n-1}} X_n$$

and nontrivial surjective endomorphisms $f_i\colon X_i \to X_i$ for $0 \leq i \leq n$ such that

(1) $\mu_0 = \mu$, $f_0 = f$, $\mu_i \circ f_i = f_{i+1} \circ \mu_i$ for $0 \leq i \leq n$,
(2) $\mu_{i-1}\colon X_{i-1} \to X_i$ is (the inverse of) the blowing up along an elliptic curve C_i on X_i with $f_i^{-1}(C_i) = C_i$ for $1 \leq i \leq n$,
(3) X_n is a smooth minimal model.

Definition 3.4. The final endomorphism $f_n\colon X_n \to X_n$ is called a minimal reduction of $f\colon X \to X$.

Corollary 3.5. *Let* X *be a smooth non-minimal projective* 3*-fold with* $\kappa(X) \geq 0$ *admitting a nontrivial surjective endomorphism* $f\colon X \to X$. *Then* $\mathrm{Fix}(f^k) \neq \emptyset$ *for a suitable power* f^k.

Proof. Let $\mu = \mu_0\colon X = X_0 \to X_1$ be the blowing up and $f_1\colon X_1 \to X_1$ be the endomorphism above. Then $f_1|_{C_1}\colon C_1 \to C_1$ is a nontrivial surjective endomorphism of the elliptic curve C_1. In particular, $\mathrm{Fix}(f_1) \cap C_1 \neq \emptyset$. For a point $x \in \mathrm{Fix}(f_1) \cap C_1$, $f|_{\mu^{-1}(x)} : \mu^{-1}(x) \to \mu^{-1}(x)$ is a surjective endomorphism of $\mu^{-1}(x) \simeq \mathbb{P}^1$. Hence $\mathrm{Fix}(f) \cap \mu^{-1}(x) \neq \emptyset$. □

The abundance theorem for 3-folds (cf. [11], [12], [8]) says that K_{X_n} is semi-ample. In particular, the Iitaka fibration $\varphi \colon X \to W$ is holomorphic for the canonical model

$$W = \mathrm{Proj} \bigoplus\nolimits_{m \geq 0} \mathrm{H}^0(X, \mathcal{O}_X(mK_X)),$$

where $\varphi = \varphi_n \circ \mu_{n-1} \circ \cdots \circ \mu_0$ for the Iitaka fibration $\varphi_n \colon X_n \to W$. There is an automorphism $h \in \mathrm{Aut}(W)$ with $\varphi \circ f = h \circ \varphi$, since

$$f^* \colon \mathrm{H}^0(X, \mathcal{O}_X(mK_X)) \to \mathrm{H}^0(X, \mathcal{O}_X(mK_X))$$

is isomorphic for any m. The canonical model W is denoted by C when it is one-dimensional, i.e., $\kappa(X) = 1$.

Lemma 3.6. *Suppose that $\kappa(X) = 1$. Let $\Gamma \subset X$ be a smooth curve such that $f^{-1}\Gamma = \Gamma$ for the endomorphism f of X. Let $\mu \colon \widehat{X} \to X$ be the blowing up along Γ. Then Γ is an elliptic curve contained in a fiber of the Iitaka fibration of X and f induces an endomorphism $\widehat{f}$ of $\widehat{X}$ with $\mu \circ \widehat{f} = f \circ \mu$.*

Proof. If Γ dominates C, then we have $\deg(f^{-1}\Gamma/C) = (\deg f)\deg(\Gamma/C)$ by $\varphi \circ f = h \circ \varphi$; this contradicts $f^{-1}(\Gamma) = \Gamma$ and $\deg f > 1$. Moreover, Γ is an elliptic curve, since f induces a nontrivial surjective endomorphism of Γ. Let $\mathcal{I}$ be the defining ideal of Γ in X. Then $f^*\mathcal{I}$ is the defining ideal of $f^{-1}\Gamma = \Gamma$. Hence, we have a morphism $\widehat{f} \colon \widehat{X} \to \widehat{X}$ with $\mu \circ \widehat{f} = f \circ \mu$ by the universality of blowing up. □

In particular, the center C_i of the i-th blowing-up $\mu_{i-1} \colon X_{i-1} \to X_i$, which appears at the sequence $X \to X_1 \to \cdots \to X_n$ connecting X and the minimal reduction X_n, is contained in a fiber of the Iitaka fibration $X_i \to C$.

3.3. **The class of smooth projective 3-folds of our interest.** In order to prove Main Theorem, it is enough to show the implication (A) ⇒ (B), by Theorem 2.26. To begin with, we shall show it for smooth projective 3-folds classified in our previous paper [2].

Let X be a smooth projective 3-fold with $\kappa(X) \geq 0$ admitting a nontrivial surjective endomorphism. In [2], the following cases are treated:

(1) $\kappa(X) = 0$.
(2) $\kappa(X) = 1$ and the general fiber of the Iitaka fibration of X is a hyperelliptic surface.
(3) $\kappa(X) = 2$.

If X belongs to one of the cases, then, by [2, MAIN THEOREM (A)],

- X has an abelian 3-fold as a finite étale covering, or
- X has a structure of Seifert elliptic fibration over a surface.

Hence, X satisfies the condition (B) by Lemma 2.4.

Thus, in what follows, we consider smooth projective 3-folds X satisfying the following three conditions:

(*1) There is a nontrivial surjective endomorphism $f\colon X \to X$;
(*2) $\kappa(X) = 1$;
(*3) A general fiber of the Iitaka fibration $\varphi\colon X \to C$ is an abelian surface.

As is explained in Section 3.2, there is a birational morphism $X \to X_{\min}$ to a smooth minimal model $X_{\min}$ which is described as a succession of blowups along elliptic curves contained in fibers of the Iitaka fibrations. Therefore, the Iitaka fibration $\varphi\colon X \to C$ is holomorphic and is isomorphic to the Iitaka fibration $X_{\min} \to C$ over C outside finitely many points of C.

Let X_t be the fiber $\varphi^{-1}(t)$ over a point $t \in C$. Let $h \in \mathrm{Aut}(C)$ be the automorphism determined by $\varphi \circ f = h \circ \varphi$ (cf. Proposition 3.1).

