Indefinite higher Riesz transforms

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March 30, 2007

Abstract

Stein's higher Riesz transforms are translation invariant operators on $L^2(\mathbb{R}^n)$ built from multipliers whose restrictions to the unit sphere are eigenfunctions of the Laplace–Beltrami operators. In this article, generalizing Stein's higher Riesz transforms, we construct a family of translation invariant operators by using discrete series representations for hyperboloids associated to the indefinite quadratic form of signature (p, q). We prove that these operators extend to L^r-bounded operators for $1 < r < \infty$ if the parameter of discrete series representations is generic.

*Partially supported by Grant-in-Aid for Scientific Research 18340037, Japan Society for the Promotion of Science

†Partially supported by Japan Society for the Promotion of Science

2000 Mathematics Subject Classification. Primary 22E46, Secondary 42B15

Key words and phrases. Riesz transforms, spherical harmonics, discrete series representations, unitary representations, indefinite orthogonal group

1 Introduction and statement of main results

For a measurable bounded function m on $\mathbf{R}^n$, the *multiplier operator* $T_m : f \mapsto \mathcal{F}^{-1}(m\mathcal{F}f)$ defines a continuous, translation invariant operator on $L^2(\mathbf{R}^n)$, where $\mathcal{F}$ is the Fourier transform.

Classic examples are the Riesz transforms

$$(R_j f)(x) := \lim_{\varepsilon \to 0} \frac{\Gamma(\frac{n+2}{2})}{\pi^{\frac{n+2}{2}}} \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)dy \qquad (1 \leq j \leq n)$$

which are associated to the multipliers $m_j(\xi) := |\xi|^{-1}\xi_j$. One of the important properties of the Riesz transforms is that R_j extends to a continuous operator on the Banach space $L^r(\mathbf{R}^n)$ for any $1 < r < n$. From a group theoretic view point, the space $\mathcal{M}_2(\mathbf{R}^n)$ of all continuous, translation invariant operators on $L^2(\mathbf{R}^n)$ is naturally a representation space of the general linear group $\mathrm{GL}(n, \mathbf{R})$ by

$$\mathcal{M}_2(\mathbf{R}^n) \to \mathcal{M}_2(\mathbf{R}^n), \quad T \mapsto L_g \circ T \circ L_g^{-1} \quad (g \in \mathrm{GL}(n, \mathbf{R})),$$

where $(L_g f)(x) := f(g^{-1}x)$. Then, it is noteworthy that the Riesz transforms R_j $(1 \leq j \leq n)$ span the simplest non-trivial representation (an irreducible n-dimensional representation) of the orthogonal group $\mathrm{O}(n)$.

More generally, E. M. Stein introduced a family of bounded translation invariant operators T_m (*higher Riesz transforms*) associated to multipliers m with the following two properties:

$$m \text{ is a homogeneous function of degree } 0, \tag{1.1}$$

$$m|_{S^{n-1}} \in \mathcal{H}^k(\mathbf{R}^n). \tag{1.2}$$

The condition (1.1) is equivalent to the fact that m is constant on rays emanating from the origin. Thus, m is completely determined by its restriction to the unit sphere. The condition (1.2) concerns this restriction. Here, $\mathcal{H}^k(\mathbf{R}^n)$ denotes the space of spherical harmonics of degree k defined by

$$\mathcal{H}^k(\mathbf{R}^n) := \{f \in C^\infty(S^{n-1}) : \Delta_{S^{n-1}} f = -k(k+n-2)f\}, \tag{1.3}$$

where $\Delta_{S^{n-1}}$ is the Laplace–Beltrami operator on the unit sphere S^{n-1}. Then, we have

Fact 1.1 (see [6, II, Theorem 3]). *Suppose $k \in \mathbf{N}$. Then, for any m satisfying (1.1) and (1.2), T_m extends to a continuous operator on the Banach space $L^r(\mathbf{R}^n)$ for any $1 < r < \infty$.*

We note that T_m corresponds to the identity operator for $k = 0$, and to the Riesz transforms for $k = 1$. For general k, $\{T_m : m$ satisfies (1.1) and (1.2)$\}$ forms an irreducible $\mathrm{O}(n)$ submodule in $\mathcal{M}_2(\mathbf{R}^n)$.

In the previous paper [4], we analyzed translation invariant operators from group theoretic view points, and found the following phenomenon: L^2-bounded translation invariant operators with 'large symmetries' are mostly unbounded on $L^r(\mathbf{R}^n)$ $(r \neq 2)$ except for the cases that they are built from higher Riesz transforms, as far as 'large symmetries' are defined by *finite* dimensional representations of affine subgroups (e.g. [4, Theorem 9]).

The aim of this paper is to construct a family of L^r-bounded translation invariant operators $(1 < r < \infty)$ with 'large symmetries' by using *infinite* dimensional representations.

