AN INSTABILITY CRITERION FOR ACTIVATOR-INHIBITOR SYSTEMS IN A TWO-DIMENSIONAL BALL II

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Abstract. Let B be a two-dimensional ball with radius R. We continue to study the shape of the stable steady states to

$$u_t = D_u\Delta u + f(u,\xi) \ \text{ in } \ \mathrm{B}\times\mathbb{R}_+, \qquad \tau\xi_t = \frac{1}{|B|}\iint_B g(u,\xi)dxdy \ \text{ in } \ \mathbb{R}_+,$$
$$\partial_\nu u = 0 \ \text{ on } \ \partial B\times\mathbb{R}_+,$$

where f and g satisfy the following: $f_\xi(u,\xi) < 0$, $g_\xi(u,\xi) < 0$, and there is a function $k(\xi)$ such that $g_u(u,\xi) = k(\xi)f_\xi(u,\xi)$. This system includes a special case of the Gierer-Meinhardt system and the shadow system with the FitzHugh-Nagumo type nonlinearity. We show that, if the steady state (u,ξ) is stable for some $\tau > 0$, then the maximum (minimum) of u is attained at exactly one point on ∂B and u has no critical point in $B\backslash\partial B$. In proving this results, we prove a nonlinear version of the "hot spots" conjecture of J. Rauch in the case of B.

1. Introduction and the main results

This is a continuation of [Mi06a]. We study the shape of the stable steady states of shadow reaction-diffusion systems of an activator-inhibitor type
(SS_Ω)

$$u_t = D_u\Delta u + f(u,\xi) \ \text{ in } \ \Omega\times\mathbb{R}_+ \ \text{ and } \ \tau\xi_\tau = \frac{1}{|\Omega|}\iint_\Omega g(u,\xi)dxdy \ \text{ in } \ \mathbb{R}_+,$$
$$\partial_\nu u = 0 \ \text{ on } \ \partial\Omega\times\mathbb{R}_+,$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain. Here D_u and τ are positive constants. $|\Omega|$ denotes the area of Ω, and ∂_ν denotes the outer normal derivative on the boundary. In theoretical biology, the unknowns $u = u(x,t)$ and $\xi = \xi(t)$ stand for the concentrations of biochemicals called *the short range activator* and *the long range inhibitor*, respectively. Two concrete examples of (SS_Ω) are given at the end of this section. We consider the case when Ω is a two-dimensional ball B, centered at the origin, with radius R.

Throughout the present paper, we assume that

(N)
$$f(\,\cdot\,,\cdot\,),\ g(\,\cdot\,,\cdot\,) \text{ are of class } C^2,\ f_\xi < 0,\ g_\xi < 0, \text{ and}$$
$$\text{there is a function } k(\xi)\in C^0 \text{ such that } g_u(u,\xi) = k(\xi)f_\xi(u,\xi).$$

This class of reaction-diffusion systems includes a special case of the shadow system of the Gierer-Meinhardt system (Example 1.5 below) and the shadow system with the FitzHugh-Nagumo type nonlinearity (Example 1.6 below).

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In order to state our main results, we introduce some notation. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $\mathrm{int}(\Omega)$ denote the set consisting of all the interior points of Ω. Let $\xi(\zeta)$ and $\eta(\zeta)$ be functions satisfying $(\xi(\zeta), \eta(\zeta)) \in \partial\Omega$ parameterized by the arc length parameter ζ of $\partial\Omega$. Let $(u, \xi) \in (C^2(\mathrm{int}(\Omega)) \cap C^1(\Omega \cup \partial\Omega)) \times \mathbb{R}$ be a steady state to (SS_Ω). We define

$$U(\zeta) := u(\xi(\zeta), \eta(\zeta)), \qquad \zeta \in \mathbb{R}/L\mathbb{Z},$$

where L is the arc length of $\partial\Omega$. For example, $U(\zeta) = u(R\cos(\zeta/R), R\sin(\zeta/R))$ in the case that $\Omega = B$. Let $\mathcal{Z}[\,\cdot\,]$ denote the cardinal number of the zero level set of L-periodic functions. Specifically,

$$\mathcal{Z}[w(\,\cdot\,)] := \sharp\{\zeta;\ w(\zeta) = 0,\ \zeta \in \mathbb{R}/L\mathbb{Z}\},$$

where $w(\zeta) \in C^0(\mathbb{R}/L\mathbb{Z})$. For example, $\mathcal{Z}[\sin(2\pi\zeta/L)] = 2$.

