TOPICS IN ABSOLUTE ANABELIAN GEOMETRY I: GENERALITIES

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Abstract. This paper forms the first part of a three-part series in which we treat various topics in *absolute anabelian geometry* from the point of view of developing *abstract algorithms*, or *"software"*, that may be applied to abstract profinite groups that "just happen" to arise as [quotients of] étale fundamental groups from algebraic geometry. In the present paper, after studying various *abstract combinatorial properties* of profinite groups that typically arise as absolute Galois groups or geometric fundamental groups in anabelian geometry over number fields, mixed-characteristic local fields, or finite fields, we take a more detailed look at certain *p-adic Hodge-theoretic* aspects of the absolute Galois groups of mixed-characteristic local fields. This allows us, for instance, to derive, from a certain result communicated orally to the author by A. Tamagawa, a *"semi-absolute"* Hom-*version* of the anabelian conjecture for hyperbolic curves over mixed-characteristic local fields. Finally, we generalize to the case of *varieties of arbitrary dimension over arbitrary sub-p-adic fields* certain techniques developed by the author in previous papers over mixed-characteristic local fields for group-theoretically constructing the étale fundamental group of one hyperbolic curve from the étale fundamental group of another hyperbolic curve.

Contents:

Introduction

The present paper is the first in a series of three papers, in which we continue our study of *absolute anabelian geometry* in the style of the following papers: [Mzk6], [Mzk7], [Mzk8], [Mzk9], [Mzk14], [Mzk10], [Mzk11]. If X is a [geometrically integral] *variety* over a field k, and $\Pi_X \stackrel{\mathrm{def}}{=} \pi_1(X)$ is the *étale fundamental group* of X [for some choice of basepoint], then roughly speaking, *"anabelian geometry"* may be

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summarized as the study of the extent to which properties of X — such as, for instance, the *isomorphism class* of X — may be "recovered" from [various quotients of] the profinite group Π_X. One form of anabelian geometry is *"relative anabelian geometry"* [cf., e.g., [Mzk3]], in which instead of starting from [various quotients of] the profinite group Π_X, one starts from the profinite group Π_X equipped with the *natural augmentation* $\Pi_X \twoheadrightarrow G_k$ to the *absolute Galois group* of k. By constrast, *"absolute anabelian geometry"* refers to the study of properties of X as reflected *solely in the profinite group* Π_X. Moreover, one may consider various "intermediate variants" between relative and absolute anabelian geometry such as, for instance, *"semi-absolute anabelian geometry"*, which refers to the situation in which one starts from the profinite group Π_X equipped with the *kernel* of the natural augmentation $\Pi_X \twoheadrightarrow G_k$.

The *new point of view* that underlies the various "topics in absolute anabelian geometry" treated in the present three-part series may be summarized as follows. In the past, research in anabelian geometry typically centered around the establishment of *"fully faithfulness"* results — i.e., *"Grothendieck Conjecture-type"* results — concerning some sort of "fundamental group functor $X \mapsto \Pi_X$" from varieties to profinite groups. In particular, the term *"group-theoretic"* was typically used to refer to properties preserved, for instance, by some isomorphism of profinite groups $\Pi_X \xrightarrow{\sim} \Pi_Y$ [i.e., between the étale fundamental groups of varieties X, Y]. By contrast:

> In the present series, the focus of our attention is on the development of **"algorithms"** — i.e., **"software"** — which are *"group-theoretic"* in the sense that they are phrased in *language* that only depends on the structure of the *input data* as, for instance, a *profinite group*. Here, the "input data" is a profinite group that *"just happens to arise"* from scheme theory as an étale fundamental group, but which is only of concern to us in its capacity as an **abstract profinite group**. That is to say, the algorithms in question allow one to construct various objects reminiscent of objects that arise in scheme theory, but the issue of **"eventually returning to scheme theory"** — e.g., of showing that some isomorphism of profinite groups arises from an isomorphism of schemes — is **no longer an issue of primary interest**.

This point of view may already be seen in the theory of *pro-l cuspidalizations* given in [Mzk14], §3, in which "cuspidalized geometrically pro-l fundamental groups" are "group-theoretically constructed" from geometrically pro-l fundamental groups of proper hyperbolic curves *without* ever addressing the issue of whether or not the original curve [i.e., scheme] may be reconstructed from the given geometrically pro-l fundamental group [of a proper hyperbolic curve]. In some sense, this abstract, algorithmic point of view is taken even further in [Mzk13], where one works with certain types of *purely combinatorial objects* — i.e., "semi-graphs of anabelioids" — whose definition "just happens to be" motivated by stable curves in algebraic geometry. On the other hand, the results obtained in [Mzk13] are results concerning the *abstract combinatorial geometry* of these abstract combinatorial objects — i.e., one is never concerned with the issue of "eventually returning" to, for instance, scheme-theoretic morphisms.

The *main results* of the present paper are, to a substantial extent, *"generalities"* that will be of use to us in the further development of the theory in the latter two papers of the present three-part series. These main results may be summarized as follows:

(1) In §1, we study various notions associated to abstract profinite groups such as *RTF-quotients* [i.e., quotients obtained by successive formation of torsion-free abelianizations — cf. Definition 1.1, (i)], *slimness* [i.e., the property that all open subgroups are center-free], and *elasticity* [i.e., the property that every nontrivial topologically finitely generated closed normal subgroup of an open subgroup is itself open — cf. Definition 1.1, (ii)] in the context of the *absolute Galois groups* that typically appear in anabelian geometry [cf. Proposition 1.5, Theorem 1.7].

(2) In §2, we begin by formulating the *terminology* that we shall use in our discussion of the *anabelian geometry of varieties of arbitrary dimension* [cf. Definition 2.1]. We then apply the theory of slimness and elasticity developed in §1 to study various variants of the notion of *"semi-absoluteness"* [cf. Proposition 2.5]. Moreover, in the case of arithmetic base fields that typically appear in anabelian geometry, we give various *"group-theoretic algorithms"* for constructing the quotient of an arithmetic fundamental group determined by the absolute Galois group of the base field [cf. Theorem 2.6]. Finally, in the case of *hyperbolic orbicurves*, we apply the theory of *maximal pro-RTF-quotients* developed in §1 to give quite *explicit "group-theoretic algorithms"* for constructing these quotients [cf. Theorem 2.11].

(3) In §3, we generalize the main result of [Mzk1] concerning the geometricity of arbitrary isomorphisms of absolute Galois groups of mixed-characteristic local fields that preserve the ramification filtration [cf. Theorem 3.5]. This generalization allows one to replace the condition of "preserving the ramification filtration" by various more general conditions, certain of which were motivated by a result orally communicated to the author by A. Tamagawa [cf. Remark 3.8.1]. Moreover, unlike the main result of [Mzk1], this generalization may be applied [in certain cases] to *arbitrary open homomorphisms* between absolute Galois groups of mixed-characteristic local fields, hence implies certain *semi-absolute* Hom-*versions* [cf. Corollary 3.8, 3.9] of the relative Hom-versions of the Grothendieck Conjecture given in [Mzk3], Theorems A, B. Also, we observe, in Example 2.13, that the corresponding *absolute* Hom-*version* of these results is *false* in general. Indeed, it was precisely the discovery of this *counterexample* to the "absolute Hom-version" that led the author to the detailed investigation of the "gap between absolute and semi-absolute" that forms the content of §2.

(4) In §4, we study various *"fundamental operations"* for passing from one

algebraic stack to another. In the case of arbitrary dimension, these operations are the operations of *"passing to a finite étale covering"* and *"passing to a finite étale quotient"*; in the case of hyperbolic orbicurves, we also consider the operations of *"forgetting a cusp"* and *"coarsifying a non-scheme-like point"*. Our main result asserts that if one assumes certain *relative anabelian results* concerning the varieties under consideration, then the corresponding *absolute anabelian* operations on arithmetic fundamental groups may be described *"entirely group-theoretically"* [cf. Theorem 4.7]. This theory, which generalizes the theory of [Mzk9], §2, and [Mzk14], §2, may be applied not only to *hyperbolic orbicurves over sub-p-adic fields* [cf. Example 4.8], but also to *"iso-poly-hyperbolic orbisurfaces" over sub-p-adic fields* [cf. Example 4.9]. We also give a *tempered version* of this theory [cf. Theorem 4.12].

Finally, in an Appendix, we review, for lack of an appropriate reference, various well-known facts concerning the theory of *Albanese varieties* that will play an important role in the portion of the theory of §2 concerning varieties of arbitrary dimension. Much of this theory of Albanese varieties is contained in such *classical references* as [NS], [Serre1], [Chev], which are written from a somewhat classical point of view. Thus, in the Appendix, we give a *modern scheme-theoretic treatment* of this classical theory, but without resorting to the introduction of *motives and derived categories*, as in [BS], [SS].

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Section 0: Notations and Conventions

Numbers:

The notation $\mathbb{Q}$ will be used to denote the field of *rational numbers*. The notation $\mathbb{Z} \subseteq \mathbb{Q}$ will be used to denote the set, group, or ring of *rational integers*. The notation $\mathbb{N} \subseteq \mathbb{Z}$ will be used to denote the set or monoid of *nonnegative rational integers*. The *profinite completion* of the group $\mathbb{Z}$ will be denoted $\widehat{\mathbb{Z}}$. Write

$$\mathfrak{Primes}$$

for the *set of prime numbers*. If $p \in \mathfrak{Primes}$, then the notation $\mathbb{Q}_p$ (respectively, $\mathbb{Z}_p$) will be used to denote the *p-adic completion* of $\mathbb{Q}$ (respectively, $\mathbb{Z}$). Also, we shall write

$$\mathbb{Z}_p^{(\times)} \subseteq \mathbb{Z}_p^{\times}$$

for the subgroup $1 + p\mathbb{Z}_p \subseteq \mathbb{Z}_p^{\times}$ if $p > 2$, $1 + p^2\mathbb{Z}_p \subseteq \mathbb{Z}_p^{\times}$ if $p = 2$. Thus, we have isomorphisms of *topological groups*

$$\mathbb{Z}_p^{(\times)} \times (\mathbb{Z}_p^{\times}/\mathbb{Z}_p^{(\times)}) \xrightarrow{\sim} \mathbb{Z}_p^{\times}; \quad \mathbb{Z}_p^{(\times)} \xrightarrow{\sim} \mathbb{Z}_p$$

— where the second isomorphism is the isomorphism determined by the *p-adic logarithm*; $\mathbb{Z}_p^{\times}/\mathbb{Z}_p^{(\times)} \xrightarrow{\sim} \mathbb{F}_p^{\times}$ if $p > 2$, $\mathbb{Z}_p^{\times}/\mathbb{Z}_p^{(\times)} \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ if $p = 2$.

A finite field extension of $\mathbb{Q}$ will be referred to as a *number field*, or *NF*, for short. A finite field extension of $\mathbb{Q}_p$ for some $p \in \mathfrak{Primes}$ will be referred to as a *mixed-characteristic nonarchimedean local field*, or *MLF*, for short. A field of finite cardinality will be referred to as a *finite field*, or *FF*, for short. We shall regard the set of symbols $\{\mathrm{NF}, \mathrm{MLF}, \mathrm{FF}\}$ as being equipped with a *linear ordering*

$$\mathrm{NF} > \mathrm{MLF} > \mathrm{FF}$$

and refer to an element of this set of symbols as a *field type*.

Topological Groups:

Let G be a *Hausdorff topological group*, and $H \subseteq G$ a *closed subgroup*. Let us write

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot h = h \cdot g, \ \forall\, h \in H\}$$

for the *centralizer* of H in G;

$$N_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\}$$

for the *normalizer* of H in G; and

$$C_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid (g \cdot H \cdot g^{-1}) \bigcap H \text{ has finite index in } H,\ g \cdot H \cdot g^{-1}\}$$

for the *commensurator* of H in G. Note that: (i) $Z_G(H)$, $N_G(H)$ and $C_G(H)$ are *subgroups of* G; (ii) we have *inclusions* H, $Z_G(H) \subseteq N_G(H) \subseteq C_G(H)$; (iii) H is *normal* in $N_G(H)$. If $H = N_G(H)$ (respectively, $H = C_G(H)$), then we shall say that H is *normally terminal* (respectively, *commensurably terminal*) in G. Note that $Z_G(H)$, $N_G(H)$ are *always closed in* G, while $C_G(H)$ is *not necessarily closed in* G. Also, we shall write $Z(G) \stackrel{\text{def}}{=} Z_G(G)$ for the *center* of G.

Let G be a *topological group*. Then [cf. [Mzk12], §0] we shall refer to a normal open subgroup $H \subseteq G$ such that the quotient group G/H is a free discrete group as *co-free*. We shall refer to a co-free subgroup $H \subseteq G$ as *minimal* if every co-free subgroup of G contains H. Thus, any minimal co-free subgroup of G is necessarily *unique* and *characteristic*.

We shall refer to a continuous homomorphism between topological groups as *dense* (respectively, *of DOF-type* [cf. [Mzk10], Definition 6.2, (iii)]; *of OF-type*) if its image is dense (respectively, dense in some open subgroup of finite index; an open subgroup of finite index). Let Π be a *topological group*; Δ a *normal closed subgroup* such that every characteristic open subgroup of finite index $H \subseteq \Delta$ admits a *minimal co-free subgroup* $H^{\text{co-fr}} \subseteq H$. Write $\widehat{\Pi}$ for the *profinite completion* of Π. Let

$$\widehat{\Pi} \twoheadrightarrow Q$$

be a *quotient of profinite groups*. Then we shall refer to as the (Q, Δ)*-co-free completion of* Π, or *co-free completion of* Π *with respect to [the quotient* $\widehat{\Pi} \twoheadrightarrow$*]* Q *and [the subgroup]* $\Delta \subseteq \Pi$ — where we shall often omit mention of Δ when it is fixed throughout the discussion — the inverse limit

$$\Pi^{Q/\text{co-fr}} \stackrel{\text{def}}{=} \varprojlim_{H} \ \text{Im}_Q(\Pi/H^{\text{co-fr}})$$

— where $H \subseteq \Delta$ ranges over the characteristic open subgroups of Δ of finite index; $\widehat{H}^{\text{co-fr}} \subseteq \widehat{\Pi}$ is the closure of the image of $H^{\text{co-fr}}$ in $\widehat{\Pi}$; $\widehat{H}^{\text{co-fr}}_Q \subseteq Q$ is the image of $\widehat{H}^{\text{co-fr}}$ in Q; "$\text{Im}_Q(-)$" denotes the image in $Q/\widehat{H}^{\text{co-fr}}_Q$ of the group in parentheses. Thus, we have a *natural dense homomorphism* $\Pi \to \Pi^{Q/\text{co-fr}}$.

We shall say that a profinite group G is *slim* if for every open subgroup $H \subseteq G$, the centralizer $Z_G(H)$ is trivial. Note that every *finite normal closed subgroup* $N \subseteq G$ of a slim profinite group G is *trivial*. [Indeed, this follows by observing that for any normal open subgroup $H \subseteq G$ such that $N \bigcap H = \{1\}$, consideration of the inclusion $N \hookrightarrow G/H$ reveals that the conjugation action of H on N is *trivial*, i.e., that $N \subseteq Z_G(H) = \{1\}$.]

We shall say that a profinite group G is *decomposable* if there exists an isomorphism of profinite groups $H_1 \times H_2 \xrightarrow{\sim} G$, where H_1, H_2 are nontrivial profinite groups. If a profinite group G is not decomposable, then we shall say that it is *indecomposable*.

We shall write G^{ab} for the *abelianization* of a profinite group G, i.e., the quotient of G by the closure of the commutator subgroup of G, and

$$G^{\text{ab-t}}$$

for the *torsion-free abelianization* of G, i.e., the quotient of G^{ab} by the closure of the torsion subgroup of G^{ab}. Note that the formation of G^{ab}, $G^{\text{ab-t}}$ is *functorial* with respect to arbitrary continuous homomorphisms of profinite groups.

We shall denote the group of automorphisms of a profinite group G by $\text{Aut}(G)$. Conjugation by elements of G determines a homomorphism $G \to \text{Aut}(G)$ whose image consists of the *inner automorphisms* of G. We shall denote by $\text{Out}(G)$ the quotient of $\text{Aut}(G)$ by the [normal] subgroup consisting of the inner automorphisms. In particular, if G is *center-free*, then we have an *exact sequence* $1 \to G \to \text{Aut}(G) \to \text{Out}(G) \to 1$. If, moreover, G is *topologically finitely generated*, then it follows immediately that the topology of G admits a basis of *characteristic open subgroups*, which thus determine a *topology* on $\text{Aut}(G)$, $\text{Out}(G)$ with respect to which the exact sequence $1 \to G \to \text{Aut}(G) \to \text{Out}(G) \to 1$ becomes an exact sequence of *profinite groups*.

Algebraic Stacks:

We refer to [FC], Chapter I, §4.10, for a discussion of the *coarse space* associated to an algebraic stack. We shall say that an algebraic stack is *scheme-like* if it is, in fact, a scheme. We shall say that an algebraic stack is *generically scheme-like* if it admits an open dense substack which is a scheme.

Curves:

We shall use the following terms, as they are defined in [Mzk14], §0: *hyperbolic curve*, *family of hyperbolic curves*, *cusp*, *tripod*. Also, we refer to [Mzk6], the proof of Lemma 2.1; [Mzk6], the discussion following Lemma 2.1, for an explanation of the terms *"stable reduction"* and *"stable model"* applied to a hyperbolic curve over an MLF.

If X is a *generically scheme-like algebraic stack* over a field k that admits a *finite étale Galois covering* $Y \to X$, where Y is a hyperbolic curve over a finite extension of k, then we shall refer to X as a *hyperbolic orbicurve* over k. [Thus, when k is of *characteristic zero*, this definition *coincides* with the definition of a "hyperbolic orbicurve" in [Mzk14], §0, and *differs* from, but is *equivalent* to, the definition of a "hyperbolic orbicurve" given in [Mzk7], Definition 2.2, (ii). We refer to [Mzk14], §0, for more on this *equivalence*.] Note that the notion of a "cusp of a hyperbolic curve" given in [Mzk14], §0, generalizes immediately to the notion of *"cusp of a hyperbolic orbicurve"*. If $X \to Y$ is a dominant morphism of hyperbolic orbicurves, then we shall refer to $X \to Y$ as a *partial coarsification morphism* if the morphism induced by $X \to Y$ on *associated coarse spaces* is an *isomorphism*.

Let X be a *hyperbolic orbicurve* over an algebraically closed field; denote its *étale fundamental group* by Δ_X. We shall refer to the order of the [manifestly finite!] decomposition group of a closed point x of X as the *order of* x. We shall refer to the [manifestly finite!] *least common multiple* of the orders of the closed points of X as the *order of* X. Thus, it follows immediately from the definitions that X is a *hyperbolic curve* if and only if the order of X is equal to 1.

Section 1: Some Profinite Group Theory

We begin by discussing certain aspects of *abstract profinite groups*, as they relate to the *Galois groups* of *finite fields*, *mixed-characteristic nonarchimedean local fields*, and *number fields*. In the following, let G be a *profinite group*.

Definition 1.1.

(i) In the following, "RTF" is to be understood as an abbreviation for *"recursively torsion-free"*. If $H \subseteq G$ is a normal open subgroup that arises as the kernel of a continuous surjection $G \twoheadrightarrow Q$, where Q is a finite abelian group, that *factors* through the torsion-free abelianization $G \twoheadrightarrow G^{\text{ab-t}}$ of G [cf. §0], then we shall refer to (G, H) as an *RTF-pair*. If for some integer $n \geq 1$, a sequence of open subgroups

$$G_n \subseteq G_{n-1} \subseteq \ldots G_1 \subseteq G_0 = G$$

of G satisfies the condition that, for each nonnegative integer $j \leq n-1$, (G_j, G_{j+1}) is an RTF-pair, then we shall refer to this sequence of open subgroups as an *RTF-chain [from G_n to G]*. If $H \subseteq G$ is an open subgroup such that there exists an RTF-chain from H to G, then we shall refer to $H \subseteq G$ as an *RTF-subgroup [of G]*. If the kernel of a continuous surjection $\phi : G \twoheadrightarrow Q$, where Q is a finite group, is an RTF-subgroup of G, then we shall say that $\phi : G \twoheadrightarrow Q$ is an *RTF-quotient* of G. If $\phi : G \twoheadrightarrow Q$ is a continuous surjection of profinite groups such that the topology of Q admits a basis of normal open subgroups $\{N_\alpha\}_{\alpha \in A}$ satisfying the property that each composite $G \twoheadrightarrow Q \twoheadrightarrow Q/N_\alpha$ [for $\alpha \in A$] is an RTF-quotient, then we shall say that $\phi : G \twoheadrightarrow Q$ is a *pro-RTF-quotient*. If G is a finite (respectively, profinite) group such that the identity map of G forms an RTF-quotient (respectively, pro-RTF-quotient), then we shall say that G is an *RTF-group* (respectively, a *pro-RTF-group*).

(ii) We shall say that G is *elastic* if it holds that every topologically finitely generated closed normal subgroup $N \subseteq H$ of an open subgroup $H \subseteq G$ of G is either trivial or of finite index in G. If G is elastic, but *not* topologically finitely generated, then we shall say that G is *very elastic*.

(iii) Let $\Sigma \subseteq \mathfrak{Primes}$ [cf. §0] be a *set of prime numbers*. If G admits an open subgroup which is pro-Σ, then we shall say that G is *almost pro-Σ*. We shall refer to a quotient $G \twoheadrightarrow Q$ as *almost pro-Σ-maximal* if for some normal open subgroup $N \subseteq G$ with maximal pro-Σ quotient $N \twoheadrightarrow P$, we have $\text{Ker}(G \twoheadrightarrow Q) = \text{Ker}(N \twoheadrightarrow P)$. [Thus, any almost pro-$\Sigma$-maximal quotient of G is almost pro-Σ.] When $\Sigma \stackrel{\text{def}}{=} \mathfrak{Primes} \backslash \{p\}$ for some $p \in \mathfrak{Primes}$, then we shall write *"pro-$(\neq p)$"* for "pro-Σ". Write

$$\widehat{\mathbb{Z}}^{(\neq p)}$$

for the *maximal pro-$(\neq p)$ quotient* of $\widehat{\mathbb{Z}}$. We shall say that G is *pro-omissive* (respectively, *almost pro-omissive*) if it is pro-$(\neq p)$ for some $p \in \mathfrak{Primes}$ (respectively, if it admits a pro-omissive open subgroup). We shall say that G is *augmented pro-p* if there exists an exact sequence of profinite groups $1 \to N \to G \to \widehat{\mathbb{Z}}^{(\neq p)} \to 1$, where N is *pro-p*; in this case, the image of N in G is *uniquely determined* [i.e., as the maximal pro-p subgroup of G]; the quotient $G \twoheadrightarrow \widehat{\mathbb{Z}}^{(\neq p)}$ [which is well-defined up to automorphisms of $\widehat{\mathbb{Z}}^{(\neq p)}$] will be referred to as the *augmentation* of the augmented pro-p group G. We shall say that G is *augmented pro-prime* if it is augmented pro-p for some [*not necessarily unique*!] $p \in \mathfrak{Primes}$. If $\Sigma = \{p\}$ for some *unspecified* $p \in \mathfrak{Primes}$, we shall write *"pro-prime"* for "pro-Σ". When $\mathcal{C}$ is the *"full formation"* [cf., e.g., [FJ], p. 343] of finite solvable Σ-groups, then we shall refer to a pro-$\mathcal{C}$ group as a *pro-Σ-solvable group*.