Proposition 3.7. *The automorphism h is of finite order.*

Proof. If $\kappa(C) = 1$, then the automorphism group of C is finite. If φ is smooth, then $\kappa(X) = \kappa(C) = 1$ by Lemma 2.1. Thus we may assume that φ admits at least one singular fiber. Thus the discriminant locus $\Delta = \Delta_\varphi$ is not empty. If C is an elliptic curve, then h preserves the finite set $\Delta \neq \emptyset$, and hence h is of finite order. If C is a smooth rational curve, then Δ consists of at least three points by Corollary 2.12; thus h is of finite order. □

By Proposition 3.7, by taking a power of f, we may replace the condition (*1) with the following stronger condition:

(*1′) There exists a nontrivial surjective endomorphism $f\colon X \to X$ over the curve C, i.e., $\varphi \circ f = \varphi$.

Thus it is enough to consider only the endomorphisms f defined over C. For such an f, let $f_t\colon X_t \to X_t$ denote the restriction of f to the fiber $X_t = \varphi^{-1}(t)$ for $t \in C$.

Lemma 3.8. *The image of the natural homomorphism $\pi_1(X_t) \to \pi_1(X)$ of fundamental groups is not finite for a general fiber X_t.*

Proof. Assume the contrary. Let $U_t \to X_t$ be the finite étale covering associated with the kernel of $\pi_1(X_t) \to \pi_1(X)$. Since $\pi_1(U_t) \to \pi_1(X)$ is a zero map, the the fiber product $U_t \times_{X, f^k} X$ by any power $f^k\colon X \to X$ for $k \geq 1$ is a disjoint union of copies of U_t. Since

$f^{-1}(X_t) = X_t$ is connected, we have natural inclusions $\pi_1(U_t) \subset f_{t*}\pi_1(X_t) \subset \pi_1(X_t)$. Iterating f, we have a sequence of inclusions

$$\pi_1(U_t) \subset f_{t*}^k\pi_1(X_t) \subset \cdots \subset f_{t*}\pi_1(X_t) \subset \pi_1(X_t).$$

However, the mapping degree of the power $f_t^k\colon X_t \to X_t$ and the index of the subgroup $f_{t\,*}^k\pi_1(X_t) \subset \pi_1(X_t)$ coincide with $k \deg f > 1$. Since the index of the subgroup $\pi_1(U_t)$ in $\pi_1(X_t)$ is finite, we have a contradiction. □

Corollary 3.9. *Suppose that the Iitaka fibration $\varphi\colon X \to C$ is an imprimitive abelian fibration. Let $\widetilde{H}$ be the variation of Hodge structure $R_1\varphi_*\mathbb{Z}_X|_{C^\star}$ defined on $C^\star = C \setminus \Delta_\varphi$. Then there is uniquely a variation of Hodge substructure $H \subset \widetilde{H}$ of rank two such that the stalk H_t is contained in the kernel of $\mathrm{H}_1(X_t, \mathbb{Z}) = \pi_1(X_t) \to \pi_1(X)$ and $f_{t*}^{-1}H_t = H_t$ for any $t \in C^\star$.*

Proof. Lemma 3.8 implies that the Hodge substructure H_t of $\mathrm{H}_1(X_t, \mathbb{Z})$ contained in the kernel of $\pi_1(X_t) \to \pi_1(X)$ is uniquely determined. In particular, $f_{t*}^{-1}H_t = H_t$ for the endomorphism f over C. Since the Hodge substructure is preserved by the action of monodromy, it defines a variation of Hodge substructure $H \subset \widetilde{H}$ over $C^\star$. □

Suppose that the Iitaka fibration $\varphi\colon X \to C$ is a primitive abelian fibration. By Theorem 2.9 and Lemma 2.1, there is a finite morphism $\tau\colon \widetilde{C} \to C$ from a smooth curve $\widetilde{C}$ of genus ≥ 2 such that the normalization $\widetilde{X}$ of $X \times_C \widetilde{C}$ is smooth over $\widetilde{C}$. A nontrivial surjective endomorphism f of X satisfying $\varphi \circ f = f$ induces a nontrivial surjective endomorphism $\tilde{f}$ of $\widetilde{X}$ over $\widetilde{C}$.

4. The primitive case

In this section, we shall prove Main Theorem in the primitive case, i.e., the case where X is a smooth projective 3-fold admitting a nontrivial surjective endomorphism with $\kappa(X) = 1$ such that the Iitaka fibration $X \to C$ is a primitive abelian fiber space. We fix a smooth minimal model $Y = X_{\min}$ of X with a minimal reduction $g = f_{\min}\colon Y \to Y$ of powers of f. For the Iitaka fibration $\varphi_Y\colon Y \to C$, we assume that $\varphi_Y \circ g = g$ (cf. Section 3.3).

4.1. **The case of simple abelian fibration.** Suppose that the Iitaka fibration $\varphi\colon X \to C$ is a simple abelian fibration. Then (A) ⇒ (B) in Main Theorem in this case is derived from Lemma 2.4 and:

Theorem 4.1. *Let X be a smooth projective 3-folds of $\kappa(X) = 1$ admitting a nontrivial surjective endomorphism. If the Iitaka fibration $\varphi\colon X \to C$ is a simple abelian fibration, then X is minimal and φ is a Seifert fibration.*

Proof. By Corollary 2.11 and Lemma 2.4, we infer that φ_Y is a Seifert fibration. In particular, for a finite ramified covering $\widetilde{C} \to C$, the normalization $\widetilde{Y}$ of $Y \times_C \widetilde{C}$ is smooth over $\widetilde{C}$ and is étale over Y. Here, $\widetilde{Y} \to \widetilde{C}$ is a smooth abelian fibration whose very general fiber is a simple abelian surface. Since $\varphi_Y \circ g = g$, there exists a nontrivial surjective endomorphism $\tilde{g}\colon \widetilde{Y} \to \widetilde{Y}$ with $g \circ \tau = \tau \circ \tilde{g}$ for the étale covering $\tau\colon \widetilde{Y} \to Y$. Therefore, any fiber of $\widetilde{Y} \to \widetilde{C}$ does not contain any elliptic curve $\widetilde{E}$ with $\tilde{g}^{-1}(\widetilde{E}) = \widetilde{E}$ by Theorem 2.24. The birational morphism $\Psi\colon X = X_0 \to X_1 \to \cdots \to X_n = Y$ explained in Section 3.2 is a succession of blowups along elliptic curves contained in fibers over C. However, every fiber of $Y \to C$ does not contain any elliptic curve E with $g^{-1}E = E$. In fact, the pullback of the elliptic curve by the étale morphism τ is a union of elliptic curves which are preserved by a power of $\tilde{g}$. Therefore, $X \simeq Y$. □

4.2. **The case of non-simple abelian fibration.** Suppose next that the Iitaka fibration $\varphi\colon X \to C$ is a non-simple abelian fibration.