Let us explain the activator-inhibitor system. The activator-inhibitor system is a mathematical model describing the interaction between the activator and the inhibitor. The activator activates the production rate of the inhibitor ($g_u > 0$), and the inhibitor suppresses the production rate of the activator ($f_\xi < 0$). The production rate of the inhibitor is decreased as the inhibitor increases ($g_\xi < 0$). However, we do not impose a monotonicity assumption on f with respect to u, because the activator may react autocatalytically and f may not be monotone in u. We call (SS_Ω) the shadow system of the activator-inhibitor type if f and g satisfy

$$\text{(AI)} \qquad f_\xi < 0, \quad g_u > 0, \quad \text{and} \quad g_\xi < 0.$$

The time constant of the inhibitor τ which appears in (SS_Ω) means the ratio of the reaction speeds between the activator and the inhibitor. If τ is large, then the inhibitor reacts slowly, and the system behaves like a scalar reaction-diffusion equation. In this case, we can expect and show that, if the domain is convex, then every inhomogeneous steady state is unstable for large $\tau > 0$ [Y06, E06]. On the contrary, if τ is small, then the inhibitor reacts quickly, and the system tends to be stable. Hence, an inhomogeneous stable steady state can exist. There is a possibility that a steady state that is unstable for large $\tau > 0$ is stable when $\tau > 0$ is small. (A Hopf bifurcation occurs as τ increases. See [NTY01, WW03] for the case of the shadow Gierer-Meinhardt system.) Therefore, it is important to obtain a sufficient condition, which can be determined by the shape, for steady states to be *unstable* not only in the case for large $\tau > 0$ but also in the case for *all* $\tau > 0$, because the contrapositive of the sufficient condition becomes a necessary condition for steady states to be stable for *some* $\tau > 0$. In other words, we know the shape of all the stable steady states. A partial result in this research direction is the following:

Proposition 1.1 ([Mi06a, Corollary B]). *Suppose that* (N) *holds. Let* (u, ξ) *be an inhomogeneous steady state to* (SS_B). *If* (u, ξ) *is stable for some* $\tau > 0$, *then* $\mathcal{Z}[U_\zeta(\,\cdot\,)] = 2$.

We know by Proposition 1.1 the shape of u on the boundary. However, we cannot obtain information about u in the interior of the domain. One of the main results of author's previous paper [Mi06b, Theorem 4.7] is a partial answer of this question. In [Mi06b], we show that, if $\sup_{(\rho_1,\rho_2)\in\mathbb{R}^2} f_u(\rho_1, \rho_2) < D_u\kappa_4$, then the conclusion of Theorem A below holds. Here, κ_4 is the forth eigenvalue of the Neumann Laplacian in B. In the present paper, we remove this assumption which seems to be technical. The main result of this paper is

Theorem A. *Suppose that* (N) *holds. Let* (u, ξ) *be an inhomogeneous steady state to* (SS_B). *If* (u, ξ) *is stable for some* $\tau > 0$, *then the maximum (minimum) of* u *is attained at exactly one point on* ∂B, *and there is no critical point of* u *in* $\mathrm{int}(B)$. *Here, we call* $p \in B$ *a critical point of* u *if* $u_x(p) = u_y(p) = 0$.

Note that we do not assume smallness or largeness of D_u.

From Theorem A we see that every stable steady state of (SS_B) does not have interior spikes or spots. Combining Theorem A and Proposition 1.1, we see that only the steady states whose shape are like a boundary one-spike layer can be stable.

Combining the results of [LT01, NT91, NTY01], we see that the shadow Gierer-Meinhardt system in B, which is (GM) below, has a stable boundary one-spike layer and that this inhomogeneous stable steady state satisfies that $\mathcal{Z}\left[U_\zeta(\,\cdot\,)\right] = 2$ and that the maximum of u is attained at exactly one point on ∂B. Thus their results are consistent with Proposition 1.1 and Theorem A.

