THE ALGEBRAIC AND ANABELIAN GEOMETRY OF CONFIGURATION SPACES

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Abstract. In this paper, we study the *pro-Σ fundamental groups of configuration spaces*, where Σ is either the set of all prime numbers or a set consisting of a single prime number. In particular, we show, via two somewhat distinct approaches, that, in many cases, the *"fiber subgroups"* of such fundamental groups arising from the various natural projections of a configuration space to lower-dimensional configuration spaces may be *characterized group-theoretically.*

Contents:

Introduction

Let $n \geq 1$ be an integer; X a *hyperbolic curve* of type (g, r) [where $2g-2+r > 0$] over an algebraically closed field k of characteristic 0. Denote by

$$X_n \subseteq P_n$$

the *n-th configuration space* associated to X, i.e., the open subscheme of the direct product P_n of n copies of X obtained by removing the various *diagonals* from P_n

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[cf. Definition 2.1, (i)]. By omitting the factors corresponding to various subsets of the set of n copies of X, we obtain various *natural projection morphisms*

$$X_n \to X_m$$

for nonnegative integers $m \leq n$ [cf. Definition 2.1, (ii)]. Next, let $\Sigma_{\mathcal{C}}$ be either the set of all prime numbers or a set consisting of a single prime number. Write $\mathcal{C}$ for the class of all finite groups of order a product of primes $\in \Sigma_{\mathcal{C}}$. Then by considering the *maximal pro-$\mathcal{C}$ quotient of the étale fundamental group*, which we denote by "$\pi_1^{\mathcal{C}}(-)$", we obtain various *natural surjections*

$$\pi_1^{\mathcal{C}}(X_n) \twoheadrightarrow \pi_1^{\mathcal{C}}(X_m)$$

arising from the natural projection morphisms considered above. We shall refer to the kernel of such a surjection $\pi_1^{\mathcal{C}}(X_n) \twoheadrightarrow \pi_1^{\mathcal{C}}(X_m)$ as a *fiber subgroup* of $\pi_1^{\mathcal{C}}(X_n)$ of *length* $n - m$ and *co-length* m [cf. Definition 2.3, (iii)]. Also, we shall refer to a closed subgroup of $\pi_1^{\mathcal{C}}(X_n)$ that arises as the inverse image of a closed subgroup of $\pi_1^{\mathcal{C}}(P_n)$ via the natural surjection $\pi_1^{\mathcal{C}}(X_n) \twoheadrightarrow \pi_1^{\mathcal{C}}(P_n)$ [induced by the inclusion $X_n \hookrightarrow P_n$] as *product-theoretic* [cf. Definition 2.3, (ii)].

The present paper is concerned with the issue of the *group-theoretic characterization* of these *fiber subgroups.* Our *main results* [cf. Corollaries 4.8, 6.3] may be summarized as follows:

(i) Suppose that $g \geq 2$. Let $H \subseteq \pi_1^{\mathcal{C}}(X_n)$ be a *product-theoretic open subgroup.* Then the subgroups $H \bigcap F$ of H — where F ranges over the various *fiber subgroups* of $\pi_1^{\mathcal{C}}(X_n)$ — may be characterized *group-theoretically* [cf. Corollary 4.8].

(ii) Suppose that (g, r) is not equal to $(0, 3)$ or $(1, 1)$. Then the *fiber subgroups* of $\pi_1^{\mathcal{C}}(X_n)$ may be characterized *group-theoretically* [cf. Corollary 6.3].

The proof of (i) relies on a certain group-theoretic description of *abelian torsion-free quotients* of H by *product-theoretic* normal closed subgroups of H [cf. Theorem 4.7]; this description is based on a slightly complicated computation involving *Chern classes* [cf. §4], together with the well-known fact that the action of the Galois group of a finite Galois covering of a curve of genus ≥ 2 on the Tate module of the Jacobian of the covering curve contains the *regular representation* [cf. Proposition 1.3]. This *geometric* approach, due to the first author, does not [as was pointed out to the first author by the second author! — cf. Remark 3.3.2] require any "deep group theory". On the other hand, the proof of (ii), due to the second author, requires the use of a group-theoretic result due to *Lubotzky-Melnikov-van den Dries* [cf. Theorem 1.5] and makes essential use to the notion of a *"nearly abelian group"*, i.e., a profinite group G which admits a normal closed subgroup $N \subseteq G$ which is *topologically normally generated by a single element* $\in G$ such that G/N contains an open abelian subgroup [cf. Definition 6.1]. It is worth noting that at the time of writing, we are unable to prove *either* an analogue of (i) for $g < 2$ *or* an analogue of (ii) when (g, r) is equal to $(0, 3)$ or $(1, 1)$.

The original proof of (i) [due to the first author] given in §4 may be regarded as a consequence of various *explicit group-theoretic manifestations* of certain *algebro-geometric* properties. This proof of (i) motivated the second author to develop a more direct approach to understanding these essentially *purely algebro-geometric* properties. This approach, which is exposed in §5, allows one to prove a *stronger version* [cf. Theorem 5.6] of Theorem 4.7 and, moreover, implies certain interesting consequences concerning the *non-existence of units on a sufficiently generic hyperbolic curve* [cf. Corollary 5.7].

The contents of the present paper may be summarized as follows: Basic well-known facts concerning the profinite fundamental groups of *hyperbolic curves* and *configuration spaces* are reviewed in §1, §2, respectively. In §3, we discuss the *group-theoreticity* of direct product decompositions of profinite groups. In §4, §6, we present the proofs, via somewhat different techniques, of the main results (i), (ii) discussed above. In §5, we discuss the algebraic geometry of *divisors* and *units* on configuration spaces, a theory which yields an alternate approach to the theory of §4. Finally, in §7, we observe that these results (i), (ii) imply a certain *discrete analogue* [cf. Corollary 7.4] of (i), (ii).

Section 0: Notations and Conventions

Numbers:

The notation $\mathbb{Q}$ will be used to denote the field of *rational numbers*. The notation $\mathbb{Z} \subseteq \mathbb{Q}$ will be used to denote the set, group, or ring of *rational integers*. The notation $\mathbb{N} \subseteq \mathbb{Z}$ will be used to denote the set or [additive] monoid of *nonnegative integers*. If l is a prime number, then the notation $\mathbb{Q}_l$ (respectively, $\mathbb{Z}_l$) will be used to denote the *l-adic completion* of $\mathbb{Q}$ (respectively, $\mathbb{Z}$). The [topological] field of complex numbers will be denoted $\mathbb{C}$.

Topological Groups:

Let G be a *Hausdorff topological group*, and $H \subseteq G$ a *closed subgroup*. Let us write

$$Z_G(H) \stackrel{\text{def}}{=} \{g \in G \mid g \cdot h = h \cdot g, \ \forall \ h \in H\}$$

for the *centralizer* of H in G. Also, we shall write $Z(G) \stackrel{\text{def}}{=} Z_G(G)$ for the *center* of G.

We shall say that a profinite group G is *slim* if for every open subgroup $H \subseteq G$, the centralizer $Z_G(H)$ is trivial. Note that every *finite normal closed subgroup* $N \subseteq G$ of a slim profinite group G is *trivial*. [Indeed, this follows by observing that for any normal open subgroup $H \subseteq G$ such that $N \bigcap H = \{1\}$, consideration of the inclusion $N \hookrightarrow G/H$ reveals that the conjugation action of H on N is *trivial*, i.e., that $N \subseteq Z_G(H) = \{1\}$.]

We shall write G^{ab} for the *abelianization* of G, i.e., the quotient of G by the closure of the commutator subgroup of G. We shall denote the group of automorphisms of G by $\mathrm{Aut}(G)$. Conjugation by elements of G determines a homomorphism $G \to \mathrm{Aut}(G)$ whose image consists of the *inner automorphisms* of G. We shall denote by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ by the [normal] subgroup consisting of the inner automorphisms. In particular, if G is *center-free*, then we have an *exact sequence* $1 \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$.

Curves:

Suppose that $g \geq 0$ is an *integer*. Then if S is a scheme, a *family of curves of genus* g

$$X \to S$$

is defined to be a smooth, proper, geometrically connected morphism of schemes $X \to S$ whose geometric fibers are curves of genus g.

Suppose that $g, r \geq 0$ are *integers* such that $2g - 2 + r > 0$. We shall denote the *moduli stack of* r*-pointed stable curves of genus* g over $\mathbb{Z}$ (where we assume the points to be *ordered*) by $\overline{\mathcal{M}}_{g,r}$ [cf. [DM], [Knud] for an exposition of the theory of such curves]. The open substack $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ of smooth curves will be referred to as the *moduli stack of smooth* r*-pointed stable curves of genus* g or, alternatively, as the *moduli stack of hyperbolic curves of type* (g, r). The *divisor at infinity* $\overline{\mathcal{M}}_{g,r} \backslash \mathcal{M}_{g,r}$ of $\overline{\mathcal{M}}_{g,r}$ is a *divisor with normal crossings* on the $\mathbb{Z}$-smooth algebraic stack $\overline{\mathcal{M}}_{g,r}$, hence determines a *log structure* on $\overline{\mathcal{M}}_{g,r}$; denote the resulting log stack by $\overline{\mathcal{M}}_{g,r}^{\log}$. For any integer $r' > r$, the operation of "forgetting the last $r' - r$ points" determines a [1-]morphism of log algebraic stacks

$$\overline{\mathcal{M}}_{g,r'}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$$

which factors as a composite of structure morphisms of various tautological log stable curves [cf. [Knud]], hence is *log smooth*.

A *family of hyperbolic curves of type* (g, r)

$$X \to S$$

is defined to be a morphism which factors $X \hookrightarrow Y \to S$ as the composite of an open immersion $X \hookrightarrow Y$ onto the complement $Y \backslash D$ of a relative divisor $D \subseteq Y$ which is finite étale over S of relative degree r, and a family $Y \to S$ of curves of genus g. One checks easily that, if S is *normal*, then the pair (Y, D) is *unique up to canonical isomorphism*. We shall refer to Y (respectively, D) as the *compactification* (respectively, *divisor of cusps*) of X. A *family of hyperbolic curves* $X \to S$ is defined to be a morphism $X \to S$ such that the restriction of this morphism to each connected component of S is a *family of hyperbolic curves of type* (g, r) for some integers (g, r) as above. A family of hyperbolic curve of type $(0, 3)$ will be referred to as a *tripod*.

Section 1: Surface Groups

In the present §1, we discuss various well-known preliminary facts concerning the sorts of profinite groups that arise from étale fundamental groups of *hyperbolic curves.*

Definition 1.1. Let $\mathcal{C}$ be a family of finite groups containing the trivial group; Σ a set of prime numbers.

(i) We shall refer to a finite group as a *Σ-group* if every prime dividing its order belongs to Σ. We shall refer to a finite group belonging to $\mathcal{C}$ as a *$\mathcal{C}$-group* and to a profinite group every finite quotient of which is a $\mathcal{C}$-group as a *pro-$\mathcal{C}$ group.* We shall refer to $\mathcal{C}$ as a *full formation* [cf. [FJ], p. 343] if it is closed under taking quotients, subgroups, and extensions.

(ii) Suppose that $\mathcal{C}$ is a full formation; write $\Sigma_{\mathcal{C}}$ for the set of primes p such that $\mathbb{Z}/p\mathbb{Z}$ is a $\mathcal{C}$-group and $\widehat{\mathbb{Z}} \twoheadrightarrow \widehat{\mathbb{Z}}_{\mathcal{C}}$ for the maximal pro-$\mathcal{C}$ quotient of $\widehat{\mathbb{Z}}$. Then we shall say that the formation $\mathcal{C}$ is *nontrivial* if there exists a nontrivial $\mathcal{C}$-group [or, equivalently, if $\Sigma_{\mathcal{C}}$ is nonempty]. We shall say that the formation $\mathcal{C}$ is *primary* if $\Sigma_{\mathcal{C}}$ is of cardinality one. We shall say that the formation $\mathcal{C}$ is *solvable* if every $\mathcal{C}$-group is solvable. We shall say that the formation $\mathcal{C}$ is *total* if every finite group is a $\mathcal{C}$-group. We shall say that $\mathcal{C}$ is a *PT-formation* if it is *either* primary *or* total. We shall say that $\mathcal{C}$ is *invertible on a scheme S* if every prime of $\Sigma_{\mathcal{C}}$ is invertible on S.

(iii) Suppose that $\mathcal{C}$ is a full formation; let G be a profinite group. If G admits an open subgroup which is abelian, then we shall say that G is *almost abelian.* If G admits an open subgroup which is pro-$\mathcal{C}$, then we shall say that G is *almost pro-$\mathcal{C}$*. We shall refer to a quotient $G \twoheadrightarrow Q$ as *almost pro-$\mathcal{C}$-maximal* if for some normal open subgroup $N \subseteq G$ with maximal pro-$\mathcal{C}$ quotient [cf. [FJ], p. 344] $N \twoheadrightarrow P$, we have $\mathrm{Ker}(G \twoheadrightarrow Q) = \mathrm{Ker}(N \twoheadrightarrow P)$. [Thus, any almost pro-$\mathcal{C}$-maximal quotient of G is almost pro-$\mathcal{C}$.] If G is topologically finitely generated, and, moreover, the abelianization H^{ab} of every open subgroup $H \subseteq G$ is torsion-free, then we shall say that G is *strongly torsion-free.*

Remark 1.1.1. The notion of a full formation is a special case of the notion of a *Melnikov formation* [cf. [FJ], p. 343]. In the present paper, [partly for the sake of simplicity] we restrict ourselves to full formations.

Remark 1.1.2. Let $\mathcal{C}$ be a full formation. Then [it follows immediately from the definitions that] a *solvable* finite group is a $\Sigma_{\mathcal{C}}$-group [cf. Definition 1.1, (ii)] if and only if it is a $\mathcal{C}$-group. In particular, if $\mathcal{C}$ is *solvable*, then it is *completely determined* by the set of primes $\Sigma_{\mathcal{C}}$.

Remark 1.1.3. Recall that every finite group whose order is a prime power is *nilpotent*, hence, in particular, *solvable.* Thus, [cf. Remark 1.1.2] a *primary* full formation $\mathcal{C}$ is *completely determined* by the unique prime number $\in \Sigma_{\mathcal{C}}$.

Definition 1.2. Let $\mathcal{C}$ be a full formation. We shall say that a profinite group is a *[pro-$\mathcal{C}$] surface group* (respectively, an *almost pro-$\mathcal{C}$-surface group*) if it is isomorphic to the maximal pro-$\mathcal{C}$ quotient (respectively, to some almost pro-$\mathcal{C}$-maximal quotient) of the étale fundamental group of a hyperbolic curve [cf. §0] over an algebraically closed field of characteristic zero [or, equivalently, the profinite completion of the topological fundamental group of a hyperbolic Riemann surface of finite type]. We shall refer to an almost pro-$\mathcal{C}$-surface group as *open* (respectively, *closed*) if it admits (respectively, does not admit) a pro-$\mathcal{C}$ free [cf. [FJ], p. 345] open subgroup.

Remark 1.2.1. Thus, in the notation of Definition 1.2, every pro-$\mathcal{C}$ surface group is an almost pro-$\mathcal{C}$-surface group. On the other hand, if $\mathcal{C}$ is *not total*, there one verifies immediately that there exist almost pro-$\mathcal{C}$-surface groups which are not pro-$\mathcal{C}$ surface groups. Nevertheless, every almost pro-$\mathcal{C}$-surface group admits a *normal open subgroup* which is a *pro-$\mathcal{C}$ surface group.*

Remark 1.2.2. We recall that if Π is a *pro-$\mathcal{C}$ surface group* arising from a hyperbolic curve [cf. Definition 1.2] of type (g, r), then Π is *topologically generated* by $2g + r$ generators *subject to a single [well-known!] relation*, and Π^{ab} [cf. §0] is a *free abelian pro-$\mathcal{C}$ group* of rank $2g - 1 + r$ (if $r > 0$), $2g$ (if $r = 0$). In particular, [since every open subgroup of Π is again a pro-$\mathcal{C}$ surface group, it follows that] Π is *strongly torsion-free.* Moreover, for any $l \in \Sigma_{\mathcal{C}}$, the *$l$-cohomological dimension* of Π is equal to 1 (if $r > 0$), 2 (if $r = 0$); $\dim_{\mathbb{Q}_l}(H^2(\Pi, \mathbb{Q}_l)) = \dim_{\mathbb{F}_l}(H^2(\Pi, \mathbb{F}_l))$ is equal to 0 (if $r > 0$), 1 (if $r = 0$). In particular, the quantity

$$\chi(\Pi) = \sum_{i=0}^{2} (-1)^i \cdot \dim_{\mathbb{Q}_l}(H^i(\Pi, \mathbb{Q}_l)) = \sum_{i=0}^{2} (-1)^i \cdot \dim_{\mathbb{F}_l}(H^i(\Pi, \mathbb{F}_l)) = 2 - 2g - r$$

is a *group-theoretic invariant* of Π which [as is well-known] satisfies the property that

$$\chi(\Pi_1) = [\Pi : \Pi_1] \cdot \chi(\Pi)$$

for any open subgroup $\Pi_1 \subseteq \Pi$. Finally, we recall that this formula admits a *representation-theoretic generalization*, which will play a crucial role in §4 below, in the form of the following elementary consequence:

Proposition 1.3. **(Inclusion of the Regular Representation)** *Let $Y \to X$ be a finite* **Galois** *covering of* **smooth proper hyperbolic curves** *over an algebraically closed field k of characteristic prime to the order of $G \stackrel{\mathrm{def}}{=} \mathrm{Gal}(Y/X)$; l a prime number that is invertible in k. Write V for the G-module determined by the first étale cohomology module $H^1_{\text{ét}}(Y, \mathbb{Q}_l)$. Then the G-module V contains the* **regular representation** *of G as a direct summand.*

Proof. Indeed, this follows immediately from the computation of the Galois module V in [Milne], p. 187, Corollary 2.8 [cf. also [Milne], p. 187, Remark 2.9], in light of our assumption that X is proper hyperbolic, hence of *genus* ≥ 2. ○

Proposition 1.4. **(Slimness)** *Let $\mathcal{C}$ be a* **nontrivial** *full formation. Then every* **almost pro-$\mathcal{C}$-surface group** Π *is* **slim**.

Proof. Indeed, this follows immediately by considering the *conjugation action* of Π/N on $N^{\mathrm{ab}} \otimes \mathbb{Z}_l$, where $l \in \Sigma_{\mathcal{C}}$, for sufficiently small normal open subgroups $N \subseteq \Pi$ [cf. Remark 1.2.1]. That is to say, in light of the interpretation of a certain quotient of $N^{\mathrm{ab}} \otimes \mathbb{Z}_l$ as the Tate module arising from the l-power torsion points of the Jacobian of the compactification of the covering determined by N of any hyperbolic curve that gives rise to Π [cf. the proof of [Mzk3], Lemma 1.3.1], it follows that this conjugation action is *faithful.* Another [earlier] approach to the *slimness* of surface groups may be found in [Naka], Corollary 1.3.4. ○

Remark 1.4.1. The property involving the *regular representation* discussed in Proposition 1.3 may be regarded as a *stronger version* [in the case of coverings of curves of genus ≥ 2] of the *faithfulness* of the action of Π/N on [a certain quotient of] $N^{\mathrm{ab}} \otimes \mathbb{Z}_l$ that was applied in the proof of Proposition 1.4, hence, in particular, as a stronger version of the *slimness* of surface groups.

The following result is a mild generalization to arbitrary surface groups of a well-known result for free pro-$\mathcal{C}$ groups due to *Lubotzky-Melnikov-van den Dries*:

Theorem 1.5. **(Normal Closed Subgroups of Surface Groups)** *Let $\mathcal{C}$ be a full formation;* Π *an* **almost pro-$\mathcal{C}$-surface group***;* $N \subseteq \Pi$ *a* **topologically finitely generated** *normal closed subgroup. Then N is either* **trivial** *or of* **finite index**.

