Group invariance and L^p-bounded operators

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Abstract

In this paper we consider translation invariant operators with additional symmetry coming from group actions. As the classic Hilbert and Riesz transforms can be characterized up to scalar by means of relative invariance of conformal transformation groups, certain multiplier operators are characterized by relative invariance of some other affine subgroups. In this article, we formalize a geometric condition that characterizes specific multiplier operators uniquely up to scalar, and provide several examples of multiplier operators having 'large symmetry'. Finally, we classify which of these examples are L^p-bounded.

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Contents

1 Introduction

Our object of study is translation invariant operators bounded on $L^p(\mathbf{R}^n)$ from the viewpoint of group invariance, with emphasis on 'maximal symmetry' that is satisfied by specific operators.

1.1 Hilbert and Riesz transforms

Classic examples of translation invariant singular integrals are the Hilbert transform H, which is defined on (a dense subspace of) $L^2(\mathbf{R}^n)$ by

$$Hf(x) := \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{f(x-y)}{y} \, dy,$$

and the Riesz transforms R_j as its higher dimensional generalization:

$$R_j f(x) := \lim_{\epsilon \to 0} \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n+2}{2}}} \int_{|y| \geq \epsilon} \frac{y_j}{|y|^{n+1}} f(x-y)\, dy \quad (1 \leq j \leq n). \tag{1.1.1}$$

Hilbert and Riesz transforms have been used in various aspects of analysis such as

1) (harmonic analysis) L^p-convergence of Fourier series,
2) (differential equations) regularity properties of solutions to the Laplace equation.

For more details of applications and perspectives of these operators in analysis, we refer the reader to the survey papers [F2] and [S2].

On the other hand, these translation operators enjoy further group symmetry. We begin with an observation that the Hilbert transform satisfies the following two properties:

$$\tau_a \circ T = T \circ \tau_a \qquad \text{for all } a \in \mathbf{R}, \tag{1.1.2}$$

$$D_\eta \circ T = \operatorname{sgn}(\eta) T \circ D_\eta \qquad \text{for all } \eta \in \mathbf{R}^*. \tag{1.1.3}$$

Here we have used the notation:

$$(\tau_a \circ f)(x) := f(x-a) \quad \text{for } a \in \mathbf{R},$$
$$(D_\eta f)(x) := f(\eta x) \quad \text{for } \eta \in \mathbf{R}^*,$$

for translation and dilation, respectively. By translation invariant operators we mean a bounded operator satisfying the condition (1.1.2). The condition (1.1.3) is regarded as an additional group invariance, on which we shall focus in this article. The viewpoint here is that the group invariance (1.1.3) is strong enough to characterize the Hilbert transform in the sense that any translation invariant operator acting on $L^2(\mathbf{R})$ and satisfying (1.1.3) must be the Hilbert transform up to scalar multiple (see [S, Section 3.1] or [EG, Section 6.8]).

The Riesz transforms can be also characterized in a similar manner. For $f \in L^2(\mathbf{R}^n)$ and $g \in \mathrm{O}(n)$, we set $l_g(f)(x) := f(g^{-1}x)$. Let $\pi(g)$ be the standard representation of $\mathrm{O}(n)$ on $\mathbf{R}^n$. We then have:

Fact 1.1 ([S, Section 3.1, Proposition 2]). [2] *A family of translation invariant operators* $\bar{T} = {}^t(T_1, \ldots, T_n)$ *bounded on* $L^2(\mathbf{R}^n)$ *and commuting with positive dilations, satisfies the identity* $l_{g^{-1}} \circ \bar{T} \circ l_g = \pi(g) \circ \bar{T}$ *for* $g \in \mathrm{O}(n)$*, if and only if, up to a constant multiple, it is the family of Riesz transforms.*

We shall come back to these examples in Subsection 2.3 after we formalize a general framework to work in.

1.2 Strategies

In light of the aforementioned invariance properties of the Hilbert and Riesz transforms, we may expect that 'nice translation invariant operators' ought to enjoy additional symmetry. To reveal such invariance conditions arising from the affine transformation group, we propose the following strategies:

Strategy 1 (Characterization of translation invariant operators). *Suppose we are given a translation invariant operator bounded on* $L^2(\mathbf{R}^n)$*(or a family of such operators):*

Step 1. Find a (maximal) group of relative invariance of this operator.

Step 2. Conversely, find all bounded translation invariant operators that satisfy the same condition of relative invariance.

We are particularly interested in the case where Step 2 yields a finite dimensional (or even preferably, one dimensional) space of operators. Then, we might say that Strategy 1 gives a characterization of the original operator.

This idea could be used in reverse to find new operators by starting from group invariance:

Strategy 2 (Finding nice operators). *Suppose we are given an invariance condition by means of a subgroup of the affine transformation group* $\mathrm{Aff}(\mathbf{R}^n)$*:*

Step 1. Find explicitly all solutions that satisfy the invariance conditions.

Step 2. Choose the solutions that yield L^2*-bounded (or* L^p*-bounded) operators.*

[2] It was stated as $l_g \circ \bar{T} \circ l_{g^{-1}} = \pi(g) \circ \bar{T}$ in our notation, but g should read as g^{-1} in the left-hand side.

The point of Strategy 2 is to find a nice invariance condition such that the resulting space of operators in Step 2 is one dimensional, or at least non-zero and finite dimensional.

We will give a rigorous formulation in Theorem 1 in Subsection 2.1 to pursue Strategies 1 and 2. The aforementioned characterization of the Hilbert and Riesz transforms (Fact 1.1) is reexamined in Subsection 2.2. Furthermore, Stein's higher Riesz transforms (see Example 2.2.1 (2)) are obtained in this framework. Relative invariants of Sato's prehomogeneous vector spaces (see [Sa]) are also examples of the solutions in Strategy 2. The formalization of Theorem 1 is built on 'vector valued relative invariants' of prehomogeneous vector spaces.

In Sections 3 and 4, we shall illustrate our general framework (Theorem 1) by the examples of translation invariant operators with additional group invariance defined by the following affine subgroups

$$
\begin{aligned}
&(\mathbf{R}_+^* \times \mathrm{SO}(p,q)) \ltimes \mathbf{R}^{p+q} \qquad \subset \mathrm{Aff}(\mathbf{R}^{p+q}) \quad \text{(see Theorems 2, 3 and 4)},\\
&(\mathrm{O}(m) \times \mathrm{GL}_+(k,\mathbf{R})) \ltimes \mathbf{R}^{km} \subset \mathrm{Aff}(\mathbf{R}^{km}) \quad \text{(see Theorem 5)}.
\end{aligned}
$$

The latter example reproduces the Riesz transforms when $k = 1$.

In Subsection 5.2, we shall determine which of the L^2-bounded invariant operators obtained in Theorems 2, 3, 4 and 5 give L^p-bounded operators ($1 < p < \infty$). The classification is given in Theorems 6, 7, 8 and 9, respectively.

Generalizations:

Theorem 1 deals with invariance conditions of operators defined by finite dimensional representations of (almost) connected subgroups of the affine transformation groups. In subsequent papers we shall consider two directions of generalization of our strategies:

1) (A generalization from continuous to discrete)
   In [KN], we shall consider the relative invariance for semigroup actions in place of the relative invariance for group actions. This generalization allows us, for example, to give a characterization of *discrete* Riesz transforms on $\mathbf{T}^n$ and $\mathbf{Z}^n$, extending previous work by Edwards and Gaudry [EG].

2) (A generalization from finite dimensional to infinite dimensional representations)

In [KN2], we shall use unitary representations in place of finite dimensional representations for the 'symmetry' of operators. This generalization yields much more bounded invariant multiplier operators. Besides, we also generalize the formulation of our strategies by means of differential equations rather than the group action itself. A typical example is when the group $\mathbf{R}_+ \times \mathrm{O}(p,q)$ acts on $\mathbf{R}^n$ by the natural action and the discrete series representations for hyperboloids play an important role in constructing invariant L^p-bounded operators.

2 Formulation of relative invariance

2.1 Affine actions and translation invariant operators

In this section we will generalize the setting of the Introduction, and introduce the notion of translation invariant operators with additional symmetry by using group representations.