Theorem A can be obtained by Proposition 1.1 and the contrapositive of the following instability criterion:

Theorem B. *Suppose that* (N) *holds. Let* (u, ξ) *be an inhomogeneous steady state to* (SS_B). *If there is a point* $p \in \mathrm{int}(B)$ *such that* $u_x(p) = u_y(p) = 0$, *then* (u, ξ) *is unstable for all* $\tau > 0$.

Remark 1.2 (An instability criterion for 1D shadow systems). In the case of one-dimensional intervals, every inhomogeneous steady state (u, ξ) of certain classes of shadow systems is unstable for all $\tau > 0$ if u has a critical point in the interior of the interval [N94, NPY01, FR01]. We see by the contrapositive that u should be monotone if the steady state is stable for some $\tau > 0$. Theorem B can be seen as a two-dimensional version of their result.

In order to state the main technical lemma, we consider a scalar elliptic equation on a bounded and convex domain

$$(\mathrm{NP}_\Omega) \qquad \Delta u + N(u) = 0 \ \text{ in } \ \Omega, \qquad \partial_\nu u = 0 \ \text{ on } \ \partial\Omega,$$

where $N(\,\cdot\,)$ is a function of class C^2. Let u be a solution of (NP_Ω). Let $\{(\mu_n(\Omega), \phi_n)\}_{n\geq 1}$ denote the set of the eigenpairs of the problem

$$(\mathrm{EP}_\Omega) \qquad \Delta\phi + N'(u)\phi = \mu\phi \ \text{ in } \ \Omega, \qquad \partial_\nu \phi = 0 \ \text{ on } \ \partial\Omega.$$

The main technical lemma of this paper is

Lemma C. *Let* u *be a non-constant solution to* (NP_B). *If there is a point* $p \in \mathrm{int}(B)$ *such that* $u_x(p) = u_y(p) = 0$, *then* $\mu_2(B) > 0$, *where* $\mu_2(B)$ *is the second eigenvalue of* (EP_B).

Note that no assumption of the nonlinear term $N(\,\cdot\,)$ is imposed except the regularity.

Lemma C is the positive answer of the following conjecture in the case of B:

Conjecture 1.3 ([Y06]). *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded and convex domain with smooth boundary, and let* u *be a non-constant solution to* (NP_Ω). *If there is a point* $p \in \mathrm{int}(\Omega)$ *such that* $u_x(p) = u_y(p) = 0$, *then* $\mu_2(\Omega) > 0$.

This is a nonlinear version of the "hot spots" conjecture of J. Rauch [R74]. The "hot spots" conjecture immediately follows from Conjecture 1.3. If Conjecture 1.3 holds, then Theorem B holds for all the two-dimensional bounded convex domains with smooth boundary. See also Proposition 2.6.

Remark 1.4 (An instability criterion for scalar equations). The following sufficient condition for the first eigenvalue to be positive is well-known: In the case when Ω is a bounded and convex domain in $\mathbb{R}^N$ with smooth boundary, and if a solution to (NP_Ω) is not constant, then $\mu_1(\Omega) > 0$. Therefore, the contrapositive is the following: Every stable steady state is constant in the case of convex domains. See [Ch75] for the one-dimensional case and [CH78, Ma79] for the multi-dimensional case.

As announced previously, we give two examples.

Example 1.5 ([GM72]). The shadow system of the Gierer-Meinhardt model [GM72] is the following:

$$\text{(GM)} \qquad u_t = D_u\Delta u - u + \frac{u^p}{\xi^q} \quad \text{and} \quad \tau\xi_t = \frac{1}{|\Omega|}\iint_\Omega \left(-\xi + \frac{u^r}{\xi^s}\right) dxdy,$$

where (p, q, r, s) satisfy $p > 1$, $q > 0$, $r > 0$, $s \geq 0$ and $0 < (p-1)/q < r/(s+1)$. The assumption on (p, q, r, s) comes from a biological reason. (AI) always holds. If $p = r - 1$, then (N) holds. This system is a model describing the head formation of hydra, which is a small creature. Specifically, [GM72] show experimentally that the head appears at the point where the activator u attains the local maximum. It is known that this system has steady states having various shapes (see [NT91, NT93, GW00, MM02] for example). Theorem A says that, if a steady state is stable, then exactly one local (hence global) maximum of u is attained on the boundary when $\Omega = B$. This result can be interpreted as follows: The head appears at exactly one point on the edge of the body.