Proof. Since Π is *slim*, hence does not contain any *nontrivial finite normal closed subgroups* [cf. §0], it follows that we may always replace Π by an open subgroup of Π. In particular, [cf. Remark 1.2.1] we may assume, without loss of generality, that Π is a *pro-$\mathcal{C}$ surface group.* When Π is an *open* surface group, Theorem 1.5 follows formally from the *theorem of Lubotzky-Melnikov-van den Dries* [cf., e.g., [FJ], Proposition 24.10.3; [FJ], Proposition 24.10.4, (a)]. Thus, we may assume, without loss of generality, that Π is a *closed* surface group.

Suppose that N is *nontrivial* and *of infinite index.* Then there exists an $l \in \Sigma_{\mathcal{C}}$ such that N contains a nontrivial subgroup $A \subseteq N$ which is a quotient of $\mathbb{Z}_l$. In particular, there exists a normal open subgroup $\Pi_1 \subseteq \Pi$ such that the image of A in Π/Π_1 is nontrivial. Now set $\Pi_A \stackrel{\mathrm{def}}{=} \Pi_1 \cdot A \subseteq \Pi$, $N_A \stackrel{\mathrm{def}}{=} N \bigcap \Pi_A$ [so Π_A, N_A are open subgroups of Π, N, respectively]. Then N_A is a *topologically finitely generated normal closed subgroup of infinite index* of Π_A such that $A \subseteq N_A$ *surjects* onto the [nontrivial, abelian!] image of Π_A in Π/Π_1. In particular, by replacing $N \subseteq \Pi$ by $N_A \subseteq \Pi_A$, we may assume without loss of generality that the image of N in Π^{ab} is *nontrivial.*

Since Π is *topologically finitely generated*, there exists a descending sequence of normal open subgroups

$$\ldots \subseteq H_n \subseteq \ldots \subseteq \Pi$$

[where n ranges over the positive integers] of Π which is, moreover, *exhaustive*, i.e., $\bigcap_n H_n = \{1\}$. Thus, if we set $N_n \overset{\text{def}}{=} H_n \cdot N$ [for $n \geq 1$], then [one verifies immediately that] we obtain a descending sequence of normal open subgroups

$$\ldots \subseteq N_n \subseteq \ldots \subseteq \Pi$$

[where n ranges over the positive integers] of Π such that $\bigcap_n N_n = N$ [cf. the fact that N is *closed*!]. Since N is of *infinite index* in Π, it follows that $[\Pi : N_n] \to \infty$ as $n \to \infty$, hence [cf. Remark 1.2.2] that $|\chi(N_n)| \to \infty$ as $n \to \infty$. In particular, there exists an n such that the rank of N_n^{ab} is $\geq s + 2$, where we write s for any positive integer such that there exist s elements of N that topologically generate N. Since, moreover, the image of N in Π^{ab}, hence *a fortiori* in N_n^{ab} is *nontrivial*, it follows that there exists, for some $l \in \Sigma_{\mathcal{C}}$, a nontrivial homomorphism $\mathbb{Z}_l \to N_n^{\text{ab}}$ that factors through N. Now write

$$N_n \twoheadrightarrow \Pi^*$$

for the *maximal pro-l quotient* of N_n [so Π^* is a *pro-l closed surface group*], $N^* \subseteq \Pi^*$ for the image of N in Π^*. Thus, $N^* \subseteq \Pi^*$ is a *topologically finitely generated normal closed subgroup* whose image in [the free $\mathbb{Z}_l$-module of finite rank] $(\Pi^*)^{\text{ab}}$ is a *nontrivial* $\mathbb{Z}_l$-submodule $M \subseteq (\Pi^*)^{\text{ab}}$ whose rank is $\leq s$, hence $\leq$ the rank of $(\Pi^*)^{\text{ab}}$ minus 2. In particular, there exists an element $x \in \Pi^*$ such that if we denote by $F^* \subseteq \Pi^*$ the [necessarily *topologically finitely generated*!] closed subgroup topologically generated by N^* and x, then we obtain inclusions of closed subgroups

$$N^* \subseteq F^* \subseteq \Pi^*$$

such that N^* is *of infinite index* in F^*, and F^* is *of infinite index* in Π^* [as may be seen by considering the *ranks* of the images of these subgroups in Π^{ab}].

Now observe that for any two open subgroups $J_2 \subseteq J_1 \subseteq \Pi^*$, the induced morphism $H^2(J_1, \mathbb{Z}_l) \to H^2(J_2, \mathbb{Z}_l)$ maps a generator of $H^2(J_1, \mathbb{Z}_l) \cong \mathbb{Z}_l$ to $[J_1 : J_2]$ times a generator $H^2(J_2, \mathbb{Z}_l) \cong \mathbb{Z}_l$. [Indeed, this follows immediately by thinking about degrees of coverings of proper hyperbolic curves! We refer to Remark 4.1.1; Lemma 4.2, (i) [and its proof], below, for more details on this well-known circle of ideas.] In particular, since F^* is a subgroup of infinite index in Π^*, it follows immediately [by considering open subgroups $J \subseteq \Pi^*$ containing F^*] that F^* is a *pro-l group* whose *[l-]cohomological dimension is* ≤ 1. Thus, by [RZ], Theorem 7.7.4, F^* is a [topologically finitely generated] *free pro-l group*, and $N^* \subseteq F^*$ is a *nontrivial topologically finitely generated closed normal subgroup of infinite index* — in contradiction to the *theorem of Lubotzky-Melnikov-van den Dries* [cf., e.g., [FJ], Proposition 24.10.3]. ○

Section 2: Configuration Space Groups

In the present §2, we discuss various well-known preliminary facts concerning the sorts of profinite groups that arise from étale fundamental groups of *configuration spaces* associated to hyperbolic curves.

First, let us suppose that we have been given a log scheme

$$Z^{\log}$$

which is *log regular* [cf., [Kato2], Definition 2.1]; write $U_Z \subseteq Z$ for the *interior* of $Z^{\log}$ [i.e., the open subscheme on which the log structure of $Z^{\log}$ is *trivial*]. By abuse of notation, we shall often use the notation for a scheme to denote the log scheme with trivial log structure determined by the scheme. If $\mathcal{C}$ is a *full formation* that is *invertible* on Z, then we shall write

$$\pi_1^{\mathcal{C}}(Z^{\log})$$

for the *maximal pro-$\mathcal{C}$ quotient* of the *étale fundamental group* [obtained by considering *Kummer log étale coverings*, for some choice of basepoint — cf. [Ill] for more details] of $Z^{\log}$. Thus, by the *log purity theorem* of Fujiwara-Kato [cf. [Ill]; [Mzk1], Theorem B], the natural morphism $U_Z \to Z^{\log}$ induces a [continuous outer] *isomorphism* $\pi_1^{\mathcal{C}}(U_Z) \to \pi_1^{\mathcal{C}}(Z^{\log})$.

Next, suppose that S is a *regular scheme*, and that

$$X \to S$$

is a *family of hyperbolic curves of type* (g, r) over S, with *compactification* $X \hookrightarrow Y \to S$ and *divisor of cusps* $D \subseteq Y$ [cf. §0]. For simplicity, we assume that the finite étale covering $D \to S$ is *split*. Let $n \in \mathbb{N}$.

Definition 2.1.

(i) For positive integers $i, j \leq n$ such that $i < j$, write

$$\pi_{i,j} : P_n \overset{\text{def}}{=} X \times_S \ldots \times_S X \to X \times_S X$$

for the projection of the product P_n of n copies of $X \to S$ to the i-th and j-th factors. Write E for the set [of cardinality n] of factors of P_n. Then we shall refer to as the *n-th configuration space associated to $X \to S$* the S-scheme

$$X_n \to S$$

which is the open subscheme determined by the complement in P_n of the union of the various inverse images via the $\pi_{i,j}$ [as (i, j) ranges over the pairs of positive integers $\leq n$ such that $i < j$] of the image of the diagonal embedding $X \hookrightarrow X \times_S X$.

We shall refer to as the *n-th log configuration space associated to $X \to S$* the [log smooth] log scheme over S

$$Z_n^{\log} \to S$$

obtained by pulling back the [log smooth] [1-]morphism $\overline{\mathcal{M}}_{g,r+n}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ given by "forgetting the last n points" [cf. §0] via the classifying [1-]morphism $S \to \overline{\mathcal{M}}_{g,r}^{\log}$ determined [up to a permutation of the r remaining points] by $X \to S$. We shall refer to E as the *index set* of the configuration space X_n, or, alternatively, of the log configuration space $Z_n^{\log}$.

(ii) In the notation of (i), let $E' \subseteq E$ be a subset of cardinality n'; $E'' \overset{\text{def}}{=} E \backslash E'$; $n'' \overset{\text{def}}{=} n - n'$. Then by *"forgetting"* the factors of E that belong to E', we obtain a *natural projection morphism*

$$p_{E'} = p^{E''} : X_n \to X_{n''}$$

[and similarly in the logarithmic case], which we shall refer to as the *projection morphism of profile E'*, or, alternatively, the *projection morphism of co-profile E''*. Also, in this situation, we shall refer to n' (respectively, n'') as the *length* (respectively, *co-length*) of this projection morphism.

Remark 2.1.1. One verifies immediately that in the notation of Definition 2.1, (i), X_n may be naturally identified with the *interior* of $Z_n^{\log}$.

Remark 2.1.2. One verifies immediately that in the notation of Definition 2.1, (ii), each projection morphism $p_{E'} = p^{E''} : X_n \to X_{n''}$ is itself the *n'-th configuration space* associated to a *family of hyperbolic curves of type $(g, r + n'')$* over $X_{n''}$ that embeds as a dense open subscheme of the pull-back via $X_{n''} \to S$ of the original family of hyperbolic curves $X \to S$.

Proposition 2.2. (Fundamental Groups of Configuration Spaces) *In the notation of the above discussion, suppose further that the following conditions hold:*

(a) S is **connected***;*

(b) $\mathcal{C}$ is a **PT-formation** *which is* **invertible** *on S;*

(c) for each $l \in \Sigma_{\mathcal{C}}$, the **images** *of the* **cyclotomic character** *$\pi_1(S) \to \mathbb{F}_l^{\times}$ and the natural Galois action*

$$\pi_1(S) \to \mathrm{Aut}(\pi_1(Y_{\bar{s}})^{\mathrm{ab}} \otimes \mathbb{F}_l)$$

arising from the family of curves $Y \to S$ are **$\mathcal{C}$-groups** *[a condition which is vacuous if $\mathcal{C}$ is* **total***].*

Let $n \geq 1$ be an integer, $\overline{s}$ a **geometric point** *of S, and $\overline{x}$ a* **geometric point** *of X_{n-1}; we shall denote the fibers over geometric points by means of subscripts. Then:*

(i) Any projection morphism $X_n \to X_{n-1}$ of length one determines a **natural exact sequence**

$$1 \to \pi_1^{\mathcal{C}}((X_n)_{\overline{x}}) \to \pi_1^{\mathcal{C}}(X_n) \to \pi_1^{\mathcal{C}}(X_{n-1}) \to 1$$

[where we write $X_0 \overset{\mathrm{def}}{=} S$].

(ii) The profinite group $\pi_1^{\mathcal{C}}((X_n)_{\overline{s}})$ is **slim** *and* **topologically finitely generated***.*

(iii) The natural sequence

$$1 \to \pi_1^{\mathcal{C}}((X_n)_{\overline{s}}) \to \pi_1^{\mathcal{C}}(X_n) \to \pi_1^{\mathcal{C}}(S) \to 1$$

is **exact***.*

(iv) Suppose that $S = \mathrm{Spec}(R)$ is a **trait***; that $\overline{s}$ arises from an algebraic closure of the residue field of R; and that $\overline{\eta}$ is a geometric point of S that arises from an algebraic closure of the quotient field K of R. Then the operation of specialization of the normalization of X in a covering of $X_K \overset{\mathrm{def}}{=} X \times_R K$ determines an* **isomorphism** *$\pi_1^{\mathcal{C}}((X_n)_{\overline{\eta}}) \xrightarrow{\sim} \pi_1^{\mathcal{C}}((X_n)_{\overline{s}})$.*

Proof. First, let us observe that since the kernel of the natural surjection $\pi_1^{\mathcal{C}}(X_{\overline{s}}) \twoheadrightarrow \pi_1^{\mathcal{C}}(Y_{\overline{s}})$ is topologically normally generated by the inertia groups of the cusps [which are isomorphic to $\widehat{\mathbb{Z}}_{\mathcal{C}}(1)$, where the "(1)" denotes a "Tate twist", and "$\widehat{\mathbb{Z}}_{\mathcal{C}}$" is as in Definition 1.1, (ii)], condition (c) [together with our assumption that the divisor of cusps of $X \to S$ is *split*] implies that for each $l \in \Sigma_{\mathcal{C}}$, the *image* of the natural Galois action

$$\pi_1(S) \to \mathrm{Aut}(\pi_1(X_{\overline{s}})^{\mathrm{ab}} \otimes \mathbb{F}_l)$$

arising from the family of hyperbolic curves $X \to S$ is a *$\mathcal{C}$-group.*

Now we *claim* that to complete the proof of Proposition 2.2, it suffices to verify assertion (iv). Indeed, let us assume that assertion (iv) *holds* and reason by induction on $n \geq 1$. [That is to say, if $n \geq 2$, then we assume that assertions (i), (ii), and (iii) have already been verified for "$n - 1$".] Now observe that [in light of Remark 2.1.2; the easily verified fact that the family $X_n \to X_{n-1}$ also satisfies conditions (a), (b), (c)] assertion (i) is a special case of assertion (iii) for "$n = 1$"; thus, [by applying the induction hypothesis] we may assume that assertion (i) holds if $n \geq 2$. Since, moreover, the property of being a slim topologically finitely generated profinite group holds for a profinite group which is an extension of a profinite group G_1 by a profinite group G_2 whenever it holds for G_1 and G_2, assertion (ii) [for "n"] follows immediately, by applying the induction hypothesis, from assertion (i) (when $n \geq 2$) and Proposition 1.4. As for assertion (iii), let us first observe that by assertion (iv) [and various standard arguments in elementary algebraic geometry], we may assume without loss of generality that $\overline{s}$ arises from an algebraic closure of the quotient field K of S. Thus, by considering the natural action of $G_K \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{s}/\mathrm{Spec}(K))$ on $\overline{s}$, we obtain a *natural outer action*

$$G_K \to \mathrm{Out}(\pi_1^{\mathcal{C}}((X_n)_{\overline{s}}))$$

which is *compatible* with the natural outer action of G_K on $\pi_1^{\mathcal{C}}((P_n)_{\overline{s}})$ [which may be identified with the product of n copies of $\pi_1^{\mathcal{C}}(X_{\overline{s}})$], relative to the natural inclusion $X_n \hookrightarrow P_n$ [cf. Definition 2.1, (i)]. In particular, since the kernel of the natural surjection $\pi_1^{\mathcal{C}}((X_n)_{\overline{s}}) \twoheadrightarrow \pi_1^{\mathcal{C}}((P_n)_{\overline{s}})$ is topologically normally generated by the inertia groups of the cusps [which are isomorphic to $\widehat{\mathbb{Z}}_{\mathcal{C}}(1)$], condition (c) [together with the observation at the beginning of the present proof] implies that for each $l \in \Sigma_{\mathcal{C}}$, the *image* of the natural Galois action

$$G_K \to \mathrm{Aut}(\pi_1((X_n)_{\overline{s}})^{\mathrm{ab}} \otimes \mathbb{F}_l)$$

is a *$\mathcal{C}$-group*, hence [cf. Remark 1.1.3 when $\mathcal{C}$ is *primary*] that the homomorphism $G_K \to \mathrm{Out}(\pi_1^{\mathcal{C}}((X_n)_{\overline{s}}))$ *factors* through the *maximal pro-$\mathcal{C}$ quotient* $G_K^{\mathcal{C}}$ of G_K. Note, moreover, that by *Zariski-Nagata purity* [i.e., the classical non-logarithmic version of the "log purity theorem" quoted above], the kernel of the natural surjection $G_K^{\mathcal{C}} \twoheadrightarrow \pi_1^{\mathcal{C}}(S)$ is topologically normally generated by the various *inertia groups* determined by the prime divisors of S. On the other hand, by assertion (iv), the images of these inertia groups in $\mathrm{Out}(\pi_1^{\mathcal{C}}((X_n)_{\overline{s}}))$ are *trivial*. Thus, we obtain a homomorphism $\pi_1^{\mathcal{C}}(S) \to \mathrm{Out}(\pi_1^{\mathcal{C}}((X_n)_{\overline{s}}))$, hence — by pulling back the natural exact sequence

$$1 \to \pi_1^{\mathcal{C}}((X_n)_{\overline{s}}) \to \mathrm{Aut}(\pi_1^{\mathcal{C}}((X_n)_{\overline{s}})) \to \mathrm{Out}(\pi_1^{\mathcal{C}}((X_n)_{\overline{s}})) \to 1$$

[cf. assertion (ii); §0] via this homomorphism — an exact sequence as in assertion (iii). This completes the proof of the *claim*.

Finally, we consider assertion (iv). First, we remark that assertion (iv) is a special case of the more general result of [Vid], Théorème 2.2; since, however, [Vid] has yet to be published at the time of writing, we give a self-contained [modulo published results] proof of assertion (iv), as follows. We begin by observing that by the *log purity theorem*, we have natural isomorphisms

$$\pi_1^{\mathcal{C}}((X_n)_{\overline{s}}) \xrightarrow{\sim} \pi_1^{\mathcal{C}}((Z_n^{\log})_{\overline{s}}); \quad \pi_1^{\mathcal{C}}((X_n)_{\overline{\eta}}) \xrightarrow{\sim} \pi_1^{\mathcal{C}}((Z_n^{\log})_{\overline{\eta}})$$

[cf. Definition 2.1, (i); Remark 2.1.1]. Now suppose that $W_0^{\log} \to (Z_n^{\log})_{\overline{s}}$ is a connected *Kummer log étale covering*. Since $(Z_n^{\log})_{\overline{s}}$ is *log regular*, it thus follows that $W_0^{\log}$ is also log regular, hence, in particular, *normal*. By the definition of "log étale", one may deform this covering to a *formal* Kummer log étale covering over the $\mathfrak{m}_R$-completion [where $\mathfrak{m}_R$ is the maximal ideal of R] of $Z_n^{\log}$. Moreover, the underlying scheme of this formal covering may be *algebrized* [cf. [EGA III], Théorème 5.4.5; the easily verified fact that Z_n is *projective*], hence determines a *finite morphism* $W \to Z_n$. Now it follows from the well-known local structure of

Kummer log étale coverings that the formal covering that gave rise to W is *S-flat*, hence that W itself is *S-flat*, with *normal* special fiber $W_{\overline{s}} \cong W_0$. Since S is, of course, normal, we thus conclude [cf. [EGA IV], Corollaire 6.5.4, (ii)] that W is *normal* and *connected*, hence *irreducible.* By considering the formal covering that gave rise to W at completions of closed points of Z_n lying in the *interior* $X_n \subseteq Z_n$, it follows, moreover, that $W \to Z_n$ is *generically étale.* Thus, it makes sense to speak of the *ramification divisor* in Z_n of $W \to Z_n$. On the other hand, again by considering the formal covering that gave rise to W, it follows immediately that this ramification divisor is contained in the complement of X_n in Z_n, hence [by the *log purity theorem*!] that $W \to Z_n$ determines a *Kummer log étale covering* $W^{\log} \to Z_n^{\log}$ whose special fiber $W_{\overline{s}}^{\log} \to (Z_n^{\log})_{\overline{s}}$ may be naturally identified with the given covering $W_0^{\log} \to (Z_n^{\log})_{\overline{s}}$. Thus, by *algebrizing* morphisms between formal Kummer log étale coverings [cf. [EGA III], Théorème 5.4.1], we conclude that the deformation and algebrization procedure just described determines an *equivalence of categories* between the categories of Kummer log étale coverings of $(Z_n^{\log})_{\overline{s}}$, $Z_n^{\log}$. In particular, we obtain a *natural isomorphism* $\pi_1^{\mathcal{C}}((Z_n^{\log})_{\overline{s}}) \xrightarrow{\sim} \pi_1^{\mathcal{C}}((Z_n^{\log}))$.