Proposition 1.2. **(Basic Properties of Pro-RTF-quotients)** *Let*

$$\phi : G_1 \to G_2$$

be a continuous homomorphism of profinite groups. Then:

(i) If $H \subseteq G_2$ is an **RTF-subgroup** *of G_2, then $\phi^{-1}(H) \subseteq G_1$ is an* **RTF-subgroup** *of G_1.*

(ii) If $H, J \subseteq G$ are **RTF-subgroups** *of G, then so is $H \bigcap J$.*

(iii) If $H \subseteq G$ is an **RTF-subgroup** *of G, then there exists a* **normal** *[open] RTF-subgroup $J \subseteq G$ of G such that $J \subseteq H$.*

(iv) Every RTF-quotient $G \twoheadrightarrow Q$ of G factors through the quotient

$$G \twoheadrightarrow G^{\mathrm{RTF}} \stackrel{\mathrm{def}}{=} \varprojlim_{N} G/N$$

— where N ranges over the normal [open] RTF-subgroups of G. We shall refer to this quotient $G \twoheadrightarrow G^{\mathrm{RTF}}$ as the **maximal pro-RTF-quotient***.*

(v) There exists a commutative diagram

$$\begin{array}{ccc} G_1 & \stackrel{\phi}{\longrightarrow} & G_2 \\ \downarrow & & \downarrow \\ G_1^{\mathrm{RTF}} & \stackrel{\phi^{\mathrm{RTF}}}{\longrightarrow} & G_2^{\mathrm{RTF}} \end{array}$$

— where the vertical arrows are the natural morphisms, and the continuous homomorphism ϕ^{RTF} is uniquely determined by the condition that the diagram commute.

Proof. Assertion (i) follows immediately from the definitions, together with the *functoriality* of the torsion-free abelianization [cf. §0]. To verify assertion (ii), one observes that an RTF-chain from $H \bigcap J$ to G may be obtained by *concatenating* an RTF-chain from $H \bigcap J$ to J [whose existence follows from assertion (i) applied to the natural inclusion homomorphism $J \hookrightarrow G$] with an RTF-chain from J to G. Assertion (iii) follows by applying assertion (ii) to some finite intersection of conjugates of H. Assertion (iv) follows immediately from assertions (ii), (iii). Assertion (v) follows immediately from assertions (i), (iv). ○

Proposition 1.3. **(Basic Properties of Elasticity)**

(i) Let $H \subseteq G$ be an open subgroup of the profinite group G. Then the **elasticity** *of G implies that of H. If G is* **slim**, *then the elasticity of H implies that of G.*

(ii) Suppose that G is **nontrivial**. *Then G is* **very elastic** *if and only if it holds that every topologically finitely generated closed normal subgroup $N \subseteq H$ of an open subgroup $H \subseteq G$ of G is trivial.*

Proof. Assertion (i) follows immediately from the definitions, together with the fact that a slim profinite group has no normal closed finite subgroups [cf. §0]. The *necessity* portion of assertion (ii) follows from the fact that the existence of a topologically finitely generated open subgroup of G implies that G itself is topologically finitely generated; the *sufficiency* portion of assertion (ii) follows immediately by taking $N \stackrel{\mathrm{def}}{=} G \neq \{1\}$. ◯

Next, we consider *Galois groups.*

Definition 1.4. We shall refer to a field k as *solvably closed* if, for every finite abelian field extension k' of k, it holds that $k' = k$.

Remark 1.4.1. Note that if $\widetilde{k}$ is a *solvably closed Galois extension* of a field k of *type MLF or FF* [cf. §0], then $\widetilde{k}$ is an *algebraic closure* of k. Indeed, this follows from the well-known fact that the absolute Galois group of a field of type MLF or FF is *pro-solvable* [cf., e.g., [NSW], Chapter VII, §5].

Proposition 1.5. **(Pro-RTF-quotients of MLF Galois Groups)** *Let $\overline{k}$ be an algebraic closure of an* **MLF** *[cf. §0] k of residue characteristic p; $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; $G_k \twoheadrightarrow G_k^{\mathrm{RTF}}$ the* **maximal pro-RTF-quotient** *[cf. Proposition 1.2, (iv)] of G_k. Then:*

(i) G_k^{RTF} is **slim**.

(ii) There exists an exact sequence $1 \to P \to G_k^{\mathrm{RTF}} \to \widehat{\mathbb{Z}} \to 1$, where P is a **pro-p group** *whose image in G_k^{RTF} is equal to the image of the* **inertia subgroup** *of G_k in G_k^{RTF}. In particular, G_k^{RTF} is* **augmented pro-p**.

Proof. Recall from *local class field theory* [cf., e.g., [Serre2]] that for any open subgroup $H \subseteq G_k$, corresponding to a subfield $k_H \subseteq \overline{k}$, we have a natural isomorphism

$$(k_H^{\times})^{\wedge} \stackrel{\sim}{\to} H^{\mathrm{ab}}$$

[where the "$\wedge$" denotes the profinite completion of an abelian group; "$\times$" denotes the group of units of a ring]; moreover, H^{ab} fits into an *exact sequence*

$$1 \to \mathcal{O}_{k_H}^{\times} \to H^{\mathrm{ab}} \to \widehat{\mathbb{Z}} \to 1$$

[where $\mathcal{O}_{k_H} \subseteq k_H$ is ring of integers] in which the image of $\mathcal{O}_{k_H}^{\times}$ in H^{ab} *coincides* with the image of the *inertia subgroup* of H. Observe, moreover, that the quotient of the abelian profinite group $\mathcal{O}_{k_H}^{\times}$ by its torsion subgroup is a *pro-p group*. Thus, assertion (ii) follows immediately from this observation, together with the definition of the *maximal pro-RTF-quotient*. Next, let us observe that by applying the *natural isomorphism* $(\mathcal{O}_{k_H}^{\times}) \otimes \mathbb{Q}_p \xrightarrow{\sim} k_H$, it follows that whenever H is *normal* in G_k, G_k/H acts *faithfully* on $H^{\text{ab-t}}$. Thus, assertion (i) follows immediately. ◯

The following result is well-known.

Proposition 1.6. **(Maximal Pro-p Quotients of MLF Galois Groups)** *Let $\overline{k}$ be an algebraic closure of an* **MLF** *k of residue characteristic p; $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$; $G_k \twoheadrightarrow G_k^{(p)}$ the* **maximal pro-p-quotient** *of G_k. Then:*

(i) Any **almost pro-p-maximal quotient** *$G_k \twoheadrightarrow Q$ of G_k is* **slim**.

(ii) Suppose further that k contains a **primitive p-th root of unity**. *Then for any finite module M annihilated by p equipped with a continuous action by $G_k^{(p)}$ [which thus determines a continuous action by G_k], the natural homomorphism $G_k \twoheadrightarrow G_k^{(p)}$ induces an* **isomorphism**

$$H^j(G_k^{(p)}, M) \xrightarrow{\sim} H^j(G_k, M)$$

on Galois cohomology modules for all integers $j \geq 0$.

(iii) If k contains (respectively, does not contain) a **primitive p-th root of unity**, *then any closed subgroup of* **infinite index** *(respectively, any closed subgroup of arbitrary index) $H \subseteq G_k^{(p)}$ is a* **free pro-p group**.

Proof. Assertion (i) follows from the argument applied to verify Proposition 1.5, (i). To verify assertion (ii), it suffices to show that the cohomology module

$$H^j(J, M) \cong \varinjlim_{k'} H^j(G_{k'}, M)$$

[where $J \stackrel{\text{def}}{=} \mathrm{Ker}(G_k \twoheadrightarrow G_k^{(p)})$; k' ranges over the finite Galois extensions of k such that $[k' : k]$ is a power of p; $G_{k'} \subseteq G_k$ is the open subgroup determined by k'] *vanishes* for $j \geq 1$. By "dévissage", we may assume that $M \cong \mathbb{F}_p$ with the trivial G_k-action. Since the *cohomological dimension* of $G_{k'}$ is equal to 2 [cf. [NSW], Theorem 7.1.8, (i)], it suffices to consider the cases $j = 1, 2$. For $j = 2$, since $H^2(G_{k'}, \mathbb{F}_p) \cong \mathbb{F}_p$ [cf. [NSW], Theorem 7.1.8, (ii); our hypothesis that k contains a *primitive p-th root of unity*], it suffices, by the well-known *functorial behavior* of $H^2(G_{k'}, \mathbb{F}_p)$ [cf. [NSW], Corollary 7.1.4], to observe that k' always admits a cyclic Galois extension of degree p [arising, for instance, from an extension of the residue field of k']. On the other hand, for $j = 1$, the desired vanishing is a *tautology*, in light of the definition of the quotient $G_k \twoheadrightarrow G_k^{(p)}$. This completes the proof of assertion (ii).

Finally, we consider assertion (iii). If k does *not* contain a *primitive* p*-th root of unity*, then $G_k^{(p)}$ itself is a *free pro-*p *group* [cf. [NSW], Theorem 7.5.8, (i)], so any closed subgroup $H \subseteq G_k^{(p)}$ is also free pro-p [cf., e.g., [RZ], Corollary 7.7.5]. Thus, let us assume that k contains a *primitive* p*-th root of unity*, so we may apply the *isomorphism of assertion (ii)*. In particular, if $J \subseteq G_k^{(p)}$ is an open subgroup such that $H \subseteq J$, and $k_J \subseteq \overline{k}$ is the subfield determined by J, then one verifies immediately that the quotient $G_{k_J} \twoheadrightarrow J$ may be identified with the quotient $G_{k_J} \twoheadrightarrow G_{k_J}^{(p)}$, so we obtain an isomorphism $H^2(J, \mathbb{F}_p) \xrightarrow{\sim} H^2(G_{k_J}, \mathbb{F}_p)$ [where $\mathbb{F}_p$ is equipped with the trivial Galois action]. Thus, to complete the proof that H is *free pro-*p, it suffices [by a well-known cohomological criterion for *free pro-*p *groups* — cf., e.g., [RZ], Theorem 7.7.4] to show that the cohomology module

$$H^2(H, \mathbb{F}_p) \cong \varinjlim_{k_J} H^2(G_{k_J}, \mathbb{F}_p)$$

[where $\mathbb{F}_p$ is equipped with the trivial Galois action; k_J ranges over the finite extensions of k arising from open subgroups $J \subseteq G_k^{(p)}$ such that $H \subseteq J$] *vanishes.* As in the proof of assertion (ii), this vanishing follows from the well-known *functorial behavior* of $H^2(G_{k_J}, \mathbb{F}_p)$, together with the observation that, by our assumption that H is of *infinite index* in $G_k^{(p)}$, k_J always admits an extension of degree p arising from an open subgroup of J [where $J \subseteq G_k^{(p)}$ corresponds to k_J] containing H. ○

Theorem 1.7. (Slimness and Elasticity of Arithmetic Galois Groups) *Let* $\widetilde{k}$ *be a* **solvably closed Galois extension** *of a field* k*; write* $G_k \stackrel{\text{def}}{=} \text{Gal}(\widetilde{k}/k)$. *Then:*

(i) If k *is an* **FF***, then* $G_k \cong \widehat{\mathbb{Z}}$ *is* **neither** *elastic* **nor** *slim.*

(ii) If k *is an* **MLF***, then* G_k*, as well as any* **almost pro-*p*-maximal quotient** $G_k \twoheadrightarrow Q$ *of* G_k*, is* **elastic** *and* **slim***.*

(iii) If k *is an* **NF***, then* G_k *is* **very elastic** *and* **slim***.*

Proof. Assertion (i) is immediate from the definitions; assertion (iii) is the content of [Mzk11], Corollary 2.2; [Mzk11], Theorem 2.4. The *slimness* portion of assertion (ii) for G_k is shown, for instance, in [Mzk6], Theorem 1.1.1, (ii) [via the same argument as the argument applied to prove Proposition 1.5, (i); Proposition 1.6, (i)]; the *slimness* portion of assertion (ii) for Q is precisely the content of Proposition 1.6, (i).

To show the *elasticity* portion of assertion (ii) for Q, let $N \subseteq H$ be a closed normal subgroup of *infinite index* of an open subgroup $H \subseteq Q$ such that N is topologically generated by r elements, where $r \geq 1$ is an integer. Then it suffices to show that N is *trivial.* Since Q has already been shown to be *slim* [hence has no nontrivial finite normal closed subgroups — cf. §0], we may always replace k by a finite extension of k. In particular, we may assume that $H = Q$, and that Q is *maximal pro-p*. Since $[Q : N]$ is *infinite*, it follows that there exists an open subgroup $J \subseteq Q$, corresponding to a subfield $k_J \subseteq \overline{k}$, such that $N \subseteq J$, and $[k_J : \mathbb{Q}_p] \geq r + 1$. Here, we recall from our discussion of *local class field theory* in the proof of Proposition 1.5 that $\dim_{\mathbb{Q}_p}(J^{\text{ab}} \otimes \mathbb{Q}_p) = [k_J : \mathbb{Q}_p] + 1$ $(\geq r + 2)$. In particular, we conclude that N is necessarily a subgroup of *infinite index* of some topologically finitely generated closed subgroup $P \subseteq J$ such that $[J : P]$ is infinite. [For instance, one may take P to be the subgroup of J topologically generated by N, together with an element of J that maps to a non-torsion element of the quotient of J^{ab} by the image of N^{ab}.] Thus, we conclude from Proposition 1.6, (iii), that P is a *free pro-p group* which contains a topologically finitely generated closed normal subgroup $N \subseteq P$ of infinite index. On the other hand, by [a rather easy special case of] the *theorem of Lubotzky-Melnikov-van den Dries* [cf., e.g., [FJ], Proposition 24.10.3; [MT], Theorem 1.5], this implies that N is *trivial.* This completes the proof of the elasticity portion of assertion (ii) for Q.

To show the *elasticity* portion of assertion (ii) for G_k, let $N \subseteq H$ be a closed normal subgroup of *infinite index* of an open subgroup $H \subseteq G_k$ such that N is topologically generated by r elements, where $r \geq 1$ is an integer. Then it suffices to show that N is *trivial.* As in the proof of the elasticity of "Q", we may assume that $H = G_k$; also, since $[G_k : N]$ is *infinite*, by passing to a finite extension of k corresponding to an open subgroup of G_k containing N, we may assume that $[k : \mathbb{Q}_p] \geq r$. But this implies that the image of N in $G_k^{\text{ab}} \otimes \mathbb{Z}_p$ [which is of rank $[k : \mathbb{Q}_p]+1 \geq r+1$] is of infinite index, hence that the image of N in *any almost pro-p-maximal quotient $G_k \twoheadrightarrow Q$ is of infinite index.* Thus, by the elasticity for "Q", we conclude that such images are *trivial.* Since, moreover, the natural surjection

$$G_k \twoheadrightarrow \varprojlim_{Q} Q$$

[where Q ranges over the almost pro-p-maximal quotients of G_k] is [by the definition of the term "almost pro-p-maximal quotient"] an *isomorphism*, this is enough to conclude that N is *trivial*, as desired. ◯

Section 2: Semi-absolute Anabelian Geometry

In the present §2, we consider the problem of *characterizing "group-theoretically"* the quotient morphism to the *Galois group of the base field* of the arithmetic fundamental group of a variety. In particular, the theory of the present §2 refines the theory of [Mzk6], Lemma 1.1.4 in *two respects*: We extend this theory to the case of varieties of *arbitrary dimension* [cf. Corollary 2.8], and in the case of *hyperbolic orbicurves*, we give a *"group-theoretic version"* of the numerical criterion of [Mzk6], Lemma 1.1.4, via the theory of *maximal pro-RTF-quotients* developed in §1 [cf. Corollary 2.12]. The theory of the present §2 depends on the *general theory of Albanese varieties*, which we review in the Appendix, for the convenience of the reader.

Suppose that:

(1) k is a *perfect field*, $\overline{k}$ an *algebraic closure* of k, $\widetilde{k} \subseteq \overline{k}$ a *solvably closed Galois extension* of k, and $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{k}/k)$.

(2) $X \to \mathrm{Spec}(k)$ is a *geometrically connected, smooth, separated algebraic stack of finite type* over k.

(3) $Y \to X$ is a *connected finite étale Galois covering* which is a [necessarily separated, smooth, and of finite type over k] *k-scheme* such that $\mathrm{Gal}(Y/X)$ acts *freely* on some nonempty open subscheme of Y [so X is *generically scheme-like* — cf. §0].

(4) $Y \hookrightarrow \overline{Y}$ is an open immersion into a *connected proper k-scheme* $\overline{Y}$ such that $\overline{Y}$ is the underlying scheme of a log scheme $\overline{Y}^{\log}$ that is *log smooth* over k [where we regard $\mathrm{Spec}(k)$ as equipped with the trivial log structure], and the image of Y in $\overline{Y}$ coincides with the *interior* of the log scheme $\overline{Y}^{\log}$.

Thus, it follows from the *log purity* theorem [which is exposed, for instance, in [Mzk4] as "Theorem B"] that the *condition* that a finite étale covering $Z \to Y$ be *tamely ramified* over the height one primes of $\overline{Y}$ is equivalent to the condition that the normalization $\overline{Z}$ of $\overline{Y}$ in Z determine a *log étale morphism* $\overline{Z}^{\log} \to \overline{Y}^{\log}$ [whose underlying morphism of schemes is $\overline{Z} \to \overline{Y}$]; in particular, one concludes immediately that the condition that $Z \to Y$ be tamely ramified over the height one primes of $\overline{Y}$ is *independent* of the choice of "log smooth log compactification" $\overline{Y}^{\log}$ for Y. Thus, one verifies immediately [by considering the various $\mathrm{Gal}(Y/X)$-*conjugates* of the "log compactification" $\overline{Y}^{\log}$] that the finite étale coverings of X whose pull-backs to Y are *tamely ramified* over [the height one primes of] $\overline{Y}$ form a *Galois category*, whose associated profinite group [relative to an appropriate choice of basepoint for X] we denote by $\pi_1^{\mathrm{tame}}(X, Y)$, or simply

$$\pi_1^{\mathrm{tame}}(X)$$

when $Y \to X$ is fixed. In particular, if we use the subscript "$\overline{k}$" to denote base-change from k to $\overline{k}$, then by choosing a connected component of $\overline{Y}_{\overline{k}}$, we obtain a subgroup $\pi_1^{\text{tame}}(X_{\overline{k}}) \subseteq \pi_1^{\text{tame}}(X)$ which fits into a *natural exact sequence* $1 \to \pi_1^{\text{tame}}(X_{\overline{k}}) \to \pi_1^{\text{tame}}(X) \to \text{Gal}(\overline{k}/k) \to 1$.

Next, let $\Sigma \subseteq \mathfrak{Primes}$ be a *set of prime numbers*; $\pi_1^{\text{tame}}(X_{\overline{k}}) \twoheadrightarrow \Delta_X$ an *almost pro-Σ-maximal quotient* of $\pi_1^{\text{tame}}(X_{\overline{k}})$ whose kernel is *normal* in $\pi_1^{\text{tame}}(X)$, hence determines a quotient $\pi_1^{\text{tame}}(X) \twoheadrightarrow \Pi_X$; we also assume that the quotient $\pi_1^{\text{tame}}(X) \twoheadrightarrow \text{Gal}(Y/X)$ admits a *factorization* $\pi_1^{\text{tame}}(X) \twoheadrightarrow \Pi_X \twoheadrightarrow \text{Gal}(Y/X)$, and that the kernel of the resulting homomorphism $\Delta_X \to \text{Gal}(Y/X)$ is *pro-Σ*. Thus, $\text{Ker}(\Delta_X \to \text{Gal}(Y/X))$ may be identified with the *maximal pro-Σ quotient* of $\text{Ker}(\pi_1^{\text{tame}}(X_{\overline{k}}) \to \text{Gal}(Y/X))$; we obtain a *natural exact sequence*

$$1 \to \Delta_X \to \Pi_X \to \text{Gal}(\overline{k}/k) \to 1$$

— which may be thought of as an *extension* of the profinite group $\text{Gal}(\overline{k}/k)$.

Definition 2.1.

(i) We shall refer to any profinite group Δ which is isomorphic to the profinite group Δ_X constructed in the above discussion for some choice of data $(k, X, Y \hookrightarrow \overline{Y}, \Sigma)$ as a profinite group *of [almost pro-Σ] GFG-type* [where "GFG" is to be understood as an abbreviation for *"geometric fundamental group"*]. In this situation, we shall refer to any surjection $\pi_1^{\text{tame}}(X_{\overline{k}}) \twoheadrightarrow \Delta$ obtained by composing the surjection $\pi_1^{\text{tame}}(X_{\overline{k}}) \twoheadrightarrow \Delta_X$ with an isomorphism $\Delta_X \xrightarrow{\sim} \Delta$ as a *scheme-theoretic envelope* for Δ; we shall refer to $(k, X, Y \hookrightarrow \overline{Y}, \Sigma)$ as a collection of *construction data* for Δ. [Thus, given a profinite group of GFG-type, there are, in general, *many possible choices* of construction data for the profinite group.]

(ii) We shall refer to any extension $1 \to \Delta \to \Pi \to G \to 1$ of profinite groups which is isomorphic to the extension $1 \to \Delta_X \to \Pi_X \to \text{Gal}(\overline{k}/k) \to 1$ constructed in the above discussion for some choice of data $(k, X, Y \hookrightarrow \overline{Y}, \Sigma)$ as an *extension of [geometrically almost pro-Σ] AFG-type* [where "AFG" is to be understood as an abbreviation for *"arithmetic fundamental group"*]. In this situation, we shall refer to any surjection $\pi_1^{\text{tame}}(X) \twoheadrightarrow \Pi$ (respectively, $\pi_1^{\text{tame}}(X_{\overline{k}}) \twoheadrightarrow \Delta$; $\text{Gal}(\overline{k}/k) \twoheadrightarrow G$) obtained by composing the surjection $\pi_1^{\text{tame}}(X) \twoheadrightarrow \Pi_X$ (respectively, the surjection $\pi_1^{\text{tame}}(X_{\overline{k}}) \twoheadrightarrow \Delta_X$; the identity $\text{Gal}(\overline{k}/k) = \text{Gal}(\overline{k}/k)$) with an isomorphism $\Pi_X \xrightarrow{\sim} \Pi$ (respectively, $\Delta_X \xrightarrow{\sim} \Delta$; $\text{Gal}(\overline{k}/k) \xrightarrow{\sim} G$) arising from an isomorphism of the extensions $1 \to \Delta \to \Pi \to G \to 1$, $1 \to \Delta_X \to \Pi_X \to \text{Gal}(\overline{k}/k) \to 1$ as a *scheme-theoretic envelope* for Π (respectively, Δ; G); we shall refer to $(k, X, Y \hookrightarrow \overline{Y}, \Sigma)$ as a collection of *construction data* for this extension. [Thus, given an extension of AFG-type, there are, in general, *many possible choices* of construction data for the extension.]