For $f \in L^2(\mathbf{R}^n)$ we define $(l_g f)(t) = f(g^{-1}t)$, for $g \in \mathrm{GL}(n,\mathbf{R})$. Let H be a subgroup of $\mathrm{GL}(n,\mathbf{R})$ and take a finite dimensional representation (π, V) of H. We write (π^*, V^*) for the contragredient representation of (π, V). As H acts on $\mathbf{R}^n$, so does it on the character group $(\mathbf{R}^n)^*$ by the contragredient action: $\lambda \mapsto {}^t h^{-1}\lambda$. We will assume that H acts on $(\mathbf{R}^n)^*$ with finitely many open orbits, $\mathcal{O}_1, \dots, \mathcal{O}_N$ such that their union is conull in $(\mathbf{R}^n)^*$. The orbits $\mathcal{O}_j$ are expressed as homogeneous spaces H/H_j. Let $C_{\mathrm{bdd}}(\mathcal{O}_j)$ denote the complex vector space consisting of bounded continuous functions on $\mathcal{O}_j$, on which the group H acts by pullback of functions. By $\mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n))$ we denote the vector space consisting of bounded, translation invariant operators $T : L^2(\mathbf{R}^n) \to V \otimes L^2(\mathbf{R}^n)$ satisfying

$$\begin{array}{ccc} L^2(\mathbf{R}^n) & \xrightarrow{\quad T \quad} & V \otimes L^2(\mathbf{R}^n) \\ {\scriptstyle l_g}\downarrow & & \downarrow{\scriptstyle \pi(g)\otimes l_g} \\ L^2(\mathbf{R}^n) & \xrightarrow{\quad T \quad} & V \otimes L^2(\mathbf{R}^n) \end{array} \tag{2.1.1}$$

for all $g \in H$.

Theorem 1 (Description of invariant bounded operators).

1) There is a natural isomorphism of vector spaces:

$$\mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n)) \cong \bigoplus_{j=1}^{N} \mathrm{Hom}_H(V^*, C_{\mathrm{bdd}}(\mathcal{O}_j)). \tag{2.1.2}$$

2) The left-hand side of (2.1.2) *is one dimensional if H acts on* $(\mathbf{R}^n)^*$ *with an open dense orbit $\mathcal{O}_1$ and if*

$$\dim \mathrm{Hom}_H(V^*, C_{\mathrm{bdd}}(\mathcal{O}_1)) = 1.$$

3) (Upper bound) Let $V^{H_j} := \{v \in V : \pi(h)v = v$ for any $h \in H_j\}$.

$$\dim \mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n)) \le \sum_{j=1}^{N} \dim V^{H_j}. \tag{2.1.3}$$

Corollary 2.1.1. *If* $\dim V = 1$ *then* $\dim \mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n)) \le N$*. In particular, the operator is unique, up to a scalar, on each orbit if it exists.*

Proof. On each open orbit $\mathcal{O}_j$, we have

$$\dim \mathrm{Hom}_H(V^*, C_{\mathrm{bdd}}(\mathcal{O}_j)) \le \dim \mathrm{Hom}_H(V^*, \mathcal{C}(\mathcal{O}_j)) \le 1. \tag{2.1.4}$$

□

It is natural to seek for geometric conditions to ensure that $\dim \mathrm{Hom}_H(V^*, \mathcal{C}(\mathcal{O}_j)) \le 1$, even if $\dim V > 1$. Here is a sufficient condition:

Corollary 2.1.2. *Suppose H is a reductive Lie group. If $\mathcal{O}_j$ is a symmetric space of H, then*

$$\dim \mathrm{Hom}_H(V^*, \mathcal{C}(\mathcal{O}_j)) \le 1.$$

If all the orbits $\mathcal{O}_j$ are symmetric spaces, then $\dim \mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n)) \le N$ *for any irreducible finite dimensional representation (π, V) of H.*

Proof. Suppose $\mathcal{O}_j \simeq H/H_j$ is a reductive symmetric space. Then, for any irreducible finite dimensional representation (π, V), we have $\dim V^{H_j} \le 1$ by a theorem of É. Cartan ([C, Sect. 17], see also [K, Fact 29] and references therein for related results). The result then follows from (2.1.3). □

2.2 Classic examples

Example 2.2.1.

1) *(Riesz transforms) Stein's theorem (Fact 1.1) can be explained in the framework of Theorem 1 where* $H = \mathbf{R}_+ \times \mathrm{O}(n)$, π *is the trivial extension of the standard representation of* $\mathrm{O}(n)$ *to* H, *and* $V = \mathbf{R}^n$. *Then,* H *has an open dense orbit* $\mathcal{O}_1 = (\mathbf{R}^n)^* \setminus \{0\}$ *in* $(\mathbf{R}^n)^*$, *and therefore* $N = 1$. $\mathcal{O}_1 \simeq H/\mathrm{O}(n-1)$ *is a symmetric space.*

2) *(Higher Riesz transforms) Observe that the standard representation of* $\mathrm{O}(n)$ *on* $\mathbf{R}^n$ *is equivalent to the spherical harmonics representation of degree one. This observation leads us to a family of invariant operators that enjoy the same symmetry as spherical harmonics representations. Stein called these operators higher Riesz transforms. Higher Riesz transforms appear in the algebra generated by the Riesz transforms. See [S, Section 3.3 and 3.4.8] for further properties on these operators.*

Remark 2.2.2. *Fact 1.1 still holds if we replace* $\mathrm{O}(n)$ *by* $\mathrm{SO}(n)$ *for* $n \geq 3$.

Remark 2.2.3. *Suppose we are in the setting of Example 2.2.1 (1), but we let* π, *instead of being the trivial extension, be the extension taking elements of* $(r, \phi) \in R_+ \times O(n)$ *to* $r^a \pi(\phi)$. *If* $0 < a \leq n/2$ *then we have*

$$\mathrm{Hom}_H(L^2(\mathbf{R}^n), \mathbf{R}^n \otimes L^2(\mathbf{R}^n)) = 0,$$
$$\dim \mathrm{Hom}_H(L^2(\mathbf{R}^n), \mathbf{R}^n \otimes L^p(\mathbf{R}^n)) = 1 \quad (1/p = 1/2 - a/n).$$

This follows essentially from the proof of Fact 1.1 (or Theorem 1) and the Hardy–Littlewood–Sobolev theorem, see [S, Theorem 1.2.1].

Example 2.2.4. *Assume* $\dim V = 1$. *In the theory of prehomogeneous vector spaces, a non-trivial function on* $\mathcal{O}_j$ *contained in the image of* $\mathrm{Hom}_H(V^*, \mathcal{C}(\mathcal{O}_j))$, *is called a* **relative invariant**. *The corresponding one dimensional representation* (π^*, V^*) *defines a function on* H *by* $h \mapsto \pi^*(h)$, *which is called Sato–Bernstein's* b**-function**, *see [Sa] for more details. We shall give some examples in Subsections 3.1 and 3.2.*

The above three examples treat the cases where either $\dim V = 1$ or the orbits $\mathcal{O}_j$ are symmetric spaces. Corollaries 2.1.1 and 2.1.2 ensured that the dimension of $\mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n))$ does not exceed the number of open H-orbits on $\mathbf{R}^n$, giving a characterization of such operators by group invariance. However, there are also interesting examples where $\dim \mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n)) \leq 1$, even though $\dim V$ can be greater than one and the orbit is not a symmetric space. In Subsection 4.2 we will provide such an example where $\mathrm{O}(k) \times \mathrm{GL}(m, \mathbf{R})$ is acting on $\mathbf{R}^{mk}$.

Example 2.2.5. *Consider the action on $\mathbf{R}^n$ by the group $H = \mathbf{R}_+ \times \mathrm{O}(p,q)$, $pq > 0$. Let π be the standard representation of $\mathrm{O}(p,q)$ on $V := \mathbf{C}^{p+q}$ extended trivially to $\mathbf{R}_+$. In this case H has two open orbits, namely, $\mathcal{O}_1 = \mathbf{R}_+ \times \mathrm{O}(p,q)/\mathrm{O}(p-1,q)$ and $\mathcal{O}_2 = \mathbf{R}_+ \times \mathrm{O}(p,q)/\mathrm{O}(p,q-1)$. Both quotients are reductive symmetric spaces and the representation π appears in $\mathcal{C}(\mathcal{O}_1)$ as well as $\mathcal{C}(\mathcal{O}_2)$. Hence, Example 2.1.2 tells us that $\dim \mathrm{Hom}_H(V, \mathcal{C}(\mathcal{O}_1) \oplus \mathcal{C}(\mathcal{O}_2)) = 2$. However, in this case, the space $\mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n))$ is in fact trivial.*

Example 2.2.5 shows a typical feature of the action of a non-compact group. See Proposition 2.3.1.

2.3 L^2-boundedness and unitarizability

Suppose $\pi : H \to \mathrm{GL}_{\mathbf{C}}(V)$ is a finite dimensional representation of a group H. We say (π, V) is *unitarizable* if there exists an H-invariant Hermitian inner product on V, and is *non-unitarizable* if not.

For compact H, any finite dimensional representation is unitarizable. However, this is not the case for noncompact H. For example, if $H = \mathrm{SL}(n, \mathbf{R})$ and π is the natural representation of H on $V = \mathbf{R}^n$, then (π, V) is non-unitarizable for $n > 1$.

Proposition 2.3.1. *Retain the notation of Theorem 1.*

1) If (π, V) is a unitarizable representation, then we have

$$\dim \mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n)) = \sum_{j=1}^{N} \dim V^{H_j}.$$

2) If (π, V) is a non-unitarizable representation of a reductive Lie group H, then $\mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n)) = \{0\}$.