To be more precise, we take a quadratic form

$$Q(\xi) := \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2$$

of a general signature (p, q) $(p > 1)$ and work on the hyperboloid

$$X_{p,q} := \{\xi \in \mathbf{R}^{p+q} : Q(\xi) = 1\},$$

which is a (non-singular) submanifold in the open domain $\mathbf{R}^{p,q}_+ := \{\xi \in \mathbf{R}^{p+q} : Q(\xi) > 0\}$ of $\mathbf{R}^{p+q}$. We endow $X_{p,q}$ with the standard pseudo-Riemannian structure of signature $(p-1, q)$, and introduce a dense subspace of L^2-eigenfunctions of the Laplace–Beltrami operator $\Delta \equiv \Delta_{X_{p,q}}$ as follows:

$$\mathcal{H}^k(\mathbf{R}^{p,q}) := \{f \in L^2(X_{p,q}) : \Delta f = -k(k+p+q-2)f\}_{K\text{-finite}}.$$

See Section 2 for more details. In the case $(p, q) = (n, 0)$, we note $X_{p,q} = S^{n-1}$, $\mathbf{R}^{p,q}_+ = \mathbf{R}^n$, and $\mathcal{H}^k(\mathbf{R}^{p,q}) = \mathcal{H}^k(\mathbf{R}^n)$.

In our setting for general p, q, we replace (1.2) with

$$m|_{X_{p,q}} \in \mathcal{H}^k(\mathbf{R}^{p,q}) \quad \text{and} \quad \operatorname{Supp} m \subset \overline{\mathbf{R}^{p,q}_+}. \tag{1.4}$$

Then, the following subspace of $\mathcal{M}_2(\mathbf{R}^{p+q})$:

$$\{T_m : m \text{ satisfies (1.1) and (1.4)}\}$$

forms a dense subspace of an irreducible (infinite dimensional) unitary representation of the indefinite orthogonal group $\mathrm{O}(p, q)$ if $p > 1$ and $q > 0$. Our L^r-boundedness theorem is now stated as follows:

Theorem 1. *Suppose $k > 4$ ($p+q$: even) or $k > 3$ ($p+q$: odd). Then, for any m satisfying* (1.1) *and* (1.4)*, the multiplier operator T_m extends to a continuous operator on $L^r(\mathbf{R}^{p+q})$ for any $1 < r < \infty$.*

Remark 1.2. *In place of $X_{p,q} \subset \mathbf{R}_+^{p,q}$, we can also consider the open domain $\mathbf{R}_-^{p,q} := \{\xi \in \mathbf{R}^{p+q} : Q(\xi) < 0\}$ and L^2-eigenfunctions on another hyperboloid $X'_{p,q} := \{\xi \in \mathbf{R}^{p+q} : Q(\xi) = -1\}$. Then, for m supported on $\overline{\mathbf{R}_-^{p,q}}$ an analogous result also holds by switching (p,q) to (q,p) because $X'_{p,q} \simeq X_{q,p}$ and $\mathbf{R}_-^{p,q} \simeq \mathbf{R}_+^{q,p}$.*

The operators T_m with m satisfying (1.1) and (1.4) may be regarded as a generalization of Stein's higher Riesz transforms in the following sense:

$$\begin{array}{ccl} \text{spherical harmonics on } S^{n-1} & \Rightarrow & \text{discrete series for } X_{p,q}, \\ \mathrm{O}(n) & \Rightarrow & \text{indefinite orthogonal group } \mathrm{O}(p,q). \end{array}$$

We shall call T_m *indefinite Riesz transforms.* Then, Theorem 1 for indefinite Riesz transforms is a generalization of Fact 1.1

A distinguishing feature of our generalization is that the restriction of the multiplier m to the unit sphere is no more infinitely differentiable. Our multiplier m has the following property:

$$m(\xi) = 0 \quad \text{if} \quad \xi_1^2 + \cdots + \xi_p^2 \le \xi_{p+1}^2 + \cdots + \xi_{p+q}^2.$$

Unlike Fefferman's ball multiplier theorem [2] and its descendants [4] for multiplier operators with 'large symmetries', the indefinite Riesz transforms T_m remain L^r-bounded for any r ($1 < r < \infty$).

The crucial point of the proof of Theorem 1 is the asymptotic estimate of the multiplier $m(\xi)$ together with its differentials as ξ approaches the boundary of $\mathbf{R}_+^{p,q}$. This estimate is carried out by using techniques of infinite dimensional representation theory of $\mathrm{O}(p,q)$ and non-commutative harmonic analysis (e.g. [1, 5, 7]).

Notation: $\mathbf{R}_+ := \{x \in \mathbf{R} : x > 0\}$, $\mathbf{N} = \{0, 1, 2, \ldots,\}$, and $\mathbf{N}_+ := \{1, 2, \ldots\}$.

2 Basic properties of discrete series for $X_{p,q}$

In this section, after a quick review of some of the fundamental facts concerning discrete series representations for hyperboloids $X_{p,q}$, we introduce the linear vector space $\mathcal{V}_k^\infty$ consisting of smooth functions on an open domain $\mathbf{R}_+^{p,q}$ with certain decay condition. This space $\mathcal{V}_k^\infty$ will bridge discrete series for $X_{p,q}$ and 'indefinite higher multipliers'.