(iii) Let $1 \to \Delta^* \to \Pi^* \to G^* \to 1$ be an extension of AFG-type; $N \subseteq G^*$ the inverse image of the kernel of the quotient $\text{Gal}(\overline{k}/k) \twoheadrightarrow G_k$ relative to some scheme-theoretic envelope $G^* \xrightarrow{\sim} \text{Gal}(\overline{k}/k)$. Suppose further that Δ^* is *slim*, and that the outer action of N on Δ^* [arising from the extension structure] is *trivial*. Thus, every element of $N \subseteq G^*$ lifts to a unique element of Π^* that *commutes* with Δ^*. In particular, N lifts to a *closed normal subgroup* $N_\Pi \subseteq \Pi^*$. We shall refer to any extension $1 \to \Delta \to \Pi \to G \to 1$ of profinite groups which is isomorphic to an extension of the form $1 \to \Delta^* \to \Pi^*/N_\Pi \to G^*/N \to 1$ just constructed as *an extension of [geometrically almost pro-*Σ*] GSAFG-type* [where "GSAFG" is to be understood as an abbreviation for *"geometrically slim arithmetic fundamental group"*]. In this situation, we shall refer to any surjection $\pi_1^{\text{tame}}(X) \twoheadrightarrow \Pi$ (respectively, $\pi_1^{\text{tame}}(X_{\overline{k}}) \twoheadrightarrow \Delta$; $\text{Gal}(\overline{k}/k) \twoheadrightarrow G$) obtained by composing a scheme-theoretic envelope $\pi_1^{\text{tame}}(X) \twoheadrightarrow \Pi^*$ (respectively, $\pi_1^{\text{tame}}(X_{\overline{k}}) \twoheadrightarrow \Delta^*$; $\text{Gal}(\overline{k}/k) \xrightarrow{\sim} G^*$) with the surjection $\Pi^* \twoheadrightarrow \Pi$ (respectively, $\Delta^* \twoheadrightarrow \Delta$; $G^* \twoheadrightarrow G$) in the above discussion as a *scheme-theoretic envelope* for Π (respectively, Δ; G); we shall refer to $(k, \widetilde{k}, X, Y \hookrightarrow \overline{Y}, \Sigma)$ as a collection of *construction data* for this extension. [Thus, given an extension of GSAFG-type, there are, in general, *many possible choices* of construction data for the extension.]

(iv) Given *construction data* "$(k, X, Y \hookrightarrow \overline{Y}, \Sigma)$" or "$(k, \widetilde{k}, X, Y \hookrightarrow \overline{Y}, \Sigma)$" as in (i), (ii), (iii), we shall refer to "k" as the *construction data field*, to "X" as the *construction data base-stack* [or *base-scheme*, if X is a scheme], to "Y" as the *construction data covering*, to "$\overline{Y}$" as the *construction data covering compactification*, and to "Σ" as the *construction prime set*. Also, we shall refer to a portion of the construction data "$(k, X, Y \hookrightarrow \overline{Y}, \Sigma)$" or "$(k, \widetilde{k}, X, Y \hookrightarrow \overline{Y}, \Sigma)$" as in (i), (ii), (iii), as *partial construction data*. If every prime dividing the order of a finite quotient group of Δ is *invertible* in k, then we shall refer to the construction data in question as *base-prime*.

The following result is well-known, but we give the proof below for lack of an appropriate reference in the case where [in the notation of the above discussion] X is *not necessarily proper*.

Proposition 2.2. **(Topological Finite Generation)** *Any profinite group* Δ **of GFG-type** *is* **topologically finitely generated**.

Proof. Write $(k, X, Y \hookrightarrow \overline{Y}, \Sigma)$ for a choice of construction data for Δ. Since a profinite fundamental group is topologically finitely generated if and only if it admits an open subgroup that is topologically finitely generated, we may assume that $X = Y$; moreover, by applying de Jong's theory of *alterations* [as reviewed, for instance, in Lemma A.10 of the Appendix], we may assume that $\overline{Y}$ is *projective* [and even k-smooth]. Since we are only concerned with Δ, we may assume that k is *algebraically closed*, hence, in particular, *infinite*. Thus, since $\overline{Y}$ is *normal* and *projective* [over k], it follows that there exists a *connected, k-smooth, one-dimensional closed subscheme* $\overline{C} \subseteq \overline{Y}$ obtained by intersecting $\overline{Y}$ with $\dim_k(\overline{Y}) - 1$ sufficiently general hyperplane sections such that $C \stackrel{\text{def}}{=} \overline{C} \bigcap Y \neq \emptyset$, and $\overline{Y}$ is *k-smooth* at the points of $\overline{C}$. Now if $Z \to Y$ is any *connected finite étale covering* that is *tamely ramified* over the divisor $D \stackrel{\text{def}}{=} \overline{Y} \backslash Y$ [equipped with the reduced induced structure], then write $\overline{Z} \to \overline{Y}$ for the *normalization* of $\overline{Y}$ in Z. Thus, since $\overline{Z}$ is *tamely ramfied* over D [so one may apply "Abhyankar's lemma" to describe the local structure of $\overline{Z} \to \overline{Y}$], and D intersects $\overline{C}$ *transversely*, it follows immediately that $\overline{Z}_{\overline{C}} \stackrel{\text{def}}{=} \overline{Z} \times_{\overline{Y}} \overline{C}$ is *k-smooth*. On the other hand, since the closed subscheme $\overline{Z}_{\overline{C}} \subseteq \overline{Z}$ is obtained by forming the intersection of the zero locus $\dim_k(\overline{Y}) - 1$ sections of an ample line bundle on $\overline{Z}$, it thus follows [cf., e.g., [SGA2], XII, 2.4] that $\overline{Z}_{\overline{C}}$ is *connected*. But this connectedness for arbitrary choices of the covering $Z \to Y$ implies that the natural morphism $\pi_1^{\text{tame}}(C) \to \pi_1^{\text{tame}}(Y)$ is *surjective*. Thus, it suffices to prove Proposition 2.2 in the case where $\overline{Y}$ is a *curve*. But in this case, [as is well-known] Proposition 2.2 follows by *deforming* $Y \hookrightarrow \overline{Y}$ to a curve in characteristic zero, in which case, the desired topological finite generation follows from the well-known structure of the *topological fundamental group of a Riemann surface of finite type*. ◯

Proposition 2.3. **(Slimness and Elasticity for Hyperbolic Orbicurves)**

(i) Let Δ be a profinite group **of GFG-type** *that admits partial construction data (k, X, Σ) [consisting of the construction data field, construction data base-stack, and construction data prime set] such that X is a* **hyperbolic orbicurve** *[cf. §0], and Σ contains a* **prime invertible in** *k. Then Δ is* **slim** *and* **elastic**.

(ii) Let $1 \to \Delta \to \Pi \to G \to 1$ be an **extension of GSAFG-type** *that admits partial construction data (k, X, Σ) [consisting of the construction data field, construction data base-stack, and construction data prime set] such that X is a* **hyperbolic orbicurve**, *$\Sigma \neq \emptyset$, and k is either an* **MLF** *or an* **NF**. *Then Π is* **slim**, *but* **not elastic**.

Proof. Assertion (i) is the easily verified *"generalization to orbicurves over fields of arbitrary characteristic"* of [MT], Proposition 1.4; [MT], Theorem 1.5 [cf. also the technique of proof applied to the *elasticity* portion of Theorem 1.7, (ii)]. The *slimness* portion of assertion (ii) follows immediately from the slimness portion of assertion (i), together with the slimness portion of Theorem 1.7, (ii), (iii); the fact that Π is *not elastic* follows from the existence of the nontrivial, topologically finitely generated [cf. Proposition 2.2], closed, normal, infinite index subgroup $\Delta \subseteq \Pi$. ◯

Definition 2.4. For $i = 1, 2$, let

$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$

be an extension which is either *of AFG-type* or *of GSAFG-type*. Suppose that

$$\phi : \Pi_1 \to \Pi_2$$

is a *continuous homomorphism of profinite groups*. Then:

(i) We shall say that ϕ is *absolute* if ϕ is *open* [i.e., has open image].

(ii) We shall say that ϕ is *semi-absolute* (respectively, *pre-semi-absolute*) if ϕ is absolute, and, moreover, the image of $\phi(\Delta_1)$ in G_2 is *trivial* (respectively, either *trivial* or *of infinite index* in G_2).

(iii) We shall say that ϕ is *strictly semi-absolute* (respectively, *pre-strictly semi-absolute*) if ϕ is semi-absolute, and, moreover, the subgroup $\phi(\Delta_1) \subseteq \Delta_2$ is *open* (respectively, either *open* or *nontrivial*).

Proposition 2.5. **(First Properties of Absolute Homomorphisms)** *For $i = 1, 2$, let*

$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$

be an extension which is either **of AFG-type** *or* **of GSAFG-type***; (k_i, X_i, Σ_i)* **partial construction data** *for $\Pi_i \twoheadrightarrow G_i$ [consisting of the construction data field, construction data base-stack, and construction data prime set]. Suppose that*

$$\phi : \Pi_1 \to \Pi_2$$

is a **continuous homomorphism of profinite groups***. Then:*

(i) The following implications hold:

$$\phi \text{ strictly semi-absolute} \implies \phi \text{ pre-strictly semi-absolute} \implies \phi \text{ semi-absolute}$$
$$\implies \phi \text{ pre-semi-absolute} \implies \phi \text{ absolute.}$$

(ii) Suppose that k_2 is an **NF***. Then "ϕ semi-absolute" $\iff$ "ϕ pre-semi-absolute" $\iff$ "ϕ absolute".*

(iii) Suppose that k_2 is an **MLF***. Then "ϕ semi-absolute" $\iff$ "ϕ pre-semi-absolute".*

(iv) Suppose that k_1 either an **FF** *or an* **MLF***; that X_2 is a* **hyperbolic orbicurve***; and that Σ_2 is of* **cardinality** *> 1. Then "ϕ pre-strictly semi-absolute" $\iff$ "ϕ semi-absolute".*

(v) Suppose that X_2 is a **hyperbolic orbicurve***, and that Σ_2 contains a* **prime invertible in** *k_2. Then "ϕ strictly semi-absolute" $\iff$ "ϕ pre-strictly semi-absolute".*

Proof. Assertion (i) follows immediately from the definitions. Since Δ_1 is *topologically finitely generated* [cf. Proposition 2.2], assertion (ii) (respectively, (iii)) follows immediately, in light of assertion (i), from the fact that G_2 is *very elastic* [cf. Theorem 1.7, (iii)] (respectively, *elastic* [cf. Theorem 1.7, (ii)]). To verify assertion (iv), it suffices, in light of assertion (i), to consider the case where ϕ is *semi-absolute*, but *not pre-strictly semi-absolute*. Then since Δ_2 is *elastic* [cf. the hypothesis on Σ_2; Proposition 2.3, (i)], and Δ_1 is *topologically finitely generated* [cf. Proposition 2.2], it follows that the subgroup $\phi(\Delta_1) \subseteq \Delta_2$ is either *open* or *trivial*.

Since ϕ is *not pre-strictly semi-absolute*, we thus conclude that $\phi(\Delta_1) = \{1\}$, so ϕ induces an *open homomorphism* $G_1 \to \Pi_2$. That is to say, every sufficiently small open subgroup $\Delta_2^* \subseteq \Delta_2$ admits a *surjection* $H_1 \twoheadrightarrow \Delta_2^*$ for some closed subgroup $H_1 \subseteq G_1$. On the other hand, since X_2 is a *hyperbolic orbicurve*, and Σ_2 is of *cardinality* > 1, it follows [e.g., from the well-known structure of *topological fundamental groups of hyperbolic Riemann surfaces of finite type*] that we may take Δ_2^* such that Δ_2^* admits quotients $\Delta_2^* \twoheadrightarrow F'$, $\Delta_2^* \twoheadrightarrow F''$, where F' (respectively, F'') is a *nonabelian* free pro-p' (respectively, pro-p'') group, for *distinct* $p', p'' \in \Sigma_2$. But this contradicts the *well-known structure of* G_1, when k_1 is either an *FF* or an *MLF* — i.e., the fact that G_1, hence also H_1, may be written as an extension of a *meta-abelian group* by a *pro-p subgroup*, for some prime p. [Here, we recall that this fact is immediate if k_1 is an *FF*, in which case G_1 is *abelian*, and follows, for instance, from [NSW], Theorem 7.5.2; [NSW], Corollary 7.5.6, (i), when k_1 is a *MLF*.] Assertion (v) follows immediately from the *elasticity* of Δ_2 [cf. Proposition 2.3, (i)], together with the *topological finite generation* of Δ_1 [cf. Proposition 2.2]. ○

Theorem 2.6. **(Field Types and Group-theoreticity)** *Let*

$$1 \to \Delta \to \Pi \to G \to 1$$

be an extension which is either **of AFG-type** *or* **of GSAFG-type**; (k, X, Σ) **partial construction data** *[consisting of the construction data field, construction data base-stack, and construction data prime set] for* $\Pi \twoheadrightarrow G$. *Suppose further that* k *is either an* **FF**, *an* **MLF**, *or an* **NF**, *and that every prime* $\in \Sigma$ *is* **invertible** *in* k. *If* H *is a profinite group,* $j \in \{1, 2\}$, *and* $l \in \mathfrak{Primes}$, *write*

$$\delta_l^j(H) \stackrel{\text{def}}{=} \dim_{\mathbb{Q}_l}(H^j(H, \mathbb{Q}_l)) \in \mathbb{N} \bigcup \{\infty\}$$
$$\epsilon_l^j(\Pi) \stackrel{\text{def}}{=} \sup_{J \subseteq \Pi} \{\delta_l^j(J)\} \in \mathbb{N} \bigcup \{\infty\}$$
$$\theta^j(\Pi) \stackrel{\text{def}}{=} \{l \mid \epsilon_l^j(\Pi) \geq 3 - j\} \subseteq \mathfrak{Primes}$$

[where J *ranges over the open subgroups* Π*]; also, we set*

$$\zeta(H) \stackrel{\text{def}}{=} \sup_{p, p' \in \mathfrak{Primes}} \{\delta_p^1(H) - \delta_{p'}^1(H)\} \in \mathbb{Z} \bigcup \{\infty\}$$

whenever $\delta_l^1(H) < \infty$, $\forall l \in \mathfrak{Primes}$. *Then:*

(i) Suppose that k *is an* **FF**. *Then* Π *is* **topologically finitely generated**; *the natural surjections*

$$\Pi^{\text{ab-t}} \twoheadrightarrow G^{\text{ab-t}}; \quad G \twoheadrightarrow G^{\text{ab-t}}$$

are **isomorphisms**. *In particular, the kernel of the quotient* $\Pi \twoheadrightarrow G$ *may be characterized [*"**group-theoretically**"*] as the kernel of the quotient* $\Pi \twoheadrightarrow \Pi^{\text{ab-t}}$. *Moreover, for every open subgroup* $H \subseteq \Pi$, *and every prime number* l, $\delta_l^1(H) = 1$.

(ii) Suppose that k *is an* **MLF** *of residue characteristic* p. *Then* Π *is* **topologically finitely generated**; *in particular, for every open subgroup* $H \subseteq \Pi$, *and every*

prime number l, $\delta^1_l(H)$ is **finite**. *Moreover, $\delta^1_l(G) = 1$ if $l \neq p$, $\delta^1_p(G) = [k : \mathbb{Q}_p]+1$; the quantity*

$$\delta^1_l(\Pi) - \delta^1_l(G)$$

is $= 0$ if $l \notin \Sigma$, and is **independent** *of l if $l \in \Sigma$. Finally, $\epsilon^1_p(\Pi) = \infty$; in particular, the* **cardinality** *of $\theta^1(\Pi)$ is always ≥ 1.*

(iii) Let k be as in (ii). Then $\theta^2(\Pi) \subseteq \Sigma$. If, moreover, the cardinality of $\theta^1(\Pi)$ is ≥ 2, then $\theta^2(\Pi) = \Sigma$.

(iv) Let k be as in (ii). Then every **almost pro-omissive** *topologically finitely generated closed normal subgroup of Π is contained in Δ. If, moreover, $\Sigma \neq \mathfrak{Primes}$, then the kernel of the quotient $\Pi \twoheadrightarrow G$ may be characterized [***"group-theoretically"***] as the* **maximal almost pro-omissive** *topologically finitely generated closed normal subgroup of Π.*

(v) Let k be as in (ii). If $\theta^2(\Pi) \neq \mathfrak{Primes}$, then write

$$\Theta \subseteq \Pi$$

for the **maximal almost pro-omissive** *topologically finitely generated closed normal subgroup of Π, whenever a unique such maximal subgroup exists; if $\theta^2(\Pi) = \mathfrak{Primes}$, or there does not exist a unique such maximal subgroup, set $\Theta \stackrel{\mathrm{def}}{=} \{1\} \subseteq \Pi$. Then*

$$\underline{\zeta}(\Pi) \stackrel{\mathrm{def}}{=} \zeta(\Pi/\Theta) = [k : \mathbb{Q}_p]$$

*[cf. the finiteness portion of (ii)]. In particular, the kernel of the quotient $\Pi \twoheadrightarrow G$ may be characterized [***"group-theoretically"*** — since "$\theta^2(-)$", "$\zeta(-)$", "$\underline{\zeta}(-)$" are "group-theoretic"] as the intersection of the open subgroups $H \subseteq \Pi$ such that $\underline{\zeta}(H)/\underline{\zeta}(\Pi) = [\Pi : H]$.*

(vi) Suppose that k is an **NF**. *Then the natural surjection $\Pi^{\text{ab-t}} \twoheadrightarrow G^{\text{ab-t}}$ is an* **isomorphism**. *The kernel of the quotient $\Pi \twoheadrightarrow G$ may be characterized [***"group-theoretically"***] as the maximal topologically finitely generated closed normal subgroup of Π. In particular, Π is* **not topologically finitely generated**.

Proof. Write $X \to A$ for the *Albanese morphism* associated to X. [We refer to the Appendix for a review of the theory of Albanese varieties — cf., especially, Corollary A.11, Remark A.11.2.] Thus, A is a *torsor* over a *semi-abelian variety* over k such that the morphism $X \to A$ induces an *isomorphism*

$$\Delta^{\text{ab-t}} \otimes \mathbb{Z}_l \xrightarrow{\sim} T_l(A)$$

onto the *l-adic Tate module* $T_l(A)$ of A for all $l \in \Sigma$. Note, moreover, that A admits a *rational point* over some finite extension of k. Thus, by applying the Galois section arising from such a rational point, we conclude that for $l \in \Sigma$, the *image* of Δ in $\Pi^{\text{ab-t}} \otimes \mathbb{Z}_l$ is given by the quotient

$$\Delta^{\text{ab-t}} \otimes \mathbb{Z}_l \xrightarrow{\sim} T_l(A) \twoheadrightarrow T_l(A)/G$$

— where we use the notation "$/G$" to denote the *maximal torsion-free quotient* on which G acts *trivially*.

Next, whenever k is an *MLF*, let us write, for $l \in \Sigma$,

$$\Delta^{\text{ab-t}} \twoheadrightarrow \Delta^{\text{ab-t}} \otimes \mathbb{Z}_l \xrightarrow{\sim} T_l(A) \twoheadrightarrow R_l \stackrel{\text{def}}{=} R \otimes \mathbb{Z}_l \twoheadrightarrow Q_l \stackrel{\text{def}}{=} Q \otimes \mathbb{Z}_l$$

for the pro-l portion of the *quotients* $T(A) \twoheadrightarrow R \twoheadrightarrow Q$ of Lemma 2.7, (i), (ii), below [in which we take "k" to be k and "B" to be the semi-abelian variety over which A is a torsor]. [Put another way, Q_l is simply the quotient $T_l(A)/G$ considered above.] Thus, the $\mathbb{Z}_l$-ranks of R_l, Q_l are *independent* of $l \in \Sigma$.

The *topological finite generation* portion of assertion (i) follows immediately from the fact that $G \cong \widehat{\mathbb{Z}}$, together with the topological finite generation of Δ [cf. Proposition 2.2]. The remainder of assertion (i) follows immediately from the fact that $T_l(A)/G = 0$ [a consequence of the "Riemann hypothesis for abelian varieties over finite fields" — cf., e.g., [Mumf], p. 206]. In a similar vein, assertion (vi) follows immediately from the fact that $T_l(A)/G = 0$ [again a consequence of the "Riemann hypothesis for abelian varieties over finite fields"], together with the fact that G is *very elastic* [cf. Theorem 1.7, (iii)].

To verify assertion (ii), let us first observe that the topological finite generation of Π follows from that of Δ [cf. Proposition 2.2], together with that of G [cf. [NSW], Theorem 7.5.10]. Next, let us recall the well-known fact that

$$\delta^1_l(G) = 1 \text{ if } l \neq p, \quad \delta^1_p(G) = [k : \mathbb{Q}_p] + 1$$

[cf. our our discussion of *local class field theory* in the proofs of Proposition 1.5; Theorem 1.7, (ii)]; in particular, $\zeta(G) = [k : \mathbb{Q}_p]$. Moreover, the existence of a rational point of A over some finite extension of k [which determines a Galois section of the étale fundamental group of A over some open subgroup of G] implies that

$$\delta^1_l(\Pi) = \delta^1_l(G) + \dim_{\mathbb{Q}_l}(Q_l \otimes \mathbb{Q}_l)$$

[where we recall that $\dim_{\mathbb{Q}_l}(Q_l \otimes \mathbb{Q}_l)$ is *independent* of l] for $l \in \Sigma$, $\delta^1_l(\Pi) = \delta^1_l(G)$ for $l \notin \Sigma$. Thus, by considering extensions of k of arbitrarily large degree, we obtain that $\epsilon^1_p(\Pi) = \infty$. This completes the proof of assertion (ii).

Next, we consider assertion (iii). First, let us consider the "E_2-term" of the Leray spectral sequence of the group extension $1 \to \Delta \to \Pi \to G \to 1$. Since G is of *cohomological dimension* 2 [cf., e.g., [NSW], Theorem 7.1.8, (i)], and $\delta^2_l(G) = 0$ for all $l \in \mathfrak{Primes}$ [cf., e.g., [NSW], Theorem 7.2.6], the spectral sequence yields an *equality* $\delta^2_l(\Pi) = 0$ if $l \notin \Sigma$, and a pair of *injections*

$$H^1(G, \text{Hom}(R_l, \mathbb{Q}_l)) \hookrightarrow H^1(G, \text{Hom}(\Delta^{\text{ab-t}}, \mathbb{Q}_l)) \hookrightarrow H^2(\Pi, \mathbb{Q}_l)$$

if $l \in \Sigma$ [cf. Lemma 2.7, (iii), below]. By applying the analogue of this conclusion for an arbitrary open subgroup $H \subseteq \Pi$, we thus obtain that $\delta^2_l(H) = 0$ if $l \notin \Sigma$, i.e., that $\epsilon^2_l(H) = 0$ if $l \notin \Sigma$; this already implies that if $l \notin \Sigma$, then $l \notin \theta^2(\Pi)$, i.e., that $\theta^2(\Pi) \subseteq \Sigma$. If the cardinality of $\theta^1(\Pi)$ is ≥ 2, then there exists some open subgroup $H \subseteq \Pi$ and some $l \in \mathfrak{Primes}$ such that $\delta^1_l(H) \geq 2$, $l \neq p$. Now we may assume without loss of generality that H acts *trivially* on the quotient R; also to simplify notation, we may *replace* Π by H and assume that $H = \Pi$. Then [since $\delta^1_l(G) = 1$, by assertion (ii)] the fact that $\delta^1_l(\Pi) \geq 2$ implies that $l \in \Sigma$, and $\dim_{\mathbb{Q}_l}(R_l \otimes \mathbb{Q}_l) \geq 1$ [cf. our computation in the proof of assertion (ii)]. But this implies that *for any* $l' \in \Sigma$, we have $\dim_{\mathbb{Q}_{l'}}(R_{l'} \otimes \mathbb{Q}_{l'}) \geq 1$, hence that $H^1(G, \mathrm{Hom}(R_{l'}, \mathbb{Q}_{l'})) = H^1(G, \mathbb{Q}_{l'}) \otimes \mathrm{Hom}(R_{l'}, \mathbb{Q}_{l'}) \neq 0$. Thus, by the *injections* discussed above, we conclude that $\epsilon^2_{l'}(\Pi) \geq \delta^2_{l'}(\Pi) \geq 1$, so $l' \in \theta^2(\Pi)$. This completes the proof of assertion (iii).

Assertion (iv) follows immediately from the existence of a *surjection* $G \twoheadrightarrow \widehat{\mathbb{Z}}$ [cf., e.g., Proposition 1.5, (ii)], together with the *elasticity* of G [cf. Theorem 1.7, (ii)], and the *topological finite generation* of Δ [cf. Proposition 2.2].