Example 1.6 ([F61, NAY62]). The shadow system with the FitzHugh-Nagumo type nonlinearity [F61, NAY62] is the following:

$$u_t = D_u\Delta u + f_0(u) - \alpha\xi \quad \text{and} \quad \tau\xi_t = \frac{1}{|\Omega|}\iint_\Omega (\beta u - \gamma\xi)\, dxdy,$$

where α, β and γ are positive constants and $f_0(u)$ is the so-called cubic-like function. A typical example of f_0 is $u(1-u)(u-\delta)$ $(0 < \delta < 1)$. (AI) and (N) hold.

This paper consists of three sections. Section 2 has two subsections. In Subsection 2.1, we recall known results about the zero level set of the eigenfucntions, which we call *the nodal curves*. In Subsection 2.2, we recall known results about eigenvalues related to shadow systems satisfying (N). In Section 3, we prove the main results (Theorems A and B and Lemma C).

2. Preliminaries

2.1. **Known results on the nodal curves.** In this subsection, we recall known results about the nodal curves which are our main tools in Section 3.

Proposition 2.1 ([Ca33, HW53]). *Let $\Omega \subset \mathbb{R}^2$ be a domain, and let $V(x, y) \in C^0(\Omega)$. If ϕ satisfies $\Delta\phi + V\phi = 0$, then the nodal curves $\{\phi = 0\}$ consist of either the whole domain Ω or C^1-curves and intersections among those curves. If several curves intersect at one point, then they meet at equal angles.*

Let $\phi(x, y) \in C^1(\Omega)$. We say that $p \in \mathrm{int}(\Omega)$ is *a degenerate point* of ϕ if $\phi(x_0, y_0) = \phi_x(x_0, y_0) = \phi_y(x_0, y_0) = 0$.

A slight modification of the Carleman-Hartman-Wintner theorem [HW53] is

Proposition 2.2. *Let $V(x,y) \in C^0(\Omega)$, and let $\phi(x,y)$ be a function such that $\Delta\phi + V\phi = 0$ in Ω. If there exists a degenerate point $(x_0, y_0) \in \mathrm{int}(\Omega)$, then either* (i) *or* (ii) *holds:*
(i) $\phi \equiv 0$ *in* Ω,
(ii) *the nodal curves* $\{\phi = 0\}$ *have at least four branches at* (x_0, y_0)*. In this case, the measure of any connected component of* $\{\phi \neq 0\}$ *is not zero.*

Proposition 2.3 ([Mi06a, Lemma 4.3])**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary of class C^2, and let $V \in C^0(\Omega)$. Let ϕ be a non-trivial solution to*

$$\Delta\phi + V\phi = 0 \quad \text{in} \quad \Omega, \qquad \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega.$$

Suppose that there is a point $(x_1, y_1) \in \partial\Omega$ such that $\phi(x_1, y_1) = 0$ and that $\{\phi = 0\}$ is isolated in $\partial\Omega$ near (x_1, y_1). Then there is a nodal curve of ϕ connecting to (x_1, y_1).

Proposition 2.4 ([Mi06a, Lemma C])**.** *Let u be a solution to* (NP_B)*. If there is an open interval $\gamma \subset \partial B$ such that $U_\zeta \equiv 0$ on γ, then u is radially symmetric. In particular, u is constant on ∂B.*

Remark 2.5. From Proposition 2.4 we see

$$\mathcal{Z}\left[U_\zeta(\,\cdot\,)\right] = \begin{cases} n \in \mathbb{N}\backslash\{1\} & \text{if } u \text{ is not radially symmetric;} \\ \aleph_1 & \text{if } u \text{ is radially symmetric.} \end{cases}$$

2.2. **Known results on eigenvalues related to shadow systems.** In this subsection, we recall an abstract instability criterion.

Proposition 2.6 ([Mi06a, Lemma 3.2 (i)])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Suppose that* (N) *holds. Let (u, ξ) be a steady state to* (SS_Ω)*. If the second eigenvalue of the eigenvalue problem*

$$D_u\Delta\phi + f_u(u,\xi)\phi = \lambda\phi \quad \text{in} \quad \Omega, \qquad \partial_\nu\phi = 0 \quad \text{on} \quad \partial\Omega \tag{2.1}$$

is positive, then (u, ξ) is unstable for all $\tau > 0$. Specifically, the linearized operator of (SS_Ω) *at (u, ξ) has an eigenvalue with positive real part.*

Roughly speaking, shadow systems of the activator-inhibitor type have an effect removing the first eigenvalue of (2.1) [Ma05]. Hence, to determine the sign of the second eigenvalue is important for studying the stability.