On the other hand, again by the *log purity theorem*, it follows immediately that we obtain an isomorphism

$$\pi_1^{\mathcal{C}}((Z_n^{\log})_{\overline{\eta}}) \xrightarrow{\sim} \varprojlim_{S'} \pi_1^{\mathcal{C}}(Z_n^{\log} \times_S S')$$

[where S' ranges over the normalizations of S in the various finite extensions of K in the function field of $\overline{\eta}$], hence, by applying the isomorphisms

$$\pi_1^{\mathcal{C}}(Z_n^{\log} \times_S S') \xrightarrow{\sim} \pi_1^{\mathcal{C}}((Z_n^{\log})_{\overline{s}})$$

[where we regard $\overline{s}$ as a geometric point of the various S'] obtained above, we obtain an isomorphism $\pi_1^{\mathcal{C}}((Z_n^{\log})_{\overline{\eta}}) \xrightarrow{\sim} \pi_1^{\mathcal{C}}((Z_n^{\log})_{\overline{s}})$, as desired. ○

Remark 2.2.1. Another proof of Proposition 2.2, (iii), in the case $n = 1$ may be found in [Stix], Proposition 2.3.

Definition 2.3. Let $\mathcal{C}$ be a PT-formation.

(i) We shall say that a profinite group is a *[pro-$\mathcal{C}$] configuration space group* if it is isomorphic to the maximal pro-$\mathcal{C}$ quotient of the étale fundamental group

$$\pi_1^{\mathcal{C}}(X_n)$$

of the n-th configuration space X_n for some $n \geq 1$ [cf. Definition 2.1, (i)] of a hyperbolic curve X over an algebraically closed field of characteristic $\notin \Sigma_{\mathcal{C}}$ [where we note that in this situation, the conditions (a), (b), (c) of Proposition 2.2 are satisfied].

(ii) Let X be a hyperbolic curve over an algebraically closed field of characteristic $\notin \mathcal{C}$; X_n the n-th configuration space [for some $n \geq 1$] associated to X. Then we shall refer to a closed subgroup $H \subseteq \pi_1^{\mathcal{C}}(X_n)$ as being *product-theoretic* if H arises as the inverse image via the natural surjection

$$\pi_1^{\mathcal{C}}(X_n) \twoheadrightarrow \pi_1^{\mathcal{C}}(P_n)$$

[cf. Definition 2.1, (i)] of a closed subgroup of $\pi_1^{\mathcal{C}}(P_n)$.

(iii) Let X, X_n be as in (ii); write E for the *index set* of X_n. Let $E' \subseteq E$ be a subset of cardinality n'; $E'' \overset{\text{def}}{=} E \backslash E'$; $n'' \overset{\text{def}}{=} n - n'$; $p_{E'} = p^{E''} : X_n \to X_{n''}$ the projection morphism of profile E'. Then we shall refer to the *kernel*

$$F \subseteq \pi_1^{\mathcal{C}}(X_n)$$

of the induced *surjection* $\pi_1^{\mathcal{C}}(X_n) \twoheadrightarrow \pi_1^{\mathcal{C}}(X_{n''})$ [cf. Remark 2.1.2; Proposition 2.2, (iii)] as the *fiber subgroup* of $\pi_1^{\mathcal{C}}(X_n)$ of *profile* E', or, alternatively, as the *fiber subgroup* of $\pi_1^{\mathcal{C}}(X_n)$ of *co-profile* E''. Also, we shall refer to n' (respectively, n'') as the *length* (respectively, *co-length*) of F.

Proposition 2.4. **(Fiber Subgroups of Configuration Spaces)** *Let* $\mathcal{C}$ *be a* **PT-formation***;* X *a* **hyperbolic curve** *over an algebraically closed field of characteristic* $\notin \Sigma_{\mathcal{C}}$*;* X_n *the* **n-th configuration space** *[for some* $n \geq 1$*] associated to* X*;* E *the* **index set** *of* X_n*;* $\Pi \overset{\text{def}}{=} \pi_1^{\mathcal{C}}(X_n)$*;* $E'_1, E'_2 \subseteq E$ *subsets whose respective complements we denote by* $E''_1, E''_2 \subseteq E$*;* $F_1, F_2 \subseteq \Pi$ *the* **fiber subgroups** *with respective* **profiles** $E'_1, E'_2 \subseteq E$*. Then:*

(i) The description of Remark 2.1.2 determines on F_2 *(respectively,* Π/F_2*) a structure of* **configuration space group** *with* **index set** E'_2 *(respectively,* E''_2*).*

(ii) $F_1 \subseteq F_2$ *if and only if* $E'_1 \subseteq E'_2$*. Moreover, in this situation,* $F_1 \subseteq F_2$ *is the* **fiber subgroup** *of* F_2 *with profile* $E'_1 \subseteq E'_2$ *[i.e., relative to the structure of* F_2 *as the "*$\pi_1^{\mathcal{C}}(-)$*" of a configuration space that arises from the description given in Remark 2.1.2].*

(iii) The **image** *of* F_1 *in* Π/F_2 *is the* **fiber subgroup** *of* Π/F_2 *with profile* $E'_1 \bigcap E''_2 \subseteq E''_2$ *[i.e., relative to the structure of* Π/F_2 *as the "*$\pi_1^{\mathcal{C}}(-)$*" of a configuration space that arises from the description given in Remark 2.1.2].*

(iv) The subgroup of Π **topologically generated** *by* F_1*,* F_2 *is the* **fiber subgroup** F_3 *with profile* $E'_3 \overset{\text{def}}{=} E'_1 \bigcup E'_2$*. In particular, if* E''_1*,* E''_2 *are* **disjoint** *and of* **cardinality one***, then* F_1*,* F_2 *topologically generate* Π*.*

(v) In the situation of (iv), suppose that the **length** *of* F_1*,* F_2 *is equal to* 1*. Then there exists a normal closed subgroup* $K \subseteq \Pi$ *satisfying the following properties: (a)* $K \subseteq F_3$*; (b)* K *is topologically normally generated in* F_3 *by a* **single element***; (c) the images of* F_1*,* F_2 *in* F_3/K **commute***.*

(vi) F_2 *is* **topologically generated** *by the fiber subgroups [of* Π*] of length* 1 *whose profiles are contained in* E'_2*. In particular,* Π *is* **topologically generated** *by its fiber subgroups of length* 1*.*

Proof. Assertions (i), (ii) are immediate from the definitions [and Remark 2.1.2]. Next, let us consider assertion (vi). In light of assertions (i), (ii), it suffices to verify assertion (vi) in the case where $F_2 = \Pi$; also, we may assume without loss of generality that F_1 is of *length* 1. Then, by induction on n [cf. also assertion (i)], Π/F_1 is topologically generated by its fiber subgroups of length 1. Since the inverse image in Π of any fiber subgroup of length 1 of Π/F_1 is clearly a fiber subgroup of length 2, it follows [cf. assertions (i), (ii)] that we may assume without loss of generality that $n = 2$. But then it suffices to observe that if $F_\alpha, F_\beta \subseteq \Pi$ are fiber subgroups whose profiles $E'_\alpha, E'_\beta \subseteq E$ are disjoint subsets of length 1, then the natural morphism $F_\alpha \subseteq \Pi \twoheadrightarrow \Pi/F_\beta$ [which is simply the morphism induced on "$\pi_1^{\mathcal{C}}(-)$'s" by an open immersion of hyperbolic curves] is a *surjection.* This completes the proof of assertion (vi). Now assertion (iv) follows formally from assertion (vi); also, in light of assertion (vi), assertion (iii) follows immediately from the definitions.

Finally, we consider assertion (v). First, let us observe that when $n = 2$, assertion (v) follows by observing that the kernel of the natural surjection $\pi_1^{\mathcal{C}}(X_2) \twoheadrightarrow \pi_1^{\mathcal{C}}(P_2)$ [cf. Definition 2.3, (ii)] is topologically normally generated by the inertia group of the *diagonal divisor* of X_2, which is isomorphic to $\widehat{\mathbb{Z}}_{\mathcal{C}}(1)$ [hence topologically generated by a *single element*]. Now assertion (v) follows immediately for arbitrary n, by applying assertions (i), (ii), (iv). ○

Remark 2.4.1. Note that it follows immediately from Proposition 2.2, (ii); Proposition 2.4, (i) [or, alternatively, (vi)], that the *fiber subgroups* of $\pi_1^{\mathcal{C}}(X_n)$ are *topologically finitely generated normal closed subgroups.*

Section 3: Direct Products of Profinite Groups

In the present §3, we study *quotients of products of profinite groups.* In particular, we show that, in certain cases, the product decomposition of a direct product of profinite groups is *"group-theoretic"*.

Definition 3.1. We shall say that a profinite group G is *indecomposable* if, for any isomorphism of profinite groups $G \xrightarrow{\sim} H \times J$, where H, J are profinite groups, it follows that either H or J is the trivial group.

Proposition 3.2. **(The Indecomposability of Surface Groups)** *Let $\mathcal{C}$ be a* **nontrivial** *full formation. Then every* **almost pro-$\mathcal{C}$-surface group** Π *is* **indecomposable**.

Proof. Suppose that we have an isomorphism of profinite groups $\Pi \cong H \times J$, where H, J are *nonabelian* [since Π is *slim* — cf. Proposition 1.4!] *infinite* [again since Π is *slim*, hence does not contain any *nontrivial finite normal closed subgroups* — cf.

§0] *profinite groups*. Note that since H, J, are *infinite*, it follows that for any open subgroup Π_1, we may always replace Π by an open subgroup of Π_1. In particular, [cf. Remark 1.2.1] we may assume, without loss of generality, that Π is a *pro-$\mathcal{C}$ surface group* arising from a curve *of genus* ≥ 2. Now we *claim* that for every prime number $l \in \Sigma_{\mathcal{C}}$, there exist finite quotients $H \twoheadrightarrow Q_H$, $J \twoheadrightarrow Q_J$ such that l divides the order of Q_H, Q_J. Indeed, suppose that l does *not* divide the order of any finite quotient of H. Then there exists a *proper* normal open subgroup $N_H \subseteq H$ such that if we set $N \stackrel{\mathrm{def}}{=} N_H \times J \subseteq \Pi$, then the conjugation action of $\Pi/N \cong H/N_H$ on $N^{\mathrm{ab}} \otimes \mathbb{Z}_l \cong (N_H^{\mathrm{ab}} \otimes \mathbb{Z}_l) \times (J^{\mathrm{ab}} \otimes \mathbb{Z}_l) \cong J^{\mathrm{ab}} \otimes \mathbb{Z}_l$ is *trivial*, which, as was seen in the proof of Proposition 1.4, leads to a contradiction. This completes the proof of the *claim*.

Thus, by replacing Π by the *maximal pro-l quotient* of Π for some $l \in \Sigma_{\mathcal{C}}$ [and replacing $\mathcal{C}$ by the primary formation determined by l], we may assume without loss of generality that Π, H, J are *pro-l groups*. Note, moreover, that since H, J are *nonabelian* pro-l groups, it follows that $\dim_{\mathbb{F}_l}(H^{\mathrm{ab}} \otimes \mathbb{F}_l) \geq 2$, $\dim_{\mathbb{F}_l}(J^{\mathrm{ab}} \otimes \mathbb{F}_l) \geq 2$ [cf., e.g., [RZ], Proposition 7.7.2]. On the other hand, observe that the cup product morphism

$$H^1(H, \mathbb{F}_l) \otimes H^1(J, \mathbb{F}_l) \to H^2(\Pi, \mathbb{F}_l)$$

is an *injection*. [Indeed, this follows immediately by considering the *spectral sequences* associated to the surjections $\Pi \cong H \times J \twoheadrightarrow J$, $H \twoheadrightarrow \{1\}$, where we note that the latter surjection may be regarded as a *quotient* of the former surjection.] But this implies that $\dim_{\mathbb{F}_l}(H^2(\Pi, \mathbb{F}_l)) \geq 2$, which [cf. Remark 1.2.2] is absurd. This completes the proof of Proposition 3.2. ◯

Proposition 3.3. **(Quotients of Direct Products)** *Let $G_1, \dots, G_n$ be* **profinite groups***, where $n \geq 1$ is an integer;*

$$\phi : \Pi \stackrel{\mathrm{def}}{=} \prod_{i=1}^{n} G_i \twoheadrightarrow Q$$

a **surjection** *of profinite groups. Then there exist normal closed subgroups $H_i \subseteq G_i$ [for $i = 1, \dots, n$], $N \subseteq Q$ such that $N \subseteq Z(Q)$ [cf. §0], and the composite $\Pi \twoheadrightarrow Q/N$ of ϕ with the surjection $Q \twoheadrightarrow Q/N$ induces an* **isomorphism**

$$\overline{\Pi} \stackrel{\mathrm{def}}{=} \prod_{i=1}^{n} \overline{G}_i \xrightarrow{\sim} Q/N$$

— where we write $\overline{G}_i \stackrel{\mathrm{def}}{=} G_i/H_i$. In particular, if Q is **center-free***, then we obtain an isomorphism $\overline{\Pi} \xrightarrow{\sim} Q$; if Q is* **center-free** *and* **indecomposable***, then we obtain an isomorphism $\overline{G}_i \xrightarrow{\sim} Q$ for some $i \in \{1, \dots, n\}$.*

Proof. Indeed, write $I \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\phi) \subseteq \Pi$; $I_i \subseteq G_i$ for the inverse image of I via the *natural injection* $\iota_i : G_i \hookrightarrow \Pi$ into the i-th factor; $H_i \subseteq G_i$ for the image of I under the *natural projection* $\pi_i : \Pi \twoheadrightarrow G_i$ to the i-th factor [where $i \in \{1, \dots, n\}$]. Thus, we have inclusions

$$\Pi_I \overset{\text{def}}{=} \prod_{i=1}^{n} I_i \subseteq I \subseteq \Pi_H \overset{\text{def}}{=} \prod_{i=1}^{n} H_i \subseteq \Pi$$

inside Π. Now observe that the *commutator* of any element

$$(1, \dots, 1, g_i, 1, \dots, 1) \in \Pi$$

[i.e., all of whose components, except possibly the i-th element $g_i \in G_i$, are equal to 1] with an element $h \in I$ yields an element of I [since I is *normal* in Π] which lies in the image of ι_i, hence determines an element of $I_i \subseteq G_i$, which is in fact equal to the commutator $[g_i, \pi_i(h)] \in G_i$ [where we observe that $\pi_i(h) \in H_i$] computed in G_i. In particular, since $g_i \in G_i$ is *arbitrary*, and any element of H_i arises as such a "$\pi_i(h)$", it follows that the commutator subgroup $[G_i, H_i]$ is contained in I_i. But this implies that the commutator subgroup $[\Pi, \Pi_H]$ is normally generated in Π by elements of $\Pi_I \subseteq I$, hence [since I is *normal* in Π] is contained in I. Put another way, if we set $N \subseteq Q$ equal to the image in $\Pi/I \xrightarrow{\sim} Q$ of Π_H, then it follows that $N \subseteq Z(Q)$. On the other hand, it is immediate from the definitions that ϕ determines an isomorphism $\prod_{i=1}^{n} (G_i/H_i) \xrightarrow{\sim} Q/N$, as desired. ○

Remark 3.3.1. Proposition 3.3 may be regarded as being motivated by the following elementary fact concerning *products of rings*: If $R_1, \dots, R_n$ [where $n \geq 1$ is an integer] are *[not necessarily commutative] rings with unity* and

$$\phi : R \overset{\text{def}}{=} \prod_{i=1}^{n} R_i \twoheadrightarrow Q$$

is a *surjection* of rings with unity, then there exist two-sided ideals $I_i \subseteq R_i$ [for $i = 1, \dots, n$] such that ϕ induces an *isomorphism*

$$\overline{R} \overset{\text{def}}{=} \prod_{i=1}^{n} \overline{R}_i \xrightarrow{\sim} Q$$

— where we write $\overline{R}_i \overset{\text{def}}{=} R_i/I_i$. [Indeed, this follows immediately by observing that if, for $i = 1, \dots, n$, we write $e_i \in R$ for the element whose i-th component is 1 and whose other components are 0, then any element $f \in \text{Ker}(\phi)$ may be written in the form $f = f \cdot e_1 + \ldots + f \cdot e_n$, where each $f \cdot e_i \in \text{Ker}(\phi)$ [since $\text{Ker}(\phi)$ is a *two-sided ideal*!].]

Remark 3.3.2. Proposition 3.3 is due to the *second author*. We observe in passing that when, in the notation of Proposition 3.3, Q is an *almost pro-$\mathcal{C}$-surface group* for some nontrivial full formation $\mathcal{C}$ [hence *slim* and *indecomposable* — cf. Propositions 1.4, 3.2], and the G_i are *topologically finitely generated*, one may give a different proof of Proposition 3.3 by applying Theorem 1.5 to the images J_i of the various composites of ϕ with the natural inclusions $\iota_i : G_i \hookrightarrow \Pi$ — which allows one to conclude [in light of the *slimness* of Q!] that *only one* of the J_i [as i ranges over the integers $1, \dots, n$] can be *nontrivial.* In fact, this argument was the approach originally taken by the *first author* to proving Proposition 3.3 and, moreover, underlies the proof of the main result of this paper via the approach of the *second author* given in §6 below. On the other hand, this argument [unlike the *very elementary* proof of Proposition 3.3 given above!] has the drawback that it depends on the result of *Lubotzky-Melnikov-van den Dries* that was applied in the proof of Theorem 1.5. This drawback was pointed out by the second author to the first author when the first author first informed the second author of this restricted version of Proposition 3.3 and, indeed, served to motivate the second author to obtain the more elementary proof of Proposition 3.3 given above. Perhaps somewhat ironically, this simplification due to the second author rendered the proofs of the main results of this paper via the approach of the *first author* [cf. Theorem 4.7, Corollary 4.8 below] *free of any dependence* on the theorem of Lubotzky-Melnikov-van den Dries [cf. Remark 4.8.1] — in sharp contrast to the *essential dependence* on the theorem of Lubotzky-Melnikov-van den Dries in the approach of the *second author* exposed in Corollary 6.3 below!

Corollary 3.4. **(Group-theoreticity of Product Decompositions)** *Let $\mathcal{C}$ be a* **nontrivial** *full formation; $n, m \geq 1$ integers;*

$$G_1, \dots, G_n; \quad H_m, \dots, H_m$$

almost pro-$\mathcal{C}$-surface groups*;*

$$G \subseteq \Pi_G \stackrel{\text{def}}{=} \prod_{i=1}^{n} G_i; \quad H \subseteq \Pi_H \stackrel{\text{def}}{=} \prod_{j=1}^{m} H_j$$

open subgroups*;*

$$\alpha : G \xrightarrow{\sim} H$$

an **isomorphism** *of profinite groups. For $i = 1, \dots, n$; $j = 1, \dots, m$, write $G_i^{\twoheadrightarrow} \subseteq G_i$, $H_j^{\twoheadrightarrow} \subseteq H_j$ for the respective images of G, H via the natural projections $\Pi_G \twoheadrightarrow G_i$, $\Pi_H \twoheadrightarrow H_j$;*

$$G_i^{=} \subseteq \Pi_G; \quad H_j^{=} \subseteq \Pi_H$$

for the respective intersections of G, H with the images of the natural injections $G_i \hookrightarrow \Pi_G$, $H_j \hookrightarrow \Pi_H$;

$$G_i^{\neq} \subseteq \Pi_G; \quad H_j^{\neq} \subseteq \Pi_H$$

for the respective intersections of G, H with the kernels of the natural projections $\Pi_G \twoheadrightarrow G_i$, $\Pi_H \twoheadrightarrow H_j$. Then $n = m$; there exist a **unique permutation** *σ of the set $\{1, \dots, n\}$ and* **unique isomorphisms** *of profinite groups $\alpha_i : G_i^{\twoheadrightarrow} \xrightarrow{\sim} H_{\sigma(i)}^{\twoheadrightarrow}$ [for $i = 1, \dots, n$] such that the restriction of [the composite with the inclusion into Π_H of] the isomorphism*

$$(\alpha_1, \dots, \alpha_n) : \Big(\Pi_G \supseteq\Big) \prod_{i=1}^{n} (G_i^{\twoheadrightarrow}) \xrightarrow{\sim} \prod_{i=1}^{n} (H_{\sigma(i)}^{\twoheadrightarrow}) \Big(\subseteq \Pi_H\Big)$$

to G coincides with [the composite with the inclusion into Π_H of] α.