Proposition 4.2. *Suppose that φ_Y is a smooth non-simple abelian fiber space. If X is not minimal, then $\varphi = \beta \circ \alpha$ for elliptic fibrations $\alpha\colon X \to S$ and $\beta\colon S \to C$ satisfying the following properties*:

(1) *$\alpha\colon X \to S$ is an elliptic bundle over a smooth projective surface with $\kappa(S) = 1$;*
(2) *$\beta\colon S \to C$ is an elliptic fibration whose relative minimal model is an elliptic bundle over C;*
(3) *$\alpha \circ f^k = v \circ \alpha$ for an automorphism v of S and for a positive integer k.*

Proof. We replace f freely with a power f^k of f. Let $\mu_i\colon X_i \to X_{i+1}$ for $0 \le i \le n-1$ and $f_i\colon X_i \to X_i$ for $0 \le i \le n$ be the blowups and endomorphisms explained in Section 3.2 for the minimal reduction of f. Note that the center C_i of μ_{i-1} is an elliptic curve contained in a fiber of $X_i \to C$ by Lemma 3.6. Applying Theorem 2.23 to $X_n = Y \to C$, $f_n = g$, and to the elliptic curve $C_n \subset X_n$, we have a factorization $X_n \to S_n \to C$ such that $\alpha_n\colon X_n \to S_n$ and $\beta_n\colon S_n \to C$ are smooth elliptic fibrations, and that C_n is a fiber of α_n. Moreover $\alpha_n \circ f_n = v_n \circ \alpha_n$ for an automorphism $v_n \in \mathrm{Aut}(S_n)$ fixing the point $b_n = \alpha_n(C_n)$.

For the blowing up $S_{n-1} \to S_n$ at b_n, the induced rational map $\alpha_{n-1}\colon X_{n-1} \to S_{n-1}$ is also a smooth elliptic fibration and the induced birational map $v_{n-1}\colon S_{n-1} \to S_{n-1}$ by v_n is also holomorphic. Then $\alpha_{n-1} \circ f_{n-1} = v_{n-1} \circ \alpha_{n-1}$. If C_{n-1} is not contained in a fiber of α_{n-1}, then we have a contradiction concerning with the degree of $C_{n-1} \to \alpha_{n-1}(C_{n-1})$ as in the proof of Lemma 3.6. Thus C_{n-1} is a fiber of α_{n-1} and the point $b_{n-1} = \alpha_{n-1}(C_{n-1})$ is fixed by v_{n-1}. By continuing the same argument, we have a smooth elliptic fibration

$\alpha\colon X = X_0 \to S$, a birational morphism $S \to S_n$, and an automorphism $v \in \mathrm{Aut}(S)$ such that $\alpha \circ f = v \circ \alpha$. □

In the primitive non-simple case, Main Theorem is derived from:

Theorem 4.3. *Let X be a smooth projective 3-fold of $\kappa(X) = 1$ admitting a nontrivial surjective endomorphism. Suppose that the Iitaka fibration $\varphi\colon X \to C$ is a primitive non-simple abelian fiber space.*

(1) *If X is minimal, then $\varphi\colon X \to C$ is a Seifert abelian fibration. Furthermore, there exist a non-simple abelian surface A and a finite ramified Galois covering $\widetilde{C} \to C$ such that the normalization of $X \times_C \widetilde{C}$ is étale over X and is isomorphic to $A \times \widetilde{C}$ over $\widetilde{C}$.*

(2) *If X is not minimal, then there exist a smooth projective surface S of $\kappa(S) = 1$, an elliptic curve E, and a finite étale Galois covering $\tau\colon S \times E \to X$ such that the action of the Galois group of τ on $S \times E$ is compatible with the projection $S \times E \to S$.*

Proof. (1) follows from Corollary 2.11 and Lemma 2.21.

(2): Let $\Psi\colon X \to Y$ be the birational morphism giving the minimal reduction $g\colon Y \to Y$ of a nontrivial surjective endomorphism f of X. Here, $\varphi_Y \circ g = \varphi_Y$. By (1), there is a finite Galois covering $\widetilde{C} \to C$ such that the normalization $\widetilde{Y}$ of $Y \times_C \widetilde{C}$ is smooth over $\widetilde{C}$ and is étale over Y. Note that the Galois group $G = \mathrm{Gal}(\widetilde{Y}/Y)$ is isomorphic to $\mathrm{Gal}(\widetilde{C}/C)$. Thus, for the abelian fibration $\tilde{\varphi}\colon \widetilde{Y} \to \widetilde{C}$, we have $\sigma \circ \tilde{\varphi} = \tilde{\varphi} \circ \sigma$ for $\sigma \in G$. Let $\tilde{g}$ be the induced endomorphism of $\widetilde{Y}$ from $g \times_C \mathrm{id}_{\widetilde{C}}$. Then $\sigma \circ \tilde{g} = \tilde{g} \circ \sigma$ for any $\sigma \in G$. Let $\widetilde{X} \to X$ be the pullback of the étale Galois covering $\widetilde{Y} \to Y$ by $\Psi\colon X \to Y$. Then $\tilde{g}$ induces a nontrivial surjective endomorphism $\tilde{f}$ of $\widetilde{X}$, and $\tilde{g}$ is regarded as the minimal reduction of $\tilde{f}$. By Proposition 4.2, there exist a smooth elliptic fibration $\alpha\colon \widetilde{X} \to S$ and an automorphism $v \in \mathrm{Aut}(S)$ such that $\alpha \circ \tilde{f} = v \circ \alpha$. Since $\sigma \circ \tilde{f} = \tilde{f} \circ \sigma$ for $\sigma \in G$, the condition (B) of Main Theorem is satisfied by Proposition 2.29. □

5. The imprimitive case

In this section, we treat the imprimitive case. Before proving Main Theorem, we prepare some results on non-Seifert elliptic surfaces in Section 5.1.