Proof. Theorem 1 shows that

$$\mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n)) \cong \bigoplus_{j=1}^{N} \mathrm{Hom}_H(V^*, C_{\mathrm{bdd}}(\mathcal{O}_j)).$$

In general we have the following:

Lemma 2.3.2. *Suppose $\mathcal{O} \simeq H/H_0$ is a homogeneous space of H. Then, there is a natural isomorphism between the two vector spaces $\mathrm{Hom}_H(V^*, \mathcal{C}(\mathcal{O}))$ and V^{H_0}.*

Proof of Lemma. Let ϕ be an element in $\mathrm{Hom}_H(V^*, \mathcal{C}(\mathcal{O}))$. We define a V-valued function $F : \mathcal{O} \to V$ by the relation $\phi(v^*)(x) = \langle F(x), v^* \rangle$, for $x \in \mathcal{O}$ and $v^* \in V^*$. Then, F is a V-valued continuous function satisfying the relation

$$F(h^{-1}x) = \pi(h)F(x) \quad \text{for } h \in H \text{ and } x \in \mathcal{O}.$$

We denote by $\mathcal{C}(\mathcal{O}, V)^H$ the vector space of such V-valued continuous functions on $\mathcal{O}$. Next, let $o := eH_0 \in \mathcal{O} \simeq H/H_0$. Then, $u := F(o)$ satisfies

$$\phi(v^*)(h^{-1}o) = \langle F(h^{-1}o), v^* \rangle = \langle \pi(h)u, v^* \rangle \quad \text{for any } v^* \in V^*. \qquad (2.3.1)$$

In particular, $\pi(h)u = u$ if $h \in H_0$. Then, ϕ is recovered from $u \in V^{H_0}$ by the relation $\phi(v^*)(x) = \langle \pi(h^{-1})u, v^* \rangle$ if $x = h \cdot o$ (the right-hand side does not depend on the choice of $h \in H$ such that $x = h \cdot o$). It is now readily seen that the correspondence $\phi \mapsto F \mapsto F(o)$ gives the following bijections:

$$\mathrm{Hom}_H(V^*, \mathcal{C}(\mathcal{O})) \xrightarrow{\sim} \mathcal{C}(\mathcal{O}, V)^H \xrightarrow{\sim} V^{H_0}.$$

□

1) Suppose (π, V) is a unitary representation. Then, any matrix coefficient is bounded because the operator norm $\|\pi(g)\| = 1$. Thus from (2.3.1) and Lemma 2.3.2 it follows that

$$\mathrm{Hom}_H(V^*, C_{\mathrm{bdd}}(\mathcal{O})) \cong \mathrm{Hom}_H(V^*, \mathcal{C}(\mathcal{O})) \cong V^{H_0}.$$

2) Suppose (π, V) is non-unitarizable. Then, we have

$$\mathrm{Hom}_H(V^*, C_{\mathrm{bdd}}(\mathcal{O})) = \{0\}.$$

This is a consequence of the lemma below which shows that matrix coefficients of a non-unitary representation of a reductive Lie group are unbounded. □

Lemma 2.3.3. *Let G be a simple, connected and non-compact Lie group, and (π, V) a finite dimensional representation of G. Assume further that all the matrix-coefficients of π are bounded functions on G. Then π is trivial.*

Proof of Lemma. By replacing G with $\pi(G)(\subset \mathrm{GL}_{\mathbf{C}}(V))$ if necessary, we may and do assume that G is a linear group contained in its complexification $G_{\mathbf{C}}$. We extend the representation holomorphically to $G_{\mathbf{C}}$. Let K be a maximal compact subgroup of G, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, and G_U a maximal compact subgroup of $G_{\mathbf{C}}$ containing K. As G_U is compact, there exists a G_U-invariant inner product on V. (In fact, for an arbitrary inner product on V, the new inner product defined by taking the average over G_U becomes G_U-invariant.)

Let A be a maximally split subgroup of G, and $\mathfrak{a}$ its Lie algebra, and $T := \exp(i\mathfrak{a}) \subset G_U$. As $\pi|_T$ is unitarizable the differential $d\pi|_{\mathfrak{a}}$ has only real eigenvalues. The assumption that the matrix coefficients are bounded implies that $\pi|_A$ is trivial. This means also that π is trivial on $\exp(\mathfrak{p})$ because all elements in $\exp(\mathfrak{p})$ are conjugate to an element in A. As $\exp(\mathfrak{p})$ builds up G as a group, we obtain that $\pi|_G$ is trivial. This is what we wanted. □

2.4 Multipliers

We write $\mathcal{F} : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ for the Fourier transform

$$\mathcal{F}(f)(\lambda) = \int_{\mathbf{R}^n} e^{-2\pi i \langle x, \lambda \rangle} f(x)\, dx,$$

and $\mathcal{F}^{-1}$ for its inverse. For a bounded measurable function $m(\lambda)$, we set

$$T_m(f) = \mathcal{F}^{-1}(m(\cdot)\mathcal{F}(f)(\cdot)).$$

Then, $T_m : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ is a bounded translation invariant operator. Such an operator T_m is called a *multiplier operator* associated to the *multiplier* m. Conversely, any bounded translation invariant operator

$$T : L^p(\mathbf{R}^n) \to L^p(\mathbf{R}^n), \qquad 1 < p < \infty$$

is bounded on $L^2(\mathbf{R}^n)$ as well, and has the form T_m with a bounded function $m(\lambda)$. In other words, $f \mapsto \mathcal{F} \circ T \circ \mathcal{F}^{-1}(f)$ is given by a multiplication of a bounded measurable function $m(\lambda)$ if T is a bounded translation invariant operator.

From now on, we shall identify bounded translation invariant operators with multiplier operators.

2.5 Proof of Theorem 1

Proof. Suppose $T : L^2(\mathbf{R}^n) \to V \otimes L^2(\mathbf{R}^n)$ is a bounded translation invariant operator. Then, there exists a bounded measurable V-valued function m on $\mathbf{R}^n$ such that

$$m(\lambda)(\mathcal{F}f)(\lambda) = (\mathrm{id} \otimes \mathcal{F}) \circ (Tf)(\lambda) \quad \text{for } f \in L^2(\mathbf{R}^n), \tag{2.5.1}$$

that is, the multiplication by m gives the multiplier operator (we use the same letter m):

$$m = (\mathrm{id} \otimes \mathcal{F}) \circ T \circ \mathcal{F}^{-1}.$$

We recall that the Fourier transform $\mathcal{F}$ satisfies

$$(\mathcal{F}f(g^{-1}\cdot))(\lambda) = \int_{\mathbf{R}^n} f(g^{-1}x)e^{-2\pi i\langle x,\lambda\rangle}dx = |\det g|\,(\mathcal{F}f)({}^t g\lambda),$$

that is,

$$\mathcal{F} \circ l_g = |\det g|\, l_{{}^t g^{-1}} \circ \mathcal{F}$$

for $g \in \mathrm{GL}(n, \mathbf{R})$. Suppose now $T \in \mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n))$. Then, we have

$$\begin{aligned} m \circ |\det g|\, l_{{}^t g^{-1}} \circ \mathcal{F} &= m \circ \mathcal{F} \circ l_g \\ &= (\mathrm{id} \otimes \mathcal{F}) \circ T \circ l_g \\ &= (\mathrm{id} \otimes \mathcal{F}) \circ (\pi(g) \otimes l_g) \circ T \\ &= (\pi(g) \otimes (\mathcal{F} \circ l_g)) \circ T \\ &= (\pi(g) \otimes |\det g|\, l_{{}^t g^{-1}} \circ \mathcal{F}) \circ T \\ &= |\det g|\,(\pi(g) \otimes l_{{}^t g^{-1}}) \circ (\mathrm{id} \otimes \mathcal{F}) \circ T \\ &= |\det g|\,(\pi(g) \otimes l_{{}^t g^{-1}}) \circ m \circ \mathcal{F} \end{aligned}$$

for $g \in H$. Cancelating the determinant factor on both sides, we obtain

$$m(\lambda) = \pi(g)m({}^t g\lambda) \quad \text{for } g \in H, \tag{2.5.2}$$

where we have kept the same notation, m, for the bounded vector valued function corresponding to the multiplier operator. Also, any bounded vector valued function satisfying (2.5.2) gives rise to a translation invariant operator, T, satisfying (2.1.1). Thus, we have proved the following isomorphism of vector spaces:

$$\begin{aligned} &\mathcal{B}_H(L^2(\mathbf{R}^n), V \otimes L^2(\mathbf{R}^n)) \\ \simeq &\{m : \mathbf{R}^n \to V, \text{ bounded and satisfying (2.5.2)}\}. \end{aligned}$$