Proof. First, we observe that the *uniqueness* assertions follow immediately from the *nontriviality* of $\mathcal{C}$. Thus, it suffices to verify the *existence* of σ and the α_i. Now we *claim* that for each $j = 1, \dots, m$, the kernel of the composite

$$\psi_j : G \to H_j$$

of α with the natural projection $(H \subseteq)\ \Pi_H \twoheadrightarrow H_j$ contains $G_i^{\neq}$, for a *unique* $i \in \{1, \dots, n\}$. Indeed, since the image of ψ_j is *open*, hence *slim* [cf. Proposition 1.4], it follows [cf. §0] that this image has *no nontrivial finite normal closed subgroups*; since the $G_i^{\neq}$ are *normal closed subgroups* of G, it thus suffices to prove that the kernel of the *restriction* of ψ_j to the open subgroup of $G \subseteq \Pi_G$ determined by the *direct product* of the $G_{i'}^{=}$ [for $i' = 1, \dots, n$] contains the intersection of this open subgroup with $G_i^{\neq}$, for a *unique* i. But [since open subgroups of H_j are *slim* and *indecomposable* — cf. Propositions 1.4, 3.2] this follows formally from Proposition 3.3. This completes the proof of the *claim*.

Note, moreover, that [in the notation of the *claim*] the assignment $j \mapsto i$ determines a map $\{1, \dots, m\} \to \{1, \dots, n\}$, which, in light of the *injectivity* of α, is easily verified to be *surjective*. But this implies that $m \geq n$; thus, by applying this argument to α^{-1}, we obtain that $m = n$. In particular, the map $\{1, \dots, m\} \to \{1, \dots, n\}$ considered above is a *bijection*, whose inverse we denote by σ. By rearranging the indices, we may assume without loss of generality that σ is the *identity*.

Now it follows from the definition of [the map that gave rise to] σ that we obtain a *surjection*

$$\alpha_i : G_i^{\twoheadrightarrow} \twoheadrightarrow H_i^{\twoheadrightarrow}$$

for each $i = 1, \dots, n$, such that the restriction of [the composite with the inclusion into Π_H of] the *surjection*

$$(\alpha_1, \dots, \alpha_n) : \Big(\Pi_G \supseteq\Big) \prod_{i=1}^{n} (G_i^{\twoheadrightarrow}) \twoheadrightarrow \prod_{i=1}^{n} (H_i^{\twoheadrightarrow}) \ \Big(\subseteq \Pi_H\Big)$$

to G coincides with [the composite with the inclusion into Π_H of] α. In particular, since α is injective, it follows that the kernel of each α_i is a *finite closed normal subgroup* of an open subgroup of G_i. Thus, by the *slimness* of G_i [cf. Proposition 1.4], we conclude [cf. §0] that the α_i are *injective*, as desired. ○

Section 4: Product-theoretic Quotients

In the present §4, we show that in the case of genus ≥ 2, the [closure of the] *commutator subgroup* of a *product-theoretic* open subgroup of a configuration space group is, up to torsion, again *product-theoretic* [cf. Theorem 4.7]. This result, combined with the theory of §3, implies a rather strong result, in the case of genus ≥ 2, concerning the *group-theoreticity* of the various *fiber subgroups* associated to a configuration space group [cf. Corollary 4.8].

Let Y be a *connected smooth variety* over an *algebraically closed field* k which [for simplicity] we assume to be *of characteristic zero.*

Definition 4.1. Let $j \geq 1$ be an integer. Then we shall refer to Y as *j-good* if for every positive integer $j' \leq j$ and every class

$$\eta \in H^{j'}_{\text{ét}}(Y, \mathbb{Z}/N\mathbb{Z})$$

[where "$H^{j'}_{\text{ét}}(-)$" denotes étale cohomology, and $N \geq 1$ is an integer], there exists a finite étale covering $Y' \to Y$ such that $\eta|_{Y'} = 0$.

Remark 4.1.1. As is well-known, it follows immediately from the Hochschild-Serre spectral sequence in étale cohomology [cf., e.g., [Milne], p. 105, Theorem 2.20] that one has a *natural isomorphism*

$$H^{j'}(\pi_1(Y), \widehat{\mathbb{Z}}) \xrightarrow{\sim} H^{j'}_{\text{ét}}(Y, \widehat{\mathbb{Z}})$$

for all nonnegative integers $j' \leq j$ whenever Y is *j-good.* Also, we observe that it is immediate from the definitions that the condition "1-good" is *vacuous.*

Let

$$f : Z \to Y$$

be a *family of hyperbolic curves* over Y; $y \in Y(k)$. We shall denote fibers over y by means of a subscript "y". Suppose that we have also been given a *section*

$$s : Y \to Z$$

of f, whose image we denote by $D_s \subseteq Z$. Write $U_Z \subseteq Z$ for the open subscheme given by the *complement* of D_s; $\mathcal{L} \overset{\text{def}}{=} \mathcal{O}_Z(D_s)$; $\mathbb{L}^\times \to Z$ for the complement of the zero section of the geometric line bundle determined by $\mathcal{L}$;

$$U_Z \to \mathbb{L}^\times$$

for the morphism determined by the natural inclusion $\mathcal{O}_Z \hookrightarrow \mathcal{O}_Z(D_s) = \mathcal{L}$. Thus, $U_Z \to Y$ is also a *family of hyperbolic curves.* Now if we denote by "$\pi_1(-)$" the *étale fundamental group* [for an appropriate choice of basepoint], then we have a *natural commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1((U_Z)_y) \arrow[r] \arrow[d] & \pi_1(U_Z) \arrow[r] \arrow[d] & \pi_1(Y) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(\mathbb{L}^\times_y) \arrow[r] \arrow[d] & \pi_1(\mathbb{L}^\times) \arrow[r] \arrow[d] & \pi_1(Y) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(Z_y) \arrow[r] & \pi_1(Z) \arrow[r] & \pi_1(Y) \arrow[r] & 1
\end{tikzcd}$$

in which the first and third horizontal sequences are *exact* [cf. Proposition 2.2, (iii)]. Write $I_s \subseteq \pi_1(U_Z)$ for the *inertia group* [well-defined up to conjugation in $\pi_1(U_Z)$] associated to the divisor D_s. Thus, $I_s \cong \widehat{\mathbb{Z}}(1)$ [where the "(1)" denotes a *"Tate twist"*].

Lemma 4.2. **(The Line Bundle Associated to a Cusp)** *In the notation of the above discussion, suppose further that* Y *is* ***j*-good**, *for some integer* $j \geq 2$. *Then:*

(i) Z *is* ***j*-good**.

(ii) $\pi_1(\mathbb{L}^\times)$ *fits into an short exact sequence:*

$$1 \to \widehat{\mathbb{Z}}(1) \to \pi_1(\mathbb{L}^\times) \to \pi_1(Z) \to 1$$

Moreover, the resulting extension class $\in H^2(\pi_1(Z), \widehat{\mathbb{Z}}(1)) \cong H^2_{\text{ét}}(Z, \widehat{\mathbb{Z}}(1))$ *[cf. (i); Remark 4.1.1] is the* **first Chern class** *of the line bundle* $\mathcal{L}$.

(iii) The sequence $1 \to \pi_1(\mathbb{L}_y^\times) \to \pi_1(\mathbb{L}^\times) \to \pi_1(Y) \to 1$ *of the above commutative diagram is* **exact**.

(iv) The morphism of fundmental groups $\pi_1(U_Z) \to \pi_1(\mathbb{L}^\times)$ *induces an* **isomorphism** $I_s \xrightarrow{\sim} \text{Ker}(\pi_1(\mathbb{L}^\times) \to \pi_1(Z))$. *In particular, the vertical arrows of the commutative diagram of the above discussion are* **surjections**.

(v) Write $\pi_1(U_Z/Z) \overset{\text{def}}{=} \text{Ker}(\pi_1(U_Z) \twoheadrightarrow \pi_1(Z)) \subseteq \pi_1((U_Z)_y)$. *Then the quotient of* $\pi_1(U_Z/Z)$ *by*

$$\pi_1(U_Z/\mathbb{L}^\times) \overset{\text{def}}{=} \text{Ker}(\pi_1(U_Z) \to \pi_1(\mathbb{L}^\times)) \subseteq \pi_1(U_Z/Z) \quad (\subseteq \pi_1(U_Z))$$

is the **maximal quotient** *of* $\pi_1(U_Z/Z)$ *on which the conjugation action by* $\pi_1((U_Z)_y)$ *is* **trivial**.

Proof. First, we consider assertion (i). In light of the exact sequence $1 \to \pi_1(Z_y) \to \pi_1(Z) \to \pi_1(Y) \to 1$ [together with the Leray-Serre spectral sequence for $Z \to Y$], it follows immediately that to show that Z is j-good, it suffices to show that Z_y is j-good. But this follows immediately from the fact that the cohomological dimension of Z_y is equal to 1 when Z_y is *affine* [cf., e.g., [Milne], p. 253, Theorem 7.2] and from the well-known isomorphism $H^2_{\text{ét}}(Z_y, \mathbb{Z}/N\mathbb{Z}) \cong (\mathbb{Z}/N\mathbb{Z})(-1)$ determined by considering fundamental classes of points [together with the fact that the cohomological dimension of Z_y is equal to 2 — cf., e.g., [Milne], p. 276, Theorem 11.1], when Z_y is *proper.* This completes the proof of assertion (i).

In light of assertion (i), assertion (ii) follows from [Mzk2], Lemmas 4.4, 4.5. Assertion (iii) follows immediately by considering the natural commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \widehat{\mathbb{Z}}(1) \arrow[r] \arrow[d] & \pi_1(\mathbb{L}_y^\times) \arrow[r] \arrow[d] & \pi_1(Z_y) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \widehat{\mathbb{Z}}(1) \arrow[r] & \pi_1(\mathbb{L}^\times) \arrow[r] & \pi_1(Z) \arrow[r] & 1
\end{tikzcd}$$

[in which the rows are *exact*, by assertion (ii); the vertical arrow on the left is an *isomorphism*], together with the exact sequence $1 \to \pi_1(Z_y) \to \pi_1(Z) \to \pi_1(Y) \to 1$. Assertion (iv) (respectively, (v)) follows immediately from the argument of the proof of [Mzk4], Lemma 4.2, (ii) (respectively, [Mzk4], Lemma 4.2, (iii)). $\bigcirc$

Now let l be a *prime number*; suppose that Y is 2-*good*. Also, let us suppose that, for $i = 1, \dots, m$ [where $m \geq 1$ is an integer], we have been given a *section*

$$s_i : Y \to Z$$

of f, whose image we denote by $D_{s_i} \subseteq Z$. Write $U_i \subseteq Z$ for the open subscheme given by the *complement* of D_{s_i}; $W_Z \stackrel{\mathrm{def}}{=} \bigcap_{i=1}^m U_i \subseteq Z$; $\mathcal{L}_i \stackrel{\mathrm{def}}{=} \mathcal{O}_Z(D_{s_i})$; $\mathbb{L}_i^\times \to Z$ for the complement of the zero section of the geometric line bundle determined by $\mathcal{L}_i$;

$$W_Z \to \mathbb{L}_i^\times$$

for the morphism determined by the natural inclusion $\mathcal{O}_Z \hookrightarrow \mathcal{O}_Z(D_{s_i}) = \mathcal{L}_i$. Also, let us suppose that $W_Z \to Y$ is a *family of hyperbolic curves* [i.e., that the images of the s_i do not intersect]. By forming the quotient of the exact sequence of Lemma 4.2, (ii), by the pro-prime-to-l portion of $\widehat{\mathbb{Z}}(1)$, we obtain *extensions*

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_l(1) \arrow[r] \arrow[d] & \mathbb{E}_{i,y} \arrow[r] \arrow[d] & \pi_1(Z_y) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z}_l(1) \arrow[r] & \mathbb{E}_i \arrow[r] & \pi_1(Z) \arrow[r] & 1
\end{tikzcd}$$

for $i = 1, \dots, m$. Also, let us write

$$\kappa_i \in H^2_{\text{ét}}(Z, \mathbb{Z}_l(1))$$

for the *fundamental class* associated to D_{s_i} [i.e., the *first Chern class* of the line bundle $\mathcal{L}_i$ — cf. Lemma 4.2, (ii)].

Lemma 4.3. **(Multi-section Splittings)** *In the notation of the above discussion:*

(i) The natural homomorphism

$$\pi_1(W_Z) \to \prod_{i=1}^m \mathbb{E}_i$$

[where the product is a fiber product over $\pi_1(Z)$] is **surjective**.

(ii) The natural quotient $\pi_1(W_Z) \twoheadrightarrow \pi_1(W_Z)^{\mathrm{ab}} \otimes \mathbb{Z}_l$ **factors** *through the quotient determined by the surjection of (i).*

(iii) For $i = 1, \ldots, m$, let $\lambda_i \in \mathbb{Z}_l$. Then there **exists** *a surjection $\pi_1(W_Z) \twoheadrightarrow \mathbb{Z}_l(1)$ — which, by (ii), necessarily* **factors** *through the surjection of (i), hence determines a* **surjection**

$$\prod_{i=1}^{m} \mathbb{E}_i \twoheadrightarrow \mathbb{Z}_l(1)$$

— that restricts to multiplication by λ_i on the copy of $\mathbb{Z}_l(1)$ in $\mathbb{E}_i$ if and only if the class

$$\sum_{i=1}^{m} \lambda_i \cdot \kappa_i \in H^2_{\text{ét}}(Z, \mathbb{Z}_l(1))$$

vanishes.

Proof. First, we consider assertion (i). In light of the exact sequences of Proposition 2.2, (iii), and Lemma 4.2, (iii), it suffices to show the surjectivity of $\pi_1((W_Z)_y) \to \prod_{i=1}^{m} \mathbb{E}_{i,y}$. But this follows immediately, in light of Lemma 4.2, (iv), by considering the various *inertia groups* $\subseteq \pi_1((W_Z)_y)$ of the cusps of $(W_Z)_y$. This completes the proof of assertion (i). Assertion (ii) follows immediately, in light of Lemma 4.2, (iv), from the fact that the kernel of the natural surjection $\pi_1(W_Z) \twoheadrightarrow \pi_1(Z)$ is *topologically normally generated* by the *inertia groups of cusps*. Finally, we observe that assertion (iii) follows immediately from the definitions. ○

Lemma 4.4. (The Section Arising from the Graph of a Morphism) *In the notation of the above discussion, suppose further that $Z \to Y$ is given by the projection to the second factor $C \times_k C \to C$, where we write $C \overset{\text{def}}{=} Z_y$, that C is* **proper**, *and that $s : Y \to Z$ is given by the* **graph** *of a k-morphism $\sigma : C \to C$. Then the component of the first Chern class of $\mathcal{L}$ in the middle direct summand of*

$$H^2_{\text{ét}}(Z, \mathbb{Z}_l(1)) \cong H^2_{\text{ét}}(C, \mathbb{Z}_l(1)) \oplus (H^1_{\text{ét}}(C, \mathbb{Z}_l) \otimes H^1_{\text{ét}}(C, \mathbb{Z}_l(1))) \oplus H^2_{\text{ét}}(C, \mathbb{Z}_l(1))$$

[cf. the Künneth isomorphism in étale cohomology, discussed, e.g., in [Milne], p. 258, Theorem 8.5] is given by applying the endomorphism $\sigma^* \otimes \text{id}$ of the module $H^1_{\text{ét}}(C, \mathbb{Z}_l) \otimes H^1_{\text{ét}}(C, \mathbb{Z}_l(1))$ to the inverse of the bilinear form arising from the cup product $H^1_{\text{ét}}(C, \mathbb{Z}_l) \otimes H^1_{\text{ét}}(C, \mathbb{Z}_l(1)) \to H^2_{\text{ét}}(C, \mathbb{Z}_l(1)) \cong \mathbb{Z}_l$ in étale cohomology.

Proof. Indeed, this follows immediately from [Milne], p. 287, Lemma 12.2. ○

Lemma 4.5. (Linear Independence for Vector Spaces) *Let G be a* **finite group**, *whose order we denote by $|G|$; K a field; V a finite-dimensional K-vector equipped with a linear action by G such that the G-module V contains the* **regular representation** *of G as a direct summand; $N \geq 1$ an integer. Write*

$$W \overset{\text{def}}{=} V \oplus \ldots \oplus V$$

for the direct sum of N copies of V; $\iota_i \in \text{Hom}_K(V, W)$ [where $i = 1, \ldots, N$] for the inclusion $V \hookrightarrow W$ into the i-th factor. Then the $N \cdot |G|$ elements

$$\iota_i \circ g$$

[where $i = 1, \dots, N$; $g \in G$] of $\mathrm{Hom}_K(V, W)$ *are* **linearly independent**.