5.1. **Remarks on non-Seifert elliptic surfaces.** Let $S \to C$ be a minimal elliptic fibration over a smooth projective curve. Suppose that $S \to C$ is not a Seifert elliptic fibration. Then any surjective étale endomorphism of S is an automorphism (cf. [2]). In fact, the existence of a nontrivial surjective étale endomorphism implies that $\chi_{\mathrm{top}}(S) = 0$, but $\chi_{\mathrm{top}}(S) \neq 0$ for any non-Seifert elliptic surface S. We also have the following results on the automorphism group of the non-Seifert elliptic surface.

Theorem 5.1. *Let $f\colon S \to C$ be a non-Seifert projective elliptic surface with $\kappa(S) \geq 0$. Then $\mathrm{Aut}^0(S)$ is trivial.*

For the proof, we recall the following:

Proposition 5.2 (cf. [7])**.** *Let X be a normal compact complex space. Suppose that there exists a compact complex Lie subgroup G of $\mathrm{Aut}^0 X$. Then, for every $x \in X$, the orbit map $\varphi_x\colon G \to X$ defined by $\sigma \mapsto \sigma \cdot x$ for $\sigma \in G$ is a finite morphism.*

Proof. Assume the contrary. Then there exists a point $x_0 \in X$ such that the isotropy subgroup $T := G_{x_0}$ of G at x_0 is positive-dimensional. Let $\psi\colon T \times X \to X$ be the evaluation map defined by $(t, x) \mapsto t \cdot x$ for $t \in T$, $x \in X$. Then $\psi(p^{-1}(x_0)) = \{x_0\}$ for the second projection $p\colon T \times X \to X$. Since T is compact, the rigidity lemma implies that $\psi(p^{-1}(x))$ is a point for any $x \in X$. Hence ψ factors through X, and T is zero-dimensional. This is a contradiction. □

Corollary 5.3. $\dim \mathrm{Aut}^0(X) \leq \dim X$ *for any normal compact complex space X in the class $\mathcal{C}$ with $\kappa(X) \geq 0$. If the equality holds, then X has a complex torus as its finite unramified covering.*

Proof. By [4], $\mathrm{Aut}^0(X)$ is a complex torus and hence is compact. □

Proof of Theorem 5.1. Assume the contrary. By Corollary 5.3, $E := \mathrm{Aut}^0(S)$ is at most 2-dimensional. Moreover, if $\dim E = 2$, then S is covered by a 2-dimensional complex torus and contains no rational curves. This is a contradiction. Hence E is an elliptic curve. By [4, (5.1)], there exists a natural complex space structure on the orbit space $V := S/E$ such that the natural projection $p\colon S \to V$ gives a Seifert elliptic fibration. Thus $\chi_{\mathrm{top}}(S) = 0$. This is a contradiction. □

Lemma 5.4. *Let X_1 and X_2 be smooth projective varieties and let f be a surjective endomorphism of $X_1 \times X_2$. Assume that the following conditions are satisfied:*

(1) *$p_1 \circ f = f_1 \circ p_1$ for the first projection p_1 and for an endomorphism f_1 of X_1.*
(2) *Any surjective endomorphism of X_2 is an automorphism.*
(3) *$\mathrm{Aut}^0(X_2)$ is trivial.*

Then $f = f_1 \times f_2$ for an automorphism f_2 of X_2.

Proof. By (1), f is written as

$$X_1 \times X_2 \ni (x, y) \mapsto (f_1(x), \Phi_x(y)),$$

where Φ_x is a surjective endomorphism of X_2. Then Φ_x is an automorphism by (2). The map $x \mapsto \Phi_x$ gives rise to a holomorphic map $X_1 \to \mathrm{Aut}(X_2)$, which is constant by (3). Thus $f = f_1 \times f_2$ for an automorphism $f_2 \in \mathrm{Aut}(X_2)$. □

Corollary 5.5. *Let $S \to C$ be a non-Seifert elliptic surface with $\kappa(S) \geq 0$. Let f be a surjective endomorphism of $Z \times S$ for a smooth projective variety Z such that $p_1 \circ f = f_1 \circ p_1$ for the first projection p_1 and for an endomorphism f_1 of Z. Then $f = f_1 \times f_2$ for an automorphism $f_2 \in \mathrm{Aut}(S)$.*

5.2. **Structure of an H-factorization.** The following result is not related to the existence of nontrivial surjective endomorphisms:

Theorem 5.6. *Let X be a smooth minimal projective 3-fold of $\kappa(X) = 1$ whose Iitaka fibration $\varphi\colon X \to C$ is an imprimitive abelian fibration. Let $C^\star$ be the complement of the discriminant locus $\Delta_\varphi \subset C$ of φ and let H be the variation of Hodge substructure of $R_1\varphi_*\mathbb{Z}_X|_{C^\star}$ defined in* Corollary 3.9. *Then there exist equi-dimensional elliptic fibrations $\pi\colon Y \to T$ and $q\colon T \to C$ for a smooth minimal projective 3-fold Y and for a normal projective surface T satisfying the following conditions*:

(1) *$K_Y \sim_{\mathbb{Q}} \pi^*(K_T + \Lambda)$ for a $\mathbb{Q}$-divisor Λ with (T, Λ) log-terminal.*
(2) *π is a non-Seifert elliptic fibration.*
(3) *$\varphi\colon X \to C$ and $q \circ \pi\colon Y \to C$ are birationally equivalent to each other over C. Moreover, these are isomorphic to each other over $C^\star$.*
(4) *$Y \to T \to C$ gives the H-factorization of φ* (cf. Proposition 2.20).

Here, the surface T above is uniquely determined up to isomorphism.