Since $\bigcup_{j=1}^{N} \mathcal{O}_j$ is conull in V, the right-hand side is isomorphic to

$$\bigoplus_{j=1}^{N} \{m : \mathcal{O}_j \to V, \text{bounded and satisfying (2.5.2)}\}.$$

A function satisfying (2.5.2) is continuous on each orbit $\mathcal{O}_j$(to be more precise, we identify functions which coincide almost everywhere). Hence, we obtain

$$\simeq \bigoplus_{j=1}^{N} (C_{\text{bdd}}(\mathcal{O}_j) \otimes V)^H,$$

the subspace of H-fixed vectors in the tensor product representation. By duality this can be rewritten as

$$\simeq \bigoplus_{j=1}^{N} \text{Hom}_H(V^*, C_{\text{bdd}}(\mathcal{O}_j)).$$

This proves the first statement of the theorem. The second statement is clear from (2.1.2). The upper estimate (2.1.3) follows from Lemma 2.3.2. Thus the theorem follows. □

3 Examples of invariant multipliers ($\dim V$=1)

Our main results in this section, Theorems 2, 3 and 4, all exemplify Strategy 2 of Introduction. In Subsection 3.1 we consider the case when the group $GL(2, \mathbf{R})$ is acting on $\mathbf{R}^3$. In Subsection 3.2, the group is $GL(2) \times GL(2)$ and the space is $\mathbf{R}^4$. In Subsection 3.3 these two examples are generalized by considering the group $SO_0(p, q) \times \mathbf{R}_+$ acting on $\mathbf{R}^{p+q}$. In these three examples the dimension of the representation space, V, is 1. By Corollary 2.1.1 we know that the dimension of the space of invariant operators will be at most 1 for each orbit. Hence, in order to determine all the invariant operators, it will be enough to find a single operator which satisfies the given invariance condition for each orbit. This will be carried out in Theorems 2, 3 and 4.

We will consider L^p-boundedness of the operators characterized in these examples later in Subsection 5.2. In contrast to this section, Section 4 provides an example where the dimension of the representation space is greater than 1.

3.1 Invariant multipliers for $(\mathrm{GL}(2,\mathbf{R}),\mathbf{R}^3)$

We will identify the set of real symmetric matrices $S = \mathrm{Symm}(2)$ with $\mathbf{R}^3$ by the map

$$\begin{pmatrix} x & z \\ z & y \end{pmatrix} \mapsto (x, y, z).$$

We define three open subsets in the dual space $S^* \cong \mathbf{R}^3$ by

$$\begin{aligned} \mathcal{O}_{++} &= \{\lambda = (\lambda_1, \lambda_2, \lambda_3) : \lambda_1 + \lambda_2 > 0, \lambda_1\lambda_2 - \lambda_3^2 > 0\}, \\ \mathcal{O}_{+-} &= \{\lambda = (\lambda_1, \lambda_2, \lambda_3) : \lambda_1\lambda_2 - \lambda_3^2 < 0\}, \\ \mathcal{O}_{--} &= \{\lambda = (\lambda_1, \lambda_2, \lambda_3) : \lambda_1 + \lambda_2 < 0, \lambda_1\lambda_2 - \lambda_3^2 > 0\}. \end{aligned}$$

Their union $\mathcal{O}_{++} \cup \mathcal{O}_{+-} \cup \mathcal{O}_{--}$ is open dense.

We let $\mathrm{GL}(2,\mathbf{R})$ act on S by $l_g : X \mapsto gX{}^tg$. For simplicity, we shall write $\mathrm{GL}(2)$ instead of $\mathrm{GL}(2,\mathbf{R})$. Consider the contragredient representation of $\mathrm{GL}(2)$ on $S^* \simeq \mathbf{R}^3$.

For $\beta \in \mathbf{R}$ and $\delta \in \{++, +-, --\}$ we define a function supported on the orbit $\mathcal{O}_\delta$ by

$$m_\delta^\beta(\lambda) = \begin{cases} |\lambda_1\lambda_2 - \lambda_3^2|^{-\frac{i\beta}{2}} & (\lambda \in \mathcal{O}_\delta), \\ 0 & (\lambda \notin \mathcal{O}_\delta). \end{cases}$$

The group $\mathrm{GL}(2)$ has two natural families of one dimensional unitary representations:

$$\pi_{\epsilon,\alpha} : g \mapsto \mathrm{sgn}(\det g)^\epsilon |\det g|^{i\alpha}, \tag{3.1.1}$$

where $\epsilon \in \{0, 1\}$ and $\alpha \in \mathbf{R}$.

Theorem 2. *Fix a one dimensional unitary representation* $\pi_{\epsilon,\alpha} : \mathrm{GL}(2) \to \mathbf{C}^*$. *Let* $T : L^2(\mathbf{R}^3) \to L^2(\mathbf{R}^3)$ *be a bounded, translation invariant operator, which satisfies*

$$T \circ l_g = \pi_{\epsilon,\alpha}(g)\, l_g \circ T \tag{3.1.2}$$

for all $g \in GL(2)$.

1) If $\epsilon = 0$, *then* T *is a multiplier operator associated to* $m(\lambda)$ *of the form*

$$m(\lambda) = C_1 m_{++}^\alpha(\lambda) + C_2 m_{+-}^\alpha(\lambda) + C_3 m_{--}^\alpha(\lambda),$$

for some $C_1, C_2, C_3 \in \mathbf{C}$.

2) If $\epsilon = 1$, *then* $T = 0$.

Proof. By using the bilinear map

$$\langle\ ,\ \rangle : \mathrm{Symm}(2) \times \mathrm{Symm}(2) \mapsto \mathbf{R}, \quad (u, v) \mapsto \mathrm{Trace}(uv),$$

we shall identify S^* with Symm(2), and hence also with $\mathbf{R}^3$. The contragredient representation of GL(2) on S^* is given by

$$l_g^* \lambda = {}^t g^{-1} \lambda g^{-1},$$

for $\lambda \in \mathrm{Symm}(2)$. Via the isomorphism $S \simeq S^*$, $\mathcal{O}_{++}$ corresponds to symmetric matrices with both eigenvalues positive, $\mathcal{O}_{--}$ to those with both eigenvalues negative and $\mathcal{O}_{+-}$ to those with eigenvalues of different signature. Then, each of $\mathcal{O}_{++}$, $\mathcal{O}_{+-}$, and $\mathcal{O}_{--}$ is a single orbit of GL(2), since matrices with the same signature are conjugate. We note that

$$\langle l_g u, l_g^* \lambda \rangle = \langle u, \lambda \rangle.$$

For $\delta \in \{++, +-, --\}$ and $\alpha \in \mathbf{R}$, we claim:

$$\mathrm{Hom}_{\mathrm{GL}(2)}(\pi^*_{0,\alpha}, C_{\mathrm{bdd}}(\mathcal{O}_\delta)) \simeq \mathbf{C} m^\alpha_\delta, \tag{3.1.3}$$
$$\mathrm{Hom}_{\mathrm{GL}(2)}(\pi^*_{1,\alpha}, C_{\mathrm{bdd}}(\mathcal{O}_\delta)) = \{0\}. \tag{3.1.4}$$

First we note that the dimension of the left-hand side is at most one dimensional by (2.1.4) in the proof of Corollary 2.1.1.

To see (3.1.3), it is now sufficient to show m^α_δ belongs to the left-hand side of (3.1.3). For $\lambda \in \mathcal{O}_\delta$ and $g \in \mathrm{GL}(2)$, we have from the definition of m^α_δ:

$$\begin{aligned} \pi^*_{0,\alpha}(g) m^\alpha_\delta(\lambda) &= |\det g|^{-i\alpha} m^\alpha_\delta(\lambda) \\ &= |\det({}^t g \lambda g)|^{-\frac{i\alpha}{2}} \\ &= m^\alpha_\delta({}^t g \lambda g) \\ &= m^\alpha_\delta(l^*_{g^{-1}} \lambda), \end{aligned}$$

whence (3.1.3). Hence the result for $\epsilon = 0$ follows from Theorem 1.