In what follows, we shall use the notation

$$\begin{aligned} \xi &= (\xi', \xi'') \in \mathbf{R}^{p+q}, \\ Q(\xi) &= |\xi'|^2 - |\xi''|^2 = \xi_1^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \cdots - \xi_{p+q}^2, \\ |\xi|^2 &= |\xi'|^2 + |\xi''|^2 = \xi_1^2 + \cdots + \xi_p^2 + \xi_{p+1}^2 + \cdots + \xi_{p+q}^2. \end{aligned}$$

Then, the indefinite orthogonal group

$$\mathrm{O}(p,q) := \{g \in GL(p+q; \mathbf{R}) : Q(g\xi) = Q(\xi) \quad \text{for any} \quad \xi \in \mathbf{R}^{p+q}\}$$

is non-compact if $p, q > 0$. Throughout this paper, we shall write $G := \mathrm{O}(p,q)$, and denote by $\mathfrak{g}$ the Lie algebra $\mathfrak{o}(p,q)$ of G. The group G contains

$$K := \{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in \mathrm{O}(p), B \in \mathrm{O}(q)\} \simeq \mathrm{O}(p) \times \mathrm{O}(q)$$

as a maximal compact subgroup. We note that in the case $q = 0$, $G = K$ is nothing but the orthogonal group $\mathrm{O}(p)$.

We denote by $\mathbf{R}^{p,q}$ the Euclidean space $\mathbf{R}^{p+q}$ equipped with the flat pseudo-Riemannian structure $ds^2 = d\xi_1^2 + \cdots + d\xi_p^2 - d\xi_{p+1}^2 - \cdots - d\xi_{p+q}^2$. Then, ds^2 is non-degenerate when restricted to the submanifold $X_{p,q}$, and defines a pseudo-Riemannian structure $g_{X_{p,q}}$ of signature $(p-1, q)$ on $X_{p,q}$. Obviously, the group $\mathrm{O}(p,q)$ acts on $\mathbf{R}^{p,q}$ and $X_{p,q}$, respectively, as isometries.

The action of $\mathrm{O}(p,q)$ on $X_{p,q}$ is transitive, and the isotropy subgroup at ${}^t(1, 0, \ldots, 0)$ is identified with $\mathrm{O}(p-1, q)$. Thus, $X_{p,q}$ is realized as a homogeneous space:

$$\mathrm{O}(p,q)/\mathrm{O}(p-1,q) \simeq X_{p,q}.$$

The group G acts on the space of functions on $\mathbf{R}^{p,q}$, $\mathbf{R}_+^{p,q}$ and also on $X_{p,q}$ by translations:

$$\pi(g) : f \mapsto f(g^{-1} \cdot).$$

In particular, G acts unitarily on the Hilbert space $L^2(X_{p,q})$ consisting of square integrable functions on $X_{p,q}$ with respect to the measure induced by $g_{X_{p,q}}$.

The differential of π, denoted by $d\pi$, is formally defined by

$$d\pi(Y)f := \frac{d}{dt}\Big|_{t=0} f(e^{-tY}\cdot) \quad \text{for } Y \in \mathfrak{g}.$$

Next, we consider L^2-eigenfunctions of the Laplace–Beltrami operator $\Delta \equiv \Delta_{X_{p,q}}$ on $X_{p,q}$:

$$L^2_k(X_{p,q}) := \{f \in L^2(X_{p,q}) : \Delta f = -k(k+p+q-2)f\}.$$

Here, the differential equation is interpreted as that of distributions. Then, $L^2_k(X_{p,q})$ is a closed subspace of the Hilbert space $L^2(X_{p,q})$ (possibly, equal to zero). Since Δ commutes with the G-action, $L^2_k(X_{p,q})$ is a G-invariant subspace.

Suppose $f \in L^2_k(X_{p,q})$. We say f is *K-finite* if $\mathbf{C}$-span$\{\pi(k)f : k \in K\}$ is finite dimensional. We set the vector space consisting of K-finite vectors as

$$\mathcal{H}^k(\mathbf{R}^{p,q}) := L^2_k(X_{p,q})_{K\text{-finite}}. \tag{2.1}$$

We note that any function of $\mathcal{H}^k(\mathbf{R}^{p,q})$ is real analytic although the differential operator Δ is not elliptic. We also note that if $q = 0$ then $G = K$ and $\mathcal{H}^k(\mathbf{R}^{p,0}) = L^2_k(S^{p-1})_{K\text{-finite}} = L^2_k(S^{p-1})$.

We set

$$\rho := \frac{p+q-2}{2},$$

$$\Lambda_+(p,q) := \begin{cases} \{k \in \mathbf{Z} : k > -\rho\} & (q \neq 0), \\ \{k \in \mathbf{Z} : k \geq 0\} & (q = 0). \end{cases}$$

We collect below some known results on discrete series representations for hyperboloids $X_{p,q}$. See [1] and [7] for the pioneering work. See also [5, Fact 5.4] for a survey on algebraic, geometric, and analytic aspects of these representations from modern representation theory:

Fact 2.1. *Suppose $p > 1$.*

1) *If $k \in \Lambda_+(p,q)$ then $L^2_k(X_{p,q})$ is non-zero. It is irreducible as a representation of G. Conversely, any (non-zero) irreducible closed subspace of $L^2(X_{p,q})$ is of the form $L^2_k(X_{p,q})$ for some $k \in \Lambda_+(p,q)$.*