Next, we consider assertion (v). First, let us observe that whenever $\Sigma = \mathfrak{Primes}$, it follows from assertion (ii) that $\zeta(\Pi) = \zeta(G) = [k : \mathbb{Q}_p]$.

Now *we consider the case* $\theta^2(\Pi) = \mathfrak{Primes}$. In this case, $\Theta = \{1\}$ [by definition], and $\theta^2(\Pi) = \Sigma = \mathfrak{Primes}$ [by assertion (iii)]. Thus, we obtain that $\zeta(\Pi) = \zeta(\Pi/\Theta) = [k : \mathbb{Q}_p]$, as desired [cf. [Mzk6], Lemma 1.1.4, (ii)]. Next, *we consider the case* $\theta^1(\Pi) \neq \{p\}$ [i.e., $\theta^1(\Pi)$ is of cardinality ≥ 2 — cf. assertion (ii)], $\theta^2(\Pi) \neq \mathfrak{Primes}$. In this case, by assertion (iii), we conclude that $\Sigma = \theta^2(\Pi) \neq \mathfrak{Primes}$. Thus, by assertion (iv), $\Theta = \Delta$, so $\zeta(\Pi/\Theta) = \zeta(G) = [k : \mathbb{Q}_p]$, as desired.

Finally, *we consider the case* $\theta^1(\Pi) = \{p\}$ [i.e., $\theta^1(\Pi)$ is of cardinality one], $\theta^2(\Pi) \neq \mathfrak{Primes}$. If $\Sigma \neq \mathfrak{Primes}$, then it follows from the definition of Θ, together with assertion (iv), that $\Theta = \Delta$, hence that $\zeta(\Pi/\Theta) = \zeta(G) = [k : \mathbb{Q}_p]$, as desired. If, on the other hand, $\Sigma = \mathfrak{Primes}$, then since $\theta^1(\Pi) = \{p\}$, it follows [cf. the computation in the proof of assertion (ii)] that $\dim_{\mathbb{Q}_l}(Q_l \otimes \mathbb{Q}_l) = 0$ for all primes $l \neq p$, hence that $\dim_{\mathbb{Q}_p}(Q_p \otimes \mathbb{Q}_p) = 0$; but this implies that $\delta^1_l(\Pi) = \delta^1_l(G)$ for all $l \in \mathfrak{Primes}$. Now since $\Theta \subseteq \Delta$ [by assertion (iv)], it follows that $\delta^1_l(\Pi) \geq \delta^1_l(\Pi/\Theta) \geq \delta^1_l(G)$ for all $l \in \mathfrak{Primes}$, so we obtain that $\delta^1_l(\Pi) = \delta^1_l(\Pi/\Theta) = \delta^1_l(G)$ for all $l \in \mathfrak{Primes}$. But this implies that $\zeta(\Pi) = \zeta(\Pi/\Theta) = \zeta(G) = [k : \mathbb{Q}_p]$, as desired. This completes the proof of assertion (v). ○

Remark 2.6.1. When [in the notation of Theorem 2.6] X is a *smooth proper variety*, the portion of Theorem 2.6, (ii), concerning "$\delta^1_l(\Pi) - \delta^1_l(G)$" is essentially equivalent to the main result of [Yoshi].

Lemma 2.7. (Combinatorial Quotients of Tate Modules) *Suppose that k is an* **MLF** *[so $\overline{k} = \widetilde{k}$]. Let B be a* **semi-abelian variety** *over k. Write*

$$T(B) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, B(\overline{k}))$$

for the Tate module of B. Then:

(i) The maximal torsion-free **quotient** *module $T(B) \twoheadrightarrow Q$ of $T(B)$ on which G_k acts* **trivially** *is a finitely generated* **free** *$\widehat{\mathbb{Z}}$-module.*

(ii) There exists a **quotient** *G_k-module $T(B) \twoheadrightarrow R$ such that the following properties hold: (a) R is a finitely generated* **free** *$\widehat{\mathbb{Z}}$-module; (b) the action of G_k on R factors through a finite quotient; (c) no nonzero torsion-free subquotient S of the G_k-module $N \stackrel{\mathrm{def}}{=} \mathrm{Ker}(T(B) \twoheadrightarrow R)$ satisfies the property that the resulting action of G_k on S factors through a finite quotient.*

(iii) If R is as in (ii), then the natural map

$$H^1(G_k, \mathrm{Hom}(R, \widehat{\mathbb{Z}})) \to H^1(G_k, \mathrm{Hom}(T(B), \widehat{\mathbb{Z}}))$$

is **injective**.

Proof. Assertion (i) is literally the content of [Mzk6], Lemma 1.1.5. Assertion (ii) follows immediately from the proof of [Mzk6], Lemma 1.1.5 [more precisely, the [*"combinatorial"*] quotient "T_{com}" of *loc. cit.*]. Assertion (iii) follows by considering the long exact cohomology sequence associated to the short exact sequence $0 \to \mathrm{Hom}(R, \widehat{\mathbb{Z}}) \to \mathrm{Hom}(T(B), \widehat{\mathbb{Z}}) \to \mathrm{Hom}(N, \widehat{\mathbb{Z}}) \to 0$, since the fact that N has no nonzero torsion-free subquotients on which G_k acts through a finite quotient implies that $H^0(G_k, \mathrm{Hom}(N, \widehat{\mathbb{Z}})) = 0$. ○

Corollary 2.8. **(Field Types and Absolute Homomorphisms)** *For $i = 1, 2$, let $1 \to \Delta_i \to \Pi_i \to G_i \to 1$, k_i, X_i, Σ_i, $\phi : \Pi_1 \to \Pi_2$ be as in Proposition 2.5. Suppose further that k_i is either an* **FF**, *an* **MLF**, *or an* **NF**, *and that every prime $\in \Sigma_i$ is* **invertible** *in k_i. Then:*

(i) The **field type** *of k_1 is $\geq$ [cf. §0] the field type of k_2.*

(ii) Suppose further that ϕ is an **isomorphism**. *Then the* **field types** *of k_1, k_2* **coincide**, *and ϕ is* **strictly semi-absolute**. *If, moreover, for $i = 1, 2$, k_i is an* **MLF** *of residue characteristic p_i, then $p_1 = p_2$.*

Proof. Assertion (i) follows immediately from the topological finite generation portions of Theorem 2.6, (i), (ii), (vi), together with the estimates of "$\delta^1_l(-)$", "$\epsilon^1_l(-)$" in Theorem 2.6, (i), (ii). Next, we consider assertion (ii). The fact that the *field types* of k_1, k_2 coincide follows from assertion (i) applied to ϕ, ϕ^{-1}. To verify that ϕ is *strictly semi-absolute*, let us first observe that *every* semi-absolute isomorphism whose inverse is also semi-absolute is necessarily strictly semi-absolute. Thus, since the inverse to ϕ satisfies the same hypotheses as ϕ, to complete the proof of Corollary 2.8, it suffices to verify that ϕ is *semi-absolute.* If k_1, k_2 are FF's (respectively, MLF's; NF's), then this follows immediately from the *"group-theoretic" characterizations* of $\Pi_i \twoheadrightarrow G_i$ in Theorem 2.6, (i) (respectively, Theorem 2.6, (v); Theorem 2.6, (vi)). Finally, if, for $i = 1, 2$, k_i is an *MLF* of residue characteristic p_i, then since ϕ induces an isomorphism $G_1 \xrightarrow{\sim} G_2$, the fact that $p_1 = p_2$ follows, for instance, from [Mzk6], Proposition 1.2.1, (i). ○

Remark 2.8.1. In the situation of Corollary 2.8, suppose further that k_2 is an *MLF* of residue characteristic p_2, and that $\Sigma_2 \subseteq \{p_2\}$. Then it is not clear to the author at the time of writing [but of interest in the context of the theory of the present §2!] whether or not it is possible for there to exist a *continuous surjective homomorphism*

$$G_1 \twoheadrightarrow \Pi_2$$

[in which case, by Corollary 2.8, (i), k_1 is either an NF or an MLF].

The general theory discussed so far for *arbitrary* X becomes substantially simpler and more explicit, when X is a *hyperbolic orbicurve*.

Definition 2.9. Let G be a *profinite group*. Then we shall refer to as an *aug-free decomposition* of G any pair of closed subgroups $H_1, H_2 \subseteq G$ that determine an isomorphism of profinite groups

$$H_1 \times H_2 \xrightarrow{\sim} G$$

such that H_1 is a *slim*, *topologically finitely generated*, *augmented pro-prime* [cf. Definition 1.1, (iii)] profinite group, and H_2 is either *trivial* or a *nonabelian pro-*Σ*-solvable free group* for some set $\Sigma \subseteq \mathfrak{Primes}$ of *cardinality* ≥ 2. In this situation, we shall refer to H_1 as the *augmented subgroup* of this aug-free decomposition and to H_2 as the *free subgroup* of this aug-free decomposition. If G admits an aug-free decomposition, then we shall say that G is *of aug-free type*. If G is of aug-free type, with *nontrivial* free subgroup, then we shall say that G is *of strictly aug-free type*.

Proposition 2.10. **(First Properties of Aug-free Decompositions)** *Let*

$$H_1 \times H_2 \xrightarrow{\sim} G$$

be an **aug-free decomposition** *of a profinite group* G*, in which* H_1 *is the augmented subgroup, and* H_2 *is the free subgroup. Then:*

(i) Let J *be a* **topologically finitely generated**, **augmented pro-prime** *group;* $\phi : J \to G$ *a continuous homomorphism of profinite groups such that* $\phi(J)$ *is* **normal** *in some open subgroup of* G*. Then* $\phi(J) \subseteq H_1$*.*

(ii) Aug-free decompositions are **unique** *— i.e., if* $J_1 \times J_2 \xrightarrow{\sim} G$ *is any aug-free decomposition of* G*, in which* J_1 *is the augmented subgroup, and* J_2 *is the free subgroup, then* $J_1 = H_1$*,* $J_2 = H_2$*.*

Proof. First, we consider assertion (i). Suppose that $\phi(J)$ is *not* contained in H_1. Then the image $I \subseteq H_2$ of $\phi(J)$ in H_2 is a *nontrivial*, *topologically finitely generated* closed subgroup which is *normal* in an open subgroup of H_2. Since H_2 is *elastic* [cf. [MT], Theorem 1.5], it follows that I is *open* in H_2, hence that I is a *nonabelian pro-*Σ *free group* for some set $\Sigma \subseteq \mathfrak{Primes}$ of *cardinality* ≥ 2. On the other hand, since I is a quotient of the *augmented pro-prime* group J, it follows that there exists a $p \in \mathfrak{Primes}$ such that the *maximal pro-*$(\neq p)$ *quotient* of I is *abelian*. But this implies that $\Sigma \subseteq \{p\}$, a contradiction. Next, we consider assertion (ii). By assertion (i), $J_1 \subseteq H_1$, $H_1 \subseteq J_1$. Thus, $H_1 = J_1$. Now since $H_1 = J_1$ is *slim*, it follows that the *centralizer* $Z_{H_1}(G)$ (respectively, $Z_{J_1}(G)$) is equal to H_2 (respectively, J_2), so $H_2 = J_2$, as desired. ◯

Theorem 2.11. **(Maximal Pro-RTF-quotients for Hyperbolic Orbicurves)** *Let*

$$1 \to \Delta \to \Pi \to G \to 1$$

be an extension **of AFG-type***;* (k, X, Σ) **partial construction data** *[consisting of the construction data field, construction data base-stack, and construction data prime set] for* $\Pi \twoheadrightarrow G$*. Suppose that* k *is an* **MLF** *of residue characteristic* p*;* X *is a* **hyperbolic orbicurve***;* $\Sigma \neq \emptyset$*. For* $l \in \mathfrak{Primes}$*, write*

$$\Pi[l] \subseteq \Pi$$

for the **maximal almost pro-l** *topologically finitely generated closed normal subgroup of* Π*, whenever a unique such maximal subgroup exists; if there does not exist a unique such maximal subgroup, then set* $\Pi[l] \stackrel{\mathrm{def}}{=} \{1\}$*.*

In the following, we shall use a subscript "G*" to denote the quotient of a closed subgroup of* Π *determined by the quotient* $\Pi \twoheadrightarrow G$*; we shall use the superscript "*RTF*" to denote the* **maximal pro-RTF-quotient** *and the superscripts "*RTF-aug*", "*RTF-free*" to denote the* **augmented** *and* **free** *subgroups of the maximal pro-RTF-quotient whenever this maximal pro-RTF-quotient is* **of aug-free type***. Then:*

(i) Suppose that $\Pi[l] \neq \{1\}$ *for some* $l \in \mathfrak{Primes}$*. Then* $\Pi[l] = \Delta$*,* $\Sigma = \{l\}$*;* $\Pi[l'] = \{1\}$ *for all* $l' \in \mathfrak{Primes}$ *such that* $l' \neq l$*.*

(ii) Suppose that $\Pi[l] = \{1\}$ *for all* $l \in \mathfrak{Primes}$*. Then* Σ *is of* **cardinality** ≥ 2*. Moreover, for every open subgroup* $J \subseteq \Pi$*, there exists an open subgroup* $H \subseteq J$ *which is* **characteristic** *as a subgroup of* Π *such that* H^{RTF} *is* **of aug-free type***. In particular, the subquotients* $H^{\text{RTF-aug}}$*,* $H^{\text{RTF-free}}$ *of* Π *are* **characteristic***.*

(iii) Suppose that $\Pi[l] = \{1\}$ *for all* $l \in \mathfrak{Primes}$*. Suppose, moreover, that* $H \subseteq \Pi$ *is an open subgroup that corresponds to a* **finite étale covering** $Z \to X$*, where* Z *is a* **hyperbolic curve***, defined over a finite extension* k_Z *of* k *such that* Z *has* **stable reduction** *[cf. §0] over the ring of integers* $\mathcal{O}_{k_Z}$ *of* k_Z*; that* $Z(k_Z) \neq \emptyset$*; that the* **dual graph** Γ_Z *of the geometric special fiber of the resulting model [cf. §0] over* $\mathcal{O}_{k_Z}$ *has either* **trivial** *or* **nonabelian** *topological fundamental group; and that the Galois action of* G *on* Γ_Z *is* **trivial***. Thus, the finite Galois coverings of the graph* Γ_Z *of degree a product of primes* $\in \Sigma$ *determine a pro-*Σ **"combinatorial"** *quotient* $H \twoheadrightarrow \Delta^{\mathrm{com}}$*; write* $\Delta^{\mathrm{com}} \twoheadrightarrow \Delta^{\text{com-sol}}$ *for the* **maximal pro-solvable quotient** *of* Δ^{com}*. Then the quotient*

$$H \twoheadrightarrow H_G^{\mathrm{RTF}} \times \Delta^{\text{com-sol}}$$

may be identified with the **maximal pro-RTF-quotient** $H \twoheadrightarrow H^{\mathrm{RTF}}$ *of* H*; moreover, this product decomposition determines an* **aug-free decomposition** *of* H^{RTF}*. Finally, for any open subgroup* $J \subseteq \Pi$*, there exists an open subgroup* $H \subseteq J$ *which is* **characteristic** *as a subgroup of* Π *and, moreover, satisfies the above hypotheses on "*H*".*

(iv) Suppose that $\Pi[l] = \{1\}$ *for all* $l \in \mathfrak{Primes}$*. Let* $H \subseteq J \subseteq \Pi$ *be open subgroups of* Π *such that* H^{RTF}*,* J^{RTF} *are* **of aug-free type***. Then we have* **isomorphisms**

$$J^{\text{RTF-aug}} \xrightarrow{\sim} J_G^{\mathrm{RTF}}; \quad J^{\text{RTF-free}} \xrightarrow{\sim} \mathrm{Ker}(J^{\mathrm{RTF}} \twoheadrightarrow J_G^{\mathrm{RTF}})$$

[arising from the natural morphisms involved]; the open homomorphism $H^{\mathrm{RTF}} \to J^{\mathrm{RTF}}$ *induced by* ϕ *maps* $H^{\text{RTF-aug}}$ *(respectively,* $H^{\text{RTF-free}}$*) onto an open subgroup of* $J^{\text{RTF-aug}}$ *(respectively,* $J^{\text{RTF-free}}$*).*

Proof. Since Δ is *elastic* [cf. Proposition 2.3, (i)], every nontrivial topologically finitely generated closed normal subgroup of Δ is *open*, hence almost pro-Σ' for $\Sigma' \subseteq \mathfrak{Primes}$ if and only if $\Sigma' \supseteq \Sigma$. Also, let us observe that by Theorem 2.6, (iv), $\Pi[l] \subseteq \Delta$ for all $l \in \mathfrak{Primes}$. Thus, if $\Pi[l] \neq \{1\}$ for any $l \in \mathfrak{Primes}$, then it follows that $\Sigma = \{l\}$, $\Pi[l] = \Delta$, and that $\Pi[l']$ is *finite*, hence *trivial* [since Δ is *slim* — cf. Proposition 2.3, (i)] for primes $l' \neq l$. Also, we observe that if Σ is of *cardinality one*, i.e., $\Sigma = \{l\}$ for some $l \in \mathfrak{Primes}$, then $\Delta = \Pi[l] \neq \{1\}$ [cf. Theorem 2.6, (iv)]. This completes the proof of assertion (i), as well as of the portion of assertion (ii) concerning Σ. Also, we observe that the remainder of assertion (ii) follows immediately from assertion (iii).

Next, we consider assertion (iii). Suppoes that $H \subseteq \Pi$ satisfies the hypotheses given in the statement of assertion (iii). Thus, one has the quotient $H \twoheadrightarrow \Delta^{\mathrm{com}}$, where Δ^{com} is either *trivial* or a *nonabelian pro-*Σ *free group*, where Σ is of *cardinality* ≥ 2 [cf. the portion of assertion (ii) concerning Σ]. Write $\Delta^{\mathrm{ab}} = \Delta^{\text{ab-t}} \twoheadrightarrow R$ for the maximal pro-Σ quotient of the quotient "R" of Lemma 2.7, (ii), associated to the Albanese variety of Z.

Now I *claim* that the quotient $\Delta \twoheadrightarrow R$ *coincides* with the quotient $\Delta \twoheadrightarrow (\Delta^{\mathrm{com}})^{\mathrm{ab}}$. First, let us observe that by the definition of R [cf. Lemma 2.7, (ii)], it follows that the quotient $\Delta \twoheadrightarrow (\Delta^{\mathrm{com}})^{\mathrm{ab}}$ *factors* through the quotient $\Delta \twoheadrightarrow R$. In particular, since, for $l \in \Sigma$, the modules $R \otimes \mathbb{Z}_l$, $(\Delta^{\mathrm{com}})^{\mathrm{ab}} \otimes \mathbb{Z}_l$ are $\mathbb{Z}_l$-free modules of rank *independent* of $l \in \Sigma$ [cf. Lemma 2.7, (ii); the fact that Δ^{com} is pro-Σ free], it suffices to show that these two ranks are *equal*, for some $l \in \Sigma$. Moreover, let us observe that for the purpose of verifying this *claim*, we may *enlarge* Σ. Thus, it suffices to show that the two ranks are *equal* for some $l \in \Sigma$ such that $l \neq p$. But then the *claim* follows immediately from the [well-known] fact that by the *"Riemann hypothesis for abelian varieties over finite fields"* [cf., e.g., [Mumf], p. 206], all powers of the Frobenius element in the absolute Galois group of the residue field of k act with *eigenvalues* $\neq 1$ on the pro-l abelianizations of the fundamental groups of the geometric irreducible components of the smooth locus of the special fiber of the stable model of Z over $\mathcal{O}_{k_Z}$. This completes the proof of the *claim*.

Now let us write $H \twoheadrightarrow H^{\text{com}}$ for the quotient of H by $\text{Ker}(\Delta \twoheadrightarrow \Delta^{\text{com}})$. Then by applying the above *claim* to various open subgroups of H, we conclude that the quotient $H \twoheadrightarrow H^{\text{RTF}}$ *factors* through the quotient $H \twoheadrightarrow H^{\text{com}}$ [i.e., we have a natural isomorphism $H^{\text{RTF}} \xrightarrow{\sim} (H^{\text{com}})^{\text{RTF}}$]. On the other hand, since $Z(k_Z) \neq \emptyset$, it follows that $H \twoheadrightarrow H_G$, hence also $H^{\text{com}} \twoheadrightarrow H_G$ *admits a section* $s : H_G \to H^{\text{com}}$ whose image lies in the *kernel* of the quotient $H^{\text{com}} \twoheadrightarrow \Delta^{\text{com}}$ [cf. the situation of [Mzk3], Lemma 1.4]. In particular, we conclude that the conjugation action of H_G on $\Delta^{\text{com}} = \text{Ker}(H^{\text{com}} \twoheadrightarrow H_G) \subseteq H^{\text{com}}$ arising from s is *trivial*. Thus, s determines a *direct product decomposition*

$$H^{\text{com}} \xrightarrow{\sim} H_G \times \Delta^{\text{com}}$$

— hence a direct product decomposition $H^{\text{RTF}} \xrightarrow{\sim} (H^{\text{com}})^{\text{RTF}} \xrightarrow{\sim} H_G^{\text{RTF}} \times (\Delta^{\text{com}})^{\text{RTF}}$. Moreover, since Δ^{com} is either *trivial* or *nonabelian pro-*Σ *free*, it follows immediately that the quotient $\Delta^{\text{com}} \twoheadrightarrow (\Delta^{\text{com}})^{\text{RTF}}$ may be identified with the quotient $\Delta^{\text{com}} \twoheadrightarrow \Delta^{\text{com-sol}}$, where $\Delta^{\text{com-sol}}$ is either *trivial* or *nonabelian pro-*Σ*-solvable free.* Since H_G^{RTF} is *slim*, *augmented pro-prime*, and *topologically finitely generated* [cf. Proposition 1.5, (i), (ii); Theorem 2.6, (ii)], we thus conclude that we have obtained an *aug-free decomposition* of H^{RTF}, as asserted in the statement of assertion (iii).

Finally, given an open subgroup $J \subseteq \Pi$, the existence of an open subgroup $H \subseteq J$ which satisfies the hypotheses on "H" in the statement of assertion (iii) follows immediately from well-known facts concerning stable curves over discretely valued fields [cf., e.g., the *"stable reduction theorem"* of [DM]; the fact that Σ is of cardinality ≥ 2, so that one may assume that Γ_Z is as large as one wishes by passing to admissible coverings]. The fact that one can choose H to be *characteristic* follows immediately from the characteristic nature of Δ [cf., e.g., Corollary 2.8, (ii)], together with the fact that Δ, Π are *topologically finitely generated* [cf., e.g. Proposition 2.2; Theorem 2.6, (ii)]. This completes the proof of assertion (iii).