To see (3.1.4) for $\epsilon = 1$, we just note that

$$g_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

satisfies $\det(g_0) = -1$ and that g_0 leaves some element λ_0 of each orbit invariant. Since $m({}^t g\lambda g) = m(l^*_{g^{-1}}\lambda) = \pi^*_{1,\alpha}(g)m(\lambda) = \mathrm{sgn}(\det g)|\det g|^{-i\alpha}m(\lambda)$, we then have

$$m({}^t g_0\lambda_0 g_0) = -m(\lambda_0).$$

For $\mathcal{O}_{++}$ we observe

$${}^t g_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which implies that $-m(1,1,0) = m(1,1,0)$, i.e. m has to be equal to zero on $\mathcal{O}_{++}$. For $\mathcal{O}_{+-}$ we observe

$${}^t g_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which implies that $-m(0,0,1) = m(0,0,1)$, i.e. m has to be equal to zero on $\mathcal{O}_{+-}$. Finally, the case $\mathcal{O}_{--}$ is similar to $\mathcal{O}_{++}$, and $-m(-1,-1,0) = m(-1,-1,0)$ shows $m = 0$. □

3.2 Invariant multipliers for $(\mathrm{GL}(2) \times \mathrm{GL}(2), \mathbf{R}^4)$

Next, we consider the action of the direct product group $\mathrm{GL}(2) \times \mathrm{GL}(2)$ on the set $\mathrm{M}(2)$ of 2×2 matrices by

$$X \mapsto g_1 X {}^t g_2$$

for $g = (g_1, g_2) \in \mathrm{GL}(2) \times \mathrm{GL}(2)$. Via the isomorphism

$$\mathrm{M}(2) \simeq \mathbf{R}^4 \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \mapsto {}^t(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \tag{3.2.1}$$

we then have a group homomorphism

$$\mathrm{GL}(2) \times \mathrm{GL}(2) \to \mathrm{GL}(4),$$

whose kernel is $K = \{(sI_2, s^{-1}I_2) : s \in \mathbf{R}^*\}$. Here, I_2 is the 2×2 identity matrix.

By using the non-degenerate bilinear symmetric form

$$\langle\ ,\ \rangle : \mathrm{M}(2) \times \mathrm{M}(2) \to \mathbf{R}, \quad (X, Y) \mapsto \mathrm{Trace}\, X {}^t Y,$$

we identify the dual space $\mathrm{M}(2)^*$ with $\mathrm{M}(2)$. Then the contragredient representation is given by

$$Y \mapsto {}^t g_1^{-1} Y g_2^{-1}$$

because $\langle g_1 X {}^t g_2, {}^t g_1^{-1} Y g_2^{-1} \rangle = \langle X, Y \rangle$ for any $g_1, g_2 \in \mathrm{GL}(2)$. We note that any one dimensional unitary representation of $\mathrm{GL}(2) \times \mathrm{GL}(2)$ is of the form $\pi_{\epsilon_1,\alpha_1} \otimes \pi_{\epsilon_2,\alpha_2}$ for some ϵ_1, $\epsilon_2 \in \{0,1\}$ and α_1, $\alpha_2 \in \mathbf{R}$, where the representation $\pi_{\epsilon,\alpha}$ is given in (3.1.1).

Theorem 3. *Fix* ϵ_1, $\epsilon_2 \in \{0,1\}$ *and* α_1, $\alpha_2 \in \mathbf{R}$. *Let* $T : L^2(\mathbf{R}^4) \to L^2(\mathbf{R}^4)$ *be a bounded, translation invariant operator, which satisfies the relation*

$$T \circ l_{(g_1,g_2)} = \pi_{\epsilon_1,\alpha_1}(g_1) \pi_{\epsilon_2,\alpha_2}(g_2)\, l_{(g_1,g_2)} \circ T \tag{3.2.2}$$

for all $g_1, g_2 \in \mathrm{GL}(2)$. *Then* T *is non-zero if and only if* $\epsilon_1 = \epsilon_2$ *and* $\alpha_1 = \alpha_2$. *In this case, we set* $\epsilon := \epsilon_1 = \epsilon_2$ *and* $\alpha := \alpha_1 = \alpha_2$. *Then,* T *is a multiplier operator corresponding to a multiplier function of the form*

$$m(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = C \operatorname{sgn}(\lambda_1\lambda_4 - \lambda_2\lambda_3)^\epsilon |\lambda_1\lambda_4 - \lambda_2\lambda_3|^{i\alpha}, \tag{3.2.3}$$

where C *is a constant.*

Proof. We claim that any bounded, translation invariant operator T satisfying (3.2.2) must be zero if $\epsilon_1 \neq \epsilon_2$ or $\alpha_1 \neq \alpha_2$. Transferring the relation (3.2.2) to the Fourier transform side, we see that the corresponding multiplier function m must satisfy

$$m({}^t g_1^{-1}, \lambda g_2^{-1}) = (\operatorname{sgn}(\det g_1))^{\epsilon_1} (\operatorname{sgn}(\det g_2))^{\epsilon_2} |\det g_1|^{i\alpha_1} |\det g_2|^{i\alpha_2} m(\lambda), \tag{3.2.4}$$

for almost everywhere $\lambda \in \mathrm{M}(2)$ for each $g_1, g_2 \in \mathrm{GL}(2)$.

Since $\mathrm{GL}(2) \times \mathrm{GL}(2)$ has an open dense orbit $\mathrm{GL}(2)$ on $\mathrm{M}(2)$, we may and do assume that m is continuous on $\mathrm{GL}(2)$. Then, the condition (3.2.4) applied to $\lambda := I_2$ and $(g_1, g_2) := ({}^t g^{-1}, I_2)$ or (I_2, g^{-1}) amounts to

$$m(g) = (\operatorname{sgn}(\det g))^{\epsilon_1} |\det g|^{-i\alpha_1} m(I_2),$$
$$m(g) = (\operatorname{sgn}(\det g))^{\epsilon_2} |\det g|^{-i\alpha_2} m(I_2),$$

respectively. Hence, if m is not identically zero, we must have

$$\epsilon_1 = \epsilon_2 \quad \text{and} \quad \alpha_1 = \alpha_2.$$

Thus, from now on we consider the case where $\epsilon := \epsilon_1 = \epsilon_2$ and $\alpha := \alpha_1 = \alpha_2$. The identity (3.2.4) then becomes

$$m({}^t g_1^{-1} \lambda g_2^{-1}) = (\mathrm{sgn}(\det g_1 g_2))^{\epsilon} |\det g_1 g_2|^{i\alpha} m(\lambda). \tag{3.2.5}$$

On the other hand, it is obvious that the function in (3.2.3) satisfies (3.2.5). By Corollary 2.1.1, the proof of Theorem 3 is completed. This completes the proof. □

3.3 Invariant multipliers for $(\mathrm{SO}_0(p,q) \times \mathbf{R}_+, \mathbf{R}^{p+q})$

In light of local isomorphisms of Lie groups

$$\mathrm{SL}(2,\mathbf{R}) \approx \mathrm{SO}_0(2,1),$$

$$\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R}) \approx \mathrm{SO}_0(2,2),$$

the previous two examples can be extended to a more general setting by using the indefinite orthogonal group $\mathrm{O}(p,q)$ as follows.

For $p, q \geq 1$, we let $G_1 := \mathrm{SO}_0(p,q)$, the identity component of the indefinite orthogonal group

$$\mathrm{O}(p,q) = \{g \in \mathrm{GL}(p+q,\mathbf{R}) : Q(gx) = Q(x) \text{ for any } x \in \mathbf{R}^{p+q}\},$$

where Q is the quadratic form given by

$$Q(x) := x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2\ .$$

We shall consider a direct product group

$$G := G_1 \times \mathbf{R}_+,$$

the group acting conformally on the standard flat pseudo-Riemannian manifold $\mathbf{R}^{p,q}$ equipped with the indefinite metric $ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$.

We define a family of one dimensional unitary representations of G by

$$\pi_\alpha : G \to \mathbf{C}^\times, \quad (h,a) \mapsto a^{i\alpha} \qquad (p+q \geq 3) \tag{3.3.1}$$

$$\pi_{\alpha,\beta} : G \to \mathbf{C}^\times, \quad \left(\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, a \right) \mapsto a^{i\alpha} e^{it\beta} \qquad (p+q = 2) \tag{3.3.2}$$

for $\alpha, \beta \in \mathbf{R}$.

