THE ÉTALE THETA FUNCTION AND ITS FROBENIOID-THEORETIC MANIFESTATIONS

Shinichi Mochizuki

September 2006

Abstract. We develop the theory of the *tempered anabelian* and *Frobenioid-theoretic* aspects of the *"étale theta function"*, i.e., the Kummer class of the classical formal algebraic theta function associated to a Tate curve over a nonarchimedean local field. In particular, we consider a certain natural "environment" for the study of the étale theta function, which we refer to as a *"mono-theta environment"*, and show that this mono-theta environment satisfies certain *remarkable rigidity properties* involving *cyclotomes*, *discreteness*, and *constant multiples*, all in a fashion that is *compatible* with the *topology* of the tempered fundamental group and the *extension structure* of the associated tempered Frobenioid.

Contents:

Introduction

The *fundamental goal* of the present paper is to study the *tempered anabelian* [cf. [André], [Mzk14]] and *Frobenioid-theoretic* [cf. [Mzk16], [Mzk17]] aspects of the *theta function of a Tate curve* over a nonarchimedean local field. The motivation for this approach to the theta function arises from the long-term goal of overcoming various obstacles that occur when one attempts to apply the *Hodge-Arakelov theory of elliptic curves* [cf. [Mzk4], [Mzk5]; [Mzk6], [Mzk7], [Mzk8], [Mzk9], [Mzk10]] to the diophantine geometry of elliptic curves over number fields. That is to say, the theory of the present paper is motivated by the expectation that these obstacles may be overcome by *translating* the [essentially] *scheme-theoretic* formulation of Hodge-Arakelov theory into the language of the *geometry of categories* [e.g., the "temperoids" of [Mzk14], and the "Frobenioids" of [Mzk16], [Mzk17]]. In certain respects, this situation is reminiscent of the well-known classical solution to the problem of relating the dimension of the first cohomology group of the structure sheaf of a smooth proper variety in positive characteristic to the dimension of its Picard variety — a problem whose solution remained elusive until the foundations of the algebraic geometry of *varieties* were reformulated in the language of *schemes* [i.e., one allows for the possibility of nilpotent sections of the structure sheaf].

Since Hodge-Arakelov theory centers around the theory of the *theta function* of an elliptic curve with bad multiplicative reduction [i.e., a *"Tate curve"*], it is natural to attempt to begin such a translation by concentrating on such theta functions on Tate curves, as is done in the present paper. Here, we recall that although classically, the arithmetic theory of theta functions on Tate curves is developed in the language of formal schemes in, for instance, [Mumf][cf. [Mumf], p. 289], this theory only addresses the *"slope zero portion"* of the theory — i.e., the portion of the theory that involves the quotient of the fundamental group of the generic fiber of the Tate curve that *extends* to an étale covering in positive characteristic. From this point of view, the relation of the theory of §1, §2 of the present paper to the theory of [Mumf] may be regarded as roughly analogous to the relation of the theory of p-adic uniformizations of hyperbolic curves developed in [Mzk1] to Mumford's theory of Schottky uniformations of hyperbolic curves [cf., e.g., [Mzk1], Introduction, §0.1; cf. also Remark 5.10.2 of the present paper].

Frequently in classical scheme-theoretic constructions, such as those that appear in the *scheme-theoretic* formulation of Hodge-Arakelov theory, there is a tendency to make various *arbitrary choices* in situations where, a priori, some sort of *indeterminacy* exists, without providing any sort of *intrinsic justification* for these choices. Typical examples of such choices involve the *choice* of a particular rational function or section of a line bundle among various possibilities related by a *constant multiple*, or the *choice* of a natural identification between various *"cyclotomes"* [i.e., isomorphic copies of the module of N-th roots of unity, for $N \geq 1$ an integer] appearing in a situation [cf., e.g., [Mzk13], Theorem 4.3, for an example of a crucial *rigidity* result in anabelian geometry concerning this sort of "choice"].

One important theme of the present paper — cf. the theory of [Mzk16], [Mzk17] — is the study of the interaction of *"[tempered] étale-like"* and *"Frobenius-like"* phenomena involving the theta function. Typically, such interaction phenomena revolve around some sort of *"extraordinary rigidity"* [cf. Grothendieck's famous use of this expression in describing his anabelian philosophy]. The various examples of "extraordinary rigidity" that appear in the theory of the étale theta function may be thought of as examples of *intrinsic, category-theoretic "justifications"* for the *arbitrary choices* that appear in classical scheme-theoretic discussions. Such category-theoretic "justifications" depend heavily on the *"proper category-theoretic formulation"* of various scheme-theoretic "venues". In the theory of the present paper, one central such category-theoretic formulation is a mathematical structure that we shall refer to as a *mono-theta environment* [cf. Definition 2.14, (ii)]. A mono-theta environment may be thought of as a sort of *common core* for, or *bridge* between, the [tempered] étale- and Frobenioid-theoretic approaches to the étale theta function [cf. Remarks 2.18.1, 5.10.1, 5.10.2, 5.10.3].

We are now ready to discuss the *main results* of the present paper. These results may be summarized as follows:

> A *mono-theta environment* is a *category/group-theoretic invariant* of both the *tempered étale fundamental group* [cf. Corollary 2.18] associated to [certain coverings of] a punctured elliptic curve over a nonarchimedean local field [satisfying certain properties] and a certain *tempered Frobenioid* [cf. Theorem 5.10, (iii)] associated to such a curve. Moreover, a mono-theta environment satisfies the following *rigidity* properties:
>
> (a) **cyclotomic rigidity** [cf. Theorem 2.15, (ii); Remark 2.15.7];
>
> (b) **discrete rigidity** [cf. Theorem 2.15, (iii); Remarks 2.15.1, 2.15.7];
>
> (c) **constant multiple rigidity** [cf. Theorem 2.15, (iii); Corollary 2.8, (i); Remarks 5.12.3, 5.12.5].
>
> — all in a fashion that is *compatible* with the **topology** of the tempered fundamental group as well as with the **extension structure** of the tempered Frobenioid [cf. Corollary 5.12 and the discussion of the following remarks].

In particular, the phenomenon of *"cyclotomic rigidity"* gives a *"category-theoretic"* explanation for the *special role* played by the *first power* [i.e., as opposed to the M-th power, for $M > 1$ an integer] of [the l-th root, when one works with l-torsion points, for $l \geq 1$ an odd integer, of] the *theta function* [cf. Remark 2.15.5] — a phenomenon which may also be seen in "scheme-theoretic" Hodge-Arakelov theory.

Here, it is important to note that although the above three rigidity properties may be stated and understood to a certain extent *without reference to the Frobenioid-theoretic portion of the theory* [cf. §2], certain aspects of the *interdependence* of these rigidity properties, as well as the meaning of establishing these rigidity properties under the condition of *compatibility* with the *topology of the tempered fundamental group* as well as with the *extension structure of the tempered Frobenioid*, may only be understood in the context of the theory of *Frobenioids* [cf. Corollary 5.12 and the discussion of the following remarks]. Indeed, one important theme of the theory of the present paper — which may, roughly, be summarized as the idea that sometimes

> "*less* (respectively, *more*) data yields *more* (respectively, *less*) information"

[cf. Remark 5.12.8] — is precisely the study of the rather *intricate* way in which these various rigidity/compatibility properties are related to one another.

The contents of the present paper are organized as follows. In §1, we discuss the purely *tempered étale-theoretic anabelian* aspects of the theta function and show, in particular, that the *"étale theta function"* — i.e., the *Kummer class* of the usual formal algebraic theta function — is *preserved* by isomorphisms of the tempered fundamental group [cf. Theorems 1.6; 1.10]. In §2, after studying various coverings and quotient coverings of a punctured elliptic curve, we introduce the notions of a *mono-theta environment* and a *bi-theta environment* [cf. Definition 2.14] and study the *rigidity properties* of these notions [cf. Theorem 2.15]. In §3, we define the *"tempered Frobenioids"* in which we shall develop the Frobenioid-theoretic approach to the étale theta function; in particular, we show that these tempered Frobenioids satisfy various nice properties which allow one to apply the extensive theory of [Mzk16], [Mzk17] [cf. Theorem 3.7; Corollary 3.8; Proposition 3.9]. In §4, we develop *"bi-Kummer theory"* — i.e., a sort of generalization of the "Kummer class associated to a rational function" to the *"bi-Kummer data"* associated to a pair of sections [corresponding to the numerator and denominator of a rational function] of a line bundle — in a *category-theoretic* fashion [cf. Theorem 4.4] for fairly general tempered Frobenioids. Finally, in §5, we specialize the theory of §3, §4 to the case of the *étale theta function*, as discussed in §1. In particular, we observe that a *mono-theta environment* may also be regarded as a mathematical structure naturally associated to a certain tempered Frobenioid [cf. Theorem 5.10, (iii)]. Also, we discuss certain aspects of the *constant multiple rigidity* [as well as, to a lesser extent, of the *cyclotomic* and *discrete rigidity*] of a mono-theta environment that may only be understood in the context of the *Frobenioid-theoretic* approach to the étale theta function [cf. Corollary 5.12 and the discussion of the following remarks].

Acknowledgements:

I would like to thank *Akio Tamagawa*, *Makoto Matsumoto*, *Minhyong Kim*, *Seidai Yasuda*, *Kazuhiro Fujiwara*, and *Fumiharu Kato* for many helpful comments concerning the material presented in this paper.

Section 0: Notations and Conventions

In addition to the *"Notations and Conventions"* of [Mzk16], §0, we shall employ the following "Notations and Conventions" in the present paper:

Ordered Sets:

Let E be a *partially ordered set*. Then [cf. [Mzk16], §0] we shall denote by

$$\mathrm{Order}(E)$$

the category whose *objects* are elements $e \in E$, and whose *morphisms* $e_1 \to e_2$ [where $e_1, e_2 \in E$] are the relations $e_1 \leq e_2$.

A subset $E' \subseteq E$ will be called *orderwise connected* if for every $c \in E$ such $a < c < b$ for some $a, b \in E'$, it follows that $c \in E'$. A partially ordered set which is isomorphic [as a partially ordered set] to an orderwise connected subset of the set of rational integers $\mathbb{Z}$, equipped with its usual ordering, will be referred to as a *countably ordered set*. If E is a countably ordered set, then any choice of an isomorphism of E with an orderwise connected subset $E' \subseteq \mathbb{Z}$ allows one to define [in a fashion independent of the choice of E'], for *non-maximal* (respectively, *non-minimal*) $e \in E$ [i.e., e such that there exists an $f \in E$ that is $> e$ (respectively, $< e$)], an element "$e + 1$" (respectively, "$e - 1$") of E.

Monoids:

We shall denote by $\mathbb{N}_{\geq 1}$ the *multiplicative monoid* of [rational] integers ≥ 1 [cf. [Mzk16], §0].

Let Q be a *commutative monoid* [with unity]; $P \subseteq Q$ a *submonoid*. If Q is *integral* [so Q embeds into its *groupification* Q^{gp}], then we shall refer to the submonoid

$$P^{\mathrm{gp}} \bigcap Q \ (\subseteq Q^{\mathrm{gp}})$$

of Q as the *group-saturation* of P in Q; if P is equal to its group-saturation in Q, then we shall say that P is *group-saturated* in Q. If Q is *torsion-free* [so Q embeds into its *perfection* Q^{pf}], then we shall refer to the submonoid

$$P^{\mathrm{pf}} \bigcap Q \ (\subseteq Q^{\mathrm{pf}})$$

of Q as the *perf-saturation* of P in Q; if P is equal to its perf-saturation in Q, then we shall say that P is *perf-saturated* in Q.

Topological Groups:

Let Π be a *topological group*. Then let us write

$$\mathcal{B}^{\mathrm{temp}}(\Pi)$$

for the *category* whose *objects* are *countable* [i.e., of cardinality $\leq$ the cardinality of the set of natural numbers], *discrete* sets equipped with a continuous Π-action and whose *morphisms* are morphisms of Π-sets [cf. [Mzk14], §3]. If Π may be written as an inverse limit of an inverse system of surjections of countable discrete topological groups, then we shall say that Π is *tempered* [cf. [Mzk14], Definition 3.1, (i)].

We shall refer to a normal open subgroup $H \subseteq \Pi$ such that the quotient group Π/H is free as *co-free*. We shall refer to a co-free subgroup $H \subseteq \Pi$ as *minimal* if every co-free subgroup of Π contains H. Thus, a minimal co-free subgroup of Π is necessarily *unique* and *characteristic*.

Categories:

Let $\mathcal{C}$ be a *category*; $A \in \mathrm{Ob}(\mathcal{C})$. Then we shall write

$$\mathcal{C}_A$$

for the category whose *objects* are morphisms $B \to A$ of $\mathcal{C}$ and whose *morphisms* [from an object $B_1 \to A$ to an object $B_2 \to A$] are A-morphisms $B_1 \to B_2$ in $\mathcal{C}$ [cf. [Mzk16], §0] and

$$\mathcal{C}[A] \subseteq \mathcal{C}$$

for the *full subcategory* of $\mathcal{C}$ determined by the objects of $\mathcal{C}$ that admit a morphism to A. Given two arrows $f_i : A_i \to B_i$ (where $i = 1, 2$) in $\mathcal{C}$, we shall refer to a commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\sim} & A_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{\sim} & B_2 \end{array}$$

— where the horizontal arrows are isomorphisms in $\mathcal{C}$ — as an *abstract equivalence* from f_1 to f_2. If there exists an abstract equivalence from f_1 to f_2, then we shall say that f_1, f_2 are *abstractly equivalent*.

Let $\Phi : \mathcal{C} \to \mathcal{D}$ be a *faithful functor* between categories $\mathcal{C}$, $\mathcal{D}$. Then we shall say that Φ is *isomorphism-full* if every isomorphism $\Phi(A) \xrightarrow{\sim} \Phi(B)$ of $\mathcal{D}$, where $A, B \in \mathrm{Ob}(\mathcal{C})$, arises by applying Φ to an isomorphism $A \xrightarrow{\sim} B$ of $\mathcal{C}$. Suppose that Φ is *isomorphism-full*. Then observe that the *objects* of $\mathcal{D}$ that are isomorphic to objects in the image of Φ, together with the *morphisms* of $\mathcal{D}$ that are abstractly equivalent to morphisms in the image of Φ, form a *subcategory* $\mathcal{C}' \subseteq \mathcal{D}$ such that Φ induces an *equivalence of categories* $\mathcal{C} \xrightarrow{\sim} \mathcal{C}'$. We shall refer to this subcategory $\mathcal{C}' \subseteq \mathcal{D}$ as the *essential image* of Φ. [Thus, the terminology is consistent with the usual terminology of "essential image" in the case where Φ is *fully faithful*.]

Curves:

We refer to [Mzk14], §0, for generalities concerning *[families of] hyperbolic curves*, *smooth log curves*, *stable log curves*, *divisors of cusps*, and *divisors of marked points*.

If $C^{\log} \to S^{\log}$ is a *stable log curve*, and, moreover, S is the spectrum of a *field*, then we shall say that $C^{\log}$ is *split* if each of the irreducible components and nodes of C is *geometrically irreducible* over S.

A morphism of log stacks

$$C^{\log} \to S^{\log}$$

for which there exists an étale surjection $S_1 \to S$, where S_1 is a scheme, such that $C_1^{\log} \stackrel{\text{def}}{=} C^{\log} \times_S S_1$ may be obtained as the result of forming the quotient [in the sense of log stacks!] of a stable (respectively, smooth) log curve $C_2^{\log} \to S_1^{\log} \stackrel{\text{def}}{=} S^{\log} \times_S S_1$ by the *action of a finite group* of automorphisms of $C_2^{\log}$ over $S_1^{\log}$ will be referred to as a *stable log orbicurve* (respectively, *smooth log orbicurve*) over $S^{\log}$. Thus, the divisor of cusps of $C_2^{\log}$ determines a *divisor of cusps* of $C_1^{\log}$, $C^{\log}$. Here, if $C_2^{\log} \to S_1^{\log}$ is of type $(1,1)$, and the finite group of automorphisms is given by the *action of "*± 1*"* [i.e., relative to the group structure of the underlying elliptic curve of $C_2^{\log} \to S_1^{\log}$], then the resulting stable log orbicurve will be referred to as being of *type* $(1,1)_{\pm}$.

If $S^{\log}$ is the spectrum of a field, equipped with the trivial log structure, then a *hyperbolic orbicurve* $X \to S$ is defined to be the algebraic [log] stack [with trivial log structure] obtained by removing the divisor of cusps from some *smooth log orbicurve* $C^{\log} \to S^{\log}$ over $S^{\log}$. If X (respectively, Y) is a *hyperbolic orbicurve over a field* K (respectively, L), then we shall say that X is *isogenous* to Y if there exists a hyperbolic curve Z over a field M together with *finite étale morphisms* $Z \to X$, $Z \to Y$.

Section 1: The Anabelian Absoluteness of the Étale Theta Function

In this §, we construct a certain *cohomology class* in the group cohomology of the tempered fundamental group of a once-punctured elliptic curve which may be regarded as a sort of *analytic representation of the theta function.* We then discuss various properties of this *"étale theta function"*. In particular, we apply the theory of [Mzk14], §6, to show that it is preserved by arbitrary *automorphisms of the tempered fundamental group.*

Let K be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$; $\overline{K}$ an algebraic closure of K; $\mathfrak{S}$ the formal scheme given by the *p-adic completion* of $\mathrm{Spec}(\mathcal{O}_K)$; $\mathfrak{S}^{\log}$ the formal log scheme obtained by equipping $\mathfrak{S}$ with the log structure determined by the unique closed point of $\mathrm{Spec}(\mathcal{O}_K)$; $\mathfrak{X}^{\log}$ a *stable log curve* over $\mathfrak{S}^{\log}$ of type $(1,1)$. Also, we assume that the *special fiber* of $\mathfrak{X}$ is *singular* and *split* [cf. §0], and that the *generic fiber* of the algebrization of $\mathfrak{X}^{\log}$ is a *smooth log curve.* Write $X^{\log} \stackrel{\mathrm{def}}{=} \mathfrak{X}^{\log} \times_{\mathcal{O}_K} K$ for the ringed space with log structure obtained by tensoring the structure sheaf of $\mathfrak{X}$ over $\mathcal{O}_K$ with K. In the following discussion, we shall often [by abuse of notation] use the notation $X^{\log}$ also to denote the generic fiber of the algebrization of $\mathfrak{X}^{\log}$.

Let us write

$$\underline{\Pi}_X^{\mathrm{tm}}$$

for the *tempered fundamental group* associated to $X^{\log}$ [cf. [André], §4; the group "$\pi_1^{\mathrm{temp}}(X_K^{\log})$" of [Mzk14], Examples 3.10, 5.6]. Here, despite the fact that the [tempered] fundamental group in question is best thought of not as "the fundamental group of X" but rather as "the fundamental group of $X^{\log}$", we use the notation "$\underline{\Pi}_X^{\mathrm{tm}}$" rather than "$\underline{\Pi}_{X^{\log}}^{\mathrm{tm}}$" in order to *minimize the number of subscripts and superscripts* that appear in the notation [cf. the discussion to follow in the remainder of the present paper!]; thus, the reader should think of the notation "$\underline{\Pi}_{(-)}^{\mathrm{tm}}$" as an abbreviation for the "*logarithmic* tempered fundamental group of the scheme $(-)$, equipped with the log structure currently under consideration", i.e., an abbreviation for "$\pi_1^{\mathrm{temp}}((-)^{\log})$".

Denote by $\underline{\Delta}_X^{\mathrm{tm}} \subseteq \underline{\Pi}_X^{\mathrm{tm}}$ the *"geometric tempered fundamental group"*. Thus, we have a natural *exact sequence*:

$$1 \to \underline{\Delta}_X^{\mathrm{tm}} \to \underline{\Pi}_X^{\mathrm{tm}} \to G_K \to 1$$

[where $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{K}/K)$].

Since special fiber of $\mathfrak{X}$ is *split*, it follows that the universal graph-covering of the dual graph of this special fiber determines [up to composition with an element of $\mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$] a natural *surjection*

$$\underline{\Pi}_X^{\mathrm{tm}} \twoheadrightarrow \mathbb{Z}$$

whose kernel, which we denote by $\underline{\Pi}_Y^{\mathrm{tm}}$, determines an infinite étale covering

$$\mathfrak{Y}^{\log} \to \mathfrak{X}^{\log}$$

— i.e., $\mathfrak{Y}^{\log}$ is a p-adic formal scheme equipped with a log structure; the special fiber of $\mathfrak{Y}$ is an *infinite chain of copies of the projective line*, joined at 0 and ∞; write $Y^{\log} \stackrel{\mathrm{def}}{=} \mathfrak{Y}^{\log} \times_{\mathcal{O}_K} K$; $\underline{\mathbb{Z}} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(Y/X)$ $(\cong \mathbb{Z})$.

Write $\underline{\Pi}_X \stackrel{\mathrm{def}}{=} (\underline{\Pi}_X^{\mathrm{tm}})^\wedge$; $\underline{\Delta}_X \stackrel{\mathrm{def}}{=} (\underline{\Delta}_X^{\mathrm{tm}})^\wedge$ [where the "$\wedge$" denotes the profinite completion]. Then the abelianization $\underline{\Delta}_X^{\mathrm{ell}} \stackrel{\mathrm{def}}{=} \underline{\Delta}_X^{\mathrm{ab}} = \underline{\Delta}_X/[\underline{\Delta}_X, \underline{\Delta}_X]$ fits into a well-known *natural exact sequence*:

$$1 \to \widehat{\mathbb{Z}}(1) \to \underline{\Delta}_X^{\mathrm{ell}} \to \widehat{\mathbb{Z}} \to 1$$

Since $\underline{\Delta}_X$ is a *profinite free group on* 2 *generators*, the quotient

$$\underline{\Delta}_X^{\Theta} \stackrel{\mathrm{def}}{=} \underline{\Delta}_X/[\underline{\Delta}_X, [\underline{\Delta}_X, \underline{\Delta}_X]]$$

fits into a *natural exact sequence*:

$$1 \to \wedge^2 \ \underline{\Delta}_X^{\mathrm{ell}} \ (\cong \widehat{\mathbb{Z}}(1)) \to \underline{\Delta}_X^{\Theta} \to \underline{\Delta}_X^{\mathrm{ell}} \to 1$$

Let us denote the *image* of $\wedge^2 \ \underline{\Delta}_X^{\mathrm{ell}}$ in $\underline{\Delta}_X^{\Theta}$ by $(\widehat{\mathbb{Z}}(1) \cong)$ $\underline{\Delta}_{\Theta} \subseteq \underline{\Delta}_X^{\Theta}$. Similarly, if we write

$$\underline{\Delta}_X^{\mathrm{tm}} \twoheadrightarrow (\underline{\Delta}_X^{\mathrm{tm}})^{\Theta} \twoheadrightarrow (\underline{\Delta}_X^{\mathrm{tm}})^{\mathrm{ell}}$$

for the quotients induced by the quotients $\underline{\Delta}_X \twoheadrightarrow \underline{\Delta}_X^{\Theta} \twoheadrightarrow \underline{\Delta}_X^{\mathrm{ell}}$, then we obtain *natural exact sequences*:

$$1 \to \widehat{\mathbb{Z}}(1) \to (\underline{\Delta}_X^{\mathrm{tm}})^{\mathrm{ell}} \to \underline{\mathbb{Z}} \to 1$$

$$1 \to \underline{\Delta}_{\Theta} \to (\underline{\Delta}_X^{\mathrm{tm}})^{\Theta} \to (\underline{\Delta}_X^{\mathrm{tm}})^{\mathrm{ell}} \to 1$$

Also, we shall write

$$\underline{\Pi}_X^{\mathrm{tm}} \twoheadrightarrow (\underline{\Pi}_X^{\mathrm{tm}})^{\Theta} \twoheadrightarrow (\underline{\Pi}_X^{\mathrm{tm}})^{\mathrm{ell}}$$

for the quotients whose kernels are the kernels of the quotients $\underline{\Delta}_X^{\mathrm{tm}} \twoheadrightarrow (\underline{\Delta}_X^{\mathrm{tm}})^{\Theta} \twoheadrightarrow (\underline{\Delta}_X^{\mathrm{tm}})^{\mathrm{ell}}$ and

$$\underline{\Pi}_Y^{\mathrm{tm}} \twoheadrightarrow (\underline{\Pi}_Y^{\mathrm{tm}})^{\Theta} \twoheadrightarrow (\underline{\Pi}_Y^{\mathrm{tm}})^{\mathrm{ell}}; \quad \underline{\Delta}_Y^{\mathrm{tm}} \twoheadrightarrow (\underline{\Delta}_Y^{\mathrm{tm}})^{\Theta} \twoheadrightarrow (\underline{\Delta}_Y^{\mathrm{tm}})^{\mathrm{ell}}$$

for the quotients of $\underline{\Pi}_Y^{\mathrm{tm}}$, $\underline{\Delta}_Y^{\mathrm{tm}}$ induced by the quotients of $\underline{\Pi}_X^{\mathrm{tm}}$, $\underline{\Delta}_X^{\mathrm{tm}}$ with similar superscripts. Thus, $(\underline{\Delta}_Y^{\mathrm{tm}})^{\mathrm{ell}} \cong \widehat{\mathbb{Z}}(1)$; we have a natural exact sequence of *abelian* profinite groups $1 \to \underline{\Delta}_{\Theta} \to (\underline{\Delta}_Y^{\mathrm{tm}})^{\Theta} \to (\underline{\Delta}_Y^{\mathrm{tm}})^{\mathrm{ell}} \to 1$.

Next, let us write $q_X \in \mathcal{O}_K$ for the *q-parameter* of the underlying elliptic curve of $X^{\log}$. If $N \geq 1$ is an integer, set

$$K_N \stackrel{\mathrm{def}}{=} K(\zeta_N, q_X^{1/N}) \subseteq \overline{K}$$

[where ζ_N is a primitive N-th root of unity]. Then any decomposition group of a cusp of $Y^{\log}$ determines a section $G_K \to (\underline{\Pi}_Y^{\mathrm{tm}})^{\mathrm{ell}}$ of the natural surjection

$(\underline{\Pi}_Y^{\mathrm{tm}})^{\mathrm{ell}} \twoheadrightarrow G_K$ whose restriction to the open subgroup $G_{K_N} \subseteq G_K$ determines an open immersion

$$G_{K_N} \hookrightarrow (\underline{\Pi}_Y^{\mathrm{tm}})^{\mathrm{ell}}/N \cdot (\underline{\Delta}_Y^{\mathrm{tm}})^{\mathrm{ell}}$$

the *image* of which is *stabilized* by the conjugation action of $\underline{\Pi}_X^{\mathrm{tm}}$. [Indeed, this follows from the fact that G_{K_N} acts trivially on $(\underline{\Delta}_X^{\mathrm{tm}})^{\mathrm{ell}}/N \cdot (\underline{\Delta}_Y^{\mathrm{tm}})^{\mathrm{ell}}$.] Thus, this image determines a *Galois covering*

$$Y_N \to Y$$

such that the resulting surjection $\underline{\Pi}_Y^{\mathrm{tm}} \twoheadrightarrow \mathrm{Gal}(Y_N/Y)$, whose kernel we denote by $\underline{\Pi}_{Y_N}^{\mathrm{tm}}$, induces a natural exact sequence $1 \to (\underline{\Delta}_Y^{\mathrm{tm}})^{\mathrm{ell}} \otimes \mathbb{Z}/N\mathbb{Z} \to \mathrm{Gal}(Y_N/Y) \to \mathrm{Gal}(K_N/K) \to 1$. Also, we shall write

$$\underline{\Pi}_{Y_N}^{\mathrm{tm}} \twoheadrightarrow (\underline{\Pi}_{Y_N}^{\mathrm{tm}})^{\Theta} \twoheadrightarrow (\underline{\Pi}_{Y_N}^{\mathrm{tm}})^{\mathrm{ell}}; \quad \underline{\Delta}_{Y_N}^{\mathrm{tm}} \twoheadrightarrow (\underline{\Delta}_{Y_N}^{\mathrm{tm}})^{\Theta} \twoheadrightarrow (\underline{\Delta}_{Y_N}^{\mathrm{tm}})^{\mathrm{ell}}$$

for the quotients of $\underline{\Pi}_{Y_N}^{\mathrm{tm}}$, $\underline{\Delta}_{Y_N}^{\mathrm{tm}}$ induced by the quotients of $\underline{\Pi}_Y^{\mathrm{tm}}$, $\underline{\Delta}_Y^{\mathrm{tm}}$ with similar superscripts and

$$Y_N^{\log}$$

for the object obtained by equipping Y_N with the log structure determined by the K_N-valued points of Y_N lying over the cusps of Y. Set

$$\mathfrak{Y}_N \to \mathfrak{Y}$$

equal to the *normalization* of $\mathfrak{Y}$ in Y_N. One verifies easily that the special fiber of $\mathfrak{Y}_N$ is an *infinite chain of copies of the projective line*, joined at 0 and ∞; each of these points "0" and "∞" is a *node* on $\mathfrak{Y}_N$; each projective line in this chain maps to a projective line in the special fiber of $\mathfrak{Y}$ by the "N-th power map" on the copy of "$\mathbb{G}_{\mathrm{m}}$" obtained by removing the nodes; if we choose some irreducible component of the special fiber of $\mathfrak{Y}$ as a *"basepoint"*, then the natural action of $\underline{\mathbb{Z}}$ on $\mathfrak{Y}$ allows one to think of the projective lines in the special fiber of $\mathfrak{Y}$ as being *labeled by elements of* $\underline{\mathbb{Z}}$. In particular, it follows that the isomorphism class of a line bundle on $\mathfrak{Y}_N$ is completely determined by the degree of the restriction of the line bundle to each of these copies of the projective line. That is to say, *these degrees determine an isomorphism*

$$\mathrm{Pic}(\mathfrak{Y}_N) \xrightarrow{\sim} \mathbb{Z}^{\underline{\mathbb{Z}}}$$

[where $\mathbb{Z}^{\underline{\mathbb{Z}}}$ denotes the module of functions $\underline{\mathbb{Z}} \to \mathbb{Z}$; the additive structure on this module is induced by the additive structure on the codomain "$\mathbb{Z}$"]. Write

$$\mathfrak{L}_N$$

for the line bundle on $\mathfrak{Y}_N$ determined by the *constant function* $\underline{\mathbb{Z}} \to \mathbb{Z}$ whose value is 1. Also, we observe that it follows immediately from the above explicit description of the special fiber of $\mathfrak{Y}_N$ that $\Gamma(\mathfrak{Y}_N, \mathcal{O}_{\mathfrak{Y}_N}) = \mathcal{O}_{K_N}$.

Next, write:

$$J_N \stackrel{\mathrm{def}}{=} K_N(a^{1/N})_{a \in K_N} \subseteq \overline{K}$$

Note that [since $K_N^\times$ is topologically finitely generated] J_N is a *finite Galois extension* of K_N. Observe, moreover, that by the construction of Y_N, we obtain an exact sequence:

$$1 \to \underline{\Delta}_\Theta \otimes \mathbb{Z}/N\mathbb{Z} \ (\cong \mathbb{Z}/N\mathbb{Z}(1)) \to (\underline{\Pi}_{Y_N}^{\text{tm}})^\Theta / N \cdot (\underline{\Delta}_Y^{\text{tm}})^\Theta \to G_{K_N} \to 1$$

Since *any two splittings* of this exact sequence differ by a cohomology class $\in H^1(G_{K_N}, \mathbb{Z}/N\mathbb{Z}(1))$, it follows [by the definition of J_N] that *all splittings* of this exact sequence determine the *same* splitting over G_{J_N}. Thus, the *image* of the resulting open immersion

$$G_{J_N} \hookrightarrow (\underline{\Pi}_{Y_N}^{\text{tm}})^\Theta / N \cdot (\underline{\Delta}_Y^{\text{tm}})^\Theta$$

is *stabilized* by the conjugation action of $\underline{\Pi}_X^{\text{tm}}$, hence determines a *Galois covering*

$$Z_N \to Y_N$$

such that the resulting surjection $\underline{\Pi}_{Y_N}^{\text{tm}} \twoheadrightarrow \text{Gal}(Z_N/Y_N)$, whose kernel we denote by $\underline{\Pi}_{Z_N}^{\text{tm}}$, induces a natural exact sequence $1 \to \underline{\Delta}_\Theta \otimes \mathbb{Z}/N\mathbb{Z} \to \text{Gal}(Z_N/Y_N) \to \text{Gal}(J_N/K_N) \to 1$. Also, we shall write

$$\underline{\Pi}_{Z_N}^{\text{tm}} \twoheadrightarrow (\underline{\Pi}_{Z_N}^{\text{tm}})^\Theta \twoheadrightarrow (\underline{\Pi}_{Z_N}^{\text{tm}})^{\text{ell}}; \quad \underline{\Delta}_{Z_N}^{\text{tm}} \twoheadrightarrow (\underline{\Delta}_{Z_N}^{\text{tm}})^\Theta \twoheadrightarrow (\underline{\Delta}_{Z_N}^{\text{tm}})^{\text{ell}}$$

for the quotients of $\underline{\Pi}_{Z_N}^{\text{tm}}$, $\underline{\Delta}_{Z_N}^{\text{tm}}$ induced by the quotients of $\underline{\Pi}_{Y_N}^{\text{tm}}$, $\underline{\Delta}_{Y_N}^{\text{tm}}$ with similar superscripts and

$$Z_N^{\log}$$

for the object obtained by equipping Z_N with the log structure determined by the [manifestly] J_N-valued points of Z_N lying over the cusps of Y. Set

$$\mathfrak{Z}_N \to \mathfrak{Y}_N$$

equal to the *normalization* of $\mathfrak{Y}$ in Z_N. Since $\mathfrak{Y}$ is *"generically of characteristic zero"* [i.e., Y is of characteristic zero], it follows that $\mathfrak{Z}_N$ is *finite* over $\mathfrak{Y}$.

Next, let us observe that there exists a section

$$s_1 \in \Gamma(\mathfrak{Y} = \mathfrak{Y}_1, \mathfrak{L}_1)$$

— well-defined up to an $\mathcal{O}_K^\times$-multiple — whose zero locus on $\mathfrak{Y}$ is precisely the *divisor of cusps* of $\mathfrak{Y}$. Also, let us fix an *isomorphism* of $\mathfrak{L}_N^{\otimes N}$ with $\mathfrak{L}_1|_{\mathfrak{Y}_N}$, which we use to *identify* these two bundles. Note that there is a *natural action* of $\text{Gal}(Y/X)$ on $\mathfrak{L}_1$ which is *uniquely* determined by the condition that it preserve s_1. Thus, we obtain a *natural action* of $\text{Gal}(Y_N/X)$ on $\mathfrak{L}_1|_{\mathfrak{Y}_N}$.

Proposition 1.1. (Theta Action of the Tempered Fundamental Group)

(i) The section

$$s_1|_{\mathfrak{Y}_N} \in \Gamma(\mathfrak{Y}_N, \mathfrak{L}_1|_{\mathfrak{Y}_N} \cong \mathfrak{L}_N^{\otimes N})$$

admits an N**-th root** $s_N \in \Gamma(\mathfrak{Z}_N, \mathfrak{L}_N|_{\mathfrak{Z}_N})$ *over* $\mathfrak{Z}_N$. *In particular, if we denote associated "geometric line bundles" by the notation "*$\mathbb{V}(-)$*", then we obtain a commutative diagram*

$$\begin{array}{ccccccc} \mathfrak{Z}_N & \longrightarrow & \mathfrak{Y}_N & = & \mathfrak{Y}_N & \longrightarrow & \mathfrak{Y}_1 \\ \downarrow & & & & \downarrow & & \downarrow \\ \mathbb{V}(\mathfrak{L}_N|_{\mathfrak{Z}_N}) & \longrightarrow & \mathbb{V}(\mathfrak{L}_N) & \longrightarrow & \mathbb{V}(\mathfrak{L}_N^{\otimes N}) & \longrightarrow & \mathbb{V}(\mathfrak{L}_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{Z}_N & \longrightarrow & \mathfrak{Y}_N & = & \mathfrak{Y}_N & \longrightarrow & \mathfrak{Y}_1 \end{array}$$

where the horizontal morphisms in the first and last lines are the natural morphisms; in the second line of horizontal morphisms, the first and third horizontal morphisms are the pull-back morphisms, while the second morphism is given by raising to the N*-th power; in the first row of vertical morphisms, the morphism on the left (respectively, in the middle; on the right) is that determined by* s_N *(respectively,* $s_1|_{\mathfrak{Y}_N}$*;* s_1*); the vertical morphisms in the second row of vertical morphisms are the natural morphisms; the vertical composites are the identity morphisms.*

(ii) There is a **unique** *action of* $\underline{\Pi}_X^{\mathrm{tm}}$ *on* $\mathfrak{L}_N \otimes_{\mathcal{O}_{K_N}} \mathcal{O}_{J_N}$ *[a line bundle on* $\mathfrak{Y}_N \times_{\mathcal{O}_{K_N}} \mathcal{O}_{J_N}$*] that is* **compatible** *with the morphism* $\mathfrak{Z}_N \to \mathbb{V}(\mathfrak{L}_N \otimes_{\mathcal{O}_{K_N}} \mathcal{O}_{J_N})$ *determined by* s_N *[hence induces the* **identity** *on* $s_N^{\otimes N} = s_1|_{\mathfrak{Z}_N}$*]. Moreover, this action of* $\underline{\Pi}_X^{\mathrm{tm}}$ *factors through* $\underline{\Pi}_X^{\mathrm{tm}}/\underline{\Pi}_{Z_N}^{\mathrm{tm}} = \mathrm{Gal}(Z_N/X)$*, and, in fact, induces a* **faithful** *action of* $\underline{\Delta}_X^{\mathrm{tm}}/\underline{\Delta}_{Z_N}^{\mathrm{tm}}$ *on* $\mathfrak{L}_N \otimes_{\mathcal{O}_{K_N}} \mathcal{O}_{J_N}$.

Proof. First, observe that by the discussion above [concerning the structure of the special fiber of $\mathfrak{Y}_N$], it follows that the action of $\underline{\Pi}_X^{\mathrm{tm}}$ ($\twoheadrightarrow \mathrm{Gal}(Y_N/X)$) on $\mathfrak{Y}_N$ preserves the isomorphism class of the line bundle $\mathfrak{L}_N$, hence also the isomorphism class of the line bundle $\mathfrak{L}_N^{\otimes N}$ [i.e., "the *identification* of $\mathfrak{L}_N^{\otimes N}$ with $\mathfrak{L}_1|_{\mathfrak{Y}_N}$, up to multiplication by an element of $\Gamma(\mathfrak{Y}_N, \mathcal{O}_{\mathfrak{Y}_N}^\times) = \mathcal{O}_{K_N}^\times$"]. In particular, if we denote by $\mathcal{G}_N$ the *group of automorphisms* of the pull-back of $\mathfrak{L}_N$ to $Y_N \times_{K_N} J_N$ that *lie over* the J_N-linear automorphisms of $Y_N \times_{K_N} J_N$ induced by elements of $\underline{\Delta}_X^{\mathrm{tm}}/\underline{\Delta}_{Y_N}^{\mathrm{tm}} \subseteq \mathrm{Gal}(Y_N/X)$ and whose N-th tensor power *fixes* the pull-back of $s_1|_{\mathfrak{Y}_N}$, then one verifies immediately [by recalling the *definition* of J_N] that $\mathcal{G}_N$ fits into a natural exact sequence

$$1 \to \boldsymbol{\mu}_N(J_N) \to \mathcal{G}_N \to \underline{\Delta}_X^{\mathrm{tm}}/\underline{\Delta}_{Y_N}^{\mathrm{tm}} \to 1$$

[where $\boldsymbol{\mu}_N(J_N) \subseteq J_N^\times$ denotes the group of N-th roots of unity in J_N].

Now I *claim* that the kernel $\mathcal{H}_N \subseteq \mathcal{G}_N$ of the composite surjection

$$\mathcal{G}_N \twoheadrightarrow \underline{\Delta}_X^{\mathrm{tm}}/\underline{\Delta}_{Y_N}^{\mathrm{tm}} \twoheadrightarrow \underline{\Delta}_X^{\mathrm{tm}}/\underline{\Delta}_Y^{\mathrm{tm}} \cong \underline{\mathbb{Z}}$$

[where we note that $\mathrm{Ker}(\underline{\Delta}_X^{\mathrm{tm}}/\underline{\Delta}_{Y_N}^{\mathrm{tm}} \twoheadrightarrow \underline{\Delta}_X^{\mathrm{tm}}/\underline{\Delta}_Y^{\mathrm{tm}}) = \underline{\Delta}_Y^{\mathrm{tm}}/\underline{\Delta}_{Y_N}^{\mathrm{tm}} \cong \mathbb{Z}/N\mathbb{Z}(1)$] is an *abelian group annihilated by multiplication by* N. Indeed, one verifies immediately, by considering various relevant line bundles on "$\mathbb{G}_{\mathrm{m}}$", that [if we write U for the standard multiplicative coordinate on $\mathbb{G}_m$ and ζ for a primitive N-th root of unity, then] this follows from the *identity of functions* on "$\mathbb{G}_m$"

$$\prod_{j=0}^{N} f(\zeta^{-j} \cdot U) = 1$$

— where $f(U) \stackrel{\mathrm{def}}{=} (U-1)/(U-\zeta)$ represents an element of $\mathcal{H}_N$ that maps to a generator of $\underline{\Delta}_Y^{\mathrm{tm}}/\underline{\Delta}_{Y_N}^{\mathrm{tm}}$.

Now consider the *tautological* $\mathbb{Z}/N\mathbb{Z}(1)$*-torsor* $\mathfrak{R}_N \to \mathfrak{Y}_N$ obtained by extracting an N-th root of s_1. More explicitly, $\mathfrak{R}_N \to \mathfrak{Y}_N$ may be thought of as the finite $\mathfrak{Y}_N$-scheme associated to the $\mathcal{O}_{\mathfrak{Y}_N}$-algebra

$$\bigoplus_{j=0}^{N-1} \mathfrak{L}_N^{\otimes -j}$$

where the "algebra structure" is defined by the morphism $\mathfrak{L}_N^{\otimes -N} \to \mathcal{O}_{\mathfrak{Y}_N}$ given by multiplying by $s_1|_{\mathfrak{Y}_N}$. In particular, it follows immediately from the *definition* of $\mathcal{G}_N$ that $\mathcal{G}_N$ *acts naturally* on $(\mathfrak{R}_N)_{J_N} \stackrel{\mathrm{def}}{=} \mathfrak{R}_N \times_{\mathcal{O}_{K_N}} J_N$. Since $s_1|_{Y_N}$ has *zeroes of order* 1 at each of the cusps of Y_N, it thus follows immediately that $(\mathfrak{R}_N)_{J_N}$ is *Galois* over $X_{J_N} \stackrel{\mathrm{def}}{=} X \times_K J_N$, and that this action determines an isomorphism:

$$\mathcal{G}_N \xrightarrow{\sim} \mathrm{Gal}((\mathfrak{R}_N)_{J_N}/X_{J_N})$$

Since the abelian group $\underline{\Delta}_X^{\mathrm{tm}}/\underline{\Delta}_{Y_N}^{\mathrm{tm}}$ acts *trivially* on $\boldsymbol{\mu}_N(J_N)$, and $\mathcal{H}_N$ is *annihilated by* N, it thus follows formally from the *definition* of $(\underline{\Delta}_X)^{\Theta}$ [i.e., as the quotient by a certain "double commutator subgroup"] that at least *"geometrically"*, there exists a map from $\mathfrak{Z}_N$ to $\mathfrak{R}_N$. More precisely, there is a morphism $\mathfrak{Z}_N \times_{\mathcal{O}_{J_N}} \overline{K} \to \mathfrak{R}_N$ over $\mathfrak{Y}_N$. That this morphism in fact factors through $\mathfrak{Z}_N$ — inducing an *isomorphism*

$$\mathfrak{Z}_N \xrightarrow{\sim} \mathfrak{R}_N \times_{\mathcal{O}_{K_N}} \mathcal{O}_{J_N}$$

— follows from the *definition* of the open immersion $G_{J_N} \hookrightarrow (\underline{\Pi}_{Y_N}^{\mathrm{tm}})^{\Theta}/N \cdot (\underline{\Delta}_Y^{\mathrm{tm}})^{\Theta}$ whose image was used to define $\mathfrak{Z}_N \to \mathfrak{Y}_N$ [together with the fact that $s_1|_{Y_N}$ is defined over K_N]. This completes the proof of assertion (i).

Next, we consider assertion (ii). Since the *natural action* of $\underline{\Pi}_X^{\mathrm{tm}}$ on $\mathfrak{L}_1|_{\mathfrak{Y}_N} \cong \mathfrak{L}_N^{\otimes N}$ *preserves* $s_1|_{\mathfrak{Y}_N}$, and the action of $\underline{\Pi}_X^{\mathrm{tm}}$ on $\mathfrak{Y}_N$ preserves the *isomorphism class* of the line bundle $\mathfrak{L}_N$, the existence and uniqueness of the desired action of $\underline{\Pi}_X^{\mathrm{tm}}$ on $\mathfrak{L}_N \otimes_{\mathcal{O}_{K_N}} \mathcal{O}_{J_N}$ follow immediately from the definitions [cf. especially the definition of J_N]. Moreover, since s_N is defined over Z_N, it is immediate that this action factors through $\underline{\Pi}_X^{\mathrm{tm}}/\underline{\Pi}_{Z_N}^{\mathrm{tm}}$. Finally, the asserted *faithfulness* follows from the fact that s_1 has *zeroes of order* 1 at the cusps of Y_N [together with the tautological fact that $\underline{\Delta}_X^{\mathrm{tm}}/\underline{\Delta}_{Y_N}^{\mathrm{tm}}$ acts faithfully on Y_N]. ◯

Next, let us set:

$$\ddot{K}_N \stackrel{\text{def}}{=} K_{2N}; \quad \ddot{J}_N \stackrel{\text{def}}{=} \ddot{K}_N(a^{1/N})_{a\in \ddot{K}_N} \subseteq \overline{K}$$

$$\ddot{\mathfrak{Y}}_N \stackrel{\text{def}}{=} \mathfrak{Y}_{2N} \times_{\mathcal{O}_{\ddot{K}_N}} \mathcal{O}_{\ddot{J}_N}; \quad \ddot{Y}_N \stackrel{\text{def}}{=} Y_{2N} \times_{\ddot{K}_N} \ddot{J}_N; \quad \ddot{\mathfrak{L}}_N \stackrel{\text{def}}{=} \mathfrak{L}_N|_{\ddot{\mathfrak{Y}}_N} \cong \mathfrak{L}_{2N}^{\otimes 2} \otimes_{\mathcal{O}_{\ddot{K}_N}} \mathcal{O}_{\ddot{J}_N}$$

Also, we shall write $\ddot{Z}_N$ for the composite of the coverings $\ddot{Y}_N$, Z_N of Y_N; $\ddot{\mathfrak{Z}}_N$ for the *normalization* of $\mathfrak{Z}_N$ in $\ddot{Z}_N$; $\ddot{Y} \stackrel{\text{def}}{=} \ddot{Y}_1 = Y_2$; $\ddot{\mathfrak{Y}} \stackrel{\text{def}}{=} \ddot{\mathfrak{Y}}_1 = \mathfrak{Y}_2$; $\ddot{K} \stackrel{\text{def}}{=} \ddot{K}_1 = \ddot{J}_1 = K_2$. Thus, we have a *cartesian commutative diagram*:

$$\begin{array}{ccc} \mathbb{V}(\ddot{\mathfrak{L}}_N) & \longrightarrow & \mathbb{V}(\mathfrak{L}_N) \\ \big\downarrow & & \big\downarrow \\ \ddot{\mathfrak{Y}}_N & \longrightarrow & \mathfrak{Y}_N \end{array}$$

Note that here, $\underline{\Pi}_X^{\text{tm}}$ acts compatibly on $\ddot{\mathfrak{Y}}_N$, $\mathfrak{Y}_N$, and [by Proposition 1.1, (ii)] on $\mathfrak{L}_N \otimes_{\mathcal{O}_{K_N}} \mathcal{O}_{J_N}$. Thus, since this diagram is cartesian [and $J_N \subseteq \ddot{J}_N$], we obtain a *natural action of* $\underline{\Pi}_X^{\text{tm}}$ *on* $\ddot{\mathfrak{L}}_N$ which *factors* through $\underline{\Pi}_X^{\text{tm}}/\underline{\Pi}_{\ddot{Z}_N}^{\text{tm}}$. Moreover, we have a *natural exact sequence*

$$1 \to \underline{\Pi}_{Z_N}^{\text{tm}}/\underline{\Pi}_{\ddot{Z}_N}^{\text{tm}} \to \underline{\Pi}_Y^{\text{tm}}/\underline{\Pi}_{\ddot{Z}_N}^{\text{tm}} \to \text{Gal}(Z_N/Y) \to 1$$

[where $\underline{\Pi}_{Z_N}^{\text{tm}}/\underline{\Pi}_{\ddot{Z}_N}^{\text{tm}} \hookrightarrow \text{Gal}(\ddot{Y}_N/Y_N)$] which is *compatible* with the conjugation actions by $\underline{\Pi}_X^{\text{tm}}$ on each of the terms in the exact sequence.

Next, let us choose an *orientation* on the dual graph of the special fiber of $\mathfrak{Y}$. Such an orientation determines a specific isomorphism $\underline{\mathbb{Z}} \xrightarrow{\sim} \mathbb{Z}$, hence a *label* $\in \mathbb{Z}$ for each irreducible component of the special fiber of $\mathfrak{Y}$. Note that this choice of labels determines a *label* $\in \mathbb{Z}$ for each irreducible component of the special fiber of $\mathfrak{Y}_N$, $\ddot{\mathfrak{Y}}_N$. Now we define $\mathfrak{D}_N$ to be the *effective divisor* on $\ddot{\mathfrak{Y}}_N$ which is supported on the special fiber and corresponds to the *function*

$$\mathbb{Z} \ni j \mapsto j^2 \cdot \log(q_X)/2N$$

— i.e., at the irreducible component labeled j, the divisor $\mathfrak{D}_N$ is equal to the divisor given by the schematic zero locus of $q_X^{j^2/2N}$. Note that since the completion of $\ddot{\mathfrak{Y}}_N$ at each node of its special fiber is isomorphic to the ring

$$\mathcal{O}_{\ddot{J}_N}[[u,v]]/(uv - q_X^{1/2N})$$

it follows that this divisor $\mathfrak{D}_N$ is *Cartier*. Moreover, a simple calculation of degrees reveals that we have an isomorphism of line bundles on $\ddot{\mathfrak{Y}}_N$:

$$\mathcal{O}_{\ddot{\mathfrak{Y}}_N}(\mathfrak{D}_N) \cong \ddot{\mathfrak{L}}_N$$

Thus, we obtain a section, well-defined up to an $\mathcal{O}_{\ddot{J}_N}^\times$-multiple, $\in \Gamma(\ddot{\mathfrak{Y}}_N, \ddot{\mathfrak{L}}_N)$ whose zero locus is precisely the divisor $\mathfrak{D}_N$. That is to say, we have a commutative diagram

$$\begin{array}{ccccccccc} \ddot{\mathfrak{Y}}_N & \xrightarrow{\tau_N} & \mathbb{V}(\ddot{\mathfrak{L}}_N) & \longrightarrow & \mathbb{V}(\mathfrak{L}_N) & \longrightarrow & \mathbb{V}(\mathfrak{L}_N^{\otimes N}) & \longrightarrow & \mathbb{V}(\mathfrak{L}_1) \\ \big\downarrow \text{id} & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \ddot{\mathfrak{Y}}_N & = & \ddot{\mathfrak{Y}}_N & \longrightarrow & \mathfrak{Y}_N & = & \mathfrak{Y}_N & \longrightarrow & \mathfrak{Y}_1 \end{array}$$

in which the second square is the cartesian commutative diagram discussed above; the third and fourth squares are the lower second and third squares of the diagram of Proposition 1.1, (i); τ_N — which we shall refer to as the *theta trivialization* of $\ddot{\mathfrak{L}}_N$ — is the section whose zero locus is equal to $\mathfrak{D}_N$. Moreover, since the action of $\underline{\Pi}_Y^{\mathrm{tm}}$ on $\ddot{\mathfrak{Y}}_N$ clearly *fixes the divisor* $\mathfrak{D}_N$, we conclude that the action of $\underline{\Pi}_Y^{\mathrm{tm}}$ on $\ddot{\mathfrak{Y}}_N$, $\mathbb{V}(\ddot{\mathfrak{L}}_N)$ always *preserves* τ_N*, up to an* $\mathcal{O}^{\times}_{\ddot{J}_N}$*-multiple.*

Next, let $M \geq 1$ be an *integer that divides* N. Then $\mathfrak{Y}_M \to \mathfrak{Y}$ (respectively, $\mathfrak{Z}_M \to \mathfrak{Y}$; $\ddot{\mathfrak{Y}}_M \to \mathfrak{Y}$) may be regarded as a *subcovering* of $\mathfrak{Y}_N \to \mathfrak{Y}$ (respectively, $\mathfrak{Z}_N \to \mathfrak{Y}$; $\ddot{\mathfrak{Y}}_N \to \mathfrak{Y}$). Moreover, we have natural isomorphisms $\mathfrak{L}_M|_{\mathfrak{Y}_N} \cong \mathfrak{L}_N^{\otimes N/M}$; $\ddot{\mathfrak{L}}_M|_{\ddot{\mathfrak{Y}}_N} \cong \ddot{\mathfrak{L}}_N^{\otimes N/M}$. Thus, we obtain a *diagram*

$$\begin{array}{ccccc} \ddot{\mathfrak{Y}}_N & \stackrel{\tau_N}{\longrightarrow} & \mathbb{V}(\ddot{\mathfrak{L}}_N) & \longrightarrow & \ddot{\mathfrak{Y}}_N \\ \downarrow & & \downarrow (-)^{N/M} & & \downarrow \\ \ddot{\mathfrak{Y}}_M & \stackrel{\tau_M}{\longrightarrow} & \mathbb{V}(\ddot{\mathfrak{L}}_M) & \longrightarrow & \ddot{\mathfrak{Y}}_M \end{array}$$

in which the second square consists of the natural morphisms, hence commutes; the first square *"commutes up to an* $\mathcal{O}^{\times}_{\ddot{J}_N}$*-,* $\mathcal{O}^{\times}_{\ddot{J}_M}$*-multiple"*, i.e., commutes up to composition, at the upper right-hand corner of the square, with an *automorphism of* $\mathbb{V}(\ddot{\mathfrak{L}}_N)$ arising from multiplication by an element of $\mathcal{O}^{\times}_{\ddot{J}_N}$, and, at the lower right-hand corner of the square, with an *automorphism of* $\mathbb{V}(\ddot{\mathfrak{L}}_M)$ arising from multiplication by an element of $\mathcal{O}^{\times}_{\ddot{J}_M}$. [Indeed, this last commutativity follows from the *definition* of $\ddot{J}_N$, and the easily verified fact that there exist "τ_N's" which are *defined over* $\mathfrak{Y}_{2N}$.]