This type of the results are obtained by several authors. In [Y02], the gradient case ($k(\xi) = 1$) and the skew-gradient case ($k(\xi) = -1$) are proven. The case of inhomogeneous media is also considered. In [E01], an argument similar to the proof of Proposition 2.6 appears in the case of some specific systems.

Suppose that ξ is fixed. Then the first equation of (SS_Ω) is a reaction-diffusion equation in homogeneous media. Specifically, f does not depend on x explicitly. (2.1) can be treated as an eigenvalue problem of scalar equations in homogeneous media. For simplicity, we do not write ξ in the nonlinear term in Section 3.

Thanks to Proposition 2.6, what we have to do is obtain a sufficient condition for the second eigenvalue of (2.1) to be positive.

Without loss of generality, we can assume that $D_u = 1$, because the sign of each eigenvalue of (2.1) does not change when Ω is rescaled to $\Omega/\sqrt{D_u}$.

3. Proofs of the main results

In this section, we mainly prove Lemma C. Specifically, we will show that $\mu_2(B) > 0$ if the solution of (NP_B) is not constant.

In proving the positiveness of the second eigenvalue $\mu_2(\Omega)$, we use a variational characterization of $\mu_2(\Omega)$. It is convenient to define a functional $\mathcal{H}[\,\cdot\,]$ by

$$\mathcal{H}[\psi] := \iint_{\Omega} \left(-|\nabla\psi|^2 + N'(u)\psi^2 \right) dxdy.$$

Lemma 3.1. *Let u be a non-constant solution of (NP_B). Then one of the following holds:*
(i) $\mathcal{H}[u_x] > 0$ *or* $\mathcal{H}[u_y] > 0$,
(ii) *u is radially symmetric.*

Proof. This lemma is well-known. We sketch the proof. The key ingredient is the following:

$$-\partial_\nu |\nabla u|^2 = \frac{2}{R^3} u_\theta^2 \quad \text{on} \quad \partial B, \tag{3.1}$$

where $u_\theta := -yu_x + xu_y$. We have

$$\begin{aligned}
&\mathcal{H}[u_x] + \mathcal{H}[u_y] \\
&\quad = \iint_B \left(-|\nabla u_x|^2 + N'(u)u_x^2 \right) dxdy + \iint_B \left(-|\nabla u_y|^2 + N'(u)u_y^2 \right) dxdy \\
&\quad = \iint_B (\Delta u_x + N'(u)u_x)\, u_x dxdy + \iint_B (\Delta u_y + N'(u)u_y)\, u_y dxdy \\
&\qquad\qquad - \int_{\partial B} (u_x \partial_\nu u_x + u_y \partial_\nu u_y)\, d\sigma.
\end{aligned}$$

Since $\Delta u_x + N'(u)u_x = 0$ and $\Delta u_y + N'(u)u_y = 0$, we see by (3.1) that $\mathcal{H}[u_x] + \mathcal{H}[u_y] \ge 0$. We show that $\mathcal{H}[u_x] + \mathcal{H}[u_y] \ne 0$ if u is not radially symmetric. Suppose the contrary, namely, $\mathcal{H}[u_x] + \mathcal{H}[u_y] = 0$. Then $u_\theta \equiv 0$ on ∂B. We see by Proposition 2.4 that u is radially symmetric. This is a contradiction. We see that $\mathcal{H}[u_x] + \mathcal{H}[u_y] > 0$ and that (i) holds if u is not radially symmetric. □

See [CH78, Ma79] for a result similar to Lemma 3.1 in the case of bounded convex domains in $\mathbb{R}^N$.

Lemma 3.2 ([Mi06a, Lemma 3.5])**.** *Let u be a non-constant solution to (NP_B). If u is radially symmetric, then $\mu_2(B) > 0$.*

Proof. See the proof of Lemma 3.5 in [Mi06a]. We omit the proof. □

Because of Lemma 3.2, we do not need to consider the case that u is radially symmetric. Hence, we can assume that (i) of Lemma 3.1 always occurs.