We also define bounded functions on $\mathbf{R}^{p+q}$ by

$$Q_+(\lambda)^{i\alpha} := \begin{cases} Q(\lambda)^{i\alpha} & \text{if } Q(\lambda) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Q_-(\lambda)^{i\alpha} := \begin{cases} |Q(\lambda)|^{i\alpha} & \text{if } Q(\lambda) < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$Q_+^{(\pm)}(\lambda)^{i\alpha} := \begin{cases} Q(\lambda)^{i\alpha} & \text{if } Q(\lambda) > 0 \text{ and } \pm\lambda_1 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

We also use the following notation for $\epsilon = +$ or $-$.

$$a_\epsilon^{i\alpha} = \begin{cases} |a|^{i\alpha} & \text{if } \epsilon a > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4. *Let $p, q \geq 1$. Let $T : L^2(\mathbf{R}^{p+q}) \to L^2(\mathbf{R}^{p+q})$ be a bounded translation invariant operator, which satisfies the following relation*

$$T \circ l_g = \begin{cases} \pi_\alpha(g) l_g \circ T & (p+q \geq 2) \\ \pi_{\alpha,\beta}(g) l_g \circ T & (p+q = 2) \end{cases}$$

for all $g \in \mathrm{SO}_0(p,q) \times \mathbf{R}_+$, where the representations π_α and $\pi_{\alpha,\beta}$ are defined by (3.3.1) and (3.3.2) respectively. Then T is a multiplier operator associated to the multiplier of the form:

$$m(\lambda) \tag{3.3.3}$$
$$= \begin{cases} c_1 Q_+(\lambda)^{-\frac{1}{2}i\alpha} + c_2 Q_-(\lambda)^{-\frac{1}{2}i\alpha} & (p, q \geq 2) \\ c_1 Q_+^{(+)}(\lambda)^{-\frac{1}{2}i\alpha} + c_2 Q_+^{(-)}(\lambda)^{-\frac{1}{2}i\alpha} + c_3 Q_-(\lambda)^{-\frac{1}{2}i\alpha} & (p = 1, q \geq 2) \\ \displaystyle\sum_{\varepsilon_1 = \pm, \varepsilon_2 = \pm} c_{\varepsilon_1,\varepsilon_2} (\lambda_1 + \lambda_2)_{\varepsilon_1}^{-\frac{1}{2}i(\alpha+\beta)} (\lambda_1 - \lambda_2)_{\varepsilon_2}^{-\frac{1}{2}i(\alpha-\beta)} & (p = q = 1) \end{cases}$$

for some constants $c_1, c_2, c_3, c_{\varepsilon_1,\varepsilon_2} \in \mathbf{C}$. The case $p \geq 2$ and $q = 1$ is similar to the second case.

Remark 3.3.1. *Here we have treated the connected group $\mathrm{SO}_0(p,q)$. The cases $\mathrm{SO}(p,q)$ and $\mathrm{O}(p,q)$ can be reduced to this one. However, the number of orbits are different for $p = 1$ or $q = 1$*

Proof. Consider the natural action of $G = \mathrm{SO}_0(p,q) \times \mathbf{R}_+$ on $\mathbf{R}^{p+q}$. Then, the following unions of open G-orbits

$$\begin{aligned}
&\mathcal{O}_+ \cup \mathcal{O}_- && (p,q \geq 3),\\
&\mathcal{O}_+^{(+)} \cup \mathcal{O}_+^{(-)} \cup \mathcal{O}_- && (p=1, q \geq 2),\\
&\mathcal{O}_+^{(+)} \cup \mathcal{O}_+^{(-)} \cup \mathcal{O}_-^{(+)} \cup \mathcal{O}_-^{(-)} && (p=q=1)
\end{aligned}$$

are dense in $\mathbf{R}^{p+q}$, respectively, where we set

$$\begin{aligned}
\mathcal{O}_\pm &:= \{\lambda \in \mathbf{R}^{p+q} : \pm Q(\lambda) > 0\},\\
\mathcal{O}_+^{(\pm)} &:= \{\lambda \in \mathcal{O}_+ : \pm \lambda_1 > 0\} && (p=1),\\
\mathcal{O}_-^{(\pm)} &:= \{\lambda \in \mathcal{O}_- : \pm \lambda_{p+1} > 0\} && (q=1).
\end{aligned}$$

Owing to Corollary 2.1.1, Theorem 4 follows if we show that the function m in equation (3.3.3) satisfies the relation

$$m({}^t g\lambda) = \begin{cases} \pi_{-\alpha}(g)m(\lambda) & (p+q \geq 2)\\ \pi_{-\alpha,-\beta}(g)m(\lambda) & (p+q=2) \end{cases}$$

for any $g \in G$ on each orbit. A simple computation shows that this is indeed the case. □

4 Invariant multipliers for $(\mathrm{O}(m)\times\mathrm{GL}_+(k,\mathbf{R}), \mathbf{R}^{mk})$

This section provides an example of Theorem 1 where the invariance conditions determine multiplier operators up to scalar, even in the setting that (π, V) is not one dimensional. The main result of this section is Theorem 5. It shall be noted that the open H-orbits are not symmetric in this example.

4.1 Exterior Riesz transforms

Let $n = mk$ $(m \geq k)$, and

$$H := G_1 \times G_2 = \mathrm{O}(m) \times \mathrm{GL}_+(k, \mathbf{R}).$$

Then H acts on $\mathbf{R}^n \simeq \mathrm{M}(m,k;\mathbf{R})$ in the following manner: for $(a,b) \in H$,

$$X \mapsto aXb^{-1}.$$

We define a subset of $M(m,k;\mathbf{R})$ by

$$\mathcal{O} = \{X \in \mathrm{M}(m,k;\mathbf{R}) : \operatorname{rank} X = k\}.$$

Then $\mathcal{O}$ is open dense in $\mathrm{M}(m,k;\mathbf{R})$. Furthermore, if $X \in \mathcal{O}$, then the $k \times k$ matrix ${}^tXX$ is positive definite, and in particular $\det({}^tXX) > 0$.

Associated to a subset $I \subset \{1,2,\ldots,m\}$ with $|I| = k$, we define a function

$$m_I : \mathcal{O} \to \mathbf{R},\ X \mapsto \frac{\det(X_{ij})_{i \in I, 1 \le j \le k}}{\det({}^tXX)^{\frac{1}{2}}}, \tag{4.1.1}$$

where X_{ij} is the (i,j) component of the matrix X.

Let $\{e_1,\ldots,e_m\}$ be the standard basis of $\mathbf{R}^m$. For $I = \{i_1,\ldots,i_k\}$ $(1 \le i_1 < \cdots < i_k \le m)$, we set $e_I := e_{i_1} \wedge \cdots \wedge e_{i_k}$. Then, $\{e_I : |I| = k\}$ forms a basis of the kth exterior tensor space $\wedge^k(\mathbf{R}^m)$. Thus, we regard a family of functions $m = \{m_I\}$ as a $\wedge^k(\mathbf{R}^m)$-valued function on $\mathcal{O}$. Let π be the standard representation of $\mathrm{O}(n)$ on $\mathbf{R}^m$. We use the same letter π to denote the kth exterior tensor representation of $\mathrm{O}(m)$ on $\wedge^k(\mathbf{R}^m)$. Then, the function $m : \mathcal{O} \to \wedge^k(\mathbf{R}^m)$ satisfies

$$m(aXb^{-1}) = \pi(a)m(X) \quad \text{for } a \in \mathrm{O}(m) \text{ and } b \in \mathrm{GL}_+(k,\mathbf{R}).$$

We extend the representation $(\pi, \wedge^k(\mathbf{R}^m))$ of $\mathrm{O}(m)$ to H by letting $\mathrm{GL}_+(k,\mathbf{R})$ act trivially on $\wedge^k(\mathbf{R}^m)$. With this notation, we have

$$m(gX) = \pi(g)m(X) \quad \text{for } g \in H.$$

We now recall a minor summation formula (see [B, exercise III.8.6] for instance):

$$\sum_I (\det(X_{ij})_{i \in I, 1 \le j \le k})^2 = \det({}^tXX).$$

Hence, $|m_I(X)| \le 1$ for any $X \in \mathcal{O}$ and any I. As $\mathcal{O}$ is open dense in $\mathrm{M}(m,k;\mathbf{R}) \simeq \mathbf{R}^n$, we shall regard m_I as a bounded function on $\mathbf{R}^n$ and m as a $\wedge^k(\mathbf{R})$-valued bounded function on $\mathrm{M}(m,k;\mathbf{R})$.

Theorem 5. *Let $H = \mathrm{O}(m) \times \mathrm{GL}_+(k,\mathbf{R})$ $(m \ge k)$, π the representation of H on $\wedge^k(\mathbf{R}^m)$ as above, and $n = mk$. Then the set of multipliers $\{m_I\}$ defines a bounded translation invariant operator*

$$T : L^2(\mathbf{R}^n) \to \wedge^k(\mathbf{R}^m) \otimes L^2(\mathbf{R}^n).$$

This operator satisfies the invariance condition

$$(\pi(g) \otimes l_g)T = T \circ l_g \quad \text{for } g \in H. \tag{4.1.2}$$

Conversely, any bounded translation invariant operator $L^2(\mathbf{R}^n) \to \wedge^k(\mathbf{R}^m) \otimes L^2(\mathbf{R}^n)$ *satisfying (4.1.2) is a scalar multiple of* T.