Thus, since by the *classical theory of the theta function* [cf., e.g., [Mumf], p. 289; the relation "$\ddot{\Theta}(-\ddot{U}) = -\ddot{\Theta}(\ddot{U})$" given in Proposition 1.4, (ii), below], it follows that one may choose τ_1 so that the natural action of $\underline{\Pi}_Y^{\mathrm{tm}}$ on $\mathbb{V}(\ddot{\mathfrak{L}}_1)$ [arising from the fact that $\ddot{\mathfrak{L}}_1$ is the pull-back of the line bundle $\mathfrak{L}_1$ on $\mathfrak{Y}$; cf. the action of Proposition 1.1, (ii)] *preserves* $\pm\tau_1$, we conclude, in light of the definition of $\ddot{J}_N$, the following:

Lemma 1.2. **(Compatibility of Theta Trivializations)** *By modifying the various* τ_N *by suitable* $\mathcal{O}^{\times}_{\ddot{J}_N}$*-multiples, one may assume that* $\tau_{N_1}^{\otimes N_1/N_2} = \tau_{N_2}$*, for all positive integers* N_1*,* N_2 *such that* $N_2|N_1$*. In particular, there exists a* **compatible system** *[as* N *varies over the positive integers] of* **actions** *of* $\underline{\Pi}_Y^{\mathrm{tm}}$ *(respectively,* $\underline{\Pi}_{\ddot{Y}}^{\mathrm{tm}}$*) on* $\ddot{\mathfrak{Y}}_N$*,* $\mathbb{V}(\ddot{\mathfrak{L}}_N)$ *which* **preserve** τ_N*. Finally, each action of this system* **differs** *from the action determined by the action of Proposition 1.1, (ii), by multiplication by a(n)* $2N$**-th root of unity** *(respectively,* N**-th root of unity***).*

Thus, by taking the τ_N to be as in Lemma 1.2 and applying the *natural isomorphism* $\underline{\Delta}_{\Theta} \cong \widehat{\mathbb{Z}}(1)$ to the *difference* between the actions of $\underline{\Pi}_Y^{\mathrm{tm}}$, $\underline{\Pi}_{\ddot{Y}}^{\mathrm{tm}}$ arising from Proposition 1.1, (ii), and Lemma 1.2, we obtain the following:

Proposition 1.3. **(The Étale Theta Class)** *The natural action of* $\underline{\Pi}_Y^{\text{tm}}$ *on [constant multiples of]* τ_N *determines a* **cohomology class**

$$\eta_N^{\Theta} \in H^1(\underline{\Pi}_Y^{\text{tm}}, (\frac{1}{2}\mathbb{Z}/N\mathbb{Z})(1)) \cong H^1(\underline{\Pi}_Y^{\text{tm}}, \underline{\Delta}_{\Theta} \otimes (\frac{1}{2}\mathbb{Z}/N\mathbb{Z}))$$

which arises from a cohomology class $\in H^1(\underline{\Pi}_Y^{\text{tm}}/\underline{\Pi}_{\ddot{Z}_N}^{\text{tm}}, \underline{\Delta}_{\Theta} \otimes (\frac{1}{2}\mathbb{Z}/N\mathbb{Z}))$ *whose restriction to*

$$H^1(\underline{\Delta}_{\ddot{Y}_N}^{\text{tm}}/\underline{\Delta}_{\ddot{Z}_N}^{\text{tm}}, \underline{\Delta}_{\Theta} \otimes (\frac{1}{2}\mathbb{Z}/N\mathbb{Z})) = \text{Hom}(\underline{\Delta}_{\ddot{Y}_N}^{\text{tm}}/\underline{\Delta}_{\ddot{Z}_N}^{\text{tm}}, \underline{\Delta}_{\Theta} \otimes (\frac{1}{2}\mathbb{Z}/N\mathbb{Z}))$$

is the composite of the natural isomorphism $\underline{\Delta}_{\ddot{Y}_N}^{\text{tm}}/\underline{\Delta}_{\ddot{Z}_N}^{\text{tm}} \xrightarrow{\sim} \underline{\Delta}_{\Theta} \otimes \mathbb{Z}/N\mathbb{Z}$ *with the natural inclusion* $\underline{\Delta}_{\Theta} \otimes (\mathbb{Z}/N\mathbb{Z}) \hookrightarrow \underline{\Delta}_{\Theta} \otimes (\frac{1}{2}\mathbb{Z}/N\mathbb{Z})$. *Moreover, if we set*

$$\mathcal{O}_{K/\ddot{K}}^{\times} \stackrel{\text{def}}{=} \{a \in \mathcal{O}_{\ddot{K}}^{\times} \mid a^2 \in K\}$$

and regard $\mathcal{O}_{K/\ddot{K}}^{\times}$ *as acting on* $H^1(\underline{\Pi}_Y^{\text{tm}}, (\frac{1}{2}\mathbb{Z}/N\mathbb{Z})(1))$ *via the composite*

$$\mathcal{O}_{K/\ddot{K}}^{\times} \to H^1(G_K, (\frac{1}{2}\mathbb{Z}/N\mathbb{Z})(1)) \to H^1(\underline{\Pi}_Y^{\text{tm}}, (\frac{1}{2}\mathbb{Z}/N\mathbb{Z})(1))$$

[where the first map is the evident generalization of the Kummer map, which is compatible with the Kummer map $\mathcal{O}_K^{\times} \to H^1(G_K, (\mathbb{Z}/N\mathbb{Z})(1))$ *relative to the natural inclusion* $\mathcal{O}_K^{\times} \hookrightarrow \mathcal{O}_{K/\ddot{K}}^{\times}$ *and the morphism induced on cohomology by the natural inclusion* $(\mathbb{Z}/N\mathbb{Z}) \hookrightarrow (\frac{1}{2}\mathbb{Z}/N\mathbb{Z})$*; the second map is the natural map], then the* **set of cohomology classes**

$$\mathcal{O}_{K/\ddot{K}}^{\times} \cdot \eta_N^{\Theta} \in H^1(\underline{\Pi}_Y^{\text{tm}}, \underline{\Delta}_{\Theta} \otimes (\frac{1}{2}\mathbb{Z}/N\mathbb{Z}))$$

is **independent of the choices** *of* s_1, s_N, τ_N. *In particular, by allowing* N *to* **vary** *among all positive integers, we obtain a* **set of cohomology classes**

$$\mathcal{O}_{K/\ddot{K}}^{\times} \cdot \eta^{\Theta} \in H^1(\underline{\Pi}_Y^{\text{tm}}, \frac{1}{2}\underline{\Delta}_{\Theta})$$

each of which is a cohomology class $\in H^1((\underline{\Pi}_Y^{\text{tm}})^{\Theta}, \frac{1}{2}\underline{\Delta}_{\Theta})$ *whose restriction to*

$$H^1((\underline{\Delta}_Y^{\text{tm}})^{\Theta}, \frac{1}{2}\underline{\Delta}_{\Theta}) = \text{Hom}((\underline{\Delta}_Y^{\text{tm}})^{\Theta}, \frac{1}{2}\underline{\Delta}_{\Theta})$$

is the composite of the natural isomorphism $(\underline{\Delta}_Y^{\text{tm}})^{\Theta} \xrightarrow{\sim} \underline{\Delta}_{\Theta}$ *with the natural inclusion* $\underline{\Delta}_{\Theta} \hookrightarrow \frac{1}{2}\underline{\Delta}_{\Theta}$. *Moreover, the restricted classes*

$$\mathcal{O}_{K/\ddot{K}}^{\times} \cdot \eta^{\Theta}|_{\ddot{Y}} \in H^1(\underline{\Pi}_{\ddot{Y}}^{\text{tm}}, \frac{1}{2}\underline{\Delta}_{\Theta})$$

arise naturally from classes

$$\mathcal{O}^{\times}_{\ddot{K}} \cdot \ddot{\eta}^{\Theta} \in H^1(\underline{\Pi}^{\mathrm{tm}}_{\ddot{Y}}, \underline{\Delta}_{\Theta})$$

[where $\mathcal{O}^{\times}_{\ddot{K}}$ *acts via the composite of the Kummer map* $\mathcal{O}^{\times}_{\ddot{K}} \to H^1(G_{\ddot{K}}, \underline{\Delta}_{\Theta})$ *with the natural map* $H^1(G_{\ddot{K}}, \underline{\Delta}_{\Theta}) \to H^1(\underline{\Pi}^{\mathrm{tm}}_{\ddot{Y}}, \underline{\Delta}_{\Theta})$*] "without denominators". By abuse of the definite article, we shall refer to any element of the sets* $\mathcal{O}^{\times}_{K/\ddot{K}} \cdot \eta^{\Theta}_{N}$, $\mathcal{O}^{\times}_{K/\ddot{K}} \cdot \eta^{\Theta}$, $\mathcal{O}^{\times}_{\ddot{K}} \cdot \ddot{\eta}^{\Theta}$ *as the* **"étale theta class"***.*

Remark 1.3.1. Note that the *denominators* "$\frac{1}{2}$" in Proposition 1.3 are by no meas superfluous: Indeed, this follows immediately from the fact that the divisor $\mathfrak{D}_2$ on $\ddot{\mathfrak{Y}}$ clearly does not descend to $\mathfrak{Y}$.

Let us denote by

$$\mathfrak{U} \subseteq \mathfrak{Y}$$

the *open formal subscheme* obtained by removing the nodes from the irreducible component of the special fiber labeled $0 \in \mathbb{Z}$. If we take the unique cusp lying in $\mathfrak{U}$ as the *origin*, then — as is well-known from the theory of the *Tate curve* [cf., e.g., [Mumf], p. 289] — the group structure on the underlying elliptic curve of $X^{\log}$ determines a *group structure* on $\mathfrak{U}$, together with a *unique* [in light of our choice of an *orientation* on the dual graph of the special fiber of $\mathfrak{Y}$] *isomorphism* of $\mathfrak{U}$ with the p-adic formal completion of $\mathbb{G}_{\mathrm{m}}$ over $\mathcal{O}_K$. In particular, this isomorphism determines a *multiplicative coordinate*:

$$U \in \Gamma(\mathfrak{U}, \mathcal{O}^{\times}_{\mathfrak{U}})$$

Moreover, it is immediate from the definitions that U admits a *square root*

$$\ddot{U} \in \Gamma(\ddot{\mathfrak{U}}, \mathcal{O}^{\times}_{\ddot{\mathfrak{U}}})$$

on $\ddot{\mathfrak{U}} \stackrel{\mathrm{def}}{=} \mathfrak{U} \times_{\mathfrak{Y}} \ddot{\mathfrak{Y}}$.

Proposition 1.4. **(Relation to the Classical Theta Function)** *Set*

$$\ddot{\Theta} = \ddot{\Theta}(\ddot{U}) \stackrel{\mathrm{def}}{=} q_X^{-\frac{1}{8}} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot q_X^{\frac{1}{2}(n+\frac{1}{2})^2} \cdot \ddot{U}^{2n+1} \in \Gamma(\ddot{\mathfrak{U}}, \mathcal{O}_{\ddot{\mathfrak{U}}})$$

so $\ddot{\Theta}$ *extends uniquely to a* **meromorphic function** *on* $\ddot{\mathfrak{Y}}$ *[cf., e.g., [Mumf], p. 289]. Then:*

(i) The **zeroes** *of* $\ddot{\Theta}$ *on* $\ddot{\mathfrak{Y}}$ *are precisely the the cusps of* $\ddot{\mathfrak{Y}}$*; each zero has multiplicity* 1*. The divisor of* **poles** *of* $\ddot{\Theta}$ *on* $\ddot{\mathfrak{Y}}$ *is precisely the divisor* $\mathfrak{D}_1$*.*

(ii) We have:

$$\ddot{\Theta}(\ddot{U}) = -\ddot{\Theta}(\ddot{U}^{-1}); \quad \ddot{\Theta}(-\ddot{U}) = -\ddot{\Theta}(\ddot{U});$$
$$\ddot{\Theta}(q_X^{a/2}\ddot{U}) = (-1)^a \cdot q_X^{-a^2/2} \cdot \ddot{U}^{-2a} \cdot \ddot{\Theta}(\ddot{U})$$

for $a \in \mathbb{Z}$.

(iii) The classes

$$\mathcal{O}^{\times}_{\ddot{K}} \cdot \ddot{\eta}^{\Theta} \in H^1(\underline{\Pi}^{\text{tm}}_{\ddot{Y}}, \underline{\Delta}_{\Theta})$$

of Proposition 1.3 are precisely the **"Kummer classes"** *associated to* $\mathcal{O}^{\times}_{\ddot{K}}$*-multiples of* $\ddot{\Theta}$*, regarded as a regular function on* $\ddot{Y}$*. In particular, if* L *is a finite extension of* $\ddot{K}$*,* $y \in \ddot{Y}(L)$ *is a* **non-cuspidal** *point, then the* **restricted classes**

$$\mathcal{O}^{\times}_{\ddot{K}} \cdot \ddot{\eta}^{\Theta}|_y \in H^1(G_L, \underline{\Delta}_{\Theta}) \cong H^1(G_L, \widehat{\mathbb{Z}}(1)) \cong (L^{\times})^{\wedge}$$

[where the "$\wedge$*" denotes the profinite completion] lie in* $L^{\times} \subseteq (L^{\times})^{\wedge}$ *and are equal to the* **values** $\mathcal{O}^{\times}_{\ddot{K}} \cdot \ddot{\Theta}(y)$ *of* $\ddot{\Theta}$ *and its* $\mathcal{O}^{\times}_{\ddot{K}}$*-multiples at* y*. A similar statement holds if* $y \in \ddot{Y}(L)$ *is a* **cusp***, if one restricts first to* D_y *and then to a section* $G_L \hookrightarrow D_y$ *compatible with the* **canonical integral structure** *[cf. [Mzk13], Definition 4.1, (iii)] on* D_y*. In light of this relationship between the cohomology classes of Proposition 1.3 and the values of* $\ddot{\Theta}$*, we shall sometimes refer to these classes as* **"the étale theta function"**.

Proof. Assertion (ii) is a routine calculation involving the series used to define $\ddot{\Theta}$. A similar calculation shows that $\ddot{\Theta}(\pm 1) = 0$. The formula given for $\ddot{\Theta}(q_X^{a/2}\ddot{U})$ in assertion (ii) shows that the portion of the divisor of poles supported in the special fiber of $\ddot{\mathfrak{Y}}$ is equal to $\mathfrak{D}_1$. This formula also shows that to complete the proof of assertion (i), it suffices to show that the given description of the zeroes and poles of $\ddot{\Theta}$ is accurate over the irreducible component of the special fiber of $\ddot{\mathfrak{Y}}$ labeled 0. But this follows immediately from the fact that the restriction of $\ddot{\Theta}$ to this irreducible component is the rational function $\ddot{U} - \ddot{U}^{-1}$. Finally, in light of assertion (i) [and the fact, observed above, that $\Gamma(\mathfrak{Y}_N, \mathcal{O}^{\times}_{\mathfrak{Y}_N}) = \mathcal{O}^{\times}_{K_N}$], assertion (iii) is a formal consequence of the construction of the classes $\mathcal{O}^{\times}_{\ddot{K}} \cdot \ddot{\eta}^{\Theta}$. ◯

Proposition 1.5. **(Theta Cohomology)**

(i) The Leray-Serre spectral sequences associated to the filtration of closed subgroups

$$\underline{\Delta}_{\Theta} \subseteq (\underline{\Delta}^{\text{tm}}_{Y})^{\Theta} \subseteq (\underline{\Pi}^{\text{tm}}_{Y})^{\Theta}$$

determine a natural **filtration** $0 \subseteq F^2 \subseteq F^1 \subseteq F^0 = H^1((\underline{\Pi}^{\text{tm}}_{Y})^{\Theta}, \underline{\Delta}_{\Theta})$ *on the cohomology module* $H^1((\underline{\Pi}^{\text{tm}}_{Y})^{\Theta}, \underline{\Delta}_{\Theta})$ *with subquotients given as follows:*

$$F^0/F^1 = \text{Hom}(\underline{\Delta}_{\Theta}, \underline{\Delta}_{\Theta}) = \widehat{\mathbb{Z}} \cdot \log(\Theta)$$
$$F^1/F^2 = \text{Hom}((\underline{\Delta}^{\text{tm}}_{Y})^{\text{ell}}/\underline{\Delta}_{\Theta}, \underline{\Delta}_{\Theta}) = \widehat{\mathbb{Z}} \cdot \log(U)$$
$$F^2 = H^1(G_K, \underline{\Delta}_{\Theta}) \xrightarrow{\sim} H^1(G_K, \widehat{\mathbb{Z}}(1)) \xrightarrow{\sim} (K^{\times})^{\wedge}$$

[where we use the symbol $\log(\Theta)$ *to denote the identity morphism* $\underline{\Delta}_\Theta \to \underline{\Delta}_\Theta$ *and the symbol* $\log(U)$ *to denote the standard isomorphism* $(\underline{\Delta}_Y^{\text{tm}})^{\text{ell}}/\underline{\Delta}_\Theta \xrightarrow{\sim} \widehat{\mathbb{Z}}(1) \xrightarrow{\sim} \underline{\Delta}_\Theta$*].*

(ii) Similarly, the Leray-Serre spectral sequences associated to the filtration of closed subgroups

$$\underline{\Delta}_\Theta \subseteq (\underline{\Delta}_{\ddot{Y}}^{\text{tm}})^\Theta \subseteq (\underline{\Pi}_{\ddot{Y}}^{\text{tm}})^\Theta$$

determine a natural **filtration** $0 \subseteq \ddot{F}^2 \subseteq \ddot{F}^1 \subseteq \ddot{F}^0 = H^1((\underline{\Pi}_{\ddot{Y}}^{\text{tm}})^\Theta, \underline{\Delta}_\Theta)$ *on the cohomology module* $H^1((\underline{\Pi}_{\ddot{Y}}^{\text{tm}})^\Theta, \underline{\Delta}_\Theta)$ *with subquotients given as follows:*

$$\ddot{F}^0/\ddot{F}^1 = \text{Hom}(\underline{\Delta}_\Theta, \underline{\Delta}_\Theta) = \widehat{\mathbb{Z}} \cdot \log(\Theta)$$

$$\ddot{F}^1/\ddot{F}^2 = \text{Hom}((\underline{\Delta}_{\ddot{Y}}^{\text{tm}})^{\text{ell}}/\underline{\Delta}_\Theta, \underline{\Delta}_\Theta) = \widehat{\mathbb{Z}} \cdot \log(\ddot{U})$$

$$\ddot{F}^2 = H^1(G_{\ddot{K}}, \underline{\Delta}_\Theta) \xrightarrow{\sim} H^1(G_{\ddot{K}}, \widehat{\mathbb{Z}}(1)) \xrightarrow{\sim} (\ddot{K}^\times)^\wedge$$

[where we write $\log(\ddot{U}) \stackrel{\text{def}}{=} \frac{1}{2} \cdot \log(U)$*].*

(iii) Any class $\ddot{\eta}^\Theta \in H^1(\underline{\Pi}_{\ddot{Y}}^{\text{tm}}, \underline{\Delta}_\Theta)$ *arises from a unique class [which, by abuse of notation, we shall denote by]* $\ddot{\eta}^\Theta \in H^1((\underline{\Pi}_{\ddot{Y}}^{\text{tm}})^\Theta, \underline{\Delta}_\Theta)$ *that maps to* $\log(\Theta)$ *in the quotient* $\ddot{F}^0/\ddot{F}^1$ *and on which* $a \in \mathbb{Z} \cong \underline{\mathbb{Z}} \cong \underline{\Pi}_X^{\text{tm}}/\underline{\Pi}_Y^{\text{tm}}$ *acts as follows:*

$$\ddot{\eta}^\Theta \mapsto \ddot{\eta}^\Theta - 2a \cdot \log(\ddot{U}) - \frac{a^2}{2} \cdot \log(q_X) + \log(\mathcal{O}_{\ddot{K}}^\times)$$

[where we use the notation "$\log$*" to express the fact that we wish to write the group structure of* $(\ddot{K}^\times)^\wedge$ **additively***]. Similarly, any* **inversion automorphism** ι *of* $\underline{\Pi}_Y^{\text{tm}}$ *— i.e., an automorphism lying over the action of "–1" on the underlying elliptic curve of* $X^{\log}$ *which fixes the irreducible component of the special fiber of* $\mathfrak{Y}$ *labeled* 0 *—* **fixes** $\ddot{\eta}^\Theta + \log(\mathcal{O}_{\ddot{K}}^\times)$*, but maps* $\log(\ddot{U}) + \log(\mathcal{O}_{\ddot{K}}^\times)$ *to* $-\log(\ddot{U}) + \log(\mathcal{O}_{\ddot{K}}^\times)$*.*

Proof. Assertions (i), (ii) follows immediately from the definitions. Here, in (i) (respectively, (ii)), we note that the fact that F^0 (respectively, $\ddot{F}^0$) *surjects* onto $\text{Hom}(\underline{\Delta}_\Theta, \underline{\Delta}_\Theta)$ follows, for instance, by observing that if $D_y \subseteq (\underline{\Pi}_Y^{\text{tm}})^\Theta$ (respectively, $D_y \subseteq (\underline{\Pi}_{\ddot{Y}}^{\text{tm}})^\Theta$) is a decomposition group of a cusp, then any splitting of the natural surjection $D_y \twoheadrightarrow G_K$ (respectively, $D_y \twoheadrightarrow G_{\ddot{K}}$) [where we recall that it is well-known that such splittings exist — cf., e.g., the discussion at the beginning of [Mzk13], §4] determines, a *splitting* of the natural surjection $(\underline{\Pi}_Y^{\text{tm}})^\Theta \twoheadrightarrow G_K$ (respectively, $(\underline{\Pi}_{\ddot{Y}}^{\text{tm}})^\Theta \twoheadrightarrow G_{\ddot{K}}$). Assertion (iii) follows from Propositions 1.3; 1.4, (ii), (iii). ○

Theorem 1.6. (Tempered Anabelian Absoluteness of the Étale Theta Function) *Let* $X_\alpha^{\log}$ *(respectively,* $X_\beta^{\log}$*) be a* **smooth log curve** *of type* $(1,1)$ *over a finite extension* K_α *(respectively,* K_β*) of* $\mathbb{Q}_p$*; we assume that* $X_\alpha^{\log}$ *(respectively,* $X_\beta^{\log}$*) has* **stable reduction** *over* $\mathcal{O}_{K_\alpha}$ *(respectively,* $\mathcal{O}_{K_\beta}$*), and that the resulting special fibers are* **singular** *and* **split***. We shall use similar notation for objects* *associated to* $X_\alpha^{\log}$, $X_\beta^{\log}$ *[but with a subscript* α *or* β*] to the notation that was used for objects associated to* $X^{\log}$*. Let*

$$\gamma : \underline{\Pi}_{X_\alpha}^{\mathrm{tm}} \xrightarrow{\sim} \underline{\Pi}_{X_\beta}^{\mathrm{tm}}$$

be an **isomorphism of topological groups***. Then:*

(i) We have: $\gamma(\underline{\Pi}_{\ddot{Y}_\alpha}^{\mathrm{tm}}) = \underline{\Pi}_{\ddot{Y}_\beta}^{\mathrm{tm}}$*.*

(ii) γ *induces an isomorphism*

$$(\underline{\Delta}_\Theta)_\alpha \xrightarrow{\sim} (\underline{\Delta}_\Theta)_\beta$$

that is **compatible** *with the surjections*

$$H^1(G_{\ddot{K}_\alpha}, (\underline{\Delta}_\Theta)_\alpha) \xrightarrow{\sim} H^1(G_{\ddot{K}_\alpha}, \widehat{\mathbb{Z}}(1)) \xrightarrow{\sim} (\ddot{K}_\alpha^\times)^\wedge \twoheadrightarrow \widehat{\mathbb{Z}}$$
$$H^1(G_{\ddot{K}_\beta}, (\underline{\Delta}_\Theta)_\beta) \xrightarrow{\sim} H^1(G_{\ddot{K}_\beta}, \widehat{\mathbb{Z}}(1)) \xrightarrow{\sim} (\ddot{K}_\beta^\times)^\wedge \twoheadrightarrow \widehat{\mathbb{Z}}$$

determined by the **valuations** *on* $\ddot{K}_\alpha$, $\ddot{K}_\beta$*. That is to say,* γ *induces an isomorphism* $H^1(G_{\ddot{K}_\alpha}, (\underline{\Delta}_\Theta)_\alpha) \xrightarrow{\sim} H^1(G_{\ddot{K}_\beta}, (\underline{\Delta}_\Theta)_\beta)$ *that preserves both the kernel of these surjections and the elements "*$1 \in \widehat{\mathbb{Z}}$*" in the resulting quotients.*

(iii) The isomorphism of cohomology groups induced by γ **maps** *the classes*

$$\mathcal{O}_{\ddot{K}_\alpha}^\times \cdot \ddot{\eta}_\alpha^\Theta \in H^1(\underline{\Pi}_{\ddot{Y}_\alpha}^{\mathrm{tm}}, (\underline{\Delta}_\Theta)_\alpha)$$

of Proposition 1.3 for X_α *to some* $\underline{\Pi}_{X_\beta}^{\mathrm{tm}}/\underline{\Pi}_{Y_\beta}^{\mathrm{tm}} \cong \underline{\mathbb{Z}}$**-conjugate** *of the corresponding classes*

$$\mathcal{O}_{\ddot{K}_\beta}^\times \cdot \ddot{\eta}_\beta^\Theta \in H^1(\underline{\Pi}_{\ddot{Y}_\beta}^{\mathrm{tm}}, (\underline{\Delta}_\Theta)_\beta)$$

of Proposition 1.3 for X_β*.*

Proof. Assertion (i) is immediate from the definitions; the *discreteness* of the topological group "$\mathbb{Z}$"; and the fact that γ preserves *decomposition groups of cusps* [cf. [Mzk14], Theorem 6.5, (iii)]. As for assertion (ii), the fact that γ induces an isomorphism $(\underline{\Delta}_\Theta)_\alpha \xrightarrow{\sim} (\underline{\Delta}_\Theta)_\beta$ is immediate from the definitions. The asserted *compatibility* then follows from [Mzk14], Theorem 6.12; [Mzk2], Proposition 1.2.1, (iv), (vi), (vii).

Next, we consider assertion (iii). By composing γ with an appropriate *inner automorphism* of $\underline{\Pi}_{X_\beta}^{\mathrm{tm}}$, it follows from [Mzk14], Theorem 6.8, (ii), that we may assume that the isomorphism $\underline{\Pi}_{\ddot{Y}_\alpha}^{\mathrm{tm}} \xrightarrow{\sim} \underline{\Pi}_{\ddot{Y}_\beta}^{\mathrm{tm}}$ is *compatible with suitable "inversion automorphisms"* ι_α, ι_β [cf. Proposition 1.5, (iii)] on both sides. Next, let us observe that it is a *tautology* that γ is compatible with the *symbols* "$\log(\Theta)$" of Proposition 1.5, (i), (ii). On the other hand, by Proposition 1.5, (ii), (iii), the property of "mapping to $\log(\Theta)$ in the quotient $\ddot{F}^0/\ddot{F}^1$ and being fixed, up to a unit multiple, under an inversion automorphism" *completely determines* the classes $\ddot{\eta}^{\Theta}$ up to a $(\ddot{K}^{\times})^{\wedge}$-multiple. Thus, to complete the proof, it suffices to *reduce* this *"indeterminacy up to a* $(\ddot{K}^{\times})^{\wedge}$*-multiple"* to an *"indeterminacy up to a* $\mathcal{O}^{\times}_{\ddot{K}}$*-multiple"*.

This *"reduction of indeterminacy from* $(\ddot{K}^{\times})^{\wedge}$ *to* $\mathcal{O}^{\times}_{\ddot{K}}$*"* may be achieved [in light of the *compatibility* shown in assertion (ii)] by *evaluating* the classes $\ddot{\eta}^{\Theta}$ at a *cusp* that maps to the irreducible component of the special fiber of $\ddot{\mathfrak{Y}}$ *labeled* 0 [e.g., a cusp that is preserved by the inversion automorphism] via the *canonical integral structure*, as in Proposition 1.4, (iii), and applying the fact that, by [Mzk14], Theorem 6.5, (iii); [Mzk14], Corollary 6.11, γ *preserves* both *the decomposition groups* and the *canonical integral structures* [on the decomposition groups] of cusps. ◯

Remark 1.6.1. In the proof of Theorem 1.6, (iii), we eliminated the "indeterminacy" in question by *restricting to cusps, via the canonical integral structure.* Another way to eliminate this indeterminacy is to restrict to *non-cuspidal torsion points*, which are *temp-absolute* by [Mzk14], Theorem 6.8, (iii). This latter approach amounts to invoking the theory of [Mzk13], §2, which is, in some sense, less elementary [for instance, in the sense that it makes use, in a *much more essential way*, of the main result of [Mzk11]] than the theory of [Mzk13], §4 [which one is, in effect, applying if one uses *cusps*].

Remark 1.6.2. One way of thinking about isomorphisms of the tempered fundamental group is that they arise from *variation of the basepoint, or underlying set theory*, relative to which one considers the associated *"temperoids"* [cf. [Mzk14], §3]. Indeed, this is the point of view taken in [Mzk12], in the case of *anabelioids.* From this point of view, the content of Theorem 1.6 may be interpreted as stating that:

> The *étale theta function* is *preserved* by arbitrary *"changes of the underlying set theory"* relative to which one considers the tempered fundamental group in question.

When viewed in this way, Theorem 1.6 may be thought of — especially if one takes the *non-cuspidal* approach of Remark 1.6.1 — as a sort of *nonarchimedean analogue* of the so-called *functional equation* of the classical *complex theta function*, which also states that the "theta function" is preserved, in effect, by "changes of the underlying set theory" relative to which one considers the *integral singular cohomology* of the elliptic curve in question, i.e., more concretely, by the action of the *modular group* $SL_2(\mathbb{Z})$. Note that in the complex case, it is *crucial*, in order to prove the functional equation, to have not only the "Schottky uniformization" $\mathbb{C}^{\times} \twoheadrightarrow \mathbb{C}^{\times}/q^{\mathbb{Z}}$ by $\mathbb{C}^{\times}$ — which naturally gives rise to the analytic series representation of the theta function, but is *not*, however, preserved by the action of the modular group — but also the *full uniformization of an elliptic curve by* $\mathbb{C}$ [which *is* preserved by the action of the modular group]. This "preservation of the full uniformization by $\mathbb{C}$" in the complex case may be regarded as being analogous to the preservation of the *non-cuspidal torsion points* in the approach to proving Theorem 1.6 discussed in Remark 1.6.1.

Remark 1.6.3. The interpretation of Theorem 1.6 given in Remark 1.6.2 is reminiscent of the discussion given in the Introduction of [Mzk7], in which the author expresses his hope, in effect, that some sort of *p-adic analogue of the functional equation of the theta function* could be developed.

Remark 1.6.4. One verifies immediately that there are [easier] *profinite versions* of the constructions given in the present §1: That is to say, if we denote by

$$(\ddot{Y}^{\log})^{\wedge} \to (Y^{\log})^{\wedge} \to X^{\log}$$

the *profinite étale coverings* determined by the tempered coverings $\ddot{Y}^{\log} \to Y^{\log} \to X^{\log}$ [so $\underline{\Pi}_X/\underline{\Pi}_{Y^{\wedge}} \cong \widehat{\mathbb{Z}}$], then the set of classes $\mathcal{O}^{\times}_{\ddot{K}} \cdot \ddot{\eta}^{\Theta} \in H^1(\underline{\Pi}^{\mathrm{tm}}_{\ddot{Y}}, \underline{\Delta}_{\Theta})$ determines, by profinite completion, a set of classes

$$\mathcal{O}^{\times}_{\ddot{K}} \cdot (\ddot{\eta}^{\Theta})^{\wedge} \in H^1(\underline{\Pi}_{\ddot{Y}^{\wedge}}, \underline{\Delta}_{\Theta})$$

on which any $\underline{\Pi}_X/\underline{\Pi}_{Y^{\wedge}} \cong \widehat{\mathbb{Z}} \ni a$ acts via

$$(\ddot{\eta}^{\Theta})^{\wedge} \mapsto (\ddot{\eta}^{\Theta})^{\wedge} - 2a \cdot \log(\ddot{U}) - \frac{a^2}{2} \cdot \log(q_X) + \log(\mathcal{O}^{\times}_{\ddot{K}})$$

[cf. Proposition 1.5, (iii)]. Moreover, given X_{α}, X_{β} as in Theorem 1.6, any isomorphism

$$\underline{\Pi}_{X_{\alpha}} \xrightarrow{\sim} \underline{\Pi}_{X_{\beta}}$$

preserves these *profinite étale theta functions.*

In fact, it is possible to *eliminate the* $\mathcal{O}^{\times}_{\ddot{K}}$*-indeterminacy* of Theorem 1.6, (iii), to a substantial extent [cf. [Mzk13], Corollary 4.12]. For simplicity, let us assume in the following discussion that the following two conditions hold:

(I) $K = \ddot{K}$.

(II) The hyperbolic curve determined by $X^{\log}$ is *not arithmetic over* K [cf., e.g., [Mzk3], Remark 2.1.1].

As is well-known, condition (II) amounts, relative to the j-invariant of the elliptic curve underlying X, to the assertion that we *exclude four exceptional* j*-invariants* [cf. [Mzk3], Proposition 2.7].

Now let us write $\ddot{X}^{\log} \to X^{\log}$ for the Galois [by condition (I)] covering of degree 4 determined by the *"multiplication by* 2*" map* on the elliptic curve underlying X; write $X^{\log} \to C^{\log}$ for the *stack-theoretic quotient* of $X^{\log}$ by the natural action of ± 1 on [the underlying elliptic curve of] X. Thus, [by condition (II)] the hyperbolic orbicurve determined by $C^{\log}$ is a *K-core* [cf. [Mzk3], Remark 2.1.1]. Observe that the covering $\ddot{X}^{\log} \to C^{\log}$ is *Galois*, with Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. Moreover, we have two natural automorphisms

$$\epsilon_\mu \in \mathrm{Gal}(\ddot{X}/X) \subseteq \mathrm{Gal}(\ddot{X}/C); \quad \epsilon_\pm \in \mathrm{Gal}(\ddot{X}/C)$$

— i.e., respectively, the unique nontrivial element of $\mathrm{Gal}(\ddot{X}/X)$ that acts trivially on the set of irreducible components of the special fiber; the unique nontrivial element of $\mathrm{Gal}(\ddot{X}/C)$ that acts trivially on the set of cusps of $\ddot{X}$.

Now suppose that we are given a nontrivial element

$$\epsilon_{\mathbb{Z}} \in \mathrm{Gal}(\ddot{X}/X)$$

which is $\neq \epsilon_\mu$. Then $\epsilon_{\mathbb{Z}}$ determines a *commutative diagram*

$$\begin{array}{ccccccc}
\ddot{Y}^{\log} & \longrightarrow & \ddot{X}^{\log} & \longrightarrow & \dot{X}^{\log} & \longrightarrow & X^{\log} \\
 & & & & \downarrow & & \downarrow \\
 & & & & \dot{C}^{\log} & \longrightarrow & C^{\log}
\end{array}$$

— where $\ddot{Y}^{\log} \to \ddot{X}^{\log}$, $X^{\log} \to C^{\log}$ are the natural morphisms; $\ddot{X}^{\log} \to \dot{X}^{\log}$ is the quotient by the action of $\epsilon_{\mathbb{Z}}$; $\dot{X}^{\log} \to X^{\log}$ is the quotient by the action of ϵ_μ; $\dot{X}^{\log} \to \dot{C}^{\log}$ is the [stack-theoretic] quotient by the action of $\epsilon_\pm \cdot \epsilon_\mu$; the square is *cartesian*.

Definition 1.7. Let K be a finite extension of $\mathbb{Q}_p$ satisfying condition (I) above. Then we shall refer to a smooth log orbicurve over K that arises, up to isomorphism, as the smooth log orbicurve $\dot{X}^{\log}$ (respectively, $\dot{C}^{\log}$) constructed above for some choice of $\epsilon_{\mathbb{Z}}$ as being *of type* $(1, \boldsymbol{\mu}_2)$ (respectively, $(1, \boldsymbol{\mu}_2)_\pm$). We shall also apply this terminology to the associated hyperbolic orbicurves.

Proposition 1.8. (Characteristic Nature of Coverings) *For* $\Box = \alpha, \beta$, *let* $\dot{X}^{\log}_\Box$ *be a* **smooth log curve** *of type* $(1, \boldsymbol{\mu}_2)$ *over a finite extension* $K_\Box$ *of* $\mathbb{Q}_p$*; write* $\ddot{Y}^{\log}_\Box$, $\ddot{X}^{\log}_\Box$, $X^{\log}_\Box$, $\dot{C}^{\log}_\Box$, $C^{\log}_\Box$ *for the related smooth log orbicurves [as in the above discussion]. Then any isomorphism of topological groups*

$$\gamma : \underline{\Pi}^{\mathrm{tm}}_{\dot{X}_\alpha} \xrightarrow{\sim} \underline{\Pi}^{\mathrm{tm}}_{\dot{X}_\beta} \ \textit{(respectively, } \gamma : \underline{\Pi}^{\mathrm{tm}}_{\dot{C}_\alpha} \xrightarrow{\sim} \underline{\Pi}^{\mathrm{tm}}_{\dot{C}_\beta}\textit{)}$$

induces an **isomorphism** *between the commutative diagrams of outer homomorphisms of topological groups*

$$\begin{array}{ccccccc}
\underline{\Pi}^{\mathrm{tm}}_{\ddot{Y}_\Box} & \longrightarrow & \underline{\Pi}^{\mathrm{tm}}_{\ddot{X}_\Box} & \longrightarrow & \underline{\Pi}^{\mathrm{tm}}_{\dot{X}_\Box} & \longrightarrow & \underline{\Pi}^{\mathrm{tm}}_{X_\Box} \\
 & & & & \downarrow & & \downarrow \\
 & & & & \underline{\Pi}^{\mathrm{tm}}_{\dot{C}_\Box} & \longrightarrow & \underline{\Pi}^{\mathrm{tm}}_{C_\Box}
\end{array}$$

[where $\Box = \alpha, \beta$]. A similar statement holds when "$\underline{\Pi}^{\text{tm}}$" is replaced by "$\underline{\Pi}$".

Proof. First, we consider the *tempered case*. By [Mzk14], Theorem 6.8, (ii) [cf. also [Mzk3], Theorem 2.4], it follows from condition (II) that γ induces an isomorphism $\underline{\Pi}^{\text{tm}}_{C_\alpha} \xrightarrow{\sim} \underline{\Pi}^{\text{tm}}_{C_\beta}$ that is compatible with γ. Also, by [Mzk2], Lemma 1.3.8, it follows that γ is compatible with the respective projections to $G_{K_\Box}$. Since [as is easily verified] $\underline{\Delta}^{\text{tm}}_{X_\Box} \subseteq \underline{\Delta}^{\text{tm}}_{C_\Box}$ may be characterized as the unique open subgroup of index 2 that corresponds to a double covering which is a *scheme* [i.e., open subgroup of index 2 that contains no torsion elements], it follows that γ determines an isomorphism $\underline{\Delta}^{\text{tm}}_{X_\alpha} \xrightarrow{\sim} \underline{\Delta}^{\text{tm}}_{X_\beta}$, hence also an isomorphism $(\underline{\Delta}^{\text{tm}}_{X_\alpha})^{\text{ell}} \xrightarrow{\sim} (\underline{\Delta}^{\text{tm}}_{X_\beta})^{\text{ell}}$. Moreover, by considering the *discreteness* of $\text{Gal}(Y_\Box/X_\Box) \cong \underline{\mathbb{Z}}_\Box$, or, alternatively, the *triviality* of the action of $G_{K_\Box}$ on $\text{Gal}(Y_\Box/X_\Box)$, it follows that this last isomorphism determines an isomorphism $\underline{\Delta}^{\text{tm}}_{X_\alpha}/\underline{\Delta}^{\text{tm}}_{Y_\alpha} \xrightarrow{\sim} \underline{\Delta}^{\text{tm}}_{X_\beta}/\underline{\Delta}^{\text{tm}}_{Y_\beta} \cong \underline{\mathbb{Z}}_\beta$, hence [by considering the kernel of the action of $\underline{\Pi}^{\text{tm}}_{C_\Box}$ on $\underline{\Delta}^{\text{tm}}_{X_\Box}/\underline{\Delta}^{\text{tm}}_{Y_\Box}$] an isomorphism $\underline{\Pi}^{\text{tm}}_{X_\alpha} \xrightarrow{\sim} \underline{\Pi}^{\text{tm}}_{X_\beta}$. Since $\dot{X}^{\log}_\Box \to \dot{C}^{\log}_\Box$ may be characterized as the quotient by the unique automorphism of $\dot{X}^{\log}_\Box$ over $C^{\log}_\Box$ that acts nontrivially on the cusps of $\dot{X}^{\log}_\Box$ [where we recall that γ preserves decomposition group of cusps — cf. [Mzk14], Theorem 6.5, (iii)] but does not lie over $X^{\log}_\Box$, we thus conclude that γ induces isomorphisms between the respective $\underline{\Pi}^{\text{tm}}_{\dot{X}_\Box}$, $\underline{\Pi}^{\text{tm}}_{\dot{C}_\Box}$, $\underline{\Pi}^{\text{tm}}_{X_\Box}$, $\underline{\Pi}^{\text{tm}}_{C_\Box}$ that are compatible with the natural inclusions among these subgroups [for a fixed "$\Box$"]. Moreover, since γ preserves the decomposition groups of cusps of $\underline{\Pi}^{\text{tm}}_{X_\Box}$ [cf. [Mzk14], Theorem 6.5, (iii)], we conclude immediately that γ is also compatible with the subgroups $\underline{\Pi}^{\text{tm}}_{\ddot{Y}_\Box} \subseteq \underline{\Pi}^{\text{tm}}_{\ddot{X}_\Box} \subseteq \underline{\Pi}^{\text{tm}}_{X_\Box}$, as desired. The *profinite case* is proven similarly [or may be derived from the tempered case via [Mzk14], Theorem 6.6]. ○

Next, let us suppose that:

$$\sqrt{-1} \in K$$

Note that "$\sqrt{-1}$" determines a 4-*torsion point* τ of [the underlying elliptic curve of] $\dot{X}$ whose restriction to the special fiber lies in the interior of [i.e., avoids the nodes of] the unique irreducible component of the special fiber; the 4-torsion point "τ^{-1}" determined by "$-\sqrt{-1}$" admits a similar description. Let

$$\ddot{\eta}^{\Theta} \in H^1(\underline{\Pi}^{\text{tm}}_{\ddot{Y}}, \underline{\Delta}_\Theta)$$

be a class as in Proposition 1.3; write $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$ for the $\underline{\Pi}^{\text{tm}}_{\ddot{X}}/\underline{\Pi}^{\text{tm}}_{\ddot{Y}} \cong \underline{\mathbb{Z}}$-*orbit* of $\ddot{\eta}^{\Theta}$.

Definition 1.9. Suppose that $\sqrt{-1} \in K$.

(i) We shall refer to either of the following two sets of values [cf. Proposition 1.4, (iii)] of $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$

$$\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}|_\tau, \quad \ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}|_{\tau^{-1}} \subseteq K^\times$$

as a *standard set of values of* $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$.

(ii) If $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$ satisfies the property that the unique value $\in 2 \cdot \mathcal{O}_K^\times$ [cf. the value at $\sqrt{-1}$ of the series representation of $\ddot{\Theta}$ given in Proposition 1.4] contained in some standard set of values of $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$ is equal to ± 2, then we shall say that $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$ is *of standard type.*

Remark 1.9.1. Observe that it is immediate from the definitions that any inner automorphism of $\underline{\Pi}_{\dot{C}}^{\mathrm{tm}}$ arising from $\underline{\Pi}_{\dot{X}}^{\mathrm{tm}}$ acts *trivially* on $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$, and that the automorphisms $(\epsilon_\mu)_\Box$, $(\epsilon_\pm)_\Box$ map $\ddot{\eta}_\Box^{\Theta,\underline{\mathbb{Z}}} \mapsto -\ddot{\eta}_\Box^{\Theta,\underline{\mathbb{Z}}}$ [cf. Proposition 1.4, (ii)]. In particular, any *inner automorphism* of $\underline{\Pi}_{\dot{C}}^{\mathrm{tm}}$ maps $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}} \mapsto \ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$.

The point of view of Remark 1.6.1 motivates the following result:

Theorem 1.10. **(Constant Multiple Rigidity of the Étale Theta Function)** *For $\Box = \alpha, \beta$, let $C_\Box$ be a* **smooth log curve** *of type $(1,1)_\pm$ over a finite extension $K_\Box$ of $\mathbb{Q}_p$ that contains a square root of -1. Let*

$$\gamma : \underline{\Pi}_{\dot{C}_\alpha}^{\mathrm{tm}} \xrightarrow{\sim} \underline{\Pi}_{\dot{C}_\beta}^{\mathrm{tm}}$$

be an isomorphism of topological groups. Then:

(i) The isomorphism γ preserves [cf. Theorem 1.6, (iii)] the property that $\ddot{\eta}_\Box^{\Theta,\underline{\mathbb{Z}}}$ be **of standard type**, *a property that determines this collection of classes up to* **multiplication by** ± 1.

(ii) The isomorphism

$$K_\alpha^\times \xrightarrow{\sim} K_\beta^\times$$

[where we regard $K_\Box^\times \subseteq (K_\Box^\times)^\wedge$ as a subset of $(K_\Box^\times)^\wedge \cong H^1(G_{K_\Box}, (\underline{\Delta}_\Theta)_\Box) \subseteq H^1(\underline{\Pi}_{\dot{C}_\Box}^{\mathrm{tm}}, (\underline{\Delta}_\Theta)_\Box)$] induced by [an arbitrary] γ preserves the **standard sets of values** *of $\ddot{\eta}_\Box^{\Theta,\underline{\mathbb{Z}}}$.*

(iii) Suppose that $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$ is **of standard type**. *Then $\frac{1}{2} \cdot \ddot{\eta}_\Box^{\Theta,\underline{\mathbb{Z}}}$ determines a* **$\{\pm 1\}$-structure** *[cf. [Mzk13], Corollary 4.12; Remark 1.10.3 below] on the $(K_\Box^\times)^\wedge$-torsor at the unique cusp of $\dot{C}_\Box^{\log}$ that is compatible with the* **canonical integral structure** *and, moreover,* **preserved** *by [arbitrary] γ.*

Proof. First, observe that assertions (i), (iii) follow formally from assertion (ii) [cf. also the series representation of Proposition 1.4, concerning the *factor* of $\frac{1}{2}$]. Now we verify assertion (ii), as follows. By [Mzk14], Theorem 6.8, (iii), and the fact that γ induces an automorphism of the dual graph of the special fiber of $\dot{X}$ [cf., Proposition 1.8; [Mzk2], Lemma 2.3; [Mzk14], Lemma 5.5], it follows that γ maps [the decomposition group of] τ to [the decomposition group of] $\tau^{\pm 1}$. Now assertion (ii) follows immediately. ◯

Remark 1.10.1. The "$\pm$-indeterminacy" of Theorem 1.10, (i), is *reminiscent of, but stronger than*, the indeterminacy up to multiplication by a 12-th root of unity of

[Mzk13], Corollary 4.12. Also, we note that from the point of view of the technique of the proof of *loc. cit.*, applied in the present context of *"Tate curves"*, it is the fact that there is a *"special* 2*-torsion point"*, i.e., the 2-torsion point whose image in the special fiber lies in the same irreducible component as the origin, that allows one to reduce the "$12 = 2 \cdot (3!)$" of *loc. cit.* to "2".

Remark 1.10.2. Relative to the issue of *strengthening* Theorem 1.6 in the fashion of Theorem 1.10, one observation that one might make is that since Theorem 1.6 depends on the theory of [Mzk13], §2 [cf. Remark 1.6.1], one natural approach to strengthening Theorem 1.6 is by applying the *"absolute p-adic version of the Grothendieck Conjecture"* of [Mzk15], Corollary 2.12, which may be regarded as a "strengthening" of the theory of [Mzk13], §2. One problem here, however, is that unlike the portion of the theory of [Mzk13], §2, that concerns [non-cuspidal] torsion points of once-punctured elliptic curves [i.e., [Mzk13], Corollary 2.6], the "absolute p-adic version of the Grothendieck Conjecture" of [Mzk15], Corollary 2.12, only holds for elliptic curves which are *defined over number fields.* Moreover, even if, in the future, this hypothesis should be eliminated, the [somewhat weaker] theory of [Mzk13], §2, follows "formally" [cf. [Mzk13], Remark 2.8.1] from certain *"general nonsense"-type* arguments that hold over any base over which the *relative isomorphism version of the Grothendieck Conjecture* [i.e., the isomorphism portion of [Mzk11], Theorem A], together with the *absolute preservation of cuspidal decomposition groups* [cf. [Mzk13], Theorem 1.3, (iii)], holds. In particular, by restricting our attention to *consequences of this "general nonsense"* in the style of [Mzk13], §2, one may hope to generalize the results discussed in the present §1 to *much more general bases* [such as, for instance, $\mathbb{Z}_p[[q]][q^{-1}] \otimes \mathbb{Q}_p$, where q is an indeterminate intended to suggest the "q-parameter of a Tate curve"], or, for instance, to the case of *"pro-Σ versions of the tempered fundamental group"* [i.e., where Σ is a set of primes containing p which is not necessarily the set of all prime numbers] — situations in which it is by no means clear [at least at the time of writing] whether or not it is possible to prove an "absolute version of the Grothendieck Conjecture".

Remark 1.10.3. We take this opportunity to remark that in [Mzk13], Corollary 4.12, the author omitted the hypothesis that "K contain a primitive 12-th root of unity". The author apologizes for this omission.

Section 2: The Theory of Theta Environments

In this §, we begin by discussing various "general nonsense" complements to the theory of étale theta functions of §1. This discussion leads naturally to the theory of the *cyclotomic envelope* and the associated *mono-* and *bi-theta environments*, which we shall use in §5, below, to relate the theory of the present §2 to the theory to be discussed in §3, §4, below.