We define a rotational derivative of u with center (x_0, y_0) by

$$(\partial_\theta^{(x_0,y_0)} u)(x,y) := -(y-y_0)u_x(x,y) + (x-x_0)u_y(x,y).$$

Hereafter in this section, we consider the case when the nodal curves $\{\partial_\theta^{(x_0,y_0)} u = 0\}$ have a loop in the closure of the domain. We define ω by the area enclosed by the loop. Therefore, $\partial\omega$ is the loop. We define a function $z(x,y)$ by

$$z(x,y) := \begin{cases} (\partial_\theta^{(x_0,y_0)} u)(x,y) & \text{if} \quad (x,y) \in \omega; \\ 0 & \text{if} \quad (x,y) \in \Omega \backslash \omega. \end{cases}$$

Note that $\partial_\theta^{(x_0,y_0)}\Delta_{(x,y)} = \Delta_{(x,y)}\partial_\theta^{(x_0,y_0)}$.

We consider the case that $\partial_\theta^{(x_0,y_0)}u \equiv 0$. Then u is radially symmetric.

Suppose that Ω is not ball. There is a point (x_1, y_1) on $\partial\Omega$ such that the vector $(x_1 - x_0, y_1 - y_0)$ is not parallel to ν, where ν is an outer normal vector on the boundary. Therefore, there is a neighborhood Γ of (x_1, y_1) in $\partial\Omega$ such that $u_x = u_y = 0$ on Γ. Since u is radially symmetric and the vector $(x_1 - x_0, y_1 - y_0)$ is not perpendicular to the tangent line of $\partial\Omega$ at (x_1, y_1), u is constant on Γ and there is an open set in Ω such that u is constant. Thus the value of u at a point in the open set, say c, is a root of f, specifically $f(c) = 0$. Thus $\psi = u - c$ satisfies $\Delta\psi + V\psi = 0$, where $V := (f(u) - f(c))/(u - c)$, and ψ vanishes in the open set. We see by the strong unique continuation at an interior point that $u \equiv c$ in Ω. This case does not occur if u is not constant.

Suppose that Ω is ball. If (x_0, y_0) is not the center of B, then we see by the same argument that u is constant. If (x_0, y_0) is the center of B, then u is radially symmetric. Thus, u is constant or $\mu_2(B) > 0$ (Lemma 3.2). We do not need to consider the case that $\partial_\theta^{(x_0,y_0)}u \equiv 0$ in B.

When $\partial_\theta^{(x_0,y_0)}u \not\equiv 0$, we see that the measure of ω is not zero, that $z = 0$ on $\partial\omega$ and that

$$z > 0 \ \text{ in } \ \mathrm{int}(\omega) \qquad \text{or} \qquad z < 0 \ \text{ in } \ \mathrm{int}(\omega).$$

Lemma 3.3. (i) $\mathcal{H}[z] = 0$.
(ii) *Let* $u_\alpha := \cos\alpha u_x + \sin\alpha u_y$. *Then* $\iint_\Omega (-\nabla u_\alpha\cdot\nabla z + N'(u)u_\alpha z)\,dxdy = 0$.

Proof. We prove (i). We have

$$\begin{aligned}\mathcal{H}[z] &= \iint_\Omega \left(-|\nabla z|^2 + N'(u)z^2\right) dxdy = \iint_\omega \left(-|\nabla z|^2 + N'(u)z^2\right) dxdy\\ &= \iint_\omega (\Delta z + N'(u)z)\, z dxdy - \int_{\partial\omega} z\partial_\nu z d\sigma = 0,\end{aligned}$$

because $\Delta z + N'(u)z = 0$ in $\mathrm{int}(\omega)$ and $z = 0$ on $\partial\omega$.

We prove (ii). We have

$$\begin{aligned}&\iint_\Omega (-\nabla u_\alpha\cdot\nabla z + N'(u)u_\alpha z)\,dxdy = \iint_\omega (-\nabla u_\alpha\cdot\nabla z + N'(u)u_\alpha z)\,dxdy\\ &= \iint_\omega (\Delta u_\alpha + N'(u)u_\alpha)\, z dxdy - \int_{\partial\omega} z\partial_\nu u_\alpha d\sigma = 0,\end{aligned}$$

because $\Delta u_\alpha + N'(u)u_\alpha = 0$ and $z = 0$ on $\partial\omega$. □

Let ∂_τ denote a tangential derivative along $\partial\Omega$.