2) $\mathcal{H}^k(\mathbf{R}^{p,q})$ *is a K-invariant dense subspace of $L^2_k(X_{p,q})$. As a representation of K,*

$$\mathcal{H}^k(\mathbf{R}^{p,q}) \simeq \begin{cases} \displaystyle\bigoplus_{\substack{a,b\in\mathbf{N}\\ a-b\geq k+q\\ a-b\equiv k+q \bmod 2}} \mathcal{H}^a(\mathbf{R}^p)\otimes\mathcal{H}^b(\mathbf{R}^q) & (q>0),\\ \mathcal{H}^k(\mathbf{R}^p) & (q=0).\end{cases}$$

3) *Suppose further $q>0$. If $f\in\mathcal{H}^k(\mathbf{R}^{p,q})$, then there exists $a(\omega,\eta)\in C^\infty(S^{p-1}\times S^{q-1})$ such that*

$$f(\omega\cosh t,\eta\sinh t) = a(\omega,\eta)e^{-(k+2\rho)t}(1+e^{-2t}\,\mathrm{O}(t)) \quad \textit{as } t\to\infty.$$

4) $d\pi(Y)\mathcal{H}^k(\mathbf{R}^{p,q})\subset\mathcal{H}^k(\mathbf{R}^{p,q})$ *for any $Y\in\mathfrak{g}$.*

The irreducible unitary representation of G realized on a closed subspace of $L^2(X_{p,q})$ is called a *discrete series representation* for the hyperboloid $X_{p,q}$. Discrete series representations for $X_{p,q}$ exist if $p>1$. By Fact 2.1 1), $\Lambda_+(p,q)$ is the parameter space of discrete series representations for $X_{p,q}$.

Remark 2.2. 1) *In the literature, the normalization of the parameter is often taken to be*

$$\lambda := k+\rho \ (= k+\frac{p+q-2}{2}).$$

Then, for $p>1$ and $q>0$, $k\in\Lambda_+(p,q)$ if and only if

$$\lambda>0, \quad \lambda\in\mathbf{Z}+\frac{p+q}{2}.$$

2) *If $f\in\mathcal{H}^k(\mathbf{R}^{p,q})$ belongs to the K-type $\mathcal{H}^a(\mathbf{R}^p)\otimes\mathcal{H}^b(\mathbf{R}^q)$ (see Fact 2.1 2)), then we have an explicit formula of f as follows:*

$$f(\omega\cosh t,\eta\sinh t) = h_a(\omega)h_b(\eta)(\cosh t)^a(\sinh t)^b\varphi_{i\lambda}^{(b+\frac{q}{2}-1,a+\frac{p}{2}-1)}(t), \quad (2.2)$$

for some $h_a\in\mathcal{H}^a(\mathbf{R}^p)$ and $h_b\in\mathcal{H}^b(\mathbf{R}^q)$. Here $\lambda=k+\rho$, and $\varphi_{i\lambda}^{(\lambda'',\lambda')}(t)$ $(\lambda''\neq-1,-2,\dots)$ is the Jacobi function which is the unique solution to the following differential equation:

$$(L+(\lambda'+\lambda''+1)^2-\lambda^2)\varphi=0, \quad \varphi(0)=1$$

if we set $L := \frac{d^2}{dt^2} + ((2\lambda'+1)\tanh t + (2\lambda''+1)\coth t)\frac{d}{dt}$*. Equivalently, in terms of the hypergeometric function* ${}_2F_1$*, we have*

$$\varphi_{i\lambda}^{(\lambda'',\lambda')}(t) = {}_2F_1\left(\frac{\lambda'+\lambda''+1-\lambda}{2}, \frac{\lambda'+\lambda''+1+\lambda}{2}; \lambda''+1; -\sinh^2 t\right).$$

With these preparations, let us investigate the asymptotic behavior of the multiplier m near the boundary of $\mathbf{R}_+^{p,q}$. For this, we fix $\kappa \in \mathbf{R}_+$ and let

$$\mathcal{V}_\kappa := \Big\{ f \in C^\infty(\mathbf{R}_+^{p,q}) : \quad 1) \quad f \text{ is a homogeneous function of degree } 0$$
$$2) \quad \sup_{\xi \in \mathbf{R}_+^{p,q}} |f(\xi)| \left(\frac{Q(\xi)}{|\xi|^2}\right)^{-\kappa} < \infty \Big\}. \tag{2.3}$$

Remark 2.3. *Obviously,* $\mathcal{V}_\kappa \subset \mathcal{V}_{\kappa'}$ *if* $\kappa > \kappa'$*.*

For any $g \in G$, we set $c := \max(\|g\|, \|g^{-1}\|)$ where $\|g\|$ denotes the operator norm of g. This means that

$$c^{-1}|\xi| \le |g\xi| \le c|\xi| \quad (\xi \in \mathbf{R}^{p+q}).$$

Further, $Q(g\xi) = Q(\xi)$. Hence, $\mathcal{V}_\kappa$ is a G-invariant subspace of $C^\infty(\mathbf{R}_+^{p,q})$. We define

$$\mathcal{V}_\kappa^\infty := \{ f \in \mathcal{V}_\kappa : d\pi(X_1) \circ \cdots \circ d\pi(X_l) f \in \mathcal{V}_\kappa,$$
$$\text{for any } l = 0, 1, \dots \text{ and } X_1, \dots, X_l \in \mathfrak{g}\}.$$