Let $X^{\log}$ be a *smooth log curve of type* $(1,1)$ over a field K of *characteristic zero*. For simplicity, we assume that the hyperbolic curve determined by $X^{\log}$ is *not* K*-arithmetic* [i.e., admits a K*-core*]. As in §1, we shall denote the *(profinite) étale fundamental group* of $X^{\log}$ by $\underline{\Pi}_X$. Thus, we have a natural *exact sequence*:

$$1 \to \underline{\Delta}_X \to \underline{\Pi}_X \to G_K \to 1$$

[where $G_K \stackrel{\text{def}}{=} \text{Gal}(\overline{K}/K)$; $\underline{\Delta}_X$ is defined so as to make the sequence exact]. Since $\underline{\Delta}_X$ is a *profinite free group on* 2 *generators*, the quotient

$$\underline{\Delta}_X^{\Theta} \stackrel{\text{def}}{=} \underline{\Delta}_X/[\underline{\Delta}_X, [\underline{\Delta}_X, \underline{\Delta}_X]]$$

fits into a *natural exact sequence*:

$$1 \to \underline{\Delta}_{\Theta} \to \underline{\Delta}_X^{\Theta} \to \underline{\Delta}_X^{\text{ell}} \to 1$$

[where $\underline{\Delta}_X^{\text{ell}} \stackrel{\text{def}}{=} \underline{\Delta}_X^{\text{ab}} = \underline{\Delta}_X/[\underline{\Delta}_X, \underline{\Delta}_X]$; we write $\underline{\Delta}_{\Theta}$ for the *image* of $\wedge^2 \underline{\Delta}_X^{\text{ell}}$ in $\underline{\Delta}_X^{\Theta}$]. Also, we shall write $\underline{\Pi}_X \twoheadrightarrow \underline{\Pi}_X^{\Theta}$ for the quotient whose kernel is the kernel of the quotient $\underline{\Delta}_X \twoheadrightarrow \underline{\Delta}_X^{\Theta}$.

Now let $l \geq 1$ be an *integer*. One verifies easily by considering the well-known structure of $\underline{\Delta}_X^{\Theta}$ that the subgroup of $\underline{\Delta}_X^{\Theta}$ generated by l-th powers of elements of $\underline{\Delta}_X^{\Theta}$ is *normal*. We shall write $\underline{\Delta}_X^{\Theta} \twoheadrightarrow \overline{\Delta}_X$ for the quotient of $\underline{\Delta}_X^{\Theta}$ by this normal subgroup. Thus, the above exact sequence for $\underline{\Delta}_X^{\Theta}$ determines a quotient exact sequence:

$$1 \to \overline{\Delta}_{\Theta} \to \overline{\Delta}_X \to \overline{\Delta}_X^{\text{ell}} \to 1$$

[where $\overline{\Delta}_{\Theta} \cong (\mathbb{Z}/l\mathbb{Z})(1)$; $\overline{\Delta}_X^{\text{ell}}$ is a free $(\mathbb{Z}/l\mathbb{Z})$-module of rank 2]. Also, we shall write $\underline{\Pi}_X \twoheadrightarrow \overline{\Pi}_X$ for the quotient whose kernel is the kernel of the quotient $\underline{\Delta}_X \twoheadrightarrow \overline{\Delta}_X$ and $\overline{\Pi}_X^{\text{ell}} \stackrel{\text{def}}{=} \overline{\Pi}_X/\overline{\Delta}_{\Theta}$.

Let us write x for the *unique cusp* of $X^{\log}$. Then there is a natural injective (outer) homomorphism

$$D_x \hookrightarrow \underline{\Pi}_X^{\Theta}$$

[where $D_x \subseteq \underline{\Pi}_X$ is the decomposition group associated to x] which maps the inertia group $I_x \subseteq D_x$ isomorphically onto $\underline{\Delta}_{\Theta}$. Write $\overline{D}_x \subseteq \overline{\Pi}_X$ for the image of D_x in $\overline{\Pi}_X$. Thus, we have exact sequences:

$$1 \to \overline{\Delta}_X \to \overline{\Pi}_X \to G_K \to 1; \quad 1 \to \overline{\Delta}_{\Theta} \to \overline{D}_x \to G_K \to 1$$

Next, let $\overline{\Pi}_X^{\text{ell}} \twoheadrightarrow Q$ be a *quotient* onto a free $(\mathbb{Z}/l\mathbb{Z})$-module Q of rank 1 such that the restricted map $\overline{\Delta}_X^{\text{ell}} \to Q$ is still *surjective*, but the restricted map $D_x \to Q$ is *trivial*. Denote the corresponding *covering* by $\underline{X}^{\log} \to X^{\log}$; write $\overline{\Pi}_{\underline{X}} \subseteq \overline{\Pi}_X$, $\overline{\Delta}_{\underline{X}} \subseteq \overline{\Delta}_X$, $\overline{\Delta}_{\underline{X}}^{\text{ell}} \subseteq \overline{\Delta}_X^{\text{ell}}$ for the corresponding open subgroups. Observe that our assumption that the restricted map $D_x \to Q$ is trivial implies that every cusp of $\underline{X}^{\log}$ is *K-rational*. Let us write ι (respectively, $\underline{\iota}$) for the *automorphism* of $X^{\log}$ (respectively, $\underline{X}^{\log}$) determined by "multiplication by -1" on the underlying elliptic curve relative to choosing the unique cusp of $X^{\log}$ (respectively, relative to some choice of a cusp of $\underline{X}^{\log}$) as the origin. Thus, if we denote the *stack-theoretic quotient* of $X^{\log}$ (respectively, $\underline{X}^{\log}$) by the action of ι (respectively, $\underline{\iota}$) by $C^{\log}$ (respectively, $\underline{C}^{\log}$), then we have a cartesian commutative diagram:

$$\begin{array}{ccc} \underline{X}^{\log} & \longrightarrow & X^{\log} \\ \downarrow & & \downarrow \\ \underline{C}^{\log} & \longrightarrow & C^{\log} \end{array}$$

We shall write $\Pi_{\underline{C}}$, Π_C for the respective (profinite) étale fundamental groups of $\underline{C}^{\log}$, $C^{\log}$. Thus, we obtain subgroups $\Delta_{\underline{C}} \subseteq \Pi_{\underline{C}}$, $\Delta_C \subseteq \Pi_C$ [i.e., the kernels of the natural surjections to G_K]; moreover, by forming the quotient by the kernels of the quotients $\Pi_{\underline{X}} \twoheadrightarrow \overline{\Pi}_{\underline{X}}$, $\Pi_X \twoheadrightarrow \overline{\Pi}_X$, we obtain quotients $\Pi_{\underline{C}} \twoheadrightarrow \overline{\Pi}_{\underline{C}}$, $\Pi_C \twoheadrightarrow \overline{\Pi}_C$, $\Delta_{\underline{C}} \twoheadrightarrow \overline{\Delta}_{\underline{C}}$, $\Delta_C \twoheadrightarrow \overline{\Delta}_C$. Similarly, the quotient $\overline{\Delta}_X \twoheadrightarrow \overline{\Delta}_X^{\text{ell}}$ determines a quotient $\overline{\Delta}_C \twoheadrightarrow \overline{\Delta}_C^{\text{ell}}$. Let $\epsilon_{\underline{\iota}} \in \overline{\Delta}_{\underline{C}}$ be an element that lifts the nontrivial element of $\text{Gal}(\underline{X}/\underline{C}) \cong \mathbb{Z}/2\mathbb{Z}$.

Definition 2.1. We shall refer to a smooth log orbicurve over K that arises, up to isomorphism, as the smooth log orbicurve $\underline{X}^{\log}$ (respectively, $\underline{C}^{\log}$) constructed above for some choice of $\overline{\Pi}_X^{\text{ell}} \twoheadrightarrow Q$ as being *of type* $(1, l\text{-tors})$ (respectively, $(1, l\text{-tors})_{\pm}$). We shall also apply this terminology to the associated hyperbolic orbicurves.

Remark 2.1.1. Note that although $\underline{X}^{\log} \to X^{\log}$ is [by construction] *Galois*, with $\text{Gal}(\underline{X}/X) \cong Q$, the covering $\underline{C}^{\log} \to C^{\log}$ *fails to be Galois* in general. More precisely, *no* nontrivial automorphism $\in \text{Gal}(\underline{X}/X)$ of, say, *odd* order descends to an automorphism of $\underline{C}^{\log}$ over $C^{\log}$. Indeed, this follows from the fact that $\underline{\iota}$ acts on Q by *multiplication by* -1.

Proposition 2.2. **(The Inversion Automorphism)** *Suppose that l is* **odd**. *Then:*

*(i) The conjugation action of $\epsilon_{\underline{\iota}}$ on the rank two $(\mathbb{Z}/l\mathbb{Z})$-***module** $\overline{\Delta}_{\underline{X}}$ *determines a direct product* **decomposition**

$$\overline{\Delta}_{\underline{X}} \cong \overline{\Delta}_{\underline{X}}^{\text{ell}} \times \overline{\Delta}_{\Theta}$$

into eigenspaces, with eigenvalues -1 *and* 1*, respectively, that is compatible with the conjugation action of* $\overline{\Pi}_{\underline{X}}$*. Denote by*

$$s_\iota : \overline{\Delta}^{\mathrm{ell}}_{\underline{X}} \to \overline{\Delta}_{\underline{X}}$$

the resulting splitting of the natural surjection $\overline{\Delta}_{\underline{X}} \twoheadrightarrow \overline{\Delta}^{\mathrm{ell}}_{\underline{X}}$*.*

(ii) In the notation of (i), the normal subgroup $\mathrm{Im}(s_\iota) \subseteq \overline{\Pi}_{\underline{X}}$ *induces an* **isomorphism***:*

$$\overline{D}_x \xrightarrow{\sim} \overline{\Pi}_{\underline{X}}/\mathrm{Im}(s_\iota)$$

In particular, any section of the $H^1(G_K, \overline{\Delta}_\Theta) \cong K^\times/(K^\times)^l$*-torsor of splittings* $\overline{D}_x \twoheadrightarrow G_K$ *determines a covering*

$$\underline{\underline{X}}^{\log} \to \underline{X}^{\log}$$

whose corresponding open subgroup we denote by $\overline{\Pi}_{\underline{\underline{X}}} \subseteq \overline{\Pi}_{\underline{X}}$*. Here, the "geometric portion"* $\overline{\Delta}_{\underline{\underline{X}}}$ *of* $\overline{\Pi}_{\underline{\underline{X}}}$ *maps* **isomorphically** *onto* $\overline{\Delta}^{\mathrm{ell}}_{\underline{X}}$ *[hence is a cyclic group of order* l*].*

(iii) There exists a unique coset $\in \overline{\Delta}_{\underline{C}}/\mathrm{Im}(s_\iota)$ *such that* $\epsilon_{\underline{\iota}}$ *has order* 2 *if and only if it belongs to this coset. If we choose* $\epsilon_{\underline{\iota}}$ *to have order* 2*, then the open subgroup generated by* $\overline{\Pi}_{\underline{\underline{X}}}$ *and* $\epsilon_{\underline{\iota}}$ *in* $\overline{\Pi}_{\underline{C}}$ *[or, alternatively, the open subgroup generated by* $G_K \cong \overline{\Pi}_{\underline{\underline{X}}}/\mathrm{Im}(s_\iota)$ *and* $\epsilon_{\underline{\iota}}$ *in* $\overline{\Pi}_{\underline{C}}/\mathrm{Im}(s_\iota)$*] determines a* **double covering** $\underline{\underline{X}}^{\log} \to \underline{\underline{C}}^{\log}$ *which fits into a cartesian commutative diagram*

$$\begin{array}{ccc} \underline{\underline{X}}^{\log} & \longrightarrow & \underline{X}^{\log} \\ \Big\downarrow & & \Big\downarrow \\ \underline{\underline{C}}^{\log} & \longrightarrow & \underline{C}^{\log} \end{array}$$

[where $\underline{\underline{X}}^{\log}$ *is as in (ii)].*

Proof. Assertions (i) and (ii) are immediate from the definitions. To verify assertion (iii), we observe that $D_x \cong \overline{\Pi}_{\underline{X}}/\mathrm{Im}(s_\iota)$ is of index 2 in $\overline{\Pi}_{\underline{C}}/\mathrm{Im}(s_\iota)$. Thus, $\epsilon_{\underline{\iota}}$ normalizes $D_x \cong \overline{\Pi}_{\underline{X}}/\mathrm{Im}(s_\iota)$. Since, moreover, l is *odd*, and conjugation by $\epsilon_{\underline{\iota}}$ induces the identity on $\overline{\Delta}_\Theta$ and G_K, it follows that $\epsilon_{\underline{\iota}}$ *centralizes* $D_x \cong \overline{\Pi}_{\underline{X}}/\mathrm{Im}(s_\iota)$, hence [*a fortiori*] $G_K \cong \overline{\Pi}_{\underline{\underline{X}}}/\mathrm{Im}(s_\iota)$. Now assertion (iii) follows immediately. ○

Remark 2.2.1. We shall not discuss the case of *even* l in detail here. Nevertheless, we pause briefly to observe that if $l = 2$, then [since $\overline{\Delta}_\Theta$ lies in the *center* of $\overline{\Delta}_X$] the automorphism $\epsilon_\pm \in \mathrm{Gal}(\ddot{X}/C) \cong \overline{\Delta}^{\mathrm{ell}}_C$ of §1 *acts naturally* on the exact sequence $1 \to \overline{\Delta}_\Theta \to \overline{\Delta}_X \to \overline{\Delta}^{\mathrm{ell}}_X \to 1$. Since this action is clearly trivial on $\overline{\Delta}_\Theta$, $\overline{\Delta}^{\mathrm{ell}}_X$, one verifies immediately that this action determines a homomorphism $\overline{\Delta}^{\mathrm{ell}}_X \to \overline{\Delta}_\Theta$, i.e., in effect, a 2-*torsion point* [so long as the homomorphism is nontrivial] of the elliptic curve underlying $X^{\log}$. Thus, by considering the case where K is the *field of moduli* of this elliptic curve [so that G_K permutes the 2-torsion points *transitively*], we conclude that this homomorphism must be *trivial*, i.e., that every element of $\overline{\Delta}_X^{\text{ell}}$ admits an $\epsilon_{\pm}$-*invariant lifting* to $\overline{\Delta}_X$.

Definition 2.3. We shall refer to a smooth log orbicurve over K that arises, up to isomorphism, as the smooth log orbicurve $\underline{\underline{X}}^{\log}$ (respectively, $\underline{\underline{C}}^{\log}$) constructed in Proposition 2.2 above as being *of type* $(1, l\text{-tors}^{\Theta})$ (respectively, $(1, l\text{-tors}^{\Theta})_{\pm}$). We shall also apply this terminology to the associated hyperbolic orbicurves.

Remark 2.3.1. Thus, one may think of the "single underline" in the notation $\underline{X}^{\log}$, $\underline{C}^{\log}$ as denoting the result of *"extracting a single copy of $\mathbb{Z}/l\mathbb{Z}$"*, and the "double underline" in the notation $\underline{\underline{X}}^{\log}$, $\underline{\underline{C}}^{\log}$ as denoting the result of *"extracting two copies of $\mathbb{Z}/l\mathbb{Z}$"*.

Proposition 2.4. **(Characteristic Nature of Coverings)** *For* $\Box = \alpha, \beta$, *let* $\underline{\underline{X}}_{\Box}^{\log}$ *be a* **smooth log curve** *of type* $(1, l\text{-tors}^{\Theta})$ *over a finite extension* $K_{\Box}$ *of* $\mathbb{Q}_p$, *where* l *is* **odd**; *write* $\underline{\underline{C}}_{\Box}^{\log}$, $\underline{X}_{\Box}^{\log}$, $\underline{C}_{\Box}^{\log}$, $X_{\Box}^{\log}$, $C_{\Box}^{\log}$ *for the related smooth log orbicurves [as in the above discussion]. Assume further that* $X_{\Box}^{\log}$ *has* **stable reduction** *over* $\mathcal{O}_{K_{\Box}}$, *with* **singular** *and* **split** *special fiber. Then any isomorphism of topological groups*

$$\gamma : \underline{\Pi}_{\underline{\underline{X}}_{\alpha}} \xrightarrow{\sim} \underline{\Pi}_{\underline{\underline{X}}_{\beta}} \ \textit{(respectively, } \gamma : \underline{\Pi}_{\underline{X}_{\alpha}} \xrightarrow{\sim} \underline{\Pi}_{\underline{X}_{\beta}};$$
$$\gamma : \underline{\Pi}_{\underline{\underline{C}}_{\alpha}} \xrightarrow{\sim} \underline{\Pi}_{\underline{\underline{C}}_{\beta}};\ \gamma : \underline{\Pi}_{\underline{C}_{\alpha}} \xrightarrow{\sim} \underline{\Pi}_{\underline{C}_{\beta}}\textit{)}$$

induces isomorphisms compatible with the various natural maps between the respective "$\underline{\Pi}$'s" of $\underline{X}_{\Box}^{\log}$, $X_{\Box}^{\log}$, $C_{\Box}^{\log}$, $\ddot{Y}_{\Box}^{\log}$ *(respectively,* $X_{\Box}^{\log}$, $C_{\Box}^{\log}$, $\ddot{Y}_{\Box}^{\log}$; $\underline{C}_{\Box}^{\log}$, $C_{\Box}^{\log}$, $\underline{\underline{X}}_{\Box}^{\log}$, $\underline{X}_{\Box}^{\log}$, $X_{\Box}^{\log}$, $\ddot{Y}_{\Box}^{\log}$; $C_{\Box}^{\log}$, $\underline{X}_{\Box}^{\log}$, $X_{\Box}^{\log}$, $\ddot{Y}_{\Box}^{\log}$*) where* $\Box = \alpha, \beta$.

Proof. As in the proof of Proposition 1.8, it follows from our assumption that the hyperbolic curve determined by $X_{\Box}^{\log}$ admits a $K_{\Box}$-*core* that γ induces an isomorphism $\underline{\Pi}_{C_{\alpha}} \xrightarrow{\sim} \underline{\Pi}_{C_{\beta}}$ [cf. [Mzk3], Theorem 2.4] which [cf. [Mzk2], Lemma 1.3.8] induces an isomorphism $\underline{\Delta}_{C_{\alpha}} \xrightarrow{\sim} \underline{\Delta}_{C_{\beta}}$; moreover, this last isomorphism induces [by considering open subgroups of index 2 that contain no torsion elements] an isomorphism $\underline{\Delta}_{X_{\alpha}} \xrightarrow{\sim} \underline{\Delta}_{X_{\beta}}$, hence also [by considering the conjugation action of $\underline{\Pi}_{C_{\Box}}$ on an appropriate abelian quotient of $\underline{\Delta}_{X_{\Box}}$ as in the proof of Proposition 1.8] an isomorphism $\underline{\Pi}_{X_{\alpha}} \xrightarrow{\sim} \underline{\Pi}_{X_{\beta}}$, which preserves the *decomposition groups of cusps* — cf. [Mzk13], Theorem 1.3, (iii). Also, by the definition of $\overline{\Delta}_{X_{\Box}}$, the isomorphism $\underline{\Delta}_{X_{\alpha}} \xrightarrow{\sim} \underline{\Delta}_{X_{\beta}}$ determines an isomorphism $\overline{\Delta}_{X_{\alpha}} \xrightarrow{\sim} \overline{\Delta}_{X_{\beta}}$. In light of these observations, the various assertions of Proposition 2.4 follow immediately from the definitions. ◯

Now, we return to the discussion of §1. In particular, we assume that K is a *finite extension of* $\mathbb{Q}_p$.

Definition 2.5. Suppose that l is *odd* and $K = \ddot{K}$ [cf. Definition 1.7].

(i) Suppose, in the situation of Definitions 2.1, 2.3, that the quotient $\overline{\Pi}_X^{\mathrm{ell}} \twoheadrightarrow Q$ *factors* through the natural quotient $\overline{\Pi}_X^{\mathrm{ell}} \twoheadrightarrow \widehat{\mathbb{Z}}$ discussed at the beginning of §1, and that the *choice of a splitting* of $\overline{D}_x \rightarrow G_K$ [cf. Proposition 2.2, (ii)] that determined the covering $\underline{\underline{X}}^{\log} \rightarrow \underline{X}^{\log}$ is *compatible* with the *"$\{\pm 1\}$-structure"* of Theorem 1.10, (iii). Then we shall say that the orbicurve of type $(1, l\text{-tors})$ (respectively, $(1, l\text{-tors}^{\Theta})$; $(1, l\text{-tors})_{\pm}$; $(1, l\text{-tors}^{\Theta})_{\pm}$) under consideration is *of type* $(1, \mathbb{Z}/l\mathbb{Z})$ (respectively, $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$; $(1, \mathbb{Z}/l\mathbb{Z})_{\pm}$; $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})_{\pm}$).

(ii) In the notation of the above discussion and the discussion at the end of §1, we shall refer to a smooth log orbicurve isomorphic to the smooth log orbicurve

$$\underline{\dot{X}}^{\log} \text{ (respectively, } \underline{\underline{\dot{X}}}^{\log}; \underline{\dot{C}}^{\log}; \underline{\underline{\dot{C}}}^{\log})$$

obtained by taking the composite of the covering

$$\underline{X}^{\log} \text{ (respectively, } \underline{\underline{X}}^{\log}; \underline{C}^{\log}; \underline{\underline{C}}^{\log})$$

of $C^{\log}$ with the covering $\dot{C}^{\log} \rightarrow C^{\log}$, as being *of type* $(1, \boldsymbol{\mu}_2 \times \mathbb{Z}/l\mathbb{Z})$ (respectively, $(1, \boldsymbol{\mu}_2 \times (\mathbb{Z}/l\mathbb{Z})^{\Theta})$; $(1, \boldsymbol{\mu}_2 \times \mathbb{Z}/l\mathbb{Z})_{\pm}$; $(1, \boldsymbol{\mu}_2 \times (\mathbb{Z}/l\mathbb{Z})^{\Theta})_{\pm}$).

Remark 2.5.1. Thus, the *irreducible components* of the special fiber of $\underline{X}$, $\underline{\dot{X}}$ (respectively, $\underline{C}$, $\underline{\dot{C}}$) may be *naturally identified* with the elements of $\mathbb{Z}/l\mathbb{Z}$ (respectively, $(\mathbb{Z}/l\mathbb{Z})/\{\pm 1\}$).

Proposition 2.6. **(Characteristic Nature of Coverings)** *For $\square = \alpha, \beta$, let us assume that we have* **smooth log orbicurves** *as in the above discussion, over a finite extension $K_{\square}$ of $\mathbb{Q}_p$. Then any isomorphism of topological groups*

$$\gamma : \underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\dot{X}}}_{\alpha}} \xrightarrow{\sim} \underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\dot{X}}}_{\beta}} \text{ (respectively, } \gamma : \underline{\Pi}^{\mathrm{tm}}_{\underline{\dot{X}}_{\alpha}} \xrightarrow{\sim} \underline{\Pi}^{\mathrm{tm}}_{\underline{\dot{X}}_{\beta}};$$
$$\gamma : \underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\dot{C}}}_{\alpha}} \xrightarrow{\sim} \underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\dot{C}}}_{\beta}}; \gamma : \underline{\Pi}^{\mathrm{tm}}_{\underline{\dot{C}}_{\alpha}} \xrightarrow{\sim} \underline{\Pi}^{\mathrm{tm}}_{\underline{\dot{C}}_{\beta}})$$

induces isomorphisms compatible with the various natural maps between the respective "$\underline{\Pi}^{\mathrm{tm}}$'s" of $\underline{\underline{X}}^{\log}_{\square}$ (respectively, $\underline{X}^{\log}_{\square}$; $\underline{\underline{C}}^{\log}_{\square}$; $\underline{C}^{\log}_{\square}$) and $\dot{C}^{\log}_{\square}$, where $\square = \alpha, \beta$. A similar statement holds when "$\underline{\Pi}^{\mathrm{tm}}$" is replaced by "$\underline{\Pi}$".

Proof. The proof is entirely similar to the proofs of Propositions 1.8, 2.4. ○

Remark 2.6.1. Suppose, for simplicity, that K contains a *primitive l-th root of unity.* Then we observe in passing that by applying the Propositions 2.4, 2.6

to *"isomorphisms of fundamental groups arising from isomorphisms of the orbicurves in question"* [cf. also Remark 2.1.1], one computes easily that the groups of *K-linear automorphisms* "$\mathrm{Aut}_K(-)$" of the various smooth log orbicurves under consideration are given as follows:

$$\mathrm{Aut}_K(\underline{\underline{X}}^{\log}) = \boldsymbol{\mu}_l \times \{\pm 1\}; \quad \mathrm{Aut}_K(\underline{X}^{\log}) = \mathbb{Z}/l\mathbb{Z} \rtimes \{\pm 1\}$$
$$\mathrm{Aut}_K(\underline{\underline{C}}^{\log}) = \boldsymbol{\mu}_l; \quad \mathrm{Aut}_K(\underline{C}^{\log}) = \{1\}$$

[where $\boldsymbol{\mu}_l$ denotes the group of l-th roots of unity in K, and the semi-direct product "$\rtimes$" is with respect to the natural multiplicative action of ± 1 on $\mathbb{Z}/l\mathbb{Z}$]; the "$\mathrm{Aut}_K(-)$'s" of the various "once-dotted versions" of these orbicurves [cf. Definition 2.5, (ii)] are given by taken the direct product of the "$\mathrm{Aut}_K(-)$'s" listed above with $\mathrm{Gal}(\dot{C}^{\log}/C^{\log}) \cong \{\pm 1\}$.

Next, we consider *étale theta functions*. First, let us observe that the covering $\ddot{Y}^{\log} \to C^{\log}$ *factors naturally* through $\underline{\dot{X}}^{\log}$. Thus, the class [which is only well-defined up to a $\mathcal{O}_K^\times$-multiple]

$$\ddot{\eta}^\Theta \in H^1(\underline{\Pi}^{\mathrm{tm}}_{\underline{\ddot{Y}}}, \underline{\Delta}_\Theta)$$

of §1 — as well as the corresponding $\underline{\Pi}^{\mathrm{tm}}_{\underline{\dot{X}}}/\underline{\Pi}^{\mathrm{tm}}_{\underline{\ddot{Y}}} \cong \underline{\mathbb{Z}}$-*orbit* $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$ — may be thought of as objects associated to the "$\underline{\Pi}^{\mathrm{tm}}$" of $\underline{\dot{X}}^{\log}$, $\underline{\dot{C}}^{\log}$, $\underline{X}^{\log}$, $\underline{C}^{\log}$. On the other hand, the *composites* of the coverings $\ddot{Y}^{\log} \to C^{\log}$, $Y^{\log} \to C^{\log}$ with $\underline{\underline{C}}^{\log} \to C^{\log}$ determine *new coverings*

$$\underline{\underline{\ddot{Y}}}^{\log} \to \ddot{Y}^{\log}; \quad \underline{\underline{Y}}^{\log} \to Y^{\log}$$

of degree l. Moreover, the *choice of a splitting* of $\overline{D}_x \to G_K$ [cf. Proposition 2.2, (ii)] that determined the covering $\underline{\underline{X}}^{\log} \to \underline{X}^{\log}$ determines [by considering the natural map $D_x \to \underline{\Pi}^{\mathrm{tm}}_{\underline{\ddot{Y}}} \to (\underline{\Pi}^{\mathrm{tm}}_{\underline{\ddot{Y}}})^\Theta$ — cf. Proposition 1.5, (ii)] a *specific class* $\in H^1(\underline{\Pi}^{\mathrm{tm}}_{\underline{\ddot{Y}}}, \underline{\Delta}_\Theta \otimes \mathbb{Z}/l\mathbb{Z})$, which may be thought of as a *choice of* $\ddot{\eta}^\Theta$ *up to an* $(\mathcal{O}_K^\times)^l$-*multiple* [i.e., as opposed to only up to a $\mathcal{O}_K^\times$-multiple]. Now it is a *tautology* that, upon restriction to the covering $\underline{\underline{\ddot{Y}}}^{\log} \to \ddot{Y}^{\log}$ [which was determined, in effect, by the *choice of a splitting* of $\overline{D}_x \to G_K$], the class $\ddot{\eta}^\Theta$ determines a class

$$\underline{\underline{\ddot{\eta}}}^\Theta \in H^1(\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\ddot{Y}}}}, l \cdot \underline{\Delta}_\Theta)$$

— as well as a corresponding $l \cdot \underline{\Pi}^{\mathrm{tm}}_{\underline{\dot{X}}}/\underline{\Pi}^{\mathrm{tm}}_{\underline{\ddot{Y}}} \cong l \cdot \underline{\mathbb{Z}}$-*orbit* $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\underline{\mathbb{Z}}}$ — which may be thought of as objects associated to the "$\underline{\Pi}^{\mathrm{tm}}$" of $\underline{\underline{\dot{X}}}^{\log}$, $\underline{\underline{\dot{C}}}^{\log}$, $\underline{\underline{X}}^{\log}$, $\underline{\underline{C}}^{\log}$, and which satisfy the following property:

$$H^1(\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\ddot{Y}}}}, l \cdot \underline{\Delta}_\Theta) \ni \underline{\underline{\ddot{\eta}}}^\Theta \mapsto \ddot{\eta}^\Theta|_{\underline{\underline{\ddot{Y}}}} \in H^1(\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\ddot{Y}}}}, \underline{\Delta}_\Theta)$$

[relative to the natural inclusion $l \cdot \underline{\Delta}_\Theta \hookrightarrow \underline{\Delta}_\Theta$]. That is to say, at a more intuitive level, $\underline{\underline{\ddot{\eta}}}^\Theta$ may be thought of as an *"l-th root of the étale theta function"*. In the following, we shall also consider the $l \cdot \underline{\mathbb{Z}}$-*orbit* $\ddot{\eta}^{\Theta,l\cdot\underline{\mathbb{Z}}}$ of $\ddot{\eta}^\Theta$.

Definition 2.7. If $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$ is of standard type, then we shall also refer to $\ddot{\eta}^{\Theta,l\cdot\underline{\mathbb{Z}}}$, $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\underline{\mathbb{Z}}}$ as being *of standard type.*

Corollary 2.8. (Constant Multiple Rigidity of Roots of the Étale Theta Function) *For* $\Box = \alpha, \beta$, *let us assume that we have* **smooth log orbicurves** *as in the above discussion, over a finite extension* $K_\Box$ *of* $\mathbb{Q}_p$. *Let*

$$\gamma : \Pi^{\text{tm}}_{\underline{\underline{X}}_\alpha} \xrightarrow{\sim} \Pi^{\text{tm}}_{\underline{\underline{X}}_\beta} \ (\textit{respectively, } \gamma : \Pi^{\text{tm}}_{\underline{X}_\alpha} \xrightarrow{\sim} \Pi^{\text{tm}}_{\underline{X}_\beta};$$
$$\gamma : \Pi^{\text{tm}}_{\underline{\underline{C}}_\alpha} \xrightarrow{\sim} \Pi^{\text{tm}}_{\underline{\underline{C}}_\beta}; \ \gamma : \Pi^{\text{tm}}_{\underline{C}_\alpha} \xrightarrow{\sim} \Pi^{\text{tm}}_{\underline{C}_\beta})$$

be an isomorphism of topological groups. Then:

(i) The isomorphism γ **preserves** *the property that* $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\underline{\mathbb{Z}}}_\Box$ *(respectively,* $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}_\Box$; $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\underline{\mathbb{Z}}}_\Box$; $\ddot{\eta}^{\Theta,l\cdot\underline{\mathbb{Z}}}_\Box$*) be* **of standard type** — *a property that determines this collection of classes up to* **multiplication by a root of unity** *of order* $2l$ *(respectively,* 2; $2l$; 2*).*

(ii) Suppose further that the cusps of $\underline{\underline{X}}_\Box$ *are* **rational** *over* $K_\Box$, *and that* $K_\Box$ *contains a* **primitive** l**-th root of unity**. *Then the* $\{\pm 1\}$*-[i.e.,* $\boldsymbol{\mu}_2$*-] structure of Theorem 1.10, (iii), determines a* $\boldsymbol{\mu}_{2l}$ **(respectively,** $\boldsymbol{\mu}_2$**;** $\boldsymbol{\mu}_{2l}$**;** $\boldsymbol{\mu}_2$**)-structure** *[cf. [Mzk13], Corollary 4.12] on the* $(K^\times_\Box)^\wedge$*-torsor at the cusps of* $\underline{\underline{X}}^{\log}_\Box$ *(respectively,* $\underline{X}^{\log}_\Box$; $\underline{\underline{C}}^{\log}_\Box$; $\underline{C}^{\log}_\Box$*). Moreover, this* $\boldsymbol{\mu}_{2l}$ *(respectively,* $\boldsymbol{\mu}_2$; $\boldsymbol{\mu}_{2l}$; $\boldsymbol{\mu}_2$*)-structure is compatible with the* **canonical integral structure** *[cf. [Mzk13], Definition 4.1, (iii)] determined by the stable model of* $X^{\log}_\Box$ *and* **preserved** *by* γ.

(iii) If the data for $\Box = \alpha, \beta$ *are* **equal**, *and* γ *arises [cf. Proposition 2.6] from an* **inner automorphism** $\Pi^{\text{tm}}_{\underline{\underline{\ddot{X}}}_\Box}$ *(respectively,* $\Pi^{\text{tm}}_{\underline{\ddot{X}}_\Box}$; $\Pi^{\text{tm}}_{\underline{\underline{\dot{C}}}_\Box}$; $\Pi^{\text{tm}}_{\underline{\dot{C}}_\Box}$*), then* γ **preserves** $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\underline{\mathbb{Z}}}$ *(respectively,* $\ddot{\eta}^{\Theta,\underline{\mathbb{Z}}}$; $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\underline{\mathbb{Z}}}$; $\ddot{\eta}^{\Theta,l\cdot\underline{\mathbb{Z}}}$*) [i.e., without any constant multiple indeterminacy].*

Proof. First, let us recall the *characteristic nature* of the various coverings involved [cf. Propositions 2.4]. Now assertion (i) follows immediately from Theorem 1.10, (i), and the definitions; assertion (ii) follows immediately from Theorem 1.10, (iii), and the definitions; assertion (iii) follows immediately from Remark 1.9.1. ◯

Before proceeding, we pause to take a closer look at the *cusps* of the various smooth log orbicurves under consideration. First, we recall from the discussion preceding Lemma 1.2 that the irreducible components of the special fiber of $\mathfrak{Y}^{\log}$ may be assigned *labels* $\in \mathbb{Z}$, in a natural fashion. These labels thus determine labels $\in \mathbb{Z}$ for the *cusps* of $Y^{\log}$ [i.e., by considering the irreducible component of the special fiber of $\mathfrak{Y}^{\log}$ that contains the closure in $\mathfrak{Y}$ of the cusp in question]. Moreover, by considering the covering $\underline{\underline{\ddot{Y}}}^{\log} \to Y^{\log}$, we thus obtain labels $\in \mathbb{Z}$ for the cusps of $\underline{\underline{\ddot{Y}}}^{\log}$. Since the various smooth log orbicurves

$$\underline{\underline{\dot{X}}}^{\log}; \quad \underline{\dot{X}}^{\log}; \quad \underline{\underline{\dot{C}}}^{\log}; \quad \underline{\dot{C}}^{\log}; \quad \underline{\underline{X}}^{\log}; \quad \underline{X}^{\log}; \quad \underline{\underline{C}}^{\log}; \quad \underline{C}^{\log}$$

all appear as *subcoverings* of the covering $\underline{\underline{\ddot{Y}}}^{\log} \to X^{\log}$, we thus obtain labels $\in \mathbb{Z}$ for the cusps of these smooth log orbicurves, which are well-defined up to a certain indeterminacy. If we write

$$(\mathbb{Z}/l\mathbb{Z})_{\pm}$$

for the quotient of the set $\mathbb{Z}/l\mathbb{Z}$ by the natural multiplicative action of ± 1, then it follows immediately from the construction of these smooth log orbicurves that this indeterminacy is such that the labels for the cusps of these smooth log orbicurves may be thought of as *well-defined* elements of $(\mathbb{Z}/l\mathbb{Z})_{\pm}$.

Corollary 2.9. (Labels of Cusps) *Suppose that K contains a* **primitive l-th root of unity**. *Then for each of the smooth log orbicurves*

$$\underline{\underline{\dot{X}}}^{\log}; \quad \underline{\underline{\dot{C}}}^{\log}; \quad \underline{\dot{C}}^{\log}; \quad \underline{\underline{X}}^{\log}; \quad \underline{\underline{C}}^{\log}; \quad \underline{C}^{\log}$$

[as defined in the above discussion], the labels of the above discussion determine a **bijection** *of the set*

$$(\mathbb{Z}/l\mathbb{Z})_{\pm}$$

with the set of "$\mathrm{Aut}_K(-)$*"-***orbits** *[cf. Remark 2.6.1] of the cusps of the smooth log orbicurve. Moreover, in the case of $\underline{\underline{X}}^{\log}$, $\underline{\underline{C}}^{\log}$, and $\underline{C}^{\log}$, these bijections are* **preserved** *by arbitrary isomorphisms of topological groups "γ" as in Corollary 2.8.*

Proof. The asserted *bijections* follow immediately by tracing through the definitions of the various smooth log orbicurves [cf. Remark 2.6.1]. The fact that these bijections are *preserved* by "γ" as in Corollary 2.8 follows immediately from the definition of the labels in question in the discussion above, together with the fact that such γ always *preserve the dual graphs* of the special fibers of the orbicurves in question [cf. [Mzk2], Lemma 2.3]. ◯

Remark 2.9.1. We observe in passing that a bijection as in Corollary 2.9 *fails* to hold for $\underline{\dot{X}}^{\log}$, $\underline{X}^{\log}$ — cf. Remarks 2.1.1, 2.6.1.

Remark 2.9.2. In the situation of Corollary 2.8, (ii), we make the following observation, relative to the *labels* of Corollary 2.9: The $2l$ (respectively, 2; $2l$; 2) trivializations of the $(K^{\times}_{\Box})^{\wedge}$-torsor at a cusp *labeled* 0 (respectively, an *arbitrary* cusp; a cusp *labeled* 0; a cusp *labeled* 0) of $\underline{\underline{\dot{X}}}^{\log}_{\Box}$ (respectively, $\underline{\dot{X}}^{\log}_{\Box}$; $\underline{\underline{\dot{C}}}^{\log}_{\Box}$; $\underline{\dot{C}}^{\log}_{\Box}$) determined by the $\boldsymbol{\mu}_{2l}$ (respectively, $\boldsymbol{\mu}_2$; $\boldsymbol{\mu}_{2l}$; $\boldsymbol{\mu}_2$)-structure under discussion are *permuted transitively* by the subgroup of $\mathrm{Aut}_{K_{\Box}}(\underline{\underline{\dot{X}}}^{\log}_{\Box})$ (respectively, $\mathrm{Aut}_{K_{\Box}}(\underline{\dot{X}}^{\log}_{\Box})$; $\mathrm{Aut}_{K_{\Box}}(\underline{\underline{\dot{C}}}^{\log}_{\Box})$; $\mathrm{Aut}_{K_{\Box}}(\underline{\dot{C}}^{\log}_{\Box})$) [cf. Remarks 2.1.1, 2.6.1, 2.9.1] that stabilizes the cusp. In the case of $\underline{\underline{\dot{X}}}^{\log}_{\Box}$, $\underline{\underline{\dot{C}}}^{\log}_{\Box}$, at cusps with *nonzero labels*, the subgroup of the corresponding "$\mathrm{Aut}_{K_{\Box}}(-)$" that stabilizes the cusp permutes the $2l$ trivializations under consideration via the *action of* $\boldsymbol{\mu}_l$ [hence has precisely *two* orbits]. Similarly, in the case of $\underline{\dot{X}}^{\log}_{\Box}$, $\underline{\dot{C}}^{\log}_{\Box}$, at cusps with *nonzero labels*, the subgroup of "$\mathrm{Aut}_{K_{\Box}}(-)$"

that stabilizes the cusp acts *trivially* on the 2 trivializations under consideration [hence has precisely *two* orbits].

Next, let $N \geq 1$ be an *integer*; set:

$$\Delta_{\boldsymbol{\mu}_N} \stackrel{\mathrm{def}}{=} (\mathbb{Z}/N\mathbb{Z}(1)); \quad \Pi_{\boldsymbol{\mu}_N,K} \stackrel{\mathrm{def}}{=} \Delta_{\boldsymbol{\mu}_N} \rtimes G_K$$

Thus, we have a natural exact sequence: $1 \to \Delta_{\boldsymbol{\mu}_N} \to \Pi_{\boldsymbol{\mu}_N,K} \to G_K \to 1$.

Definition 2.10. If $\Pi \twoheadrightarrow G_K$ is a *topological group* equipped with an *augmentation* [i.e., a surjection] to G_K, then we shall write

$$\Pi[\boldsymbol{\mu}_N] \stackrel{\mathrm{def}}{=} \Pi \times_{G_K} \Pi_{\boldsymbol{\mu}_N,K}$$

and refer to $\Pi[\boldsymbol{\mu}_N]$ as the *cyclotomic envelope* of $\Pi \twoheadrightarrow G_K$ [or Π, for short]. Also, if $\Delta \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi \twoheadrightarrow G_K)$, then we shall write

$$\Delta[\boldsymbol{\mu}_N] \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi[\boldsymbol{\mu}_N] \twoheadrightarrow G_K)$$

— so $\Delta[\boldsymbol{\mu}_N] = \Delta \times \Delta_{\boldsymbol{\mu}_N}$; we have a natural exact sequence: $1 \to \Delta[\boldsymbol{\mu}_N] \to \Pi[\boldsymbol{\mu}_N] \to G_K \to 1$. Note that, by construction, we have a *tautological section* $G_K \to \Pi_{\boldsymbol{\mu}_N,K}$ of $\Pi_{\boldsymbol{\mu}_N,K} \twoheadrightarrow G_K$, which determines a section

$$s^*_\Pi : \Pi \to \Pi[\boldsymbol{\mu}_N]$$

of $\Pi[\boldsymbol{\mu}_N] \twoheadrightarrow \Pi$, which we shall also call *tautological.* We shall refer to a collection of subgroups of $\Pi[\boldsymbol{\mu}_N]$ which is obtained as the $\boldsymbol{\mu}_N$-orbit [relative to the action of $\boldsymbol{\mu}_N$ by conjugation] of a single subgroup of $\Pi[\boldsymbol{\mu}_N]$ as a *$\boldsymbol{\mu}_N$-conjugacy class of subgroups* of $\Pi[\boldsymbol{\mu}_N]$.

Proposition 2.11. (General Properties of the Cyclotomic Envelope) *For $\square = \alpha, \beta$, let $\Pi_\square \twoheadrightarrow G_{K_\square}$ be an* **open subgroup** *of either the* **tempered** *or the* **profinite** *fundamental group of a hyperbolic orbicurve over a finite extension $K_\square$ of $\mathbb{Q}_p$; write $\Delta_\square$ for the kernel of the natural morphism $\Pi_\square \to G_{K_\square}$. Then:*

(i) The kernel of the natural surjection $\Delta_\square[\boldsymbol{\mu}_N] \twoheadrightarrow \Delta_\square$ is equal to the **center** *of $\Delta_\square[\boldsymbol{\mu}_N]$. In particular, any isomorphism of topological groups $\Delta_\alpha[\boldsymbol{\mu}_N] \xrightarrow{\sim} \Delta_\beta[\boldsymbol{\mu}_N]$ is* **compatible** *with the natural surjections $\Delta_\square[\boldsymbol{\mu}_N] \twoheadrightarrow \Delta_\square$.*

(ii) The kernel of the natural surjection $\Pi_\square[\boldsymbol{\mu}_N] \twoheadrightarrow \Pi_\square$ is equal to the **union of the centralizers** *of the open subgroups of $\Pi_\square[\boldsymbol{\mu}_N]$. In particular, any isomorphism of topological groups $\Pi_\alpha[\boldsymbol{\mu}_N] \xrightarrow{\sim} \Pi_\beta[\boldsymbol{\mu}_N]$ is* **compatible** *with the natural surjections $\Pi_\square[\boldsymbol{\mu}_N] \twoheadrightarrow \Pi_\square$.*

Proof. Assertions (i), (ii) follow immediately from the *"temp-slimness"* [i.e., the triviality of the centralizers of all open subgroups of] $\Delta_\square$, $\Pi_\square$ [cf. [Mzk14], Example 3.10]. ◯

Next, let us write:

$$\underline{\Pi}_C^{\Theta} \stackrel{\mathrm{def}}{=} \underline{\Pi}_C/\mathrm{Ker}(\underline{\Delta}_X \twoheadrightarrow \underline{\Delta}_X^{\Theta}); \quad \underline{\Delta}_C^{\Theta} \stackrel{\mathrm{def}}{=} \underline{\Delta}_C/\mathrm{Ker}(\underline{\Delta}_X \twoheadrightarrow \underline{\Delta}_X^{\Theta})$$

These quotients thus determine quotients of the *tempered* and *profinite* fundamental groups of $\underline{\underline{\dot{X}}}^{\log}$, $\underline{\dot{X}}^{\log}$, $\dot{X}^{\log}$, $\underline{\underline{X}}^{\log}$, $\underline{X}^{\log}$, $X^{\log}$, $\underline{\underline{\dot{C}}}^{\log}$, $\underline{\dot{C}}^{\log}$, $\dot{C}^{\log}$, $\underline{\underline{C}}^{\log}$, $\underline{C}^{\log}$, $C^{\log}$, which we shall also denote by means of a *superscript "*Θ*"*. Similarly, if we define

$$\underline{\Pi}_C^{\mathrm{ell}} \stackrel{\mathrm{def}}{=} \underline{\Pi}_C/\mathrm{Ker}(\underline{\Delta}_X \twoheadrightarrow \underline{\Delta}_X^{\mathrm{ell}}); \quad \underline{\Delta}_C^{\mathrm{ell}} \stackrel{\mathrm{def}}{=} \underline{\Delta}_C/\mathrm{Ker}(\underline{\Delta}_X \twoheadrightarrow \underline{\Delta}_X^{\mathrm{ell}})$$

then we obtain various induced quotients, which we shall also denote by means of a *superscript "*ell*"*.

Proposition 2.12. (The Cyclotomic Envelope of the Theta Quotient) *Let* Δ_* *be one of the following topological groups:*

$$\underline{\Delta}_{\underline{\underline{X}}}^{\mathrm{tm}}; \quad \underline{\Delta}_{\underline{\underline{\dot{C}}}}^{\mathrm{tm}}; \quad \underline{\Delta}_{\underline{\underline{C}}}^{\mathrm{tm}}; \quad \underline{\Delta}_{\underline{\underline{X}}}; \quad \underline{\Delta}_{\underline{\underline{\dot{C}}}}; \quad \underline{\Delta}_{\underline{\underline{C}}}$$

Then:

(i) We have:

$$\mathrm{Ker}\Big(\Delta_*^{\Theta} \twoheadrightarrow \Delta_*^{\mathrm{ell}}\Big) = l \cdot \underline{\Delta}_{\Theta} \subseteq \Big[\Delta_*^{\Theta}, \Delta_*^{\Theta}\Big]$$

(ii) The intersection

$$\Big[\Delta_*^{\Theta}[\boldsymbol{\mu}_N], \Delta_*^{\Theta}[\boldsymbol{\mu}_N]\Big] \bigcap (l \cdot \underline{\Delta}_{\Theta})[\boldsymbol{\mu}_N] \subseteq (l \cdot \underline{\Delta}_{\Theta})[\boldsymbol{\mu}_N] \subseteq \Delta_*^{\Theta}[\boldsymbol{\mu}_N]$$

coincides with the image of the restriction of the **tautological section** *of* $\Delta_*^{\Theta}[\boldsymbol{\mu}_N] \twoheadrightarrow \Delta_*^{\Theta}$ *to* $l \cdot \underline{\Delta}_{\Theta}$*.*

Proof. First, we consider the inclusion of assertion (i). Now since l is *odd*, the prime-to-2 portion of this inclusion then follows immediately from the well-known *structure of the "theta group"* $(\underline{\Delta}_X^{\mathrm{tm}})^{\Theta}$ [cf. also the definition of the covering $\underline{\underline{\dot{C}}}^{\log} \to C^{\log}$]; in the case of $\underline{\underline{X}}^{\log}$, $\underline{\underline{C}}^{\log}$, the pro-2 portion of this inclusion follows similarly. On the other hand, in the case of $\underline{\underline{\dot{C}}}^{\log}$, the pro-2 portion of this inclusion follows from the fact that, in the notation of Remark 2.2.1 [i.e., more precisely, when "$l = 2$"], if we denote by $\epsilon_{\mathbb{Z}}^{\Delta}, \epsilon_{\mu}^{\Delta} \in \overline{\Delta}_X$ $(\subseteq \overline{\Delta}_C)$, $\epsilon_{\pm}^{\Delta} \in \overline{\Delta}_C$ liftings to $\overline{\Delta}_C$ of the elements of $\overline{\Delta}_C^{\mathrm{ell}}$ determined by the automorphisms "$\epsilon_{\mathbb{Z}}$", "ϵ_{μ}", "$\epsilon_{\pm}$" of the discussion preceding Definition 1.7, then $\epsilon_{\pm}^{\Delta}$ *commutes* with $\epsilon_{\mathbb{Z}}^{\Delta}$, ϵ_{μ}^{Δ} [cf. the observation of Remark 2.2.1], so the *commutator*

$$[\epsilon_{\mathbb{Z}}^{\Delta}, \epsilon_{\pm}^{\Delta} \cdot \epsilon_{\mu}^{\Delta}] = [\epsilon_{\mathbb{Z}}^{\Delta}, \epsilon_{\mu}^{\Delta}]$$

is a *nonzero* element of $\overline{\Delta}_{\Theta}$. Assertion (ii) follows formally from assertion (i). ○

Remark 2.12.1. Note that the inclusion of Proposition 2.12, (i) — which will be *crucial* in the theory to follow — *fails to hold* if one replaces $\underline{\underline{X}}^{\log}$, $\underline{\underline{\dot{C}}}^{\log}$, $\underline{\underline{C}}^{\log}$ by $\underline{\underline{\dot{X}}}^{\log}$, $\dot{X}^{\log}$ [one has problems at the prime 2]; $\underline{\dot{X}}^{\log}$ [one has problems at the prime 2 and the primes dividing l]; $\underline{X}^{\log}$, $\underline{\dot{C}}^{\log}$, $\underline{C}^{\log}$ [one has problems at the primes dividing l]. (There is no problem, however, if one replaces $\underline{\underline{X}}^{\log}$, $\underline{\underline{\dot{C}}}^{\log}$, $\underline{\underline{C}}^{\log}$ by $X^{\log}$, $\dot{C}^{\log}$, $C^{\log}$ since this just corresponds to the case $l = 1$.) Indeed, the original motivation for the introduction of the slightly complicated coverings $\underline{\underline{X}}^{\log}$, $\underline{\underline{\dot{C}}}^{\log}$, $\underline{\underline{C}}^{\log}$ was precisely to avoid these problems.