Lemma 3.4 ([Mi06b, Lemma 4.4])**.** *Let* $\Omega(\subset \mathbb{R}^2)$ *be a bounded convex domain with boundary of class* C^2*, and let* u *be a solution to* (NP_Ω)*. Suppose that* $(x_1, y_1) \in \partial\Omega$*. Then*

$$(\partial_\tau u)(x_1, y_1) = 0 \quad \textit{if and only if} \quad (\partial_\theta^{(x_0,y_0)}u)(x_1, y_1) = 0 \textit{ for all } (x_0, y_0) \in \mathrm{int}(\Omega).$$

In particular,

$$(\partial_\theta^{(x_0,y_0)}u)(x_1, y_1) = 0 \textit{ for some } (x_0, y_0) \in \mathrm{int}(\Omega) \quad \textit{if and only if} \quad (\partial_\theta^{(x_0,y_0)}u)(x_1, y_1) = 0 \textit{ for all } (x_0, y_0) \in \mathrm{int}(\Omega).$$

Moreover, if the nodal curves $\{\partial_\theta^{(x_0,y_0)}u=0\}$ *connect to* (x_1,y_1) *on the boundary, then* $(\partial_\tau u)(x_1,y_1)=0$*, hence,* $(x_1,y_1)\in\{U_\zeta=0\}$*.*

Proof. The tangent line of $\partial\Omega$ at (x_1,y_1) is not parallel to the vector (x_1-x_0,y_1-y_0), because Ω is convex. Hence if $(\partial_\theta^{(x_0,y_0)}u)(x_1,y_1)=(\partial_\nu u)(x_1,y_1)=0$, then $u_x(x_1,y_1)=u_y(x_1,y_1)=0$. Therefore $(\partial_\tau u)(x_1,y_1)=0$. Conversely, if $(\partial_\tau u)(x_1,y_1)=(\partial_\nu u)(x_1,y_1)=0$, then $u_x(x_1,y_1)=u_y(x_1,y_1)=0$. Thus $(\partial_\theta^{(x_0,y_0)}u)(x_1,y_1)=-(y_1-y_0)u_x(x_1,y_1)+(x_1-x_0)u_y(x_1,y_1)=0$ for all $(x_0,y_0)\in\mathrm{int}(\Omega)$. The latter half part of the statements is clear. □

Lemma 3.5. *Let* u *be a non-constant solution of* (NP_B)*. If the nodal curves* $\{\partial_\theta^{(x_0,y_0)}u=0\}$ *have a loop in the closure of* B*, then* $\mu_2(B)>0$*, where* $\mu_2(B)$ *is the second eigenvalue of* (EP_B)*.*

Proof. Because of Lemma 3.1, there is $\alpha\in\mathbb{R}/2\pi\mathbb{Z}$ such that $\mathcal{H}[u_\alpha]>0$. Let ϕ_1 denote the first eigenfunction of (EP_B). We define ψ_0 by

$$\psi_0:=u_\alpha+az,$$

where a is chosen so that $\langle\psi_0,\phi_1\rangle=0$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the usual L^2-inner product. Specifically, $a=-\langle u_\alpha,\phi_1\rangle/\langle z,\phi_1\rangle$. We see that $\langle z,\phi_1\rangle\neq0$, since ϕ_1 and z are continuous and do not change signs on the interior of the support set of z and the measure of the area enclosed by the loop is not zero.

We have

$$\begin{aligned}\mathcal{H}[\psi_0]&=\iint_B\left\{-|\nabla(u_\alpha+az)|^2+N'(u)(u_\alpha+az)^2\right\}dxdy\\&=\mathcal{H}[u_\alpha]+2a\iint_B(-\nabla u_\alpha\cdot\nabla z+N'(u)u_\alpha z)\,dxdy+a^2\mathcal{H}[z]=\mathcal{H}[u_\alpha]>0,\end{aligned}$$

where we use (i) and (ii) of Lemma 3.3. Therefore,

$$\mu_2(B):=\sup_{\psi\in\left(\mathrm{span}\langle\phi_1\rangle^\perp\cap H^1\right)}\frac{\mathcal{H}[\psi]}{\|\psi\|_2^2}\geq\frac{\mathcal{H}[\psi_0]}{\|\psi_0\|_2^2}>0,$$

where H^1 denotes the Sobolev space of order 1, $\|\cdot\|_2$ denotes the usual L^2-norm and $\mathrm{span}\,\langle\phi_1\rangle^\perp:=\{v\in L^2;\ \langle v,\phi_1\rangle=0\}$. □