Lemma 2.4. *Let* m *be as in Theorem 1. Then* $m|_{\mathbf{R}_+^{p,q}} \in \mathcal{V}_{\frac{k}{2}+\rho}^\infty$*.*

Proof of Lemma. Suppose $\xi \in \mathbf{R}_+^{p,q}$. Then, $Q(\xi) > 0$ and $Q(\xi)^{-\frac{1}{2}}\xi \in X_{p,q}$. Hence, we can find $\omega \in S^{p-1}$, $\eta \in S^{q-1}$ and $t \in \mathbf{R}$ such that

$$Q(\xi)^{-\frac{1}{2}}\xi = (\omega\cosh t, \eta\sinh t).$$

This means that

$$Q(\xi)^{-1}|\xi|^2 = \cosh^2 t + \sinh^2 t = \cosh 2t.$$

If m satisfies (1.1), then $m(\xi) = m(Q(\xi)^{-\frac{1}{2}}\xi) = m(\omega\cosh t, \eta\sinh t)$. Therefore, we have

$$\sup_{\xi \in \mathbf{R}_+^{p,q}} \left(\frac{|\xi|^2}{Q(\xi)}\right)^{\frac{k}{2}+\rho} m(\xi) < \infty$$

by Fact 2.1 3). Hence $m|_{\mathbf{R}_+^{p,q}} \in \mathcal{V}_{\frac{k}{2}+\rho}$. Hence, Lemma follows by iterating Fact 2.1 4). $\square$

3 Proof of L^p-boundedness

For an open subset V in $\mathbf{R}^n$, we write $C^k(V)$ for the space of functions on V with continuous derivatives up to order k.

We recall from [6, Section IV, Theorem 3] the Hörmander–Michlin condition for L^r-multipliers:

Fact 3.1. *Suppose $m \in C^{[\frac{n}{2}]+1}(\mathbf{R}^n \setminus \{0\})$ satisfies*

$$\sup_{\xi \in \mathbf{R}^n \setminus \{0\}} |\xi|^{|\alpha|} \left| \frac{\partial^\alpha m(\xi)}{\partial \xi^\alpha} \right| < \infty \tag{3.1}$$

for all multi-indices α such that $|\alpha| \leq [\frac{n}{2}] + 1$. Then, the multiplier operator T_m extends to a continuous operator on $L^r(\mathbf{R}^n)$ for any r $(1 < r < \infty)$.

In Section 5, we shall show:

Proposition 3.2. *If $f \in \mathcal{V}_\kappa^\infty$, then*

$$\sup_{\xi \in \mathbf{R}_+^{p,q}} |\xi|^{|\alpha|} \left| \frac{\partial^\alpha f}{\partial \xi^\alpha} \right| < \infty \tag{3.2}$$

for any multi-index $\alpha \in \mathbf{N}^{p+q}$ with $|\alpha| \leq \kappa$.

Proposition 3.3. *For $f \in \mathcal{V}_\kappa^\infty$ let F be the extension by zero of f to all of $\mathbf{R}^n$. Let N be any non-negative integer such that $N < \kappa$. Then $F \in C^N(\mathbf{R}^n \setminus \{0\})$. In particular, F satisfies* (3.1) *for any α with $|\alpha| < \kappa$.*

Admitting Propositions 3.2 and 3.3 for a while, let us complete the proof of Theorem 1.

Proof of Theorem 1. Suppose m is as in Theorem 1. Then $m|_{\mathbf{R}_+^{p,q}} \in \mathcal{V}_{\frac{k}{2}+\rho}^\infty$ by Lemma 2.4. Hence, m satisfies (3.1) for any multi-index α with $|\alpha| < \frac{k}{2} + \rho$ by Proposition 3.3.

Since the assumption $k > 4$ ($p+q$: even) or $k > 3$ ($p+q$: odd) implies

$$\frac{k}{2} + \rho > \left[\frac{p+q}{2} \right] + 1$$

the Hörmander–Michlin condition for m is fulfilled. Therefore, the operator is bounded on $L^r(\mathbf{R}^n)$ by Fact 3.1. $\square$

4 Differential operators along $\mathrm{O}(p,q)$-orbits

The vector space $\mathcal{V}_\kappa^\infty$ in which our multiplier lives (see Lemma 2.4) is stable under the action of the Lie algebra $\mathfrak{g}$ and the Euler operator $E = \sum_{i=1}^{p+q} \xi_i \frac{\partial}{\partial \xi_i}$. In this section, we shall give a formula of the standard derivatives $\frac{\partial}{\partial \xi_i}$ $(1 \le i \le p+q)$ by means of $d\pi(Y)$ $(Y \in \mathfrak{g})$ and E. The main result of this section is Proposition 4.3, and we shall study the space H_1 of coefficients (or more generally H_N; see (4.11)) in Section 5.