Next, let us observe that, by *subtracting* [i.e., if we treat cohomology classes additively] the reduction modulo N of *any member* of the collection of [cocycles determined by the collection of] classes $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\mathbb{Z}}$ in $H^1(\Pi^{\text{tm}}_{\underline{\underline{\ddot{Y}}}}, l\cdot\underline{\Delta}_\Theta)$ from the [composite with the inclusion into $\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ of the] *tautological section*

$$s^*_{\underline{\underline{\ddot{Y}}}} \stackrel{\text{def}}{=} s^*_{\Pi^{\text{tm}}_{\underline{\underline{\ddot{Y}}}}} : \Pi^{\text{tm}}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{\text{tm}}_{\underline{\underline{\ddot{Y}}}}[\boldsymbol{\mu}_N] \hookrightarrow \Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$$

[where we apply the natural isomorphism $\boldsymbol{\mu}_N \cong (l \cdot \underline{\Delta}_\Theta) \otimes (\mathbb{Z}/N\mathbb{Z})$] yields a *new homomorphism*:

$$s^\Theta_{\underline{\underline{\ddot{Y}}}} : \Pi^{\text{tm}}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$$

Now since the tautological section $s^*_{\underline{\underline{\ddot{Y}}}}$ extends to a *tautological section* $s^*_{\Pi^{\text{tm}}_{\underline{\underline{C}}}} : \Pi^{\text{tm}}_{\underline{\underline{C}}} \to \Pi^{\text{tm}}_{\underline{\underline{C}}}[\boldsymbol{\mu}_N]$ [where we regard $\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ as a subgroup of $\Pi^{\text{tm}}_{\underline{\underline{C}}}[\boldsymbol{\mu}_N]$], it follows that the *natural outer action*

$$\text{Gal}(\underline{\underline{Y}}/\underline{\underline{C}}) \cong \Pi^{\text{tm}}_{\underline{\underline{C}}}/\Pi^{\text{tm}}_{\underline{\underline{Y}}} \cong \Pi^{\text{tm}}_{\underline{\underline{C}}}[\boldsymbol{\mu}_N]/\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N] \hookrightarrow \text{Out}(\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$$

of $\text{Gal}(\underline{\underline{Y}}/\underline{\underline{C}})$ on $\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ *fixes* the image of $s^*_{\underline{\underline{\ddot{Y}}}}$, up to conjugation by an element of $\boldsymbol{\mu}_N$. In particular, it follows immediately from the definitions that the *various* $s^\Theta_{\underline{\underline{\ddot{Y}}}}$ that arise from *different choices* of [a cocycle contained in] a class $\in \underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\mathbb{Z}}$ are obtained as $l \cdot \underline{\mathbb{Z}} \cong \text{Gal}(\underline{\underline{Y}}/\underline{\underline{X}})$*-conjugates* [where $\text{Gal}(\underline{\underline{Y}}/\underline{\underline{X}}) \subseteq \text{Gal}(\underline{\underline{Y}}/\underline{\underline{C}})$] of any given $s^\Theta_{\underline{\underline{\ddot{Y}}}}$. [Here, we note that "conjugation by an element of $\boldsymbol{\mu}_N$" corresponds precisely to modifying a cocycle by a *coboundary*.] Note, moreover, that we have a *natural outer action*

$$K^\times \twoheadrightarrow (K^\times)/(K^\times)^N \xrightarrow{\sim} H^1(G_K, \boldsymbol{\mu}_N) \hookrightarrow H^1(\Pi^{\text{tm}}_{\underline{\underline{Y}}}, \boldsymbol{\mu}_N) \to \text{Out}(\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$$

[where the " $\xrightarrow{\sim}$ " is the Kummer map] of $K^\times$ on $\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ [which induces the trivial outer action on both the quotient $\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N] \twoheadrightarrow \Pi^{\text{tm}}_{\underline{\underline{Y}}}$ and the kernel of this quotient]. Relative to this outer action, replacing $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\mathbb{Z}}$ by an $\mathcal{O}^\times_K$*-multiple* of $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\mathbb{Z}}$ [cf. Proposition 1.3] corresponds to replacing $s^\Theta_{\underline{\underline{\ddot{Y}}}}$ by an $\mathcal{O}^\times_K$*-conjugate* of $s^\Theta_{\underline{\underline{\ddot{Y}}}}$. Write

$$\mathcal{D}_{\underline{\underline{Y}}} \subseteq \text{Out}(\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$$

for the subgroup of $\mathrm{Out}(\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$ generated by the image of $K^\times$, $\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}})$ ($\cong l \cdot \underline{\mathbb{Z}}$).

Proposition 2.13. (Nonexistence of a Mono-theta-theoretic Basepoint) *In the notation of the above discussion:*

(i) The subset

$$\{\gamma(\beta)\cdot\beta^{-1} \in (\underline{\Delta}^{\mathrm{tm}}_{\underline{\underline{Y}}})^{\Theta}[\boldsymbol{\mu}_N] \mid \beta \in (\underline{\Delta}^{\mathrm{tm}}_{\underline{\underline{Y}}})^{\Theta}[\boldsymbol{\mu}_N], \ \ \gamma \in \mathrm{Aut}(\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]) \ \textit{s.t.} \ \gamma|_{\boldsymbol{\mu}_N} = \mathrm{id}_{\boldsymbol{\mu}_N} \ \textit{and } \gamma \textit{ maps to an element of } \mathcal{D}_{\underline{\underline{Y}}}\} \bigcap (l \cdot \underline{\Delta}_{\Theta})[\boldsymbol{\mu}_N] \subseteq (\underline{\Delta}^{\mathrm{tm}}_{\ddot{\underline{\underline{Y}}}})^{\Theta}[\boldsymbol{\mu}_N]$$

coincides with the image of the **tautological section** *of* $(l \cdot \underline{\Delta}_{\Theta})[\boldsymbol{\mu}_N] \twoheadrightarrow (l \cdot \underline{\Delta}_{\Theta})$.

(ii) Let

$$t^{\Theta}_{\ddot{\underline{\underline{Y}}}} : \Pi^{\mathrm{tm}}_{\ddot{\underline{\underline{Y}}}} \to \Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$$

be a section obtained as a **conjugate** *of* $s^{\Theta}_{\ddot{\underline{\underline{Y}}}}$, *relative to the actions of* $\mathcal{O}^{\times}_{K}$, $(l \cdot \underline{\mathbb{Z}})$. *Write* δ *for the* **cocycle** *of* $\Pi^{\mathrm{tm}}_{\ddot{\underline{\underline{Y}}}}$ *with coefficients in* $\boldsymbol{\mu}_N$ *obtained by subtracting* $s^{\Theta}_{\ddot{\underline{\underline{Y}}}}$ *from* $t^{\Theta}_{\ddot{\underline{\underline{Y}}}}$ *and*

$$\ddot{\alpha}_{\delta} \in \mathrm{Aut}(\Pi^{\mathrm{tm}}_{\ddot{\underline{\underline{Y}}}}[\boldsymbol{\mu}_N])$$

for the **automorphism** *of the topological group* $\Pi^{\mathrm{tm}}_{\ddot{\underline{\underline{Y}}}}[\boldsymbol{\mu}_N]$ *obtained by* **"shifting"** *by* δ *[which induces the identity on both the quotient* $\Pi^{\mathrm{tm}}_{\ddot{\underline{\underline{Y}}}}[\boldsymbol{\mu}_N] \twoheadrightarrow \Pi^{\mathrm{tm}}_{\ddot{\underline{\underline{Y}}}}$ *and the kernel of this quotient]. Then* $\ddot{\alpha}_{\delta}$ *extends to an automorphism* $\alpha_{\delta} \in \mathrm{Aut}(\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$ *which induces the* **identity** *on both the quotient* $\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N] \twoheadrightarrow \Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}$ *and the kernel of this quotient; conjugation by* α_{δ} *maps* $s^{\Theta}_{\ddot{\underline{\underline{Y}}}}$ *to* $t^{\Theta}_{\ddot{\underline{\underline{Y}}}}$ *and* **preserves** *the subgroup* $\mathcal{D}_{\underline{\underline{Y}}} \subseteq \mathrm{Out}(\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$.

Proof. First, we consider assertion (i). Since the automorphisms of $\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ arising from $K^\times$ restrict to the *identity* on $\underline{\Delta}^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$, and $\boldsymbol{\mu}_N$ lies in the *center* of $\underline{\Delta}^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$, one computes easily that assertion (i) follows immediately from Proposition 2.12, (ii), in the case where $\Delta_* = \underline{\Delta}^{\mathrm{tm}}_{\underline{\underline{X}}}$.

Next, we consider assertion (ii). It is immediate from the definitions that conjugation by $\ddot{\alpha}_{\delta}$ maps $s^{\Theta}_{\ddot{\underline{\underline{Y}}}}$ to $t^{\Theta}_{\ddot{\underline{\underline{Y}}}}$. Since the action of $\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}})$ on $\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ *fixes* the section $s^{*}_{\ddot{\underline{\underline{Y}}}}$, up to conjugation by an element of $\boldsymbol{\mu}_N$, it follows that the difference cocycle δ under consideration determines a cohomology class of

$$H^1(\Pi^{\mathrm{tm}}_{\ddot{\underline{\underline{Y}}}}, \boldsymbol{\mu}_N)$$

that lies in the submodule generated by the Kummer classes of $K^\times$ and "$\ddot{U}^{2l\cdot(1/l)} = \ddot{U}^{2}$" [cf. Proposition 1.5, (ii), (iii)]. Here, we note that the factor of "$1/l$" in the exponent of $\ddot{U}$ arises from the fact that to work with $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\underline{\mathbb{Z}}}$ amounts to working with *l-th roots of theta functions* [cf. the discussion preceding Definition 2.7]; the factor of "l" arises from the factor of l in "$l \cdot \underline{\mathbb{Z}}$".

Since, moreover, the meromorphic function "$\ddot{U}^2$" on $\underline{\underline{\ddot{Y}}}$ *descends* to $\underline{\underline{Y}}$, we thus conclude that δ extends to a cocycle of $\Pi^{\text{tm}}_{\underline{\underline{Y}}}$ with coefficients in $\boldsymbol{\mu}_N$, hence that $\ddot{\alpha}_\delta$ extends to an automorphism $\alpha_\delta \in \text{Aut}(\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$ which induces the *identity* on both the quotient $\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N] \twoheadrightarrow \Pi^{\text{tm}}_{\underline{\underline{Y}}}$ and the kernel of this quotient. Since the action by an element of $\text{Gal}(\underline{\underline{Y}}/\underline{\underline{X}})$ clearly maps $\ddot{U}^2$ to a $K^\times$-multiple of $\ddot{U}^2$, it thus follows that conjugation by α_δ preserves $\mathcal{D}_{\underline{\underline{Y}}} \subseteq \text{Out}(\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$ [cf. the definition of $\mathcal{D}_{\underline{\underline{Y}}}$!], as desired. This completes the proof of assertion (ii). ○

Remark 2.13.1. Note that, in the notation of Proposition 2.13, (ii), although the automorphism $\ddot{\alpha}_\delta$ extends to an automorphism α_δ of $\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$, the automorphism α_δ *fails to extend* to $\Pi^{\text{tm}}_{\underline{\underline{X}}}[\boldsymbol{\mu}_N]$ [i.e., since $\ddot{U}^2$ fails to descend from $\underline{\underline{Y}}$ to $\underline{\underline{X}}$!]; thus, it is essential to work with maps $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}, t^{\Theta}_{\underline{\underline{\ddot{Y}}}} : \Pi^{\text{tm}}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$, as opposed to *composites* of such maps with the natural inclusion $\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N] \hookrightarrow \Pi^{\text{tm}}_{\underline{\underline{X}}}[\boldsymbol{\mu}_N]$.

Remark 2.13.2. Note that if, in the situation of Proposition 2.13, one tries to replace $\text{Gal}(\underline{\underline{Y}}/\underline{\underline{X}})$ by $\text{Gal}(\underline{\underline{Y}}/\underline{\underline{C}})$, then one must contend with the *"inversion automorphism"* [cf. Proposition 1.5, (iii)], which maps $\ddot{U} \mapsto \ddot{U}^{-1}$. This obliges one — if one is to retain the property that "conjugation by α_δ preserves $\mathcal{D}_{\underline{\underline{Y}}}$" — to *enlarge* "$\mathcal{D}_{\underline{\underline{Y}}}$" so as to include the outer automorphisms of $\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ that arise from Kummer classes of integral powers of $\ddot{U}^4 = (\ddot{U}^2) \cdot (\ddot{U}^{-2})^{-1}$. On the other hand, if one enlarges $\mathcal{D}_{\underline{\underline{Y}}}$ in this fashion, then one verifies easily [cf. the description of the Kummer class of $\ddot{U}$ in Proposition 1.5, (ii)] that the subset considered in Proposition 2.13, (i), is *no longer contained* in the image of the *tautological section* of $(l \cdot \underline{\Delta}_\Theta)[\boldsymbol{\mu}_N] \twoheadrightarrow (l \cdot \underline{\Delta}_\Theta)$.

Remark 2.13.3. Note that if, in the situation of Proposition 2.13, (ii), $t^{\Theta}_{\underline{\underline{\ddot{Y}}}}$ is obtained as an $N \cdot (l \cdot \underline{\mathbb{Z}})$*-conjugate* of $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$, then the cocycle δ is a *coboundary*; in particular, [in this case] the automorphism α_δ preserves the $\boldsymbol{\mu}_N$*-conjugacy classes of subgroups* determined by the images of $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$, $t^{\Theta}_{\underline{\underline{\ddot{Y}}}}$, $s^{*}_{\underline{\underline{\ddot{Y}}}}$.

Definition 2.14.

(i) We shall refer to $s^{*}_{\underline{\underline{\ddot{Y}}}} : \Pi^{\text{tm}}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ as the *[mod N] algebraic section.* We shall refer to $s^{\Theta}_{\underline{\underline{\ddot{Y}}}} : \Pi^{\text{tm}}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ as the *[mod N] theta section.*

(ii) We shall refer to as a *[mod N] mono-theta environment* the following [ordered] collection of data: (a) the *topological group* $\Pi^{\text{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$; (b) the *subgroup*

$\mathcal{D}_{\underline{\underline{Y}}} \subseteq \mathrm{Out}(\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$; (c) the $\boldsymbol{\mu}_N$-conjugacy class of subgroups $\subseteq \Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ determined by the image of the *theta section* $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$. We shall also refer to as a *[mod N] mono-theta environment* any collection of data consisting of a topological group Π, a subgroup $\mathcal{D}_\Pi \subseteq \mathrm{Out}(\Pi)$, and a collection of subgroups s^{Θ}_{Π} of Π such that there exists an isomorphism of topological groups $\Pi \xrightarrow{\sim} \Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ [cf. (a)] mapping $\mathcal{D}_\Pi \subseteq \mathrm{Out}(\Pi)$ to $\mathcal{D}_{\underline{\underline{Y}}}$ [cf. (b)] and s^{Θ}_{Π} to the $\boldsymbol{\mu}_N$-conjugacy class of (c). We shall refer to as an *isomorphism of [mod N] mono-theta environments* between two [mod N] mono-theta environments

$$(\Pi, \mathcal{D}_\Pi, s^{\Theta}_{\Pi}); \quad (\Pi', \mathcal{D}_{\Pi'}, s^{\Theta}_{\Pi'})$$

any isomorphism of topological groups $\Pi \xrightarrow{\sim} \Pi'$ that maps $\mathcal{D}_\Pi \mapsto \mathcal{D}_{\Pi'}$, $s^{\Theta}_{\Pi} \mapsto s^{\Theta}_{\Pi'}$.

(iii) We shall refer to as a *[mod N] bi-theta environment* the following [ordered] collection of data: (a) the *topological group* $\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$; (b) the *subgroup* $\mathcal{D}_{\underline{\underline{Y}}} \subseteq \mathrm{Out}(\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N])$; (c) the $\boldsymbol{\mu}_N$-conjugacy class of subgroups $\subseteq \Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ determined by the image of the *theta section* $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$; (d) the $\boldsymbol{\mu}_N$-conjugacy class of subgroups $\subseteq \Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ determined by the image of the *algebraic section* $s^{*}_{\underline{\underline{\ddot{Y}}}}$. We shall also refer to as a *[mod N] bi-theta environment* any collection of data consisting of a topological group Π, a subgroup $\mathcal{D}_\Pi \subseteq \mathrm{Out}(\Pi)$, and an ordered pair of collections of subgroups s^{Θ}_{Π}, s^{*}_{Π} of Π such that there exists an isomorphism of topological groups $\Pi \xrightarrow{\sim} \Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}[\boldsymbol{\mu}_N]$ [cf. (a)] mapping $\mathcal{D}_\Pi \subseteq \mathrm{Out}(\Pi)$ to $\mathcal{D}_{\underline{\underline{Y}}}$ [cf. (b)], s^{Θ}_{Π} to the $\boldsymbol{\mu}_N$-conjugacy class of (c), and s^{*}_{Π} to the $\boldsymbol{\mu}_N$-conjugacy class of (d). We shall refer to as an *isomorphism of [mod N] bi-theta environments* between two [mod N] bi-theta environments

$$(\Pi, \mathcal{D}_\Pi, s^{\Theta}_{\Pi}, s^{*}_{\Pi}); \quad (\Pi', \mathcal{D}_{\Pi'}, s^{\Theta}_{\Pi'}, s^{\Theta}_{\Pi'})$$

any isomorphism of topological groups $\Pi \xrightarrow{\sim} \Pi'$ that maps $\mathcal{D}_\Pi \mapsto \mathcal{D}_{\Pi'}$, $s^{\Theta}_{\Pi} \mapsto s^{\Theta}_{\Pi'}$, $s^{*}_{\Pi} \mapsto s^{*}_{\Pi'}$.

(iv) In the situation of (iii), if $\underline{\underline{\ddot{\eta}}}^{\Theta, l\cdot\mathbb{Z}}$ is of standard type, then we shall refer to the resulting [mod N] bi-theta environment as being *of standard type*.

Remark 2.14.1. The existence of *"shifting automorphisms"* as in Proposition 2.13, (ii), may be interpreted as the *"nonexistence of a mono-theta-theoretic basepoint"* [cf. the discussion preceding Proposition 1.1 concerning *"labels"*] relative to the $l \cdot \underline{\mathbb{Z}}$ action on $\underline{Y}$ — i.e., the nonexistence of a *"distinguished irreducible component of the special fiber of $\mathfrak{Y}$"* associated to the data constituted by a mod N mono-theta environment. On the other hand, the description of the *poles of the theta function* [cf. Proposition 1.4, (i)] reveals that the data constituted by a *mod N bi-theta environment* [which includes, by considering the "difference" between the subgroups of Definition 2.14, (iii), (c), (d), a choice of a "specific étale theta function modulo N"] *does determine*, in effect, a *"basepoint modulo N"*, i.e., a distinguished irreducible component of the special fiber of $\mathfrak{Y}$, up to the action of $N \cdot (l \cdot \underline{\mathbb{Z}})$.

The following is the *main result* of the present §2:

Theorem 2.15. (Rigidity Properties of Mono-theta Environments) *Let $N \geq 1$ be an integer. For $\square = \alpha, \beta$, let $\underline{\underline{X}}^{\log}_{\square}$ be a* **smooth log curve** *of type $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$, where l is* **odd**, *over a finite extension $K_{\square}$ of $\mathbb{Q}_p$ such that $K_{\square} = \ddot{K}_{\square}$, and $\underline{\underline{\ddot{\eta}}}^{\Theta, l\cdot\underline{\mathbb{Z}}}_{\square}$ an associated orbit of l-th roots of* **étale theta functions** *which is* **of standard type** *[cf. Definition 2.7]; write $\underline{\underline{Y}}^{\log}_{\square} \to \underline{\underline{X}}^{\log}_{\square}$, $\underline{\underline{\ddot{Y}}}^{\log}_{\square} \to \underline{\underline{Y}}^{\log}_{\square}$ for the corresponding coverings [as in the above discussion],*

$$\Delta^{[\boldsymbol{\mu}_N]}_{\square} \stackrel{\text{def}}{=} \text{Ker}((l \cdot \underline{\Delta}_{\Theta})_{\square}[\boldsymbol{\mu}_N] \to (l \cdot \underline{\Delta}_{\Theta})_{\square})$$

[i.e., for the "$\boldsymbol{\mu}_N$" of "$[\boldsymbol{\mu}_N]$"],

$$\underline{\underline{\ddot{\eta}}}^{\Theta, l\cdot\underline{\mathbb{Z}}}[\boldsymbol{\mu}_N]_{\square}$$

for the collection of classes of $H^1(\Pi^{\text{tm}}_{\underline{\underline{\ddot{Y}}}_{\square}}, \Delta^{[\boldsymbol{\mu}_N]}_{\square})$ obtained by applying the natural surjection $(l \cdot \underline{\Delta}_{\Theta})_{\square} \twoheadrightarrow \Delta^{[\boldsymbol{\mu}_N]}_{\square}$ to $\underline{\underline{\ddot{\eta}}}^{\Theta, l\cdot\underline{\mathbb{Z}}}_{\square}$, and

$$s^{*}_{\underline{\underline{\ddot{Y}}}_{\square}} : \Pi^{\text{tm}}_{\underline{\underline{\ddot{Y}}}_{\square}} \to \Pi^{\text{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N]; \quad s^{\Theta}_{\underline{\underline{\ddot{Y}}}_{\square}} : \Pi^{\text{tm}}_{\underline{\underline{\ddot{Y}}}_{\square}} \to \Pi^{\text{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N]$$

for the resulting mod N **algebraic** *and* **theta** *sections. [Thus, the topological group $\Pi^{\text{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N]$ is equipped with a natural mod N* **mono-** *(respectively,* **bi-***)* **theta environment** *structure.] Let*

$$\gamma : \Pi^{\text{tm}}_{\underline{\underline{Y}}_{\alpha}}[\boldsymbol{\mu}_N] \stackrel{\sim}{\to} \Pi^{\text{tm}}_{\underline{\underline{Y}}_{\beta}}[\boldsymbol{\mu}_N]$$

be an **isomorphism of mono-theta environments**. *Then:*

(i) **(Compatibility with Quotients)** *The isomorphism γ is* **compatible** *with [i.e., preserves the kernel of] the quotients*

$$\Pi^{\text{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N] \twoheadrightarrow \Pi^{\text{tm}}_{\underline{\underline{Y}}_{\square}}; \quad \Pi^{\text{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N] \twoheadrightarrow (\Pi^{\text{tm}}_{\underline{\underline{Y}}_{\square}})^{\Theta}[\boldsymbol{\mu}_N]$$

[where $\square = \alpha, \beta$], hence induces isomorphisms of topological groups $\gamma_{\underline{\underline{Y}}} : \Pi^{\text{tm}}_{\underline{\underline{Y}}_{\alpha}} \stackrel{\sim}{\to} \Pi^{\text{tm}}_{\underline{\underline{Y}}_{\beta}}$, $\gamma^{\Theta} : (\Pi^{\text{tm}}_{\underline{\underline{Y}}_{\alpha}})^{\Theta}[\boldsymbol{\mu}_N] \stackrel{\sim}{\to} (\Pi^{\text{tm}}_{\underline{\underline{Y}}_{\beta}})^{\Theta}[\boldsymbol{\mu}_N]$. Moreover, there exists a **unique isomorphism**

$$\Pi^{\text{tm}}_{\underline{\underline{X}}_{\alpha}} \stackrel{\sim}{\to} \Pi^{\text{tm}}_{\underline{\underline{X}}_{\beta}}$$

that is **compatible** *with $\gamma_{\underline{\underline{Y}}}$, as well as with the quotients $\Pi^{\text{tm}}_{\underline{\underline{X}}_{\square}} \twoheadrightarrow G_{K_{\square}}$ and the subquotients $(l \cdot \underline{\Delta}_{\Theta})_{\square}$ of $\Pi^{\text{tm}}_{\underline{\underline{X}}_{\square}}$ [cf. Corollary 2.18 below for a sort of converse to this statement].*

(ii) **(Cyclotomic Rigidity)** *The isomorphism* γ^{Θ} *induces an isomorphism between the respective subgroups*

$$(l \cdot \underline{\Delta}_{\Theta})_{\square}[\boldsymbol{\mu}_N] \subseteq (\underline{\Delta}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}})^{\Theta}[\boldsymbol{\mu}_N] \subseteq (\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}})^{\Theta}[\boldsymbol{\mu}_N]$$

[cf. Proposition 2.12, (ii)] which is **compatible** *with the kernels of the* **natural surjections** $(l \cdot \underline{\Delta}_{\Theta})_{\square}[\boldsymbol{\mu}_N] \twoheadrightarrow (l \cdot \underline{\Delta}_{\Theta})_{\square}$*, as well as with the splittings of these surjections determined by the* **tautological section** *[cf. Definition 2.10]. In particular, we obtain* **compatible direct product decompositions**

$$(l \cdot \underline{\Delta}_{\Theta})_{\square}[\boldsymbol{\mu}_N] \overset{\sim}{\to} (l \cdot \underline{\Delta}_{\Theta})_{\square} \times \Delta_{\square}^{[\boldsymbol{\mu}_N]}$$

which allow us to regard the [necessarily compatible] resulting restrictions of the given **theta sections** $s^{\Theta}_{\ddot{\underline{\underline{Y}}}_{\square}}$ *as graphs of* **compatible surjections** $(l \cdot \underline{\Delta}_{\Theta})_{\square} \twoheadrightarrow \Delta_{\square}^{[\boldsymbol{\mu}_N]}$ *which are* **preserved** *by* γ*. Put another way, although* $\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N]$ *acts* **nontrivially** *by conjugation on both* $(l \cdot \underline{\Delta}_{\Theta})_{\square}$ *and* $\Delta_{\square}^{[\boldsymbol{\mu}_N]}$*, any choice of a generator of* $(l \cdot \underline{\Delta}_{\Theta})_{\square}$ *determines a generator of* $\Delta_{\square}^{[\boldsymbol{\mu}_N]}$ *in a fashion that is always* **compatible with** γ*.*

(iii) **(Discrete and Constant Multiple Rigidity)** *The isomorphism* γ **preserves** *the collection of classes* $\underline{\underline{\ddot{\eta}}}^{\Theta, l \cdot \underline{\underline{\mathbb{Z}}}}[\boldsymbol{\mu}_N]_{\square}$ *of* $H^1(\underline{\Pi}^{\mathrm{tm}}_{\ddot{\underline{\underline{Y}}}_{\square}}, \Delta_{\square}^{[\boldsymbol{\mu}_N]})$*, up to addition [i.e., "multiplication", if one works multiplicatively — cf. Corollary 2.8, (i)] of a* **root of unity** *of order 2l. Moreover, for* **any member** $\zeta_{\square}$ *of this collection of classes* $\underline{\underline{\ddot{\eta}}}^{\Theta, l \cdot \underline{\underline{\mathbb{Z}}}}[\boldsymbol{\mu}_N]_{\square}$*, there exists an* **automorphism** *of* $\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N]$ *arising from a cocycle of* $\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}}$ *with coefficients in* $\Delta_{\square}^{[\boldsymbol{\mu}_N]}$ *which determines an* **isomorphism of bi-theta environments** *between the bi-theta environments determined by the following two collections of data:*

(a) $\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N]$, $\mathcal{D}_{\underline{\underline{Y}}_{\square}}$*, the [ordered]* **pair** *of* $\boldsymbol{\mu}_N$*-conjugacy classes of subgroups of* $\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N]$ *arising from the given theta section* $s^{\Theta}_{\ddot{\underline{\underline{Y}}}_{\square}}$ *and the section*

$$s^{\Theta}_{\ddot{\underline{\underline{Y}}}_{\square}} + \zeta_{\square}$$

[well-defined up to $\boldsymbol{\mu}_N$**-conjugacy***], obtained by* **adding** $\zeta_{\square}$ *to the given theta function;*

(b) $\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N]$, $\mathcal{D}_{\underline{\underline{Y}}_{\square}}$*, the [ordered]* **pair** *of* $\boldsymbol{\mu}_N$*-conjugacy classes of subgroups of* $\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N]$ *that appear in some mod N* **bi-theta environment of standard type** *[cf. Definition 2.14, (iii), (c), (d); (iv)].*

In particular, for each such $\zeta_{\square}$*, there exists a* **unique lifting** *[relative to the natural morphism* $\mathrm{Out}(\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N]) \to \mathrm{Out}(\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}})$ *— cf. Proposition 2.11, (ii)] of the subgroup* $\underline{\underline{\mathbb{Z}}}_{\square} \subseteq \mathrm{Out}(\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}})$ *to a subgroup of* $\mathcal{D}_{\underline{\underline{Y}}_{\square}} \subseteq \mathrm{Out}(\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}_{\square}}[\boldsymbol{\mu}_N])$ *that* **stabilizes** $s^{\Theta}_{\ddot{\underline{\underline{Y}}}_{\square}} + \zeta_{\square}$*.*

(iv) **(Bi-theta Environments)** *Suppose further that γ is an* **"isomorphism of bi-theta environments"**. *Then the isomorphisms induced by γ are* **compatible** *with the $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\underline{\mathbb{Z}}}[\boldsymbol{\mu}_N]_\square$, and, in fact, with the* **specific members** *of these collections that give rise to $s^{\Theta}_{\underline{\underline{\ddot{Y}}}_\square}$, up to the possible action of an element of $N\cdot l\cdot\underline{\mathbb{Z}}_\square$. Conversely, $N\cdot l\cdot\underline{\mathbb{Z}}_\square$-conjugate members of these collections yield* **isomorphic bi-theta environments** *that are isomorphic via an isomorphism $\Pi^{\rm tm}_{\underline{\underline{Y}}_\alpha}[\boldsymbol{\mu}_N] \xrightarrow{\sim} \Pi^{\rm tm}_{\underline{\underline{Y}}_\beta}[\boldsymbol{\mu}_N]$ that determines [cf. (i)] the same isomorphism $\Pi^{\rm tm}_{\underline{\underline{Y}}_\alpha} \xrightarrow{\sim} \Pi^{\rm tm}_{\underline{\underline{Y}}_\beta}$ as γ.*

Proof. First, we consider assertion (i). The *compatibility* of γ with the quotient $\Pi^{\rm tm}_{\underline{\underline{Y}}_\square}[\boldsymbol{\mu}_N] \twoheadrightarrow \Pi^{\rm tm}_{\underline{\underline{Y}}_\square}$ follows from Proposition 2.11, (ii); the existence of a *unique isomorphism* $\Pi^{\rm tm}_{\underline{\underline{X}}_\alpha} \xrightarrow{\sim} \Pi^{\rm tm}_{\underline{\underline{X}}_\beta}$ that is compatible with $\gamma_{\underline{\underline{Y}}}$ then follows from the *"temp-slimness"* of $\Pi^{\rm tm}_{\underline{\underline{X}}_\square}$ [cf. the proof of Proposition 2.11; [Mzk14], Example 3.10], together with condition (1) of the statement of Theorem 2.15. In particular, it follows from Proposition 2.4 [and the definitions] that $\gamma_{\underline{\underline{Y}}}$ is compatible with the quotients $\Pi^{\rm tm}_{\underline{\underline{Y}}_\square} \twoheadrightarrow (\Pi^{\rm tm}_{\underline{\underline{Y}}_\square})^\Theta$ and the subgroups $(l\cdot\underline{\Delta}_\Theta)_\square \subseteq (\Pi^{\rm tm}_{\underline{\underline{Y}}_\square})^\Theta$, as well as with the quotients $\Pi^{\rm tm}_{\underline{\underline{X}}_\square} \twoheadrightarrow G_{K_\square}$ [cf. [Mzk2], Lemma 1.3.8], for $\square = \alpha, \beta$. Moreover, it follows from the construction of the *theta section* $s^{\Theta}_{\underline{\underline{\ddot{Y}}}_\square}$ [i.e., more precisely the fact that the difference between the theta section and the tautological section *vanishes* on $\mathrm{Ker}(\Pi^{\rm tm}_{\underline{\underline{Y}}_\square} \twoheadrightarrow (\Pi^{\rm tm}_{\underline{\underline{Y}}_\square})^\Theta)$ — cf. Proposition 1.3; the discussion preceding Proposition 2.13] that the kernel of the quotient $\Pi^{\rm tm}_{\underline{\underline{Y}}_\square}[\boldsymbol{\mu}_N] \twoheadrightarrow (\Pi^{\rm tm}_{\underline{\underline{Y}}_\square})^\Theta[\boldsymbol{\mu}_N]$ [where $\square = \alpha, \beta$] may be recovered as the *intersection* of the *image* of $s^{\Theta}_{\underline{\underline{\ddot{Y}}}_\square}$ with the *inverse image* via the quotient $\Pi^{\rm tm}_{\underline{\underline{Y}}_\square}[\boldsymbol{\mu}_N] \twoheadrightarrow \Pi^{\rm tm}_{\underline{\underline{Y}}_\square}$ of $\mathrm{Ker}(\Pi^{\rm tm}_{\underline{\underline{Y}}_\square} \twoheadrightarrow (\Pi^{\rm tm}_{\underline{\underline{Y}}_\square})^\Theta)$. Thus, by condition (2) of the statement of Theorem 2.15, we conclude that γ is compatible with the quotient $\Pi^{\rm tm}_{\underline{\underline{Y}}_\square}[\boldsymbol{\mu}_N] \twoheadrightarrow (\Pi^{\rm tm}_{\underline{\underline{Y}}_\square})^\Theta[\boldsymbol{\mu}_N]$ [where $\square = \alpha, \beta$], as desired.

In light of assertion (i), assertion (ii) follows immediately from Proposition 2.13, (i). In light of assertion (ii), the portion of assertion (iii) concerning *"preservation"* follows immediately from Corollary 2.8, (i); the portion of assertion (iii) concerning the *"existence of automorphisms"* follows immediately from Proposition 2.13, (ii).

Finally, the portion of assertion (iv) concerning *compatibility* with the collection of classes $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\underline{\mathbb{Z}}}[\boldsymbol{\mu}_N]_\square$ (respectively, with specific members of $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\underline{\mathbb{Z}}}[\boldsymbol{\mu}_N]_\square$) follows immediately from the definitions (respectively, the definitions and Proposition 1.5, (ii), (iii)); the *"converse"* portion of assertion (iv) follows immediately from Proposition 2.13, (ii), applied in the case discussed in Remark 2.13.3. ○

Remark 2.15.1. Observe that the properties discussed in Theorem 2.15, (iii), (iv), are indicative of a *fundamental qualitative difference* between *mono-* and *bi-theta* environments. Indeed, if one allows the integer $N \geq 1$ to *vary* [multiplicatively, i.e., in $\mathbb{N}_{\geq 1}$], then the various resulting mono- and bi-theta environments naturally determine *projective systems*. Moreover, it is natural to think of each of the mod

N mono- or bi-theta environments appearing in these projective systems as only being known *up to isomorphism* [cf. Remarks 5.12.1, 5.12.2 in §5 below for more on this point]. From this point of view, Theorem 2.15, (iii), asserts, in effect, that:

> If one works with this projective system of *mono-theta environments*, then in light of the *compatibility* of the various [collections of subgroups determined by the image of the] *theta sections* of the mono-theta environments in the projective system, the various mod N étale theta classes determine, in the projective limit, a
>
> $$\textit{single “discrete” } l \cdot \underline{\mathbb{Z}}\textit{-torsor}$$
>
> whose reduction modulo N [i.e., the result of applying a change of structure group via the homomorphism $l \cdot \underline{\mathbb{Z}} \twoheadrightarrow l \cdot \underline{\mathbb{Z}}/N \cdot l \cdot \underline{\mathbb{Z}}$] appears [cf. the various "$s^{\Theta}_{\underline{\underline{Y}}_\square} + \zeta_\square$" of Theorem 2.15, (iii), (a)] in the mod N mono-theta environment.

By contrast, Theorem 2.15, (iv), implies that if one tries to carry out such a construction in the case of *bi-theta environments*, then since the projective system in question gives rise to a *"basepoint indeterminacy"* [cf. Remark 2.14.1; Theorem 2.15, (iv)], for the mod N bi-theta environment of the system, given precisely by the action of the group $N \cdot l \cdot \underline{\mathbb{Z}}$, the resulting projective limit necessarily leads to a *"torsor of possible basepoints"* [each of which is completely determined, i.e., "determined up to zero indeterminacy"] over the *"non-discrete"* profinite limit group $l \cdot \widehat{\underline{\mathbb{Z}}} \stackrel{\text{def}}{=} l \cdot \underline{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$. Put another way, the crucial *"shifting symmetry"* that exists in the case of a *mono-theta environment* [cf. Proposition 2.13; Remark 2.14.1] gives rise to a "*constant* [i.e., independent of N] $l \cdot \underline{\mathbb{Z}}$*-indeterminacy*", hence implies precisely that, in the mono-theta case, the problem of *"finding a common basepoint"* for the various $(l \cdot \underline{\mathbb{Z}}/N \cdot l \cdot \underline{\mathbb{Z}})$-torsors that appear in the projective system amounts to the issue of *trivializing a torsor over the projective limit*

$$\varprojlim_N \, (l \cdot \underline{\mathbb{Z}}/l \cdot \underline{\mathbb{Z}}) \cong \{0\}$$

— which remains *"discrete"* — whereas in the case of a *bi-theta environment*, the corresponding torsor is a *torsor over the projective limit*

$$\varprojlim_N \, (l \cdot \underline{\mathbb{Z}}/N \cdot l \cdot \underline{\mathbb{Z}}) \cong l \cdot \widehat{\underline{\mathbb{Z}}}$$

— which is "essentially profinite", hence, in particular, *"non-discrete"*. We refer to Corollary 2.16 below for a more explicit description of this *non-discrete-ness* phenomenon in the case of *bi-theta environments*.

Remark 2.15.2. Although the present paper is essentially only concerned with the *"local theory"* of the theta function [i.e., over finite extensions of $\mathbb{Q}_p$], frequently in applications [cf. [Mzk4], [Mzk5]] it is of interest to develop the local theory in such a way that it may be *related naturally to the "global theory"* [i.e., over number fields]. In such situations, one is typically obligated to contend with some sort of *homomorphism of topological groups*

$$\phi : \Pi^{\mathrm{tm}}_{\underline{\underline{X}}} \to \Pi_{\underline{\underline{X}}_F}$$

relating the *tempered fundamental group* of the smooth log curve $\underline{\underline{X}}^{\log}$ [appearing in the theory of the present paper] to the *profinite fundamental group* of a smooth log curve $\underline{\underline{X}}^{\log}_F$ over a number field F such that $\underline{\underline{X}}^{\log}$ is obtained from $\underline{\underline{X}}^{\log}_F$ by base-changing to some completion $K = F_v$ of F at a finite prime v. Moreover, typically, one must assume that ϕ is only given *up to composition with an inner automorphism* [i.e., as an "outer homomorphism"]. Alternatively, one may think of ϕ *"category-theoretically"* via its associated *morphism of temperoids* [cf. [Mzk14], Proposition 3.2]

$$\mathcal{T} \to \mathcal{T}_F$$

— i.e., a *functor* $\mathcal{T}_F \to \mathcal{T}$ [obtained by associating to a $\Pi_{\underline{\underline{X}}_F}$-set the $\Pi^{\mathrm{tm}}_{\underline{\underline{X}}}$-set determined by composing with ϕ], which is typically only determined *up to isomorphism*. In this situation, connected tempered coverings of $\underline{\underline{X}}^{\log}$ [e.g., a finite étale covering of $\underline{\underline{Y}}^{\log}$], which correspond to open subgroups $H \subseteq \Pi^{\mathrm{tm}}_{\underline{\underline{X}}}$, are *subject to an indeterminacy* with respect to *conjugation* by elements of the *normalizer*

$$N_{\Pi_{\underline{\underline{X}}_F}}(\mathrm{Im}(H))$$

of the image of H in $\Pi_{\underline{\underline{X}}_F}$ — i.e., as opposed to just the "weaker" indeterminacy with respect to conjugation by elements of the normalizer $N_{\Pi^{\mathrm{tm}}_{\underline{\underline{X}}}}(H)$, which arises from working with the topological group $\Pi^{\mathrm{tm}}_{\underline{\underline{X}}}$ up to inner automorphism. In this situation, since the two normalizers in question in fact *coincide* — i.e., we have

$$N_{\Pi^{\mathrm{tm}}_{\underline{\underline{X}}}}(H) = N_{\Pi_{\underline{\underline{X}}_F}}(\mathrm{Im}(H))$$

[by Lemma 2.17 below; the well-known fact that the absolute Galois group G_{F_v} is equal to its own normalizer in the absolute Galois group G_F — cf. [Mzk2], Theorem 1.1.1, (i)] — in the present situation, this state of affairs does *not* in fact result in any further indeterminacy [by comparison to the strictly local situation]. Moreover, we observe that *replacing* $\Pi^{\mathrm{tm}}_{\underline{\underline{X}}}$ by H, for instance, does *not* result in any reduction in the indeterminacy to which H is subject. Thus, when H corresponds to a finite étale covering of $\underline{\underline{Y}}^{\log}$, the corresponding covering will always be subject to an indeterminacy with respect to the action of *some finite index subgroup of* $\underline{\mathbb{Z}}$ [an indeterminacy which results in the sort of situation discussed in Remark 2.15.1]. Thus, in summary:

> The indeterminacy which results in the phenomena discussed in Remark 2.15.1 may be regarded as the *inevitable result* of attempting to *accommodate simultaneously* the "two mutually alien copies of $\mathbb{Z}$" constituted by the *geometric* Galois group $\underline{\mathbb{Z}}$ and the *arithmetic* global base $\mathbb{Z} \subseteq \mathbb{Q} \subseteq F$.

[Here, we remark that the *"mutual exclusivity"* of these two copies of $\mathbb{Z}$ arises from the fact that [non-finite] *tempered* coverings only exist *p-adically*, hence *fail to descend* to coverings defined over a number field.]

Remark 2.15.3. One way to try to eliminate the indeterminacy discussed in Remarks 2.15.1, 2.15.2 is to attempt to work with *profinite coverings* of $X^{\log}$ that correspond to the covering $\underline{\underline{X}}^{\log} \to X^{\log}$ for "*l infinite*". On the other hand, such coverings amount to taking *N-th roots* [for all integers $N \geq 1$] of the theta function. In particular, when N is a power of p, this has the effect of *annihilating the differentials* of the curve under consideration. Since the differentials of the curve play an *essential role* in the proof of the main result of [Mzk11], it thus seems unrealistic [at least at the time of writing] to expect to generalize the main result of [Mzk11] [hence also the theory of §1, which depends on this result of [Mzk11] in an essential way] so as to apply to such profinite coverings.

Remark 2.15.4. Relative to the analogy between *Galois group actions* and *differentials* [cf. the discussion of [Mzk4]], the equality of the normalizers discussed in Remark 2.15.2 may be thought of as a sort of *group-theoretic* version of the condition that the map from a finite prime of a number field to the global number field be *"unramified"*.

Remark 2.15.5. The *"cyclotomic rigidity"* of Theorem 2.15, (ii), is a consequence of the *theta section* portion of the data that constitutes a *mono-theta environment* [cf. Definition 2.14, (ii), (c)], together with the subtle property of the *commutator* $[-,-]$ discussed in Proposition 2.12 [which takes the place of the *algebraic section*, which does *not* appear in a mono-theta environment]. Note that this subtle property depends in an essential way on the fact that the étale theta class in question determines an *isomorphism* between the subquotient $\underline{\Delta}_{\Theta}$ of the tempered fundamental group and the cyclotomic coefficients under consideration [cf. Proposition 1.3]. In particular:

> This subtle property *fails to hold* if, instead of considering $\underline{\underline{\ddot{\eta}}}^{\Theta,l\cdot\underline{\mathbb{Z}}}$ over $\underline{\underline{\ddot{Y}}}^{\log}$ [i.e., the *first power* of an l-th root of the theta function — cf. the discussion preceding Definition 2.7], one attempts to use some *M-th power* of the l-th root of the theta function for $M > 1$.

Put another way, if one tries to work with such an M-th power, where $M > 1$, then one ends up only being able to assert the desired "cyclotomic rigidity" for the submodule $M \cdot \boldsymbol{\mu}_N \subseteq \boldsymbol{\mu}_N$ [for, say, N divisible by M]; that is to say, the "remainder" of $\boldsymbol{\mu}_N$ is *not rigid*, but rather *subject to an indeterminacy* with respect to the action of $\mathrm{Ker}((\mathbb{Z}/N\mathbb{Z})^{\times} \twoheadrightarrow (\mathbb{Z}/(N/M)\mathbb{Z})^{\times})$. Alternatively, if, instead of working with *torsion* coefficients [i.e., $\boldsymbol{\mu}_N$] one works with *$\widehat{\mathbb{Z}}$-flat coefficients* [e.g., the inverse limit of the $\boldsymbol{\mu}_N$, as N ranges over the integers ≥ 1], then one may still obtain the [$\widehat{\mathbb{Z}}$-flat analogue of the] desired "cyclotomic rigidity" property of Theorem 2.15, (ii), for $M > 1$, but only at the cost of working with *"profinite coverings"* whose finite subcoverings are *"immune" to automorphism indeterminacy*, which [cf. Corollary 5.12 and the following remarks in §5 below] appears to be somewhat *unnatural.*

Remark 2.15.6. If one thinks in terms of the projective systems discussed in Remark 2.15.1, and one writes $\Delta_{\square}^{[\boldsymbol{\mu}_\infty]}$ for the inverse limit of the $\Delta_{\square}^{[\boldsymbol{\mu}_N]}$ [as N ranges over the integers ≥ 1], then one may think of the *isomorphism*

$$(l \cdot \underline{\Delta}_{\Theta})_{\square} \xrightarrow{\sim} \Delta_{\square}^{[\boldsymbol{\mu}_\infty]}$$

arising from the *"cyclotomic rigidity"* [i.e., the *compatible surjections*] of Theorem 2.15, (ii), as determining a sort of *"integral structure"*, i.e., a sort of *"basepoint"* corresponding to the *first power* of the l-th root of the theta function, relative to the *various M-th powers* of the l-th root of the theta function [cf. Remark 2.15.5] obtained by composing this isomorphism with the map $\Delta_{\square}^{[\boldsymbol{\mu}_\infty]} \to \Delta_{\square}^{[\boldsymbol{\mu}_\infty]}$ on $\Delta_{\square}^{[\boldsymbol{\mu}_\infty]}$ given by multiplication by M. Put another way:

> To work in the *absence* of such a "basepoint" amounts to sacrificing the datum of an *intrisically defined "origin"*, or *"fixed reference point"*, in the system
>
> $$\dots \xrightarrow{M\cdot} \Delta_{\square}^{[\boldsymbol{\mu}_\infty]} \xrightarrow{M\cdot} \Delta_{\square}^{[\boldsymbol{\mu}_\infty]} \xrightarrow{M\cdot} \Delta_{\square}^{[\boldsymbol{\mu}_\infty]} \xrightarrow{M\cdot} \dots$$
>
> obtained by multiplication by M on the cyclotome $\Delta_{\square}^{[\boldsymbol{\mu}_\infty]}$.

Put another way, there is *no intrinsic way to distinguish* "$\Delta_{\square}^{[\boldsymbol{\mu}_\infty]}$" from "$M \cdot \Delta_{\square}^{[\boldsymbol{\mu}_\infty]}$" — i.e., the distinction between these two objects is entirely a matter of *arbitrary labels* [which are typically *implicit* in classical discussions of arithmetic geometry — cf. the discussion of the Introduction to the present paper].

Remark 2.15.7. Finally, before proceeding, it is natural to pause and reflect on the topic of *precisely what one gains* from the discrete and cyclotomic rigidity of Theorem 2.15. On the one hand, *discrete rigidity* assures one that, when one works with the projective systems discussed in Remark 2.15.1, one may restrict to the *$\underline{\mathbb{Z}}$-translates* of [an l-th root of] the theta function without having to worry about confusion with *arbitrary $\widehat{\underline{\mathbb{Z}}}$-translates*, which are *"unnatural"*. At the level of *theta values* [cf., e.g., Proposition 1.4, (iii)], this means that one obtains *values in $K^\times$*, as opposed to $(K^\times)^\wedge$; in particular, it makes sense to perform [not just multiplication operations, but also] *addition operations* involving these values in $K^\times \subseteq K$, which is *not* possible with arbitrary elements of $(K^\times)^\wedge$. On the other hand, *cyclotomic rigidity* assures one that one may work with the *first power* of [an l-th root of] the theta function without having to worry that this first power might be *"confused with some arbitrary λ-th power"*, for $\lambda \in \widehat{\mathbb{Z}}^\times$. At the level of *theta values* [cf., e.g., Proposition 1.4, (iii)], this means that one need not worry about confusion between the "original desired values" in $K^\times \subseteq (K^\times)^\wedge$ and arbitrary λ-th powers of such values in $(K^\times)^\wedge$, for $\lambda \in \widehat{\mathbb{Z}}^\times$ — where again it is useful to recall that raising to the

λ-th power on $(K^\times)^\wedge$ [for $\lambda \in \widehat{\mathbb{Z}}^\times$] is *not a ring homomorphism* [i.e., not compatible with *addition*] unless $\lambda = 1$.

Corollary 2.16. **(Profinite Non-discrete-ness of Bi-theta Environments)** *Fix some member*

$$\underline{\underline{\ddot{\eta}}}^{\Theta}$$

of the collection of [cocycles determined by the collection of] classes $\underline{\underline{\ddot{\eta}}}^{\Theta, l \cdot \underline{\mathbb{Z}}}$ *[cf. the discussion preceding Proposition 2.13] in* $H^1(\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\ddot{Y}}}}, l \cdot \underline{\Delta}_{\Theta})$*. For* $M \in \mathbb{N}_{\geq 1}$*, write*

$$\mathbb{B}_M$$

for the **bi-theta environment** *obtained [cf. the discussion preceding Proposition 2.13] by reducing this* $\underline{\underline{\ddot{\eta}}}^{\Theta}$ *modulo* M*;* $\Pi_M[\boldsymbol{\mu}_M] \twoheadrightarrow \Pi_M$ *for the portion of the data* $\mathbb{B}_M$ *constituted by the topological group [together with its natural surjection] — cf. Definition 2.14, (iii), (a) [so* Π_M *may be thought of as a copy of* $\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{Y}}}$*];* $\ddot{\Pi}_M \subseteq \Pi_M$ *for the subgroup which is the image in* Π_M *of the theta section — cf. Definition 2.14, (iii), (c) [so* $\ddot{\Pi}_M$ *may be thought of as a copy of* $\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\ddot{Y}}}}$*]. Let* $E \subseteq \mathbb{N}_{\geq 1}$ *be a* **cofinal***,* **totally ordered** *subset of* $\mathbb{N}_{\geq 1}$*, relative to the order relation on* $\mathbb{N}_{\geq 1}$ *arising from the monoid structure of* $\mathbb{N}_{\geq 1}$ *[i.e., "*$M \leq M'$*" if and only if* M *divides* M'*] such that* $1 \in E$ *. Thus, we obtain a natural projective system of topological groups*

$$\ldots \longrightarrow \Pi_{M'}[\boldsymbol{\mu}_{M'}] \stackrel{\beta_{M',M}}{\longrightarrow} \Pi_M[\boldsymbol{\mu}_M] \longrightarrow \ldots$$

[where $M, M' \in E$*;* M *divides* M'*]. Let* $j_\infty \in l \cdot \widehat{\underline{\mathbb{Z}}}$*. Then there exists a* **projective system of topological groups**

$$\ldots \longrightarrow \Pi_{M'}[\boldsymbol{\mu}_{M'}] \stackrel{\gamma_{M',M}}{\longrightarrow} \Pi_M[\boldsymbol{\mu}_M] \longrightarrow \ldots$$

[where $M, M' \in E$*;* M *divides* M'*] such that the following properties hold: (a) each* $\gamma_{M',M}$ *is* **compatible** *with the [collections of subgroups constituted by the image of the] algebraic and theta sections of* $\mathbb{B}_M$*; (b) for each* $\gamma_{M',M}$*, there exist* **automorphisms** α**,** α' **of the bi-theta environments** $\mathbb{B}_M$*,* $\mathbb{B}_{M'}$*, respectively, such that* $\gamma_{M',M} = \alpha \circ \beta_{M',M} \circ \alpha'$*; (c) the class of*

$$H^1(\ddot{\Pi}_1, l \cdot \underline{\Delta}_{\Theta}) = H^1(\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\ddot{Y}}}}, l \cdot \underline{\Delta}_{\Theta})$$

obtained by transporting the difference of the algebraic and theta sections of the $\mathbb{B}_M$ *down to* $\ddot{\Pi}_1$ *via the isomorphism* $\ddot{\Pi}_M \stackrel{\sim}{\rightarrow} \ddot{\Pi}_1$ *induced by the* $\gamma_{M',M}$ *is equal to the* j_∞**-conjugate** *of* $\underline{\underline{\ddot{\eta}}}^{\Theta}$*.*

Proof. Indeed, choose a sequence $\{j_M\}_{M \in E}$ of elements of $l \cdot \underline{\mathbb{Z}}$ such that j_M maps to the image of j_∞ in $(l \cdot \underline{\mathbb{Z}}/M \cdot l \cdot \underline{\mathbb{Z}})$, and $j_1 = 0$. Then for $M, M' \in E$ such that M divides M', we take $\gamma_{M',M}$ to be the composite of $\beta_{M',M}$ with the automorphism of $\mathbb{B}_M$ determined by the action of $j_{M'} - j_M \in M \cdot l \cdot \underline{\mathbb{Z}}$ on $\mathbb{B}_M$ [cf. the outer action of $l \cdot \underline{\mathbb{Z}}$ on $\Pi_M[\boldsymbol{\mu}_M]$ appearing in the discussion preceding Proposition 2.13; Remark 2.13.3; Theorem 2.15, (iv)]. Now it is immediate from the construction of the $\gamma_{M',M}$ that the properties (a) [which, in fact, follows formally from (b)], (b), (c) of the statement of Corollary 2.16 are satisfied. ◯

Lemma 2.17. **(Discrete Normalizers)** *If G_1 is a subgroup of a group G_2, then write $N_{G_2}(G_1)$ for the* **normalizer** *of G_1 in G_2. Then:*

(i) Let F be a **free discrete group of finite rank**, *$H \subseteq F$ a* **nonabelian** *subgroup. Write $\widehat{F}$ for the* **profinite completion** *of F. Then $N_{\widehat{F}}(H) = N_F(H)$.*

(ii) Let Π be the **tempered** *fundamental group of a hyperbolic orbicurve over a finite extension $K_\Box$ of $\mathbb{Q}_p$, $H \subseteq \Pi$ an* **open subgroup**. *Write $\widehat{\Pi}$ for the* **profinite completion** *of Π. Then $N_{\widehat{\Pi}}(H) = N_\Pi(H)$.*

Proof. The proof of assertion (i) is similar to [but slightly more involved than] the proof of the case $H = F$ discussed in [André], Corollary 6.2.2: Let $\{x_i\}_{i \in I}$ [where I is some index set of *cardinality* ≥ 2] be a set of generators of H, $a \in N_{\widehat{F}}(H)$. Now let us fix *two distinct elements* $i_1, i_2 \in I$ [so x_{i_1}, x_{i_2} generate a free subgroup of F of rank 2]. Then there exists a subgroup $J \subseteq F$ of *finite index* such that $x_{i_1}, x_{i_2} \in J$, and, moreover, x_{i_1}, x_{i_2} appear in some collection of free generators of J [cf. [Mzk14], Corollary 1.6, (ii)]. In particular, for each $j = 1, 2$, the *centralizer* of x_{i_j} in the profinite completion $\widehat{J}$ of J is topologically generated by x_{i_j}. Moreover, since J is of finite index in F, it follows that for each $j = 1, 2$, there exists a $b_j \in F$ such that $a \in b_j \cdot \widehat{J}$ $(\subseteq \widehat{F})$. Now by a classical result of P. Stebe [cf. [LynSch], Proposition 4.9], it follows that J is *"conjugacy-separated"*, hence [by applying the equality $F \bigcap \widehat{J} = J$] that for each $j = 1, 2$, there exists an $a_j \in b_j \cdot J$ $(\subseteq F)$ such that $a x_{i_j} a^{-1} = a_j x_{i_j} a_j^{-1}$. Thus, for each $j = 1, 2$, $c_j \stackrel{\mathrm{def}}{=} a^{-1} a_j$ belongs to the *centralizer* of x_{i_j} in $\widehat{J}$, hence is of the form $x_{i_j}^{\lambda_j}$, for some $\lambda_j \in \widehat{\mathbb{Z}}$. But this implies that $c_2^{-1} c_1 = a_2^{-1} a_1 \in F \bigcap \widehat{J} = J$, hence [for instance, by considering the image of c_1, c_2 in the *abelianization* of $\widehat{J}$] that $c_1, c_2 \in J \subseteq F$, so $a \in F$, as desired. Assertion (ii) now follows immediately from assertion (i) by applying assertion (i) to free subgroups of finite index of quotients of Π by characteristic open subgroups of Π. ◯

Finally, we observe the following result:

Corollary 2.18. **(Group-theoretic Constructibility of Mono-theta Environments)** *For $\Box = \alpha, \beta$, let $\underline{\underline{X}}_\Box^{\log}$, $s^{\Theta}_{\underline{\underline{\ddot{Y}}}_\Box}$ be as in Theorem 2.15. Then for every isomorphism of topological groups*

$$\Pi^{\mathrm{tm}}_{\underline{\underline{X}}_\alpha} \xrightarrow{\sim} \Pi^{\mathrm{tm}}_{\underline{\underline{X}}_\beta}$$

there exists a **compatible** *[cf. Theorem 2.15, (i)]* **isomorphism of mono-theta environments** $\underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\ddot{Y}}}_\alpha}[\boldsymbol{\mu}_N] \xrightarrow{\sim} \underline{\Pi}^{\mathrm{tm}}_{\underline{\underline{\ddot{Y}}}_\beta}[\boldsymbol{\mu}_N]$.