Lemma 3.6 ([Mi06a, Lemmas 3.4 and 3.5])**.** *Let* u *be a non-constant solution of* (NP_B)*. If* $\mathcal{Z}[U_\zeta(\,\cdot\,)]\geq3$*, then* $\mu_2(B)>0$*, where* $\mu_2(B)$ *is the second eigenvalue of* (EP_B)*.*

Proof. Let $w(x,y):=(\partial_\theta^{(0,0)}u)(x,y)$. Since

$$\Delta w+N'(u)w=0\ \text{ in }\ B,\qquad \partial_\nu w=0\ \text{ on }\ \partial B,$$

0 is an eigenvalue of (EP_B). There are two cases. One case is that $\mathcal{Z}[U_\theta(\,\cdot\,)]\in\mathbb{N}\backslash\{1,2\}$. There is a nodal curve $\{w=0\}$ connecting to each of $\{U_\zeta=0\}$, because of Proposition 2.3. Thus w has at least three points on the boundary which nodal curves connect to. It follows from an elementary topological argument of two-dimensional domains that w has at least three nodal domains. Courant's nodal theorem says that 0 is not the first or second eigenvalue. This means that $\mu_2(B)$ cannot be 0 or negative. Thus $\mu_2(B)>0$. The other case is that $\mathcal{Z}[U_\zeta(\,\cdot\,)]=\aleph_1$. Because of Remark 2.5, u should be radially symmetric. In this case, we see by Lemma 3.2 that $\mu_2(B)>0$. □

Proof of Lemma C. There are two cases. One case is that $\mathcal{Z}[U_\zeta(\,\cdot\,)] \geq 3$. We see that $\mu_2(B) > 0$, using Lemma 3.6.

The other case is that $\mathcal{Z}[U_\zeta(\,\cdot\,)] = 2$. Let $p = (x_0, y_0)$ be an interior point of B such that $u_x(p) = u_y(p) = 0$. Let $w(x,y) := (\partial_\theta^{(x_0,y_0)} u)(x,y)$. Since

$$\begin{aligned} w(x,y) &= -(y-y_0)u_x + (x-x_0)u_y, \\ w_x(x,y) &= -(y-y_0)u_{xx} + u_y + (x-x_0)u_{yx} \quad \text{and} \\ w_y(x,y) &= -u_x - (y-y_0)u_{xy} + (x-x_0)u_{yy}, \end{aligned}$$

we see that $w(x_0,y_0) = w_x(x_0,y_0) = w_y(x_0,y_0) = 0$. Therefore, $p = (x_0,y_0)$ is a degenerate point of w. Because of Proposition 2.2, there are at least four branches of the nodal curves $\{w = 0\}$ at p, otherwise, $w \equiv 0$ in B and we already showed that $\mu_2(B) > 0$ or u is constant. Each branch should connect to one of the branches or the boundary of the domain. If there is a branch connecting to one of the branches, then there exists a loop, and Lemma 3.5 says that $\mu_2(B) > 0$. We consider the case that all the branches connect to the boundary of the domain. Because of Lemma 3.4, all the branches connect to one of the zero set $\{U_\zeta = 0\}$. However, it is impossible that this occurs without loop, because $\mathcal{Z}[U_\zeta(\,\cdot\,)] = 2$ and there are at least four branches at p. Thus there is a loop of $\{w = 0\}$, and we see by Lemma 3.5 that $\mu_2(B) > 0$. □

Proof of Theorem B. Because of the assumption, u has a critical point in $\mathrm{int}(B)$. From Lemma C we see that the second eigenvalue of (2.1) is positive. We see by Proposition 2.6 that (u, ξ) is unstable. □

Proof of Theorem A. Because of the contrapositive of Theorem B, u has no critical point in int(B), hence, the maximum (minimum) of u is attained on ∂B. We see by Proposition 1.1 that the maximum (minimum) point of u should be unique. □

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