Proof. Let us take an H-factorization $X \dashrightarrow S \to C$ of φ. Note that $X \dashrightarrow S$ and $S \to C$ are smooth elliptic fibrations over the open subset $C^\star$. Thus $\widehat{X} \to S$ is an elliptic fibration for a certain blowing up $\widehat{X} \to X$. Let us consider the equi-dimensional model of a relative minimal model of $\widehat{X} \to S$ (cf. [18, Appendix A, Proposition A.6]). Then we have a birational morphism $S' \to S$ from a normal variety and a birational map $X' \dashrightarrow \widehat{X}$ such that

- X' is $\mathbb{Q}$-factorial with only terminal singularities,
- the induced map $h'\colon X' \to S'$ is an equi-dimensional elliptic fibration,
- $K_{X'} \sim_{\mathbb{Q}} h'^*(K_{S'} + D')$ for a $\mathbb{Q}$-divisor with (S', D') log-terminal.

By the same argument as in *Step* 2 of the proof of [18, Theorem B2], we have a birational morphism $S' \to T$ into a normal surface and a birational map $X' \dashrightarrow Y$ such that

- Y is $\mathbb{Q}$-factorial with only terminal singularities,
- the induced map $\pi\colon Y \to T$ is an equi-dimensional elliptic fibration,

- $K_Y \sim_{\mathbb{Q}} \pi^*(K_T + \Lambda)$ for a $\mathbb{Q}$-divisor Λ with (T, Λ) log-terminal,
- K_Y is nef.

Since X and Y are connected by a finite sequence of flops, Y is also smooth. The Iitaka fibration of Y is the composite $q \circ \pi$ for a morphism $q\colon T \to C$. Since $X \times_C C^\star \simeq Y \times_C C^\star$ and $S \times_C C^\star \simeq T \times_C C^\star$, the required conditions except for (2) are satisfied for $Y \to T$ and $T \to C$. By a property of H-factorization, $\pi_1(Y_t) \to \pi_1(Y) \simeq \pi_1(X)$ is zero for the fiber Y_t over any point $t \in T \times_C C^\star$. In particular, $\pi\colon Y \to T$ is non-Seifert, and the condition (2) is also satisfied. It remains to show the uniqueness of T. Let $\pi'\colon Y' \to T'$ and $q'\colon T' \to C$ be equi-dimensional elliptic fibrations satisfying the same conditions. By the construction of H-factorization, there is a birational map $T \dashrightarrow T'$ over C, which commutes with the birational map $Y \dashrightarrow X \dashrightarrow Y'$. Since the birational map $Y \dashrightarrow Y'$ is an isomorphism in codimension one, and since π and π' are equi-dimensional, $T \dashrightarrow T'$ is also an isomorphism in codimension one. Thus $T \simeq T'$ by the Zariski main theorem (cf. [18, Appendix A. Remark A.7]). □

In the rest of Section 5.2, we fix a smooth minimal projective 3-fold X admitting a nontrivial surjective endomorphism f such that the Iitaka fibration $\varphi\colon X \to C$ of X is an imprimitive abelian fibration over a curve C. By replacing f with its power, we assume that $\varphi \circ f = \varphi$. Let $\pi\colon Y \to T$ and $q\colon T \to C$ be the elliptic fibrations satisfying the conditions of Theorem 5.6 for X.

Proposition 5.7. *There is a nontrivial surjective endomorphism β of T over C such that β is étale in codimension one and $\pi \circ f = \beta \circ \pi$. Moreover, there is a finite Galois covering $\widetilde{C} \to C$ satisfying the following properties*:

(1) *The normalization $\widetilde{T}$ of $T \times_C \widetilde{C}$ is isomorphic over $\widetilde{C}$ to the product $E \times \widetilde{C}$ for an elliptic curve E, and $\widetilde{T} \to T$ is étale in codimension one.*

(2) *Let $\widetilde{Y}$ be the normalization of $Y \times_C \widetilde{C}$. Then $\widetilde{Y} \to Y$ is étale. Furthermore, the induced elliptic fibration $\tilde{\pi}\colon \widetilde{Y} \to \widetilde{T}$ is relatively minimal, the discriminant locus $\Delta_{\tilde{\pi}}$ is a non-empty subset of $E \times (\widetilde{C} \setminus \widetilde{C}^\star)$, where $\widetilde{C}^\star = \widetilde{C} \times_C C^\star$, and the singular fiber type of $\tilde{\pi}$ over any component of $\Delta_{\tilde{\pi}}$ is not of type ${}_m\mathrm{I}_0$.*

(3) *The lift $\tilde{\beta}$ of β to $\widetilde{T}$ is written as $\phi \times \mathrm{id}_{\widetilde{C}}$ as an endomorphism of $E \times \widetilde{C}$ for an endomorphism ϕ of E.*

Proof. Step 1: *Existence of $\beta\colon T \to T$.*

Let $\phi\colon X \dashrightarrow Y$ be the birational map over C. Then an isomorphism $\phi_*\colon \pi_1(X) \simeq \pi_1(Y)$ is induced. Let $f^{(1)}\colon Y^{(1)} \to Y$ be the finite étale covering corresponding the image of $\phi_* \circ f_*\colon \pi_1(X) \to \pi_1(X) \simeq \pi_1(Y)$. Then $\phi \circ f = f^{(1)} \circ \phi^{(1)}$ for a birational map

$\phi^{(1)}\colon X \cdots\to Y^{(1)}$. Let us consider the Stein factorization

$$Y^{(1)} \xrightarrow{\pi^{(1)}} T^{(1)} \xrightarrow{\beta^{(1)}} T$$

of $\pi \circ f^{(1)}$. Then $\pi^{(1)}$ is an elliptic fibration birational to π by $\phi^{(1)} \circ \phi^{-1}\colon Y \cdots\to Y^{(1)}$, since the variation of Hodge substructure H is preserved by the induced homomorphism $f_*\colon \widetilde{H} \to \widetilde{H}$. Let $\psi^{(1)}\colon T \cdots\to T^{(1)}$ be the birational map. Since π and $\pi^{(1)}$ are equi-dimensional and since $\phi^{(1)} \circ \phi^{-1}$ is isomorphic in codimension one, $\psi^{(1)}$ is an isomorphism by the Zariski main theorem. Therefore, we have the following commutative diagram of rational maps:

$$\begin{array}{ccccccc}
X & \xrightarrow{\phi} & Y & \xrightarrow{\pi} & T & \xrightarrow{q} & C \\
\| & & \downarrow & & \downarrow{\scriptstyle \psi^{(1)}} & & \| \\
X & \xrightarrow{\phi^{(1)}} & Y^{(1)} & \xrightarrow{\pi^{(1)}} & T^{(1)} & \xrightarrow{q^{(1)}} & C \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{(1)}} & & \downarrow{\scriptstyle \beta^{(1)}} & & \| \\
X & \xrightarrow{\phi} & Y & \xrightarrow{\pi} & T & \xrightarrow{q} & C.
\end{array}$$

Thus we have a surjective endomorphism $\beta = \beta^{(1)} \circ \psi^{(1)}\colon T \to T$. Assume that β is an isomorphism. Let γ be a rational curve contained in the fiber of π over a general point of Δ_π. Such a rational curve γ exists since π is non-Seifert. Then $(f^{(1)})^{-1}\gamma$ is a reducible curve consisting of rational components by $\deg f > 1$. Since Δ_π has finitely many components, we have a contradiction. Hence, β is not an isomorphism.