Proof. By taking the "$\boldsymbol{\mu}_N$" of "[$\boldsymbol{\mu}_N$]" to be *"another copy"* of $(l \cdot \underline{\Delta}_\Theta)_\Box \otimes (\mathbb{Z}/N\mathbb{Z})$, one verifies easily that Corollary 2.18 follows immediately from Corollary 2.8, (i); Proposition 2.13, (ii) [and the definitions]. ◯

Remark 2.18.1. Thus, in a word, Corollary 2.18 may be interpreted as asserting that a mono-theta environment may be regarded as an object naturally constructed from/associated to the *tempered fundamental group*. On the other hand, as we shall see in §5, a mono-theta environment also appears as an object this may be naturally constructed from/associated to a *certain Frobenioid*. In fact:

> One of the *main motivating reasons*, from the point of view of the author, for the introduction of the notion of a mono-theta environment was precisely the fact that it provides a convenient *common ground* for relating the *[tempered-]étale-theoretic* and *Frobenioid-theoretic* approaches to the theta function.

This point of view will be discussed in more detail in Remark 5.10.1 in §5 below.

Section 3: Frobenioids Associated to Semi-stable Curves

In the present §3, we construct certain *Frobenioids* arising from the geometry of *line bundles on tempered coverings of a p-adic curve.* After discussing various properties of these *"tempered Frobenioids"* [cf. Theorem 3.7; Corollary 3.8; Proposition 3.9], we explain how certain aspects of the theory of the *étale theta function* discussed in §1, §2 may be interpreted from the point of view of tempered Frobenioids [cf. Example 3.10].

Let L be a finite extension of $\mathbb{Q}_p$ [where p is a prime number], with ring of integers $\mathcal{O}_L$ and residue field k_L; $\mathfrak{X}$ the formal scheme given by the *p-adic completion* of $\mathrm{Spec}(\mathcal{O}_L)$; $\mathfrak{X}^{\log}$ the formal log scheme obtained by equipping $\mathfrak{X}$ with the log structure determined by the unique closed point of $\mathrm{Spec}(\mathcal{O}_L)$; $\mathfrak{Z}^{\log}$ a *stable log curve* over $\mathfrak{X}^{\log}$. Also, we assume that the *special fiber* $\mathfrak{Z}_{k_L}$ of $\mathfrak{Z}$ is *split*, and that the *generic fiber* of the algebrization of $\mathfrak{Z}^{\log}$ is a *smooth log curve.* Write $Z^{\log} \stackrel{\mathrm{def}}{=} \mathfrak{Z}^{\log} \times_{\mathcal{O}_L} L$ for the ringed space with log structure obtained by tensoring the structure sheaf of $\mathfrak{Z}$ over $\mathcal{O}_L$ with L. In the following discussion, we shall often [by abuse of notation] use the notation $Z^{\log}$ also to denote the generic fiber of the algebrization of $\mathfrak{Z}^{\log}$ [cf. §1].

The *universal covering* of the *dual graph* of the special fiber $\mathfrak{Z}^{\log}_{k_L}$ of $\mathfrak{Z}^{\log}$ determines an *infinite Galois étale covering*

$$\mathfrak{Z}^{\log}_{\infty} \to \mathfrak{Z}^{\log}$$

of $\mathfrak{Z}^{\log}$; such *"universal combinatorial coverings"* appear in the theory of the *tempered fundamental group* [cf. [André], §4; [Mzk14], Example 3.10]. Thus, $\mathfrak{Z}^{\log}_{\infty}$ is a *formal log scheme*; write $Z^{\log}_{\infty} \stackrel{\mathrm{def}}{=} \mathfrak{Z}^{\log}_{\infty} \times_{\mathcal{O}_L} L$. Also, we shall refer to the inverse image of the *divisor of cusps* of $\mathfrak{Z}^{\log}$ in $\mathfrak{Z}^{\log}_{\infty}$ as the divisor of cusps of $\mathfrak{Z}^{\log}_{\infty}$ and to $\mathfrak{Z}^{\log}_{\infty}$ as the *stable model* of $Z^{\log}_{\infty}$.

Definition 3.1.

(i) A divisor on $\mathfrak{Z}_{\infty}$ whose support lies in the special fiber $(\mathfrak{Z}_{\infty})_{k_L}$ (respectively, the divisor of cusps of $\mathfrak{Z}^{\log}_{\infty}$; the union of the special fiber and divisor of cusps of $\mathfrak{Z}^{\log}_{\infty}$) will be referred to as a *non-cuspidal log-divisor* (respectively, *cuspidal log-divisor*; *log-divisor*) on $\mathfrak{Z}^{\log}_{\infty}$. Write

$$\mathrm{DIV}(\mathfrak{Z}^{\log}_{\infty}) \text{ (respectively, } \mathrm{DIV}_+(\mathfrak{Z}^{\log}_{\infty});\ \mathrm{Div}(\mathfrak{Z}^{\log}_{\infty});\ \mathrm{Div}_+(\mathfrak{Z}^{\log}_{\infty}))$$

for the monoid of log-divisors (respectively, effective log-divisors; Cartier log-divisors; effective Cartier log-divisors) on $\mathfrak{Z}^{\log}$. Thus, we have natural inclusions

$$\mathrm{Div}_+(\mathfrak{Z}^{\log}_{\infty}) \subseteq \mathrm{DIV}_+(\mathfrak{Z}^{\log}_{\infty}) \subseteq \mathrm{DIV}(\mathfrak{Z}^{\log}_{\infty})$$

$$\mathrm{Div}_+(\mathfrak{Z}^{\log}_{\infty}) \subseteq \mathrm{Div}(\mathfrak{Z}^{\log}_{\infty}) \subseteq \mathrm{DIV}(\mathfrak{Z}^{\log}_{\infty})$$

and a natural identification $\mathrm{DIV}(\mathfrak{Z}^{\log}_{\infty}) = \mathrm{DIV}_+(\mathfrak{Z}^{\log}_{\infty})^{\mathrm{gp}}$.

(ii) A nonzero meromorphic function on $\mathfrak{Z}_\infty^{\log}$ whose divisor of zeroes and poles is a log-divisor will be referred to as a *log-meromorphic function* on $\mathfrak{Z}_\infty^{\log}$. The group of log-meromorphic functions on $\mathfrak{Z}_\infty^{\log}$ will be denoted $\mathrm{Mero}(\mathfrak{Z}_\infty^{\log})$. A log-meromorphic function arising from $L^\times$ will be referred to as *constant*.

Proposition 3.2. (Divisors and Rational Functions on Universal Combinatorial Coverings) *In the notation of the above discussion:*

(i) There exists a positive integer n such that $n \cdot \mathrm{DIV}_+(\mathfrak{Z}_\infty^{\log}) \subseteq \mathrm{Div}_+(\mathfrak{Z}_\infty^{\log})$, $n \cdot \mathrm{DIV}(\mathfrak{Z}_\infty^{\log}) \subseteq \mathrm{Div}(\mathfrak{Z}_\infty^{\log})$. In particular, there exists a **natural isomorphism**

$$\mathrm{Div}_+(\mathfrak{Z}_\infty^{\log})^{\mathrm{pf}} \xrightarrow{\sim} \mathrm{DIV}_+(\mathfrak{Z}_\infty^{\log})^{\mathrm{pf}}$$

— where $\mathrm{DIV}_+(\mathfrak{Z}_\infty^{\log})^{\mathrm{pf}}$ may be naturally identified with a **direct product** *of copies of $\mathbb{Q}_{\geq 0}$, indexed by the* **cusps** *[i.e., irreducible components of the divisor of cusps] and* **irreducible components of the special fiber** *of $\mathfrak{Z}_\infty^{\log}$.*

(ii) The structure morphism $\mathfrak{Z}_\infty^{\log} \to \mathfrak{X}^{\log}$ determines a **natural isomorphism** $\mathcal{O}_L \xrightarrow{\sim} \Gamma(\mathfrak{Z}_\infty, \mathcal{O}_{\mathfrak{Z}_\infty})$ *— i.e., "all* **regular** *functions on $\mathfrak{Z}_\infty$ are* **constant***".*

(iii) Let f be a nonzero **meromorphic function** *on $\mathfrak{Z}_\infty$ such that for every $N \in \mathbb{N}_{\geq 1}$ [cf. §0], there exists a meromorphic function g_N on $\mathfrak{Z}_\infty$ such that $g_N^N = f$. Then $f = 1$.*

Proof. To verify assertion (i), let us first observe that the completion of $\mathfrak{Z}_\infty$ along a node of $\mathfrak{Z}_\infty$ may be identified with the formal spectrum of a complete local ring of the form $\mathcal{O}_L[[x, y]]/(xy - \pi_L^e)$, where π_L is a uniformizer of $\mathcal{O}_L$, and e is a positive integer; moreover, despite the "infinite" nature of $\mathfrak{Z}_\infty$, the number of "e's" that occur at completions of $\mathfrak{Z}_\infty$ along its nodes is *finite* [cf. the definition of $\mathfrak{Z}_\infty^{\log}$ in terms of $\mathfrak{Z}^{\log}$!]. Now assertion (i) follows from the fact that the two irreducible components of the special fiber of this formal spectrum determine divisors D, E such that $e \cdot D$, $e \cdot E$ are *Cartier* [i.e., since they occur as the schematic zero loci of "x", "y"].

Next, we consider assertion (ii). Let $0 \neq f \in \Gamma(\mathfrak{Z}_\infty, \mathcal{O}_{\mathfrak{Z}_\infty})$; write $V(f)$ for the schematic zero locus of f on $\mathfrak{Z}_\infty$. Now observe that for each irreducible component C of $(\mathfrak{Z}_\infty)_{k_L}$, there exists an $e_C \in \mathbb{Z}_{\geq 0}$ such that the *meromorphic function*

$$f \cdot \pi_L^{-e_C}$$

[where π_L is a uniformizer of $\mathcal{O}_L$] has no zeroes or poles at the generic point of C. By the discrete structure of $\mathbb{Z}_{\geq 0}$, it follows that there exists an irreducible component C_1 such that $e_{C_1} \leq e_C$, for all irreducible components C of $(\mathfrak{Z}_\infty)_{k_L}$. Thus, the meromorphic function $f_1 \stackrel{\mathrm{def}}{=} f \cdot \pi_L^{-e_{C_1}}$ is *regular*, i.e., $f_1 \in \Gamma(\mathfrak{Z}_\infty, \mathcal{O}_{\mathfrak{Z}_\infty})$, and, moreover, has *nonzero* restriction to $(\mathfrak{Z}_\infty)_{k_L}$. On the other hand, since $(\mathfrak{Z}_\infty)_{k_L}$ is *connected* and *reduced*, and each irreducible component C of $(\mathfrak{Z}_\infty)_{k_L}$ is *geometrically integral*, it follows that immediately that the natural morphism $k_L \to \Gamma((\mathfrak{Z}_\infty)_{k_L}, \mathcal{O}_{(\mathfrak{Z}_\infty)_{k_L}})$

is an *isomorphism*, hence that $f_1 = \lambda + \pi_L \cdot g$, where $\lambda \in \mathcal{O}_L^\times$, $g \in \Gamma(\mathfrak{Z}_\infty, \mathcal{O}_{\mathfrak{Z}_\infty})$. Thus, by repeating this argument [with "f" replaced by "g"] and applying the p-adic completeness of $\mathfrak{Z}_\infty$, we conclude that $f \in \mathcal{O}_L$, as desired.

Finally, we consider assertion (iii). Since $\mathfrak{Z}_\infty$ is *locally noetherian*, it follows immediately from the existence of the g_N that the divisor of zeroes and poles of f is 0, hence, by assertion (ii), that f is a *constant* $\in \mathcal{O}_L^\times$. Since L is a finite extension of $\mathbb{Q}_p$, it thus follows from the well-known structure of $\mathcal{O}_L^\times$ that

$$f \in \bigcap_{N \in \mathbb{N}_{\geq 1}} (\mathcal{O}_L^\times)^N = \{1\}$$

— i.e., that $f = 1$, as desired. ○

Next, let K be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field k; K' a finite Galois extension of K [cf. Remark 3.6.5 below], with ring of integers $\mathcal{O}_{K'}$; $\mathfrak{S}$ the formal stack given by forming the stack-theoretic quotient with respect to the natural action of $\mathrm{Gal}(K'/K)$ of the *p-adic completion* of $\mathrm{Spec}(\mathcal{O}_{K'})$; $\mathfrak{S}^{\log}$ the formal log stack obtained by equipping $\mathfrak{S}$ with the log structure determined by the unique closed point of $\mathrm{Spec}(\mathcal{O}_{K'})$; $\mathfrak{X}^{\log}$ a *stable log orbicurve* [cf. §0] over $\mathfrak{S}^{\log}$. Also, we assume that the *generic fiber* $X^{\log} \stackrel{\mathrm{def}}{=} \mathfrak{X}^{\log} \times_{\mathcal{O}_K} K$ [of the algebrization] of $\mathfrak{X}^{\log}$ is a *smooth log orbicurve* [cf. §0]. Write

$$\mathcal{B}^{\mathrm{temp}}(X^{\log})$$

for the *temperoid of tempered coverings of* $X^{\log}$ [cf. [Mzk14], Example 3.10], $\mathcal{B}(\mathrm{Spec}(K))$ for the Galois category of finite étale coverings of $\mathrm{Spec}(K)$, and

$$\mathcal{D}_0 \stackrel{\mathrm{def}}{=} \mathcal{B}^{\mathrm{temp}}(X^{\log})^0; \quad \mathcal{D}_{\mathrm{cnst}} \stackrel{\mathrm{def}}{=} \mathcal{B}(\mathrm{Spec}(K))^0$$

[where the superscript "0" denotes the *full subcategory constituted by the connected objects* — cf. [Mzk16], §0, for more details]. Thus, if $\underline{\Pi}_X^{\mathrm{tm}} \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{temp}}(X^{\log})$ is the *tempered fundamental group* of $X^{\log}$ [cf. [André], §4; [Mzk14], Example 3.10], then the temperoid $\mathcal{B}^{\mathrm{temp}}(X^{\log})$ is naturally isomorphic [as a temperoid] to the temperoid $\mathcal{B}^{\mathrm{temp}}(\underline{\Pi}_X^{\mathrm{tm}})$ associated to the tempered group $\underline{\Pi}_X^{\mathrm{tm}}$ [cf. §0]. In a similar vein, the Galois category $\mathcal{B}(\mathrm{Spec}(K))$ is naturally equivalent to the Galois category $\mathcal{B}(G_K)$ associated to the *absolute Galois group* G_K of K. Also, we observe that the natural surjection $\underline{\Pi}_X^{\mathrm{tm}} \twoheadrightarrow G_K$ determines a *natural functor* $\mathcal{D}_0 \to \mathcal{D}_{\mathrm{cnst}}$ [cf. [Mzk17], Example 1.3, (ii)]. Write $\underline{\Delta}_X^{\mathrm{tm}} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\underline{\Pi}_X^{\mathrm{tm}} \twoheadrightarrow G_K)$.

Definition 3.3.

(i) Let Δ be a *tempered group* [cf. §0]. Then we shall refer to as a *tempered filter* on Δ a *countable* collection of *characteristic open subgroups of finite index*

$$\Delta^{\mathrm{fil}} = \{\Delta_i^{\mathrm{fil}}\}_{i \in I}$$

of Δ such that the following conditions are satisfied:

(a) We have: $\bigcap_{i\in I} \Delta_i^{\mathrm{fil}} = \{1\}$.

(b) Every Δ_i^{fil} admits a *minimal co-free subgroup* [cf. §0] $\Delta_i^{\mathrm{fil},\infty}$ [which is necessarily *characteristic* as a subgroup of Δ].

(c) For each open subgroup $H \subseteq \Delta$, there exists a [necessarily unique] $i_H \in I$ such that $\Delta_{i_H}^{\mathrm{fil},\infty} \subseteq H$, and, moreover, for every $i \in I$, $\Delta_i^{\mathrm{fil},\infty} \subseteq H$ implies $\Delta_i^{\mathrm{fil},\infty} \subseteq \Delta_{i_H}^{\mathrm{fil},\infty}$.

In the situation of (c), we shall refer to $\Delta_{i_H}^{\mathrm{fil},\infty}$ as the Δ^{fil}-*closure* of H in Δ.

(ii) We shall refer to a tempered filter on $\underline{\Delta}_X^{\mathrm{tm}}$ as a *tempered filter on* $X^{\log}$. Let

$$\Delta^{\mathrm{fil}} = \{\Delta_i^{\mathrm{fil}}\}_{i\in I}$$

be a tempered filter on $X^{\log}$. Suppose that $Z^{\log} \to X^{\log}$ is a finite étale Galois covering that admits a *stable model* $\mathfrak{Z}^{\log}$ over the ring of integers of the extension field of K determined by the integral closure of K in $Z^{\log}$ such that the *special fiber* of $\mathfrak{Z}^{\log}$ is *split* [i.e., $Z^{\log}$ is a curve as in the discussion at the beginning of the present §3], and, moreover, the open subgroup determined by the [geometric portion of] this covering is equal to one of the $\Delta_i^{\mathrm{fil}} \subseteq \underline{\Delta}_X^{\mathrm{tm}}$. Write $\mathfrak{Z}_\infty^{\log} \to \mathfrak{Z}^{\log}$ for the *"universal combinatorial covering"* of $\mathfrak{Z}^{\log}$ and $Z_\infty^{\log} \to Z^{\log}$ for the generic fiber of $\mathfrak{Z}_\infty^{\log} \to \mathfrak{Z}^{\log}$ [so $Z_\infty^{\log} \to Z^{\log}$ corresponds to the subgroup $\Delta_i^{\mathrm{fil},\infty} \subseteq \underline{\Delta}_X^{\mathrm{tm}}$ — cf. [André], Proposition 4.3.1; [André], the proof of Lemma 6.1.1]. Then we shall refer to $Z_\infty^{\log} \to X^{\log}$ as a Δ^{fil}-*covering of* $X^{\log}$. If, moreover, $Y^{\log} \to X^{\log}$ is a connected tempered covering, which determines an open subgroup $H \subseteq \underline{\Delta}_X^{\mathrm{tm}}$, and $\Delta_i^{\mathrm{fil}} \subseteq H$ is the Δ^{fil}-*closure* of H, then we shall refer to any covering $Z_\infty^{\log} \to Y^{\log}$ whose composite with $Y^{\log} \to X^{\log}$ is the covering $Z_\infty^{\log} \to X^{\log}$ as a Δ^{fil}-*closure* of $Y^{\log} \to X^{\log}$.

(iii) Let $\Delta^{\mathrm{fil}} = \{\Delta_i^{\mathrm{fil}}\}_{i\in I}$ be a *tempered filter* on $X^{\log}$. Then for any connected tempered covering $Y^{\log} \to X^{\log}$, it makes sense to define

$$\Phi_0(Y^{\log}) \stackrel{\mathrm{def}}{=} \varinjlim_{Z_\infty^{\log}} \ \mathrm{Div}_+(\mathfrak{Z}_\infty^{\log})^{\mathrm{Gal}(Z_\infty^{\log}/Y^{\log})}$$

$$\mathbb{B}_0(Y^{\log}) \stackrel{\mathrm{def}}{=} \varinjlim_{Z_\infty^{\log}} \ \mathrm{Mero}(Z_\infty^{\log})^{\mathrm{Gal}(Z_\infty^{\log}/Y^{\log})}$$

[where the inductive limits range over the Δ^{fil}-*closures* $Z_\infty^{\log} \to Y^{\log}$ of $Y^{\log} \to X^{\log}$; the superscript Galois groups denote the submonoids of elements fixed by the Galois group in question]. Moreover, by (i), (c), the assignments $Y^{\log} \mapsto \Phi_0(Y^{\log})$, $Y^{\log} \mapsto \mathbb{B}_0(Y^{\log})$ determine *functors*

$$\Phi_0 : \mathcal{D}_0 \to \mathfrak{Mon}; \quad \mathbb{B}_0 : \mathcal{D}_0 \to \mathfrak{Mon}$$

[where "$\mathfrak{Mon}$" is the category of commutative monoids — cf. [Mzk16], §0], together with a *natural transformation*

$$\mathbb{B}_0 \to \Phi_0^{\mathrm{gp}}$$

[given by assigning to a log-meromorphic function its log-divisor of zeroes and poles], whose image we denote by $\Phi_0^{\text{birat}} \subseteq \Phi_0^{\text{gp}}$. Also, we shall write $\mathbb{F}_0 \subseteq \mathbb{B}_0$ for the subfunctor determined by the *constant* log-meromorphic functions and $\Phi_0^{\text{cnst}} \subseteq \Phi_0^{\text{gp}}$ for the image of $\mathbb{F}_0$ in Φ_0^{gp}.

Remark 3.3.1. Note that the set of *primes* [cf. [Mzk16], §0] of the monoid

$$\mathrm{Div}_+(\mathfrak{Z}_\infty^{\log})^{\mathrm{Gal}(Z_\infty^{\log}/Y^{\log})}$$

appearing in the definition of $\Phi_0(Y^{\log})$ is in natural bijective correspondence with the set of $\mathrm{Gal}(Z_\infty^{\log}/Y^{\log})$*-orbits of prime log-divisors on* $\mathfrak{Z}_\infty^{\log}$ [cf. Proposition 3.2, (i)]. Moreover, since, by definition, different Δ^{fil}-closures $Z_\infty^{\log} \to Y^{\log}$ differ only by an extension of the base field K, it follows immediately that in the inductive limit appearing in the definition of $\Phi_0(Y^{\log})$, the maps between monoids induce *isomorphisms of monoids* on the respective *perfections*, hence that the resulting sets of primes map *bijectively* to one another.

Proposition 3.4. (Divisor and Rational Function Monoids) *In the notation of the above discussion:*

(i) $\Phi_0(Y^{\log})$*, as well as each of the monoids*

$$\mathrm{Div}_+(\mathfrak{Z}_\infty^{\log})^{\mathrm{Gal}(Z_\infty^{\log}/Y^{\log})}$$

appearing in the inductive limit defining $\Phi_0(Y^{\log})$*, is* **perf-factorial**. *Moreover, every endomorphism of* $\Phi_0(Y^{\log})$ *or one of the* $\mathrm{Div}_+(\mathfrak{Z}_\infty^{\log})^{\mathrm{Gal}(Z_\infty^{\log}/Y^{\log})}$ *induced by an endomorphism of* $Y^{\log}$ *over* $X^{\log}$ *is* **non-dilating**. *In particular, the functor* Φ_0 *defines a* **divisorial monoid** *on* $\mathcal{D}_0$ *which is, moreover,* **perf-factorial** *and* **non-dilating**.

(ii) Suppose that $Y^{\log} \to X^{\log}$ *is a connected tempered covering such that the composite morphism* $Y^{\log} \to \mathrm{Spec}(K)$ *factors through* $\mathrm{Spec}(L)$*, for some finite extension* L *of* K*, in such a way that* $Y^{\log}$ *is geometrically connected over* L*. Then we have* **natural isomorphisms of monoids**

$$\mathcal{O}_L^\times \xrightarrow{\sim} \mathrm{Ker}(\mathbb{B}_0(Y^{\log}) \to \Phi_0^{\text{gp}}(Y^{\log})) \subseteq \mathbb{B}_0(Y^{\log})$$

$$\mathcal{O}_L^\triangleright \xrightarrow{\sim} \mathbb{B}_0(Y^{\log}) \times_{\Phi_0^{\text{gp}}(Y^{\log})} \Phi_0(Y^{\log}); \quad L^\times \xrightarrow{\sim} \mathbb{F}_0(Y^{\log}) \subseteq \mathbb{B}_0(Y^{\log})$$

[where "$\mathcal{O}_L^\triangleright$*" is as in [Mzk17], Example 1.1].*

Proof. First, we consider assertion (i). Let M be one of the monoids under consideration. The fact that M is *divisorial* is immediate from the definitions. The fact that M is *perf-factorial* then follows immediately from Proposition 3.2, (i) [cf. also the description of the primes of M in terms of *"orbits of prime log-divisors"* given in Remark 3.3.1]. Now let α be an endomorphism of M that is induced by an endomorphism of $Y^{\log}$ over $X^{\log}$ such that α induces the *identity* endomorphism on the set of primes of M. Then by considering *local functions* on $\mathfrak{Z}_\infty$ that *arise from local functions on* $\mathfrak{X}$ and vanish at various primes of M, it follows that α is the identity, as desired. This completes the proof of assertion (i). Assertion (ii) follows immediately from Proposition 3.2, (ii) [and the definitions]. ◯

Lemma 3.5. **(Perfections and Realifications of Perf-factorial Submonoids)** *Let* P, Q *be* **perf-factorial** *monoids such that: (a)* P *is a* **submonoid** *of* Q*; (b)* P *is* **group-saturated** *[cf. §0] in* Q*; (c)* $\mathbb{R}$ **supports** Q *[cf. [Mzk16], Definition 2.4, (ii)]. Then:*

(i) The inclusion $P \hookrightarrow Q$ *extends* **uniquely** *to inclusions* $P^{\mathrm{pf}} \hookrightarrow Q$, $P^{\mathrm{rlf}} \hookrightarrow Q$.

(ii) Relative to the inclusions of (i), P^{pf}, P^{rlf} *are* **group-saturated** *in* Q.

Proof. Indeed, the portion of assertions (i), (ii) involving "P^{pf}" follows immediately from the definitions. Next, let $\mathfrak{p} \in \mathrm{Prime}(P)$ [where "Prime$(-)$" is as in [Mzk16], §0]. Since P is *perf-factorial*, it follows that the "primary component" $P_{\mathfrak{p}}$ associated to $\mathfrak{p}$ is isomorphic to $\mathbb{Z}_{\geq 0}$, $\mathbb{Q}_{\geq 0}$, or $\mathbb{R}_{\geq 0}$ [cf. [Mzk16], Definition 2.4, (i), (b)]. Since $\mathbb{R}_{\geq 0}$ *acts* on Q [cf. condition (c)], it thus follows that the natural homomorphism of monoids $P_{\mathfrak{p}} \hookrightarrow P \hookrightarrow Q$ *extends* [uniquely] to a homomorphism of monoids $P_{\mathfrak{p}}^{\mathrm{rlf}} \hookrightarrow Q$. Next, observe that it follows from the definition of the *realification* [cf. [Mzk16], Definition 2.4, (i)] that for every $a \in P^{\mathrm{rlf}}$, there exists an $a' \in P^{\mathrm{pf}}$ such that $a' \geq a$. In particular, it follows that for each $\mathfrak{q} \in \mathrm{Prime}(Q)$, the *sum* of the images of the various "primary components $a_{\mathfrak{p}} \in P_{\mathfrak{p}}$ of a" [as $\mathfrak{p}$ ranges over the elements of Prime(P)] in $Q_{\mathfrak{q}} \cong \mathbb{R}_{\geq 0}$ is *bounded above* [i.e., by the image in $Q_{\mathfrak{q}} \cong \mathbb{R}_{\geq 0}$ of a', which is *well-defined* since $a' \in P^{\mathrm{pf}}$]. Thus, this sum *converges* to an element of $Q_{\mathfrak{q}} \cong \mathbb{R}_{\geq 0}$. Now, letting $\mathfrak{q}$ range over the elements of Prime(Q), we conclude that we obtain a *homomorphism of monoids*

$$P^{\mathrm{rlf}} \to Q^{\mathrm{pf}}_{\mathrm{factor}} = Q^{\mathrm{rlf}}_{\mathrm{factor}}$$

[relative to the notation of [Mzk16], Definition 2.4, (i), (c)]. Since, moreover, Q is *perf-factorial*, it follows from [Mzk16], Definition 2.4, (i), (d) [together with the existence of an $a' \in P^{\mathrm{pf}}$ such that $a' \geq a$], that this homomorphism *factors* through Q, hence determines a *homomorphism of monoids*

$$\phi : P^{\mathrm{rlf}} \to Q$$

that is [easily verified to be] *uniquely* characterized by the property that it *extends* the natural homomorphism of monoids $P^{\mathrm{pf}} \hookrightarrow Q$. Write $\phi^{\mathrm{gp}} : (P^{\mathrm{rlf}})^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ for the induced homomorphism on groupifications. Next, let $a, b \in P^{\mathrm{rlf}}$ be such that $\phi^{\mathrm{gp}}(a - b) \geq 0$ [i.e., $\phi^{\mathrm{gp}}(a - b) \in Q$]. Then for any $a', b' \in P^{\mathrm{pf}}$ such that $a' \geq a$, $b' \leq b$, we obtain that $\phi^{\mathrm{gp}}(a' - b') \geq \phi^{\mathrm{gp}}(a - b) \geq 0$, hence [by the portion of assertion (ii) concerning "P^{pf}"] that $a' \geq b'$. On the other hand, since P is *perf-factorial* [cf. [Mzk16], Definition 2.4, (i), (d)], it follows immediately that if $a \not\geq b$, then there exist $a', b' \in P^{\mathrm{pf}}(A)$ such that $a' \geq a$, $b' \leq b$, $a' \not\geq b'$. Thus, we conclude that $a \geq b$. In particular, if $\phi^{\mathrm{gp}}(a - b) = 0$, then it follows that there exists a

$c \in P^{\text{rlf}}$ [i.e., $c \stackrel{\text{def}}{=} a - b$] such that $\phi(c) = 0$. On the other hand, if $c \neq 0$, then [cf. [Mzk16], Definition 2.4, (i), (d)] there exists a $c' \in P^{\text{pf}}$ such that $0 < c' \leq c$, hence that $0 \leq \phi(c') \leq 0$, so $\phi(c') = 0$, in contradiction to the *injectivity* of the natural homomorphism of monoids $P^{\text{pf}} \hookrightarrow Q$. Thus, we conclude that ϕ is *injective*. This completes the proof of the portion of assertions (i), (ii) involving "P^{rlf}". ○

Remark 3.5.1. Observe that it follows immediately from Lemma 3.5, (i), that a nonzero submonoid P of an $\mathbb{R}$-monoprime monoid Q is *perf-factorial* and *group-saturated* if and only if it is *monoprime* [cf. [Mzk16], §0].

Remark 3.5.2. Note that the *injectivity portion* of Lemma 3.5, (i), *fails to hold* if one omits the crucial hypothesis that P is *group-saturated* in Q. Indeed, this may be seen, for instance, by considering an injection $P \stackrel{\text{def}}{=} \mathbb{Z}_{\geq 0} \oplus \mathbb{Z}_{\geq 0} \hookrightarrow Q \stackrel{\text{def}}{=} \mathbb{R}_{\geq 0}$ that sends the elements $(1, 0)$; $(0, 1)$ of P to [nonzero] $\mathbb{Q}$-*linearly independent* elements of $\mathbb{R}_{\geq 0}$.

Definition 3.6. In the notation of Definition 3.3, (iii):

(i) Let Λ be a *monoid type*. Define Φ_0^{Λ}, $\mathbb{B}_0^{\Lambda}$, $\mathbb{F}_0^{\Lambda}$ as follows:

$$\Phi_0^{\mathbb{Z}} \stackrel{\text{def}}{=} \Phi_0; \quad \Phi_0^{\mathbb{Q}} \stackrel{\text{def}}{=} \Phi_0^{\text{pf}}; \quad \Phi_0^{\mathbb{R}} \stackrel{\text{def}}{=} \Phi_0^{\text{rlf}}$$

$$\mathbb{B}_0^{\mathbb{Z}} \stackrel{\text{def}}{=} \mathbb{B}_0; \quad \mathbb{B}_0^{\mathbb{Q}} \stackrel{\text{def}}{=} \mathbb{B}_0^{\text{pf}}; \quad \mathbb{B}_0^{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \cdot \Phi_0^{\text{birat}} \subseteq (\Phi_0^{\mathbb{R}})^{\text{gp}}$$

$$\mathbb{F}_0^{\mathbb{Z}} \stackrel{\text{def}}{=} \mathbb{F}_0; \quad \mathbb{F}_0^{\mathbb{Q}} \stackrel{\text{def}}{=} \mathbb{F}_0^{\text{pf}}; \quad \mathbb{F}_0^{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \cdot \Phi_0^{\text{cnst}} \subseteq (\Phi_0^{\mathbb{R}})^{\text{gp}}$$

[where Φ_0^{rlf} is as in [Mzk16], Definition 2.4, (i) — cf. Proposition 3.4, (i)].

(ii) Let $\mathcal{D}$ be a *connected, totally epimorphic* category, equipped with a functor $\mathcal{D} \to \mathcal{D}_0$;

$$\Phi \subseteq \Phi^{\mathbb{R}\text{-log}} \stackrel{\text{def}}{=} \Phi_0^{\mathbb{R}}|_{\mathcal{D}}$$

a *group-saturated* [i.e., $\Phi(A)$ is group-saturated in $\Phi^{\mathbb{R}\text{-log}}(A)$, $\forall A \in \text{Ob}(\mathcal{D})$] *subfunctor in monoids* which determines a *perf-factorial divisorial monoid* on $\mathcal{D}$ such that the image of the resulting homomorphism of group-like monoids on $\mathcal{D}$

$$\mathbb{F} \stackrel{\text{def}}{=} \mathbb{F}_0^{\Lambda}|_{\mathcal{D}} \times_{(\Phi_0^{\mathbb{R}})^{\text{gp}}|_{\mathcal{D}}} \Phi^{\text{gp}} \to \Phi^{\text{gp}}$$

determines a subfunctor in *nonzero* monoids of Φ^{gp} [i.e., for every $A \in \text{Ob}(\mathcal{D})$, the homomorphism $\mathbb{F}(A) \to \Phi^{\text{gp}}(A)$ is nonzero]. [Thus, it follows from these conditions that for every $A \in \text{Ob}(\mathcal{D})$, the image of the homomorphism $\mathbb{F}(A) \to \Phi^{\text{gp}}(A)$ contains a nonzero element of $\Phi(A)$.] Write $\mathbb{B} \stackrel{\text{def}}{=} \mathbb{B}_0^{\Lambda}|_{\mathcal{D}} \times_{(\Phi_0^{\Lambda})^{\text{gp}}|_{\mathcal{D}}} \Phi^{\text{gp}} \to \Phi^{\text{gp}}$. Thus, the data

$$(\mathcal{D}, \Phi, \mathbb{B}, \mathbb{B} \to \Phi^{\text{gp}})$$

determines a *model Frobenioid*

$$\mathcal{C}$$

[cf. [Mzk16], Theorem 5.2, (ii)]. We shall refer to a Frobenioid obtained in this way as a *tempered Frobenioid*. If $\mathcal{C}$ is of rational (respectively, strictly rational) type [a property which is *completely determined* by Φ — cf. [Mzk16], Definition 4.5, (ii)], then we shall say that Φ is *rational* (respectively, *strictly rational*).

(iii) If $A \in \mathrm{Ob}(\mathcal{D})$, then we shall say that an element of $\Phi(A)$ is *non-cuspidal* (respectively, *cuspidal*) if it arises [cf. the inductive limit that appears in the definition of Φ_0] from a non-cuspidal (respectively, cuspidal) log-divisor; we shall say that a prime $\mathfrak{p}$ of the monoid $\Phi(A)$ is *non-cuspidal* (respectively, *cuspidal*) if the primary elements of $\Phi(A)$ that are contained in $\mathfrak{p}$ are non-cuspidal (respectively, cuspidal). In the following, we shall write

$$\Phi(A)^{\mathrm{ncsp}} \subseteq \Phi(A); \quad \Phi(A)^{\mathrm{csp}} \subseteq \Phi(A)$$

$$\mathrm{Prime}(\Phi(A))^{\mathrm{ncsp}} \subseteq \mathrm{Prime}(\Phi(A)); \quad \mathrm{Prime}(\Phi(A))^{\mathrm{csp}} \subseteq \mathrm{Prime}(\Phi(A))$$

for the submonoids of non-cuspidal and cuspidal elements and the subsets of non-cuspidal and cuspidal primes, respectively. We shall refer to a pre-step of $\mathcal{C}$ as *non-cuspidal* (respectively, *cuspidal*) if its zero divisor is non-cuspidal (respectively, cuspidal).

(iv) Set:

$$\Phi^{\text{bs-fld}} \stackrel{\mathrm{def}}{=} \Phi \times_{(\Phi_0^{\mathbb{R}})^{\mathrm{gp}}|_{\mathcal{D}}} (\mathbb{R} \cdot \Phi_0^{\mathrm{cnst}})|_{\mathcal{D}} \subseteq \Phi^{\mathbb{R}\text{-log}}$$

Thus, we obtain a homomorphism of functors of monoids $\mathbb{F} \to (\Phi^{\text{bs-fld}})^{\mathrm{gp}}$ whose image determines a *monoprime* [by Remark 3.5.1 — cf. also Remark 3.6.1 below] subfunctor in *nonzero* monoids of $(\Phi^{\text{bs-fld}})^{\mathrm{gp}}$ [cf. (ii)]. In particular, the data

$$(\mathcal{D}, \Phi^{\text{bs-fld}}, \mathbb{F}, \mathbb{F} \to (\Phi^{\text{bs-fld}})^{\mathrm{gp}})$$

determines a *model Frobenioid*

$$\mathcal{C}^{\text{bs-fld}}$$

[cf. [Mzk16], Theorem 5.2, (ii)]. Moreover, the *natural inclusion* $\Phi^{\text{bs-fld}}(-) \subseteq \Phi(-)$ determines a *natural faithful functor* $\mathcal{C}^{\text{bs-fld}} \to \mathcal{C}$ which may be applied to think of $\mathcal{C}^{\text{bs-fld}}$ as a *subcategory* of $\mathcal{C}$ [cf. Remark 3.6.3 below]. Moreover, it follows immediately from the existence of the *natural functor* $\mathcal{D}_0 \to \mathcal{D}_{\mathrm{cnst}}$ that $\mathcal{C}^{\text{bs-fld}}$ is a *p-adic Frobenioid* in the sense of [Mzk17], Example 1.1, (ii). We shall refer to the Frobenioid $\mathcal{C}^{\text{bs-fld}}$ obtained in this way as the *base-field-theoretic hull* of the tempered Frobenioid $\mathcal{C}$. Also, we shall refer to a morphism of the Frobenioid $\mathcal{C}$ as *base-field-theoretic* if its zero divisor belongs to $\Phi^{\text{bs-fld}}(-) \subseteq \Phi(-)$.

(v) We shall say that Φ is *cuspidally pure* if the following conditions are satisfied: (a) for every *non-cuspidal primary element* $x \in \Phi(A)$, where $A \in \mathrm{Ob}(\mathcal{D})$, there exists an element $y \in \Phi^{\text{bs-fld}}(A)$ such that $x \leq y$; (b) we have

$$\mathrm{Prime}(\Phi(A)) = \mathrm{Prime}(\Phi(A))^{\mathrm{ncsp}} \bigcup \mathrm{Prime}(\Phi(A))^{\mathrm{csp}}$$

for every $A \in \mathrm{Ob}(\mathcal{D})$.

Remark 3.6.1. Note that the *group-saturated-ness* hypothesis of Definition 3.6, (ii), may be regarded as the condition that "divisors relative to Φ are *effective* if and only if they are effective relative to $\Phi^{\mathbb{R}\text{-log}}$, i.e., if and only if they are effective in the *usual sense*". Alternatively, this hypothesis [together with the *perf-factoriality* hypothesis of Definition 3.6, (ii)] may be regarded as the analogue in the present "tempered context" of the *monoprime-ness* hypothesis in [Mzk17], Example 1.1, (ii) — cf. Remark 3.5.1.

Remark 3.6.2. Observe that the base-field-theoretic hull of Definition 3.6, (iv), is itself a tempered Frobenioid, and, moreover, that every p-adic Frobenioid may be obtained in this way [cf. Remarks 3.5.1, 3.6.1]. In particular, it follows that *"the notion of a p-adic Frobenioid is a special case of the notion of a tempered Frobenioid"*. Also, we observe in passing that $\Phi^{\text{bs-fld}}$ is always *non-dilating* and *strictly rational*.

Remark 3.6.3. It follows immediately from Proposition 3.4, (ii), and the *explicit divisorial description* of objects and morphisms of a *model Frobenioid* given in [Mzk16], Theorem 5.2, (i) [cf. also the *equivalences of categories* of [Mzk16], Definition 1.3, (iii), (d), determined by the operation of taking the *zero divisor of a co-angular pre-step*] that the *objects* of the *essential image* [cf. §0] of the natural functor $\mathcal{C}^{\text{bs-fld}} \to \mathcal{C}$ may be described as the objects of $\mathcal{C}$ that may be "linked" to a *Frobenius-trivial* object via *base-field theoretic pre-steps*, while the *morphisms* of the *essential image* of the natural functor $\mathcal{C}^{\text{bs-fld}} \to \mathcal{C}$ may be described as the *base-field theoretic morphisms* of $\mathcal{C}$ between objects of the essential image of $\mathcal{C}^{\text{bs-fld}} \to \mathcal{C}$. In particular, the natural functor $\mathcal{C}^{\text{bs-fld}} \to \mathcal{C}$ is *isomorphism-full* [cf. §0]. Thus, no confusion arises from "identifying" $\mathcal{C}^{\text{bs-fld}}$ with its *essential image* via the natural functor $\mathcal{C}^{\text{bs-fld}} \to \mathcal{C}$ in $\mathcal{C}$ [cf. §0].

Remark 3.6.4. If Φ, $\mathcal{C}$ are as in Definition 3.6, (ii), then it follows from Lemma 3.5 [applied to the submonoid $\Phi \subseteq \Phi^{\mathbb{R}\text{-log}}$] that the respective divisor monoids Φ^{pf}, Φ^{rlf} of $\mathcal{C}^{\text{pf}}$, $\mathcal{C}^{\text{rlf}}$ also satisfy the conditions of Definition 3.6, (ii). That is to say, the *perfection* and *realification* of a tempered Frobenioids are again *tempered Frobenioids*.

Remark 3.6.5. Note that by taking the extension field K' used to define the stack structure of $\mathfrak{S}$ to be "sufficiently large", one may treat the case in which $X^{\log}$ *fails to have stable reduction* over $\mathcal{O}_K$. Moreover, although at first sight the choice of K' may appear to be somewhat arbitrary, one verifies immediately that the *category* $\mathcal{D}_0$, as well as the *monoid* Φ_0 on $\mathcal{D}_0$, are *unaffected* by replacing K' by some larger finite Galois extension of K.

Now we have the following *"tempered analogue"* of [Mzk17], Theorem 1.2:

Theorem 3.7. (Basic Properties of Tempered Frobenioids) *In the notation of Definition 3.6:*

(i) If $\Lambda = \mathbb{Z}$ *(respectively,* $\Lambda = \mathbb{R}$*), then* $\mathcal{C}$ *is of* **unit-profinite** *(respectively,* **unit-trivial***) type. For arbitrary* Λ*, the Frobenioid* $\mathcal{C}$ *is of* **isotropic**, **model** *[hence, in particular,* **birationally Frobenius-normalized***], and* **sub-quasi-Frobenius-trivial** *type, but* **not** *of group-like type.*

(ii) Suppose $\mathcal{D}$ *is of* **FSMFF-type***, and that* Φ *is* **non-dilating**. *Then* $\mathcal{C}$ *is of* **standard** *type. If, moreover,* Φ *is* **rational** *[cf. Definition 3.6, (ii)], then* $\mathcal{C}$ *is of* **rationally standard** *type.*

(iii) Let $A \in \text{Ob}(\mathcal{C})$*;* $A_{\mathcal{D}} \stackrel{\text{def}}{=} \text{Base}(A) \in \text{Ob}(\mathcal{D})$*. Write* $A_{\text{cnst}} \in \text{Ob}(\mathcal{D}_{\text{cnst}})$ *for the image of* $A_{\mathcal{D}}$ *in* $\mathcal{D}_{\text{cnst}}$*. Then the natural action of* $\text{Aut}_{\mathcal{C}}(A)$ *on* $\mathcal{O}^{\rhd}(A)$*,* $\mathcal{O}^{\times}(A)$ **factors** *through* $\text{Aut}_{\mathcal{D}_{\text{cnst}}}(A_{\text{cnst}})$*. If, moreover,* $\Lambda \in \{\mathbb{Z}, \mathbb{Q}\}$*, then this factorization determines a* **faithful** *action of the image of* $\text{Aut}_{\mathcal{C}}(A)$ *in* $\text{Aut}_{\mathcal{D}_{\text{cnst}}}(A_{\text{cnst}})$ *on* $\mathcal{O}^{\rhd}(A)$*,* $\mathcal{O}^{\times}(A)$*.*

(iv) If $\mathcal{D}$ *is* **slim** *[cf. [Mzk16], §0], and* $\Lambda \in \{\mathbb{Z}, \mathbb{R}\}$*, then* $\mathcal{C}$ *is also* **slim**.

Proof. First, we consider assertion (i). In light of the definition of $\mathcal{C}$ as a *model Frobenioid*, it follows from [Mzk16], Theorem 5.2, (ii), that $\mathcal{C}$ is of *isotropic* and *model* type; the fact that $\mathcal{C}$ is of *sub-quasi-Frobenius-trivial* type follows from [Mzk16], Proposition 1.10, (vi). By Proposition 3.4, (ii) (respectively, by the definition of the *realification* of a Frobenioid — cf. [Mzk16], Proposition 5.3), it follows that if, moreover, $\Lambda = \mathbb{Z}$ (respectively, $\Lambda = \mathbb{R}$), then $\mathcal{C}$ is of *unit-profinite* (respectively, *unit-trivial*) type; the condition imposed on $\mathbb{F}$ in Definition 3.6, (ii), implies immediately that $\mathcal{C}$ is *not* of group-like type. This completes the proof of assertion (i). As for assertion (ii), let us first observe that since $\underline{\Pi}_X^{\text{tm}}$ acts *trivially* on $K^{\times}/\mathcal{O}_K^{\times}$, it follows [cf. also the condition imposed on $\mathbb{F}$ in Definition 3.6, (ii)] that *every* object of $(\mathcal{C}^{\text{un-tr}})^{\text{birat}}$ is *Frobenius-compact*. Thus, assertion (ii) follows immediately from the definitions. Assertion (iii) follows immediately from Proposition 3.4, (ii). Assertion (iv) follows formally from [Mzk16], Proposition 1.13, (iii) [since, by assertion (i) of the present Theorem 3.2, "condition (b)" of *loc. cit.* is always satisfied by objects of $\mathcal{C}$]. This completes the proof of Theorem 3.7. ◯

Remark 3.7.1. We recall [cf. [Mzk17], §0] in passing that if $\mathcal{D}$ is of *weakly indissectible* (respectively, *strongly dissectible*; *weakly dissectible*) *type*, then so is $\mathcal{C}$.

Remark 3.7.2. We recall in passing that $\mathcal{D}_0$ is *slim* [cf. [Mzk14], Example 3.10; [Mzk14], Remark 3.4.1].

Corollary 3.8. (Preservation of Base-field-theoretic Morphisms and Hulls) *Suppose that for* $i = 1, 2$*,* $\mathcal{C}_i$ *is a* **tempered Frobenioid** *whose* **base category** $\mathcal{D}_i$ *is of* **FSMFF-type***, and whose* **divisor monoid** Φ_i *is* **non-dilating**. *Let*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

be an **equivalence of categories**. *Then:*

(i) Suppose, for $i = 1, 2$, that the **base category** $\mathcal{D}_i$ *of* $\mathcal{C}_i$ *is* **Frobenius-slim***. Then* Ψ *preserves the* **base-field-theoretic morphisms***.*

(ii) Suppose, for $i = 1, 2$, that the **base category** $\mathcal{D}_i$ *of* $\mathcal{C}_i$ *is* Div**-slim** *[relative to Φ_i]. Then* Ψ *preserves the* **base-field-theoretic morphisms** *and induces a* **compatible equivalence**

$$\mathcal{C}_1^{\text{bs-fld}} \xrightarrow{\sim} \mathcal{C}_2^{\text{bs-fld}}$$

of the subcategories $\mathcal{C}_1^{\text{bs-fld}} \subseteq \mathcal{C}_1$, $\mathcal{C}_2^{\text{bs-fld}} \subseteq \mathcal{C}_2$ *given by the respective* **base-field-theoretic hulls***.*

(iii) Suppose that Ψ *preserves the* **base-field-theoretic morphisms***, and that* Φ_1, Φ_2 *are* **cuspidally pure***. Then* Ψ *preserves the* **non-cuspidal** *and* **cuspidal** *pre-steps. If, moreover,* Φ_1, Φ_2 *are* **rational***, then the induced* **isomorphism of divisor monoids**

$$\Psi^{\Phi} : \Phi_1|_{\mathcal{C}_1} \xrightarrow{\sim} \Phi_2|_{\mathcal{C}_2}$$

[lying over Ψ] of [Mzk16], Theorem 4.9, preserves **non-cuspidal** *elements and primes, as well as* **cuspidal** *elements and primes.*

Proof. Indeed, by Theorem 3.7, (i), (ii), $\mathcal{C}_1$, $\mathcal{C}_2$ are of *standard* and *isotropic* type, but *not* of group-like type. In particular, by [Mzk16], Theorem 3.4, (ii); [Mzk16], Theorem 4.2, (i), it follows that Ψ preserves *pre-steps* and *primary steps*. Moreover, Ψ is compatible with the operation of passing to the *perfection* [cf. [Mzk16], Theorem 3.4, (iii)].