Finally, we consider assertion (iv). First, we observe that since the augmented and free subgroups of any aug-free decomposition are *slim* [cf. Definition 2.9; [MT], Proposition 1.4], hence, in particular, do not contain any *nontrivial closed normal finite subgroups*, we may always *replace* H by an open subgroup of H that satisfies the same hypotheses as H. In particular, we may assume that H is an open subgroup "H" as in assertion (iii) [which *exists*, by assertion (iii)]. Then by Proposition 2.10, (i), the image of $H^{\text{RTF-aug}}$ in J^{RTF} is *contained in* $J^{\text{RTF-aug}}$, so we obtain a morphism $H^{\text{RTF-aug}} \to J^{\text{RTF-aug}}$. By assertion (iii), $H^{\text{RTF-free}} = \text{Ker}(H^{\text{RTF}} \twoheadrightarrow H_G^{\text{RTF}})$, and the natural morphism $H^{\text{RTF-aug}} \to H_G^{\text{RTF}}$ is an *isomorphism.* Since $H_G \to J_G$, hence also $H_G^{\text{RTF}} \to J_G^{\text{RTF}}$, is clearly an *open homomorphism*, we thus conclude that the natural morphism $H^{\text{RTF-aug}} \to J_G^{\text{RTF}}$, hence also the natural morphism $J^{\text{RTF-aug}} \to J_G^{\text{RTF}}$, is *open.* Thus, the image of $J^{\text{RTF-free}}$ in J_G^{RTF} *commutes* with an open subgroup of J_G^{RTF} [i.e., the image of $J^{\text{RTF-aug}}$ in J_G^{RTF}], so by the *slimness* of J_G^{RTF} [cf. Proposition 1.5, (i)], we conclude that $J^{\text{RTF-free}} \subseteq \text{Ker}(J^{\text{RTF}} \twoheadrightarrow J_G^{\text{RTF}})$. In particular, we obtain a surjection $J^{\text{RTF-aug}} \twoheadrightarrow J_G^{\text{RTF}}$, hence an *exact sequence*

$$1 \to N \to J^{\text{RTF-aug}} \to J_G^{\text{RTF}} \to 1$$

[where we write $N \stackrel{\text{def}}{=} \text{Ker}(J^{\text{RTF-aug}} \twoheadrightarrow J_G^{\text{RTF}}) \subseteq J^{\text{RTF-aug}} \subseteq J^{\text{RTF}}$]. Note, moreover, that since J_G^{RTF} is an *augmented pro-p* group [cf. Proposition 1.5, (ii)] which admits a surjection $J_G^{\mathrm{RTF}} \twoheadrightarrow \mathbb{Z}_p \times \mathbb{Z}_p$ [cf. the computation of "$\delta_p^1(-)$" in Theorem 2.6, (ii)], it follows immediately that [the augmented pro-prime group] $J^{\text{RTF-aug}}$ is an augmented pro-p group whose augmentation *factors* through J_G^{RTF}; in particular, we conclude that N is *pro-p*. Also, we observe that since the composite $H^{\text{RTF-free}} \to H_G^{\mathrm{RTF}} \to J_G^{\mathrm{RTF}}$ is *trivial*, it follows that the projection under the quotient $J^{\mathrm{RTF}} \twoheadrightarrow J^{\text{RTF-aug}}$ of the image of $H^{\text{RTF-free}}$ in J^{RTF} is *contained in N*.

Now I *claim* that to complete the proof of assertion (iv), it suffices to verify that $N = \{1\}$ [or, equivalently, since $J^{\text{RTF-aug}}$ is *slim*, that N is *finite*]. Indeed, if $N = \{1\}$, then we obtain immediately the isomorphisms $J^{\text{RTF-aug}} \xrightarrow{\sim} J_G^{\mathrm{RTF}}$, $J^{\text{RTF-free}} \xrightarrow{\sim} \mathrm{Ker}(J^{\mathrm{RTF}} \twoheadrightarrow J_G^{\mathrm{RTF}})$. Moreover, by the above discussion, if $N = \{1\}$, then it follows that the image of $H^{\text{RTF-free}}$ in J^{RTF} is *contained in* $J^{\text{RTF-free}}$. Since the homomorphism $H^{\mathrm{RTF}} \to J^{\mathrm{RTF}}$ is open, this implies that the open homomorphism $H^{\mathrm{RTF}} \twoheadrightarrow J^{\mathrm{RTF}}$ induced by ϕ maps $H^{\text{RTF-aug}}$ (respectively, $H^{\text{RTF-free}}$) onto an open subgroup of $J^{\text{RTF-aug}}$ (respectively, $J^{\text{RTF-free}}$), as desired. This completes the proof of the *claim*.

Next, let $\underline{J} \subseteq J$ be an *open subgroup* that arises as the inverse image in J of an [open] *RTF-subgroup* $\underline{J}_G \subseteq J_G$ [so the notation "$\underline{J}_G$" does not lead to any contradictions]. Then one verifies immediately from the definitions that any RTF-subgroup of $\underline{J}_G$ (respectively, $\underline{J}$) determines an RTF-subgroup of J_G (respectively, J). Thus, the natural morphisms

$$\underline{J}_G^{\mathrm{RTF}} \to J_G^{\mathrm{RTF}}; \quad \underline{J}^{\mathrm{RTF}} \to J^{\mathrm{RTF}}$$

are *injective*. Moreover, the subgroups $J^{\text{RTF-aug}} \bigcap \underline{J}^{\mathrm{RTF}}$, $J^{\text{RTF-free}}$ of $\underline{J}^{\mathrm{RTF}}$ clearly determine an *aug-free decomposition* of $\underline{J}^{\mathrm{RTF}}$. Thus, from the point of view of *verifying the finiteness of* N, we may *replace* J by $\underline{J}$ [and H by an appropriate smaller open subgroup contained in $\underline{J}$ and satisfying the hypotheses of the "H" of (iii)]. In particular, since [by the definition of "RTF" and of the subgroup N!] there exists a $\underline{J}$ such that $N \subseteq \underline{J}^{\text{RTF-aug}}$ has *nontrivial image* in $(\underline{J}^{\text{RTF-aug}})^{\text{ab-t}}$, we may assume without loss of generality that N has *nontrivial image* in $(J^{\text{RTF-aug}})^{\text{ab-t}}$. Thus, we have

$$(\delta_p^1(J) \geq)\ \delta_p^1(J^{\text{RTF-aug}}) > \delta_p^1(J_G^{\mathrm{RTF}}) = \delta_p^1(J_G)$$

[cf. the notation of Theorem 2.6], i.e., $s_J \stackrel{\mathrm{def}}{=} \delta_p^1(J^{\text{RTF-aug}}) - \delta_p^1(J_G^{\mathrm{RTF}}) > 0$. By Theorem 2.6, (ii), this already implies that $p \in \Sigma$.

In a similar vein, let $\underline{J} \subseteq J$ be an *open subgroup* that arises as the inverse image in J of an [open] *RTF-subgroup* $\underline{J}^{\text{RTF-free}} \subseteq J^{\text{RTF-free}}$. Then one verifies immediately from the definitions that any RTF-subgroup of $\underline{J}$ determines an RTF-subgroup of J. Thus, the natural morphism $\underline{J}^{\mathrm{RTF}} \to J^{\mathrm{RTF}}$ is *injective*, with image equal to $J^{\text{RTF-aug}} \times \underline{J}^{\text{RTF-free}}$. Moreover, the subgroups $J^{\text{RTF-aug}}$, $\underline{J}^{\text{RTF-free}}$ of $\underline{J}^{\mathrm{RTF}}$ clearly determine an *aug-free decomposition* of $\underline{J}^{\mathrm{RTF}}$ [so the notation "$\underline{J}^{\text{RTF-free}}$" does not lead to any contradictions]. Since [by the above discussion applied to $\underline{J}$ instead of J] $\underline{J}^{\text{RTF-free}}$ maps to the identity in $\underline{J}_G^{\mathrm{RTF}}$, we thus obtain a pro-RTF quotient $\underline{J}^{\mathrm{RTF}} \twoheadrightarrow \underline{J}^{\text{RTF-aug}} = J^{\text{RTF-aug}} \twoheadrightarrow \underline{J}_G^{\mathrm{RTF}}$, hence a pro-RTF quotient

$J^{\text{RTF}} \twoheadrightarrow J^{\text{RTF-aug}} \twoheadrightarrow \underline{J}_G^{\text{RTF}}$ in which the image of $J \bigcap \Delta$ is a *finite* normal closed subgroup, hence *trivial* [since $\underline{J}_G^{\text{RTF}}$ is *slim* — cf. Proposition 1.5, (i)]. That is to say, the pro-RTF quotient $J \twoheadrightarrow J^{\text{RTF-aug}} \twoheadrightarrow \underline{J}_G^{\text{RTF}}$ factors through J_G, hence through J_G^{RTF}. Thus, we obtain a surjection $J_G^{\text{RTF}} \twoheadrightarrow \underline{J}_G^{\text{RTF}}$ whose composite $J_G^{\text{RTF}} \twoheadrightarrow \underline{J}_G^{\text{RTF}} \to J_G^{\text{RTF}}$ with the natural morphism induced by the inclusion $\underline{J} \hookrightarrow J$ is the *identity* [since all of these maps "lie under a fixed $J^{\text{RTF-aug}}$"]. But this implies that the natural morphism $\underline{J}_G^{\text{RTF}} \to J_G^{\text{RTF}}$ is an *isomorphism*. In particular, we have an isomorphism of kernels $\text{Ker}(\underline{J}^{\text{RTF-aug}} \twoheadrightarrow \underline{J}_G^{\text{RTF}}) \xrightarrow{\sim} \text{Ker}(J^{\text{RTF-aug}} \twoheadrightarrow J_G^{\text{RTF}})$. Thus, from the point of view of *verifying the finiteness of* N, we may *replace* J by $\underline{J}$ [and H by an appropriate smaller open subgroup contained in $\underline{J}$ and satisfying the hypotheses of the "H" of (iii)]. In particular, since $\underline{J}^{\text{RTF-aug}} \xrightarrow{\sim} J^{\text{RTF-aug}}$, we may assume without loss of generality that the rank r_J of the *pro-*Σ_J*-solvable free group* $J^{\text{RTF-free}}$ [for some subset $\Sigma_J \subseteq \mathfrak{Primes}$ of cardinality ≥ 2] is either 0 or $> \delta_p^1(J^{\text{RTF-aug}})$. In particular, if $l \in \Sigma_J$, then either $r_J = 0$ or $r_J = \delta_l^1(J^{\text{RTF-free}}) > \delta_p^1(J^{\text{RTF-aug}}) \geq s_J$.

Now we *compute*: Since Σ is of *cardinality* ≥ 2, let $l \in \Sigma$ be a prime $\neq p$. Then:

$$\begin{aligned}\delta_l^1(J^{\text{RTF-free}}) &= \delta_l^1(J^{\text{RTF-free}}) + \delta_l^1(J^{\text{RTF-aug}}) - \delta_l^1(J_G^{\text{RTF}})\\ &= \delta_l^1(J^{\text{RTF}}) - \delta_l^1(J_G^{\text{RTF}}) = \delta_l^1(J) - \delta_l^1(J_G)\\ &= \delta_p^1(J) - \delta_p^1(J_G) = \delta_p^1(J^{\text{RTF}}) - \delta_p^1(J_G^{\text{RTF}})\\ &= \delta_p^1(J^{\text{RTF-free}}) + \delta_p^1(J^{\text{RTF-aug}}) - \delta_p^1(J_G^{\text{RTF}}) = \delta_p^1(J^{\text{RTF-free}}) + s_J\end{aligned}$$

[where we apply the *"independence of l"* of Theorem 2.6, (ii)]. Thus, we conclude that $s_J = \delta_l^1(J^{\text{RTF-free}}) - \delta_p^1(J^{\text{RTF-free}})$, where $\delta_l^1(J^{\text{RTF-free}})$, $\delta_p^1(J^{\text{RTF-free}}) \in \{0, r_J\}$ [depending on whether or not l, p belong to Σ_J], is a *positive integer*. But this implies that $s_J \in \{0, r_J, -r_J\}$, hence that $s_J = r_J > 0$ — in *contradiction* to the inequality $s_J < r_J$ [which holds if $r_J > 0$]. This completes the proof of assertion (iv). ○

Remark 2.11.1. One way of thinking about the content of Theorem 2.11, (iv), is that it asserts that *"***aug-free decompositions of maximal pro-RTF-quotients** *play an analogous [though somewhat more complicated] role for absolute Galois group of* **MLF***'s to the role played by* **torsion-free abelianizations** *for absolute Galois groups of* **FF***'s"* [cf. Theorem 2.6, (i)].

Corollary 2.12. (Group-theoretic Semi-absoluteness via Maximal Pro-RTF-quotients) *For $i = 1, 2$, let $1 \to \Delta_i \to \Pi_i \to G_i \to 1$, k_i, X_i, Σ_i, $\phi : \Pi_1 \to \Pi_2$ be as in Proposition 2.5. Suppose further that k_i is an* **MLF***; X_i is a* **hyperbolic orbicurve***; $\Sigma_i \neq \emptyset$. Also, for $i = 1, 2$, let us write*

$$\Theta_i \subseteq \Pi_i$$

for the **maximal almost pro-prime** *topologically finitely generated closed normal subgroup of Π_i if such a maximal subgroup exists; if such a maximal subgroup does not exist, then we set $\Theta_i \stackrel{\text{def}}{=} \{1\}$. Then:*

(i) For $i = 1, 2$, $\Theta_i \subseteq \Delta_i$; $\Theta_i \neq \{1\}$ if and only if Σ_i is of **cardinality one***; if $\Theta_i \neq \{1\}$, then $\Theta_i = \Delta_i$. Finally, $\phi(\Theta_1) \subseteq \Theta_2$ [so ϕ induces a morphism $\Pi_1/\Theta_1 \to \Pi_2/\Theta_2$].*

(ii) In the notation of Theorem 2.11, ϕ is **semi-absolute** *[or, equivalently,* **pre-semi-absolute** *— cf. Proposition 2.5, (iii)] if and only if the following [***"group-theoretic"***] condition holds:*

($*^{\text{s-ab}}$) *For $i = 1, 2$, let $H_i \subseteq \Pi_i/\Theta_i$ be an open subgroup such that H_i^{RTF} is* **of aug-free type***, and [the morphism induced by] ϕ maps H_1 into H_2. Then the open homomorphism*

$$H_1^{\text{RTF}} \to H_2^{\text{RTF}}$$

induced by ϕ maps $H_1^{\text{RTF-free}}$ into $H_2^{\text{RTF-free}}$.

(iii) If, moreover, Σ_2 is of **cardinality** ≥ 2*, then ϕ is* **semi-absolute** *if and only if it is* **strictly semi-absolute** *[or, equivalently,* **pre-strictly semi-absolute** *— cf. Proposition 2.5, (v)].*

Proof. First, we consider assertion (i). By Theorem 2.6, (iv), any almost pro-prime topologically finitely generated closed normal subgroup of Π_i — hence, in particular, Θ_i — is contained in Δ_i. Thus, by Theorem 2.11, (i), (ii), $\Theta_i \neq \{1\}$ if and only if Σ_i is of cardinality one; if $\Theta_i \neq \{1\}$, then $\Theta_i = \Delta_i$. Now to show that $\phi(\Theta_1) \subseteq \Theta_2$, it suffices to consider the case where $\phi(\Theta_1) \neq \{1\}$ [so Σ_1 is of cardinality one]. Then, by Theorem 2.6, (iv), we have $\phi(\Theta_1) \subseteq \Delta_2$. Thus, we may assume that $\Theta_2 = \{1\}$ [so Σ_2 is of cardinality ≥ 2]. But then the *elasticity* of Δ_2 [cf. Proposition 2.3, (i)] implies that $\phi(\Theta_1)$ is an *open* subgroup of Δ_2, hence that $\phi(\Theta_1)$ is *almost pro*-Σ_2 [for some Σ_2 of cardinality ≥ 2], which contradicts the fact that $\phi(\Theta_1)$ is *almost pro*-Σ_1 [for some Σ_1 of *cardinality one*]. This completes the proof of assertion (i).

Next, we consider assertion (ii). By Proposition 2.5, (iii), one may replace the term *"semi-absolute"* in assertion (ii) by the term *"pre-semi-absolute"*. By assertion (i), for $i = 1, 2$, either $\Theta_i = \{1\}$ or $\Theta_i = \Delta_i$; in either case, it follows from Theorem 2.11, (iv) [cf. also Proposition 1.5, (i), (ii)], that [in the notation of ($*^{\text{s-ab}}$)] the projection $H_i^{\text{RTF}} \twoheadrightarrow H_i^{\text{RTF-aug}}$ may be identified with the projection $H_i^{\text{RTF}} \twoheadrightarrow (H_i)_{G_i}^{\text{RTF}}$ [which is an *isomorphism* whenever $\Theta_i = \Delta_i$]. Thus, the condition ($*^{\text{s-ab}}$) may be thought of as the condition that the morphism $H_1^{\text{RTF}} \to H_2^{\text{RTF}}$ be *compatible* with the projection morphisms $H_i^{\text{RTF}} \twoheadrightarrow (H_i)_{G_i}^{\text{RTF}}$. From this point of view, it follows immediately that the *semi-absoluteness* of ϕ implies ($*^{\text{s-ab}}$), and that ($*^{\text{s-ab}}$) implies [in light of the existence of H_1, H_2 — cf. Theorem 2.11, (ii)] the *pre-semi-absoluteness* of ϕ. Assertion (iii) follows from Proposition 2.5, (iv), (v). ◯

Remark 2.12.1. The criterion of Corollary 2.12, (ii), may be thought of as a *"group-theoretic* Hom*-version"*, in the case of *hyperbolic orbicurves*, of the *numerical criterion* "$\underline{\zeta}(H)/\underline{\zeta}(\Pi) = [\Pi : H]$" of Theorem 2.6, (v).

Example 2.13. A Non-pre-semi-absolute Absolute Homomorphism.

(i) In the situation of Theorem 2.11, suppose that $\Sigma = \mathfrak{Primes}$. Fix a *natural number* N [which one wants to think of as being *"large"*]. By replacing Π by an open subgroup of Π, we may assume that Π satisfies the hypotheses of the subgroup "H" of Theorem 2.11, (iii). Thus, we have a *"combinatorial"* quotient $\Pi \twoheadrightarrow \Delta^{\mathrm{com}}$, where Δ^{com} is a *nonabelian profinite free group*. In particular, there exists an open subgroup of Δ^{com} which is a *profinite free group on* $> N$ *generators*. Thus, by replacing Π by an open subgroup of Π arising from an open subgroup of Δ^{com}, we may assume from the start that Δ^{com} is a *profinite free group on* $> N$ *generators*.

(ii) Now let

$$1 \to \Delta^* \to \Pi^* \to G^* \to 1$$

be an extension *of AFG-type* that admits a *construction data field* which is an *MLF*. Thus, Π^* is *topologically finitely generated* [cf. Theorem 2.6, (ii)], so it follows that there exists a Π as in (i), together with a *surjection of profinite groups*

$$\psi : \Pi \twoheadrightarrow \Pi^*$$

that factors through the quotient $\Pi \twoheadrightarrow \Delta^{\mathrm{com}}$. Thus, ψ is an *absolute homomorphism* which is *not pre-semi-absolute* [hence, *a fortiori*, not semi-absolute].

In light of the appearance of the *"combinatorial quotient"* in Theorem 2.11, (iii), we pause to recall the following result [cf. [Mzk6], Lemma 2.3, in the profinite case].

Theorem 2.14. (Graph-theoreticity for Hyperbolic Curves) *For* $i = 1, 2$, *let* $1 \to \Delta_i \to \Pi_i \to G_i \to 1$, k_i, X_i, Σ_i, $\phi : \Pi_1 \to \Pi_2$ *be as in Proposition 2.5. Suppose further that* k_i *is an* **MLF** *of residue characteristic* p_i*; that* Σ_i *contains a prime* $\neq p_i$*; that* ϕ *is an* **isomorphism***; and that* X_i *is a* **hyperbolic curve** *with* **stable reduction** *over the ring of integers* $\mathcal{O}_{k_i}$ *of* k_i*. Write* Γ_i *for the* **dual semi-graph with compact structure** *[i.e., the dual graph, together with additional open edges corresponding to the cusps — cf. [Mzk6], Appendix] of the geometric special fiber of the stable model of* X_i *over* $\mathcal{O}_{k_i}$*. Then:*

(i) We have $p_1 = p_2$, $\Sigma_1 = \Sigma_2$*;* ϕ *induces isomorphisms* $\Delta_1 \xrightarrow{\sim} \Delta_2$, $G_1 \xrightarrow{\sim} G_2$*;* ϕ *induces an* **isomorphism of semi-graphs** $\phi_\Gamma : \Gamma_1 \xrightarrow{\sim} \Gamma_2$ *which is* **functorial** *in* ϕ*. In particular, the natural Galois action of* G_1 *on* Γ_1 *is compatible, relative to* ϕ_Γ*, with the natural Galois action of* G_2 *on* Γ_2*.*

(ii) For $i = 1, 2$*, suppose that the action of* G_i *on* Γ_i *is* **trivial***. Write* $\Pi_i \twoheadrightarrow \Delta_i^{\mathrm{com}}$ *for the pro-*Σ_i **"combinatorial"** *quotient determined by the finite Galois coverings of the semi-graph* Γ_i *of degree a product of primes* $\in \Sigma_i$*. Then* ϕ *is* **compatible** *with the quotients* $\Pi_i \twoheadrightarrow \Delta_i^{\mathrm{com}}$*.*

Proof. First, we consider assertion (i). By Corollary 2.8, (ii), $p_1 = p_2$, and ϕ induces isomorphisms $\Delta_1 \xrightarrow{\sim} \Delta_2$, $G_1 \xrightarrow{\sim} G_2$. Since [by the well-known structure of geometric fundamental groups of hyperbolic curves] Σ_i is the *unique minimal* $\Sigma \subseteq \mathfrak{Primes}$ such that Δ_i is almost pro-Σ, we thus conclude that $\Sigma_1 = \Sigma_2$. Write $p \stackrel{\mathrm{def}}{=} p_1 = p_2$, $\Sigma \stackrel{\mathrm{def}}{=} \Sigma_1 = \Sigma_2$; let $l \in \Sigma$ be such that $l \neq p$. Then it follows immediately from the *"Riemann hypothesis for abelian varieties over finite fields"* [cf., e.g., [Mumf], p. 206] that the action of G_i on the *maximal pro-l quotient* $\Delta_i \twoheadrightarrow \Delta_i^{(l)}$ is — in the terminology of [Mzk13] — *"l-graphically full"*. Thus, by [Mzk13], Corollary 2.7, (ii), the isomorphism $\Delta_1^{(l)} \xrightarrow{\sim} \Delta_2^{(l)}$ is — again in the terminology of [Mzk13] — *"graphic"*, hence induces a *functorial isomorphism of semi-graphs* $\Gamma_1 \xrightarrow{\sim} \Gamma_2$, as desired.

Next, we consider assertion (ii). First, we observe that, by assertion (i), the condition that the action of G_i on Γ_i be *trivial* is *compatible* with ϕ. Also, let us observe that if $H_i \subseteq \Pi_i$ is an open subgroup corresponding to a finite étale covering $Z_i \to X_i$ of X_i, then the condition that Z_i have *stable reduction* is *compatible* with ϕ [cf. [Mzk6], the proof of Lemma 2.1; our assumption that there exists an $l \in \Sigma_i$ such that $l \neq \{p_i\}$]. Next, I *claim* that:

> A finite étale Galois covering $Z_i \to X_i$ of X_i arises from Δ_i^{com} if and only if Z_i has *stable reduction*, and the action of $\mathrm{Gal}(Z_i/X_i)$ on the dual semi-graph with compact structure of the geometric special fiber of the stable model of Z_i is *free*.