Proof. Indeed, any nontrivial linear relation between these elements implies — by applying the various linear morphisms $\mathrm{Hom}_K(V, W) \to \mathrm{Hom}_K(V, V)$ obtained by *projecting* onto the various factors of V in W — a nontrivial linear relation between the endomorphisms $\in \mathrm{Hom}_K(V, V)$ determined by the elements of G, in contradiction to the assumption that the G-module V contains the *regular representation* of G as a direct summand. ○

Lemma 4.6. **(Linear Independence for Configuration Spaces)** *In the notation of the above discussion, suppose further that:*

(a) there exists a commutative diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & Y \arrow[d] \\
X \times_k X_n \arrow[r] & X_n
\end{tikzcd}$$

where the upper horizontal arrow is the given morphism $Z \to Y$; the lower horizontal arrow is the projection to the second factor; $n \geq 1$ is an integer; X_n is the ***n*-th configuration space** *associated to some* **hyperbolic curve** *X over k; the vertical arrows are* **finite étale Galois coverings** *arising from the coverings of $X \times_k X_n$, X_n determined by taking the direct product of copies of a finite étale Galois covering $Z_0 \to X$ [so Z_y may be* **identified** *with Z_0];*

(b) the **genus** *of the* **compactification** *B of X is ≥ 2;*

(c) if we write $C \to B$ for the **normalization** *of B in Z_0, then we have $m = n \cdot \deg(C/B)$, and the $s_i : Y \to Z$ are the various liftings of the* ***n* tautological sections** *$X_n \to X \times_k X_n$ arising from the definition of the configuration space X_n.*

[Thus, the fact that $W_Z \to Y$ is a **family of hyperbolic curves** *follows immediately from Remark 2.1.2; the fact that X_n, hence also Y, is* **2-good** *follows, by induction on n, from Lemma 4.2, (i). Moreover, W_Z forms a finite étale covering of X_{n+1} that arises from a* **product-theoretic** *open subgroup of $\pi_1(X_{n+1})$.] Then the images of the κ_i in $H^2_{\text{ét}}(Z, \mathbb{Q}_l(1))$ are* **linearly independent** *[over $\mathbb{Q}_l$].*

Proof. Note that the projection to the first factor $X \times_k X_n \to X$ determines a morphism $Z \to Z_0$ ($\subseteq C$). Suppose that the section $s : Y \to Z$ arises from a point $\in Z_0(k)$. Write $\kappa \in H^2_{\text{ét}}(Z, \mathbb{Z}_l(1))$ for the *fundamental class* associated to D_s. For $i = 1, \dots, m$, set $\kappa'_i \overset{\text{def}}{=} \kappa_i - \kappa$. Note that since κ and the κ_i all map [cf. the Leray-Serre spectral sequence for $Z \to Y$] to the *same element* of $H^2_{\text{ét}}(Z_y, \mathbb{Q}_l(1))$ — a $\mathbb{Q}_l$-vector space of dimension 0 [cf., e.g., [Milne], p. 253, Theorem 7.2] if Z_y is *affine* and dimension 1 [cf., e.g., [Milne], p. 276, Theorem 11.1, (a)] if Z_y is *proper* — it follows that κ'_i maps to 0 in $H^2_{\text{ét}}(Z_y, \mathbb{Q}_l(1))$. Thus, we conclude that the κ'_i determine classes

$$\eta_i \in H^1_{\text{ét}}(Y, H^1_{\text{ét}}(Z_y, \mathbb{Q}_l(1))) \cong H^1_{\text{ét}}(Y, \mathbb{Q}_l) \otimes H^1_{\text{ét}}(Z_0, \mathbb{Q}_l(1))$$

[cf. the Leray-Serre spectral sequence for $Z \to Y$], and that to show the linear independence of the images of the κ_i in $H^2_{\text{ét}}(Z, \mathbb{Q}_l(1))$, it suffices to verify that the η_i are *linearly independent.*

On the other hand, it follows immediately from the definitions that the κ_i arise as pull-backs via the various projections $Y \to Z_0 \hookrightarrow C$, $Z \to Z_0 \hookrightarrow C$ of the classes [cf. Lemma 4.4] determined by the *graphs* $\subseteq C \times_k C$ of the various $\sigma : C \to C$, for $\sigma \in \text{Gal}(C/B)$. In particular, the η_i arise as pull-backs via these various projections of the classes in

$$H^1_{\text{ét}}(C, \mathbb{Q}_l) \otimes H^1_{\text{ét}}(C, \mathbb{Q}_l(1)) \ (\hookrightarrow H^1_{\text{ét}}(Y, \mathbb{Q}_l) \otimes H^1_{\text{ét}}(Z_0, \mathbb{Q}_l(1)))$$

determined [cf. Lemma 4.4] by the *graphs* of the various $\sigma \in G \overset{\text{def}}{=} \text{Gal}(C/B)$. On the other hand, by Proposition 1.3 [cf. our assumption that the genus of B is ≥ 2!], it follows that the G-module $V \overset{\text{def}}{=} H^1_{\text{ét}}(C, \mathbb{Q}_l)$ contains the *regular representation* of G as a direct summand. Note, moreover, that the n inclusions $V = H^1_{\text{ét}}(C, \mathbb{Q}_l) \hookrightarrow H^1_{\text{ét}}(Y, \mathbb{Q}_l)$ [determined up to composition with the action of G on V] arising from the n projections $X_n \to X$ determine a map of $\mathbb{Q}_l$-vector spaces

$$\Big(\bigoplus H^1_{\text{ét}}(C, \mathbb{Q}_l) \Big) \to H^1_{\text{ét}}(Y, \mathbb{Q}_l)$$

[where the direct sum is over n copies of $H^1_{\text{ét}}(C, \mathbb{Q}_l)$] which is *injective.* Thus, we are, in effect, in the situation of Lemma 4.5, so the linear independence of the η_i follows from the linear independence asserted in Lemma 4.5. ◯

Theorem 4.7. (Strongly Torsion-free Pro-solvable Product-theoreticity) *Let X be a* **hyperbolic curve** *of* **genus** ≥ 2 *over an* **algebraically closed field** k **of characteristic zero***; $n \geq 1$ an integer; X_n the* **n-th configuration space** *associated to X; $H \subseteq \pi_1(X_n)$ a* **product-theoretic open subgroup***; G a* **strongly torsion-free pro-solvable profinite group**. *Then the kernel of any continuous homomorphism*

$$H \to G$$

is **product-theoretic**.

Proof. First, we *claim* that it suffices to verify Theorem 4.7 in the case where $G = \mathbb{Z}_l$ [for some prime number l]. Indeed, since G is topologically finitely generated [cf. Definition 1.1, (iii)], and arbitrary intersections of product-theoretic closed subgroups of $\pi_1(X_n)$ are clearly product-theoretic, Theorem 4.7 for arbitrary [torsion-free] *abelian* G follows immediately from the case "$G = \mathbb{Z}_l$". Thus, by replacing H, G *successively* by appropriate open subgroups of H, G, Theorem

4.7 for arbitrary [strongly torsion-free] *pro-solvable* G follows immediately from the [torsion-free] abelian case. This completes the proof of the *claim*. Thus, in the following, we assume that $G = \mathbb{Z}_l$.

Now observe that Theorem 4.7 is vacuous for $n = 1$. Thus, by induction on n, it suffices to verify Theorem 4.7 for "$n+1$" under the assumption that it holds for "n". Next, let us observe that it follows immediately from the definition of "product-theoretic" that any covering of X_{n+1} that arises from a product-theoretic open subgroup $J \subseteq \pi_1(X_{n+1})$ is *dominated* by a covering of the form "$W_Z \rightarrow X_{n+1}$" for W_Z as in Lemma 4.6. Thus, [since $G = \mathbb{Z}_l$ is *torsion-free*] we may assume, without loss of generality, that J corresponds to a covering "$W_Z \rightarrow X_{n+1}$" for W_Z as in Lemma 4.6. In particular, by applying Lemma 4.3, (iii), to arbitrary quotients $J \twoheadrightarrow J^{\mathrm{ab}} \otimes \mathbb{Z}_l \twoheadrightarrow \mathbb{Z}_l$, in light of the *linear independence* asserted in Lemma 4.6, and the *induction hypothesis*, we conclude that the kernel of such a quotient $J \twoheadrightarrow \mathbb{Z}_l$ is *product-theoretic*, as desired. ○

Remark 4.7.1. Note that Theorem 4.7 is *false* if the genus of X is < 2 and $n \geq 2$. Indeed, to construct a counter-example for arbitrary $n \geq 2$, it suffices to construct a counter-example for $n = 2$. If, moreover, U is the hyperbolic curve determined by an open subscheme of X, then consideration of the natural morphism $U_2 \hookrightarrow X_2$ shows that the existence of a counter-example for X_2 implies the existence of a counter-example for U_2. Thus, we may assume, without loss of generality, that $n = 2$, and X is either of type $(0, 3)$ or $(1, 1)$. But in either of these cases, it is well-known that there exists a dominant map $X_2 \rightarrow X$ that extends to a map $X \times_k X \rightarrow C$ [where C is a compactification of X] that maps the diagonal of $X \times_k X$ to a single point of C. Thus, by pulling back appropriate infinite cyclic coverings of finite étale coverings of X, one obtains infinite cyclic coverings of finite étale coverings of $X \times_k X$ that are [infinitely] ramified over the diagonal of $X \times_k X$.

Corollary 4.8. (Group-theoreticity of Projections of Configuration Spaces I) *Let $\mathcal{C}$ be a* **PT-formation**. *For $\Box = \alpha, \beta$, let $X^{\Box}$ be a* **hyperbolic curve** *of* **genus ≥ 2** *over an* **algebraically closed field $k_{\Box}$ of characteristic zero**; *$n_{\Box} \geq 1$ an integer; $X^{\Box}_{n_{\Box}}$ the* **$n_{\Box}$-th configuration space** *associated to $X^{\Box}$; $E_{\Box}$ the* **index set** *of $X^{\Box}_{n_{\Box}}$; $H_{\Box} \subseteq \Pi^{\Box} \stackrel{\mathrm{def}}{=} \pi_1^{\mathcal{C}}(X^{\Box}_{n_{\Box}})$ a* **product-theoretic open subgroup**. *Let*

$$\gamma : H_{\alpha} \xrightarrow{\sim} H_{\beta}$$

be an **isomorphism of profinite groups**. *Then γ induces a* **bijection** *$\sigma : E_{\alpha} \xrightarrow{\sim} E_{\beta}$ [so $n_{\alpha} = n_{\beta}$] such that*

$$\gamma(F_{\alpha} \bigcap H_{\alpha}) = F_{\beta} \bigcap H_{\beta}$$

for all **fiber subgroups** *$F_{\alpha} \subseteq \Pi^{\alpha}$, $F_{\beta} \subseteq \Pi^{\beta}$, whose respective* **profiles** *$E'_{\alpha} \subseteq E_{\alpha}$, $E'_{\beta} \subseteq E_{\beta}$ correspond via σ.*

Proof. First, let us observe that to complete the proof of Corollary 4.8, it suffices construction a *bijection* $\sigma : E_{\alpha} \xrightarrow{\sim} E_{\beta}$ [so $n_{\alpha} = n_{\beta}$] such that

$$\gamma(F_{\alpha} \bigcap H_{\alpha}) = F_{\beta} \bigcap H_{\beta}$$

for all *fiber subgroups* $F_\alpha \subseteq \Pi^\alpha$, $F_\beta \subseteq \Pi^\beta$ *of co-length one* whose respective *profiles* correspond via σ. Indeed, this follows immediately by applying *induction* on $n \stackrel{\mathrm{def}}{=} n_\alpha = n_\beta$ [cf. also Proposition 2.4, (i), (ii)].

Next, for $j = 1, \dots, n_\Box$, let us write

$$K_j^\Box \subseteq H_\Box$$

for the intersection with $H_\Box$ of the *fiber subgroup* $\subseteq \Pi^\Box$ *of co-length one* with co-profile given by the element of $E_\Box$ labeled by j. Thus, [cf. Proposition 2.4, (iv); the fact that fiber subgroups of co-length one are *normal closed subgroups of infinite index*] for *distinct* $j, j' \in \{1, \dots, n_\Box\}$, $K_j^\Box$, $K_{j'}^\Box$ topologically generate an *open subgroup* of $\Pi^\Box$; in particular, $K_j^\Box$ is *not* contained in $K_{j'}^\Box$.

Now we *claim* that to complete the proof of Corollary 4.8, it suffices to prove that the following *statement* holds [in general]:

For each $i \in E_\alpha$, there exists a $j \in E_\beta$ such that $K_j^\beta \subseteq \gamma(K_i^\alpha)$.

Indeed, by applying this statement to γ, γ^{-1}, we conclude that for each $i \in E_\alpha$, there exist $j \in E_\beta$, $i' \in E_\alpha$ such that $\gamma(K_{i'}^\alpha) \subseteq K_j^\beta \subseteq \gamma(K_i^\alpha)$, hence that $K_{i'}^\alpha \subseteq K_i^\alpha$. But, as observed above, this implies that $i' = i$, hence that $K_j^\beta = \gamma(K_i^\alpha)$. Moreover, this relation "$K_j^\beta = \gamma(K_i^\alpha)$" determines an assignment $i \mapsto j$, hence a mapping $\sigma : E_\alpha \to E_\beta$, which is a *bijection*, relative to which intersections with H_α, H_β of fiber subgroups of co-length one with corresponding profiles correspond via γ. This completes the proof of the *claim*.

To verify the *"statement"* of the above *claim*, we reason as follows: Let $l \in \Sigma_{\mathcal{C}}$. Write $H_\alpha/K_i^\alpha \twoheadrightarrow G$ for the *maximal pro-l quotient* of H_α/K_i^α;

$$\phi : H_\beta \xrightarrow{\sim} H_\alpha \twoheadrightarrow H_\alpha/K_i^\alpha \twoheadrightarrow G$$

for the surjection determined by γ^{-1}. Then observe that since H_α/K_i^α is a *pro-$\mathcal{C}$ surface group*, it follows that G is a *pro-l surface group*, hence *strongly torsion-free* [cf. Remark 1.2.2] and *pro-solvable* [cf. Remark 1.1.3]. Thus, it follows from Theorem 4.7 that ϕ factors through the quotient $H_\beta \twoheadrightarrow Q_\beta$ determined by the quotient "$\pi_1^{\mathcal{C}}(X_{n_\beta}^\beta) \twoheadrightarrow \pi_1^{\mathcal{C}}(P_{n_\beta}^\beta)$" [i.e., the image of $H_\beta \subseteq \pi_1^{\mathcal{C}}(X_{n_\beta}^\beta)$ in $\pi_1^{\mathcal{C}}(P_{n_\beta}^\beta)$] corresponding to the quotient that was denoted "$\pi_1^{\mathcal{C}}(X_n) \twoheadrightarrow \pi_1^{\mathcal{C}}(P_n)$" in Definition 2.3, (ii). In particular, since Q_β admits an open subgroup with a direct product decomposition induced by the natural direct product decomposition of $\pi_1^{\mathcal{C}}(P_{n_\beta}^\beta)$, it thus follows [since open subgroups of G are *slim* and *indecomposable* — cf. Propositions 1.4, 3.2] from Proposition 3.3 that there exists a $j \in E_\beta$ such that the image of K_j^β in G is a *finite normal closed subgroup*, hence *trivial* [since G is *slim* — cf. Proposition 1.4, §0].

Next, let us observe that by applying the above argument to *arbitrary open subgroups* of H_α, H_β that correspond via γ, we conclude that the image of K_j^β in an *arbitrary almost pro-l-maximal quotient* $H_\alpha/K_i^\alpha \twoheadrightarrow J$ [so J is an *almost pro-l-surface group*!] is a *finite normal closed subgroup*, hence *trivial* [since J is *slim* — cf. Proposition 1.4, §0]. On the other hand, since the natural homomorphism

$$H_\alpha/K_i^\alpha \to \varprojlim_J J$$

[where J ranges over the *almost pro-l-maximal quotients* $H_\alpha/K_i^\alpha \twoheadrightarrow J$ of H_α/K_i^α] is clearly an *isomorphism*, we thus conclude that the image of K_j^β in H_α/K_i^α is *trivial*, as desired. This completes the proof of Corollary 4.8. ○

Remark 4.8.1. Observe that the proof of Corollary 4.8 given above is *entirely free of any dependence* on Theorem 1.5, hence, in particular, on the *theorem of Lubotzky-Melnikov-van den Dries* [cf. Remark 3.3.2]. Put another way, the somewhat complicated geometric computations performed in the present §4 may be regarded as a sort of *"substitute for a certain portion"* of the "group-theoretic content" of the theorem of Lubotzky-Melnikov-van den Dries.

Remark 4.8.2. The original motivation, for the first author, for developing the theory applied to prove Corollary 4.8 was the idea that by combining Corollary 4.8 with the techniques of [Mzk5], [Mzk6], one could apply Corollary 4.8 to obtain results in the *absolute anabelian geometry of configuration spaces over p-adic local fields.* It is the intention of the first author to carry out this application of Corollary 4.8 in a subsequent paper.

Section 5: Divisors and Units on Coverings of Configuration Spaces

In the present §5, we discuss a certain *generalization* [cf. Theorem 5.6; Remark 5.6.1] of Theorem 4.7 [due to the second author]. Unlike the proof of Theorem 4.7 given in §4, the proof of this generalization does not rely on the notion of *"goodness"* or properties involving the *"regular representation"*. In this sense, the approach given in the present §5 is *more efficient* and relies on direct *algebro-geometric properties* — such as the *disjointness of divisors* — of which the properties involving the *"regular representation"* applied in §4 may be thought of as a sort of *"étale-topological translation"*. On the other hand, the approach of §4 [which was discovered first, by the first author], though less efficient, has the virtue of relying on *explicit group-theoretic manifestations* of these algebro-geometric properties; it was this explicitness that served to render the approach of §4 more readily accessible to the intuition of the first author. Finally, we discuss certain consequences [cf. Corollary 5.7] of the theory of the present §5 concerning the *"non-existence of units"* on a sufficiently generic hyperbolic curve.

We begin by reviewing some essentially well-known *generalities concerning log schemes.*

Definition 5.1. Let $X^{\log}$ be a *fine log scheme* [cf. [Kato1]].

(i) Denote by M_X the étale sheaf of monoids on X that defines the log structure on $X^{\log}$. Thus, we have a natural injection $\mathcal{O}_X^{\times} \hookrightarrow M_X$, which we shall use to regard $\mathcal{O}_X^{\times}$ as a subsheaf of M_X. We shall refer to the quotient sheaf of monoids

$$M_X^{\mathrm{char}} \overset{\mathrm{def}}{=} M_X/\mathcal{O}_X^{\times}$$

as the *characteristic* of $X^{\log}$ and to the associated sheaf of groupifications

$$M_X^{\text{char-gp}}$$

as the *group-characteristic* of $X^{\log}$. Thus, [since $X^{\log}$ is *fine*] the fibers of M_X^{char} (respectively, $M_X^{\text{char-gp}}$) are finitely generated torsion-free abelian monoids (respectively, abelian groups). For $n \in \mathbb{N}$, we shall denote by

$$U_X^{[n]} \subseteq X$$

and refer to as the *n-interior* of $X^{\log}$ the subset [cf. Proposition 5.2, (i), (ii) below] of points [of the scheme X lying under geometric points of the scheme X] at which the fiber of $M_X^{\text{char-gp}}$ is of *rank* $\leq n$. Thus, $U_X^{[0]}$ is the *interior* $U_X \subseteq X$ *of* $X^{\log}$ [i.e., the open subscheme of points at which the log structure of $X^{\log}$ is *trivial*].

(ii) Let M be a finitely generated [abstract] abelian monoid; $N \geq 1$ an integer. We shall say that M is *$\mathbb{Q}$-regular [with exponent N]* if for some $n \in \mathbb{N}$, the map

$$\mathbb{N}^n \to \mathbb{N}^n$$

[where $\mathbb{N}^n$ is the monoid determined by the product of n copies of $\mathbb{N}$] given by multiplication by N *factors* as a composite of injections of monoids $\mathbb{N}^n \hookrightarrow M \hookrightarrow \mathbb{N}^n$. We shall say that $X^{\log}$ is *weakly $\mathbb{Q}$-regular* (respectively, *strongly $\mathbb{Q}$-regular*) if, for every geometric point $\overline{x}$ of X, the fiber of $M_{\overline{x}}^{\mathrm{char}}$ at $\overline{x}$ is a $\mathbb{Q}$-regular monoid (respectively, $\mathbb{Q}$-regular monoid with exponent invertible in the residue field of $\overline{x}$).

Proposition 5.2. **(Generalities on Log Schemes)** *Let $X^{\log}$ be a* **fine log scheme***; $n \in \mathbb{N}$; l a prime number invertible on X. Then:*

(i) The ***n*-interior** *$U_X^{[n]} \subseteq X$ is* **open***.*

(ii) Suppose that $X^{\log}$ is **log regular***. Then the* **complement** *of the n-interior $U_X^{[n]}$ is a closed subset of X of* **codimension** *$> n$.*

(iii) Suppose that $X^{\log}$ is **log regular** *and* **weakly $\mathbb{Q}$-regular***. Then X is* **locally $\mathbb{Q}$-factorial** *[i.e., every Weil divisor on X admits a positive multiple which is Cartier].*

(iv) Suppose that $X^{\log}$ *is* **log smooth** *over a field* k *[equipped with the trivial log structure] and* **strongly** $\mathbb{Q}$**-regular***; let* $F \subseteq X$ *be a closed subset of* **codimension** $\geq n$*. Then the natural map on étale cohomology*

$$H^j_{\text{ét}}(X, \mathbb{Q}_l) \to H^j_{\text{ét}}(X \backslash F, \mathbb{Q}_l)$$

is an **isomorphism** *for* $j \leq 2n-2$ *and an* **injection** *for* $j = 2n-1$*.*

(v) Under the assumptions of (iv), suppose further that X *is* **connected***; write*

$$D_X = \bigcup_{i \in I} D_{X,i}$$

— where I *is a finite set; the* $D_{X,i} \subseteq X$ *are* **irreducible** *divisors — for the complement* $X \backslash U_X$ *[equipped with the reduced induced scheme structure]. [Thus, we have a natural surjection of étale fundamental groups*

$$\pi_1(U_X) \twoheadrightarrow \pi_1(X)$$

whose kernel contains the **inertia groups** *of the* $D_{X,i}$*; the maximal pro-*l *quotient of each of these inertia groups is naturally isomorphic to* $\mathbb{Z}_l(1)$*.] Then we have a* **natural exact sequence**

$$\begin{aligned} 0 \to \text{Hom}(\pi_1(X), \mathbb{Q}_l(1)) &\to \text{Hom}(\pi_1(U_X), \mathbb{Q}_l(1)) \\ &\to \bigoplus_{i \in I} \mathbb{Q}_l \to H^2_{\text{ét}}(X, \mathbb{Q}_l(1)) \to H^2_{\text{ét}}(U_X, \mathbb{Q}_l(1)) \end{aligned}$$

*— where the "*Hom*'s" denote the modules of continuous homomorphisms of topological groups; the second arrow is the arrow determined by the natural surjection* $\pi_1(U_X) \twoheadrightarrow \pi_1(X)$*; the third arrow is the arrow determined by the* **inertia groups** *of the* $D_{X,i}$ *[and the natural identification of* $\mathbb{Q}_l$ *with* $\text{Hom}(\mathbb{Z}_l(1), \mathbb{Q}_l(1))$*]; the fourth arrow is the arrow that sends the* $1 \in \mathbb{Q}_l$ *in the direct summand labeled "*i*" to the* **fundamental class** $c(D_{X_i})$ *of the Weil divisor* $D_{X,i}$ *[which is well-defined, by (iii)].*

Proof. Indeed, assertion (i) follows immediately from the definition of a *fine* log scheme [cf. [Kato1], §2.1-3]. In light of assertion (i), assertion (ii) follows immediately from the inequality $\dim(\mathcal{O}_{X,\overline{x}}) \geq \text{rank}_{\mathbb{Z}}((M_X^{\text{char-gp}})_{\overline{x}})$ [where $\overline{x}$ is a geometric point of X; $\mathcal{O}_{X,\overline{x}}$ is the corresponding strict henselization of a local ring of X] — cf. the definition of "log regular" in [Kato2], Definition 2.1.