Next, we consider assertions (i), (ii). By applying [Mzk16], Theorem 3.4, (iv), in the case of assertion (i), and [Mzk16], Theorem 3.4, (ii); [Mzk16], Corollary 4.11, (ii), in the case of assertion (ii), it follows that Ψ preserves the submonoids "$\mathcal{O}^{\rhd}(-)$". Now observe [cf. the *equivalences of categories* of [Mzk16], Definition 1.3, (iii), (d), determined by the operation of taking the *zero divisor of a co-angular pre-step*] that a pre-step of $\mathcal{C}_i$ is *base-field-theoretic* if and only if its image $A \to B$ in $\mathcal{C}_i^{\text{pf}}$ may be written as a [filtered] projective limit in the category $(\mathcal{C}_i^{\text{pf}})_B^{\text{coa-pre}}$ [where "coa-pre" denotes the subcategory determined by the (necessarily co-angular) pre-steps of $\mathcal{C}_i^{\text{pf}}$] of pre-steps $A' \to B$ that are *abstractly equivalent* [cf. §0] to an endomorphism that belongs to "$\mathcal{O}^{\rhd}(-)$". Thus, Ψ preserves the *base-field-theoretic pre-steps*. Note, moreover, that Ψ preserves [cf. [Mzk16], Theorem 3.4, (ii), (iii)] the *factorization* [cf. [Mzk16], Definition 1.3, (iv), (a)] of a morphism of $\mathcal{C}_i$ into a composite of a morphism of Frobenius type, a pre-step, and a pull-back morphism. Thus, we conclude that Ψ preserves the *base-field-theoretic morphisms*. This completes the proof of assertion (i). Next, to complete the proof of assertion (ii), let us observe that, under the assumptions of assertion (ii), Ψ preserves [cf. [Mzk16], Theorem 3.4, (iii); [Mzk16], Corollary 4.11, (ii)] the *Frobenius-trivial objects*. Since Ψ preserves base-field-theoretic pre-steps and base-field-theoretic morphisms, it follows from the *explicit description* of the base-field-theoretic hull given in Remark 3.6.3 that Ψ preserves the [objects and morphisms of the] subcategories $\mathcal{C}_1^{\text{bs-fld}} \subseteq \mathcal{C}_1$, $\mathcal{C}_2^{\text{bs-fld}} \subseteq \mathcal{C}_2$, hence induces an equivalence of categories $\mathcal{C}_1^{\text{bs-fld}} \xrightarrow{\sim} \mathcal{C}_2^{\text{bs-fld}}$, as desired. This completes the proof of assertion (ii).

Finally, we consider assertion (iii). Since, by assumption, Ψ preserves the *base-field-theoretic pre-steps*, we conclude from Definition 3.6, (v), (a) [cf. also the *first equivalence of categories involving pre-steps* of [Mzk16], Definition 1.3, (iii), (d)], that Ψ preserves the *primary non-cuspidal steps*, hence, [by Definition 3.6, (v), (b)] that Ψ preserves the *primary cuspidal steps*. Thus, by considering the *"factorization morphisms"* arising from the fact that Φ_1, Φ_2 are *perf-factorial* [cf. [Mzk16], Definition 2.4, (i), (c)] in the context of the *perfections* of $\mathcal{C}_1$, $\mathcal{C}_2$, it follows that Ψ preserves the *non-cuspidal* and *cuspidal* pre-steps. The remainder of assertion (iii) now follows immediately from the isomorphism $\Psi^{\Phi} : \Phi_1|_{\mathcal{C}_1} \xrightarrow{\sim} \Phi_2|_{\mathcal{C}_2}$ of [Mzk16], Theorem 4.9. This completes the proof of assertion (iii). ◯

Remark 3.8.1. Note that in the situation of Corollary 3.8, (ii), for suitable *base categories* [i.e., of the sort that appear in [Mzk17], Theorem 2.4] one may apply to the equivalence of categories $\mathcal{C}_1^{\text{bs-fld}} \xrightarrow{\sim} \mathcal{C}_2^{\text{bs-fld}}$ induced by Ψ the theory of the *category-theoreticity of the Kummer and reciprocity maps*, as discussed in [Mzk17], Theorem 2.4.

Remark 3.8.2. In the situation of Corollary 3.8, suppose further that Ψ preserves the *base-field-theoretic morphisms*, and that Φ_1, Φ_2 are *cuspidally pure* and *rational* [cf. Corollary 3.8, (iii)]. Then observe that by considering *zero divisors of elements of* $\mathcal{O}^{\triangleright}(-)$ as in the proof of Corollary 3.8, (i), (ii), it follows that [in the notation of Corollary 3.8], for $C_1 \in \mathrm{Ob}(\mathcal{C}_1)$, $C_2 \overset{\text{def}}{=} \Psi(C_1)$, *non-cuspidal* primes $\mathfrak{p}_1$, $\mathfrak{q}_1$ of $\Phi(C_1)$ such that $\mathfrak{p}_1 \mapsto \mathfrak{p}_2$, $\mathfrak{q}_1 \mapsto \mathfrak{q}_2$ [where $\mathfrak{p}_2, \mathfrak{q}_2 \in \mathrm{Prime}(\Phi(C_2))$], we obtain, for $i = 1, 2$, *natural isomorphisms*

$$(\mathbb{R}_{\geq 0} \cong) \quad \Phi_i(C_i)^{\text{rlf}}_{\mathfrak{p}_i} \xrightarrow{\sim} \Phi_i(C_i)^{\text{rlf}}_{\mathfrak{q}_i} \quad (\cong \mathbb{R}_{\geq 0})$$

[i.e., induced by considering the zero divisors of elements of $\mathcal{O}^{\triangleright}(C_i)$] which are *compatible* with the isomorphism $\Phi_1(C_1) \xrightarrow{\sim} \Phi_2(C_2)$ induced by Ψ^{Φ}.

Next, we consider *base-field-theoretic hulls* in the context of the theory of *poly-Frobenioids*, as discussed in [Mzk17], §5.

Proposition 3.9. (Base-field-theoretic Hulls in the Context of Poly-Frobenioids) *In the notation of Definition 3.6, suppose further that $\mathcal{C}'$ is a* **Frobenioid** *of* **isotropic** *type over a base category $\mathcal{D}'$ with* **perf-factorial divisor monoid** *Φ'. Then:*

(i) If $\Psi : \mathcal{C}' \to (\mathcal{C}^{\text{bs-fld}})^{\top}$ is a **GC-admissible** *functor, then the composite functor*

$$\mathcal{C}' \to (\mathcal{C}^{\text{bs-fld}})^{\top} \hookrightarrow \mathcal{C}^{\top}$$

[where "$\hookrightarrow$" is the natural inclusion functor] is also GC-admissible.

(ii) Suppose [cf. Remark 3.9.1 below] that $\mathcal{C}'$ is of **model** *type; that Φ' is* **supported by** $\mathbb{R}$ *[i.e., $\Phi' = (\Phi')^{\text{rlf}}$ — cf. [Mzk16], Definition 2.4, (ii)]; and that*

$\Psi' : \mathcal{C}^{\text{bs-fld}} \to \mathcal{C}'$ is an **LC-admissible** *functor. Then there exists a* 1*-commutative diagram of functors*

$$\begin{array}{ccccc} \mathcal{C}^{\text{bs-fld}} & \xrightarrow{\text{id}} & \mathcal{C}^{\text{bs-fld}} & \hookrightarrow & \mathcal{C} \\ \downarrow \Psi' & & \downarrow \Psi^{\text{bs-fld}} & & \downarrow \Psi \\ \mathcal{C}' & \xrightarrow{\sim} & \underline{\mathcal{C}}^{\text{bs-fld}} & \hookrightarrow & \underline{\mathcal{C}} \end{array}$$

— where $\underline{\mathcal{C}}$ is a **tempered Frobenioid***; $\underline{\mathcal{C}}^{\text{bs-fld}}$ is the* **base-field-theoretic hull** *of $\underline{\mathcal{C}}$; the "$\hookrightarrow$'s" are the natural inclusion functors; Ψ is an* **LC-admissible** *functor; $\Psi^{\text{bs-fld}}$ is induced by Ψ; and "$\xrightarrow{\sim}$" is an equivalence of categories that is compatible with the Frobenioid structures of $\mathcal{C}'$, $\underline{\mathcal{C}}^{\text{bs-fld}}$. If, moreover, Ψ' is* **LC-unit-admissible***, then so is Ψ.*

Proof. Assertion (i) follows immediately from the definition of a "GC-admissible functor" given in [Mzk17], Definition 5.3, (i). Next, we consider assertion (ii). First, let us observe that the subfunctor of *"constant realified rational functions"* [i.e., the subfunctor denoted "Φ^{cnst}" in [Mzk17], Definition 5.3, (ii), applied to Ψ'] is necessarily contained in the subfunctor given by

$$\mathbb{R} \cdot \Phi_0^{\text{cnst}}|_{\mathcal{D}} \subseteq (\Phi^{\text{rlf}})^{\text{gp}}$$

[cf. Definition 3.6, (i), (ii)]. Thus, if we take $\underline{\Phi} \stackrel{\text{def}}{=} \Phi^{\text{rlf}}$ [cf. Lemma 3.5, (i), (ii); Remark 3.6.4] and $\underline{\mathbb{B}}$ to be the *push-forward* [group-like monoid] of the diagram

$$\begin{array}{ccc} \mathbb{F} & \hookrightarrow & \mathbb{B} \\ \downarrow & & \\ \mathbb{F}' & & \end{array}$$

[where $\mathbb{F} \hookrightarrow \mathbb{B}$ is the natural inclusion; $\mathbb{F}'$ is the group-like monoid on $\mathcal{D}$ determined by the *rational function monoid* of the Frobenioid $\mathcal{C}'$; $\mathbb{F} \to \mathbb{F}'$ is the natural transformation determined by the LC-admissible functor Ψ'], then we obtain a homomorphism of group-like monoids $\underline{\mathbb{B}} \to \underline{\Phi}^{\text{gp}}$, hence a collection of data

$$(\mathcal{D}, \underline{\Phi}, \underline{\mathbb{B}}, \underline{\mathbb{B}} \to \underline{\Phi}^{\text{gp}})$$

that determines a *tempered Frobenioid* $\underline{\mathcal{C}}$, together with an *LC-admissible* functor $\Psi : \mathcal{C} \to \underline{\mathcal{C}}$ [where we take the subfunctor denoted "Φ^{cnst}" in [Mzk17], Definition 5.3, (ii), applied to Ψ to be the subfunctor in [Mzk17], Definition 5.3, (ii), applied to Ψ'] which fits into a 1-commutative diagram as in the statement of assertion (ii). Here we note that the existence of the equivalence of categories $\mathcal{C}' \xrightarrow{\sim} \underline{\mathcal{C}}^{\text{bs-fld}}$ compatible with the Frobenioid structures follows from our assumption that $\mathcal{C}'$ is of *model* type [which implies that it suffices to check the base categories, divisor monoids, and rational function monoids of $\mathcal{C}'$, $\underline{\mathcal{C}}^{\text{bs-fld}}$ — cf. [Mzk16], Theorem 5.2], our assumption that Φ' is *supported by* $\mathbb{R}$ [which implies that Φ' is naturally isomorphic to $\underline{\Phi}^{\text{bs-fld}}$], and our definition of $\underline{\mathbb{B}}$ as a *push-forward* [which implies that $\mathbb{F}'$ is naturally isomorphic to the rational function monoid $\underline{\mathbb{F}}$ of $\underline{\mathcal{C}}^{\text{bs-fld}}$ — cf.

Proposition 3.4, (ii)]. Moreover, it is immediate from the definition of $\underline{\mathbb{B}}$ as a *push-forward* that if Ψ' is *LC-unit-admissible*, then so is Ψ. This completes the proof of assertion (ii). ◯

Remark 3.9.1. Note that the hypotheses of Proposition 3.9, (ii), are not so difficult to satisfy: Indeed, typically in applications [cf., e.g., [Mzk17], Example 5.6, (iii)] $\mathcal{C}'$ will be a *p-adic Frobenioid*, hence, in particular, of *model* type. Moreover, if $\mathcal{C}^*$ is a p-adic Frobenioid, then the rational function monoid $\mathbb{F}^*$ of $\mathcal{C}^*$, together with the realification $(\Phi^*)^{\text{rlf}}$ [which is *supported by* $\mathbb{R}$!] of the divisor monoid Φ^* of $\mathcal{C}^*$, determine a p-adic [model] Frobenioid $\mathcal{C}'$ together with an *LC-unit-admissible functor* $\mathcal{C}^* \to \mathcal{C}'$ which is, moreover, *faithful* [since $\mathcal{C}^*$, $\mathcal{C}'$ have the same rational function monoid]. Thus, by composing any *LC-admissible functor* $\mathcal{C}^{\text{bs-fld}} \to \mathcal{C}^*$ [where $\mathcal{C}^*$ is as an *arbitrary p-adic Frobenioid*] with the functor $\mathcal{C}^* \to \mathcal{C}'$ just constructed, we obtain an *LC-admissible functor* $\Psi' : \mathcal{C}^{\text{bs-fld}} \to \mathcal{C}'$ as in Proposition 3.9, (ii).

Remark 3.9.2. The following are typical examples of *LC-admissible functors* $\Psi' : \mathcal{C}^{\text{bs-fld}} \to \mathcal{C}'$ as in Proposition 3.9, (ii): (a) the functor obtained by composing the natural *perfection* functor $\mathcal{C}^{\text{bs-fld}} \to \mathcal{C}^* \stackrel{\text{def}}{=} (\mathcal{C}^{\text{bs-fld}})^{\text{pf}}$ with the functor "$\mathcal{C}^* \to \mathcal{C}'$" of Remark 3.9.1; (b) the natural *realification* functor $\mathcal{C}^{\text{bs-fld}} \to (\mathcal{C}^{\text{bs-fld}})^{\text{rlf}}$. Note that if instead one attempts to consider 1-commutative diagrams

$$\begin{array}{ccc} \mathcal{C}^{\text{bs-fld}} & \hookrightarrow & \mathcal{C} \\ \downarrow \Psi^{\text{bs-fld}} & & \downarrow \Psi \\ \underline{\mathcal{C}}^{\text{bs-fld}} & \hookrightarrow & \underline{\mathcal{C}} \end{array}$$

– where the "$\hookrightarrow$'s" are the natural inclusion functors; Ψ is the natural *perfection* or *realification* functor for $\mathcal{C}$ — then one runs into problems since Ψ *fails* [in general] to be *LC-admissible* — i.e., it violates [in general] the *crucial* [cf. the proof of [Mzk17], Proposition 5.4, (i)] condition (e) of [Mzk17], Definition 5.3, (ii)].

Remark 3.9.3. Thus, by applying Proposition 3.9, one may pass from a *collection of Frobenioid-theoretic grafting data* [cf. [Mzk17], Definition 5.3, (iii)] in which the *local components* are obtained as *base-field-theoretic hulls* of *tempered Frobenioids* [cf., e.g., [Mzk17], Example 5.6], equipped with *LC-admissible* functors to Frobenioids of *model type* with *perf-factorial* divisor monoid *supported by* $\mathbb{R}$ [cf. Proposition 3.9, (ii)], to a **new** *collection of Frobenioid-theoretic grafting data* in which these local components [i.e., the base-field-theoretic hulls] are *replaced* by the corresponding tempered Frobenioids, equipped with LC-admissible functors as in Proposition 3.9, (ii). If, moreover, the original data is of *poly-standard* (respectively, *poly-rationally standard*) type [cf. [Mzk17], Definition 5.3, (iii)], and the tempered Frobenioids in question have *non-dilating* (respectively, *non-dilating* and *rational*) divisor monoids [cf. Theorem 3.7, (ii)] and lie over base categories which are either *weakly indissectible* or *strongly dissectible* [cf. Remark 3.7.1], then the new data will also be of *poly-standard* (respectively, *poly-rationally standard*) type. We leave the routine clerical details to the interested reader.

Finally, we begin to relate the theory of *tempered Frobenioids* to the theory of the *étale theta function*, as discussed in §1, §2:

Example 3.10. Theta Functions and Tempered Frobenioids.

(i) Suppose that $X^{\log}$ is a *smooth log orbicurve* of the sort defined in Definition 2.5, (i), (ii) [i.e., one of the following smooth log orbicurves: "$\underline{X}^{\log}$", "$\underline{C}^{\log}$", "$\underline{\underline{X}}^{\log}$", "$\underline{\underline{C}}^{\log}$", "$\dot{\underline{X}}^{\log}$", "$\dot{\underline{C}}^{\log}$", "$\dot{\underline{\underline{X}}}^{\log}$", "$\dot{\underline{\underline{C}}}^{\log}$"]. Then there exists a [1-]commutative diagram of *finite log étale Galois coverings of smooth log orbicurves*

$$\begin{array}{ccc} U^{\log} & \rightarrow & X^{\log} \\ \downarrow & & \downarrow \\ Y^{\log} & \rightarrow & W^{\log} \end{array}$$

— where $U^{\log}$, $Y^{\log}$ are *smooth log curves* that arise as generic fibers of *stable log curves* $\mathfrak{U}^{\log}$, $\mathfrak{Y}^{\log}$ over [formal spectra equipped with appropriate log structures determined by] rings of integers of appropriate finite extensions of K; the diagram induces a natural isomorphism $\mathrm{Gal}(U^{\log}/X^{\log}) \xrightarrow{\sim} \mathrm{Gal}(Y^{\log}/W^{\log})$; the order of the group $\mathrm{Gal}(U^{\log}/X^{\log}) \cong \mathrm{Gal}(Y^{\log}/W^{\log})$ is ≤ 2; $Y^{\log} \rightarrow W^{\log}$ is *unramified at the cusps* of $Y^{\log}$; $Y^{\log}$ is of *genus* 1. [Thus, for instance, when $X^{\log}$ is "$\dot{\underline{\underline{C}}}^{\log}$", one may take the upper arrow of the diagram to be "$\dot{\underline{\underline{X}}}^{\log} \rightarrow \dot{\underline{\underline{C}}}^{\log}$" and the lower arrow of the diagram to be "$\dot{\underline{X}}^{\log} \rightarrow \dot{\underline{C}}^{\log}$".] Write:

$$\mathcal{D}_W \stackrel{\mathrm{def}}{=} \mathcal{B}^{\mathrm{temp}}(W^{\log})^0; \quad \mathcal{D}_X \stackrel{\mathrm{def}}{=} \mathcal{B}^{\mathrm{temp}}(X^{\log})^0 \ (= \mathcal{D}_0)$$
$$\mathcal{D}_Y \stackrel{\mathrm{def}}{=} \mathcal{B}^{\mathrm{temp}}(Y^{\log})^0; \quad \mathcal{D}_U \stackrel{\mathrm{def}}{=} \mathcal{B}^{\mathrm{temp}}(U^{\log})^0$$

Note that the above [1-]commutative diagram induces natural functors $\mathcal{D}_U \rightarrow \mathcal{D}_X$, $\mathcal{D}_U \rightarrow \mathcal{D}_Y$, $\mathcal{D}_X \rightarrow \mathcal{D}_W$, $\mathcal{D}_Y \rightarrow \mathcal{D}_W$ [obtained by regarding a tempered covering of the *domain* orbicurve of an arrow of the above commutative diagram as a tempered covering of the *codomain* curve of the arrow].

(ii) Let us write

$$\mathcal{D}_W^{\mathrm{ell}} \subseteq \mathcal{D}_W; \quad \mathcal{D}_Y^{\mathrm{ell}} \subseteq \mathcal{D}_Y$$

for the full subcategory of tempered coverings that are *unramified over the cusps* of $W^{\log}$, $Y^{\log}$ [i.e., the "tempered coverings of the underlying elliptic curve of $Y^{\log}$"]; thus, by taking the *left adjoints* to the natural inclusion functors $\mathcal{D}_W^{\mathrm{ell}} \hookrightarrow \mathcal{D}_W$, $\mathcal{D}_Y^{\mathrm{ell}} \hookrightarrow \mathcal{D}_Y$, we obtain *natural functors* $\mathcal{D}_W \rightarrow \mathcal{D}_W^{\mathrm{ell}}$, $\mathcal{D}_Y \rightarrow \mathcal{D}_Y^{\mathrm{ell}}$ [cf. [Mzk17], Example 1.3, (ii)]. Since, moreover, $Y^{\log} \rightarrow W^{\log}$ is *unramified at the cusps* of $Y^{\log}$, we obtain 1-commutative diagrams of natural functors:

$$\begin{array}{ccc} \mathcal{D}_Y^{\mathrm{ell}} & \hookrightarrow & \mathcal{D}_Y \\ \downarrow & & \downarrow \\ \mathcal{D}_W^{\mathrm{ell}} & \hookrightarrow & \mathcal{D}_W \end{array} \qquad \begin{array}{ccc} \mathcal{D}_Y & \rightarrow & \mathcal{D}_Y^{\mathrm{ell}} \\ \downarrow & & \downarrow \\ \mathcal{D}_W & \rightarrow & \mathcal{D}_W^{\mathrm{ell}} \end{array}$$

(iii) Next, let us denote by $\Phi_{\mathcal{W}}$ the monoid on $\mathcal{D}_W$ given by forming the *perfection* of the monoid "Φ_0" of Definition 3.3, (iii), for some choice of *tempered filter* on $W^{\log}$ that arises from a tempered filter on $Y^{\log}$ [i.e., whose constituent subgroups $\subseteq \underline{\Delta}_W^{\text{tm}}$ are contained in $\underline{\Delta}_Y^{\text{tm}} \subseteq \underline{\Delta}_W^{\text{tm}}$]. Now define

$$\Phi^{\text{ell}}_{W^{\text{ell}}} \subseteq \Phi_{\mathcal{W}}|_{\mathcal{D}^{\text{ell}}_W}$$

as follows: For $A \in \text{Ob}(\mathcal{D}^{\text{ell}}_W)$, we take $\Phi^{\text{ell}}_{W^{\text{ell}}}(A)$ to be the *perf-saturation* [cf. §0] in $\Phi_{\mathcal{W}}(A)$ of the submonoid

$$\varinjlim_{Z^{\log}_\infty} \text{Div}_+(\mathfrak{Z}^{\log}_\infty)^{\text{Gal}(Z^{\log}_\infty/A)} \subseteq \Phi_{\mathcal{W}}(A)$$

— where $Z^{\log}_\infty$ ranges over the connected tempered coverings $Z^{\log}_\infty \to A$ in $\mathcal{D}^{\text{ell}}_W$ such that the composite covering $Z^{\log}_\infty \to A \to W^{\log}$ arises as the generic fiber of the *"universal combinatorial covering"* $\mathfrak{Z}^{\log}_\infty$ of the stable logarithmic model $\mathfrak{Z}^{\log}$ of some finite log étale Galois covering $Z^{\log} \to W^{\log}$ [in $\mathcal{D}^{\text{ell}}_W$] with *stable*, *split* reduction over the ring of integers of a finite extension L of K; the superscript Galois group denotes the submonoid of elements fixed by the Galois group in question. Here, we pause to *observe* that the various monoids that occur in the above inductive limit are all *contained* in $\Phi_{\mathcal{W}}(A)$, and that the *induced morphisms on perf-saturations* between these monoids are *bijective* [cf. Remark 3.3.1]. Indeed, this bijectivity follows immediately from the well-known structure of the *special fibers of the "universal combinatorial coverings"* that appear in the above inductive limit [i.e., *"chains of copies of the projective line"* — cf., e.g., the discussion preceding Proposition 1.1]. Set:

$$\Phi^{\text{ell}}_W \stackrel{\text{def}}{=} \Phi^{\text{ell}}_{W^{\text{ell}}}|_{\mathcal{D}_W} \subseteq \Phi_{\mathcal{W}}$$

Now observe that Φ^{ell}_W is a *perfect* [cf. the definition of $\Phi^{\text{ell}}_{W^{\text{ell}}}$ as a *perf-saturation* inside the *perfect* monoid $\Phi_{\mathcal{W}}|_{\mathcal{D}^{\text{ell}}_W}$] and [manifestly] *group-saturated* submonoid of the monoid $\Phi_{\mathcal{W}}$ on $\mathcal{D}_W$, which is, moreover, *perf-factorial*, *non-dilating* [cf. Proposition 3.4, (i); the above observation concerning the bijectivity of induced morphisms on perf-saturations], and *cuspidally pure* [cf. the well-known structure of the special fibers of the "universal combinatorial coverings" that appear in the above inductive limit]. Also, Φ^{ell}_W is [manifestly] *independent*, up to natural isomorphism, of the choice of *tempered filter* on W used to define $\Phi_{\mathcal{W}}$.

(iv) If $\alpha : A \to B$ is any morphism of $\mathcal{D}_W$, then set:

$$\mathcal{D}_\alpha \stackrel{\text{def}}{=} (\mathcal{D}_W)_B[\alpha] \quad (\subseteq (\mathcal{D}_W)_B)$$

[where we regard α as an object of $(\mathcal{D}_W)_B$ – cf. the notational conventions of §0; [Mzk16], §0]. Thus, $\mathcal{D}^\top_\alpha$ is a *quasi-temperoid* [cf. [Mzk17], Example 1.3, (ii); [Mzk14], Definition A.1, (ii)]. Also, we observe that $\mathcal{D}_W$, $\mathcal{D}_X$, $\mathcal{D}_Y$, $\mathcal{D}_U$ are *special cases* of "$\mathcal{D}_\alpha$" [obtained by taking "α" be the identity morphism of $W^{\log}$, $X^{\log}$, $Y^{\log}$, $U^{\log}$]. Note that we have a natural functor $\mathcal{D}_\alpha \to \mathcal{D}_W$; write:

$$\Phi^{\text{ell}}_\alpha \stackrel{\text{def}}{=} \Phi^{\text{ell}}_W|_{\mathcal{D}_\alpha}$$

Now it follows immediately from the above discussion that the monoid Φ_α^{ell} — which is obtained simply by restricting the functor Φ_W^{ell} via *some* functor — is *perfect*, *perf-factorial*, *non-dilating*, and *cuspidally pure*. Note, moreover, that the existence of the *theta functions* discussed in §1 [cf. especially the description of the *zeroes* and *poles* of these theta functions given in Proposition 1.4, (i)] implies that the monoid Φ_α^{ell} is also *rational* [cf. Definition 3.6, (ii); [Mzk16], Definition 4.5, (ii)]. In particular, it follows that:

> This monoid Φ_α^{ell} [along with its *perfection* and *realification* — cf. Remark 3.6.4] gives rise to a *tempered Frobenioid of rationally standard type* [cf. Theorem 3.7, (ii)] with *perfect divisor monoid* over the *slim* [cf. Remark 3.7.2] *base category* $\mathcal{D}_\alpha$.

Section 4: General Bi-Kummer Theory

In the present §4, we apply the theory of *tempered Frobenioids* developed in §3 to discuss the analogue, for *log-meromorphic functions on tempered coverings* of smooth log orbicurves over nonarchimedean local fields, of the *Kummer theory for p-adic Frobenioids* developed in [Mzk17], §2. One important aspect [i.e., in a word, the *"bi"* portion of the term "bi-Kummer"] of the *"bi-Kummer theory"* theory developed here — by comparison, for instance, to the *Kummer theory for arbitrary Frobenioids* discussed in [Mzk17], Definition 2.1 — is that instead of just taking roots of the given log-meromorphic function, one considers roots of the *pair* of sections of a line bundle that correspond, respectively, to the *"numerator"* and *"denominator"* of the log-meromorphic function [cf. Remark 4.3.1 below]. Another important feature of the theory developed here — by comparison to the theory of [Mzk17], §2, for *p-adic Frobenioids* — is the *absence* of an analogue of the *reciprocity map* [cf. Remark 4.4.1 below]. As we shall see in §5 below, the additional "layer of complexity" that arises from the former feature has the effect of *compensating* [to a certain extent, at least in the case of the situation discussed in §2] for the *"handicap"* constituted by the latter feature. Finally, we remark that the theory developed here may be regarded as and, indeed, was *motivated* by the goal of developing a *Frobenioid-theoretic translation/generalization* — via the theory of *base-Frobenius pairs* [cf. [Mzk16], Definition 2.7, (iii); [Mzk16], Proposition 5.6] — of the scheme-theoretic constructions of §1.

Let $X^{\log}$, K, $\mathcal{D}_0 = \mathcal{B}^{\text{temp}}(X^{\log})^0$ be as in §3. In the following discussion, we fix a *tempered Frobenioid* $\mathcal{C}$ whose *monoid type* [i.e., the "Λ" of Definition 3.6] is $\mathbb{Z}$, whose *divisor monoid* Φ is *perfect*, whose *base category* $\mathcal{D}$ is of the form

$$\mathcal{D} \stackrel{\text{def}}{=} \mathcal{D}_0[D] \quad (\subseteq \mathcal{D}_0)$$

[cf. §0], where $D \in \text{Ob}(\mathcal{D}_0)$, and whose *base-field-theoretic hull* we denote by $\mathcal{C}^{\text{bs-fld}} \subseteq \mathcal{C}$. Also, we fix a *Frobenius-trivial* object $A_{\circledcirc} \in \text{Ob}(\mathcal{C})$ such that $A_{\circledcirc}^{\text{bs}} \stackrel{\text{def}}{=} \text{Base}(A_{\circledcirc}) \in \text{Ob}(\mathcal{D}_0)$ is a *Galois* [cf. [Mzk14], Definition 3.1, (iv)] *object*. Thus, $A_{\circledcirc}^{\text{bs}}$ determines *normal open subgroups*

$$H_{\circledcirc} \subseteq \underline{\Pi}_X^{\text{tm}}; \quad H_{\circledcirc}^{\text{bs-fld}} \subseteq G_K$$

[i.e., $H_{\circledcirc}^{\text{bs-fld}}$ is the image of $H_{\circledcirc}$ in G_K] of the *tempered fundamental group* $\underline{\Pi}_X^{\text{tm}}$ of $X^{\log}$ and the quotient $\underline{\Pi}_X^{\text{tm}} \twoheadrightarrow G_K$ determined by the *absolute Galois group* of K. In the following discussion, we shall use the superscript "birat" (respectively, "bs") to denote the object or arrow determined by a given object or arrow of $\mathcal{C}$ in the *birationalization* $\mathcal{C}^{\text{birat}}$ [cf. [Mzk16], Proposition 4.4] (respectively, *base category* $\mathcal{D}$) of $\mathcal{C}$.

Definition 4.1. Let $A \in \text{Ob}(\mathcal{C})$.

(i) We shall refer to an element $f \in \mathcal{O}^{\times}(A^{\text{birat}})$ as *Cartier* if the zero divisor and divisor of poles of the [log-meromorphic function determined by] f are Cartier.

Thus, if $f \in \mathcal{O}^\times(A^{\mathrm{birat}})$ is Cartier, then f coincides with the element of $\mathcal{O}^\times(A^{\mathrm{birat}})$ determined [cf. [Mzk16], Proposition 4.4; [Mzk16], Definition 1.3, (iii), (c)] by a *base-equivalent pair of pre-steps*

$$s^{\sqcap}, s^{\sqcup} : A \to B$$

such that $\mathrm{Div}(s^{\sqcap})$ is the *zero divisor* of f, and $\mathrm{Div}(s^{\sqcup})$ is the *divisor of poles* of f [cf. [Mzk16], Definition 1.3, (iii), (d)]; we shall refer to such a pair $(s^{\sqcap}, s^{\sqcup})$ as a *fraction-pair*, or, alternatively, as a *right fraction-pair [for f]*; also, we shall refer to A (respectively, B) as the *domain* (respectively, *codomain*) of the fraction-pair $(s^{\sqcap}, s^{\sqcup})$. If we denote by $f|_B \in \mathcal{O}^\times(B^{\mathrm{birat}})$ the element of $\mathcal{O}^\times(B^{\mathrm{birat}})$ determined by $(s^{\sqcap}, s^{\sqcup})$ [cf. [Mzk16], Proposition 4.4], then we shall refer to the pair $(s^{\sqcap}, s^{\sqcup})$ as a *left fraction-pair [for* $f|_B$*]*.

(ii) We shall say that A is *Galois* if $A^{\mathrm{bs}} \in \mathrm{Ob}(\mathcal{D})$ is Galois. Suppose that A is *Galois*. Then, by the definition of $\mathcal{D}$, there is a natural *surjective outer homomorphism*

$$\underline{\Pi}_X^{\mathrm{tm}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}}(A^{\mathrm{bs}})$$

[cf. the discussion of [Mzk17], Definition 2.2, (i), in the case of p-adic Frobenioids]; write

$$H_{A^{\mathrm{bs}}} \subseteq \mathrm{Aut}_{\mathcal{D}}(A^{\mathrm{bs}})$$

for the image of $H_{\circledcirc}$ via this surjection [which is well-defined, since $H_{\circledcirc}$ is *normal*] and

$$H_A \subseteq \mathrm{Aut}_{\mathcal{C}}(A)/\mathcal{O}^\times(A)$$

for the inverse image of $H_{A^{\mathrm{bs}}}$ via the natural injection $\mathrm{Aut}_{\mathcal{C}}(A)/\mathcal{O}^\times(A) \hookrightarrow \mathrm{Aut}_{\mathcal{D}}(A^{\mathrm{bs}})$. If the natural injection $H_A \hookrightarrow H_{A^{\mathrm{bs}}}$ is a bijection, then we shall say that A is *$H_{\circledcirc}$-ample*.

(iii) Suppose that A is *$H_{\circledcirc}$-ample* [hence, in particular, *Galois*], and that $f \in \mathcal{O}^\times(A^{\mathrm{birat}})$ is an element *fixed* by the natural action of H_A; let $N \in \mathbb{N}_{\geq 1}$. Then we shall say that A is *$(N, H_{\circledcirc}, f)$-saturated* if the following conditions are satisfied: (a) there exist pre-steps $A'' \to A$, $A'' \to A'$ in $\mathcal{C}$, where A' is *Frobenius-trivial* [hence determines an object of the *p-adic Frobenioid* $\mathcal{C}^{\text{bs-fld}}$], such that A' is *$(N, H_{\circledcirc}^{\text{bs-fld}})$-saturated* [cf. [Mzk17], Definition 2.2, (ii)] as an object of $\mathcal{C}^{\text{bs-fld}}$; (b) there exists a $g \in \mathcal{O}^\times(A^{\mathrm{birat}})$ such that $g^N = f$.

(iv) We shall say that a morphism $\alpha : A \to B$ of $\mathcal{C}$ is *of base-Frobenius type* if there exist a subgroup $G \subseteq \mathrm{Aut}_{\mathcal{C}_B}(\alpha) \subseteq \mathrm{Aut}_{\mathcal{C}}(A)$ and a factorization $\alpha = \alpha'' \circ \alpha'$ such that the following conditions are satisfied: (a) A is *Frobenius-trivial*, *Galois*, and *$\boldsymbol{\mu}_N$-saturated* [cf. [Mzk17], Definition 2.1, (i)], where $N \overset{\mathrm{def}}{=} \deg_{\mathrm{Fr}}(\alpha)$; (b) G maps isomorphically to $\mathrm{Gal}(A^{\mathrm{bs}}/B^{\mathrm{bs}}) \subseteq \mathrm{Aut}_{\mathcal{D}}(A^{\mathrm{bs}})$; (c) α' is a base-identity endomorphism of Frobenius type; (d) α'' is a pull-back morphism; (e) G, α', α'' arise from a *base-Frobenius pair* of $\mathcal{C}$ [cf. Theorem 3.7, (i); [Mzk16], Proposition 5.6]. In this situation, we shall refer to the subgroup G and the factorization $\alpha = \alpha'' \circ \alpha'$ as being *of base-Frobenius type*.

Remark 4.1.1. Note that in the situation of Definition 4.1, (iv), if $\alpha : A \to B$ is *of base-Frobenius type*, then [one verifies easily that] $A \to B$ is a *categorical quotient* [cf. [Mzk16], §0] of A by the subgroup $G \cdot \boldsymbol{\mu}_N(A) \subseteq \mathrm{Aut}_{\mathcal{C}}(A)$.

Proposition 4.2. (Construction of Bi-Kummer Data I: Roots of Fraction-Pairs) *In the notation of the above discussion, let* $f \in \mathcal{O}^{\times}(A^{\mathrm{birat}}_{\circledcirc})$ *be* **Cartier***;* $s^{\sqcap}, s^{\sqcup} : A_{\circledcirc} \to B_{\circledcirc}$ *a* **right fraction-pair** *for* f *[i.e., a* **left fraction-pair** *for* $f|_{B_{\circledcirc}}$*];* $N \in \mathbb{N}_{\geq 1}$*. Then:*

(i) A pair of morphisms $t^{\sqcap}, t^{\sqcup} : A_{\circledcirc} \to C_{\circledcirc}$ *is a* **right fraction-pair** *for* f *if and only if there exists a [necessarily unique] isomorphism* $v : B_{\circledcirc} \xrightarrow{\sim} C_{\circledcirc}$ *such that* $t^{\sqcap} = v \circ s^{\sqcap}$*,* $t^{\sqcup} = v \circ s^{\sqcup}$*. A pair of morphisms* $t^{\sqcap}, t^{\sqcup} : A_{\circledcirc} \to B_{\circledcirc}$ *is a* **right fraction-pair** *for* f *if and only if there exists a [necessarily unique] automorphism* $u \in \mathrm{Aut}_{\mathcal{C}}(A_{\circledcirc})$ *such that* $t^{\sqcap} = s^{\sqcap} \circ u$*,* $t^{\sqcup} = s^{\sqcup} \circ u$*,* $(u^{\mathrm{birat}})^*(f) = f$*.*

(ii) A pair of morphisms $t^{\sqcap}, t^{\sqcup} : C_{\circledcirc} \to B_{\circledcirc}$ *is a* **left fraction-pair** *for* $f|_{B_{\circledcirc}}$ *if and only if there exists a [necessarily unique] isomorphism* $v : C_{\circledcirc} \xrightarrow{\sim} A_{\circledcirc}$ *such that* $t^{\sqcap} = s^{\sqcap} \circ v$*,* $t^{\sqcup} = s^{\sqcup} \circ v$*. A pair of morphisms* $t^{\sqcap}, t^{\sqcup} : A_{\circledcirc} \to B_{\circledcirc}$ *is a* **left fraction-pair** *for* $f|_{B_{\circledcirc}}$ *if and only if there exists a [necessarily unique] automorphism* $u \in \mathrm{Aut}_{\mathcal{C}}(B_{\circledcirc})$ *such that* $t^{\sqcap} = u \circ s^{\sqcap}$*,* $t^{\sqcup} = u \circ s^{\sqcup}$*,* $(u^{\mathrm{birat}})^*(f|_{B_{\circledcirc}}) = f|_{B_{\circledcirc}}$*.*

(iii) There **exist** *commutative diagrams in* $\mathcal{C}$

$$
\begin{array}{ccc}
A_N & \xrightarrow{s^{\sqcap}_N} & B_N \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
A_{\circledcirc} & \xrightarrow{s^{\sqcap}} & B_{\circledcirc}
\end{array}
\qquad\qquad
\begin{array}{ccc}
A_N & \xrightarrow{s^{\sqcup}_N} & B_N \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
A_{\circledcirc} & \xrightarrow{s^{\sqcup}} & B_{\circledcirc}
\end{array}
$$

— where α*,* β *are* **isometries** *of* **Frobenius degree** N*;* α *is* **of base-Frobenius type***, with factorization of base-Frobenius type* $\alpha = \alpha'' \circ \alpha'$*;* $f|_{A_N} \stackrel{\mathrm{def}}{=} ((\alpha'')^{\mathrm{birat}})^*(f)$ *[cf. [Mzk16], Proposition 1.11, (v)];* A_N *is* $(N, H_{\circledcirc}, f|_{A_N})$**-saturated***;* $s^{\sqcap}_N, s^{\sqcup}_N : A_N \to B_N$ *are* **base-equivalent pre-steps***. In particular, there exists a* **Cartier** *element* $\in \mathcal{O}^{\times}(A^{\mathrm{birat}}_N)$ *(respectively,* $\in \mathcal{O}^{\times}(B^{\mathrm{birat}}_N)$*) for which* $(s^{\sqcap}_N, s^{\sqcup}_N)$ *is a* **right** *(respectively,* **left***)* **fraction-pair***, and whose* N*-th power is equal to* $f|_{A_N}$ *(respectively,* $f|_{B_N} \stackrel{\mathrm{def}}{=} (f|_{A_N})|_{B_N}$ *[cf. the notation of Definition 4.1, (i)]). In the following, we shall refer to such a pair of commutative diagrams as an* N**-th root of the fraction-pair** $(s^{\sqcap}, s^{\sqcup})$*, to* A_N *as the* N**-domain** *of this root of a fraction-pair, and to* B_N *as the* N**-codomain** *of this root of a fraction-pair.*

(iv) We continue to use the notation of (iii). Let

$$
\begin{array}{ccc}
\underline{A}_N & \xrightarrow{\underline{s}^{\sqcap}_N} & \underline{B}_N \\
\downarrow{\scriptstyle \underline{\alpha}} & & \downarrow{\scriptstyle \underline{\beta}} \\
A_{\circledcirc} & \xrightarrow{\underline{s}^{\sqcap}} & B_{\circledcirc}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\underline{A}_N & \xrightarrow{\underline{s}^{\sqcup}_N} & \underline{B}_N \\
\downarrow{\scriptstyle \underline{\alpha}} & & \downarrow{\scriptstyle \underline{\beta}} \\
A_{\circledcirc} & \xrightarrow{\underline{s}^{\sqcup}} & B_{\circledcirc}
\end{array}
$$

be **another** N*-th root of a left fraction-pair* $(\underline{s}^{\sqcap}, \underline{s}^{\sqcup})$ *for* $f|_{B_{\circledcirc}}$*;* $\delta \in \mathrm{Aut}_{\mathcal{C}}(B_{\circledcirc})$ *the unique automorphism [cf. (ii)] such that* $\underline{s}^{\sqcap} = \delta \circ s^{\sqcap}$*,* $\underline{s}^{\sqcup} = \delta \circ s^{\sqcup}$*;* $\epsilon_B : B^{\mathrm{bs}}_N \xrightarrow{\sim} \underline{B}^{\mathrm{bs}}_N$

an isomorphism of $\mathcal{D}$ *such that* $\delta^{\mathrm{bs}} \circ \beta^{\mathrm{bs}} = \underline{\beta}^{\mathrm{bs}} \circ \epsilon_B$. *Then, after possibly* **replacing** $s_N^{\sqcap}$ *by* $u \circ s_N^{\sqcap}$, *for some* $u \in \boldsymbol{\mu}_N(B_N)$ *[where "*$\boldsymbol{\mu}_N(-)$*" is as in [Mzk17], Definition 2.1, (i)], there* **exist** *isomorphisms* $\zeta_A : A_N \xrightarrow{\sim} \underline{A}_N$, $\zeta_B : B_N \xrightarrow{\sim} \underline{B}_N$ *in* $\mathcal{C}$ *which fit into commutative diagrams*

$$\begin{array}{ccc} A_N & \xrightarrow{s_N^{\sqcap}} & B_N \\ \downarrow \zeta_A & & \downarrow \zeta_B \\ \underline{A}_N & \xrightarrow{\underline{s}_N^{\sqcap}} & \underline{B}_N \end{array} \qquad \begin{array}{ccc} A_N & \xrightarrow{s_N^{\sqcup}} & B_N \\ \downarrow \zeta_A & & \downarrow \zeta_B \\ \underline{A}_N & \xrightarrow{\underline{s}_N^{\sqcup}} & \underline{B}_N \end{array}$$

and, moreover, satisfy $\alpha = \underline{\alpha} \circ \zeta_A$, $\delta \circ \beta = \underline{\beta} \circ \zeta_B$, $\zeta_B^{\mathrm{bs}} = \epsilon_B$. *Here,* ζ_A *is* **uniquely** *determined by* ζ_B*;* ζ_B *is* **uniquely** *determined by* ϵ_B*, up to composition with an element of* $\boldsymbol{\mu}_N(B_N)$. *In the following, we shall refer to such a pair* (ζ_A, ζ_B) *as an* **isomorphism** *between the two given* N*-th roots of fraction-pairs.*

Proof. The *sufficiency* portion of assertion (i) is immediate. To verify the *necessity* portion of assertion (i), observe that by [Mzk16], Definition 1.3, (iii), (d), the condition on the Div$(-)$'s of the components of a fraction-pair $t^{\sqcap}, t^{\sqcup} : A_{\circledcirc} \to C_{\circledcirc}$ implies the existence of isomorphisms $v, v' : B_{\circledcirc} \xrightarrow{\sim} C_{\circledcirc}$ such that $t^{\sqcap} = v \circ s^{\sqcap}$, $t^{\sqcup} = v' \circ s^{\sqcup}$; since, moreover, $t^{\sqcap}$, $t^{\sqcup}$ are *base-equivalent*, it follows that $v = v'' \circ v'$, for some $v'' \in \mathcal{O}^{\times}(C_{\circledcirc})$. On the other hand, since the fraction-pair $t^{\sqcap}, t^{\sqcup} : A_{\circledcirc} \to C_{\circledcirc}$ determines the same element of $\mathcal{O}^{\times}(A_{\circledcirc}^{\mathrm{birat}})$ as the fraction-pair $s^{\sqcap}, s^{\sqcup} : A_{\circledcirc} \to C_{\circledcirc}$, we thus conclude that $v'' = 1$. If $B_{\circledcirc} = C_{\circledcirc}$, then it follows from the Aut-*ampleness* of the Frobenius-trivial object $A_{\circledcirc}$ [cf. [Mzk16], Theorem 5.1, (iii)] that there exist $u \in \mathrm{Aut}_{\mathcal{C}}(A_{\circledcirc})$, $u' \in \mathcal{O}^{\times}(A_{\circledcirc})$ such that $t^{\sqcap} = s^{\sqcap} \circ u$, $t^{\sqcup} = s^{\sqcup} \circ u \circ u'$; since, moreover, $(u^{\mathrm{birat}})^*(f) = f$, we conclude that $u' = 1$, as desired. Note that the uniqueness of u follows from the *total epimorphicity* of $\mathcal{C}$; the uniqueness of v follows from the fact that pre-steps are always *monomorphisms* [cf. [Mzk16], Definition 1.3, (v), (a)]. This completes the proof of assertion (i). The proof of assertion (ii) is entirely similar.

Next, we consider assertions (iii), (iv). First, let us *observe* that, by the definition of *"log-meromorphic"* [cf. Definition 3.1, (ii)], it follows immediately that over some tempered covering of $X^{\log}$ that occurs as the "universal combinatorial covering" of a finite étale covering of $X^{\log}$ with stable, split reduction, f admits an N*-th root*, which, moreover, is *Cartier*. [Indeed, to see that such an N-th root is Cartier, it suffices to check that this N-th root is Cartier on the non-smooth locus, i.e., at the *nodes*, of the stable model of the tempered covering in question; but, since the *divisor of cusps* of the stable model is always *disjoint* from the nodes, this follows immediately from the very *existence* of such an N-th root.] Since, moreover, the divisor monoid Φ is assumed to be *perfect*, it follows that the divisors of zeroes and poles of such an N-th root belong to $\Phi(-)$ of the tempered covering in question. Now since the Frobenioid $\mathcal{C}$ is of *model* [hence, in particular, *pre-model*] type [cf. Theorem 3.7, (i)], it follows that $\mathcal{C}$ admits a *base-Frobenius pair* [cf. [Mzk16], Definition 2.7, (iii)]. Thus, the *existence* of a pair of commutative diagrams as in the statement of assertion (iii) follows by *translating* the above *"scheme-theoretic observations"* into the *language of Frobenioids* — cf. [Mzk16], Definition 1.3, (iii), (d) [on the existence of *pre-steps with prescribed zero divisor*]; [Mzk17], Remark 2.2.1 [concerning the issue of *"$(N, H_{\circledcirc}^{\text{bs-fld}})$-saturation"*]. Finally, to verify the *uniqueness up to isomorphism* of such a pair of commutative diagrams as stated in assertion (iv), let us first observe that by replacing β, $s^{\sqcap}$, $s^{\sqcup}$ by $\delta \circ \beta$, $\delta \circ s^{\sqcap}$, $\delta \circ s^{\sqcup}$, respectively, we may assume without loss of generality that δ is the *identity.* Now the existence of a ζ_A as desired follows immediately from the *uniqueness up to conjugation by a unit* of base-Frobenius pairs of A_N, $\underline{A}_N$ [cf. [Mzk16], Proposition 5.6], by thinking of α, $\underline{\alpha}$ as *categorical quotients* [cf. Remark 4.1.1] and applying the *base-triviality* and Aut-*ampleness* of the full subcategory of $\mathcal{C}$ determined by the Frobenius-trivial objects [cf. [Mzk16], Theorem 5.1, (iii)]; the existence of a ζ_B as desired follows from the *equivalences of categories determined by pre-steps* of [Mzk16], Definition 1.3, (iii), (d). The essential uniqueness of ζ_A, ζ_B as asserted follows immediately from the various conditions imposed on ζ_A, ζ_B. This completes the proof of assertion (iv). ◯

Proposition 4.3. (Construction of Bi-Kummer Data II: Bi-Kummer Roots) *In the notation of Proposition 4.2, (iii):*

(i) Let

$$s_N^{\text{trv}} : H_{A_N} \to \text{Aut}_{\mathcal{C}}(A_N)$$

be the group homomorphism arising from a **base-Frobenius pair** *of A_N [cf. Theorem 3.7, (i); [Mzk16], Proposition 5.6]. [Thus, s_N^{trv} is completely determined up to* **conjugation** *by an element of $\mathcal{O}^{\times}(A_N)$ — cf. Theorem 3.7, (i); [Mzk16], Proposition 5.6.] Then there exist unique* **group homomorphisms**

$$s_N^{\sqcap\text{-gp}} : H_{B_N} \to \text{Aut}_{\mathcal{C}}(B_N); \quad s_N^{\sqcup\text{-gp}} : H_{B_N} \to \text{Aut}_{\mathcal{C}}(B_N)$$

such that, relative to the isomorphism $H_{A_N} \xrightarrow{\sim} H_{B_N}$ determined by the [base-equivalent!] pair of morphisms $s_N^{\sqcap}$, $s_N^{\sqcup}$, we have

$$s_N^{\sqcap\text{-gp}}(h) \circ s_N^{\sqcap} = s_N^{\sqcap} \circ (s_N^{\text{trv}}|_{H_{B_N}})(h); \quad s_N^{\sqcup\text{-gp}}(h) \circ s_N^{\sqcup} = s_N^{\sqcup} \circ (s_N^{\text{trv}}|_{H_{B_N}})(h)$$

*for all $h \in H_{B_N}$. [In particular, it follows that B_N is $H_{\circledcirc}$-***ample***.] In the following, we shall refer to such a pair $(s_N^{\sqcap\text{-gp}}, s_N^{\sqcup\text{-gp}})$ as a* **bi-Kummer N-th root of the fraction-pair** *$(s^{\sqcap}, s^{\sqcup})$; also, we shall speak of A_N, B_N as being the "N-domain", "N-codomain", respectively, [cf. Proposition 4.2, (iii)] not only of the original root of a fraction-pair, also of the resulting bi-Kummer root.*

(ii) In the notation of (i), the collection of bi-Kummer N-th roots, with **N-codomain** *B_N, of a fraction-pair with domain isomorphic to $A_{\circledcirc}$, of some rational function whose pull-back to [the N-codomain] B_N is equal to $f|_{B_N}$, is equal to the collection of pairs obtained from $(s_N^{\sqcap\text{-gp}}, s_N^{\sqcup\text{-gp}})$ by* **simultaneous conjugation** *by an element*

$$\zeta_{\text{Aut}} \in \text{Aut}_{\mathcal{C}}(B_N)$$

such that $(\zeta_{\text{Aut}}^{\text{birat}})^*$ fixes $f|_{B_N}$, followed by **non-simultaneous conjugation** *[of, say, $s_N^{\sqcap\text{-gp}}$, but not $s_N^{\sqcup\text{-gp}}$] by an element $u \in \boldsymbol{\mu}_N(B_N)$ [cf. the "ζ_B", "u" of Proposition 4.2, (iv)]. In particular, if the data under consideration satisfy the condition* *that "for* $\gamma \in \mathrm{Aut}_{\mathcal{C}}(B_N)$*,* $(\gamma^{\mathrm{birat}})^*$ *fixes* $f|_{B_N}$ *if and only if* $\gamma^{\mathrm{bs}} \in H_{B_N}$*", then* ζ_{Aut} *may be taken to be* $\in \mathcal{O}^\times(B_N)$*.*

(iii) In the notation of (i), the **difference** $s_N^{\sqcap\text{-gp}} \cdot (s_N^{\sqcup\text{-gp}})^{-1}$ *determines a* **twisted homomorphism** $H_{B_N} \to \boldsymbol{\mu}_N(B_N)$*, hence an element of the cohomology module* $H^1(H_{B_N}, \boldsymbol{\mu}_N(B_N))$*, which is equal to the* **Kummer class** *[cf. [Mzk17], Definition 2.1, (ii)]*

$$\kappa_{f|_{B_N}} \in H^1(H_{B_N}, \boldsymbol{\mu}_N(B_N))$$

of $f|_{B_N}$ *[cf. the notation of Proposition 4.2, (iii)]. In particular, this cohomology class is* **independent** *of the [simultaneous and non-simultaneous] conjugation operations discussed in (ii).*

Proof. First, we consider assertion (i). To prove the existence and uniqueness of $s_N^{\sqcap\text{-gp}}$, $s_N^{\sqcup\text{-gp}}$, it follows from the general theory of Frobenioids — cf. the *first equivalence of categories involving pre-steps* of [Mzk16], Definition 1.3, (iii), (d); the fact that Frobenioids are always *totally epimorphic* — that it suffices to prove that

$$\mathrm{Div}(s_N^{\sqcap}), \mathrm{Div}(s_N^{\sqcup}) \in \Phi(A_N)$$

are *fixed* by H_{A_N}. But since

$$N \cdot \mathrm{Div}(s_N^{\sqcap}), N \cdot \mathrm{Div}(s_N^{\sqcup}) \in \Phi(A_N)$$

arise as *pull-backs* to A_N of elements of $\Phi(A_{\circledcirc})$ [i.e. $\mathrm{Div}(s^{\sqcap})$, $\mathrm{Div}(s^{\sqcup})$], this follows from the fact that [by definition] $H_{\circledcirc}$ acts *trivially* on $A_{\circledcirc}^{\mathrm{bs}}$ [together with the fact the monoid $\Phi(A_N)$ is *torsion-free*!]. This completes the proof of assertion (i). Assertion (iii) follows immediately from the definitions [cf., especially, [Mzk17], Definition 2.1, (ii)].