Indeed, the *necessity* of this criterion is clear. To verify the *sufficiency* of this criterion, observe that, by considering the *non-free* actions of inertia subgroups of the Galois covering $Z_i \to X_i$, it follows immediately that this criterion implies that all of the inertia groups arising from irreducible components of the geometric special fiber of a stable model of X_i are *trivial*, hence [cf., e.g., [Tama], Lemma 2.1, (iii)] that the covering $Z_i \to X_i$ extends to an *admissible* covering of the respective stable models. On the other hand, once one knows that the covering $Z_i \to X_i$ admits such an admissible extension, the sufficiency of this criterion is immediate. This completes the proof of the *claim*. Now assertion (ii) follows immediately, by applying the *functorial isomorphisms of semi-graphs* of assertion (i). ◯

Section 3: Absolute Open Homomorphisms of Local Galois Groups

In the present §, we give various generalizations of the main result of [Mzk1] concerning *isomorphisms between Galois groups of MLF's*. One aspect of these generalizations is the *substitution* of the condition given in [Mzk1] for such an isomorphism to arise geometrically — a condition that involves the *higher ramification filtration* — by various other conditions [cf. Theorem 3.5]. Certain of these conditions were motivated by a recent result of *A. Tamagawa* [cf. Remark 3.8.1] concerning *Lubin-Tate groups* and *abelian varieties with complex multiplication*; other conditions [cf. Corollary 3.7] were motivated by a certain application of the theory of the present §3 to be discussed in [Mzk15]. Another aspect of these generalizations is that certain of the conditions studied below allow one to prove a *"*Hom-*version"* [i.e., involving *open homomorphisms*, as opposed to just isomorphisms — cf. Theorem 3.5] of the main result of [Mzk1]. Finally, this Hom-version of the main result of [Mzk1] implies certain *semi-absolute* Hom-*versions* [cf. Corollary 3.8, 3.9 below] of the *absolute* Isom-*version* of the Grothendieck Conjecture given in [Mzk14], §2, and the *relative* Hom-*version* of the Grothendieck Conjecture *for function fields* given in [Mzk3], Theorem B.

Let k be an *MLF* of residue characteristic p; $\overline{k}$ an *algebraic closure* of k; $G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$; $\widehat{\overline{k}}$ the p-adic completion of $\overline{k}$; E an *MLF* of residue characteristic p *all of whose* $\mathbb{Q}_p$*-conjugates* are contained in k. Write $I_k \subseteq G_k$ (respectively, $I_k^{\text{wild}} \subseteq I_k$) for the *inertia subgroup* (respectively, *wild inertia subgroup*) of G_k; $G_k^{\text{tame}} \stackrel{\text{def}}{=} G_k/I_k^{\text{wild}}$; $G_k^{\text{unr}} \stackrel{\text{def}}{=} G_k/I_k$ ($\cong \widehat{\mathbb{Z}}$).

Definition 3.1.

(i) Let A be an abelian topological group; $\rho, \rho' : G_k \to A$ *characters* [i.e., continuous homomorphisms]. Then we shall write $\rho \equiv \rho'$ and say that ρ, ρ' are *inertially equivalent* if, for some open subgroup $H \subseteq I_k$, the restricted characters $\rho|_H$, $\rho'|_H$ *coincide* [cf. [Serre3], III, §A.5].

(ii) Write $\text{Emb}(E, k)$ for the set of *field embeddings* $\sigma : E \hookrightarrow k$. Let $\sigma \in \text{Emb}(E, k)$. Then if π is a *uniformizer* of k, then we shall denote by $\chi_{\sigma,\pi} : G_k \to E^\times$ the composite homomorphism

$$G_k \twoheadrightarrow G_k^{\text{ab}} \xrightarrow{\sim} (k^\times)^\wedge \xrightarrow{\sim} \mathcal{O}_k^\times \times \widehat{\mathbb{Z}} \twoheadrightarrow \mathcal{O}_k^\times \to \mathcal{O}_E^\times \subseteq E^\times$$

— where the "$\wedge$" denotes the profinite completion; the first "$\xrightarrow{\sim}$" is the isomorphism arising from *local class field theory* [cf., e.g., [Serre2]]; the second "$\xrightarrow{\sim}$" is the splitting determined by π; the second "$\twoheadrightarrow$" is the projection to the factor $\mathcal{O}_k^\times$, composed with the *inverse* automorphism on $\mathcal{O}_k^\times$ [cf. Remark 3.1.1 below]; the homomorphism $\mathcal{O}_k^\times \to \mathcal{O}_E^\times$ is the *norm map* associated to the field embedding σ. Since [as is well-known, from local class field theory] $I_k \subseteq G_k$ surjects to $\mathcal{O}_k^\times \times \{1\} \subseteq \mathcal{O}_k^\times \times \widehat{\mathbb{Z}}$, it follows immediately that the *inertial equivalence class* of

$\chi_{\sigma,\pi}$ is *independent* of the choice of π. Thus, we shall often write χ_σ to denote $\chi_{\sigma,\pi}$ for some unspecified choice of π.

(iii) Let $\rho : G_k \to E^\times$ be a character. Then we shall say that ρ is *of qLT-type* [i.e., "quasi-Lubin-Tate" type] if there exists an open subgroup $H \subseteq G_k$, corresponding to a field extension k_H of k, and a field embedding $\sigma : E \hookrightarrow k_H$ such that $\rho|_H \equiv \chi_\sigma$; in this situation, we shall refer to $[E : \mathbb{Q}_p]$ as the *dimension* of ρ. We shall say that ρ is *of 01-type* if it is *Hodge-Tate*, and, moreover, every *weight* appearing in its Hodge-Tate decomposition $\in \{0, 1\}$. Write

$$\chi_k^{\text{cyclo}} : G_k \to \mathbb{Q}_p^\times$$

for the *cyclotomic character* associated to G_k. We shall say that ρ is *of ICD-type* [i.e., "inertially cyclotomic determinant" type] if its *determinant* $\det(\rho) : G_k \to \mathbb{Q}_p^\times$ [i.e., the composite of ρ with the norm map $E^\times \to \mathbb{Q}_p^\times$] is *inertially equivalent* to χ_k^{cyclo}.

(iv) For $i = 1, 2$, let k_i be an *MLF* of residue characteristic p_i; $\overline{k}_i$ an algebraic closure of k_i; $\widehat{\overline{k}}_i$ the p_i-adic completion of $\overline{k}_i$. We shall use similar notation for the various subquotients of the absolute Galois group $G_{k_i} \stackrel{\text{def}}{=} \text{Gal}(\overline{k}_i/k_i)$ of k_i to the notation already introduced for G_k. Let

$$\phi : G_{k_1} \to G_{k_2}$$

be an *open homomorphism*. Then we shall say that ϕ is *of qLT-type* (respectively, *of 01-qLT-type*) if $p_1 = p_2$, and, moreover, for every pair of open subgroups $H_1 \subseteq G_{k_1}$, $H_2 \subseteq G_{k_2}$ such that $\phi(H_1) \subseteq H_2$, and every character $\rho : H_2 \to F^\times$ of qLT-type [where F is an MLF of residue characteristic $p_1 = p_2$ all of whose conjugates are contained in the fields determined by H_1, H_2], the restricted character $\rho|_{H_1} : H_1 \to F^\times$ [obtained by restricting via ϕ] is of qLT-type (respectively, of 01-type). We shall say that ϕ is *of HT-type* [i.e., "Hodge-Tate" type] if $p_1 = p_2$, and, moreover, the topological G_{k_1}-module [but *not necessarily the topological field*!] obtained by composing ϕ with the natural action of G_{k_2} on $\widehat{\overline{k}}_2$ is isomorphic [as a topological G_{k_1}-module] to $\widehat{\overline{k}}_1$. We shall say that ϕ is *of CHT-type* [i.e., "cyclotomic Hodge-Tate" type] if ϕ is of HT-type, and, moreover, the cyclotomic characters of G_{k_1}, G_{k_2} satisfy $\chi_{k_1}^{\text{cyclo}} = \chi_{k_2}^{\text{cyclo}} \circ \phi$. We shall say that ϕ is *geometric* if it arises from an isomorphism of fields $\overline{k}_2 \xrightarrow{\sim} \overline{k}_1$ that maps k_2 into k_1 [which implies, by considering the *divisibility* of the $k_i^\times$, that $p_1 = p_2$].

(v) Let $1 \to \Delta \to \Pi \to G_k \to 1$ be an extension *of AFG-type*. Then we shall say that this extension $\Pi \twoheadrightarrow G_k$ [or, when there is no danger of confusion, that Π] is *of A-qLT-type* [i.e., "Albanese-quasi-Lubin-Tate" type] if for every open subgroup $H \subseteq G_k$, and every character $\rho : H \to F^\times$ of qLT-type [where F is an MLF of residue characteristic p all of whose conjugates are contained in the field determined by H], there exists an open subgroup $J \subseteq \Pi \times_{G_k} (I_k \bigcap H)$ [so one has an outer action of the image J_G of J in G_k on $J_\Delta \stackrel{\text{def}}{=} J \bigcap \Delta$] such that the J_G-module V_ρ

obtained by letting J_G act on F via $\rho|_{J_G}$ is *isomorphic* to some subquotient S of the J_G-module $J_\Delta^{\mathrm{ab}} \otimes \mathbb{Q}_p$.

Remark 3.1.1. As is well-known, the ρ that arises from a *Lubin-Tate group* is of *qLT-type* — cf., e.g., [Serre3], III, §A.4, Proposition 4. This is the reason for the terminology *"quasi-Lubin-Tate"*.

We begin by reviewing some well-known facts.

Proposition 3.2. **(Characterization of Hodge-Tate Characters)** *Let* $\rho : G_k \to E^\times$ *be a* **character***; write* V_ρ *for the* G_k*-module obtained by letting* G_k *act on* E *via* ρ*. Then* ρ *is* **Hodge-Tate** *if and only if*

$$\rho \equiv \prod_{\sigma \in \mathrm{Emb}(E,k)} \chi_\sigma^{n_\sigma}$$

for some $n_\sigma \in \mathbb{Z}$*. Moreover, in this case, we have an isomorphism of* $\hat{\overline{k}}[G_k]$*-modules:*

$$V_\rho \otimes_{\mathbb{Q}_p} \hat{\overline{k}} \cong \bigoplus_{\sigma \in \mathrm{Emb}(E,k)} \hat{\overline{k}}(n_\sigma)$$

[where the "$(-)$" denotes a Tate twist].

Proof. Indeed, this criterion for the character ρ to be Hodge-Tate is precisely the content of [Serre3], III, §A.5, Corollary. The Hodge-Tate decomposition of V_ρ then follows immediately the Hodge-Tate decomposition of "V_ρ" in the case where one takes "ρ" to be χ_σ [cf. [Serre3], III, §A.5, proof of Lemma 2]. ◯

Proposition 3.3. **(Characterization of Quasi-Lubin-Tate Characters)** *Let* ρ*,* V_ρ *be as in Proposition 3.2. Then the following conditions on* ρ *are* **equivalent***:*

(i) ρ *is* **of qLT-type**.

(ii) We have an isomorphism of $\hat{\overline{k}}[G_k]$*-modules:* $V_\rho \otimes_{\mathbb{Q}_p} \hat{\overline{k}} \cong \hat{\overline{k}}(1) \oplus \hat{\overline{k}} \oplus \ldots \oplus \hat{\overline{k}}$.

(iii) ρ *is* **of ICD-type** *and* **Hodge-Tate***; the resulting* n_σ*'s of Proposition 3.2 are* $\in \{0, 1\}$.

(iv) ρ *is* **of ICD-type** *and* **of 01-type**.

Proof. The fact that (i) implies (ii) follows immediately from the description of the Hodge-Tate decomposition of "V_ρ" in the case where one takes "ρ" to be χ_σ [cf. [Serre3], III, §A.5, proof of Lemma 2]. Next, let us assume that (ii), (iii), or (iv) holds. In either of these cases, it follows that ρ, hence also the determinant

$\det(\rho) : G_k \to \mathbb{Q}_p^\times$ of ρ, is *Hodge-Tate*. Then by applying Proposition 3.2 to ρ, we obtain that the associated n_σ's are $\in \{0, 1\}$; by applying Proposition 3.2 to $\det(\rho)$ [in which case one takes "E" to be $\mathbb{Q}_p$], we obtain that $\det(\rho)$ is *inertially equivalent* to the $(\sum_\sigma n_\sigma)$-th power of χ_k^{cyclo}. But this allows one to conclude [either from the explicit Hodge-Tate decomposition of (ii), or from the assumption that ρ is *of ICD-type* in (iii), (iv)] that $\sum_\sigma n_\sigma = 1$, hence that there exists *precisely one* $\sigma \in \text{Emb}(E, k)$ such that $n_\sigma = 1$, $n_{\sigma'} = 0$ for $\sigma' \neq \sigma$. Thus, [sorting through the definitions] we conclude that (i), (ii), (iii), and (iv) hold. This completes the proof of Proposition 3.3. ◯

Proposition 3.4. **(Preservation of Tame Quotients)** *In the notation of Definition 3.1, (iv), let $\phi : G_{k_1} \to G_{k_2}$ be an* **open homomorphism**. *Then $p_1 = p_2$, and there exists a commutative diagram*

$$\begin{array}{ccc} G_{k_1} & \stackrel{\phi}{\longrightarrow} & G_{k_2} \\ \downarrow & & \downarrow \\ G_{k_1}^{\text{tame}} & \stackrel{\phi^{\text{tame}}}{\longrightarrow} & G_{k_2}^{\text{tame}} \end{array}$$

— where the vertical arrows are the natural surjections; ϕ^{tame} is an **injective** *homomorphism.*

Proof. We may assume without loss of generality that ϕ is *surjective*. Next, let $H_2 \subseteq G_{k_2}$ be an open subgroup, $H_1 \stackrel{\text{def}}{=} \phi^{-1}(H_2) \subseteq G_{k_1}$. Then if $l \stackrel{\text{def}}{=} p_1 \neq p_2$, then [since we have a surjection $H_2 \twoheadrightarrow H_1$] $1 = \delta_l^1(H_2) \geq \delta_l^1(H_1) \geq 2$ for $l = p_1$ [cf. Theorem 2.6, (ii)]; thus, we conclude that $p_1 = p_2$. Write $p \stackrel{\text{def}}{=} p_1 = p_2$. Since $G_{k_2}^{\text{tame}} \cong \widehat{\mathbb{Z}}^{(\neq p)}(1) \rtimes \widehat{\mathbb{Z}}$ [for some *faithful* action of $\widehat{\mathbb{Z}}$ on $\widehat{\mathbb{Z}}^{(\neq p)}(1)$ — cf., e.g., [NSW], Theorem 7.5.2], it follows immediately that every *closed normal pro-p subgroup* of $G_{k_2}^{\text{tame}}$ is *trivial*. Thus, the image of $\phi(I_{k_1}^{\text{wild}})$ in $G_{k_2}^{\text{tame}}$ is *trivial*, so we conclude that ϕ induces a surjection $\phi^{\text{tame}} : G_{k_1}^{\text{tame}} \twoheadrightarrow G_{k_2}^{\text{tame}}$. Since, for $i = 1, 2$, the quotient $G_{k_i}^{\text{tame}} \twoheadrightarrow G_{k_i}^{\text{unr}} \cong \widehat{\mathbb{Z}}$ may be characterized as the quotient $G_{k_i}^{\text{tame}} \twoheadrightarrow (G_{k_i}^{\text{tame}})^{\text{ab-t}}$, it thus follows immediately that ϕ^{tame} induces continuous homomorphisms

$$\widehat{\mathbb{Z}} \cong G_{k_1}^{\text{unr}} \twoheadrightarrow G_{k_2}^{\text{unr}} \cong \widehat{\mathbb{Z}}; \quad \widehat{\mathbb{Z}}^{(\neq p)}(1) \cong I_{k_1}/I_{k_1}^{\text{wild}} \to I_{k_2}/I_{k_2}^{\text{wild}} \cong \widehat{\mathbb{Z}}^{(\neq p)}(1)$$

— the first of which is *surjective*, hence an *isomorphism* [since, as is well-known, every surjective endomorphism of a profinite group is an isomorphism]. But this implies that the second displayed homomorphism is also *surjective*, hence an *isomorphism*. This completes the proof of Proposition 3.4. ◯

Theorem 3.5. **(Criteria for Geometricity)** *For $i = 1, 2$, let k_i be an* **MLF** *of residue characteristic p_i; $\overline{k}_i$ an algebraic closure of k_i; $\widehat{\overline{k}}_i$ the p_i-adic completion of $\overline{k}_i$. We shall use similar notation for the various subquotients of the absolute*

Galois group $G_{k_i} \stackrel{\text{def}}{=} \text{Gal}(\overline{k}_i/k_i)$ *of* k_i *to the notation introduced at the beginning of the present §3 for* G_k. *Let*

$$\phi : G_{k_1} \to G_{k_2}$$

be an **open homomorphism**. *Then:*

(i) The following conditions on ϕ *are* **equivalent**: *(a)* ϕ *is* **of CHT-type**; *(b)* ϕ *is* **of 01-qLT-type**; *(c)* ϕ *is* **of qLT-type**; *(d)* ϕ *is* **geometric**.

(ii) Suppose that ϕ *is an* **isomorphism**. *Then* ϕ *is* **geometric** *if and only if it is* **of HT-type**.

(iii) For $i = 1, 2$, *let* $1 \to \Delta_i \to \Pi_i \to G_{k_i} \to 1$ *be an extension* **of AFG-type**;

$$\psi : \Pi_1 \to \Pi_2$$

a **semi-absolute** *[or, equivalently,* **pre-semi-absolute** *— cf. Proposition 2.5, (iii)] homomorphism that lifts* ϕ. *Suppose that* Π_2 *is* **of A-qLT-type**. *Then* ϕ *is* **geometric**.

Proof. First, we observe that by Proposition 3.4, it follows that $p_1 = p_2$; write $p \stackrel{\text{def}}{=} p_1 = p_2$. Also, we may always assume without loss of generality that ϕ is *surjective*. In the following, we will use a superscript "G_{k_i}" [where $i = 1, 2$] to denote the *submodule of* G_{k_i}*-invariants* of a G_{k_i}-module.

Next, we consider assertion (i). First, we observe that it is immediate that condition (d) implies condition (a). Next, let us suppose that condition (a) holds. Since $k_i = \widehat{\overline{k}}_i^{G_{k_i}}$ is *finite-dimensional* over $\mathbb{Q}_p$, it follows that, for $i = 1, 2$, any G_{k_i}-module M which is finite-dimensional over $\mathbb{Q}_p$ is *Hodge-Tate with weights* $\in \{0, 1\}$ if and only if

$$\dim_{\mathbb{Q}_p}((M \otimes \widehat{\overline{k}}_i)^{G_{k_i}}) + \dim_{\mathbb{Q}_p}((M(-1) \otimes \widehat{\overline{k}}_i)^{G_{k_i}}) = \dim_{\mathbb{Q}_p}(M) \cdot \dim_{\mathbb{Q}_p}(\widehat{\overline{k}}_i^{G_{k_i}})$$

[where the tensor products are over $\mathbb{Q}_p$]. Now suppose that M is a G_{k_2}-module that arises as a "V_ρ" for some character $\rho : G_{k_2} \to E^\times$ of qLT-type [so M is *Hodge-Tate with weights* $\in \{0, 1\}$ — cf. Proposition 3.3, (i) $\Longrightarrow$ (iv)]; write M_ϕ for the G_{k_1}-module M_ϕ obtained by composing the G_{k_2}-action on M with ϕ. Thus, it follows immediately from our assumption that ϕ is *of CHT-type* that the above condition concerning $\mathbb{Q}_p$-dimensions for M implies the above condition concerning $\mathbb{Q}_p$-dimensions for M_ϕ. Applying this argument to corresponding open subgroups of G_{k_1}, G_{k_2} thus shows that ϕ is *of 01-qLT-type*, i.e., that condition (b) holds.

Next, let us assume that condition (b) holds. First, I *claim* that $\chi^{\text{cyclo}}_{k_1} \equiv \chi^{\text{cyclo}}_{k_2} \circ \phi$. Indeed, by condition (b), it follows that the character $\chi^{\text{cyclo}}_{k_2} \circ \phi : G_{k_1} \to \mathbb{Q}_p^\times$ is *of 01-type*. Thus, by Proposition 3.2, we conclude that $\chi^{\text{cyclo}}_{k_2} \circ \phi \equiv (\chi^{\text{cyclo}}_{k_1})^n$, for some $n \in \{0, 1\}$. On the other hand, the *restriction* of $\chi^{\text{cyclo}}_{k_2}$ to I_{k_2} clearly has *open image*; since ϕ is *open*, it thus follows that the restrction of $\chi^{\text{cyclo}}_{k_2} \circ \phi$ to I_{k_1} has *open image*. This rules out the possibility that $n = 0$, hence completes the proof of the *claim*. Now, by applying this claim, together with Proposition 3.3, (i) $\iff$ (iv), we conclude that ϕ is *of qLT-type*, i.e., that condition (c) holds.

Next, let us assume that condition (c) holds. First, I *claim* that this already implies that ϕ is *injective* [i.e., an *isomorphism*]. Indeed, let $\gamma \in \mathrm{Ker}(\phi) \subseteq G_{k_1}$ be such that $\gamma \neq 1$. Then there exists an *open subgroup* $J_1 \subseteq G_{k_1} \subseteq G_{\mathbb{Q}_p}$ satisfying the following conditions: (1) $\gamma \notin J_1$; (2) J_1 is *characteristic* as a subgroup $G_{\mathbb{Q}_p}$; (3) the extension E of $\mathbb{Q}_p$ determined by J_1 contains all $\mathbb{Q}_p$-conjugates of k_2. Fix an embedding $\sigma_2 : k_2 \hookrightarrow E$; write $H_2 \subseteq G_{k_2}$ for the corresponding open subgroup. Let $H_1 \subseteq J_1 \subseteq G_{k_1}$ be an open subgroup which is normal in G_{k_1} such that $\phi(H_1) \subseteq H_2$; for $i = 1, 2$, write k_{H_i} for the extension of k_i determined by H_i. Thus, the embedding $\sigma_2 : E \hookrightarrow k_{H_2}$ given by the identity $E = k_{H_2}$ (respectively, $\sigma_1 : E \hookrightarrow k_{H_1}$ determined by the inclusion $H_1 \subseteq J_1$) determines a character $\rho_2 : H_2 \to E^\times$ (respectively, $\rho_1 : H_1 \to E^\times$) *of qLT-type* [i.e., the character "χ_{σ_2}" (respectively, "χ_{σ_1}")]. Moreover, by condition (c), the character $\rho_2 \circ (\phi|_{H_1}) : H_1 \to E^\times$ is *of qLT-type*, hence is *inertially equivalent* to $\tau \circ \rho_1 : H_1 \to E^\times$ for some $\tau \in \mathrm{Gal}(E/\mathbb{Q}_p)$. In particular, by replacing σ_2 by $\sigma_2 \circ \tau$, we may assume that τ is the identity, hence that $\rho_2 \circ (\phi|_{H_1}) \equiv \rho_1$. On the other hand, since $\gamma \notin J_1$, hence acts *nontrivially* on the subfield $E \subseteq k_{H_1}$ [relative to the embedding σ_1], it follows that $\rho_1 \circ \kappa_\gamma \equiv \delta \circ \rho_1$, where we write κ_γ for the automorphism of H_1 given by conjugating by γ, and $\delta \in \mathrm{Gal}(E/\mathbb{Q}_p)$ is *not equal to the identity*. But since $\phi(\gamma) = 1 \in G_{k_2}$, we thus conclude that $\delta \circ \rho_1 \equiv \rho_1 \circ \kappa_\gamma \equiv \rho_2 \circ (\phi|_{H_1}) \circ \kappa_\gamma \equiv \rho_2 \circ (\phi|_{H_1}) \equiv \rho_1$, which [since ρ_1 has *open image*] contradicts the fact that $\delta \in \mathrm{Gal}(E/\mathbb{Q}_p)$ is *not equal to the identity*. This completes the proof of the *claim*. Thus, we may assume that ϕ is an *isomorphism of qLT-type*, i.e., we are, in effect, in the situation of [Mzk1], §4. In particular, the fact that ϕ is *geometric*, i.e., that condition (d) holds, follows immediately via the argument of [Mzk1], §4. This completes the proof of assertion (i).