To verify assertion (iii) (respectively, (iv)), let us first observe that it follows immediately from our assumptions that $X^{\log}$ is *log regular* (respectively, *log smooth* over k) and *weakly* $\mathbb{Q}$*-regular* (respectively, *strongly* $\mathbb{Q}$*-regular*) that every point of X admits an étale neighborhood $V \to X$ such that there exists a finite (respectively, Kummer log étale) dominant morphism $W^{\log} \to V^{\log}$ [where we equip V with the log structure pulled back from X] such that the scheme W is *regular* (respectively, *smooth* over k) and connected, and the log structure of $W^{\log}$ arises from a *divisor* *with normal crossings on* W. Now assertion (iii) follows immediately by pulling back a given Weil divisor on X to the *regular* scheme W [which yields a *Cartier* divisor on W] and then pushing forward via $W \to V$ [which multiplies the original divisor on V by the degree of the morphism $W \to V$].

To verify assertion (iv), let us first observe that assertion (iv) holds when X is *smooth* over k. Indeed, in this case, by applying *noetherian induction* to F and possibly base-changing to a finite inseparable extension of k, we may assume without loss of generality that F is *smooth* over k; but then the content of assertion (iv) is well-known [cf., e.g., [Milne], p. 244, Remark 5.4, (b)]. In the case of *arbitrary* $X^{\log}$, we argue as follows: Since we have already verified assertion (iv) for k-smooth X, we may assume that $F \bigcap U_X = \emptyset$. Write $\iota : X_F \overset{\text{def}}{=} X \backslash F \hookrightarrow X$ for the natural inclusion. Then [by applying a well-known exact sequence in étale cohomology] it suffices to verify that $\mathbb{R}^j \iota_{\text{ét},*}(\mathbb{Q}_l) = 0$ for $0 < j \leq 2n-1$. In particular, it suffices [cf. [Milne], p. 88, Theorem 1.15] to verify, for an arbitrary *strictly henselization* $\underline{V}$ of V at a closed point of V, that $H^j(\underline{V}_F, \mathbb{Q}_l) = 0$ for $0 < j \leq 2n-1$ [where we write $\underline{V}_F \overset{\text{def}}{=} X_F \times_X \underline{V}$]. On the other hand, let us observe that since $\zeta^{\log} : \underline{W}_F^{\log} \overset{\text{def}}{=} \underline{V}_F \times_V W^{\log} \to \underline{V}_F^{\log} \overset{\text{def}}{=} \underline{V}_F \times_V V^{\log}$ is *Kummer log étale*, it follows that one may define a *"trace morphism"*

$$\zeta_{\text{ét},*}((\mathbb{Q}_l)_{\underline{W}_F}) \to (\mathbb{Q}_l)_{\underline{V}_F}$$

[where we use the subscripts "$\underline{W}_F$", "$\underline{V}_F$" to denote the constant sheaf on $\underline{W}_F$, $\underline{V}_F$] that restricts, relative to $\zeta^*_{\text{ét}}$, to multiplication by the degree $\deg(\zeta)$ of ζ on $(\mathbb{Q}_l)_{\underline{V}_F}$. [Indeed, this is immediate for the restriction $\zeta_U : U_{\underline{W}_F} \to U_{\underline{V}_F}$ to the respective interiors, since this restriction is *finite étale*. On the other hand, since $\underline{V}_F$ is *normal*, we have a *natural isomorphism* $(\mathbb{Q}_l)_{\underline{V}_F} \xrightarrow{\sim} \theta_*\theta^*((\mathbb{Q}_l)_{\underline{V}_F})$, where we write $\theta : U_{\underline{V}_F} \hookrightarrow \underline{V}_F$ for the natural inclusion of the interior. Thus, we obtain a trace morphism as desired by restricting to the interiors, applying the trace morphism on the interiors, and then applying this natural isomorphism.] Thus, by taking étale cohomology, one obtains a *trace morphism* $\tau : H^j(\underline{W}_F, \mathbb{Q}_l) \to H^j(\underline{V}_F, \mathbb{Q}_l)$ such that the composite $\tau \circ \rho$ with the restriction morphism $\rho : H^j(\underline{V}_F, \mathbb{Q}_l) \to H^j(\underline{W}_F, \mathbb{Q}_l)$ is equal to multiplication by $\deg(\zeta)$ on $H^j(\underline{V}_F, \mathbb{Q}_l)$. Since, moreover, we have already verified assertion (iv) for k-smooth X, it follows that $H^j(\underline{W}_F, \mathbb{Q}_l) = 0$ for $0 < j \leq 2n-1$, hence that $H^j(\underline{V}_F, \mathbb{Q}_l) = 0$ for $0 < j \leq 2n-1$, as desired.

Finally, we consider assertion (v). When $X = U_X^{[1]}$ [so X, D_X are *smooth* over k], assertion (v) follows immediately by applying the well-known *Gysin sequence* in étale cohomology [cf., e.g., [Milne], p. 244, Remark 5.4, (b)]

$$0 \to H^1_{\text{ét}}(X, \mathbb{Q}_l(1)) \to H^1_{\text{ét}}(U_X, \mathbb{Q}_l(1)) \to \bigoplus_{i \in I} \mathbb{Q}_l \to H^2_{\text{ét}}(X, \mathbb{Q}_l(1)) \to H^2_{\text{ét}}(U_X, \mathbb{Q}_l(1))$$

and the natural isomorphisms $H^1_{\text{ét}}((-), \mathbb{Q}_l(1)) \cong \text{Hom}(\pi_1((-)), \mathbb{Q}_l(1))$, for "$(-)$" equal to X, U_X. For *arbitrary* $X^{\log}$, we reduce immediately to the case where $X = U_X^{[1]}$ by applying assertions (ii), (iv). ○

Remark 5.2.1. We recall in passing that Proposition 5.2, (iii), is *false* for arbitrary [not necessarily weakly $\mathbb{Q}$-regular] log regular $X^{\log}$. Indeed, such an example appears in the Remark following [Mzk1], Corollary 1.8.

Now we return to our discussion of *configuration spaces*. Let X be a *proper hyperbolic curve of genus* g_X over an *algebraically closed field* k *of characteristic zero*, $n \geq 1$ an integer, l a prime number; write $X_n \subseteq P_n$ for the associated *n-th configuration space*, $Z_n^{\log}$ for the associated *n-th log configuration space*, and E for the *index set* of X_n, $Z_n^{\log}$ [cf. Definition 2.1, (i)]. Thus, X_n may be identified with the *interior* U_{Z_n} of $Z_n^{\log}$. Also, let us suppose that we have been given a nonempty open subscheme

$$\underline{X} \subseteq X$$

which is the complement $X \backslash S$ of a finite set of closed points $S \subseteq X$. Thus, $\underline{X}$ determines an associated *n-th configuration space* $\underline{X}_n \subseteq \underline{P}_n$ and an associated *n-th log configuration space* $\underline{Z}_n^{\log}$ [with index set E]. Moreover, by "forgetting certain of the marked points of the stable curve" parametrized by $\underline{Z}_n^{\log}$, we obtain a *natural morphism*

$$\underline{Z}_n^{\log} \to Z_n^{\log}$$

that extends the natural inclusion $\underline{X}_n \hookrightarrow X_n$.

Proposition 5.3. (The Logarithmic Geometry of the Log Configuration Space) *In the notation of the above discussion: Write*

$$V \stackrel{\text{def}}{=} U_{Z_n}^{[1]}$$

for the 1*-interior of* $Z_n^{\log}$*. For* $j \geq 1$ *an integer, let us denote by*

$$\wedge^j E$$

the set of subsets of E *of cardinality* j *[so* E *may be identified with* $\wedge^1 E$*] and by*

$$\wedge^* E \stackrel{\text{def}}{=} \bigcup_{j=1}^{n} \wedge^j E$$

the [disjoint] union of the subsets of cardinality $j \geq 1$*. Then:*

(i) We shall refer to a divisor on Z_n *obtained as the pull-back via a projection morphism* $Z_n \to X$ *of co-length* 1 *[and co-profile* $e \in \wedge^1 E = E$*] of a point in* $S \subseteq X$ *as a* **fiber divisor [of co-profile** e**]** *[on* Z_n*]. Then all fiber divisors of co-profile* $e \in \wedge^1 E = E$ *determine the* **same** *fundamental class*

$$\eta_e \in H^2_{\text{ét}}(Z_n, \mathbb{Q}_l(1))$$

— which we shall refer to as the **fiber class of co-profile** e *[on* Z_n*].*

(ii) The **irreducible divisors** *on Z_n contained in the divisor D_{Z_n} defining the log structure of $Z_n^{\log}$ are in* **natural bijective correspondence** *with the elements of $(\wedge^* E)\backslash E$. That is to say, a point of V belongs to the irreducible divisor $D_\varepsilon \subseteq V$ corresponding to an element $\varepsilon \in (\wedge^* E)\backslash E$ if and only if it corresponds to a stable curve with precisely two irreducible components, one isomorphic to X, the other of genus zero, such that the marked points that lie on X are precisely the marked points determined by the factors $e \in \varepsilon' \overset{\mathrm{def}}{=} E\backslash\varepsilon$. In particular, we obtain a* **natural isomorphism of schemes**

$$D_\varepsilon \cong X_{|\varepsilon'|+1} \times Q_{|\varepsilon|-2}$$

— where $|\varepsilon|$, $|\varepsilon'|$ are the cardinalities of of ε, ε', respectively; the projection $D_\varepsilon \to X_{|\varepsilon'|+1}$ is induced by any projection $X_n \to X_{|\varepsilon'|+1}$ of co-profile ε^+, for $\varepsilon^+ \in \wedge^{|\varepsilon'|+1} E$ an element such that $\varepsilon' \subseteq \varepsilon^+$; $Q_{|\varepsilon|-2}$ is the $(|\varepsilon|-2)$-th configuration space [i.e., $\mathrm{Spec}(k)$, *when $|\varepsilon| = 2$] of "the"* **tripod** *[cf. §0] over k. In particular, the index set of the configuration space $X_{|\varepsilon'|+1}$ appearing in this isomorphism may be naturally identified with the set "E/ε" obtained from E by* **identifying** *the elements of ε to a single element $[\varepsilon] \in E/\varepsilon$. We shall refer to the irreducible divisor on Z_n contained in D_{Z_n} that corresponds to $\varepsilon \in (\wedge^* E)\backslash E$ as the* **log-prime divisor of co-profile** *ε [on Z_n]; we shall refer to the fundamental class*

$$\eta_\varepsilon \in H^2_{\text{ét}}(Z_n, \mathbb{Q}_l(1))$$

of the log-prime divisor of co-profile ε as the **log-prime class of co-profile** *ε [on Z_n].*

(iii) The **union of the fiber and log-prime divisors** *of Z_n determine a* **divisor with normal crossings** *on V. Denote the resulting log scheme by $V^{\log}$; let*

$$W^{\log} \to V^{\log}$$

be a connected **Kummer log étale covering**. *Then $W^{\log}$ is* **log smooth** *over k and* **strongly $\mathbb{Q}$-regular**, *and the 2-interior $U_W^{[2]}$ of $W^{\log}$ is equal to W. We shall also refer to irreducible divisors on W that lie over fiber divisors on Z_n as* **fiber divisors** *on W, and to fundamental classes of fiber divisors on W as* **fiber classes** *on W; in a similar vein, we shall refer to irreducible divisors on W that lie over log-prime divisors on Z_n as* **log-prime divisors** *on W, and to fundamental classes of log-prime divisors on W as* **log-prime classes** *on W. Then a fiber class on W is completely determined by its* **coprofile** *[cf. (i)]. Also, we shall refer to a divisor (respectively, class) on W or Z_n as a* **log-characteristic divisor** *(respectively,* **log-characteristic class***) if it is either a fiber divisor (respectively, class) or a log-prime divisor (respectively, class).*

(iv) Write

$$(\underline{X}_n^\times)^{\log} \overset{\mathrm{def}}{=} \underline{P}_n \times_{P_n} \underline{Z}_n^{\log}; \quad (X_n^\times)^{\log} \overset{\mathrm{def}}{=} \underline{P}_n \times_{P_n} Z_n^{\log}$$

[where, by abuse of notation, we use notation for schemes to denote the corresponding log schemes with trivial log structure]. Then the morphism $\underline{Z}_n^{\log} \to Z_n^{\log}$ induces an **isomorphism** *$(\underline{X}_n^\times)^{\log} \xrightarrow{\sim} (X_n^\times)^{\log}$.*

Proof. Assertion (i) follows, for instance, from [Milne], p. 276, Theorem 11.1, (a). Assertion (ii) follows immediately from the definition of $Z_n^{\log}$ involving the *[log] moduli stack of stable curves.* Assertion (iii) follows immediately from the isomorphism "$D_\varepsilon \cong X_{|\varepsilon'|+1} \times Q_{|\varepsilon|-2}$" of assertion (ii). Assertion (iv) follows immediately from the fact that points of $\underline{Z}_n^{\log}$ that project to $\underline{P}_n \subseteq P_n$ correspond precisely to the stable curves [which necessarily consist of precisely one irreducible component which may be naturally identified with X and whose remaining irreducible components are of genus zero] such that the points of $S \subseteq X$ are *marked points* [not nodes!] — i.e., stable curves which may be reconstructed [without any indeterminacy!] even if one *forgets* these marked points determined by the points of $S \subseteq X$. ○

Lemma 5.4. **(Line Bundles on Log-prime Divisors)** *In the notation of Proposition 5.3, (ii):*

(i) The isomorphism $D_\varepsilon \cong X_{|\varepsilon'|+1} \times Q_{|\varepsilon|-2}$ *of Proposition 5.3, (ii), determines an isomorphism of* **Picard groups** $\mathrm{Pic}(D_\varepsilon) \xrightarrow{\sim} \mathrm{Pic}(X_{|\varepsilon'|+1})$.

(ii) The **co-normal bundle** *of* D_ε *is isomorphic [cf. (i)] to the line bundle obtained by pulling back the* **canonical bundle** ω_X *of* X *via the [unique!] projection* $X_{|\varepsilon'|+1} \to X$ *that arises from a projection morphism* $X_n \to X$ *of co-length* 1 *whose co-profile is* **not** *contained in* ε'.

Proof. First, we consider assertion (i). Since $Q_{|\varepsilon|-2}$ is an open subscheme of the *affine space* [of dimension $|\varepsilon| - 2$] over k, it follows that $D_\varepsilon \cong X_{|\varepsilon'|+1} \times Q_{|\varepsilon|-2}$ is isomorphic to an open subscheme of the *affine space* [of dimension $|\varepsilon| - 2$] over $X_{|\varepsilon'|+1}$. Thus, the assertion concerning Picard groups follows immediately from elementary algebraic geometry [cf., e.g., [Fulton], Theorem 3.3, (a)].

As for assertion (ii), we observe that by the description of the stable curves parametrized by D_ε given in Proposition 5.3, (ii), implies [in light of the well-known *local structure* of a node] that the co-normal bundle in question is naturally isomorphic to the tensor product of the pull-back of the canonical bundle described in the statement of assertion (ii) with some [necessarily *trivial* — by assertion (i)] line bundle on D_ε pulled back from the natural projection to $Q_{|\varepsilon|-2}$. This completes the proof of assertion (ii). ○

Lemma 5.5. **(Linear Independence of Log-characteristic Classes)** *In the notation of Proposition 5.3, (iii): Set*

$$I_W \overset{\text{def}}{=} I_W^{\text{fiber}} \bigcup I_W^{\text{log-prime}}$$

— where we write $I_W^{\text{fiber}} \overset{\text{def}}{=} E$ *and* $I_W^{\text{log-prime}}$ *for the set of* **log-prime divisors** *on* W*. Thus, if we think of the elements of* I_W^{fiber} *as the co-profiles of fiber classes of* $H^2_{\text{ét}}(W, \mathbb{Q}_l(1))$*, then we obtain a [fiber or log-prime] class*

$$\eta_i \in H^2_{\text{ét}}(W, \mathbb{Q}_l(1))$$

for each element $i \in I_W$*. For* $i \in I_W^{\text{fiber}}$*, write*

$$\eta_i^+ \overset{\text{def}}{=} \eta_i|_{U_W^+} \in H^2_{\text{ét}}(U_W^+, \mathbb{Q}_l(1))$$

for the restriction of η_i *to* $U_W^+ \overset{\text{def}}{=} X_n \times_{P_n} W \subseteq W$ *[so* $U_W \subseteq U_W^+$*]; for* $i \in I_W^{\text{log-prime}}$*, write*

$$\eta_i^\times \overset{\text{def}}{=} \eta_i|_{U_W^\times} \in H^2_{\text{ét}}(U_W^\times, \mathbb{Q}_l(1))$$

for the restriction of η_i *to* $U_W^\times \overset{\text{def}}{=} \underline{P}_n \times_{P_n} W \subseteq W$ *[so* $U_W \subseteq U_W^\times$*]. Then:*

(i) The **log-characteristic classes** $\{\eta_i\}_{i \in I_W}$ *are* **linearly independent** *over* $\mathbb{Q}_l$*.*

(ii) The restricted **fiber classes** $\{\eta_i^+\}_{i \in I_W^{\text{fiber}}}$ *are* **linearly independent** *over* $\mathbb{Q}_l$*.*

(iii) The restricted **log-prime classes** $\{\eta_i^\times\}_{i \in I_W^{\text{log-prime}}}$ *are* **linearly independent** *over* $\mathbb{Q}_l$*.*

Proof. First, we observe that the description of the kernel of the restriction map $H^2_{\text{ét}}(W, \mathbb{Q}_l(1)) \to H^2_{\text{ét}}(U_W^+, \mathbb{Q}_l(1))$ (respectively, $H^2_{\text{ét}}(W, \mathbb{Q}_l(1)) \to H^2_{\text{ét}}(U_W^\times, \mathbb{Q}_l(1))$) given in the exact sequence of Proposition 5.2, (v), implies that assertion (ii) (respectively, (iii)) follows immediately from assertion (i). For $i \in I_W^{\text{log-prime}}$, write

$$D_i$$

for the divisor [tautologically!] determined by i. Thus,

$$D_i \bigcap D_j = \emptyset$$

for all $i, j \in I_W^{\text{log-prime}}$ such that $i \neq j$. Also, we observe that it follows immediately from the definitions that, for $i \in I_W^{\text{log-prime}}$, the *normalization*

$$\widetilde{D}_i$$

of D_i may be naturally identified with the normalization of some log-prime divisor $D_\varepsilon \cong X_{|\varepsilon'|+1} \times Q_{|\varepsilon|-2}$ [cf. Proposition 5.3, (ii)] of $V^{\log}$ in a finite étale covering of $\underline{X}_{|\varepsilon'|+1} \times Q_{|\varepsilon|-2} \subseteq X_{|\varepsilon'|+1} \times Q_{|\varepsilon|-2}$. Finally, we observe that we may assume without loss of generality that the Kummer log étale covering $W^{\log} \to V^{\log}$ is *Galois*, with Galois group $\Gamma \overset{\text{def}}{=} \text{Gal}(W^{\log}/V^{\log})$.