Step 2: *Any fiber of* $q\colon T \to C$ *is irreducible and* β *is étale in codimension one.*

Let F be a fiber of q. By replacing β with a power of β, we may assume that $\beta^*\gamma = \gamma$ for any irreducible component γ of F. Hence $(\deg \beta - 1)\gamma^2 = 0$. Therefore, $\gamma^2 = 0$ and F is irreducible. As a consequence, we infer that the canonical divisor K_T is $\mathbb{Q}$-linearly equivalent to q^*B for a $\mathbb{Q}$-divisor B on T. On the other hand, $K_T \sim_{\mathbb{Q}} \beta^*K_T + R$ for the ramification divisor R of β. Then $R = 0$ since $\beta^*q^*B = q^*B$. Therefore, β is étale in codimension one.

Step 3: T *is the quotient surface of a smooth elliptic surface by a finite group along any singular fiber.*

For the fiber $F = q^{-1}(o)$ over a point $o \in C$, let m be the multiplicity, i.e., $F = mF_{\mathrm{red}}$. Let $\mathcal{U} \subset C$ be an analytic open neighborhood biholomorphic to a unit disc $\boldsymbol{\Delta} = \{z \in \mathbb{C}; |z| < 1\}$. Let $\mathcal{U}' \simeq \boldsymbol{\Delta} \to \mathcal{U}$ be the cyclic covering given by $z' \mapsto z = z'^m$ and let $\mathcal{V}'$ be the normalization of $q^{-1}(\mathcal{U}) \times_{\mathcal{U}} \mathcal{U}'$. Then $\mathcal{V}' \to q^{-1}(\mathcal{U})$ is étale in codimension one and the fiber F' of $\mathcal{V}' \to \mathcal{U}'$ over the origin is reduced. In particular, $\mathcal{V}'$ has only quotient singularities. A nontrivial surjective endomorphism β' of $\mathcal{V}'$ over $\mathcal{U}'$ is induced from β. By the same argument as in *Step* 2, we infer that F' is irreducible and reduced.

Suppose that F' is singular. Then F' is a rational curve of arithmetic genus one. Since $\mathcal{V}'$ is nonsingular outside $\operatorname{Sing} F'$, $\beta'^{-1}(F' \setminus \operatorname{Sing} F') \to F' \setminus \operatorname{Sing} F'$ is étale. Hence, F' has no cusp but a node $P = \operatorname{Sing} F'$. Here, $\beta'^{-1}(P) = P$, since $\pi_1(F' \setminus \{P\}) \simeq \mathbb{Z}$. Let $\{\mathcal{V}_\alpha\}$ be a fundamental system of open neighborhoods of P in $\mathcal{V}'$. Then $\{\beta'^{-1}(\mathcal{V}_\alpha)\}$ is also a fundamental system of open neighborhoods of $\beta'^{-1}(P) = P$. The natural injection

$$\beta'_*\colon \pi_1(\beta'^{-1}(\mathcal{V}_\alpha) \setminus \{P\}) \hookrightarrow \pi_1(\mathcal{V}_\alpha \setminus \{P\})$$

is not an isomorphism since $\deg \beta' = \deg \beta > 1$. However, the both sides of the injection β'_* is isomorphic to the local fundamental group at P for some α. Since $(\mathcal{V}, P)$ is a quotient singularity, the local fundamental group is finite, and hence, β'_* is isomorphic. This is a contradiction.

Therefore, F' is nonsingular. Thus $\mathcal{V}'$ is also nonsingular and $\mathcal{V}' \to \mathcal{U}'$ is a smooth elliptic surface; thus T is the quotient of a smooth elliptic surface along F.

Step 4: *There is a finite covering* $\widetilde{C} \to C$ *satisfying the property* (1).

For a point $o \in C \setminus C^\star$, the local monodromy of $R_1 q_* \mathbb{Z}_T|_{C^\star}$ around o is finite by *Step* 3. In particular, the J-function associated with the elliptic surface q is constant and the image of the monodromy representation $\rho\colon \pi_1(C^\star) \to \mathrm{SL}(2, \mathbb{Z})$ is finite. Let $\tau_1^\star\colon C_1^\star \to C^\star$ be the finite étale Galois covering associated with the kernel of ρ, and let the finite Galois covering $\tau\colon C_1 \to C$ of smooth projective curves be the natural extension of $\tau^\star$. For the normalization T_1 of $T \times_C C_1$, the projection $q_1\colon T_1 \to C_1$ is an elliptic surface with trivial local monodromies and with constant period. Thus the relative minimal model of q_1 has only singular fibers of type ${}_m\mathrm{I}_0$. By the local description of singular fibers of q in *Step* 3, we infer that T_1 is nonsingular and q_1 is the relative minimal model. By Corollary 2.6, q_1 is a Seifert elliptic surface. Thus we have an expected finite covering $\widetilde{C} \to C$.

Step 5. *The rest of the proof.*

The property (2) for $\widetilde{C} \to C$ in *Step* 4 is derived from Theorem 5.6, (2), and Corollary 2.6. If we do not consider the Galois property, then, by Lemma 2.25, we can find such a finite covering $\widetilde{C} \to C$ satisfying also the property (3) by taking a further finite étale covering. Even in case $\widetilde{C} \to C$ is not Galois, the Galois closure satisfies all the required properties (1)–(3). □

Corollary 5.8. $X \simeq Y$.