In the polar coordinate for the first p-factor:

$$\mathbf{R}_+ \times S^{p-1} \times \mathbf{R}^q \to \mathbf{R}^{p+q}, \quad (r, \omega, \xi'') \mapsto (r\omega, \xi''), \tag{4.1}$$

an easy computation shows

$$\frac{\partial}{\partial \xi_i} = a_i(\omega) \frac{\partial}{\partial r} + \frac{1}{r} Y_i(\omega) \qquad (1 \le i \le p), \tag{4.2}$$

where $a_i(\omega) \in C^\infty(S^{p-1})$ and Y_i is a smooth vector field on S^{p-1}.

In order to rewrite (4.2) by using the Lie algebra action $d\pi$, we note that $\mathfrak{g} = \mathfrak{o}(p,q)$ is given in matrices as

$$\begin{aligned} \mathfrak{g} &\simeq \{ \begin{pmatrix} A & B \\ {}^t B & C \end{pmatrix} : {}^t A = -A,\ {}^t C = -C,\ B \in M(p,q;\mathbf{R}) \} \\ &= (\mathfrak{o}(p) + \mathfrak{o}(q)) + \mathfrak{p} \qquad\qquad \text{(Cartan decomposition)}, \end{aligned}$$

where we set

$$\mathfrak{p} := \{ \begin{pmatrix} 0 & B \\ {}^t B & 0 \end{pmatrix} : B \in M(p,q;\mathbf{R}) \}.$$

Let $\mathfrak{X}(S^{p-1})$ be the vector space consisting of smooth vector fields on S^{p-1}. Since $\mathrm{O}(p)$ acts transitively on S^{p-1}, the map

$$C^\infty(S^{p-1}) \otimes \mathfrak{o}(p) \to \mathfrak{X}(S^{p-1}), \quad (b, X) \mapsto b\, d\pi(X)$$

is surjective. Let $\{K_h : 1 \le h \le \frac{1}{2}p(p-1)\}$ be a basis of the Lie algebra $\mathfrak{o}(p)$. Then, we can find $b_i^h \in C^\infty(S^{p-1})$ such that

$$Y_i(\omega) = \sum_h b_i^h(\omega) d\pi(K_h). \tag{4.3}$$

Next, we set

$$Y_{ij} := E_{i,p+j} + E_{p+j,i} \qquad (1 \le i \le p,\ 1 \le j \le q).$$

Here, E_{ij} are matrix units in $M(p+q, \mathbf{R})$. By definition, Y_{ij} spans $\mathfrak{p}$ and $d\pi(Y_{ij})$ is the vector field on $\mathbf{R}^{p+q}$ given as

$$d\pi(Y_{ij}) = \xi_{p+j}\frac{\partial}{\partial \xi_i} + \xi_i \frac{\partial}{\partial \xi_{p+j}}. \tag{4.4}$$

Lemma 4.1. *For $1 \le i \le p$ we have*

$$\frac{\partial}{\partial \xi_i} = \frac{a_i(\omega)}{rQ(\xi)}\left(r^2 E - \sum_{k=1}^{p}\sum_{j=1}^{q} \xi_k \xi_{p+j} d\pi(Y_{kj})\right) + \frac{1}{r}\sum_h b_i^h(\omega) d\pi(K_h), \tag{4.5}$$

and for $1 \le j \le q$

$$\frac{\partial}{\partial \xi_{p+j}} = \frac{1}{r^2}\left(\sum_{i=1}^{p} \xi_i d\pi(Y_{ij}) - \frac{\xi_{p+j}}{Q(\xi)}\left(r^2 E - \sum_{i=1}^{p}\sum_{k=1}^{q} \xi_i \xi_{p+k} d\pi(Y_{ik})\right)\right). \tag{4.6}$$

Proof. By multiplying (4.4) by ξ_i and summing over i $(1 \le i \le p)$, we get

$$\frac{\partial}{\partial \xi_{p+j}} = \frac{1}{r^2}\left(\sum_{i=1}^{p} \xi_i d\pi(Y_{ij}) - \xi_{p+j} r \frac{\partial}{\partial r}\right), \tag{4.7}$$

where we have used that $r\frac{\partial}{\partial r} = \sum_{i=1}^{p} \xi_i \frac{\partial}{\partial \xi_i}$.

Next, we multiply (4.7) by ξ_{p+j} and sum over j $(1 \le j \le q)$, we obtain the identity for the Euler operator $E_{\xi''} = \sum_{j=1}^{q} \xi_{p+j}\frac{\partial}{\partial \xi_{p+j}}$:

$$E_{\xi''} = \frac{1}{r^2}\sum_{i=1}^{p}\sum_{j=1}^{q} \xi_i \xi_{p+j} d\pi(Y_{ij}) - \frac{|\xi''|^2}{r^2} r \frac{\partial}{\partial r}.$$

Combining with the identity

$$E_{\xi''} + r\frac{\partial}{\partial r} = E$$

we get

$$r\frac{\partial}{\partial r} = \frac{1}{Q(\xi)}\left(r^2 E - \sum_{i=1}^{p}\sum_{j=1}^{q} \xi_i \xi_{p+j} d\pi(Y_{ij})\right). \tag{4.8}$$

By (4.7), this proves (4.6).