Next, we observe that assertion (ii) follows immediately from Proposition 4.2, (iv): That is to say, the "ζ_{Aut}" of assertion (ii) arises from the possible "ζ_B" of *loc. cit.* that may occur when it happens that $B_N = \underline{B}_N$ — up to possibly replacing $s_N^{\sqcap}$ by $u \circ s_N^{\sqcap}$ for some $u \in \boldsymbol{\mu}_N(B_N)$ [i.e., the "u" of assertion (ii) corresponds precisely to the "u" of *loc. cit.*] — composed with an element $\in \mathcal{O}^\times(B_N)$ that arises from the $\mathcal{O}^\times(A_N)$*-indeterminacy* of the *base-Frobenius pair* that occurs in assertion (i). Finally, if the data under consideration satisfy the condition that "for $\gamma \in \mathrm{Aut}_{\mathcal{C}}(B_N)$, $(\gamma^{\mathrm{birat}})^*$ fixes $f|_{B_N}$ if and only if $\gamma^{\mathrm{bs}} \in H_{B_N}$", then we may write $\zeta_{\mathrm{Aut}} = \zeta' \cdot u'$, where ζ' lies in the image of $s_N^{\sqcup\text{-gp}}$, $u' \in \mathcal{O}^\times(B_N)$. Moreover, by assertion (iii), we may write $\zeta' = \zeta'' \cdot u''$, where ζ'' lies in the image of $s_N^{\sqcap\text{-gp}}$, $u'' \in \boldsymbol{\mu}_N(B_N)$. Thus, *simultaneous conjugation* by ζ' yields the *same* pair of sections as *non-simultaneous conjugation* by u''. That is to say, modulo the non-simultaneous conjugation operation by elements of $\boldsymbol{\mu}_N(B_N)$, simultaneous conjugation by ζ_{Aut} yields the same pair of sections as simultaneous conjugation by an element $\in \mathcal{O}^\times(B_N)$. This completes the proof of assertion (ii). ◯

Remark 4.3.1. Of course, even without applying the somewhat nontrivial theory of *base-Frobenius pairs* [i.e., [Mzk16], Proposition 5.6], liftings to $\mathrm{Aut}_{\mathcal{C}}(A_N)$ of *individual elements* of H_{A_N} [cf. the notation of Proposition 4.3, (i)] are completely determined up to possible *translation* by elements of $\mathcal{O}^\times(A_N)$. The *crucial difference*, however, between an indeterminacy up to translation by elements of $\mathcal{O}^\times(A_N)$ and an indeterminacy up to *conjugation* by elements of $\mathcal{O}^\times(A_N)$ is that, unlike the former, the latter allows one to work with sections [i.e., $s^{\sqcap\text{-gp}}$, $s^{\sqcup\text{-gp}}$] which are *group homomorphisms.* Of course, the Kummer class [which, by Proposition 4.3, (iii), is determined by the pair $(s^{\sqcap\text{-gp}}, s^{\sqcup\text{-gp}})$] is always a [twisted] homomorphism. On the other hand, the theory of base-Frobenius pairs allows one to work with *group homomorphisms* [i.e., $s^{\sqcap\text{-gp}}$ or $s^{\sqcup\text{-gp}}$] *even* when one is forced to restrict one's attention to *only one* of the two arrows $s^\sqcap$ or $s^\sqcup$, i.e., in situations where one is not allowed to work with the *fraction-pair as a "single entity"* — cf. the theory of §5 below.

Remark 4.3.2. Note that if N divides $N' \in \mathbb{N}_{\geq 1}$, then one verifies immediately that given *an N-th root of the fraction-pair* $(s^\sqcap, s^\sqcup)$ [e.g., as in Proposition 4.2, (iii)], there exists a *"morphism"* from a suitable *N'-th root of the fraction-pair* $(s^\sqcap, s^\sqcup)$

$$\begin{array}{ccc} A_{N'} & \xrightarrow{s^\sqcap_{N'}} & B_{N'} \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta'} \\ A_{\circledcirc} & \xrightarrow{s^\sqcap} & B_{\circledcirc} \end{array} \qquad \begin{array}{ccc} A_{N'} & \xrightarrow{s^\sqcup_{N'}} & B_{N'} \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta'} \\ A_{\circledcirc} & \xrightarrow{s^\sqcup} & B_{\circledcirc} \end{array}$$

to the given N-th root of the fraction-pair $(s^\sqcap, s^\sqcup)$, i.e., a pair of commutative diagrams

$$\begin{array}{ccc} A_{N'} & \xrightarrow{s^\sqcap_{N'}} & B_{N'} \\ \downarrow{\scriptstyle \alpha_{N,N'}} & & \downarrow{\scriptstyle \beta_{N,N'}} \\ A_N & \xrightarrow{s^\sqcap_N} & B_N \end{array} \qquad \begin{array}{ccc} A_{N'} & \xrightarrow{s^\sqcup_{N'}} & B_{N'} \\ \downarrow{\scriptstyle \alpha_{N,N'}} & & \downarrow{\scriptstyle \beta_{N,N'}} \\ A_N & \xrightarrow{s^\sqcup_N} & B_N \end{array}$$

— where $\alpha_{N,N'}$ (respectively, $\beta_{N,N'}$) is an *isometry of Frobenius degree N'/N* that is compatible with α', α (respectively, β', β); $\alpha_{N,N'}$ is *of base-Frobenius type.* For instance, such a "morphism" may be constructed by extracting an "N'/N-th root" [cf. Proposition 4.2, (iii)] of the [fraction-pair determined by the] given "N-th root" of the fraction-pair $(s^\sqcap, s^\sqcup)$. Moreover, this collection of data induces a natural morphism

$$H^1(H_{B_N}, \boldsymbol{\mu}_N(B_N)) \to H^1(H_{B_{N'}}, \boldsymbol{\mu}_N(B_{N'}))$$

that maps $\kappa_{f|_{B_N}}$ to the image of $\kappa_{f|_{B_{N'}}}$ in $H^1(H_{B_{N'}}, \boldsymbol{\mu}_N(B_{N'}))$ [i.e., via the natural surjection $\boldsymbol{\mu}_{N'}(B_{N'}) \twoheadrightarrow \boldsymbol{\mu}_N(B_{N'})$]. In particular, by allowing N to vary, we obtain

a compatible system of roots of the fraction-pair $(s^\sqcap, s^\sqcup)$

hence a *compatible system of Kummer classes*, which, by Proposition 3.2, (iii), is *sufficient to distinguish f* from other elements of $\mathcal{O}^\times(A^{\text{birat}}_{\circledcirc})$.

We conclude our discussion of *"general bi-Kummer theory"* by observing that, up to the various *indeterminacies* discussed so far, our constructions are entirely *"category-theoretic"*:

Theorem 4.4. **(Category-theoreticity of Bi-Kummer Data)** *For $i = 1, 2$, let p_i be a prime number; K_i a finite extension of $\mathbb{Q}_{p_i}$; $X_i^{\log}$ a* **smooth log orbicurve** *over K_i; $\mathcal{C}_i$ a* **tempered Frobenioid**, *whose* **monoid type** *[i.e., the "Λ" of Definition 3.6] is $\mathbb{Z}$, whose* **divisor monoid** *Φ_i is* **perfect** *and* **non-dilating**, *and whose* **base category** *$\mathcal{D}_i$ is of the form*

$$\mathcal{D}_i \stackrel{\text{def}}{=} \mathcal{B}^{\text{temp}}(X_i^{\log})[D_i]$$

where $D_i \in \text{Ob}(\mathcal{B}^{\text{temp}}(X_i^{\log}))$; $A_{\circledcirc,i} \in \text{Ob}(\mathcal{C}_i)$ a **Frobenius-trivial** *object such that $A^{\text{bs}}_{\circledcirc,i} \stackrel{\text{def}}{=} \text{Base}(A_{\circledcirc,i}) \in \text{Ob}(\mathcal{D}_i)$ is* **Galois**, *hence determines a* **normal open subgroup**

$$H_{\circledcirc,i} \subseteq \underline{\Pi}^{\text{tm}}_{X_i}$$

of the **tempered fundamental group** *$\underline{\Pi}^{\text{tm}}_{X_i}$ of $X_i^{\log}$; $N \in \mathbb{N}_{\geq 1}$. Suppose that*

$$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$$

is an **equivalence of categories** *which induces [cf. Theorem 3.7, (i), (ii); Remark 3.7.2; [Mzk16], Theorem 3.4, (v)] an equivalence $\Psi^{\text{bs}} : \mathcal{D}_1 \xrightarrow{\sim} \mathcal{D}_2$ that maps $A^{\text{bs}}_{\circledcirc,1}$ to an isomorph of $A^{\text{bs}}_{\circledcirc,2}$. Then:*

(i) Ψ maps **isomorphs of** *$A_{\circledcirc,1}$ to isomorphs of $A_{\circledcirc,2}$ and $H_{\circledcirc,1}$-***ample** *objects to $H_{\circledcirc,2}$-ample objects; Ψ^{bs} induces an* **isomorphism** *$H_{\circledcirc,1} \xrightarrow{\sim} H_{\circledcirc,2}$, which is well-defined up to composition with inner automorphisms of $\underline{\Pi}^{\text{tm}}_{X_i}$.*

*(ii) Ψ induces a 1-***compatible** *equivalence of categories $\Psi^{\text{birat}} : \mathcal{C}_1^{\text{birat}} \xrightarrow{\sim} \mathcal{C}_2^{\text{birat}}$. Moreover, Ψ^{birat} preserves* **Cartier** *elements of $\mathcal{O}^\times(-)$; Ψ preserves* **fraction-pairs**, **right fraction-pairs**, *and* **left fraction-pairs**. *Finally, Ψ maps N-***th roots of fraction-pairs** *with domain isomorphic to $A_{\circledcirc,1}$ to N-th roots of fraction-pairs with domain isomorphic to $A_{\circledcirc,2}$.*

*(iii) For $i = 1, 2$, let $A_i \in \text{Ob}(\mathcal{C}_i)$ be $H_{\circledcirc,i}$-***ample**; *$f_i \in \mathcal{O}^\times(A_i^{\text{birat}})$ an element fixed by the natural action of H_{A_i} [where "H_{A_i}" is defined as in Definition 4.1, (ii), by taking "$H_\circledcirc$" to be $H_{\circledcirc,i}$]; $N \in \mathbb{N}_{\geq 1}$. Suppose that $A_1 \mapsto A_2$ via Ψ, and that $f_1 \mapsto f_2$ via Ψ^{birat}. Then A_1 is $(N, H_{\circledcirc,1}, f_1)$-***saturated** *if and only if A_2 is $(N, H_{\circledcirc,2}, f_2)$-***saturated**. *Suppose that, for $i = 1, 2$, A_i is $(N, H_{\circledcirc,i}, f_i)$-saturated. Then the isomorphism*

$$H^1(H_{A_1}, \boldsymbol{\mu}_N(A_1)) \xrightarrow{\sim} H^1(H_{A_2}, \boldsymbol{\mu}_N(A_2))$$

maps $\kappa_{f_1} \mapsto \kappa_{f_2}$, i.e., is compatible with the **Kummer classes** *of [Mzk17], Definition 2.1, (ii).*

*(iv) For $i = 1, 2$, let $B_i \in \text{Ob}(\mathcal{C}_i)$ be an N-***codomain** *of an N-th root of a fraction-pair with domain isomorphic to $A_{\circledcirc,i}$; write*

$$s_i^{\sqcap\text{-gp}}, s_i^{\sqcup\text{-gp}} : H_{B_i} \to \text{Aut}_{\mathcal{C}_i}(B_i)$$

for the corresponding **bi-Kummer N-th root** *[cf. Proposition 4.3, (i)] and* $f_i \in \mathcal{O}^\times(B_i^{\mathrm{birat}})$ *for the restriction to* B_i *[i.e., "*$f|_{B_N}$*" in the notation of Proposition 4.2, (iii)] of the rational function determined by the original fraction-pair. Suppose that* $B_1 \mapsto B_2$ *via* Ψ*, and that* $f_1 \mapsto f_2$ *via* Ψ^{birat}*. Then, up to the [simultaneous and non-simultaneous]* **conjugation** *operations discussed in Proposition 4.3, (ii), the isomorphisms*

$$H_{B_1} \xrightarrow{\sim} H_{B_2}; \quad \mathrm{Aut}_{\mathcal{C}_1}(B_1) \xrightarrow{\sim} \mathrm{Aut}_{\mathcal{C}_2}(B_2)$$

map $s_1^{\sqcap\text{-gp}} \mapsto s_2^{\sqcap\text{-gp}}$*,* $s_1^{\sqcup\text{-gp}} \mapsto s_2^{\sqcup\text{-gp}}$*, i.e.,* Ψ *is* **compatible** *with* **bi-Kummer N-th roots**.

Proof. First, we observe that, for $i = 1, 2$, $\mathcal{C}_i$ is of *standard* and *isotropic* type, but *not* of group-like type [cf. Theorem 3.7, (i), (ii)]; the base category $\mathcal{D}_i$ of $\mathcal{C}_i$ is *slim* [cf. Remark 3.7.2]. Thus, Ψ preserves *pre-steps*, *morphisms of Frobenius type*, *Frobenius degrees*, *isometries*, and *base-Frobenius pairs* [cf. [Mzk16], Theorem 3.4, (ii), (iii); [Mzk16], Corollary 5.7, (i), (iii), (iv)]. In particular, [cf. also the existence of Ψ^{bs}] Ψ preserves *Frobenius-trivial* objects. Since the Frobenioid determined by the Frobenius-trivial objects of $\mathcal{C}_i$ is of *base-trivial* type [cf. [Mzk16], Theorem 5.1, (iii)], these observations [together with the existence of Ψ^{bs}] imply the portion of assertion (i) concerning Ψ. The portion of assertion (i) concerning Ψ^{bs} follows immediately from the theory of *[quasi-]temperoids* [cf. [Mzk14], Proposition 3.2; [Mzk14], Theorem A.4].

Next, we consider assertions (ii), (iii). The existence of Ψ^{birat} follows from [Mzk16], Corollary 4.10. Since, the property of being *"Cartier"* may be verified by considering the existence of fraction-pairs that satisfy various properties upon passing to the *perfections* of the $\mathcal{C}_i$ [i.e., in the style of [Mzk16], Proposition 4.1, (iii)], the remainder of assertions (ii), (iii) then follows immediately from the observations made above, the existence of Ψ^{bs}, the *category-theoreticity* of the operation of passing to the *perfection* [cf. Theorem 3.7, (i), (ii); [Mzk16], Theorem 3.4, (iii)], and the *"manifestly category-theoretic nature"* of "$(N, H_{\circledcirc,i}^{\text{bs-fld}})$-saturation" [cf. [Mzk17], Definition 2.2, (iii)] and Kummer classes [cf. [Mzk17], Definition 2.1, (ii)]. This completes the proof of assertions (ii), (iii).

Finally, we consider assertion (iv). Observe that Ψ preserves *base-Frobenius pairs* of Frobenius-trivial objects and [by assertion (ii)] maps *N-th roots of fraction-pairs* with domain isomorphic to $A_{\circledcirc,1}$ to N-th roots of fraction-pairs with domain isomorphic to $A_{\circledcirc,2}$; Ψ^{birat} maps $f_1 \mapsto f_2$. Thus, assertion (iv) follows formally from Proposition 4.3, (ii). ◯

Remark 4.4.1. Observe that one *crucial difference* between the *bi-Kummer theory* for tempered Frobenioids considered here and the theory of [Mzk17], §2, in the case of p-adic Frobenioids is that in the present case, there is *no reciprocity map*. This fact may be regarded as a reflection of the fact that the tempered group $\underline{\Pi}_X^{\mathrm{tm}}$ is *not* a profinite group of cohomological dimension 2 whose cohomology admits a *duality theory* of the sort that G_K does. Put another way, although at first glance, the *Kummer classes* of Proposition 4.3, (iii); Theorem 4.4, (iii), may appear to constitute a *"purely [tempered] fundamental group-theoretic presentation"* of the Frobenioid-theoretic rational functions under consideration, in fact:

> These Kummer classes still depend on a *crucial piece of Frobenioid-theoretic data* — namely, the *cyclotome* "$\boldsymbol{\mu}_N(-)$".

Of course, if one is only interested in this cyclotome *up to isomorphism* [i.e., up to multiplication by an element of $(\mathbb{Z}/N\mathbb{Z})^\times$], then the cyclotome may be thought of as being reconstuctible from $\underline{\Pi}_X^{\mathrm{tm}}$ [cf., e.g., [Mzk2], Proposition 1.2.1, (vi)]. On the other hand, working with the coefficient cyclotome up to isomorphism amounts, in effect, [at the level of rational functions] to working with rational functions *up to* $\widehat{\mathbb{Z}}^\times$*-powers* — which is typically *unacceptable in applications* [e.g., where one wants, for instance, to specify *the* theta function, not the theta function up to $\widehat{\mathbb{Z}}^\times$-powers!]. This technical issue of *"rigidity of the Frobenioid-theoretic cyclotome"* is, in fact, a *central theme of bi-Kummer theory* and will be discussed further in §5 below.

Section 5: The Étale Theta Function via Frobenioids

In the present §5, we carry out the goal of *translating* the *scheme-theoretic* constructions of §1 into the language of *Frobenioids*, by applying the *general bi-Kummer theory* of §4.

In the following discussion, we return to the situation of *Example 3.10.* Suppose further that the morphism $\alpha : A \to B$ of Example 3.10, (iv), is the *identity morphism*, and that "A" is one of the smooth log orbicurves

$$\underline{X}^{\log}; \quad \underline{C}^{\log}; \quad \underline{\underline{X}}^{\log}; \quad \underline{\underline{C}}^{\log}; \quad \underline{\dot{X}}^{\log}; \quad \underline{\dot{C}}^{\log}; \quad \underline{\underline{\dot{X}}}^{\log}; \quad \underline{\underline{\dot{C}}}^{\log}$$

[each of which is *geometrically connected* over the field $K = \ddot{K}$] of Definition 2.5, (ii), for some *odd integer* $l \geq 1$. We shall refer to the case where "A" is $\underline{X}^{\log}$, $\underline{C}^{\log}$, $\underline{\dot{X}}^{\log}$, or $\underline{\dot{C}}^{\log}$ as the *single underline* case and to the case where "A" is $\underline{\underline{X}}^{\log}$, $\underline{\underline{C}}^{\log}$, $\underline{\underline{\dot{X}}}^{\log}$, or $\underline{\underline{\dot{C}}}^{\log}$ as the *double underline* case. Now the divisor monoid $\Phi \stackrel{\text{def}}{=} \Phi^{\text{ell}}_{\alpha}$ on $\mathcal{D} \stackrel{\text{def}}{=} \mathcal{D}_{\alpha}$ [cf. Example 3.10] determines a *tempered Frobenioid*

$$\mathcal{C}$$

of monoid type $\mathbb{Z}$ over the base category $\mathcal{D}$. We would like to apply the theory of §4 in the present situation. Thus, in the single underline (respectively, double underline) case, we take the object

$$A_{\circledcirc}$$

of the theory of §4 to be the [Frobenius-trivial] object defined by the *trivial line bundle* over the object $\ddot{Y}^{\log}$ (respectively, $\underline{\underline{\ddot{Y}}}^{\log}$) of the discussion preceding Definition 2.7. Observe that this $A^{\text{bs}}_{\circledcirc}$ is *"characteristic"* — that is to say, it is *preserved* by arbitrary self-equivalences of $\mathcal{D}$ [cf. Propositions 1.8, 2.4, 2.6] — hence, in particular, *Galois*. Let

$$\Psi : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$$

be a *self-equivalence* of $\mathcal{C}$; $N \geq 1$ an *integer*. Then it follows from Example 3.10, (iv), that we have the following:

Proposition 5.1. **(Applicability of the General Theory)** *The Frobenioid $\mathcal{C}$ is a* **tempered Frobenioid** *of* **rationally standard** *type over a* **slim** *base category $\mathcal{D}$, whose monoid type is $\mathbb{Z}$, and whose divisor monoid $\Phi(-)$ is* **perfect**, **perf-factorial**, **non-dilating**, *and* **cuspidally pure**. *In particular, $\mathcal{C}$ and the self-equivalence $\Psi : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ satisfy all of the hypotheses of Corollary 3.8, (i), (ii), (iii); Theorem 4.4 [for "$\mathcal{C}_1$", "$\mathcal{C}_2$", "$\Psi : \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$"].*

Remark 5.1.1. In Proposition 5.1, as well as in the discussion of the remainder of the present §5, we restrict our attention to *self-equivalences*, instead of considering arbitrary equivalences between possibly distinct categories, as in Theorem 4.4. We do this, however, mainly to simplify the discussion [in particular, the notation]. That is to say, the *extension to the case of arbitrary equivalences* between possibly distinct categories is, for the most part, immediate. We leave the routine clerical details to the interested reader.

Next, we reconsider the discussion of §1 [where we take the "N" of §1 to be $l \cdot N$] from the point of view of the theory of §4. To this end, let us first observe that by the definition of $\underline{\underline{\ddot{Y}}}^{\log}$, $\ddot{Z}^{\log}_{l \cdot N}$ [together with the fact that $l \cdot N$ is *divisible* by l], it follows immediately that the covering $\ddot{Z}^{\log}_{l \cdot N} \to X^{\log}$ *factors* through the covering $\underline{\underline{\ddot{Y}}}^{\log} \to X^{\log}$. Thus, if we pull-back the line bundle $\ddot{\mathfrak{L}}_{l \cdot N}$ on $\ddot{\mathfrak{Y}}_{l \cdot N}$ [cf. the discussion preceding Lemma 1.2] to $\ddot{\mathfrak{Z}}_{l \cdot N}$ [or, equivalently, the line bundle $\mathfrak{L}_{l \cdot N}$ on $\mathfrak{Y}_{l \cdot N}$, first to $\mathfrak{Z}_{l \cdot N}$, and then to $\ddot{\mathfrak{Z}}_{l \cdot N}$] so as to obtain a line bundle $\ddot{\mathfrak{L}}_{l \cdot N}|_{\ddot{\mathfrak{Z}}_{l \cdot N}}$ on $\ddot{\mathfrak{Z}}_{l \cdot N}$, then the pull-backs to $\ddot{\mathfrak{Z}}_{l \cdot N}$ of

(1) the section $s_{l \cdot N} \in \Gamma(\mathfrak{Z}_{l \cdot N}, \mathfrak{L}_{l \cdot N}|_{\mathfrak{Z}_{l \cdot N}})$ of Proposition 1.1, (i);

(2) the *theta trivialization* $\tau_{l \cdot N} \in \Gamma(\ddot{\mathfrak{Y}}_{l \cdot N}, \ddot{\mathfrak{L}}_{l \cdot N})$ of Lemma 1.2

may be interpreted as morphisms

$$\mathbb{V}(\mathcal{O}_{\ddot{\mathfrak{Z}}_{l \cdot N}}) \to \mathbb{V}(\ddot{\mathfrak{L}}_{l \cdot N}|_{\ddot{\mathfrak{Z}}_{l \cdot N}})$$

— i.e., as *morphisms between objects* of $\mathcal{C}$. Now we have the following:

Proposition 5.2. **(Bi-Kummer Theory)** *In the notation of the above discussion:*

(i) The pair of morphisms of $\mathcal{C}$ determined by "$s_{l \cdot N}$", "$\tau_{l \cdot N}$" constitutes an $l \cdot N$**-th root of a right fraction-pair** *[cf. Proposition 4.2, (iii)] of [the Frobenioid-theoretic version of the log-meromorphic function constituted by] the* **theta function** $\ddot{\Theta}$ *of Proposition 1.4, or, alternatively, an* N**-th root of a right fraction-pair** *[cf. Proposition 4.2, (iii)] of [the Frobenioid-theoretic version of the log-meromorphic function constituted by] an* l**-th root of the theta function** $\ddot{\Theta}$ *[cf. Remark 4.3.2].*

(ii) The **group actions** *of Proposition 1.1, (ii); Lemma 1.2, arising from "$s_{l \cdot N}$", "$\tau_{l \cdot N}$", respectively, are precisely the actions determined by the* **bi-Kummer** $l \cdot N$**-th root** *[cf. Proposition 4.3, (i)] arising from the $l \cdot N$-th root of (i).*

(iii) The **Kummer class** *determined by the bi-Kummer $l \cdot N$-th root of (ii) [cf. Proposition 4.3, (iii)] corresponds precisely to the reduction modulo $l \cdot N$ of the class "$\ddot{\eta}^{\Theta}$" of Proposition 1.3 — i.e., to the* **"étale theta function"** *— relative to the* **natural isomorphism** *[cf. Remark 5.2.1 below] between "$\boldsymbol{\mu}_{l \cdot N}(-)$" [cf. Proposition 4.3, (iii)] and $\underline{\Delta}_{\Theta} \otimes (\mathbb{Z}/l \cdot N\mathbb{Z})$ [cf. Proposition 1.3]. Similarly, the* **Kummer class** *determined by the bi-Kummer N-th root of the N-th root of (i) [cf. Proposition 4.3, (iii)] corresponds precisely to the reduction modulo N of the*

class "$\underline{\underline{\ddot{\eta}}}^{\Theta}$" of the discussion preceding Definition 2.7 — i.e., to an ***l*-th root of the "étale theta function"** *— relative to the* **natural isomorphism** *between "$\boldsymbol{\mu}_N(-) = l \cdot \boldsymbol{\mu}_N(-)$" and $(l \cdot \underline{\Delta}_{\Theta}) \otimes (\mathbb{Z}/N\mathbb{Z})$.*

(iv) Suppose that $K = \ddot{K}$, and that "A" arises from $\underline{X}^{\log}$, $\underline{\dot{X}}^{\log}$, $\underline{\underline{X}}^{\log}$, or $\underline{\underline{\dot{X}}}^{\log}$. Then the condition in the final portion of Proposition 4.3, (ii), is **satisfied**, *i.e., the* **simultaneous conjugation** *indeterminacy of Proposition 4.3, (ii), may be taken to involve conjugation by* **units** $\in \mathcal{O}^{\times}(-)$ *of the $l \cdot N$-codomain in question.*

Proof. These assertions follow immediately from the definitions. In the case of assertion (i), we observe that the *"$(l \cdot N, H_{\circledcirc}, f|_{A_{l \cdot N}})$-saturated-ness"* condition of Proposition 4.2, (iii), follows immediately from the definition of the field $\ddot{J}_{l \cdot N}$ in §1. In the case of assertion (iv), we observe that the fact that the condition in the final portion of Proposition 4.3, (ii), is satisfied follows from the definition of "$\tau_{l \cdot N}$" [i.e., as a section whose zero divisor is *not fixed* by nontrivial translation by elements of $\underline{\mathbb{Z}} \cong \text{Gal}(\ddot{Y}/\dot{X})$], together with the "±" indeterminacy inherent in the definition of the theta function $\ddot{\Theta}$ [cf. Remark 1.3.1; Proposition 1.4, (ii)]. ○

Remark 5.2.1. This natural isomorphism constitutes a *scheme-theoretic* ingredient in the otherwise *Frobenioid-theoretic* formulation of Proposition 5.2 [cf. Remark 4.4.1]. The translation of this final scheme-theoretic ingredient into *category theory* is the topic of Proposition 5.5; Theorems 5.6, 5.7 below.

Next, we consider *divisors*. Recall that the special fiber of "$\ddot{\mathfrak{Y}}$" may be described as an *infinite chain of copies of the projective line*, joined to one another at the points "0" an "∞" [cf., e.g., the discussion preceding Proposition 1.1]. In particular, there is a *natural bijection* [cf. the discussion at the beginning of §1]

$$\text{Prime}(\Phi(A_{\circledcirc}))^{\text{ncsp}} \xrightarrow{\sim} \mathbb{Z}$$

which is well-defined, up to the operations of *translation* by an element of $\mathbb{Z}$ and *multiplication by* ±1 on $\mathbb{Z}$. Moreover, there is a *natural surjection*

$$\text{Prime}(\Phi(A_{\circledcirc}))^{\text{csp}} \twoheadrightarrow \text{Prime}(\Phi(A_{\circledcirc}))^{\text{ncsp}}$$

[i.e., given by considering the "irreducible component of the special fiber that contains the cusp determined by the cuspidal prime"]. Also, let us observe that since all the cusps of $\ddot{Y}^{\log}$ arise from $\ddot{K}$*-rational points*, it follows immediately that one has the following *cuspidal* version of Remark 3.8.2: If $\mathfrak{p}$, $\mathfrak{q}$ are *cuspidal* primes of $\Phi(A_{\circledcirc})$, then there is a *natural isomorphism between "primary components"*

$$\Phi(A_{\circledcirc})_{\mathfrak{p}} \xrightarrow{\sim} \Phi(A_{\circledcirc})_{\mathfrak{q}}$$

— determined by identifying the elements on each side that arise from [scheme-theoretic] *prime log-divisors*.

Next, let us *choose* an isomorphism $\Psi(A_{\circledcirc}) \xrightarrow{\sim} A_{\circledcirc}$ of $\mathcal{C}$ [cf. Proposition 5.1; Theorem 4.4, (i)]. Let us write

$$\Psi^{\Phi}_{A_{\circledcirc}} : \Phi(A_{\circledcirc}) \xrightarrow{\sim} \Phi(A_{\circledcirc})$$

for the automorphism of the monoid $\Phi(A_{\circledcirc})$ obtained by composing the *isomorphism of divisor monoids induced by* $\Psi : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ [cf. Corollary 3.8, (iii); [Mzk16], Theorem 4.9] with the isomorphism $\Phi(\Psi(A_{\circledcirc})) \xrightarrow{\sim} \Phi(A_{\circledcirc})$ induced by the chosen isomorphism $\Psi(A_{\circledcirc}) \xrightarrow{\sim} A_{\circledcirc}$.

Proposition 5.3. (Category-theoreticity of the Geometry of Divisors) *In the notation of the above discussion,* $\Psi^{\Phi}_{A_{\circledcirc}}$ **preserves** *the following objects:*

(i) **non-cuspidal** *and* **cuspidal** *elements;*

(ii) the **natural isomorphisms** *between distinct* **non-cuspidal primary components** *of* $\Phi(A_{\circledcirc})$ *[cf. Remark 3.8.2];*

(iii) the **natural isomorphisms** *[described in the discussion above] between distinct* **cuspidal primary components** *of* $\Phi(A_{\circledcirc})$*;*

(iv) the **natural surjection** $\mathrm{Prime}(\Phi(A_{\circledcirc}))^{\mathrm{csp}} \twoheadrightarrow \mathrm{Prime}(\Phi(A_{\circledcirc}))^{\mathrm{ncsp}}$*;*

(v) the **natural bijection** $\mathrm{Prime}(\Phi(A_{\circledcirc}))^{\mathrm{ncsp}} \xrightarrow{\sim} \mathbb{Z}$ *[up to translation by an element of* $\mathbb{Z}$ *and multiplication by* ± 1*];*

(vi) the $\mathrm{Aut}_{\mathcal{C}}(A_{\circledcirc})$*-orbit of the* **divisor of zeroes and poles** $\in \Phi(A_{\circledcirc})^{\mathrm{gp}}$ *of [the Frobenioid-theoretic version of the log-meromorphic function constituted by] the* **theta function** $\ddot{\Theta}$ *of Proposition 1.4.*

Proof. The preservation of (i) (respectively, (ii)) follows immediately from Corollary 3.8, (iii) (respectively, Remark 3.8.2). To verify the preservation of the remaining objects, it suffices to consider the well-known *"intersection theory of divisors supported on the chain of copies of the projective line"* that constitutes the special divisor of "$\ddot{\mathfrak{Y}}$" as follows:

Let us refer to pairs of elements of $\Phi(A_{\circledcirc})^{\mathrm{gp}}$ whose difference lies in the image of the *birational function monoid* of the Frobenioid $\mathcal{C}$ as *linearly equivalent.* Let us refer to an element $\in \Phi(A_{\circledcirc})^{\mathrm{gp}}$ which is linearly equivalent to 0 as *principal.* Let us refer to as *cuspidally minimal* any cuspidal element of $\Phi(A_{\circledcirc})^{\mathrm{gp}}$ whose *support* [cf. [Mzk16], Definition 2.4, (i), (d)] is a finite set of *minimal* cardinality among the cardinalities of supports of cuspidal elements of $\Phi(A_{\circledcirc})^{\mathrm{gp}}$ which are *linearly equivalent* to the given element.

Now the preservation of (iv) follows by considering the support of *primary cuspidal elements* $a \in \Phi(A_{\circledcirc})$ such that, for some $b \in \Phi(A_{\circledcirc})^{\mathrm{csp}}$ which is *co-prime* to a [i.e., a, b have disjoint supports], $b-a$ is *cuspidally minimal* and *linearly equivalent* to a primary non-cuspidal element $n \in \Phi(A_{\circledcirc})$. That is to say, [by the well-known intersection theory of divisors supported on the chain of copies of the projective line] in this situation, n is linearly equivalent to some element $\in \Phi(A_{\circledcirc})^{\mathrm{gp}}$ of the form $n_1 + n_2 - a$, where $n_1, n_2 \in \Phi(A_{\circledcirc})^{\mathrm{csp}}$ are *primary cuspidal elements* that map, respectively, via the natural surjection of (iv) to the *two* non-cuspidal primes that are *adjacent* [in the "chain of copies of the projective line"] to the cuspidal prime determined by n. Moreover, relative to the isomorphisms of (iii), the *multiplicities* of n_1, n_2 are *equal* to each other as well as to *half* the multiplicity of a. Thus, the *natural surjection* of (iv) is obtained by mapping the prime determined by a to the prime determined by n.

To verify the preservation of (v), it suffices to show that the relation of *adjacency* [in the "chain of copies of the projective line"] between elements $\mathfrak{p}, \mathfrak{q} \in \Phi(A_{\circledcirc})^{\mathrm{ncsp}}$ is preserved. But this follows [again from the well-known intersection theory of divisors supported on the chain of copies of the projective line] by observing that if $a \in \mathfrak{p}$, $b \in \mathfrak{q}$ correspond via the natural isomorphisms of (ii), then $\mathfrak{p}$, $\mathfrak{q}$ are *adjacent* (respectively, *not adjacent*) if and only if every *cuspidally minimal* $c \in \Phi(A_{\circledcirc})^{\mathrm{gp}}$ which is linearly equivalent to $a + b$ has support of cardinality 4 (respectively, 5 or 6). [Here, the numbers "4", "5", "6" correspond to the number of non-cuspidal primes that are either contained in or adjacent to a prime contained in the support of $a + b$.] Now the preservation of (iii) follows by considering the *factorization* [cf. [Mzk16], Definition 2.4, (i), (c)] of *such cuspidally minimal elements* c, in the case where a, b are *adjacent*.

Finally, the preservation of (vi), at least up to $\mathbb{Q}_{>0}$-multiples, follows immediately by considering the *principal* divisors $\in \Phi(A_{\circledcirc})^{\mathrm{gp}}$ in light of the preservation of (i), (ii), (iii), and (v) [cf. the description of the divisor of zeroes and poles of $\ddot{\Theta}$ in Proposition 1.4, (i)]; to eliminate the indeterminacy with respect to $\mathbb{Q}_{>0}$-multiples, it suffices to consider the zero divisor [cf. Remark 3.8.2] of a *generator* of $\mathcal{O}^{\triangleright}(A_{\circledcirc})/\mathcal{O}^{\times}(A_{\circledcirc}) \cong \mathbb{Z}_{\geq 0}$. ◯

Next, let us recall the *characteristic quotient* and *characteristic subgroup*

$$\underline{\Pi}_X^{\mathrm{tm}} \twoheadrightarrow (\underline{\Pi}_X^{\mathrm{tm}})^{\Theta}; \quad l \cdot \underline{\Delta}_{\Theta} \subseteq (\underline{\Pi}_X^{\mathrm{tm}})^{\Theta}$$

of the discussion at the beginning of §1. Thus, for $D \in \mathrm{Ob}(\mathcal{D})$, this quotient and subgroup determine a *quotient* and *subgroup*

$$\mathrm{Aut}_{\mathcal{D}}(D) \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}}^{\Theta}(D); \quad (l \cdot \underline{\Delta}_{\Theta})_D \subseteq \mathrm{Aut}_{\mathcal{D}}^{\Theta}(D)$$

which are *preserved* by arbitrary self-equivalences of $\mathcal{D}$ [cf. Propositions 1.8, 2.4, 2.6]. If $S \in \mathrm{Ob}(\mathcal{C})$, then we shall write $(l \cdot \underline{\Delta}_{\Theta})_S \stackrel{\mathrm{def}}{=} (l \cdot \underline{\Delta}_{\Theta})_{S^{\mathrm{bs}}}$.

Definition 5.4. We shall say that $S \in \mathrm{Ob}(\mathcal{C})$ is (l, N)-*theta-saturated* if the following conditions are satisfied: (a) S is $\boldsymbol{\mu}_{l \cdot N}$-saturated [cf. [Mzk17], Definition 2.1, (i)]; (b) $(l \cdot \underline{\Delta}_{\Theta})_S \otimes \mathbb{Z}/N\mathbb{Z}$ is of cardinality N.

Now we may begin to address the issue discussed in Remark 5.2.1:

Proposition 5.5. **(Frobenioid-theoretic Cyclotomic Rigidity)** *In the notation of the above discussion, the* **Kummer class** *of Proposition 5.2, (iii), determines an* **isomorphism**

$$(l \cdot \underline{\Delta}_{\Theta})_S \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_N(S) \ (= l \cdot \boldsymbol{\mu}_{l \cdot N}(S))$$

for all (l, N)**-theta-saturated** $S \in \mathrm{Ob}(\mathcal{C})$ *which is* **functorial** *with respect to the subcategory of* $\mathcal{C}$ *determined by the* **linear morphisms** *of* (l, N)*-theta-saturated objects].*

Proof. Indeed, if $S' \in \mathrm{Ob}(\mathcal{C})$ is an $l \cdot N$*-codomain* of an $l \cdot N$-th root of a right fraction-pair of $\ddot{\Theta}$ [i.e., as discussed in Proposition 5.2, (i)], then it follows from the detailed description of the *"étale theta class"* in Proposition 1.3 that the resulting Kummer class [cf. Proposition 5.2, (iii)] determines an isomorphism

$$(l \cdot \underline{\Delta}_{\Theta})_{S'} \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_N(S')$$

which is manifestly *"functorial"* [in the evident sense, with respect $l \cdot N$-codomains of $l \cdot N$-th roots of right fraction-pairs of $\ddot{\Theta}$]. Thus, we may transport this isomorphism from S' to an arbitrary (l, N)-theta-saturated $S \in \mathrm{Ob}(\mathcal{C})$ by means of *linear morphisms* $S'' \to S$, $S'' \to S'$ [of $\mathcal{C}$], which induce isomorphisms

$$(l \cdot \underline{\Delta}_{\Theta})_{S''} \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} (l \cdot \underline{\Delta}_{\Theta})_S \otimes \mathbb{Z}/N\mathbb{Z};$$

$$(l \cdot \underline{\Delta}_{\Theta})_{S''} \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} (l \cdot \underline{\Delta}_{\Theta})_{S'} \otimes \mathbb{Z}/N\mathbb{Z};$$

$$\boldsymbol{\mu}_N(S) \xrightarrow{\sim} \boldsymbol{\mu}_N(S''); \quad \boldsymbol{\mu}_N(S') \xrightarrow{\sim} \boldsymbol{\mu}_N(S'')$$

— hence also a [*functorial*] isomorphism $(l \cdot \underline{\Delta}_{\Theta})_S \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_N(S)$, which is *independent* of the choice of S', S'' and the linear morphisms $S'' \to S$, $S'' \to S'$ [precisely because of the original *"functoriality"* of the isomorphism for S']. ○

Theorem 5.6. **(Category-theoreticity of Frobenioid-theoretic Cyclotomic Rigidity)** *Write* $\mathcal{C}$ *for the* **tempered Frobenioid** *of monoid type* $\mathbb{Z}$ *determined by the* **divisor monoid** $\Phi \stackrel{\mathrm{def}}{=} \Phi^{\mathrm{ell}}_{\alpha}$ *on the* **base category** $\mathcal{D} \stackrel{\mathrm{def}}{=} \mathcal{D}_{\alpha}$ *of Example 3.10, (iv), where we take the morphism* $\alpha : A \to B$ *of Example 3.10, (iv), to be the identity morphism and "A" to be one of the smooth log orbicurves*

$$\underline{X}^{\log}; \quad \underline{C}^{\log}; \quad \underline{\underline{X}}^{\log}; \quad \underline{\underline{C}}^{\log}; \quad \underline{\dot{X}}^{\log}; \quad \underline{\dot{C}}^{\log}; \quad \underline{\underline{\dot{X}}}^{\log}; \quad \underline{\underline{\dot{C}}}^{\log}$$

[each of which is geometrically connected over the field $K = \ddot{K}$*] of Definition 2.5, (ii), for some* **odd integer** $l \geq 1$*. Let*

$$\Psi : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$$

be a **self-equivalence** *of* $\mathcal{C}$*;* $N \geq 1$ *an* **integer***. Then* Ψ **preserves** *the* (l, N)*-theta-saturated objects, as well as the* **natural isomorphism**

$$(l \cdot \underline{\Delta}_{\Theta})_S \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_N(S) \ (= l \cdot \boldsymbol{\mu}_{l \cdot N}(S))$$

[for (l, N)-theta-saturated $S \in \mathrm{Ob}(\mathcal{C})$] of Proposition 5.5 [i.e., Ψ transports this isomorphism for S to the corresponding isomorphism for $\Psi(S)$].

Proof. Indeed, by Proposition 5.1; [Mzk16], Theorem 3.4, (iv), (v), it follows that Ψ preserves "$\mathcal{O}^\times(-)$" and induces a 1-compatible equivalence $\Psi^{\mathrm{bs}} : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$, hence, [cf. [Mzk14], Proposition 3.2; [Mzk14], Theorem A.4] an outer automorphism of the tempered fundamental group of the smooth orbicurve in question. In particular, it follows from Propositions 2.4, 2.4 that Ψ preserves "$(l \cdot \underline{\Delta}_\Theta)_{(-)}$". Thus, it follows immediately that Ψ *preserves the (l, N)-theta-saturated objects*.

Next, let $S_2 \in \mathrm{Ob}(\mathcal{C})$ be an *(l, N)-theta-saturated $l \cdot N$-codomain* of an $l \cdot N$-th root of a right fraction-pair of $\ddot{\Theta}$ [i.e., as discussed in Proposition 5.2, (i)] such that S_2^{bs} is *"characteristic"* [i.e., its isomorphism class is preserved by arbitrary self-equivalences of $\mathcal{D}$]; write

$$s^{\sqcap}, s^{\sqcup} : S_1 \to S_2$$

for the *pair of base-equivalent pre-steps* that appear in this $l \cdot N$-th root [so S_1 is *Frobenius-trivial*]. Now since Ψ preserves *pre-steps* and *Frobenius-trivial* objects and induces a 1-compatible self-equivalence $\Psi^{\mathrm{bs}} : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$ [cf. the proof of Theorem 4.4], it follows that Ψ maps this pair of base-equivalent pre-steps to a pair of base-equivalent pre-steps

$$t^{\sqcap}, t^{\sqcup} : T_1 \to T_2$$

such that T_1 is *Frobenius-trivial*, hence *isomorphic* to S_1 [cf. our assumption that $S_1^{\mathrm{bs}} \cong S_2^{\mathrm{bs}}$ is *characteristic*; [Mzk16], Theorem 5.1, (iii)] . Moreover, since Ψ [essentially] preserves the *divisor of zeroes and poles* of $\ddot{\Theta}$ [cf. Proposition 5.3, (vi)], it follows [cf. the *first equivalence of categories involving pre-steps* of [Mzk16], Definition 1.3, (iii), (d)] that there exist *isomorphisms* $\gamma_1 : S_1 \xrightarrow{\sim} T_1$, $\gamma_2 : S_2 \xrightarrow{\sim} T_2$, $u \in \mathcal{O}^\times(T_2)$ such that $\gamma_2 \circ s^{\sqcap} = t^{\sqcap} \circ \gamma_1$, $u \circ \gamma_2 \circ s^{\sqcup} = t^{\sqcup} \circ \gamma_1$. Thus, by forming the resulting *group homomorphisms*

$$s^{\sqcap\text{-gp}}, s^{\sqcup\text{-gp}} : H_{S_2} \to \mathrm{Aut}_{\mathcal{C}}(S_2); \quad t^{\sqcap\text{-gp}}, t^{\sqcup\text{-gp}} : H_{T_2} \to \mathrm{Aut}_{\mathcal{C}}(T_2)$$

[cf. Proposition 4.3, (i)] and *Kummer classes* [cf. Proposition 4.3, (iii)], and observing that the Kummer class of the *"constant function"* u does not affect the restriction of the resulting Kummer classes to $(l \cdot \underline{\Delta}_\Theta)_{S_2}$, $(l \cdot \underline{\Delta}_\Theta)_{T_2}$, we conclude that Ψ does indeed transport the *natural isomorphism* $(l \cdot \underline{\Delta}_\Theta)_{S_2} \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_N(S_2)$ of Proposition 5.5 to the corresponding isomorphism for T_2. Moreover, by the construction applied in the proof of Proposition 5.5, this preservation of the natural isomorphism of Proposition 5.5 in the *specific case* of S_2 is *sufficient* to imply the preservation of the natural isomorphism of Proposition 5.5 for *arbitrary (l, N)-theta-saturated* $S \in \mathrm{Ob}(\mathcal{C})$. This completes the proof of Theorem 5.6. ◯

Theorem 5.7. (Category-theoreticity of the Frobenioid-theoretic Theta Function) *In the notation of Theorem 5.6, Ψ* **preserves** *right fraction-pairs of [the*

Frobenioid-theoretic version of the log-meromorphic function constituted by] an l**-th root of the theta function**

$$\ddot{\Theta}$$

of Proposition 1.4, up to possible **multiplication by a** $2l$**-th root of unity** *and possible* **translation** *by an element of* $\underline{\mathbb{Z}}$ $(\cong \mathrm{Gal}(\ddot{Y}/\dot{X}))$ *[when "*A*" arises from* $\underline{X}^{\log}$*,* $\underline{\dot{X}}^{\log}$*] or* $l \cdot \underline{\mathbb{Z}}$ *[when "*A*" arises from* $\underline{C}^{\log}$*,* $\underline{\dot{C}}^{\log}$*,* $\underline{\underline{X}}^{\log}$*,* $\underline{\underline{\dot{X}}}^{\log}$*,* $\underline{\underline{C}}^{\log}$*,* $\underline{\underline{\dot{C}}}^{\log}$*].*

Proof. Indeed, this follows immediately by considering *compatible systems* as in Remark 4.3.2 and applying the theory of the *rigidity of the étale theta function* [cf. Corollary 2.8, (i)], to the *Kummer classes* of Proposition 5.2, (iii) [cf. also Theorem 4.4, (iii); Proposition 5.3, (vi)], above, in light of the *crucial isomorphisms* $(l \cdot \underline{\Delta}_{\Theta})_S \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \boldsymbol{\mu}_N(S)$ of Proposition 5.5, which, by Theorem 5.6, are *preserved* by Ψ. ○

Remark 5.7.1. Note that the "rigidity up to ± 1" asserted in Theorem 5.7 is *substantially stronger* than [what was in effect] the preservation of $\ddot{\Theta}$ up to multiplication by an *arbitrary constant function* $\in \mathcal{O}^{\times}(-)$ [cf. the "u" appearing in the proof of Theorem 5.6], that was observed by considering *divisors* [i.e., Proposition 5.3, (vi)] in the proof of Theorem 5.6.

Next, we consider *theta environments*. For the remainder of the present §5, we suppose that:

$$\text{"}A\text{" arises from } \underline{\underline{X}}^{\log}.$$

Write

$$s_N^{\sqcap}, s_N^{\sqcup} : A_N \to B_N$$

for the pair of base-equivalent morphisms of $\mathcal{C}$ determined by the sections "$s_{l \cdot N}$", "$\tau_{l \cdot N}$" of the discussion preceding Proposition 5.2 [so $s_N^{\sqcap}$, $s_N^{\sqcup}$ constitute an N*-th root of a right fraction-pair* of an l-th root of the theta function $\ddot{\Theta}$ — cf. Proposition 5.2, (i)]. Note that the *zero divisor* $\mathrm{Div}(s_N^{\sqcap}) \in \Phi(A_N)$ of $s_N^{\sqcap}$ *descends* [relative to the *unique* morphism $A_{\circledcirc}^{\mathrm{bs}} \to A$ in $\mathcal{D}$] to $\Phi(A)$ [cf. Proposition 1.4, (i)]; in particular, it follows that B_N is Aut-*ample*. Thus, the group homomorphism $s_N^{\mathrm{trv}} : \mathrm{Aut}_{\mathcal{D}}(A_N^{\mathrm{bs}}) \to \mathrm{Aut}_{\mathcal{C}}(A_N)$ arising from a *base-Frobenius pair* of A_N [cf. Proposition 5.1; Theorem 3.7, (i); [Mzk16], Proposition 5.6], which is *completely determined* [cf. Proposition 5.1; Theorem 3.7, (i); [Mzk16], Proposition 5.6] up to *conjugation* by an element of $\mathcal{O}^{\times}(A_N)$, determines unique *group homomorphisms*

$$s_N^{\sqcap\text{-gp}} : \mathrm{Aut}_{\mathcal{D}}(B_N^{\mathrm{bs}}) \to \mathrm{Aut}_{\mathcal{C}}(B_N); \quad s_N^{\sqcup\text{-gp}} : H_{B_N} \to \mathrm{Aut}_{\mathcal{C}}(B_N)$$

such that, relative to the isomorphisms $\mathrm{Aut}_{\mathcal{D}}(A_N^{\mathrm{bs}}) \xrightarrow{\sim} \mathrm{Aut}_{\mathcal{D}}(B_N^{\mathrm{bs}})$, $H_{A_N} \xrightarrow{\sim} H_{B_N}$ determined by the [base-equivalent!] pair of morphisms $s_N^{\sqcap}$, $s_N^{\sqcup}$, we have

$$s_N^{\sqcap\text{-gp}}(g) \circ s_N^{\sqcap} = s_N^{\sqcap} \circ (s_N^{\mathrm{trv}}|_{\mathrm{Aut}_{\mathcal{D}}(B_N^{\mathrm{bs}})})(g); \quad s_N^{\sqcup\text{-gp}}(h) \circ s_N^{\sqcup} = s_N^{\sqcup} \circ (s_N^{\mathrm{trv}}|_{H_{B_N}})(h)$$

for all $g \in \mathrm{Aut}_{\mathcal{D}}(B_N^{\mathrm{bs}})$, $h \in H_{B_N}$ [cf. Propositions 4.3, (i); 5.2, (ii)]. Write

$$\mathbb{E}_N \stackrel{\mathrm{def}}{=} s_N^{\sqcap\text{-gp}}(\mathrm{Im}(\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}})) \cdot \boldsymbol{\mu}_N(B_N) \subseteq \mathrm{Aut}_{\mathcal{C}}(B_N)$$

— where $\mathrm{Im}(\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}})$ denotes the image of $\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}} \subseteq \Pi^{\mathrm{tm}}_{\underline{\underline{X}}}$ via the natural outer homomorphism $\Pi^{\mathrm{tm}}_{\underline{\underline{X}}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}}(B_N^{\mathrm{bs}})$ [cf. Definition 4.1, (ii)].