Now let us verify assertion (i) by *induction* on n. Let

$$\sum_{i \in I_W} c_i \cdot \eta_i = 0$$

[where the $c_i \in \mathbb{Q}_l$] be a linear relation among the η_i. The case $n = 1$ is immediate from the definitions [cf. also [Milne], p. 276, Theorem 11.1, (a)]. Next, we consider the case $n = 2$. Let $j \in I_W^{\text{log-prime}}$. Since $n = 2$, it follows immediately from the definitions that $\widetilde{D}_j$ is a *proper hyperbolic curve* such that the covering $W^{\log} \to V^{\log}$ induces a finite dominant morphism of $\widetilde{D}_j$ onto the diagonal of $V = X \times X$. In particular, it follows that we may restrict the above linear relation to $\widetilde{D}_j$ to obtain a linear relation

$$-c_j \cdot \eta_j|_{\widetilde{D}_j} = \sum_{i \in I_W^{\text{fiber}}} c_i \cdot \eta_i|_{\widetilde{D}_j}$$

among classes of $H^2_{\text{ét}}(\widetilde{D}_j, \mathbb{Q}_l(1)) \cong \mathbb{Q}_l$ [cf. [Milne], p. 276, Theorem 11.1, (a)]. Moreover, since the class $\eta_j|_{\widetilde{D}_j}$ is the pull-back to $\widetilde{D}_j$ of some $\mathbb{Q}_l^\times$-multiple of the fundamental class of the diagonal of $X \times X$ [cf. Lemma 5.4, (ii)], we thus conclude that $\eta_j|_{\widetilde{D}_j} \neq 0$. Next, let us observe that, for $i \in I_W^{\text{fiber}}$, the classes η_i are *fixed* by the natural action of Γ on $H^2_{\text{ét}}(W, \mathbb{Q}_l(1))$. Thus, if we identify $H^2_{\text{ét}}(\widetilde{D}_j, \mathbb{Q}_l(1))$ with $\mathbb{Q}_l$ via the natural isomorphism $H^2_{\text{ét}}(\widetilde{D}_j, \mathbb{Q}_l(1)) \cong \mathbb{Q}_l$, then we conclude that the elements $\eta_i|_{\widetilde{D}_j}, \eta_j|_{\widetilde{D}_j} \in \mathbb{Q}_l$ [where $i \in I_W^{\text{fiber}}$] are *independent* of the choice of j among all Γ-conjugates of j, hence that the element $c_j \in \mathbb{Q}_l$ is *independent* of the choice of j among all Γ-conjugates of j. But this implies that the linear relation $\sum_{i \in I_W} c_i \cdot \eta_i = 0$ arises as the pull-back to W of a similar linear relation on V. That is to say, we may assume without loss of generality that $W^{\log} = V^{\log}$. But then the $\mathbb{Q}_l$-linear independence of the unique log-prime class η_Δ and the two fiber classes η_1, η_2 follows, for instance, from the [easily verified] *non-singularity of the matrix of intersection numbers* among the classes η_Δ, η_1, η_2 [where we recall that $\eta_\Delta \cdot \eta_1 = \eta_\Delta \cdot \eta_2 = \eta_1 \cdot \eta_2 = 1$, $\eta_1 \cdot \eta_1 = \eta_2 \cdot \eta_2 = 0$, $\eta_\Delta \cdot \eta_\Delta = 2 - 2g_X$]. This completes the proof of the case $n = 2$.

Now we assume that $n \geq 3$. Let D_j [where $j \in I_W^{\text{log-prime}}$] be a *log-prime divisor of co-length* 2 — i.e., which projects to a log-prime divisor of V whose *co-profile* ε is of cardinality 2. Then $\widetilde{D}_j$ may be identified with the normalization of X_{n-1} in a finite étale covering of $\underline{X}_{n-1} \subseteq X_{n-1}$, i.e., as a certain "$U_W^+$" that arises in the case "$n-1$". Moreover, it follows immediately from Lemma 5.4, (ii), that $\eta_j|_{\widetilde{D}_j}$ is a $\mathbb{Q}_l$-multiple of the restriction to $\widetilde{D}_j$ of the *fiber class* of $Z_{n-1}^{\log}$ of co-profile $[\varepsilon] \in E/\varepsilon$ [cf. Proposition 5.3, (ii)]. On the other hand, for $i \in I_W^{\text{log-prime}}$ such that $i \neq j$, $\eta_i|_{\widetilde{D}_j} = 0$; for $i \in I_W^{\text{fiber}}$ of co-profile $e \in E$, $\eta_i|_{\widetilde{D}_j}$ is a $\mathbb{Q}_l^\times$-multiple of the restriction to $\widetilde{D}_j$ of the *fiber class* of $Z_{n-1}^{\log}$ whose co-profile is the image of e in E/ε. In particular, by restricting the linear relation $\sum_{i \in I_W} c_i \cdot \eta_i = 0$ to $\widetilde{D}_j$, it follows by applying assertion (ii) for "$n-1$" [i.e., here we apply the *induction hypothesis* on n] that $c_i = 0$ for all $i \in \varepsilon' \subseteq E = I_W^{\text{fiber}}$. Since $n \geq 3$, it follows that $\varepsilon' \neq \emptyset$. Thus, by *varying* j [i.e., varying ε], we conclude that $c_i = 0$ for all $i \in I_W^{\text{fiber}}$.

Now to complete the proof of Lemma 5.5, it suffices to show that the coefficients of any linear relation $\sum_{i \in I_W^{\text{log-prime}}} c_i \cdot \eta_i = 0$ *vanish*. On the other hand, by restricting to $\widetilde{D}_j$, for $j \in I_W^{\text{log-prime}}$, we obtain relations $c_j \cdot \eta_j|_{\widetilde{D}_j} = 0$ for each

$j \in I_W^{\text{log-prime}}$. Thus, to complete the proof of Lemma 5.5, it suffices to show that

$$\eta_j|_{\widetilde{D}_j} \neq 0$$

for $j \in I_W^{\text{log-prime}}$. On the other hand, by Lemma 5.4, (ii), it follows that $\eta_j|_{\widetilde{D}_j}$ is a $\mathbb{Q}_l^\times$-multiple of the pull-back to $\widetilde{D}_j$ of a class on $X_{|\varepsilon'|+1} \times Q_{|\varepsilon|-2}$ [for some $\varepsilon \in (\wedge^* E)\backslash E$, $\varepsilon' \stackrel{\text{def}}{=} E\backslash\varepsilon$] that arises as the pull-back to $X_{|\varepsilon'|+1} \times Q_{|\varepsilon|-2}$ via the projection morphism $X_{|\varepsilon'|+1} \times Q_{|\varepsilon|-2} \to X_{|\varepsilon'|+1}$ of the restriction to $X_{|\varepsilon'|+1}$ of a *fiber class* of $Z_{|\varepsilon'|+1}$. Since $|\varepsilon'| + 1 \leq n - 1$, it thus follows from assertion (ii) for "$|\varepsilon'|+1$" [i.e., here we apply the *induction hypothesis* on n] that the class $\eta_j|_{\widetilde{D}_j} \neq 0$, as desired. This completes the proof of Lemma 5.5. ○

Remark 5.5.1. The "+" (respectively, "×") in the notation U_W^+ (respectively, $U_W^\times$) is intended to be a sort of "rough pictorial representation" of the *fiber divisors* (respectively, *log-prime divisors* [e.g., "diagonals"!]) that are appended to the *interior* U_W to form U_W^+ (respectively, $U_W^\times$).

Theorem 5.6. **(Extendability of Coverings)** *Let X be a* **proper hyperbolic curve** *over an* **algebraically closed field** k **of characteristic zero***; $S \subseteq X$ a finite set of closed points; $\underline{X} \stackrel{\text{def}}{=} X\backslash S \subseteq X$; $n \geq 1$ an integer; $X_n \subseteq P_n$, $\underline{X}_n \subseteq \underline{P}_n$ the* **n-th configuration spaces** *associated to X, $\underline{X}$, respectively; $\underline{Z}_n^{\log}$ the* **n-th log configuration space** *associated to $\underline{X}$; $(\underline{X}_n^\times)^{\log} \stackrel{\text{def}}{=} \underline{P}_n \times_{P_n} \underline{Z}_n^{\log}$ [so we have a natural inclusion $(\underline{X}_n^\times)^{\log} \hookrightarrow \underline{Z}_n^{\log}$];*

$$Y \to \underline{X}_n$$

a **finite étale morphism***, where Y is* **connected***; $Y^\times \to \underline{X}_n^\times$ the normalization of $\underline{X}_n^\times$ in Y; G a* **strongly torsion-free pro-solvable profinite group**. *Then any continuous homomorphism*

$$\pi_1(Y) \to G$$

factors *through the natural surjection $\pi_1(Y) \twoheadrightarrow \pi_1(Y^\times)$ induced by the open immersion $Y \hookrightarrow Y^\times$.*

Proof. In the notation of the discussion preceding Theorem 5.6: By the *log purity theorem* [cf. the discussion of §2], we have a *natural isomorphism* $\pi_1(U_V) \xrightarrow{\sim} \pi_1(V^{\log})$, where we observe that [it follows immediately from the definitions that] the *interior* U_V of $V^{\log}$ may be *identified* with $\underline{X}_n$. Thus, the finite étale covering $Y \to \underline{X}_n$ may be *identified* with the *interior* U_W of a *Kummer log étale covering* $W^{\log} \to V^{\log}$. Moreover, by applying the isomorphism $(\underline{X}_n^\times)^{\log} \xrightarrow{\sim} (X_n^\times)^{\log}$ of Proposition 5.3, (iv), we obtain a finite morphism $Y^\times \to X_n^\times$ that exhibits $Y^\times$ as the normalization of $X_n^\times$ in $Y = U_W$. Thus, by the *log purity theorem*, we conclude that this finite morphism determines a *Kummer log étale covering* $(Y^{\times})^{\log} \to (X_n^{\times})^{\log}$. Thus, in summary, we obtain a commutative diagram

$$\begin{tikzcd}
W^{\log} \arrow[d] & U_W = Y \arrow[l, hook'] \arrow[r, hook] \arrow[d] & (U_W^{\times})^{\log} \arrow[r, hook] \arrow[d] & (Y^{\times})^{\log} \arrow[d] \\
V^{\log} & U_V = \underline{X}_n \arrow[l, hook'] \arrow[r, hook] & (U_V^{\times})^{\log} \arrow[r, hook] & (X_n^{\times})^{\log}
\end{tikzcd}$$

in which the "hooked horizontal arrows" are open immersions; the vertical arrows are Kummer log étale coverings; the log structures on $U_W^{\times}$, $U_V^{\times}$ are those induced by restricting the log structures of $(Y^{\times})^{\log}$, $(X_n^{\times})^{\log}$, respectively; by abuse of notation, we use notation for schemes to denote the corresponding log schemes with trivial log structure. Also, we observe that it follows immediately from Proposition 5.2, (ii), that the complements of the open immersions $U_W^{\times} \hookrightarrow Y^{\times}$, $U_V^{\times} \hookrightarrow X_n^{\times}$ are of *codimension* ≥ 2.

Next, we observe that, just as in the proof of Theorem 4.7, it suffices to verify Theorem 5.6 in the case where $G = \mathbb{Z}_l$, for some prime number l. In particular, by Proposition 5.2, (iv), we have an isomorphism $H^1_{\text{ét}}(Y^{\times}, \mathbb{Q}_l) \xrightarrow{\sim} H^1_{\text{ét}}(U_W^{\times}, \mathbb{Q}_l)$, which [since $G = \mathbb{Z}_l$ is *torsion-free*] implies that to show that the given homomorphism $\pi_1(U_W) = \pi_1(Y) \to G$ factors through $\pi_1(Y) \twoheadrightarrow \pi_1(Y^{\times})$, it suffices to show that it factors through $\pi_1(Y) \twoheadrightarrow \pi_1(U_W^{\times})$. On the other hand, the fact that the given homomorphism $\pi_1(U_W) \to G$ factors through $\pi_1(U_W) \twoheadrightarrow \pi_1(U_W^{\times})$ follows immediately from the *exact sequence* of Proposition 5.2, (v), in light of the *$\mathbb{Q}_l$-linear independence* asserted in Lemma 5.5, (iii). This completes the proof of Theorem 5.6. ○

Remark 5.6.1. Note that:

> The *"extendability of coverings"* asserted in Theorem 5.6 may be regarded as a *strengthening* of the *"product-theoreticity"* asserted in Theorem 4.7.

Indeed, suppose that the covering $Y \to \underline{X}_n$ of Theorem 5.6 arises from a *product-theoretic* open subgroup of $\pi_1(\underline{X}_n)$. Write

$$Y^* \to \underline{P}_n$$

for the normalization of $\underline{P}_n$ ($\supseteq \underline{X}_n$) in Y. Since $Y^{\times}$ is *normal* and maps to $\underline{X}_n^{\times}$, hence to $\underline{P}_n$, we thus obtain a *birational morphism* $Y^{\times} \to Y^*$. Since the morphism $\underline{Z}_n \to P_n$ is *proper*, it follows that $\underline{X}_n^{\times} \to \underline{P}_n$ is *proper*, hence that the morphism $Y^{\times} \to Y^*$ is *proper*. On the other hand, since $Y \to \underline{X}_n$ arises from a *product-theoretic* open subgroup of $\pi_1(\underline{X}_n)$, one verifies immediately that Y^* is *smooth* over k, and that the kernel of the natural surjection $\pi_1(Y) \to \pi_1(Y^*)$ is *product-theoretic*. In particular, by *Zariski-Nagata purity* [i.e., the classical non-logarithmic version of the "log purity theorem" quoted above], we conclude that the natural surjection $\pi_1(Y^{\times}) \twoheadrightarrow \pi_1(Y^*)$ is an *isomorphism*. But this implies that the given homomorphism $\pi_1(Y) \to G$ *factors* through the natural surjection $\pi_1(Y) \to \pi_1(Y^*)$, hence, in particular, is *product-theoretic* [cf. Theorem 4.7].

Corollary 5.7. **(Non-existence of Generic Units)** *In the notation of Theorem 5.6, suppose that* $S = \emptyset$*; write* $X_n \stackrel{\text{def}}{=} \underline{X}_n$*,* $P_n \stackrel{\text{def}}{=} \underline{P}_n$*,* $Z_n^{\log} \stackrel{\text{def}}{=} \underline{Z}_n^{\log}$*. Also, let us fix a projection morphism*

$$\phi_X : X_n \to B \stackrel{\text{def}}{=} X_{n-1}$$

of length 1*, which allows us to regard* X_n *as a* **family of hyperbolic curves** *over* B*. Denote the [scheme determined by the] generic point of* B *by* η*; write* $k(\eta)$ *for the residue field of* η*,* $\mathcal{X} \stackrel{\text{def}}{=} X_n \times_B \eta$*. Let* $k(\eta')$ *be an arbitrary field extension of* $k(\eta)$*,* $\eta' \to \eta$ *the resulting morphism of schemes,* $\mathcal{Y}$ *a* **hyperbolic curve** *over* η'*, and*

$$\mathcal{Y} \to \mathcal{X}_{\eta'} \stackrel{\text{def}}{=} \mathcal{X} \times_\eta \eta' = X_n \times_B \eta'$$

a **finite étale covering** *over* η'*. Then every* **unit** $u \in \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}^\times)$ *on* $\mathcal{Y}$ *is* **constant***, i.e., is contained in the image of* $k(\eta')$ *in* $\mathcal{O}_{\mathcal{Y}}$*.*

Proof. One reduces immediately by well-known elementary algebraic geometry arguments [i.e., replacing $k(\eta')$ by a finitely generated field extension of $k(\eta)$, extending $\mathcal{Y} \to \mathcal{X}_{\eta'}$ over some variety that admits $k(\eta')$ as its function field, and restricting to a closed point of this variety] to the case where the morphism $\eta' \to \eta$ is *finite étale.* Now let us observe that by the *exact sequence* of Proposition 2.2, (i), it follows that, after possibly *replacing* $\mathcal{Y}$ by an appropriate connected finite étale covering of $\mathcal{Y}$, we may assume without loss of generality that there exists a *commutative diagram*

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X_n \arrow[d] \\ C \arrow[r] & B \end{tikzcd}$$

in which the horizontal arrows are *connected finite étale coverings*; the vertical arrows are *families of hyperbolic curves*; the *divisor of cusps* $D_Y \subseteq \overline{Y}$ in the *compactification* $\overline{Y} \to C$ of $Y \to C$ forms a *split* finite étale covering of C; the covering $\mathcal{Y} \to \mathcal{X}$ *factors* through $Y_{\eta_C} \stackrel{\text{def}}{=} Y \times_C \eta_C$ [where $\eta_C \stackrel{\text{def}}{=} C \times_B \eta$] in such a way that the induced covering

$$\mathcal{Y} \to Y_{\eta_C}$$

is obtained by base-changing the curve Y_{η_C} over η_C via a morphism $\eta' \to \eta_C$. In particular, since the divisor of zeroes and poles of u on the compactification of the hyperbolic curve $\mathcal{Y}$ descends [by our *"splitness"* assumption on D_Y] to a divisor on the compactification of the hyperbolic curve Y_{η_C}, it follows immediately that u is *constant* if and only if its *norm* relative to this covering $\mathcal{Y} \to Y_{\eta_C}$ [which forms a unit on Y_{η_C}] is *constant.* Thus, in summary, we may assume without loss of generality that $\mathcal{Y} = Y_{\eta_C}$, and that *u is a unit on Y_{η_C}.*

Now since u may be regarded as a *rational function* on $\overline{Y}$, the divisor of zeroes and poles of this rational function on $\overline{Y}$ may be written in the form

$$D^{\text{cusp}} + D^{\text{base}}$$

— where D^{cusp} is a divisor supported on D_Y; D^{base} is a divisor on $\overline{Y}$ that arises as the pull-back to $\overline{Y}$ of a divisor D_C^{base} on C. In particular, we obtain a relation

$$0 = c(D^{\text{cusp}}) + c(D^{\text{base}}) \in H^2_{\text{ét}}(\overline{Y}, \mathbb{Q}_l(1))$$

— where l is a prime number; we write $c(-)$ for the *Chern class* of the line bundle determined by a divisor on a k-smooth scheme [such as Y, $\overline{Y}$, C]. Next, let us pull-back this relation via a section $s : C \to \overline{Y}$ whose image D_s is contained in D_Y. This yields a relation

$$\lambda \cdot s^*(c(D_s)) = s^*(c(D^{\text{base}})) = c(D_C^{\text{base}}) \in H^2_{\text{ét}}(C, \mathbb{Q}_l(1))$$

for some $\lambda \in \mathbb{Q}_l$. On the other hand, let us observe that, relative to the notation of the discussion preceding Theorem 5.6, if we take $U_W \to U_V$ to be the covering $Y \to X_n$, then $\overline{Y}$ may be regarded as an *open subscheme* of W, and $D_s \subseteq \overline{Y}$ as a *log-prime divisor* of W. From this point of view, it follows from Lemma 5.4, (ii), that $s^*(c(D_s))$ is a $\mathbb{Q}_l$-multiple of the pull-back to C via $C \to B = X_{n-1}$ of a *fiber class* on $B = X_{n-1}$. In particular, we conclude that $c(D_C^{\text{base}})$ is a $\mathbb{Q}_l$-multiple of the pull-back to C of a *fiber class* on $B = X_{n-1}$, hence that $c(D^{\text{base}}) \in H^2_{\text{ét}}(\overline{Y}, \mathbb{Q}_l(1))$ is a $\mathbb{Q}_l$-multiple of the pull-back to $\overline{Y} \subseteq W$ of a *fiber class* on W.