Proof. The elliptic fibration $\tilde{\pi}\colon \widetilde{Y} \to \widetilde{T}$ is a unique relative minimal model by [18, §5.3], since the discriminant locus of $\tilde{\pi}$ is nonsingular. In particular, there is no irreducible curve in $\widetilde{Y}$ giving a flop. Let $\widetilde{X} \to X$ be the étale covering corresponding to the subgroup $\pi_1(\widetilde{Y}) \subset \pi_1(Y) \simeq \pi_1(X)$. Then $\widetilde{X}$ and $\widetilde{Y}$ are nonsingular relative minimal model over $\widetilde{C}$, which are connected by a sequence of flops. Thus $\widetilde{X} \simeq \widetilde{Y}$ and $X \simeq Y$. □

Lemma 5.9. *In the situation of* Proposition 5.7, *let G be the Galois group of the covering $\widetilde{C} \to C$. Then the induced action of G on $\widetilde{T} \simeq E \times \widetilde{C}$ is expressed as a diagonal action, i.e., $\sigma \in G$ acts on $E \times \widetilde{C}$ as*

$$(x, y) \mapsto (\sigma \cdot x, \, \sigma \cdot y)$$

for a suitable action of G on E.

Proof. We fix an abelian group structure of E. Then $\sigma \in G$ acts as

$$(x, y) \mapsto \big(a_\sigma(x + f_\sigma(y)), \, \sigma \cdot y\big)$$

for a root a_σ of unity and a holomorphic map $f_\sigma \colon \widetilde{C} \to E$, since the action of G on $E \times \widetilde{C}$ is compatible with the second projection. Here, $\sigma \mapsto a_\sigma$ gives rise to a homomorphism $G \to \mathbb{C}^\star$. By Proposition 5.7, (3), the action of any $\sigma \in G$ commutes with $\phi \times \mathrm{id}_{\widetilde{C}}$. It implies that

$$\phi(f_\sigma(y)) = f_\sigma(y).$$

Hence, f_σ is a constant map for any σ. Thus G acts diagonally on $E \times \widetilde{C}$. □

Theorem 5.10. *Let $\tilde{\pi} \colon \widetilde{X} \simeq \widetilde{Y} \to \widetilde{T}$ be the elliptic fibration in* Proposition 5.7. *Then the composite of $\tilde{\pi}$ and the first projection $\widetilde{T} \simeq E \times \widetilde{C} \to E$ is a holomorphic fiber bundle. Moreover, there exist a non-Seifert minimal elliptic fibration $S \to \widetilde{C}$ and a finite étale covering $\nu \colon E' \to E$ satisfying the following conditions*:

1. *The fiber product $\widetilde{X}' = E' \times_E \widetilde{X}$ is isomorphic to $E' \times S$ over E'.*
2. *The endomorphism ϕ of E lifts to an endomorphism ϕ' of E' with $\nu \circ \phi' = \phi \circ \nu$.*
3. *The composite $\widetilde{X}' \to \widetilde{X} \to X$ is a Galois covering.*
4. *The endomorphism of $\tilde{f}$ of $\widetilde{X}$ lifts to an endomorphism of $\widetilde{X}' \simeq E' \times S$ which is written as $\phi' \times v$ for an automorphism v of S.*
5. *The Galois group* $\mathrm{Gal}(\widetilde{X}'/X)$ *acts on S and the projection $\widetilde{X}' \to S$ is equivariant.*

In particular, X satisfies the condition (B) *of* Main Theorem.

Proof. In *Step* 2 below, we shall prove (1)–(4), while in *Step* 1, we consider a special case where $\widetilde{C} \to C$ is isomorphic. The remaining (5) is proved in *Step* 3.

Step 1. *The case where the identity mapping $\widetilde{C} = C \to C$ satisfies all the properties of* Proposition 5.7: The variation of Hodge structure $H(\pi) = R_1\pi_*\mathbb{Z}_Y|_{T^\star}$ defined over $T^\star = q^{-1}(C^\star) \simeq E \times C^\star$ is isomorphic to the pullback q^*H. Here, the local system H is not trivial on $C^\star$ by Proposition 5.7, (2). Thus, $\mathrm{H}^0(C^\star, H) = 0$ by [19, Corollary 4.2.5]. The subgroups $\boldsymbol{\mu}_m := m^{-1}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ form an inductive system, and we have

$$\varinjlim_{m\to\infty} \mathrm{H}^0(C^\star, H \otimes \boldsymbol{\mu}_m) \simeq \mathrm{H}^1(C^\star, H)_{\mathrm{tor}},$$

where the right hand side is a finite abelian group. We have an isomorphism

$$\mathrm{H}^1(T^\star, H(\pi) \otimes \boldsymbol{\mu}_m) \simeq \left(\mathrm{H}^1(C^\star, H \otimes \boldsymbol{\mu}_m) \otimes \mathrm{H}^0(E, \mathbb{Z})\right) \oplus \left(\mathrm{H}^0(C^\star, H \otimes \boldsymbol{\mu}_m) \otimes \mathrm{H}^1(E, \mathbb{Z})\right).$$

Let E' be a copy of E and let $\nu\colon E' \to E$ be the multiplication map by an integer N which is divisible by the order of $\mathrm{H}^1(C^\star, H)_{\mathrm{tor}}$. Then the natural homomorphism

$$\mathrm{id} \otimes \nu^*\colon\ \mathrm{H}^1(C^\star, H)_{\mathrm{tor}} \otimes \mathrm{H}^1(E, \mathbb{Z}) \to \mathrm{H}^1(C^\star, H)_{\mathrm{tor}} \otimes \mathrm{H}^1(E', \mathbb{Z})$$

is zero. Therefore, by [19, Theorem 6.2.9] and by [19, Theorem 6.3.8], there is a minimal elliptic surface S over C such that $E' \times_E X$ is birational to $E' \times S$ over $E' \times C$. Here, $S \to C$ is not Seifert since H is not trivial. There is an isomorphism $E' \times_E X \simeq E' \times S$ since both are relatively minimal over $E' \times C$. Thus the condition (1) is satisfied. The condition (2) is satisfied for the copy ϕ' of ϕ. In fact, ϕ is an endomorphism preserving the group scheme structure of E, hence ϕ commutes with the multiplication maps. The condition (3) is trivial now, and $\mathrm{Gal}(\widetilde{X}'/X) \simeq \mathrm{Gal}(E'/E) \simeq (\mathbb{Z}/N\mathbb{Z})^{\oplus 2}$. A nontrivial surjective endomorphism of $E' \times_E X$ is induced from $\phi' \times f$. Thus, the condition (4) follows from Corollary 5.5.