To prove (4.5) we insert into (4.2) the expressions for $Y_i(\omega)$ and $\frac{\partial}{\partial r}$ obtained in (4.3) and (4.8) respectively. $\square$

To handle the coefficients of (4.5) and (4.6), we introduce the subspace, denoted by $H_{a,b,c}$, of $C^\infty(\mathbf{R}^{p,q}_+)$ for $(a,b,c) \in \mathbf{N}^3$ that consists of finite linear combinations of functions of the form

$$\frac{A(\omega)P_a(\xi'')}{r^{b-c}Q(\xi)^c} = \frac{A(\omega)P_a(\xi'')}{r^{b-c}(r^2 - |\xi''|^2)^c}\,, \tag{4.9}$$

where $A \in C^\infty(S^{p-1})$ and P_a is a homogeneous polynomial of $\xi'' = (\xi_{p+1}, \dots, \xi_{p+q}) \in \mathbf{R}^q$ of degree a. If $f \in H_{a,b,c}$ and $g \in H_{a',b',c'}$ then $fg \in H_{a+a',b+b',c+c'}$, and likewise for finite linear combinations of such terms. We state this as

$$H_{a,b,c}H_{a',b',c'} \subset H_{a+a',b+b',c+c'}. \tag{4.10}$$

We also define the space

$$H_N := \bigoplus_{\substack{a,b,c \in \mathbf{N} \\ a \le 2N, c \le N \\ b-a+c = N}} H_{a,b,c} \tag{4.11}$$

The following lemma is an immediate consequence of (4.10):

Lemma 4.2. $H_N H_{N'} \subset H_{N+N'}$.

We write $H_N d\pi(\mathfrak{g})$ for the vector space consisting of differential operators on $\mathbf{R}^{p,q}_+$ which are of the form $\sum_j f_j d\pi(X_j)$ (finite sum) for some $f_j \in H_N$ and $X_j \in \mathfrak{g}$. The point of the definition of H_N is the following:

Proposition 4.3. *On* $\mathbf{R}^{p,q}_+$,

$$\frac{\partial}{\partial \xi_i} \in H_1 d\pi(\mathfrak{g}) + C^\infty(\mathbf{R}^{p,q}_+)E \qquad (1 \le i \le p+q).$$

Proof. In light of the formulas (4.5) and (4.6), it is sufficient to show that the coefficients

$$\frac{a_i(\omega)\xi_l\xi_{p+j}}{rQ(\xi)}, \frac{b_i^h(\omega)}{r}, \frac{\xi_i}{r^2}, \frac{\xi_i\xi_{p+j}\xi_{p+k}}{r^2Q(\xi)} \in H_1$$

for any $1 \le i, l \le p$ and $1 \le j, k \le q$. In fact, these coefficients belong to

$$H_{1,1,1}, H_{0,1,0}, H_{0,1,0}, H_{2,2,1},$$

respectively, by definition. □

5 Proof of Propositions 3.2 and 3.3

Lemma 5.1. *For* $1 \leq i \leq p+q$,

$$\frac{\partial}{\partial \xi_i} H_N \subset H_{N+1}.$$

Proof. Since $H_{a,b,c}$ is spanned by functions of the form (4.9), we get

$$\begin{aligned} a_i(\omega)\frac{\partial}{\partial r}(H_{a,b,c}) &\subset H_{a,b+1,c} \oplus H_{a,b,c+1}, \\ \frac{1}{r}Y_i(\omega)H_{a,b,c} &\subset H_{a,b+1,c}. \end{aligned}$$

Thus, by using (4.2) we have

$$\frac{\partial}{\partial \xi_i} H_{a,b,c} \subset H_{a,b+1,c} \oplus H_{a,b,c+1} \quad (1 \leq i \leq p).$$

For the variables $\xi'' = (\xi_{p+1}, \ldots, \xi_{p+q})$, we obtain directly

$$\frac{\partial}{\partial \xi_{j+p}} H_{a,b,c} \subset H_{a-1,b,c} \oplus H_{a+1,b+1,c+1} \quad (1 \leq j \leq q).$$

Lemma now follows from the definition (4.11) of H_N. $\square$

We denote by $H_N \cdot \mathcal{V}_\kappa^\infty$ the subspace of $C^\infty(\mathbf{R}_+^{p,q})$ consisting of finite linear combinations of products of elements from H_N and $\mathcal{V}_\kappa^\infty$. We then have

Lemma 5.2.

$$\frac{\partial}{\partial \xi_i} \mathcal{V}_\kappa^\infty \subset H_1 \cdot \mathcal{V}_\kappa^\infty \qquad (1 \leq i \leq p+q).$$

Proof. Since the Euler operator E acts on $\mathcal{V}_\kappa^\infty$ by zero and $d\pi(X)\mathcal{V}_\kappa^\infty \subset \mathcal{V}_\kappa^\infty$ $(X \in \mathfrak{g})$, Lemma follows from Proposition 4.3. $\square$

Proposition 5.3. *For any multi-index* $\alpha \in \mathbf{N}^{p+q}$,

$$\frac{\partial^\alpha}{\partial \xi^\alpha} \mathcal{V}_\kappa^\infty \subset H_{|\alpha|} \cdot \mathcal{V}_\kappa^\infty. \tag{5.1}$$