We will say the operator characterized by this theorem is the *exterior Riesz transform.*

Remark 4.1.1. *If* $k = 1$ *then* $\det({}^tXX)^{\frac{1}{2}}$ *is nothing but the norm* $|X|$ *of a vector* $X \in \mathbf{R}^n$ *and* $m_I(X) = \frac{X_i}{|X|}$ *for* $I = \{i\}$ $(1 \leq i \leq n)$. *Thus, Theorem 5 in the case* $k = 1$ *corresponds to Stein's Theorem characterizing the usual Riesz transforms (see Fact 1.1).*

Remark 4.1.2. *Theorem 5 has the following two distinguishing features: 1) the dimension of the representation space* $\wedge^k(\mathbf{R}^m)$ *is no longer one dimensional, thus Corollary 2.1.1 does not apply; 2) the orbit is not a reductive symmetric space, thus it does not fit with Corollary 2.1.2 either. Nevertheless, Theorem 5 asserts that one can characterize invariant multipliers up to scalar by the invariance condition. The idea of the following proof is to show that there is a reductive symmetric space for which the dimension of the space of homomorphisms dominate the dimension of the space of homomorphisms for our space.*

4.2 Proof of Theorem 5

Proof of Theorem 5. We apply Theorem 1. Since $\mathcal{O}$ is open dense in $\mathbf{R}^n$, Theorem 5 is a consequence of the following multiplicity-free result: □

Lemma 4.2.1. *For a representation* π *of* $\mathrm{O}(m)$, *we shall denote by* $\widetilde{\pi}$ *its extention to* $H = \mathrm{O}(m) \times \mathrm{GL}_+(k, \mathbf{R})$ *by letting* $\mathrm{GL}_+(k, \mathbf{R})$ *act trivially.*
1) For any irreducible (finite dimensional) representation π *of* $\mathrm{O}(m)$, *we have*

$$\mathrm{Hom}_H(\widetilde{\pi}, C_{\mathrm{bdd}}(\mathcal{O})) \leq 1.$$

2) Furthermore, if π *is the natural representation of* $\mathrm{O}(m)$ *on the exterior algebra* $\wedge^k(\mathbf{R}^m)$, *then*

$$\mathrm{Hom}_H(\widetilde{\pi}, C_{\mathrm{bdd}}(\mathcal{O})) = 1,$$

and the image of $\widetilde{\pi}$ *in* $C_{\mathrm{bdd}}(\mathcal{O})$ *coincides with the complex vector space spanned by the basis* $\{m_I : |I| = k\}$.

Proof of Lemma 4.2.1. We recall $G_1 = \mathrm{O}(m)$ and $G_2 = \mathrm{GL}_+(k, \mathbf{R})$, and $H = G_1 \times G_2$. We write $\mathcal{C}(\mathcal{O})^{G_2}$ for the set of G_2-invariant continuous functions of $\mathcal{O}$. Then, $\mathcal{C}(\mathcal{O})^{G_2}$ is a G_1-submodule of $\mathcal{C}(\mathcal{O})$, and we have a natural bijection:

$$\mathrm{Hom}_H(\widetilde{\pi}, \mathcal{C}(\mathcal{O})) \simeq \mathrm{Hom}_{G_1}(\pi, \mathcal{C}(\mathcal{O})^{G_2}).$$

Let us consider the right-hand side. We begin with the H-action on $M(m, k; \mathbf{R})$. It follows from the Gram–Schmidt orthogonalization procedure that H acts transitively on $\mathcal{O}$. Let L be the isotropy subgroup at $\begin{pmatrix} I_k \\ O \end{pmatrix} \in \mathcal{O}$. Then, L is given by

$$\begin{aligned} L &= \left\{ \left(\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, b \right) : b \in \mathrm{SO}(k), c \in \mathrm{O}(m-k) \right\} \\ &\simeq \mathrm{SO}(k) \times \mathrm{O}(m-k). \end{aligned}$$

Thus, we can identify $\mathcal{O}$ with the homogeneous space H/L.

Let $\iota : G_1 \to H$, $a \mapsto (a, \mathrm{I}_k)$ be the natural injection. Then, it is not difficult to see that the pull-back ι^* induces isomorphisms of G_1-modules:

$$\mathcal{C}(\mathcal{O})^{G_2} \simeq \mathcal{C}(H/L)^{G_2} \simeq \mathcal{C}(G_1/\iota^{-1}(L(\mathrm{I}_n \times G_2))).$$

In our setting, $L(\mathrm{I}_k \times G_2) = (\mathrm{SO}(k) \times \mathrm{O}(m-k)) \times \mathrm{GL}_+(k, \mathbf{R})$, and therefore

$$\mathcal{C}(\mathcal{O})^{G_2} \simeq \mathcal{C}(\mathrm{O}(m)/(\mathrm{SO}(k) \times \mathrm{O}(m-k))).$$

Thus we have shown

$$\mathrm{Hom}_H(\widetilde{\pi}, \mathcal{C}(\mathcal{O})) \simeq \mathrm{Hom}_{\mathrm{O}(m)}(\pi, \mathcal{C}(\mathrm{O}(m)/(\mathrm{SO}(k) \times \mathrm{O}(m-k)))).$$

Since $\mathrm{O}(m)/(\mathrm{SO}(k) \times \mathrm{O}(m-k))$ is a reductive symmetric space, the dimension of the right-hand side is not greater than one by a theorem of É. Cartan. Hence,

$$\dim \mathrm{Hom}_H(\widetilde{\pi}, C_{\mathrm{bdd}}(\mathcal{O})) \leq \dim \mathrm{Hom}_H(\widetilde{\pi}, \mathcal{C}(\mathcal{O})) \leq 1.$$

This shows the first statement. We have already seen that the representation of H on $\mathbf{C}$-span $\{m_I : |I| = k\}$ is isomorphic to the kth exterior representation tensor $\wedge^k(\mathbf{R}^m)$. Hence, the second statement follows. □

In Theorem 9, we shall discuss L^p-boundedness of the exterior Riesz multipliers.

5 Classification of invariant L^p-bounded operators

We have found some explicit examples of invariant multipliers (Theorems 2, 3, 4, and 5) in the framework of Strategy 2. We shall determine for which p they define L^p-multipliers. The main results of this section are Theorems 6, 7, 8, and 9. We find a feature that L^p-bounded invariant operators arising from Strategy 2 are 'rare' if $p \neq 2$, in the sense that they are built from known examples such as Riesz transforms.

5.1 Algebra of L^p-bounded operators—quick review

Standard multiplier theory tells us that a multiplier operator bounded on $L^p(\mathbf{R}^n)$ must also be bounded on $L^2(\mathbf{R}^n)$, see for example [Ho, Corollary 1.3]. This also holds in the vector valued case, namely, for a finite dimensional vector space V, a multiplier operator bounded from $L^p(\mathbf{R}^n) \to V \otimes L^p(\mathbf{R}^n)$ must be also bounded from $L^2(\mathbf{R}^n) \to V \otimes L^2(\mathbf{R}^n)$. There are some sufficient conditions for a bounded function to be an L^p-multiplier, but there are no general criteria. Hence, we are tempted to ask for which set of p the multiplier operators we have seen remain bounded.

We begin with a brief summary of some known results. For $1 \leq p \leq \infty$ we denote by $M_p(\mathbf{R}^n)$ the set of bounded functions m on $\mathbf{R}^n$ such that the corresponding translation invariant operators T_m are bounded on $L^p(\mathbf{R}^n)$.

Fact 5.1. *Suppose* $1 < p, q < \infty$.

1) $M_p(\mathbf{R}^n) = M_q(\mathbf{R}^n)$ *if* $\frac{1}{p} + \frac{1}{q} = 1$.

2) $M_q(\mathbf{R}^n) \subset M_p(\mathbf{R}^n)$ *if* $\left|\frac{1}{p} - \frac{1}{2}\right| < \left|\frac{1}{q} - \frac{1}{2}\right|$.

3) *(deLeeuw) If* $m \in M_p(\mathbf{R}^{l+n})$ *then* $m(a, \cdot) \in M_p(\mathbf{R}^n)$ *for a.e.* $a \in \mathbf{R}^l$.

4) *(Fefferman's ball multiplier theorem)* $\chi_B \notin M_p(\mathbf{R}^n)$, *if* $p \neq 2$ *and* $n \geq 2$. *Here* χ_B *is the characteristic function of the unit ball,* B, *in* $\mathbf{R}^n$.

5) *If* $m \in M_p(\mathbf{R}^n)$ *and* $A \in \mathrm{Aff}(\mathbf{R}^n)$ *then* $m \circ A \in M_p(\mathbf{R}^n)$ *and* $A \circ m \in M_p(\mathbf{R}^n)$.

6) *If* m_1 *and* m_2 *are elements in* $M_p(\mathbf{R}^n)$ *then* $m_1 \cdot m_2 \in M_p(\mathbf{R}^n)$ *and* $m_1 + m_2 \in M_p(\mathbf{R}^n)$.

7) For $\alpha \in \mathbf{R}$ we have $\lambda_+^{i\alpha} \in M_p(\mathbf{R})$ $(1 < p < \infty)$.

8) If $m \in M_p(\mathbf{R}^n)$ then the function M, defined by $M(a, b) = m(a)$, is in $M_p(\mathbf{R}^{n+l})$.