Thus, in summary, the relation $0 = c(D^{\text{cusp}}) + c(D^{\text{base}})$ in $H^2_{\text{ét}}(\overline{Y}, \mathbb{Q}_l(1))$ constitutes a *$\mathbb{Q}_l$-linear relation* between certain *log-prime classes* [i.e., $c(D^{\text{cusp}})$] and certain *fiber classes* [i.e., $c(D^{\text{base}})$] on $\overline{Y}$. Since [it follows immediately from the definitions that] the complement of $\overline{Y}$ in W is a [disjoint] union of certain log-prime divisors on W, the description of the kernel of the restriction map $H^2_{\text{ét}}(W, \mathbb{Q}_l(1)) \to H^2_{\text{ét}}(\overline{Y}, \mathbb{Q}_l(1))$ given in the exact sequence of Proposition 5.2, (v), implies that this $\mathbb{Q}_l$-linear relation $0 = c(D^{\text{cusp}}) + c(D^{\text{base}})$ in $H^2_{\text{ét}}(\overline{Y}, \mathbb{Q}_l(1))$ arises from some *$\mathbb{Q}_l$-linear relation* in $H^2_{\text{ét}}(W, \mathbb{Q}_l(1))$ obtained by appending some $\mathbb{Q}_l$-linear combination of the log-prime classes arising from the log-prime divisors in the complement $W \backslash \overline{Y}$. By the *$\mathbb{Q}_l$-linear independence* asserted in Lemma 5.5, (i), the coefficients of such a $\mathbb{Q}_l$-linear relation necessarily *vanish.* In particular, since D^{cusp} is a $\mathbb{Z}$-linear combination of log-prime divisors of W that lie in $\overline{Y}$, we thus conclude that the coefficients $\in \mathbb{Z}$ of this $\mathbb{Z}$-linear combination *vanish*, i.e., that $D^{\text{cusp}} = 0$. But this amounts precisely to the assertion that the unit u is *constant.* This completes the proof of Corollary 5.7. ○

Section 6: Nearly Abelian Groups

In the present §6, we discuss another approach, based on the notion of a *"nearly abelian"* profinite group, to verifying the *group-theoreticity* of the various *fiber subgroups* associated to a configuration space group.

Definition 6.1. We shall say that a profinite group G is *nearly abelian* if it admits a normal closed subgroup $N \subseteq G$ which is *topologically normally generated by a single element* $\in G$ such that G/N is almost abelian.

Proposition 6.2. **(Nearly Abelian Surface Groups)** *Let $\mathcal{C}$ be a full formation. Then a* **pro-$\mathcal{C}$ surface group** Π *is* **nearly abelian** *if and only if it is a free pro-$\mathcal{C}$ group on* **two generators** *— i.e., it arises from a hyperbolic curve which is either of type* $(0,3)$ *or type* $(1,1)$ *[cf. Remark 1.2.2].*

Proof. Since the *sufficiency* of the condition given in the statement of Proposition 6.2 is immediate from the definitions, it suffices to verify the *necessity* of this condition. Thus, we suppose that Π is *nearly abelian.* Now observe that for any $l \in \Sigma_{\mathcal{C}}$, the *maximal pro-$l$ quotient* of Π is again a *nearly abelian [pro-l] surface group.* Thus, we may assume without loss of generality that $\mathcal{C}$ is *primary.* Write $\Sigma_{\mathcal{C}} = \{l\}$; $\overline{\Pi} \stackrel{\text{def}}{=} \Pi/[\Pi,[\Pi,\Pi]]$; $\Pi_1 \stackrel{\text{def}}{=} \overline{\Pi}^{\text{ab}} \cong \Pi^{\text{ab}}$; $\Pi_2 \stackrel{\text{def}}{=} [\overline{\Pi},\overline{\Pi}] \subseteq \overline{\Pi}$. Thus, we have an exact sequence

$$1 \to \Pi_2 \to \overline{\Pi} \to \Pi_1 \to 1$$

and a surjection $\wedge^2 \Pi_1 \twoheadrightarrow \Pi_2$. Here, we regard Π_1, Π_2 as *finitely generated free $\mathbb{Z}_l$-modules.* Note [cf. Remark 1.2.2] that if Π_1 is of rank d, then Π_2 is of rank $\frac{1}{2}d(d-1) - \epsilon$, where $\epsilon = 0$ if Π is *open*, and $\epsilon = 1$ if Π is *closed.* Also, we observe that $d \geq 2$ if $\epsilon = 0$, and $d \geq 4$ if $\epsilon = 1$ [cf. Remark 1.2.2]. Thus, to complete the proof of Proposition 6.2, it suffices to show that $d = 2$.

Next, let us observe that $\overline{\Pi}$ is also *nearly abelian.* Thus, there exists an element $\gamma \in \overline{\Pi}$ such that if we write $N \subseteq \overline{\Pi}$ for the subgroup topologically normally generated by γ, then $\overline{\Pi}/N$ is *almost abelian.* In particular, it follows that $N \bigcap \Pi_2 \subseteq \overline{\Pi}$ forms an *open subgroup* of Π_2 — i.e., a $\mathbb{Z}_l$-module of the same rank as Π_2. If $\gamma \in \Pi_2$, then $N \bigcap \Pi_2 = N$ is of rank ≤ 1, so we obtain that $\frac{1}{2}d(d-1) - \epsilon \leq 1$, i.e., $d(d-1) \leq 2(1+\epsilon) \leq 4$, so $d \leq 2$, i.e., $d = 2$, as desired. Thus, it remains to consider the case where $\gamma \notin \Pi_2$. In this case, one verifies immediately that $N \bigcap \Pi_2 \subseteq \Pi_2$ is given by the image of the morphism

$$[\gamma, -] : \Pi_1 \to \Pi_2$$

given by forming the *commutator* with γ. Since this morphism clearly vanishes on γ, its image is of rank $\leq d-1$. Thus, we conclude that

$$d - 1 \geq \frac{1}{2}d(d-1) - \epsilon$$

— i.e., that $2 \cdot \epsilon \geq (d-1)(d-2)$, which implies that $d \leq 2$ if $\epsilon = 0$, and $d \leq 3$ if $\epsilon = 1$. But this is enough to conclude that $d = 2$ [and $\epsilon = 0$]. This completes the proof of Proposition 6.2. ○

Corollary 6.3. **(Group-theoreticity of Projections of Configuration Spaces II)** *Let $\mathcal{C}$ be a* **PT-formation**. *For $\Box = \alpha, \beta$, let $X^{\Box}$ be a* **hyperbolic**

curve *whose type is* **neither** $(0,3)$ **nor** $(1,1)$ *over an* **algebraically closed field** $k_\Box$ **of characteristic zero***;* $n_\Box \geq 1$ *an integer;* $X^\Box_{n_\Box}$ *the* $\boldsymbol{n_\Box}$**-th configuration space** *associated to* $X^\Box$*;* $\Pi^\Box \stackrel{\mathrm{def}}{=} \pi_1^{\mathcal{C}}(X^\Box_{n_\Box})$*;* $E_\Box$ *the* **index set** *of* $X^\Box_{n_\Box}$*. Let*

$$\gamma : \Pi^\alpha \xrightarrow{\sim} \Pi^\beta$$

be an **isomorphism of profinite groups***. Then* γ *induces a* **bijection** $\sigma : E_\alpha \xrightarrow{\sim} E_\beta$ *[so* $n_\alpha = n_\beta$*] such that*

$$\gamma(F_\alpha) = F_\beta$$

for all **fiber subgroups** $F_\alpha \subseteq \Pi^\alpha$*,* $F_\beta \subseteq \Pi^\beta$*, whose respective* **profiles** $E'_\alpha \subseteq E_\alpha$*,* $E'_\beta \subseteq E_\beta$ *correspond via* σ*.*

Proof. Just as in the proof of Corollary 4.8, to complete the proof of Corollary 6.3, it suffices to verify that the image via γ of any fiber subgroup of Π^α of co-length one is contained in a fiber subgroup of Π^β of co-length one. For $j = 1, \dots, n_\Box$, let us write

$$K^\Box_j \subseteq \Pi_\Box$$

for the *fiber subgroup* $\subseteq \Pi^\Box$ *of co-length one* with co-profile given by the element of $E_\Box$ labeled by j, and

$$J^\Box_j \subseteq \Pi_\Box$$

for the *fiber subgroup* $\subseteq \Pi^\Box$ *of length one* with profile given by the element of $E_\Box$ labeled by j. Thus, [cf. Proposition 2.4, (vi)] to complete the proof of Corollary 6.3, it suffices to verify the following statement [in general]:

> For each $i \in E_\alpha$, there exists a $j \in E_\beta$ such that $J^\beta_{j'} \subseteq \gamma(K^\alpha_i)$ for all $j' \in E_\beta$ such that $j' \neq j$.

To verify this statement, we reason as follows: Write

$$\phi : \Pi^\beta \xrightarrow{\sim} \Pi^\alpha \twoheadrightarrow G \stackrel{\mathrm{def}}{=} \Pi^\alpha / K^\alpha_i$$

for the surjection determined by γ^{-1}. Then it suffices to show that there do not exist two distinct elements $j_1, j_2 \in E_\beta$ such that $J^\beta_{j_1}$, $J^\beta_{j_2}$ have *nontrivial image* under ϕ. Thus, let us suppose that the images J_1, J_2 of $J^\beta_{j_1}$, $J^\beta_{j_2}$ under ϕ are *nontrivial.* Since $J^\beta_{j_1}$, $J^\beta_{j_2}$ are *topologically finitely generated normal closed subgroups* of Π^β, it follows from Theorem 1.5 that J_1, J_2 are *open* in G [cf. Remark 3.3.2]. Moreover, by Proposition 2.4, (v), it follows that there exists a normal closed subgroup $N \subseteq G$ that is topologically normally generated by a *single element* such that the images of J_1, J_2 in G/N *commute.* Thus, G/N contains an *abelian open subgroup*, i.e., is *almost abelian.* On the other hand, since $K^\alpha_i \subseteq \Pi^\alpha$ is a fiber subgroup of co-length one, it follows that $G = \Pi^\alpha / K^\alpha_i$ is a *surface group.* Thus, in summary, we conclude that G is a *nearly abelian surface group*, which, by Proposition 6.2, contradicts our hypothesis concerning the *type of the hyperbolic curve* X^{α}. This completes the proof of Corollary 6.3. $\bigcirc$

Remark 6.3.1. Unlike the case with Corollary 4.8, it seems *unrealistic* at the time of writing to extend the technique of the proof of Corollary 6.3 to the case of *arbitrary product-theoretic open subgroups* $\subseteq \pi_1^{\mathcal{C}}(X^{\square}_{n_{\square}})$ [cf. Corollary 4.8], since this would require an analogue of Proposition 6.2 for surface groups that become almost abelian after forming the quotient by a subgroup topologically normally generated by a *very large* number of elements [roughly, on the order of the *index* of the product-theoretic open subgroups under consideration].

Section 7: A Discrete Analogue

In the present §7, we discuss various consequences of Theorems 4.7, 5.6 and Corollaries 4.8, 6.3 [cf. Corollaries 7.3, 7.4 below] for the *topological fundamental groups* of configuration spaces over the complex number field $\mathbb{C}$.

In the following discussion, if Z is a connected scheme of finite type over $\mathbb{C}$, then we shall use the notation

$$\pi_1^{\mathrm{top}}(Z)$$

to denote the *"topological fundamental group"* [i.e., the fundamental group in the usual sense of algebraic topology], for some choice of basepoint, of the topological space of $\mathbb{C}$-rational points $Z(\mathbb{C})$ [equipped with the topology determined by the topology of $\mathbb{C}$].

Let X be a *hyperbolic curve* over $\mathbb{C}$, $n \geq 1$ an integer. Write $X_n \subseteq P_n$ for the *n-th configuration space* associated to X [cf. the notation of Definition 2.1, (i)]. Since the complement in a connected complex manifold of any submanifold of [complex] codimension 1 is clearly connected, it thus follows that the inclusion $X_n \hookrightarrow P_n$ induces a *natural surjection* $\pi_1^{\mathrm{top}}(X_n) \twoheadrightarrow \pi_1^{\mathrm{top}}(P_n)$.

The following "discrete analogue" of [a certain portion of] Proposition 2.2 is well-known:

Proposition 7.1. **(Topological Fundamental Groups of Configuration Spaces)** *In the notation of the above discussion:*

(i) Any projection morphism $X_n \to X_{n-1}$ *of length one determines a* **natural exact sequence**

$$1 \to \pi_1^{\mathrm{top}}((X_n)_{\overline{x}}) \to \pi_1^{\mathrm{top}}(X_n) \to \pi_1^{\mathrm{top}}(X_{n-1}) \to 1$$

[where we write $X_0 \overset{\mathrm{def}}{=} \mathrm{Spec}(\mathbb{C})$*;* $\overline{x}$ *is a* $\mathbb{C}$*-valued geometric point of* X_{n-1}*].*

(ii) The natural morphism

$$\pi_1^{\mathrm{top}}(X_n) \to \pi_1(X_n)$$

to the étale fundamental group $\pi_1(X_n)$ *is* **injective**, *i.e.,* $\pi_1^{\mathrm{top}}(X_n)$ *is* **residually finite**.

Proof. Assertion (i) is discussed, for instance, in [Birm], Theorem 1.4. To verify assertion (ii), observe that [by induction on n] it follows from the exact sequences of assertion (i) and the analogue of assertion (i) for $\pi_1(X_n)$ [cf. Proposition 2.2, (i)], that we may assume without loss of generality that $n = 1$. Now let us recall that $\pi_1^{\mathrm{top}}(X)$ may be embedded [by considering the well-known *uniformization* of $X(\mathbb{C})$ by the *upper half-plane*] into $SL_2(\mathbb{R})$. That is to say, $\pi_1^{\mathrm{top}}(X)$ is a *"finitely generated linear group"*, so the desired *residual-finiteness* follows from a well-known *theorem of Mal'cev* [cf., e.g., [Wehr], Theorem 4.2]. ○

Definition 7.2.

(i) We shall refer to a subgroup $H \subseteq \pi_1^{\mathrm{top}}(X_n)$ as being *product-theoretic* if H arises as the inverse image via the natural surjection $\pi_1^{\mathrm{top}}(X_n) \twoheadrightarrow \pi_1^{\mathrm{top}}(P_n)$ of a subgroup of $\pi_1^{\mathrm{top}}(P_n)$.

(iii) Write E for the *index set* of X_n. Let $E' \subseteq E$ be a subset of cardinality n'; $E'' \overset{\mathrm{def}}{=} E \backslash E'$; $n'' \overset{\mathrm{def}}{=} n - n'$; $p_{E'} = p^{E''} : X_n \to X_{n''}$ the projection morphism of profile E'. Then we shall refer to the *kernel*

$$F \subseteq \pi_1^{\mathrm{top}}(X_n)$$

of the induced *surjection* $\pi_1^{\mathrm{top}}(X_n) \twoheadrightarrow \pi_1^{\mathrm{top}}(X_{n''})$ [cf. Proposition 7.1, (i)] as the *fiber subgroup* of $\pi_1^{\mathrm{top}}(X_n)$ of *profile* E'.

Remark 7.2.1. Note that by the *injectivity* of Proposition 7.1, (ii), it follows immediately that the *fiber subgroup of profile* E' of $\pi_1^{\mathrm{top}}(X_n)$ [cf. the notation of Definition 7.2, (ii)] is equal to the *inverse image* via the natural injection $\pi_1^{\mathrm{top}}(X_n) \hookrightarrow \pi_1(X_n)$ of Proposition 7.1, (ii), of the fiber subgroup of $\pi_1(X_n)$ of profile E'.

Corollary 7.3. (Discrete Strongly Torsion-free Pro-solvable Product-theoreticity and Extendability of Coverings) *Let* $n \geq 1$ *be an integer;* G *a* **strongly torsion-free pro-solvable profinite group**; X *a* **hyperbolic curve** *of* **genus** ≥ 2 *over* $\mathbb{C}$; $X_n \subseteq P_n$ *the* **n-th configuration space** *associated to* X.

(i) Let $H \subseteq \pi_1^{\mathrm{top}}(X_n)$ *be a* **product-theoretic subgroup of finite index**. *Then the kernel of any homomorphism [of abstract groups!]*

$$H \to G$$

is **product-theoretic**.

(ii) Suppose that X *is* **proper**. *Let* $S \subseteq X$ *a finite set of closed points;* $\underline{X} \stackrel{\mathrm{def}}{=} X \backslash S \subseteq X$*;* $\underline{X}_n \subseteq \underline{P}_n$ *the* **n-th configuration space** *associated to* $\underline{X}$*;* $\underline{Z}_n^{\log}$ *the* **n-th log configuration space** *associated to* $\underline{X}$*;* $(\underline{X}_n^{\times})^{\log} \stackrel{\mathrm{def}}{=} \underline{P}_n \times_{P_n} \underline{Z}_n^{\log}$ *[so we have a natural inclusion* $(\underline{X}_n^{\times})^{\log} \hookrightarrow \underline{Z}_n^{\log}$*];*

$$Y \to \underline{X}_n$$

a **finite étale morphism**, *where* Y *is* **connected**; $Y^{\times} \to \underline{X}_n^{\times}$ *the normalization of* $\underline{X}_n^{\times}$ *in* Y. *Then any homomorphism [of abstract groups!]*

$$\pi_1^{\mathrm{top}}(Y) \to G$$

factors *through the natural surjection* $\pi_1^{\mathrm{top}}(Y) \twoheadrightarrow \pi_1^{\mathrm{top}}(Y^{\times})$ *induced by the open immersion* $Y \hookrightarrow Y^{\times}$.

Proof. Indeed, since the homomorphisms in question $H \to G$, $\pi_1^{\mathrm{top}}(Y) \to G$ necessarily factor, respectively, through the *profinite completions* of H, $\pi_1^{\mathrm{top}}(Y)$, the conclusions of assertions (i), (ii) follow immediately from Theorems 4.7, 5.6. ◯

Corollary 7.4. (Group-theoreticity of Projections of Configuration Spaces III) *For* $\Box = \alpha, \beta$, *let* $X^{\Box}$ *be a* **hyperbolic curve** *over* $\mathbb{C}$ *whose type is* **neither** $(0,3)$ **nor** $(1,1)$*;* $n_{\Box} \geq 1$ *an integer;* $X^{\Box}_{n_{\Box}}$ *the* **$n_{\Box}$-th configuration space** *associated to* $X^{\Box}$*;* $E_{\Box}$ *the* **index set** *of* $X^{\Box}_{n_{\Box}}$*;* $H_{\Box} \subseteq \Pi^{\Box} \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{top}}(X^{\Box}_{n_{\Box}})$ *a* **product-theoretic subgroup of finite index**. *Let*

$$\gamma : H_{\alpha} \xrightarrow{\sim} H_{\beta}$$

be an **isomorphism of groups**. *Moreover, if either* $H_{\alpha} \neq \Pi^{\alpha}$ *or* $H_{\beta} \neq \Pi^{\beta}$, *then we assume that* $X^{\Box}$ *is of* **genus** ≥ 2, *for* $\Box = \alpha, \beta$. *Then* γ *induces a* **bijection** $\sigma : E_{\alpha} \xrightarrow{\sim} E_{\beta}$ *[so* $n_{\alpha} = n_{\beta}$*] such that*

$$\gamma(F_{\alpha} \bigcap H_{\alpha}) = F_{\beta} \bigcap H_{\beta}$$

for all **fiber subgroups** $F_{\alpha} \subseteq \Pi^{\alpha}$, $F_{\beta} \subseteq \Pi^{\beta}$, *whose respective* **profiles** $E'_{\alpha} \subseteq E_{\alpha}$, $E'_{\beta} \subseteq E_{\beta}$ *correspond via* σ.

Proof. Indeed, in light of Remark 7.2.1, Corollary 7.4 follows immediately by applying Corollaries 4.8, 6.3 to the isomorphism induced by γ between the *profinite completions* of H_{α}, H_{β}. ◯

Remark 7.4.1. There is a partial overlap between the content of Corollary 7.4 above and Theorems 1, 2 of [IIM].

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