Step 2. *General case*: Applying *Step* 1 to the situation $\widetilde{X} \to \widetilde{T} \to \widetilde{C}$, we can prove all the properties except for (3) and (5). We note the following exact sequence for the multiplication map $\nu\colon E' = E \to E$ by N:

$$1 \to \mathrm{Gal}(E'/E) \to \mathrm{Aut}(E') \to \mathrm{Aut}(E) \to 1.$$

Here, $\mathrm{Aut}(E) \simeq \mathrm{Aut}(E, 0) \ltimes E$ for the finite group $\mathrm{Aut}(E, 0)$ preserving the zero element $0 \in E$, and $\mathrm{Aut}(E') \to \mathrm{Aut}(E)$ is expressed as

$$\mathrm{Aut}(E, 0) \ltimes E \ni (a, x) \mapsto (a, Nx).$$

The Galois group G of $\widetilde{C} \to C$ acts on E by Lemma 5.9, and hence, the fiber bundle $\widetilde{X} \to E$ is G-equivariant. For the induced homomorphism $G \to \mathrm{Aut}(E)$, let $G' \to \mathrm{Aut}(E')$ be the pullback by $\mathrm{Aut}(E') \to \mathrm{Aut}(E)$. Hence, we have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}(E'/E) & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Gal}(E'/E) & \longrightarrow & \mathrm{Aut}(E') & \longrightarrow & \mathrm{Aut}(E) & \longrightarrow & 1
\end{array}$$

of exact sequences. Then G' acts on $E' \times \widetilde{X}$ and also on $\widetilde{X}' = E' \times_E \widetilde{X}$. Moreover, the quotient space of $\widetilde{X}'$ by G' is just X; hence, $\widetilde{X}' \to X$ is Galois.

Step 3. *Proof of* (5): The first projection $E' \times S \to E'$ is equivariant with respect to the action of $G' = \mathrm{Gal}(\widetilde{X}'/X)$ on $E' \times S$ and on E'. Thus G' acts diagonally on $E' \times S$ by Corollary 5.5. Thus we are done. □

5.3. **The proof of Main Theorem.**

Lemma 5.11. *Let E be an elliptic curve and S a smooth projective surface. Let ϕ be a nontrivial surjective endomorphism of E and v an automorphism of S. If $C \subset E \times S$ is an irreducible curve with $f^{-1}C = C$ for the endomorphism $f = \phi \times v$, then C is an elliptic curve written as $C = E \times \{s\}$ for a point $s \in S$ with $v(s) = s$.*

Proof. For the projection $p \colon E \times S \to S$, we have $p(C) = p(f(C)) = v(p(C))$. Thus it suffices to show that $p(C)$ is a point. Assume the contrary. Then, $p(C)$ is a curve and

$$\deg(C/p(C)) = \deg(C/f(C)) \deg(f(C)/p(C)) = \deg(C/f(C)) \deg(C/p(C)).$$

Since $\deg(C/f(C)) = \deg f > 1$, we have a contradiction. □

The proof of Main Theorem is completed by showing:

Theorem 5.12. *Let X be a smooth projective 3-fold of $\kappa(X) = 1$ admitting a nontrivial surjective endomorphism $f \colon X \to X$. Suppose that the Iitaka fibration $X \to C$ is an imprimitive abelian fiber space over a curve. Then a suitable finite étale Galois covering $\widetilde{X}$ of X is isomorphic to the product $E \times S$ of an elliptic curve E and a smooth projective surface S such that*

1. *a power f^k lifts to the endomorphism $\phi \times v$ of $E \times S$ for a nontrivial surjective endomorphism ϕ of E and an automorphism v of S,*
2. *the second projection $\widetilde{X} \to S$ is equivariant with respect to the action of the Galois group $\mathrm{Gal}(\widetilde{X}/X)$ on $\widetilde{X}$.*

In particular, X satisfies the condition (B) *of* Main Theorem.

Proof. By replacing f with a power f^k, we may assume that f is an endomorphism over C. Let $f_n \colon X_n \to X_n$ be the minimal reduction of $f \colon X \to X$ (cf. Section 3.2). By Theorem 5.10, there exist a finite étale Galois covering $\widetilde{X}_n \to X_n$, a lift $\tilde{f}_n$ of f_n as an endomorphism of $\widetilde{X}_n$, an elliptic curve E, and a minimal projective surface S_n such that

- $\widetilde{X}_n \simeq E \times S_n$,
- $\tilde{f}_n \simeq \phi \times v_n$ for a nontrivial surjective endomorphism ϕ of E and an automorphism v_n of S_n,
- the second projection $\widetilde{X}_n \to S_n$ is G-equivariant for the Galois group $G = \mathrm{Gal}(\widetilde{X}_n/X_n)$.

For the sequence $X = X_0 \to X_1 \to \cdots \to X_n$ of blowups in Section 3.2, we set $\widetilde{X}_i := X_i \times_{X_n} \widetilde{X}_n$ for $0 \le i \le n$. Then $\widetilde{X}_0 \to X$ is étale. By Lemma 5.11, the center of the blowing up $\widetilde{X}_{n-1} \to \widetilde{X}_n$ is $E \times Z_n$ for a finite set $Z_n \subset S_n$ fixed by v_n. Thus $\widetilde{X}_{n-1} \simeq E \times S_{n-1}$ for the blowing up $S_{n-1} \to S_n$ along Z_n, and the endomorphism f_{n-1} of X_{n-1} lifts to an endomorphism of $\widetilde{X}_{n-1}$ which is written as $\phi \times v_{n-1}$ for an automorphism v_{n-1} of S_{n-1}. By continuing the same argument above, we have a smooth projective surface S birational to S_n and an automorphism v of S such that $\widetilde{X} = \widetilde{X}_0$ is isomorphic to $E \times S$, and f lifts to an endomorphism of $\widetilde{X}$ written as $\phi \times v$. The G-equivariance of $\widetilde{X} \to S$ follows from Corollary 5.5. □

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