Proof. 1) and 2) See [Ho, Theorem 1.3]. 3) See [T, Theorem 2.4]. 4) See [F1]. The proofs of 5), 6) and 8) are straightforward. 7) See [S, page 96]. □

5.2 Classification of L^p-bounded operators from Sections 3 and 4

Suppose we are in the setting of Subsection 3.1.

Theorem 6 **((GL(2), $\mathbf{R}^3$) case).** *The operator characterized by Theorem 2 does not extend to a bounded operator on $L^p(\mathbf{R}^3)$ $(1 < p < \infty)$ except for $p = 2$.*

Proof. Let m_δ^β be the multiplier operator in Theorem 2. Assume $m_\delta^\beta \in M_p(\mathbf{R}^3)$ for some $\delta = ++, +-, --$ and $\beta \in \mathbf{R}$. Then, also $m_\delta^{-\beta} \in M_p(\mathbf{R}^n)$, because $m_\delta^{-\beta}$ is the complex conjugate of m_δ^β. It then follows from Fact 5.1 (6) that their product $m_\delta^\beta \cdot m_\delta^{-\beta}$ is also in $M_p(\mathbf{R}^n)$. Now we observe that the product $m_\delta^\beta \cdot m_\delta^{-\beta}$ is the characteristic function $\chi_{\mathcal{O}_\delta}$ of the orbit $\mathcal{O}_\delta$. Thus, we have proved the implication:

$$m_\delta^\beta \in M_p(\mathbf{R}^3) \quad \text{for some } \beta \in \mathbf{R} \Rightarrow \mathcal{O}_\delta \in M_p(\mathbf{R}^3). \tag{5.2.1}$$

Let us show that $\chi_{\mathcal{O}_\delta} \in M_p(\mathbf{R}^3)$ only if $p = 2$. The case $\delta = +-$ can be reduced to the others because $T_{\chi_{\mathcal{O}_{+-}}} = \mathrm{id} - T_{\chi_{\mathcal{O}_{++}}} - T_{\chi_{\mathcal{O}_{--}}}$. Fix $a > 0$. For $\delta = ++$ or $= --$, the intersection of $\mathcal{O}_\delta$ with the hyperplane $\lambda_1 + \lambda_2 = a$ $(\delta = ++)$, $= -a$ $(\delta = --)$ is the ellipse $\{(x, y) \in \mathbf{R}^2 : a^2 - x^2 - 4y^2 > 0\}$ where $x = \lambda_1 - \lambda_2$ and $y = \lambda_3$. Hence, p has to be equal to 2 by Facts 5.1 (3), (4) and (5). □

Next, we consider the setting of Subsection 3.2. In the same way we have

Theorem 7 **((GL(2) × GL(2), $\mathbf{R}^4$) case).** *The operator characterized by Theorem 3 does not extend to a bounded operator on $L^p(\mathbf{R}^4)$ $(1 < p < \infty)$ except for $p = 2$.*

In this case the relevant operator, after a suitable change of variables, is the one corresponding to the characteristic function of the set $\{\lambda : \lambda_1^2 + \lambda_2^2 \geq \lambda_3^2 + \lambda_4^2\}$. Taking the intersection with the two plane $\lambda_1 = \lambda_2 = 1$ or alike, we see that the operators are bounded on $L^p(\mathbf{R}^4)$ only if $p = 2$ by Fact 5.1 (3) and (4).

It also follows in a similar manner in the setting of Subsection 3.3:

Theorem 8 ($(\mathrm{SO}(p,q) \times \mathbf{R}_+, \mathbf{R}^{p+q})$ case).
1) *The operator characterized by Theorem 4 does not extend to a bounded operator on* $L^r(\mathbf{R}^{p+q})$ $(1 < r < \infty)$ *except for* $r = 2$ *if* $p + q \geq 3$.
2) *If* $p + q = 2$, *the operator is bounded on* $L^r(\mathbf{R}^2)$, *for all* $1 < r < \infty$.

Proof. 1) For $p + q \geq 3$ the guiding operator is the one given by the characteristic function of the set $\{\lambda : \lambda_1^2 + \cdots + \lambda_p^2 \geq \lambda_{p+1}^2 + \cdots + \lambda_{p+q}^2\}$, where we might assume that $p \geq q$. The first statement then follows as before.
2) If $p = q = 1$ we are considering the multiplier

$$\sum_{\varepsilon_1 = \pm, \varepsilon_2 = \pm} c_{\varepsilon_1,\varepsilon_2} (\lambda_1 + \lambda_2)_{\varepsilon_1}^{-\frac{1}{2}i(\alpha+\beta)} (\lambda_1 - \lambda_2)_{\varepsilon_2}^{-\frac{1}{2}i(\alpha-\beta)}.$$

We want to show that the corresponding multiplier operator is bounded on $L^r(\mathbf{R}^2)$ for all $1 < r < \infty$. To do this it is enough to consider the factors separately

$$m_{1,\epsilon}^{\alpha}(\lambda) = (\lambda_1 + \lambda_2)_{\epsilon}^{i\alpha},$$
$$m_{2,\epsilon}^{\alpha}(\lambda) = (\lambda_1 - \lambda_2)_{\epsilon}^{i\alpha},$$

because of Fact 5.1 (6). Clearly, they are all simple rotations of the multiplier

$$m(\lambda) = \begin{cases} |\lambda_1|^{i\alpha} & \text{if } \lambda_1 > 0, \\ 0 & \text{otherwise,} \end{cases}$$

which, by Facts 5.1 (7) and (8), is in $M_r(\mathbf{R}^n)$ for $1 < r < \infty$. □

Theorem 9 (exterior Riesz multipliers). *Suppose* $n = mk$ $(m \geq k)$, *and we are in the setting of Subsection 4.2. Then there exists a non-trivial bounded translation invariant operator*

$$T : L^p(\mathbf{R}^n) \to \wedge^k(\mathbf{R}^m) \otimes L^p(\mathbf{R}^n)$$

satisfying the $\mathrm{O}(m) \times \mathrm{GL}_+(k, \mathbf{R})$*-invariance condition (4.1.2), if and only if one of the following conditions are satisfied:*

- *$1 < p < \infty$ and $k = 1$ (Riesz transforms, see [S, page 57 and Theorem 3] or [T, page 269]).*
- *$p = 2$ and k arbitrary (Theorem 5).*

Proof. Since a bounded translation invariant operator on L^p is automatically bounded on L^2, it follows from Theorem 5 that T must be a scalar multiple of the exterior Riesz transform defined by the set of multipliers

$$m_I : \mathcal{O} \to \mathbf{R},\ X \mapsto \frac{\det(X_{ij})_{i \in I, 1 \leq j \leq k}}{\det({}^t XX)^{\frac{1}{2}}},$$

(see (4.1.1)) for $I \subset \{1, 2, \ldots, m\}$ with $p \neq 2$ and $|I| = k$. All we have to do is to prove that this operator is not L^p-bounded if $k \geq 2$. To see this, we restrict m_I on the two dimensional subspace of $\mathrm{M}(m, k; \mathbf{R})(\simeq \mathbf{R}^n)$ defined by the system of linear equations:

$$\begin{cases} X_{11} - X_{22} = 2 + \epsilon_1 \\ X_{12} - X_{21} = \epsilon_2 \\ X_{ii} = 1 + \epsilon_i \quad (3 \leq i \leq k) \\ X_{ij} = \epsilon_{ij} \quad\ \ \text{(if } \max(i,j) \geq 3 \text{ and } i \neq j), \end{cases}$$

where ϵ_i and ϵ_{ij} are parameters. If all of ϵ_i and ϵ_{ij} are zero this subspace admits the following coordinates

$$\begin{pmatrix} A & O_{2,k-2} \\ O_{k-2,2} & I_{k-2} \\ O_{m,2} & O_{m,k-2} \end{pmatrix},$$

where

$$A = \begin{pmatrix} x+1 & y \\ -y & x-1 \end{pmatrix}$$

and $O_{m,k}$ is the zero $m \times k$ matrix. Then $m_{\{1,\ldots,k\}}(X) = \mathrm{sgn}(x^2 + y^2 - 1)$ $(= 2\chi_B - 1)$ which does not define an L^p-bounded operator by Fefferman's ball multiplier theorem, see Fact 5.1 (4). For sufficiently small ϵ_i and ϵ_{ij}, the restriction of m_I to the corresponding two dimensional vector space is of the form $\chi_{B'} - 1$, where B' is the interior of a certain ellipse depending on the parameters ϵ_i and ϵ_{ij}. Again, it is not L^p-bounded by Fact 5.1 (4) and (5). By deLeeuw's theorem, Fact 5.1 (3), $m_{\{1,\ldots,k\}} \notin M_p(\mathbf{R}^n)$. Similarly for m_I for any I with $|I| = k$ $(k \geq 2)$. This completes the proof of Theorem 9. $\square$

Hence, we have determined for which the invariant multipliers in Sections 3 and 4 define L^p-multipliers $(1 < p < \infty)$.

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