Proof. We have already proved (5.1) for $|\alpha| = 1$ in Lemma 5.2. Suppose we have proved (5.1) for $|\alpha| \le N$. Then,

$$\begin{aligned}
\frac{\partial}{\partial \xi_i}\frac{\partial^\alpha}{\partial \xi^\alpha}\mathcal{V}_\kappa^\infty &\subset \frac{\partial}{\partial \xi_i}(H_{|\alpha|}\cdot \mathcal{V}_\kappa^\infty) \\
&\subset \left(\frac{\partial}{\partial \xi_i}H_{|\alpha|}\right)\cdot \mathcal{V}_\kappa^\infty + H_{|\alpha|}\cdot\left(\frac{\partial}{\partial \xi_i}\mathcal{V}_\kappa^\infty\right) \\
&\subset H_{|\alpha|+1}\cdot \mathcal{V}_\kappa^\infty + H_{|\alpha|}(H_1\cdot \mathcal{V}_\kappa^\infty)
\end{aligned}$$

by Lemmas 5.1 and 5.2. Since $H_{|\alpha|}H_1 \subset H_{|\alpha|+1}$ by Lemma 4.2, (5.1) holds for $|\alpha| = N+1$. Hence, Proposition 5.3 is proved by induction on $|\alpha|$. $\square$

Lemma 5.4. *Let $f \in \mathcal{V}_\kappa$ and $g \in H_{a,b,c}$. Then*

$$|f(\xi)g(\xi)| \le \frac{C}{|\xi|^{b-a+c}}\left(\frac{Q(\xi)}{|\xi|^2}\right)^{\kappa-c} \qquad (\xi \in \mathbf{R}_+^{p,q}).$$

In particular, if $f \in \mathcal{V}_\kappa$ and $g \in H_N$ such that $N \le \kappa$, then we have

$$|f(\xi)g(\xi)| \le C|\xi|^{-N} \qquad (\xi \in \mathbf{R}_+^{p,q}).$$

Proof. By the definition (2.3) of $\mathcal{V}_\kappa$, f satisfies

$$|f(\xi)| \le C_1\left(\frac{Q(\xi)}{|\xi|^2}\right)^{\kappa} \quad \text{for } \ \xi \in \mathbf{R}_+^{p,q},$$

for some constant $C_1 > 0$. Hence, in view of (4.9), there exists $C' > 0$ such that

$$|f(\xi)g(\xi)| \le C'\frac{1}{r^{b-c}}\frac{|\xi''|^a}{Q(\xi)^c}\left(\frac{Q(\xi)}{|\xi|^2}\right)^{\kappa}. \tag{5.2}$$

We note that for $\xi \in \mathbf{R}_+^{p,q}$, we have $r > |\xi''|$ and therefore $|\xi| = (r^2 + |\xi''|^2)^{\frac{1}{2}}$ satisfies

$$r < |\xi| < \sqrt{2}r.$$

Hence the first factor of (5.2) is bounded by

$$\frac{1}{r^{b-c}} \le \frac{C''}{|\xi|^{b-c}}.$$

The last two factors of (5.2) are estimated as

$$\frac{|\xi''|^a}{Q(\xi)^c}\left(\frac{Q(\xi)}{|\xi|^2}\right)^{\kappa} \le \frac{1}{|\xi|^{2c-a}}\left(\frac{Q(\xi)}{|\xi|^2}\right)^{\kappa-c}.$$

Combining these estimates, we have proved

$$|f(\xi)g(\xi)| \leq \frac{C'C''}{|\xi|^{b-a+c}} \left(\frac{Q(\xi)}{|\xi|^2}\right)^{\kappa-c}.$$

$\square$

Proof of Proposition 3.2. Suppose $f \in \mathcal{V}_\kappa^\infty$. Then $\frac{\partial^\alpha f}{\partial \xi^\alpha} \in H_{|\alpha|} \cdot \mathcal{V}_\kappa^\infty$ by Proposition 5.3. If $|\alpha| \leq \kappa$ then $\left|\frac{\partial^\alpha f}{\partial \xi^\alpha}\right| \leq C|\xi|^{-|\alpha|}$ for $\xi \in \mathbf{R}_+^{p,q}$ by Lemma 5.4. Hence, Proposition 3.2 is proved. $\square$

Proof of Proposition 3.3. Let $f \in \mathcal{V}_\kappa^\infty$. It is sufficient to prove that if $|\alpha| < \kappa$ then

$$\frac{\partial^\alpha f}{\partial \xi^\alpha}(\xi) \to 0$$

as $\xi \in \mathbf{R}_+^{p,q}$ approaches to the boundary of $\mathbf{R}_+^{p,q}$ in $\mathbf{R}^{p+q} \setminus \{0\}$, namely, the light cone $\{\xi \in \mathbf{R}^{p+q} \setminus \{0\} : Q(\xi) = 0\}$. This follows again from Lemma 5.4. Hence, Proposition 3.3 is also proved. $\square$

Thus, the proof of Theorem 1 is completed.

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