Next, we consider assertion (ii). Since ϕ is an *isomorphism*, it follows [cf. [Mzk1], Proposition 1.1; [Mzk6], Proposition 1.2.1, (vi)] that $\chi^{\mathrm{cyclo}}_{k_1} = \chi^{\mathrm{cyclo}}_{k_2} \circ \phi$. In particular, ϕ is of HT-type if and only if ϕ is of CHT-type. Thus, assertion (ii) follows from the equivalence of (a), (d) in assertion (i).

Finally, we consider assertion (iii). First, let us recall that by a well-known result of Tate [cf. [Tate], §4, Corollary 2], if $J \subseteq \Pi_1$ is an open subgroup with image $J_G \subseteq G_{k_1}$ and intersection $J_\Delta \stackrel{\mathrm{def}}{=} J \bigcap \Delta_1$, then the J_G-module $J_\Delta^{\mathrm{ab}} \otimes \mathbb{Q}_p$ is always *Hodge-Tate with weights* $\in \{0, 1\}$. Thus, the condition that Π_2 is *of A-qLT-type* implies that ϕ is *of 01-qLT-type*, hence, by assertion (i), *geometric*. This completes the proof of assertion (iii). $\bigcirc$

Definition 3.6.

(i) If $H \subseteq G_k$ is an open subgroup corresponding to an extension field k_H of k, then by *local class field theory* [cf., e.g., [Serre2]], we have a natural isomorphism

$$\mathcal{O}^\times_{k_H} \xrightarrow{\sim} \mathrm{Tor}(H)$$

— where we write $\mathrm{Tor}(H)$ [i.e., the *"toral portion of H"*] for the image of $I_k \bigcap H$ in H^{ab}. Thus, by applying the *p-adic logarithm* $\mathcal{O}_{k_H}^{\times} \to k_H$, we obtain a natural isomorphism $\lambda_H : \mathrm{Tor}(H) \otimes \mathbb{Q}_p \xrightarrow{\sim} k_H$.

(ii) We shall refer to a collection $\{N_H\}_H$, where H ranges over a collection of open subgroups of G_k that form a *basis* of the topology of G_k, as a *uniformly toral neighborhood* of G_k if there exist nonnegative integers a, b [which are *independent* of H!] such that [in the notation of (i)] $N_H \subseteq \mathrm{Tor}(H) \otimes \mathbb{Q}_p$ is an open subgroup such that $p^a \cdot \mathcal{O}_{k_H} \subseteq \lambda_H(N_H) \subseteq p^{-b} \cdot \mathcal{O}_{k_H} \subseteq k_H$.

(iii) Let $\phi : G_{k_1} \xrightarrow{\sim} G_{k_2}$ be an *isomorphism of profinite groups*. Then we shall say that ϕ is *uniformly toral* if G_{k_1} admits a uniformly toral neighborhood $\{N_H\}_H$ such that $\{\phi(N_H)\}_H$ forms a uniformly toral neighborhood of G_{k_2}. We shall say that ϕ is *RF-preserving* [i.e., "ramification filtration preserving"] if ϕ is compatible with the filtrations on G_{k_1}, G_{k_2} given by the [positively indexed] higher ramification groups in the upper numbering [cf., [Mzk1], Theorem].

Corollary 3.7. **(Uniform Torality and Geometricity)** *In the situation of Theorem 3.5, suppose further that ϕ is an* **isomorphism**. *Then the following conditions on ϕ are* **equivalent**: *(a) ϕ is* **RF-preserving**; *(b) ϕ is* **uniformly toral**; *(c) ϕ is* **geometric**.

Proof. First, we observe that by Proposition 3.4, it follows that $p_1 = p_2$; write $p \stackrel{\mathrm{def}}{=} p_1 = p_2$. Also, we observe that it is immediate that condition (c) implies condition (a). Next, we recall that the fact that condition (a) implies condition (b) is precisely the content of the discussion preceding [Mzk1], Proposition 2.2. That is to say, for $i = 1, 2$, the images of appropriate higher ramification groups in $\mathrm{Tor}(H) \otimes \mathbb{Q}_p$ [for open subgroups $H \subseteq G_{k_i}$] multiplied by appropriate integral powers of p yield a *uniformly toral neighborhood* of G_{k_i} that is *compatible* with ϕ whenever ϕ is *RF-preserving*.

Next, let us assume that condition (b) holds. For $i = 1, 2$, let $\{N_H^i\}_H$ be a *uniformly toral neighborhood* of G_{k_i}. Again, we take the point of view of the discussion preceding [Mzk1], Proposition 2.2. That is to say, we think of $\overline{k}_i$ as the inductive limit

$$I_i \stackrel{\mathrm{def}}{=} \varinjlim_{H} \ \mathrm{Tor}(H) \otimes \mathbb{Q}_p$$

— where H ranges over the open subgroups $\subseteq G_{k_i}$ involved in $\{N_H^i\}_H$; the morphisms in the inductive system are those induced by the *Verlagerung, or transfer, map*. Write $N_i \subseteq I_i$ for the subgroup generated by the $N_H^i \subseteq \mathrm{Tor}(H) \otimes \mathbb{Q}_p$. Then relative to the isomorphism [of abstract modules!] $\lambda_i : I_i \xrightarrow{\sim} \overline{k}_i$ determined by the λ_H's, we have

$$p^a \cdot \mathcal{O}_{\overline{k}_i} \subseteq \lambda_i(N_i) \subseteq p^{-b} \cdot \mathcal{O}_{\overline{k}_i} \subseteq \overline{k}_i$$

for some nonnegative integers a, b [cf. Definition 3.6, (ii)]. In particular, it follows that the *topology* on I_i determined by the submodules $p^c \cdot N_i$, where $c \geq 0$ is an integer, *coincides*, relative to λ_i, with the *p-adic topology* on $\overline{k}_i$ [i.e., the topology determined by the $p^c \cdot \mathcal{O}_{\overline{k}_i}$, where $c \geq 0$ is an integer]. Write $\widehat{I}_i$ for the *completion* of I_i relative to the topology determined by the $p^c \cdot N_i$. Thus, λ_i determines an *isomorphism of topological* G_{k_i}*-modules* $\widehat{I}_i \xrightarrow{\sim} \widehat{\overline{k}}_i$. In particular, the assumption that ϕ is *uniformly toral* implies that ϕ is *of HT-type*. Thus, by Theorem 3.5, (ii), we conclude that ϕ is *geometric*, i.e., that condition (c) holds. This completes the proof of Corollary 3.7. ○

Remark 3.7.1. In fact, one verifies immediately that the argument applied in the proof of Corollary 3.7 implies that the equivalences of Corollary 3.7 [as well as the definitions of Definition 3.6] continue to hold when ϕ is replaced by an isomorphism of profinite groups between the *maximal pro-p quotients* of the G_{k_i}. We leave the routine details to the reader.

Corollary 3.8. (Geometricity of Semi-absolute Homomorphisms for Hyperbolic Orbicurves) *For* $i = 1, 2$*, let* k_i*,* $\overline{k}_i$*,* $\widehat{\overline{k}}_i$*,* p_i*,* G_{k_i} *[and its subquotients] be as in Theorem 3.5;* $1 \to \Delta_i \to \Pi_i \to G_{k_i} \to 1$ *an extension* **of AFG-type***;* (k_i, X_i, Σ_i) **partial construction data** *[consisting of the construction data field, construction data base-stack, and construction data prime set] for* $\Pi_i \twoheadrightarrow G_{k_i}$*;* $\alpha_i : \pi_1(X_i) = \pi_1^{\mathrm{tame}}(X_i) \twoheadrightarrow \Pi_i$ *a* **scheme-theoretic envelope** *compatible with the natural projections* $\pi_1(X_i) \twoheadrightarrow G_{k_i}$*,* $\Pi_i \twoheadrightarrow G_{k_i}$*;*

$$\psi : \Pi_1 \to \Pi_2$$

a **semi-absolute** *[or, equivalently,* **pre-semi-absolute** *— cf. Proposition 2.5, (iii)] homomorphism that lifts a homomorphism* $\phi : G_1 \to G_2$*. Suppose further that* X_2 *is a* **hyperbolic orbicurve***, that* $p_2 \in \Sigma_2$*, and that one of the following conditions holds:*

(a) ϕ *is* **of CHT-type***;*

(b) ϕ *is* **of 01-qLT-type***;*

(c) ϕ *is* **of qLT-type***;*

(d) ϕ *is an* **isomorphism of HT-type***;*

(e) ϕ *is a* **uniformly toral isomorphism***;*

(f) ϕ *is an* **RF-preserving isomorphism***;*

(g) Π_2 *is* **of A-qLT-type***.*

(h) ϕ *is* **geometric***;*

Then ψ *is* **geometric***, i.e., arises [relative to the* α_i*] from a* **unique dominant morphism of schemes** $X_1 \to X_2$ *lying over a morphism* $\mathrm{Spec}(k_1) \to \mathrm{Spec}(k_2)$*.*

Proof. Indeed, by Theorem 3.5, (i), (ii), (iii); Corollary 3.7, it follows that any of the conditions (a), (b), (c), (d), (e), (f), (g), (h) implies condition (h). Thus, since X_2 is a *hyperbolic orbicurve*, and $p_2 \in \Sigma_2$, the fact that ψ is *geometric* follows from [Mzk3], Theorem A. ○

Remark 3.8.1. One important motivation for the theory of the present §3 is the following result, orally communicated to the author by *A. Tamagawa*:

($*^{\text{A-qLT}}$) Let X be a *hyperbolic orbicurve* over k that admits a finite étale covering $Y \to X$ by a hyperbolic curve Y such that Y admits a *dominant* k-*morphism* $Y \to P$, where P is the projective line minus three points over k [i.e., a *tripod* — cf. §0]. Then the arithmetic fundamental group $\pi_1(X) \twoheadrightarrow G_k$ of X is *of A-qLT-type.*

In particular, it follows that:

Corollary 3.8 may be applied [in the sense that condition (g) is satisfied] whenever X_2 satisfies the conditions placed on the hyperbolic orbicurve "X" of ($*^{\text{A-qLT}}$).

Indeed, Tamagawa's original motivation for considering ($*^{\text{A-qLT}}$) was precisely the goal of applying the methods of [Mzk1] to obtain an *"isomorphism version"* of Corollary 3.8, (g). Upon learning of these ideas of Tamagawa, the author proceeded to re-examine the theory of [Mzk1]. This led the author to the discovery of the various generalizations of [Mzk1] — and, in particular, the Hom-*version* of Corollary 3.8, (g) — given in the present §3. Tamagawa derives ($*^{\text{A-qLT}}$) from the following result:

($*^{\text{CM}}$) Given a character $\rho : G_k \to E^\times$ of qLT-type, there exists an *abelian variety with complex multiplication* A over some finite extension k_A of k such that $\rho|_{G_{k_A}}$ is *inertially equivalent* to some character whose associated G_{k_A}-module appears as a subquotient of the G_{k_A}-module given by the p-adic Tate module of A.

Indeed, to derive ($*^{\text{A-qLT}}$) from ($*^{\text{CM}}$), one reasons as follows: Every abelian variety with complex multiplication A is defined over a number field, hence arises as a quotient of a Jacobian of a smooth proper curve Z over a number field. Moreover, by considering *Belyi maps*, it follows that some open subscheme $U_Z \subseteq Z$ arises as a finite étale covering of the projective line minus three points. Thus, any Galois module that appears as a subquotient of the p-adic Tate module of A also appears as a subquotient of the p-adic Tate module of the Jacobian of some finite étale covering of the curve P of ($*^{\text{A-qLT}}$), hence, *a fortiori*, as a subquotient of the p-adic Tate module of the Jacobian of some finite étale covering of the curves Y, X of ($*^{\text{A-qLT}}$). Thus, we conclude that $\pi_1(X)$ is *of A-qLT-type*, as desired.

Corollary 3.9. (Geometricity of Strictly Semi-absolute Homomorphisms for Function Fields) *Assume that the result* ($*^{\text{A-qLT}}$) *of Remark 3.8.1 holds. For* $i = 1, 2$, *let* k_i *be an* **MLF**, K_i *a* **function field** *of transcendence degree* ≥ 1 *over* k_i *[so* k_i *is algebraically closed in* K_i*],* $\overline{K}_i$ *an algebraic closure of* K_i, $\overline{k}_i$ *the algebraic closure of* k_i *determined by* $\overline{K}_i$, $\Pi_i \stackrel{\text{def}}{=} \text{Gal}(\overline{K}_i/K_i)$, $G_i \stackrel{\text{def}}{=} \text{Gal}(\overline{k}_i/k_i)$, $\Delta_i \stackrel{\text{def}}{=} \text{Ker}(\Pi_i \twoheadrightarrow G_i)$. *Then every* **open homomorphism**

$$\psi : \Pi_1 \to \Pi_2$$

that induces an open homomorphism $\psi_\Delta : \Delta_1 \to \Delta_2$ *[hence also an open homomorphism* $\phi : G_1 \to G_2$*] is* **geometric***, i.e., arises from a* **unique** *embedding of fields* $K_2 \hookrightarrow K_1$ *that induces an embedding of fields* $k_2 \hookrightarrow k_1$ *of finite degree.*

Proof. Since every function field of transcendence degree ≥ 1 over k_2 contains the function field of a tripod over k_2, it follows from $(*^{\text{A-qLT}})$ that there exists a *hyperbolic curve* X over k_2 with function equal to K_2 such that if we write $\Pi_2 \twoheadrightarrow \Pi_3 \stackrel{\text{def}}{=} \pi_1(X)$ for the resulting surjection, then Π_3 is *of A-qLT-type.* Now we wish to apply a *"birational analogue"* of Corollary 3.8, (g), to the composite homomorphism $\Pi_1 \to \Pi_2 \twoheadrightarrow \Pi_3$ [where the first arrow is ψ].

To verify that such an analogue holds, it suffices to verify that ϕ is *of 01-qLT-type* [cf. Theorem 3.5, (i), (b) $\Longrightarrow$ (d)]. To this end, set $k_3 \stackrel{\text{def}}{=} k_2$, $G_3 \stackrel{\text{def}}{=} G_2$, $\Delta_3 \stackrel{\text{def}}{=} \text{Ker}(\Pi_3 \twoheadrightarrow G_3)$; let us suppose, for $i = 1, 3$, that $H_i \subseteq \Delta_i$, $J_i \subseteq G_i$ are *characteristic open subgroups* such that $\psi_\Delta(H_1) \subseteq H_3$, $\phi(J_1) \subseteq J_3$. Thus, if we write p for the common residue characteristic of k_1, k_3 [cf. Proposition 3.4], then we obtain a surjection $H_1^{\text{ab}} \otimes \mathbb{Q}_p \twoheadrightarrow H_3^{\text{ab}} \otimes \mathbb{Q}_p$ that is compatible with ϕ. Moreover, it follows immediately from Corollary A.11 [cf. also Proposition A.3, (v)] of the Appendix that the J_1-module $H_1^{\text{ab-t}} \otimes \mathbb{Z}_p$ admits a *quotient* J_1-module $H_1^{\text{ab-t}} \otimes \mathbb{Z}_p \twoheadrightarrow Q_1$ such that Q_1 is the p-adic Tate module of some *abelian variety* over a finite extension of k_1, and, moreover, the kernel $\text{Ker}(H_1^{\text{ab-t}} \otimes \mathbb{Z}_p \twoheadrightarrow Q_1)$ is topologically generated by topologically cyclic subgroups [i.e., "copies of $\mathbb{Z}_p$"] on which some open subgroup of J_1 [which may depend on the cyclic subgroup] acts via the *cyclotomic character*. Next, let us observe that if V_3 is any J_3-module associated to a character *of qLT-type of dimension* ≥ 2, then V_3 *does not contain any sub-*J_3*-modules of dimension* 1 *over* $\mathbb{Q}_p$. From this observation, it follows immediately that *any subquotient* [cf. Definition 3.1, (v)] of the J_3-module $H_3^{\text{ab}} \otimes \mathbb{Q}_p$ that is isomorphic to the J_3-module associated to a character *of qLT-type of dimension* ≥ 2 determines a subquotient [not only of the J_1-module $H_1^{\text{ab-t}} \otimes \mathbb{Q}_p$, but also] of the J_1-module $Q_1 \otimes \mathbb{Q}_p$. Thus, we conclude that any such subquotient of the J_1-module $Q_1 \otimes \mathbb{Q}_p$ is *Hodge-Tate with weights* $\in \{0, 1\}$. Moreover, by considering *determinants* of such subquotients, one concludes that the pull-back of the cyclotomic character $J_3 \to \mathbb{Z}_p^\times$ is a character $J_1 \to \mathbb{Z}_p^\times$ which is *Hodge-Tate*, and whose *unique weight* w is ≥ 0. If $w \geq 2$, then the fact that the J_3-module determined by the cyclotomic character of J_3 occurs as a subquotient of $H_3^{\text{ab}} \otimes \mathbb{Q}_p$ [for sufficiently small H_3], hence determines a J_1-module that occurs as a subquotient [not only of the J_1-module $H_1^{\text{ab-t}} \otimes \mathbb{Q}_p$, but also, *in light of our assumption* that $w \geq 2$!] of the J_1-module $Q_1 \otimes \mathbb{Q}_p$ leads to a *contradiction* [since the J_1-module $Q_1 \otimes \mathbb{Q}_p$ is Hodge-Tate with weights $\in \{0, 1\}$]. Thus, we conclude that $\phi : G_1 \to G_2 = G_3$ is *of 01-qLT-type*, hence *geometric*, i.e., arises from a unique embedding of fields $k_2 \hookrightarrow k_1$ of finite degree. Finally, the geometricity of ϕ implies that the geometricity of ψ may be derived from the *"relative"* result given in [Mzk3], Theorem B. ○

Remark 3.9.1. The proof given above of Corollary 3.9 shows that the "Π_2" of Corollary 3.9 may, in fact, be taken to be a "Π_2" as in Corollary 3.8, (g).

Section 4: Chains of Elementary Operations

In the present §4, we generalize [cf. Theorems 4.7, 4.12; Remarks 4.7.1, 4.12.1 below] the theory of *"categories of dominant localizations"* discussed in [Mzk9], §2 [cf. also the *tempered versions* of these categories, discussed in [Mzk10], §6], to include "localizations" obtained by more general *"chains of elementary operations"* — i.e., the operations of passing to a *finite étale covering*, passing to a *finite étale quotient*, *"de-cuspidalization"*, and *"de-orbification"* [cf. Definition 4.2 below; [Mzk14], §2] — which are applied to some given algebraic stack over a field. The field and algebraic stack under consideration are *quite general* in nature [by comparison, e.g., to the theory of [Mzk9], §2; [Mzk14], §2], but are subject to various assumptions. One *key* assumption asserts that the algebraic stack satisfies a certain *relative version of the "Grothendieck Conjecture"*.

Before proceeding, we recall the following immediate consequence of [Mzk14], Lemma 2.1; [Mzk13], Proposition 1.2, (ii).

Lemma 4.1. **(Decomposition Groups of Hyperbolic Orbicurves)** *Let* Σ *be a nonempty set of prime numbers,* Δ *a* **pro-**Σ *group* **of GFG-type** *that admits* **base-prime** *[cf. Definition 2.1, (iv)] partial construction data* (k, X, Σ) *[consisting of the construction data field, construction data base-stack, and construction data prime set] such that* X *is a* **hyperbolic orbicurve** *[cf. §0], and* k *is* **algebraically closed**. *Let* x_A *(respectively,* $x_B \neq x_A$*) be either a* **closed point** *or a* **cusp** *[cf. §0] of* X*;* $A \subseteq \Delta$ *(respectively,* $B \subseteq \Delta$*) the* **decomposition group** *[well-defined up to conjugation in* Δ*] of* x_A *(respectively,* x_B*). Then:*

(i) A, B *are* **pro-cyclic** *groups;* $A \bigcap B = \{1\}$. *If* x_A *is a* **closed point** *of* X*, and* $A \neq \{1\}$*, then* A *is a* **finite**, **normally terminal** *[cf. §0] subgroup of* Δ. *If* x_A *is a* **cusp**, *then* A *is a* **torsion-free**, **commensurably terminal** *[cf. §0]* **infinite** *subgroup of* Δ.

(ii) The order of **every finite cyclic closed subgroup** $C \subseteq \Delta$ *divides the* **order of** X *[cf. §0].*

(iii) Every **finite nontrivial closed subgroup** $C \subseteq \Delta$ *is contained in a* **decomposition group of a unique closed point** *of* X. *In particular, the nontrivial decomposition groups of closed points of* X *may be characterized [***"group-theoretically"***] as the* **maximal finite nontrivial closed subgroups** *of* Δ.

(iv) X *is a* **hyperbolic curve** *if and only if* Δ *is* **torsion-free**.

(v) Suppose that the quotient $\psi_A : \Delta \twoheadrightarrow \Delta_A$ *of* Δ *by the closed normal subgroup of* Δ *topologically generated by* A *is* **slim** *and* **nontrivial**. *If* x_A *is a* **closed point** *of* X *(respectively, a* **cusp***), then we suppose further that* $\Sigma = \mathfrak{Primes}$ *[which forces the characteristic of* k *to be zero] (respectively, that* $A \subseteq J$ *for some normal open* **torsion-free** *subgroup* J *of* Δ*). Then* Δ_A *is a profinite group* **of GFG-type** *that admits* **base-prime** *partial construction data* (k, X_A, Σ) *[consisting of the construction data field, construction data base-stack, and construction data prime set]*

such that X_A *is a* **hyperbolic orbicurve** *equipped with a dominant* k*-morphism* $\phi_A : X \to X_A$ *that is* **uniquely** *determined [up to a unique isomorphism] by the property that it induces [up to composition with an inner automorphism]* ψ_A*. Moreover, if* x_A *is a* **closed point** *of* X *(respectively, a* **cusp***), then* ϕ_A *is a* **partial coarsification morphism** *[cf. §0] which is an* **isomorphism** *either over* X_A *or over the complement in* X_A *of the point of* X_A *determined by* x_A *(respectively, is an* **open immersion** *whose image is the complement of the point of* X_A *determined by* x_A*).*

(vi) In the notation of (v), if $B \neq \{1\}$*, then* $\psi_A(B) \neq \{1\}$*.*

Proof. First, we recall that by the definition of a profinite group of *GFG-type* [cf. the discussion at the beginning of §2], it follows that there exists a normal open subgroup $H \subseteq \Delta$ such that if we write $X_H \to X$ for the corresponding Galois covering, then X_H is a *hyperbolic curve*. Next, let us observe that, in light of our assumption that the partial construction data is *base-prime*, we may lift the entire situation to *characteristic zero*, hence assume, at least for the proof of assertions (i), (ii), (iii), (iv), that k is *of characteristic zero*. Thus, assertions (i), (ii), (iii) when x_A, x_B are *closed points* (respectively, *cusps*) of X follow immediately from [Mzk14], Lemma 2.1 (respectively, [Mzk13], Proposition 1.2, (ii)). Next, we consider assertion (iv). First, we observe that the *necessity* portion of assertion (iv) follows immediately from assertion (iii). To verify *sufficiency*, let us suppose that Δ is *torsion-free*. Let $\pi_1^{\text{tame}}(X) \twoheadrightarrow \Delta$ be a *scheme-theoretic envelope* of Δ. Then since X_H is a *scheme*, it follows that the nontrivial [finite closed] subgroups of $\pi_1^{\text{tame}}(X)$ that arise as decomposition groups of closed points map *injectively*, via the composite surjection $\pi_1^{\text{tame}}(X) \twoheadrightarrow \Delta \twoheadrightarrow \Delta/H$, into Δ/H, hence, *a fortiori*, injectively via the surjection $\pi_1^{\text{tame}}(X) \twoheadrightarrow \Delta$, into Δ [which is torsion-free]. Thus, the decomposition groups in $\pi_1(X) = \pi_1^{\text{tame}}(X)$ [cf. our assumption that k is algebraically closed of characteristic zero] of closed points of X are *trivial*. But this implies [by considering, for instance, the Galois covering $X_H \to X$] that X is a *scheme*, as desired. This completes the proof of assertion (iv).