Lemma 5.8. **(Conjugation by Constants)** *In the notation of the above discussion, write*

$$(K^\times)^{1/N} \subseteq \mathcal{O}^\times(B_N^{\mathrm{birat}})$$

for the subgroup of elements whose N-th power lies in the image of the natural inclusion $K \hookrightarrow \mathcal{O}^\times(B_N^{\mathrm{birat}})$; $(\mathcal{O}_K^\times)^{1/N} \stackrel{\mathrm{def}}{=} (K^\times)^{1/N} \bigcap \mathcal{O}^\times(B_N)$. Then $(\mathcal{O}_K^\times)^{1/N}$ is equal to the set of elements of $\mathcal{O}^\times(B_N)$ that **normalize** *the subgroup $\mathbb{E}_N \subseteq \mathrm{Aut}_{\mathcal{C}}(B_N)$. In particular, we have a* **natural outer action** *of*

$$(\mathcal{O}_K^\times)^{1/N}/\boldsymbol{\mu}_N(B_N) \xrightarrow{\sim} \mathcal{O}_K^\times$$

on $\mathbb{E}_N$; this outer action extends to an outer action of $(K^\times)^{1/N}/\boldsymbol{\mu}_N(B_N) \xrightarrow{\sim} K^\times$ on $\mathbb{E}_N$.

Proof. Indeed, since $\underline{\underline{Y}}$ is *geometrically connected* over K, it follows immediately that the set of elements of $\mathcal{O}^\times(B_N)$ that *normalize* the subgroup $\mathbb{E}_N \subseteq \mathrm{Aut}_{\mathcal{C}}(B_N)$ is equal to the set of elements on which $\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}$ [i.e., G_K, via the natural surjection $\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}} \twoheadrightarrow G_K$] acts via multiplication by an element of $\boldsymbol{\mu}_N(B_N)$. But this last set is easily seen to coincide with $(\mathcal{O}_K^\times)^{1/N}$. ○

Lemma 5.9. **(First Properties of Frobenioid-theoretic Theta Environments)** *In the notation of the above discussion:*

(i) $s_N^{\sqcap\text{-gp}}$, $s_N^{\sqcup\text{-gp}}$ **factor through** $\mathbb{E}_N$.

(ii) We have a **natural exact sequence**

$$1 \to \boldsymbol{\mu}_N(B_N) \to \mathbb{E}_N \to \mathrm{Im}(\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}}) \to 1$$

— where $\mathrm{Im}(\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}})$ denotes the image of $\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}} \subseteq \Pi^{\mathrm{tm}}_{\underline{\underline{X}}}$ via the natural outer homomorphism $\Pi^{\mathrm{tm}}_{\underline{\underline{X}}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}}(B_N^{\mathrm{bs}})$ [cf. Definition 4.1, (ii)].

(iii) We have a **natural outer action**

$$l \cdot \underline{\mathbb{Z}} \xrightarrow{\sim} \Pi^{\mathrm{tm}}_{\underline{\underline{X}}}/\Pi^{\mathrm{tm}}_{\underline{\underline{Y}}} \to \mathrm{Out}(\mathbb{E}_N)$$

determined by conjugating via the composite of the natural outer homomorphism $\Pi^{\mathrm{tm}}_{\underline{\underline{X}}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}}(B_N^{\mathrm{bs}})$ with $s_N^{\sqcap\text{-gp}} : \mathrm{Aut}_{\mathcal{D}}(B_N^{\mathrm{bs}}) \to \mathrm{Aut}_{\mathcal{C}}(B_N)$.

(iv) Write

$$\mathbb{E}_N^{\Pi} \stackrel{\mathrm{def}}{=} \mathbb{E}_N \times_{\mathrm{Im}(\underline{\underline{\Pi}}_{\underline{\underline{Y}}}^{\mathrm{tm}})} \underline{\underline{\Pi}}_{\underline{\underline{Y}}}^{\mathrm{tm}}$$

[where $\mathrm{Im}(\underline{\underline{\Pi}}_{\underline{\underline{Y}}}^{\mathrm{tm}})$ *is as in (ii);* $\mathbb{E}_N$, $\mathbb{E}_N^{\Pi}$ *are equipped witht the evident topologies; the homomorphism* $\underline{\underline{\Pi}}_{\underline{\underline{Y}}}^{\mathrm{tm}} \twoheadrightarrow \mathrm{Im}(\underline{\underline{\Pi}}_{\underline{\underline{Y}}}^{\mathrm{tm}})$ *is well-defined up to conjugation by an element of* $\underline{\underline{\Pi}}_{\underline{\underline{X}}}^{\mathrm{tm}}$*]. Then the natural inclusions* $\boldsymbol{\mu}_N(B_N) \hookrightarrow \mathbb{E}_N$, $\mathrm{Im}(\underline{\underline{\Pi}}_{\underline{\underline{Y}}}^{\mathrm{tm}}) \subseteq \mathbb{E}_N$ *determine an* **isomorphism of topological groups**

$$\mathbb{E}_N^{\Pi} \xrightarrow{\sim} \underline{\underline{\Pi}}_{\underline{\underline{Y}}}^{\mathrm{tm}}[\boldsymbol{\mu}_N]$$

which is an **isomorphism of mod** N **bi-theta environments** *with respect to the mod* N *bi-theta environment structure on* $\mathbb{E}_N^{\Pi}$ *determined by the* **subgroup** *of* $\mathrm{Out}(\mathbb{E}_N^{\Pi})$ *generated by the natural outer actions of* $l \cdot \underline{\mathbb{Z}}$ *[cf. (iii)],* $K^{\times}$ *[cf. Lemma 5.8] on* $\mathbb{E}_N$*, together with the* $\boldsymbol{\mu}_N$*-conjugacy classes of subgroups given by the images of the* **homomorphisms**

$$s_N^{\sqcap\text{-}\Pi}, s_N^{\sqcup\text{-}\Pi} : \underline{\underline{\Pi}}_{\ddot{\underline{\underline{Y}}}}^{\mathrm{tm}} \to \mathbb{E}_N^{\Pi}$$

arising from $s_N^{\sqcap\text{-gp}}$, $s_N^{\sqcup\text{-gp}}$ *[cf. (i)]. In particular, omitting the homomorphism* $s_N^{\sqcap\text{-}\Pi}$ *yields a* **mod** N **mono-theta environment***.*

(v) In the situation of (iv), the **cyclotomic rigidity** *isomorphism arising from the theory of* §2 *[cf. Theorem 2.15, (ii)]* **coincides** *with the* **Frobenioid-theoretic** *isomorphism of Proposition 5.5 [where we take "S" to be* B_N*].*

Proof. Immediate from the definitions. ○

We are now ready to state our *main result* relating the theory of *theta environments* of §2 to the theory of *tempered Frobenioids* discussed in §3, §4, and the present §5:

Theorem 5.10. (Category-theoreticity of Frobenioid-theoretic Theta Environments) *In the notation of Theorem 5.6, suppose further that "A" arises from* $\underline{\underline{X}}^{\log}$*; write*

$$s_N^{\sqcap}, s_N^{\sqcup} : A_N \to B_N$$

for the pair of base-equivalent morphisms of $\mathcal{C}$ *determined by the sections "*$s_{l \cdot N}$*", "*$\tau_{l \cdot N}$*" of the discussion preceding Proposition 5.2 — so* $s_N^{\sqcap}$, $s_N^{\sqcup}$ *constitute an* **N-th root of a right fraction-pair** *of an* l*-th root of the theta function* $\ddot{\Theta}$ *[cf. Proposition 5.2, (i)];*

$$\mathbb{E}_N \subseteq \mathrm{Aut}_{\mathcal{C}}(B_N)$$

for the **subgroup** *defined in the discussion preceding Lemma 5.8;*

$$\mathbb{E}_N^{\Pi} \stackrel{\mathrm{def}}{=} \mathbb{E}_N \times_{\mathrm{Im}(\underline{\underline{\Pi}}_{\underline{\underline{Y}}}^{\mathrm{tm}})} \underline{\underline{\Pi}}_{\underline{\underline{Y}}}^{\mathrm{tm}}$$

for the **topological group** *defined in Lemma 5.9, (iv);*

$$\epsilon : \mathbb{E}_N^{\Pi} \to \mathrm{Aut}_{\mathcal{C}}(B_N)$$

for the $\boldsymbol{\mu}_N$**-outer homomorphism** *[i.e., homomorphism considered up to composition with an inner automorphism defined by an element of* $\mathrm{Ker}(\mathbb{E}_N^{\Pi} \twoheadrightarrow \underline{\underline{\Pi}}_{\underline{Y}}^{\mathrm{tm}})$ *or* $\boldsymbol{\mu}_N(B_N) \subseteq \mathrm{Aut}_{\mathcal{C}}(B_N)$*] determined by the* **natural projection** $\mathbb{E}_N^{\Pi} \to \mathbb{E}_N$ $(\subseteq \mathrm{Aut}_{\mathcal{C}}(B_N))$*. Then:*

(i) The self-equivalence $\Psi : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ *preserves the* **isomorphism classes** *of* A_N*,* B_N*.*

(ii) Let β *be an isomorphism* $\beta : \Psi(B_N) \xrightarrow{\sim} B_N$ *[cf. (i)]; write*

$$\Psi_{\mathrm{Aut}} : \mathrm{Aut}_{\mathcal{C}}(B_N) \xrightarrow{\sim} \mathrm{Aut}_{\mathcal{C}}(B_N); \quad \Psi_{\mathrm{Aut}}^{\mathrm{birat}} : \mathrm{Aut}_{\mathcal{C}^{\mathrm{birat}}}(B_N^{\mathrm{birat}}) \xrightarrow{\sim} \mathrm{Aut}_{\mathcal{C}^{\mathrm{birat}}}(B_N^{\mathrm{birat}})$$

for the automorphisms determined by applying Ψ *followed by conjugation by* β*. Then* Ψ_{Aut}*,* $\Psi_{\mathrm{Aut}}^{\mathrm{birat}}$ **preserve**

$$\mathcal{O}^{\times}(B_N); \quad (\mathcal{O}_K^{\times})^{1/N} \subseteq \mathcal{O}^{\times}(B_N); \quad \mathcal{O}^{\times}(B_N^{\mathrm{birat}}); \quad (K^{\times})^{1/N} \subseteq \mathcal{O}^{\times}(B_N^{\mathrm{birat}})$$

and **map** *the data* $\mathbb{E}_N$ $(\subseteq \mathrm{Aut}_{\mathcal{C}}(B_N))$, $\mathrm{Im}(s_N^{\sqcap\text{-gp}})$, $\mathrm{Im}(s_N^{\sqcup\text{-gp}})$ *[where "*$\mathrm{Im}(-)$*" denotes the image of the homomorphism in parentheses] to data*

$$\delta_1 \cdot \mathbb{E}_N \cdot \delta_1^{-1} (\subseteq \mathrm{Aut}_{\mathcal{C}}(B_N)); \quad \delta_1 \cdot \mathrm{Im}(s_N^{\sqcap\text{-gp}}) \cdot \delta_1^{-1};$$
$$\delta_1 \cdot \delta_2 \cdot \delta_3 \cdot \mathrm{Im}(s_N^{\sqcup\text{-gp}}) \cdot \delta_3^{-1} \cdot \delta_2^{-1} \cdot \delta_1^{-1}$$

for some $\delta_1 \in \mathcal{O}^{\times}(B_N)$*,* $\delta_2 \in \boldsymbol{\mu}_{2l \cdot N}(B_N)$ $(\subseteq \mathcal{O}^{\times}(B_N))$*,* $\delta_3 \in s_N^{\sqcap\text{-gp}}(\mathrm{Aut}_{\mathcal{D}}(B_N^{\mathrm{bs}}))$*.*

(iii) The operation of applying Ψ *followed by conjugation by* β **preserves the** $\mathrm{Aut}_{\mathcal{C}}(B_N)$**-orbit** *of* $\epsilon : \mathbb{E}_N^{\Pi} \to \mathrm{Aut}_{\mathcal{C}}(B_N)$*, in a fashion which is* **compatible** *with the* **mono-theta environment** *structure on* $\mathbb{E}_N^{\Pi}$ *involving* $s_N^{\sqcup\text{-}\Pi}$ *discussed in Lemma 5.9, (iv). More precisely: there exists a* **commutative diagram**

$$\begin{array}{ccc} \mathbb{E}_N^{\Pi} & \xrightarrow{\gamma} & \mathbb{E}_N^{\Pi} \\ \downarrow \epsilon & & \downarrow \kappa \circ \epsilon \\ \mathrm{Aut}_{\mathcal{C}}(B_N) & \xrightarrow{\Psi_{\mathrm{Aut}}} & \mathrm{Aut}_{\mathcal{C}}(B_N) \end{array}$$

where κ *is an* **inner automorphism** *of* $\mathrm{Aut}_{\mathcal{C}}(B_N)$*;* γ *is an automorphism of topological groups which determines an* **automorphism of mono-theta environments***.*

Proof. First, we observe that by Proposition 5.1, the hypotheses of Theorem 4.4 are satisfied. Next, we consider assertion (i). Since Ψ preserves *Frobenius-trivial* objects [cf. the proof of Theorem 4.4], to show that Ψ preserves the isomorphism class of A_N, it suffices to show that the equivalence $\Psi^{\mathrm{bs}} : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$ [cf. Theorem

4.4] induced by Ψ preserves the isomorphism class of the objects of $\mathcal{D}$ determined by "$\ddot{Z}^{\log}_{l \cdot N}$", "$\underline{\underline{\ddot{Y}}}^{\log}$"; but this follows immediately [in light of the definitions of the various tempered coverings involved] from Proposition 2.4. Now the fact that Ψ preserves the isomorphism class of B_N follows immediately from Proposition 5.3, (vi). This completes the proof of assertion (i).

Next, we consider assertion (ii). First, we observe that it follows from the existence of Ψ^{bs} and the fact that Ψ preserves *pre-steps* [cf. the proof of Theorem 4.4] that Ψ preserves "$\mathcal{O}^{\rhd}(-)$", as well as *base-equivalent pairs of pre-steps.* In particular, it follows that applying Ψ followed by conjugation by β *preserves* $\mathcal{O}^{\rhd}(B_N)$, $\mathcal{O}^{\times}(B_N)$, $\mathcal{O}^{\times}(B_N^{\text{birat}})$. Moreover, by Proposition 5.1, we may apply Corollary 3.8, (ii), to Ψ to conclude [by considering Galois-, i.e., $\text{Aut}_{\mathcal{C}}(B_N)$-*invariants*] that Ψ_{Aut}, $\Psi^{\text{birat}}_{\text{Aut}}$ preserves the image of the natural inclusion $K^{\times} \hookrightarrow \mathcal{O}^{\times}(B_N^{\text{birat}})$, hence also $(K^{\times})^{1/N}$, $(\mathcal{O}^{\times}_K)^{1/N}$. Finally, the portion of assertion (ii) concerning $\text{Im}(s_N^{\sqcap\text{-gp}})$, $\text{Im}(s_N^{\sqcup\text{-gp}})$, $\mathbb{E}_N$ follows by observing that Ψ preserves *N-th roots of fraction pairs* [cf. Theorem 4.4, (ii)], together with the corresponding *bi-Kummer N-th roots* [cf. Theorem 4.4, (iv)], for [the Frobenioid-theoretic version of the log-meromorphic function constituted by] *l-th roots of the theta function* [cf. Theorem 5.7]. This completes the proof of assertion (ii).

Finally, we consider assertion (iii). First, observe that it follows from the *existence of* κ that, in the following argument, we may treat ϵ as a *single fixed homomorphism*, rather than just a "$\boldsymbol{\mu}_N$-outer homomorphism". Next, we observe that by taking κ to be the inner automorphism of $\text{Aut}_{\mathcal{C}}(B_N)$ determined by conjugating by the element $\delta_1 \cdot \delta_2 \cdot \delta_3 \in \text{Aut}_{\mathcal{C}}(B_N)$ of assertion (ii), we may assume that the *restrictions* of $\Psi_{\text{Aut}} \circ \epsilon$, $\kappa \circ \epsilon$ to the image of $s_N^{\sqcup\text{-}\Pi} : \underline{\underline{\Pi}}^{\text{tm}}_{\ddot{Y}} \to \mathbb{E}_N^{\Pi}$ *coincide*, up to composition with an automorphism of the topological group $\underline{\underline{\Pi}}^{\text{tm}}_{\ddot{Y}}$ that *extends* to an automorphism of the topological group $\underline{\underline{\Pi}}^{\text{tm}}_{X}$. Thus, by applying Corollary 2.18, it follows that we may choose γ so that the *restrictions* of $\Psi_{\text{Aut}} \circ \epsilon$, $\kappa \circ \epsilon \circ \gamma$ to the image of $s_N^{\sqcup\text{-}\Pi}$ *coincide* [precisely]. Moreover, by applying the *compatible* [cf. Lemma 5.9, (v)], *category/group-theoretic* [cf. Theorems 5.6; 2.15, (ii)] *cyclotomic rigidity isomorphisms* of Proposition 5.5 and Theorem 2.15, (ii) [cf. also Remark 2.15.6], it follows that the *restrictions* of $\Psi_{\text{Aut}} \circ \epsilon$, $\kappa \circ \epsilon \circ \gamma$ to $\text{Ker}(\mathbb{E}_N^{\Pi} \twoheadrightarrow \underline{\underline{\Pi}}^{\text{tm}}_{Y})$ *coincide*. Thus, we conclude that the *restrictions* of $\Psi_{\text{Aut}} \circ \epsilon$, $\kappa \circ \epsilon \circ \gamma$ to $\underline{\underline{\Pi}}^{\text{tm}}_{\ddot{Y}}[\boldsymbol{\mu}_N] \subseteq \underline{\underline{\Pi}}^{\text{tm}}_{Y}[\boldsymbol{\mu}_N] \xrightarrow{\sim} \mathbb{E}_N^{\Pi}$ [where we apply the isomorphism of Lemma 5.9, (iv)] *coincide*. Since $[\underline{\underline{\Pi}}^{\text{tm}}_{Y} : \underline{\underline{\Pi}}^{\text{tm}}_{\ddot{Y}}] = 2$, it thus follows that the *difference* between $\Psi_{\text{Aut}} \circ \epsilon$, $\kappa \circ \epsilon \circ \gamma$ determines a *cohomology class*

$$\in H^1(\underline{\underline{\Pi}}^{\text{tm}}_{Y}/\underline{\underline{\Pi}}^{\text{tm}}_{\ddot{Y}}, \mathcal{O}^{\times}(B_N)^{G_K}) \cong H^1(\mathbb{Z}/2\mathbb{Z}, \mathcal{O}^{\times}_K) \cong \boldsymbol{\mu}_2(B_N)$$

[where the superscript "G_K" denotes the subgroup of G_K-invariants; we apply Proposition 3.4, (ii), and recall that $\ddot{K} = K$]. Thus, by composing γ with the automorphism of $\mathbb{E}_N^{\Pi} \xrightarrow{\sim} \underline{\underline{\Pi}}^{\text{tm}}_{Y}[\boldsymbol{\mu}_N]$ determined by *"shifting by this cohomology class"* [cf. the situation of Proposition 2.13, (ii)], we obtain a γ such that $\Psi_{\text{Aut}} \circ \epsilon = \kappa \circ \epsilon \circ \gamma$, as desired. This completes the proof of assertion (iii). ◯

Remark 5.10.1. Observe that Theorem 5.10, (iii), may be interpreted as asserting that a *mono-theta environment* may be *"extracted"* from the *tempered Frobenioids* under consideration in a purely *category-theoretic* fashion. In particular, by coupling this observation with Corollary 2.18 [cf. also Remark 2.18.1], we conclude that:

> A *mono-theta environment* may be *"extracted"* naturally from *both* the *tempered Frobenioids* and the *tempered fundamental groups* under consideration in a purely *category/group-theoretic* fashion.

That is to say, a mono-theta environment may be thought of as a sort of *minimal core* common to both the *[tempered-]étale-theoretic* and *Frobenioid-theoretic* approaches to the theta function [cf. Remark 2.18.1]. Put another way, from the point of view of the general theory of Frobenioids, although the "*"étale-like"* [e.g., temperoid]" and *"Frobenius-like"* portions of a Frobenioid are *fundamentally alien* to one another in nature [cf. the *"fundamental dichotomy"* discussed in [Mzk16], Remark 3.1.3], a mono-theta environment serves as a sort of *bridge*, relative to the theory of *theta function*, between these two fundamentally mutually alien aspects of the structure of a Frobenioid.

Remark 5.10.2. The structure of a Frobenioid may be thought of as consisting of a sort of *extension structure* of the *base category* by various *line bundles*. From this point of view, the *theta section* portion of a mono-theta environment may be thought as a sort of *canonical splitting* of this extension, determined by the theory of the *étale theta function* [cf. the point of view of Remark 5.10.1]. This point of view is reminiscent of the notion of a *"canonical uniformizing $\mathcal{MF}^{\nabla}$-object"* discussed in [Mzk1], Introduction, §1.3.

Remark 5.10.3. One key feature of a mono-theta environment is the inclusion in the mono-theta environment of the *"distinct cyclotome"* $\boldsymbol{\mu}_N(B_N) \cong \mathrm{Ker}(\mathbb{E}_N^{\Pi} \twoheadrightarrow \underline{\underline{\Pi}}_{\underline{\underline{Y}}}^{\mathrm{tm}})$ [i.e., a cyclotome distinct from the various cyclotomes associated to $\underline{\underline{\Pi}}_{\underline{\underline{Y}}}^{\mathrm{tm}}$]. Here, we pause to observe that:

> This *"distinct cyclotome"* may be thought of as a sort of *"Frobenius germ"* — i.e., a "germ" or "trace" of the "Frobenius structure" of the tempered Frobenioid $\mathcal{C}$ constituted by raising elements of $\mathcal{O}^{\triangleright}(-)$ to $\mathbb{N}_{\geq 1}$-powers.

Indeed, when a mono-theta environment is not considered as a separate, abstract mathematical structure, but rather as a mathematical structure *associated to the tempered Frobenioid* $\mathcal{C}$, the operation of "raising to $\mathbb{N}_{\geq 1}$-powers" elements of $\mathcal{O}^{\triangleright}(-)$ in $\mathcal{C}$ is *compatible* with the natural multiplication action of $\mathbb{N}_{\geq 1}$ on this distinct cyclotome. Moreover, these actions of $\mathbb{N}_{\geq 1}$ are *compatible* with the operation of *forming Kummer classes* [e.g., passing from the Frobenioid-theoretic version of the theta function to its Kummer class, the "étale theta function"], as well as with the consideration of *values* $\in K^{\times}$ of functions [e.g., the theta function — cf. Proposition 1.4, (iii)], relative to the *reciprocity map* on elements of $K^{\times}$ [cf. [Mzk17], Theorem 2.4]. On the other hand, it is important to note that:

> These actions of $\mathbb{N}_{\geq 1}$ only make sense *within the tempered Frobenioid* $\mathcal{C}$; that is to say, they do *not* give rise to an *action* of $\mathbb{N}_{\geq 1}$ on the *mono-theta environment* [considered as a separate, abstract mathematical structure].

Indeed, the fact that one does not obtain a natural action of $\mathbb{N}_{\geq 1}$ on the mono-theta environment may be understood, for instance, by observing that the *cyclotomic rigidity isomorphism*

$$(l \cdot \underline{\Delta}_\Theta) \otimes (\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \Delta^{[\boldsymbol{\mu}_N]}$$

[cf. Theorem 2.15, (ii); Remark 2.15.6 — where we omit the "$\Box$'s"] is *not compatible* with the endomorphism of the "distinct cyclotome" $\Delta^{[\boldsymbol{\mu}_N]}$ given by *multiplication by* $M \in \mathbb{N}_{\geq 1}$, unless $M \equiv 1$ modulo N. [Here, it is useful to recall that there is *no natural action* of $\mathbb{N}_{\geq 1}$ on $\underline{\underline{\Pi}}^{\text{tm}}_{\underline{\underline{X}}}$ that induces the multiplication action on $\mathbb{N}_{\geq 1}$ on $l \cdot \underline{\Delta}_\Theta$!] Alternatively, this property may be regarded as a restatement of the interpretation [cf. Remark 2.15.6] of the cyclotomic rigidity of a mono-theta environment as a sort of *"integral structure"*, or *"basepoint"*, relative to the action of $\mathbb{N}_{\geq 1}$. Moreover, we remark that it is precisely the presence of this rigidity property that motivates the interpretation, stated above, of the "distinct cyclotome" as a sort of *"Frobenius germ"* — i.e., something that is somewhat less than the "full Frobenius structure" present in a Frobenioid, but which nevertheless serves as a sort of "trace", or *"partial, but essential record"*, of such a "full Frobenius structure".

Before proceeding, we review the following "well-known" result in category theory:

Lemma 5.11. **(Non-category-theoreticity of Particular Morphisms)** *Let* $\mathcal{E}$ *be a* **category***;* $G, H, I \in \text{Ob}(\mathcal{E})$ **distinct** *objects of* $\mathcal{E}$*;* $f : G \to H$*,* $g : G \to G$ **morphisms** *of* $\mathcal{E}$*;* $\alpha_G \in \text{Aut}_{\mathcal{E}}(G)$*,* $\alpha_H \in \text{Aut}_{\mathcal{E}}(H)$*,* $\alpha_I \in \text{Aut}_{\mathcal{E}}(I)$ **automorphisms**. *Then there exists a* **self-equivalence** $\Xi : \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ *that induces the* **identity** *on* $\text{Ob}(\mathcal{E})$ *and is* **isomorphic to the identity self-equivalence** *via an isomorphism that maps*

$$G \mapsto \alpha_G \in \text{Aut}_{\mathcal{E}}(G); \quad H \mapsto \alpha_H \in \text{Aut}_{\mathcal{E}}(H); \quad I \mapsto \alpha_I \in \text{Aut}_{\mathcal{E}}(I);$$

$$J \mapsto \text{id}_J \in \text{Aut}_{\mathcal{E}}(J)$$

[where id_J *denotes the identity morphism* $J \to J$*], for all* $J \in \text{Ob}(\mathcal{E})$ *such that* $J \neq G, H, I$*. In particular,* Ξ *maps* $f \mapsto \alpha_H \circ f \circ \alpha_G^{-1}$*,* $g \mapsto \alpha_G \circ g \circ \alpha_G^{-1}$*.*

Proof. First, let us observe that by *composing* the self-equivalences obtained by applying Lemma 5.11 with two of the three automorphisms α_G, α_H, α_I taken to be the identity, we may assume without loss of generality that α_H, α_I are the respective *identity* automorphisms of H, I. Now [one verifies immediately that] one may define an *equivalence of categories* [that is isomorphic to the identity self-equivalence]

$$\Xi : \mathcal{E} \xrightarrow{\sim} \mathcal{E}$$

which restricts to the identity on $\mathrm{Ob}(\mathcal{E})$ and maps

$$j \mapsto j \circ (\alpha_G)^{-1}; \quad j' \mapsto \alpha_G \circ j'; \quad j'' \mapsto \alpha_G \circ j'' \circ \alpha_G^{-1}$$

for $j \in \mathrm{Hom}_{\mathcal{E}}(G, J)$, $j' \in \mathrm{Hom}_{\mathcal{E}}(J, G)$, $j'' \in \mathrm{Hom}_{\mathcal{E}}(G, G)$, where $G \neq J \in \mathrm{Ob}(\mathcal{E})$. Thus, Ξ satisfies the properties asserted in the statement of Lemma 5.11. $\bigcirc$

Finally, we apply the *"general nonsense category theory"* of Lemma 5.11 to explain certain aspects of the motivation that underlies the theory of mono-theta environments.

Corollary 5.12. **(Constant Multiple Indeterminacy of Systems)** *In the notation of Theorem 5.10, assume further that $N' \geq 1$ is an integer such that N* **divides** *N', but $M \stackrel{\mathrm{def}}{=} N'/N \neq 1$; $s^{\sqcap}_{N'}, s^{\sqcup}_{N'} : A_{N'} \to B_{N'}$ an* **N'-th root of a right fraction-pair** *of an l-th root of the theta function $\ddot{\Theta}$ such that there exists a pair of commutative diagrams*

$$\begin{array}{ccc} A_{N'} & \xrightarrow{s^{\sqcap}_{N'}} & B_{N'} \\ \downarrow{\scriptstyle \alpha_{N,N'}} & & \downarrow{\scriptstyle \beta_{N,N'}} \\ A_N & \xrightarrow{s^{\sqcap}_N} & B_N \end{array} \qquad \begin{array}{ccc} A_{N'} & \xrightarrow{s^{\sqcup}_{N'}} & B_{N'} \\ \downarrow{\scriptstyle \alpha_{N,N'}} & & \downarrow{\scriptstyle \beta_{N,N'}} \\ A_N & \xrightarrow{s^{\sqcup}_N} & B_N \end{array}$$

— where $\alpha_{N,N'}$ (respectively, $\beta_{N,N'}$) is an **isometry of Frobenius degree** *$M \neq 1$; $\alpha_{N,N'}$ is* **of base-Frobenius type** *[cf. Remark 4.3.2]. Then:*

(i) The isomorphism classes of A_N, B_N, and $B_{N'}$ are **distinct***.*

(ii) There exists a **linear** *morphism $\iota : B_{N'} \to B_N$ [cf. the proof of Proposition 5.5].*

(iii) Let

$$\zeta : B_{N'} \to B_N$$

be either $\beta_{N,N'}$ or ι. Write $(\mathcal{O}^{\times}_K)^{1/N}|_A \subseteq \mathcal{O}^{\times}(A_N)$ for the subgroup induced by $(\mathcal{O}^{\times}_K)^{1/N} \subseteq \mathcal{O}^{\times}(B_N)$ [cf. Lemma 5.8] via $s^{\sqcap}_N$ or $s^{\sqcup}_N$; $(\mathcal{O}^{\times}_K)^* \subseteq \mathcal{O}^{\times}(B_{N'})$ for the subgroup

$$((\mathcal{O}^{\times}_K)^{1/N'})^{M/\deg_{\mathrm{Fr}}(\zeta)}$$

[cf. Lemma 5.8]. [Thus, we have a natural $\boldsymbol{\mu}_{N'}$-outer action of $(\mathcal{O}^{\times}_K)^*$ on $\mathrm{Aut}_{\mathcal{C}}(B_{N'})$ that is compatible, relative to ζ, with the natural $\boldsymbol{\mu}_N$-outer action of $(\mathcal{O}^{\times}_K)^{1/N}$ on $\mathrm{Aut}_{\mathcal{C}}(B_N)$.] Then for any $\kappa_A \in (\mathcal{O}^{\times}_K)^{1/N}|_A$, $\kappa_B \in (\mathcal{O}^{\times}_K)^{1/N}$, $\kappa' \in (\mathcal{O}^{\times}_K)^*$, there exists a **self-equivalence**

$$\Xi : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$$

that is **isomorphic to the identity self-equivalence** *via an isomorphism that maps*

$$A_N \mapsto \kappa_A^{-1} \in \mathrm{Aut}_{\mathcal{C}}(A_N); \quad B_N \mapsto \kappa_B^{-1} \in \mathrm{Aut}_{\mathcal{C}}(B_N); \quad B_{N'} \mapsto (\kappa')^{-1} \in \mathrm{Aut}_{\mathcal{C}}(B_{N'})$$

and all other objects of $\mathcal{C}$ *to the corresponding identity automorphism. In particular,* Ξ *maps*

$$s_N^{\sqcap} \mapsto \kappa_B^{-1} \circ s_N^{\sqcap} \circ \kappa_A; \quad s_N^{\sqcup} \mapsto \kappa_B^{-1} \circ s_N^{\sqcup} \circ \kappa_A; \quad \zeta \mapsto \kappa_B^{-1} \circ \zeta \circ \kappa';$$

and

$$\mathrm{Im}(s_N^{\mathrm{trv}}) \ (\subseteq \mathrm{Aut}_{\mathcal{C}}(A_N)) \ \mapsto \kappa_A^{-1} \cdot \mathrm{Im}(s_N^{\mathrm{trv}}) \cdot \kappa_A \subseteq \mathrm{Aut}_{\mathcal{C}}(A_N);$$

$$\mathrm{Im}(s_N^{\sqcap\text{-gp}}) \ (\subseteq \mathrm{Aut}_{\mathcal{C}}(B_N)) \ \mapsto \kappa_B^{-1} \cdot \mathrm{Im}(s_N^{\sqcap\text{-gp}}) \cdot \kappa_B \subseteq \mathrm{Aut}_{\mathcal{C}}(B_N);$$

$$\mathrm{Im}(s_N^{\sqcup\text{-gp}}) \ (\subseteq \mathrm{Aut}_{\mathcal{C}}(B_N)) \ \mapsto \kappa_B^{-1} \cdot \mathrm{Im}(s_N^{\sqcup\text{-gp}}) \cdot \kappa_B \subseteq \mathrm{Aut}_{\mathcal{C}}(B_N)$$

[where "s_N^{trv}", "$s_N^{\sqcap\text{-gp}}$", "$s_N^{\sqcup\text{-gp}}$" are as in the discussion preceding Lemma 5.8].

Proof. First, we consider assertions (i), (ii). Since $\beta_{N,N'}$ is an *isometry of Frobenius degree* M, it follows that the pull-back via the morphism $B_{N'}^{\mathrm{bs}} \to B_N^{\mathrm{bs}}$ of $\mathcal{D}$ of the line bundle that determines the object B_N is isomorphic to the M-th tensor power of the line bundle that determines the object $B_{N'}$. Moreover, it follows immediately from the discussion of line bundles in §1 [cf. the discussion preceding Proposition 1.1] that all positive tensor powers of these line bundles are *nontrivial*, and that the isomorphism classes of these line bundles are *preserved* by arbitrary automorphisms of $B_{N'}^{\mathrm{bs}}$. In particular, we conclude immediately that B_N, $B_{N'}$ are *non-isomorphic* both to one another and to the *Frobenius-trivial object* [i.e., object defined by a trivial line bundle] A_N. Moreover, by multiplying by the $(M-1)$-th tensor power of the section of line bundles that determines either of the morphisms $s_{N'}^{\sqcap}, s_{N'}^{\sqcup} : A_{N'} \to B_{N'}$ [cf. Remark 5.12.5 below], we obtain a *linear* morphism $\iota : B_{N'} \to B_N$. This completes the proof of assertions (i), (ii). Finally, in light of assertion (i), assertion (iii) follows immediately from Lemma 5.11. ◯

Remark 5.12.1. Let $\mathcal{E}$ be a *category.* Then for any $E \in \mathrm{Ob}(\mathcal{E})$,

$$\mathrm{Aut}_{\mathcal{E}}(E)$$

has a natural *group* structure. Indeed, this group structure is precisely the group structure that allows one, for instance, to represent the *group* structure of a tempered topological group *category-theoretically* via *temperoids* or to represent [cf. Theorem 5.10, (iii)] the *group* structure portion of a mono-theta environment *category-theoretically* via *tempered Frobenioids.* On the other hand, in both of these cases, one is, in fact, not just interested in the group structure, but rather in the *topological group* structure of the various objects under consideration. From this point of view:

> *Temperoids* [or, for that matter, *Galois categories*] allow one to represent the *topological group* structure under consideration via the use of **numerous objects** *corresponding to the various open subgroups* of the topological group under consideration, as opposed to the use of a *"single universal covering object"*, whose automorphism group allows one to represent the

> entire *group* under consideration via a single object, but only at the expense of *sacrificing the additional data that constitutes the* **topology** of the topological group under consideration.

This approach of using "numerous objects corresponding to the various open subgroups" carries over, in effect, to the theory of *tempered Frobenioids*, since such tempered Frobenioids typically appear over base categories given by [the subcategory of connected objects of] a *temperoid*.

Remark 5.12.2. In general, if one tries to consider *systems* [e.g., projective systems, such as *"universal coverings"*] in the context of the *"numerous objects approach"* of Remark 5.12.1, then one must contend with the following problem: Suppose that to each object E in some given given collection $\mathcal{I}$ of isomorphism classes of objects of $\mathcal{E}$, one associates certain *data*

$$E \mapsto \mathbb{D}_E$$

[such as the group $\mathrm{Aut}_{\mathcal{E}}(E)$, or some category-theoretically determined subquotient of $\mathrm{Aut}_{\mathcal{E}}(E)$], which one may think of as a *functor* on the full subcategory of $\mathcal{E}$ consisting of objects whose isomorphism class belongs to $\mathcal{I}$. Then:

> One must contend with the fact there is **no natural, "category-theoretic" choice** [cf. Lemma 5.11] of a *particular morphism* $\zeta : E \to F$, among the various composites
>
> $$\alpha_F \circ \zeta \circ \alpha_E$$
>
> [where $\alpha_E \in \mathrm{Aut}_{\mathcal{E}}(E)$, $\alpha_F \in \mathrm{Aut}_{\mathcal{E}}(F)$], for the task of *relating* $\mathbb{D}_E$ to $\mathbb{D}_F$.

In particular:

> One *necessary condition* for the data constituted by the functor $E \mapsto \mathbb{D}_E$ to form a *coherent system* is the condition that the data $\mathbb{D}_E$ be *invariant* with respect to the various automorphisms induced by the various $\mathrm{Aut}_{\mathcal{E}}(E)$ — i.e., that $\mathrm{Aut}_{\mathcal{E}}(E)$ act as the **identity** on $\mathbb{D}_E$.

One "classical" example of this phenomenon is the *category-theoretic* reconstruction of a tempered topological group from its associated temperoid [cf. [Mzk14], Proposition 3.2; the even more classical case of Galois categories], where one is obliged to work with topological groups *up to inner automorphism*.

Remark 5.12.3. Now we return to our discussion of *Frobenioid-theoretic mono-theta environments*, in the context of Theorem 5.10, (iii). If instead of working with "*finite*" mono-theta environments, one attempts to work with the *projective system* of mono-theta environments determined by letting N *vary* [cf. Remark 2.15.1], then one must contend with the *"constant multiple indeterminacy"* of Corollary 5.12, (iii), relative to κ', of $\zeta = \beta_{N,N'}$ [i.e., where we note that $\beta_{N,N'}$ may be thought of as a typical morphism appearing in this projective system]. In particular, the existence of this indeterminacy implies that, in order to obtain the analogue of Theorem 5.10, (iii) — i.e., to describe, in a the *category-theoretic* fashion, the relationship between a *single abstract, static external projective system of mono-theta environments* and the projective system of mono-theta environments constructed inside a tempered Frobenioid — one must work with mono-theta environments *up to the indeterminacies* arising from the $\boldsymbol{\mu}_N$-outer action of

$$\mathrm{Aut}_{\mathcal{C}}(B_N) \supseteq \mathcal{O}^\times(B_N) \supseteq (\mathcal{O}_K^\times)^{1/N} \twoheadrightarrow (\mathcal{O}_K^\times)^{1/N}/\boldsymbol{\mu}_N(B_N) \xrightarrow{\sim} \mathcal{O}_K^\times$$

[cf. Lemma 5.8]. That is to say, one must assume that one only knows the *theta section* portion of a mono-theta environment [cf. Definition 2.14, (ii), (c)] *up to a constant multiple*. Put another way, one is forced to *sacrifice the constant multiple rigidity* of Theorem 2.15, (iii); Corollary 2.8, (i). Moreover, we observe in passing that attempting instead to work with *bi-theta environments* [cf. the discussion preceding Theorem 5.7] does not serve to remedy this situation, since this forces one to *sacrifice "discrete rigidity"* [cf. Remark 2.15.1]. Thus, in summary:

> A [*finite*, not profinite!] *mono-theta enviroment* serves in effect to *maximize the rigidity*, i.e., to *minimize the indeterminacy*, of the [l-th roots of the] theta function that one works with in the following three *crucial* respects:
>
> (a) **cyclotomic rigidity** [cf. Theorem 2.15, (ii); Remark 2.15.7];
>
> (b) **discrete rigidity** [cf. Theorem 2.15, (iii); Remarks 2.15.1, 2.15.7];
>
> (c) **constant multiple rigidity** [cf. Theorem 2.15, (iii); Corollary 2.8, (i); the discussion of the present Remark 5.12.3; Remark 5.12.5 below].
>
> — all in a fashion that is *compatible* with the *category-theoretic representation* of the **topology** of the tempered fundamental group discussed in Remark 5.12.1.

This *"extraordinary rigidity"* of a mono-theta environment, along with the *"bridging aspect"* discussed in Remarks 2.18.1, 5.10.1, 5.10.2, 5.10.3, were, from the point of view of the author, the *main motivating reasons* for the introduction of the notion of a mono-theta environment.

Remark 5.12.4. With regard to the *projective systems of mono-theta environments* discussed in Remark 5.12.3, if, instead of trying to relate a single abstract such system to a Frobenioid-theoretic system, one instead takes the approach of relating, at each *finite* step [i.e., at each constituent object of the system], an abstract mono-theta environment to a Frobenioid-theoretic mono-theta environment via Theorem 5.10, (iii) — hence, in particular, taking into account the *"$\mathrm{Aut}_{\mathcal{C}}(B_N)$-orbit" indeterminacies* that occur at each step when one considers such relationships — then one can indeed establish a *category-theoretic* relationship between projective systems of mono-theta environments *external* and *internal* to the Frobenioid $\mathcal{C}$

without sacrificing the *cyclotomic*, *discrete*, or *constant multiple* rigidity properties discussed in Remark 5.12.3. Put another way, the difference between "attempting to relate whole systems at once as in Remark 5.12.3" and "applying Theorem 5.10, (iii), at each finite step" may be thought of as a matter of *"how one arranges one's parentheses"*, that is to say, as the difference between

$$\begin{array}{ccccc} \Big(\ldots \longrightarrow & \text{external}_{N'} & \longrightarrow & \text{external}_{N} & \longrightarrow \ldots\Big) \\ & & \downarrow & & \\ \Big(\ldots \longrightarrow & \text{internal}_{N'} & \longrightarrow & \text{internal}_{N} & \longrightarrow \ldots\Big) \end{array}$$

[i.e., the approach discussed in Remark 5.12.3] and

$$\ldots \longrightarrow \left(\begin{array}{c} \text{external}_{N'} \\ \downarrow \\ \text{internal}_{N'} \end{array}\right) \longrightarrow \left(\begin{array}{c} \text{external}_{N} \\ \downarrow \\ \text{internal}_{N} \end{array}\right) \longrightarrow \ldots$$

[i.e., "applying Theorem 5.10, (iii), at each finite step"].

Remark 5.12.5. At this point, some readers may feel prompted to pose the following question:

> Can one not perhaps simultaneously achieve both *cyclotomic* and *constant multiple* rigidity for an *M-th power version* [where we recall that $M \neq 1$] of the mono-theta environment by supplementing such an M-th power version with the additional data constituted by the *Frobenioid-theoretic cyclotomic rigidity isomorphism* of Proposition 5.5 [which may be transported, for instance, via the morphism ι of Corollary 5.12, (ii)]?

To understand this issue, let us first recall that the isomorphism of Proposition 5.5 may only be used to relate the data $\mathrm{Aut}_{\mathcal{D}}((-)^{\mathrm{bs}})$, $\boldsymbol{\mu}_N(-)$ [or, more generally, $\mathcal{O}^{\times}(-)$] of, say, $B_{N'}$, B_N [cf. the proof of Proposition 5.5] — i.e., data which constitutes a sort of *"semi-simplification"* of the "extension structure" of a Frobenioid discussed in Remark 5.10.2. In particular, passing to this "semi-simplified data" means that one is forgetting the portion of this *"extension structure"* constituted by the exact sequence

$$1 \to \mathcal{O}^{\times}(-) \to \mathrm{Aut}_{\mathcal{C}}((-)) \to \mathrm{Aut}_{\mathcal{D}}((-)^{\mathrm{bs}}) \to 1$$

[for objects "$(-)$", such as $B_{N'}$, B_N, which are Aut-*ample* — cf. the discussion preceding Lemma 5.8]. Put another way, it amounts to working up to an *indeterminacy* with respect to [so to speak "unipotent upper-triangular"] *"shifting automorphisms"* arising from cocycles of $\mathrm{Aut}_{\mathcal{D}}((-)^{\mathrm{bs}})$ with coefficients in $\mathcal{O}^{\times}(-)$ ($\supseteq \boldsymbol{\mu}_N(-)$)

[cf. Proposition 2.13, (ii)], which includes, in particular, a *constant* [i.e., $\mathcal{O}_K^\times$-] *multiple indeterminacy* [cf. the collection of cocycles determined by the image of $(\mathcal{O}_K^\times)^{1/N}/\boldsymbol{\mu}_N(B_N) \xrightarrow{\sim} \mathcal{O}_K^\times$ in $H^1(\mathrm{Aut}_{\mathcal{D}}(B_N^{\mathrm{bs}}), \boldsymbol{\mu}_N(B_N)!]$. Thus, in summary:

> Going back and forth between the data of an M-th power version of the mono-theta environment and the data of the isomorphism of Proposition 5.5 may only be done *consistently* at the expense of admitting a *"shifting automorphism indeterminacy"* [which is somewhat stronger than a "constant multiple indeterminacy"].

In particular, this observation implies a *negative* answer to the above question. Put another way:

> Working with the *semi-simplification of an extension* certainly gives rise to a *tautological "canonical splitting"* [cf. the point of view discussed in Remark 5.10.2], but that sort of "canonical splitting" is not very interesting!

Note, moreover, that to eliminate this "shifting automorphism indeterminacy", it is necessary to relate the objects $B_{N'}$, B_N that give rise, respectively, to the M-th and first power mono-theta environments to one another via a *linear* morphism [such as the morphism ι of Corollary 5.12, (ii) — cf. the proof of Proposition 5.5] which is *simultaneously compatible* with *both* the algebraic and theta sections that appear in the M-th and first power mono-theta environments. Although compatibility with the algebraic (respectively, theta) section may be achieved by taking ι to arise from $s_{N'}^{\sqcap}$ (respectively, $s_{N'}^{\sqcup}$) [cf. the proof of Corollary 5.12, (ii)], *simultaneous compatibility* may only be achieved at the cost of working *"birationally"* [i.e., of thinking of ι as only being determined up to multiplication by some log-meromorphic function such as, say, *a power of a root of the theta function*]. But this amounts precisely to admitting a "shifting automorphism indeterminacy", which is, in effect, the *Galois/Kummer-theoretic translation* of working with line bundles *birationally*. On the other hand, the *cyclotomic rigidity isomorphism* [cf. Theorem 2.15, (ii)] arising from the *specific splitting* determined by the theta section is *not preserved* by [so to speak "unipotent upper-triangular"] *"shifting automorphisms"*. Thus, in summary:

> The approach proposed in the above question results in contradictions with the *"line bundle structure"* of the Frobenioid $\mathcal{C}$, which *precludes* any sort of *linear relation* between distinct tensor powers of a non-torsion line bundle, except at the cost of working *"birationally"*.

Finally, we observe that to *avoid* such linear relations amounts, in effect, to working with a *"pair of first and M-th power mono-theta environments related by an M-th power map"* — a situation that leads to problems of the sort discussed in Remark 2.15.6 [cf. also Remark 5.10.3].

Remark 5.12.6. At this point, it is useful to reflect [cf. Remark 2.15.7] on the significance of rigidifying the *"constant multiple indeterminacy"* and *"shifting automorphism indeterminacy"* of Remarks 5.12.3, 5.12.5. To this end, we observe that these types of indeterminacy are essentially *multiplicative* notions [cf. the cases discussed in Remark 2.15.7]. Thus, to work "modulo these sorts of indeterminacy" can only be done at the cost of sacrificing the *additive structures* [cf. Remark 2.15.7] implicit in the *ring/scheme-theoretic* origins of the various objects under consideration.

Remark 5.12.7. One way to understand the *"constant multiple indeterminacy"* phenomena observed in Remarks 5.12.3, 5.12.5 is as a manifestation of the *nontriviality* of the *extension structure of a Frobenioid* discussed in Remark 5.10.2.

Remark 5.12.8. Perhaps a sort of *"unifying principle"* underlying the *"constant multiple indeterminacy"* phenomena observed in Remarks 5.12.3, 5.12.5, on the one hand, and the *discrete rigidity* discussed in Remark 2.15.1, on the other, may be expressed in the following fashion:

> Although at first glance, *two* [or many] pieces of data may appear to be likely to yield *"more information"* than *"one"* piece of data, in fact, the *more pieces* of data that one considers the *greater the indeterminacies* are that arise in describing the internal relations between these pieces of data.

Moreover, these greater indeterminacies may [as in the case of *systems of objects of a Frobenioid* in Remarks 5.12.3, 5.12.5, or *bi-theta environments* in Remark 2.15.1] ultimately result in *"less information"* than the information resulting from a single piece of data.

Bibliography

[André] Y. André, On a Geometric Description of Gal$(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a p-adic Avatar of $\widehat{GT}$, *Duke Math. J.* **119** (2003), pp. 1-39.

[Mzk1] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, AMS/IP Studies in Advanced Mathematics **11**, American Mathematical Society/International Press (1999).

[Mzk2] S. Mochizuki, The Absolute Anabelian Geometry of Hyperbolic Curves, *Galois Theory and Modular Forms*, Kluwer Academic Publishers (2003), pp. 77-122.

[Mzk3] S. Mochizuki, The Absolute Anabelian Geometry of Canonical Curves, *Kazuya Kato's fiftieth birthday, Doc. Math. 2003, Extra Vol.*, pp. 609-640.

[Mzk4] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves I, *Arithmetic Fundamental Groups and Noncommutative Algebra, Proceedings of Symposia in Pure Mathematics* **70**, American Mathematical Society (2002), pp. 533-569.

[Mzk5] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves II, *Algebraic Geometry 2000, Azumino, Adv. Stud. Pure Math.* **36**, Math. Soc. Japan (2002), pp. 81-114.

[Mzk6] S. Mochizuki, *The Hodge-Arakelov Theory of Elliptic Curves: Global Discretization of Local Hodge Theories*, RIMS Preprint Nos. **1255**, **1256** (October 1999).

[Mzk7] S. Mochizuki, *The Scheme-Theoretic Theta Convolution*, RIMS Preprint No. **1257** (October 1999).

[Mzk8] S. Mochizuki, *Connections and Related Integral Structures on the Universal Extension of an Elliptic Curve*, RIMS Preprint **1279** (May 2000).

[Mzk9] S. Mochizuki, *The Galois-Theoretic Kodaira-Spencer Morphism of an Elliptic Curve*, RIMS Preprint No. **1287** (July 2000).

[Mzk10] S. Mochizuki, *The Hodge-Arakelov Theory of Elliptic Curves in Positive Characteristic*, RIMS Preprint No. **1298** (October 2000).

[Mzk11] S. Mochizuki, The Local Pro-p Anabelian Geometry of Curves, *Invent. Math.* **138** (1999), pp. 319-423.

[Mzk12] S. Mochizuki, The Geometry of Anabelioids, *Publ. Res. Inst. Math. Sci.* **40** (2004), pp. 819-881.

[Mzk13] S. Mochizuki, Galois Sections in Absolute Anabelian Geometry, *Nagoya Math. J.* **179** (2005), pp. 17-45.

[Mzk14] S. Mochizuki, *Semi-graphs of Anabelioids*, RIMS Preprint **1477** (October 2004); to appear in *Publ. Res. Inst. Math. Sci.*

[Mzk15] S. Mochizuki, *Absolute Anabelian Cuspidalizations of Proper Hyperbolic Curves*, RIMS Preprint **1490** (February 2005).

[Mzk16] S. Mochizuki, *The Geometry of Frobenioids I: The General Theory*, RIMS Preprint **1529** (February 2006).

[Mzk17] S. Mochizuki, *The Geometry of Frobenioids II: Poly-Frobenioids*, RIMS Preprint **1530** (February 2006).

[Mumf] D. Mumford, An Analytic Construction of Degenerating Abelian Varieties over Complete Rings, Appendix to [FC].

[LynSch] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, *Ergeb. Math. Grenzgeb.* **89**, Springer (1977).