Next, we consider assertion (v). First, let us observe that X_A admits a finite étale covering $Y_A \to X_A$ arising from a *normal open subgroup* of Δ_A such that Y_A is a *curve*, which will necessarily be *hyperbolic*, in light of the *slimness* and *nontriviality* of Δ_A. Indeed, when x_A is a *closed point* of X [so $\Sigma = \mathfrak{Primes}$; k is of characteristic zero], this follows immediately from the *equivalence* of definitions of a "hyperbolic orbicurve" discussed in §0; when x_A is a *cusp*, this follows from assertion (iv) and our assumption of the existence of the subgroup $J \subseteq \Delta$. Now the remainder of assertion (v) follows immediately from the definitions. This completes the proof of assertion (v). Finally, we consider assertion (vi). Assertion (vi) is immediate if x_B is a *cusp* [cf. assertion (i)]; thus, we may assume that x_B is a *closed point* of X. If $\psi_A(B) = \{1\}$, then it follows that the decomposition group $\subseteq \Delta_A$ of the image of x_B in X_A is *trivial*. Since [by assertion (v)] X_A admits a finite étale covering $Y_A \to X_A$ arising from an open subgroup of X_A such that Y_A is a *hyperbolic curve*, we thus conclude that X_A is *scheme-like* in a neighborhood of the image of x_B in X_A, hence [in light of the explicit description of the morphism ϕ_A in the statement of assertion (v)] that X is *scheme-like* in a neighborhood of x_B. But this implies that $B = \{1\}$. This completes the proof of assertion (vi). ○

Remark 4.1.1. Note that Lemma 4.1, (iv), is *false* if we only assume that Δ is *almost pro-*Σ. Indeed, such an example may be constructed by taking X to be a *hyperbolic curve* over an algebraically closed field k of characteristic zero, $Y \to X$ a finite étale Galois covering of degree prime to Σ, and Δ to be the quotient of $\pi_1(X)$ by the kernel of the surjection $(\pi_1(X) \supseteq)\ \pi_1(Y) \twoheadrightarrow \pi_1(Y)^{(\Sigma)}$ to the *maximal pro-*Σ *quotient* $\pi_1(Y)^{(\Sigma)}$ of $\pi_1(Y)$. Then for any prime p dividing the order of $\mathrm{Gal}(Y/X)$ [so $p \notin \Sigma$], it follows by considering Sylow p-subgroups that Δ contains an *element of order* p, despite the fact that X is a *curve*.

Definition 4.2. Let G be a *slim* profinite group;

$$1 \to \Delta \to \Pi \to G \to 1$$

an *extension of GSAFG-type* that admits *base-prime* partial construction data (k, X, Σ), where $\Sigma \neq \emptyset$; $\alpha : \pi_1^{\mathrm{tame}}(X) \twoheadrightarrow \Pi$ a *scheme-theoretic envelope*. Thus, if we write $\pi_1^{\mathrm{tame}}(X) \twoheadrightarrow G_k$ for the quotient given by the absolute Galois group G_k of k, then α determines a scheme-theoretic envelope $\beta : G_k \twoheadrightarrow G$. Write $\widetilde{X} \to X$ for the *pro-finite étale covering* of X determined by the surjection α; $\widetilde{k}$ for the resulting field extension of k. In a similar vein, we shall write $\widetilde{\Pi}$ for the *projective system of profinite groups* determined by the open subgroups of Π. [Thus, one may consider homomorphisms between $\widetilde{\Pi}$ and a profinite group by thinking of the profinite group as a trivial projective system of profinite groups — cf. the theory of *"pro-anabelioids"*, as in [Mzk8], Definition 1.2.6.] Then:

(i) We shall refer to as an *[*$\widetilde{X}/X$*-]chain [of length* n*]* [where $n \geq 0$ is an integer] any finite sequence

$$X_0 \rightsquigarrow X_1 \rightsquigarrow \ldots \rightsquigarrow X_{n-1} \rightsquigarrow X_n$$

of generically scheme-like algebraic stacks X_j [for $j = 0, \ldots, n$], each equipped with a dominant *"rigidifying morphism"* $\rho_j : \widetilde{X} \to X_j$ satisfying the following conditions:

(0_X) $X_0 = X$ [equipped with its natural rigidifying morphism $\widetilde{X} \to X$].

(1_X) There exists a [uniquely determined] morphism $X_j \to \mathrm{Spec}(k_j)$ compatible with ρ_j, where $k_j \subseteq \widetilde{k}$ is a finite extension of k such that X_j is *geometrically connected* over k_j.

(2_X) Each ρ_j determines a *maximal pro-finite étale covering* $\widetilde{X}_j \to X_j$ such that $\widetilde{X} \to X_j$ admits a factorization $\widetilde{X} \to \widetilde{X}_j \to X_j$. The kernel Δ_j of the resulting natural surjection

$$\Pi_j \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{X}_j/X_j) \twoheadrightarrow G_j \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\widetilde{k}/k_j)$$

is *slim* and *nontrivial*; every prime dividing the order of a finite quotient group of Δ_j is *invertible* in k.

(3_X) Suppose that X is a *hyperbolic orbicurve* [over k]. Then each X_j is also a *hyperbolic orbicurve* [over k_j]. Moreover, each Δ_j is a *pro*-Σ group.

(4_X) Each "$X_j \rightsquigarrow X_{j+1}$" [for $j = 0, \dots, n-1$] is an *"elementary operation"*, as defined below.

Here, an *elementary operation* "$X_j \rightsquigarrow X_{j+1}$" is defined to consist of the datum of a dominant *"operation morphism"* ϕ either from X_j to X_{j+1} or from X_{j+1} to X_j which is *compatible* with ρ_j, ρ_{j+1}, and, moreover, is of one of the following four types:

(a) *Type* $\curlywedge$: In this case, the elementary operation $X_j \rightsquigarrow X_{j+1}$ consists of a *finite étale covering* $\phi : X_{j+1} \to X_j$. Thus, ϕ determines an open immersion of profinite groups $\Pi_{j+1} \hookrightarrow \Pi_j$.

(b) *Type* $\curlyvee$: In this case, the elementary operation $X_j \rightsquigarrow X_{j+1}$ consists of a finite étale morphism $\phi : X_j \to X_{j+1}$ — i.e., a *"finite étale quotient"*. Thus, ϕ determines an open immersion of profinite groups $\Pi_j \hookrightarrow \Pi_{j+1}$.

(c) *Type* $\bullet$: This type of elementary operation is only defined if X is a *hyperbolic orbicurve*. In this case, the elementary operation $X_j \rightsquigarrow X_{j+1}$ consists of an open immersion $\phi : X_j \hookrightarrow X_{j+1}$ [so $k_j = k_{j+1}$] — i.e., a *"de-cuspidalization"* — such that the image of ϕ is the complement of a single k_{j+1}-valued point of X_{j+1} whose decomposition group in Δ_j is contained in some normal open *torsion-free* subgroup of Δ_j. Thus, ϕ determines a surjection of profinite groups $\Pi_j \twoheadrightarrow \Pi_{j+1}$.

(d) *Type* $\circledcirc$: This type of elementary operation is only defined if X is a *hyperbolic orbicurve* and $\Sigma = \mathfrak{Primes}$ [which forces the characteristic of k to be zero]. In this case, the elementary operation $X_j \rightsquigarrow X_{j+1}$ consists of a partial coarsification morphism [cf. §0] $\phi : X_j \to X_{j+1}$ [so $k_j = k_{j+1}$] — i.e., a *"de-orbification"* — such that ϕ is an *isomorphism* over the *complement* in X_{j+1} of some k_{j+1}-valued point of X_{j+1}. Thus, ϕ determines a surjection of profinite groups $\Pi_j \twoheadrightarrow \Pi_{j+1}$.

Thus, any $\widetilde{X}/X$-chain determines a *sequence of symbols* $\in \{\curlywedge, \curlyvee, \bullet, \circledcirc\}$ [corresponding to the types of elementary operations in the $\widetilde{X}/X$-chain], which we shall refer to as the *type-chain* associated to the $\widetilde{X}/X$-chain.

(ii) An *isomorphism* between two $\widetilde{X}/X$-chains with *identical type-chains* [hence of the same length]

$$(X_0 \rightsquigarrow \dots \rightsquigarrow X_n) \xrightarrow{\sim} (Y_0 \rightsquigarrow \dots \rightsquigarrow Y_n)$$

is defined to be a collection of isomorphisms of generically scheme-like algebraic stacks $X_j \xrightarrow{\sim} Y_j$ [for $j = 0, \dots, n$] that are compatible with the rigidifying morphisms. [Here, we note that the condition of *compatibility* with the rigidifying morphisms implies that every *automorphism* of a $\widetilde{X}/X$-chain is given by the *identity*, and that every isomorphism of $\widetilde{X}/X$-chains of the same length is *compatible* with the respective *operation morphisms*.] Thus, one obtains a *category*

$$\mathrm{Chain}(\widetilde{X}/X)$$

whose *objects* are the $\widetilde{X}/X$-chains [with arbitrary associated type-chain], and whose *morphisms* are the isomorphisms between $\widetilde{X}/X$-chains [with identical type-chains]. A *terminal morphism* between two $\widetilde{X}/X$-chains [with arbitrary associated type-chain]

$$(X_0 \rightsquigarrow \dots \rightsquigarrow X_n) \to (Y_0 \rightsquigarrow \dots \rightsquigarrow Y_m)$$

is defined to be a dominant k-morphism $X_n \to Y_m$. Thus, one obtains a *category*

$$\mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X)$$

whose *objects* are the $\widetilde{X}/X$-chains [with arbitrary associated type-chain], and whose *morphisms* are the *terminal morphisms* between $\widetilde{X}/X$-chains; write

$$\mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X) \subseteq \mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X)$$

for the subcategory determined by the *terminal isomorphisms* [i.e., the isomorphisms of $\mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X)$]. Thus, it follows immediately from the definitions that we obtain *natural functors* $\mathrm{Chain}(\widetilde{X}/X) \to \mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X) \to \mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X)$.

(iii) We shall refer to as an *[Π-]chain [of length n]* [where $n \geq 0$ is an integer] any finite sequence

$$\Pi_0 \rightsquigarrow \Pi_1 \rightsquigarrow \dots \rightsquigarrow \Pi_{n-1} \rightsquigarrow \Pi_n$$

of slim profinite groups Π_j [for $j = 0, \dots, n$], each equipped with an open *"rigidifying homomorphism"* $\rho_j : \widetilde{\Pi} \to \Pi_j$ [i.e., since we are working with *slim* profinite groups, an open homomorphism from some open subgroup of Π to Π_j] satisfying the following conditions:

(0_Π) $\Pi_0 = \Pi$ [equipped with its natural rigidifying homomorphism $\widetilde{\Pi} \to \Pi$].

(1_Π) There exists a [uniquely determined] surjection $\Pi_j \twoheadrightarrow G_j$ compatible with ρ_j, where $G_j \subseteq G$ is an open subgroup.

(2_Π) Each kernel

$$\Delta_j \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_j \twoheadrightarrow G_j \hookrightarrow G)$$

is *slim* and *nontrivial*; every prime dividing the order of a finite quotient group of Δ_j is *invertible* in k.

(3_{Π}) Suppose that X is a *hyperbolic orbicurve* [over k]. Then each Δ_j is a *pro*-Σ group. Also, we shall refer to as a *cuspidal decomposition group* in Δ_j any *commensurator* in Δ_j of the image via ρ_j of the inverse image in $\widetilde{\Pi}$ of the decomposition group in Δ [determined by α] of a *cusp* of X.

(4_{Π}) Each "$\Pi_j \rightsquigarrow \Pi_{j+1}$" [for $j = 0, \ldots, n-1$] is an *"elementary operation"*, as defined below.

Here, an *elementary operation* "$\Pi_j \rightsquigarrow \Pi_{j+1}$" is defined to consist of the datum of an open *"operation homomorphism"* ϕ either from Π_j to Π_{j+1} or from Π_{j+1} to Π_j which is *compatible* with ρ_j, ρ_{j+1}, and, moreover, is of one of the following four types:

(a) *Type* $\curlywedge$: In this case, the elementary operation $\Pi_j \rightsquigarrow \Pi_{j+1}$ consists of an *open immersion of profinite groups* $\phi : \Pi_{j+1} \hookrightarrow \Pi_j$.

(b) *Type* $\curlyvee$: In this case, the elementary operation $\Pi_j \rightsquigarrow \Pi_{j+1}$ consists of an *open immersion of profinite groups* $\phi : \Pi_j \hookrightarrow \Pi_{j+1}$.

(c) *Type* $\bullet$: This type of elementary operation is only defined if X is a *hyperbolic orbicurve*. In this case, the elementary operation $\Pi_j \rightsquigarrow \Pi_{j+1}$ consists of a *surjection of profinite groups* $\phi : \Pi_j \twoheadrightarrow \Pi_{j+1}$, such that $\mathrm{Ker}(\phi)$ is topologically normally generated by a *cuspidal decomposition group* C in Δ_j such that C is contained in some normal open *torsion-free* subgroup of Δ_j.

(d) *Type* $\circledcirc$: This type of elementary operation is only defined if X is a *hyperbolic orbicurve* and $\Sigma = \mathfrak{Primes}$ [which forces the characteristic of k to be zero]. In this case, the elementary operation $\Pi_j \rightsquigarrow \Pi_{j+1}$ consists of a *surjection of profinite groups* $\phi : \Pi_j \twoheadrightarrow \Pi_{j+1}$, such that $\mathrm{Ker}(\phi)$ is topologically normally generated by a *finite closed subgroup* of Δ_j.

Thus, any Π-chain determines a *sequence of symbols* $\in \{\curlywedge, \curlyvee, \bullet, \circledcirc\}$ [corresponding to the types of elementary operations in the Π-chain], which we shall refer to as the *type-chain* associated to the Π-chain.

(iv) An *isomorphism* between two Π-chains with *identical type-chains* [hence of the same length]

$$(\Pi_0 \rightsquigarrow \ldots \rightsquigarrow \Pi_n) \xrightarrow{\sim} (\Psi_0 \rightsquigarrow \ldots \rightsquigarrow \Psi_n)$$

is defined to be a collection of isomorphisms of profinite groups $\Pi_j \xrightarrow{\sim} \Psi_j$ [for $j = 0, \ldots, n$] that are compatible with the rigidifying homomorphisms. [Here, we note that the condition of *compatibility* with the rigidifying homomorphisms implies [since all of the profinite groups involved are *slim*] that every *automorphism* of a Π-chain is given by the *identity*, and that every isomorphism of Π-chains of the same length is *compatible* with the respective *operation homomorphisms*.] Thus, one obtains a *category*

$$\mathrm{Chain}(\Pi)$$

whose *objects* are the Π-chains [with arbitrary associated type-chain], and whose *morphisms* are the isomorphisms between Π-chains [with identical type-chains]. A *terminal homomorphism* between two Π-chains [with arbitrary associated type-chain]

$$(\Pi_0 \rightsquigarrow \ldots \rightsquigarrow \Pi_n) \to (\Psi_0 \rightsquigarrow \ldots \rightsquigarrow \Psi_m)$$

is defined to be an open outer homomorphism $\Pi_n \to \Psi_m$ that is *compatible* [up to composition with an inner automorphism] with the open homomorphisms $\Pi_n \to G$, $\Psi_m \to G$. Thus, one obtains a *category*

$$\mathrm{Chain}^{\mathrm{trm}}(\Pi)$$

whose *objects* are the Π-chains [with arbitrary associated type-chain], and whose *morphisms* are the *terminal homomorphisms* between Π-chains; write

$$\mathrm{Chain}^{\text{iso-trm}}(\Pi) \subseteq \mathrm{Chain}^{\mathrm{trm}}(\Pi)$$

for the subcategory determined by the *terminal isomorphisms* [i.e., the isomorphisms of $\mathrm{Chain}^{\mathrm{trm}}(\Pi)$]. Thus, it follows immediately from the definitions that we obtain *natural functors* $\mathrm{Chain}(\Pi) \to \mathrm{Chain}^{\text{iso-trm}}(\Pi) \to \mathrm{Chain}^{\mathrm{trm}}(\Pi)$.

(v) We shall use the notation

$$\mathrm{Chain}^{\text{iso-trm}}(\sim)\{-\} \subseteq \mathrm{Chain}^{\text{iso-trm}}(\sim); \quad \mathrm{Chain}^{\mathrm{trm}}(\sim)\{-\} \subseteq \mathrm{Chain}^{\mathrm{trm}}(\sim)$$

— where "$(\sim)$" is either equal to "$(\widetilde{X}/X)$" or "(Π)", and "$\{-\}$" contains some subset of the set of symbols $\{\curlywedge, \curlyvee, \bullet, \circledcirc\}$ — to denote the respective *full subcategories* determined by the chains whose associated type-chain *only contains the symbols* that belong to "$\{-\}$". In particular, we shall write:

$$\mathrm{DLoc}(\widetilde{X}/X) \stackrel{\mathrm{def}}{=} \mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X)\{\curlywedge, \bullet\}; \quad \mathrm{DLoc}(\Pi) \stackrel{\mathrm{def}}{=} \mathrm{Chain}^{\mathrm{trm}}(\Pi)\{\curlywedge, \bullet\}$$

$$\text{ÉtLoc}(\widetilde{X}/X) \stackrel{\mathrm{def}}{=} \mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X)\{\curlywedge, \curlyvee\}; \quad \text{ÉtLoc}(\Pi) \stackrel{\mathrm{def}}{=} \mathrm{Chain}^{\text{iso-trm}}(\Pi)\{\curlywedge, \curlyvee\}$$

[cf. the theory of [Mzk9], §2; Remark 4.7.1 below].

Remark 4.2.1. Thus, it follows immediately from the definitions that if, in the notation of Definition 4.2, (i),

$$X_0 \rightsquigarrow X_1 \rightsquigarrow \ldots \rightsquigarrow X_{n-1} \rightsquigarrow X_n$$

is an $\widetilde{X}/X$*-chain*, then the resulting profinite groups Π_j determine a Π-*chain*

$$\Pi_0 \rightsquigarrow \Pi_1 \rightsquigarrow \ldots \rightsquigarrow \Pi_{n-1} \rightsquigarrow \Pi_n$$

with the *same associated type-chain*. In particular, we obtain *natural functors*

$$\mathrm{Chain}(\widetilde{X}/X) \to \mathrm{Chain}(\Pi)$$

$$\mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X) \to \mathrm{Chain}^{\text{iso-trm}}(\Pi); \quad \mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X) \to \mathrm{Chain}^{\mathrm{trm}}(\Pi)$$

which are *compatible* with the natural functors of Definition 4.2, (ii), (iv).

Remark 4.2.2. Note that in the situation of Definition 4.2, (i), G_j is a *slim* profinite group; $1 \to \Delta_j \to \Pi_j \to G_j \to 1$ is an *extension of GSAFG-type* that admits *base-prime* partial construction data (k_j, X_j, Σ), where X_j is a *hyperbolic orbicurve* whenever X_0 is a hyperbolic orbicurve; α, ρ_j determine [in light of the *slimness* of Π_j] a *scheme-theoretic envelope* $\alpha_j : \pi_1^{\text{tame}}(X_j) \twoheadrightarrow \Pi_j$. That is to say, we obtain, for each j, **similar data** to the data introduced at the beginning of Definition 4.2.

Proposition 4.3. **(Re-ordering of Chains)** *In the notation of Definition 4.2, suppose that* $\Sigma = \mathfrak{Primes}$*; let* $X_0 \rightsquigarrow \ldots \rightsquigarrow X_n$ *(respectively,* $\Pi_0 \rightsquigarrow \ldots \rightsquigarrow \Pi_n$*) be a(n)* $\widetilde{X}/X$*- (respectively,* Π*-)* **chain**. *Then there exists a* **terminally isomorphic** $\widetilde{X}/X$*- (respectively,* Π*-) chain* $Y_0 \rightsquigarrow \ldots \rightsquigarrow Y_m$ *(respectively,* $\Psi_0 \rightsquigarrow \ldots \rightsquigarrow \Psi_m$*) whose associated* **type-chain** *is of the form*

$$\curlywedge, \bullet, \bullet, \ldots, \bullet, (\in \{\curlyvee, \circledcirc\}), (\in \{\curlyvee, \circledcirc\}), \ldots$$

— i.e., consists of the **symbol** $\curlywedge$*, followed by a sequence of the* **symbols** $\bullet$*, followed by a sequence of* **symbols** $\in \{\curlyvee, \circledcirc\}$.

Proof. Indeed, let us first observe that it is immediate from the *compatibility with the rigidifying [homo]morphisms* that one may always "move the symbol $\curlywedge$ to the *top* of the type-chain". Thus, one may assume without loss of generality that the remaining symbols [i.e., the symbols indexed by $j \geq 1$] of the type-chain are $\in \{\curlyvee, \bullet, \circledcirc\}$; in particular, one may assume that the *operation [homo]morphisms* indexed by $j \geq 1$ always have *domain* indexed by j and *codomain* indexed by $j+1$. Thus, we may replace the *de-cuspidalization* operations at arbitrary indices by de-cuspidalization operations on the finite étale covering indexed by 0 determined by cusps of this finite étale covering that map to cusps that give rise to de-cuspidalization operations at subsequent indices. [Note that here, it is useful to recall the *equivalence* of definitions of the notion of a "hyperbolic orbicurve" discussed in §0 — cf. our assumption that $\Sigma = \mathfrak{Primes}$.] This yields a type-chain of the desired form. ◯

On the other hand, as the following example shows, the symbols "$\curlyvee$", "$\circledcirc$" *cannot* be permuted.

Example 4.4. **Non-permutability of Étale Quotients and De-orbifications.** In the notation of Definition 4.2, let us assume further $\Sigma = \mathfrak{Primes}$ [so k is of *characteristic zero*]. Then there exists an $\widetilde{X}/X$-chain $X_0 \rightsquigarrow X_1 \rightsquigarrow X_2$ of length 2 with associated type-chain $*_1, *_2$, where $*_1, *_2 \in \{\curlyvee, \circledcirc\}$, $*_1 \neq *_2$, which is *not terminally isomorphic* to any $\widetilde{X}/X$-chain $Y_0 \rightsquigarrow Y_1 \rightsquigarrow Y_2$ of length 2 with associated type-chain $*_2, *_1$. Indeed:

(i) *The case of* **type-chain** $\curlyvee, \circledcirc$: Let X be a *hyperbolic curve* of type (g, r) over k equipped with an automorphism σ of the k-scheme X of *order* 2 that has *precisely one fixed point* $x \in X(k)$; $X_0 = X \rightsquigarrow X_1$ the elementary operation of type $\curlyvee$ given by forming the stack-theoretic quotient of X by the action of σ; $x_1 \in X_1(k)$ the image of x in X_1; $X_1 \rightsquigarrow X_2$ the elementary operation of type $\circledcirc$ determined by the point $x_1 \in X_1(k)$. Thus, we assume that X_2 is a *hyperbolic curve*, whose type we denote by (g_2, r_2). On the other hand, since X is a *scheme*, any chain $Y_0 \rightsquigarrow Y_1 \rightsquigarrow Y_2$ of length 2 with associated type-chain $\circledcirc, \curlyvee$ satisfies $Y_0 \xrightarrow{\sim} Y_1$ [compatibly with $\widetilde{X}$]. Thus, if $Y_2 \xrightarrow{\sim} X_2$ over k, then the coverings $X = X_0 \to X_2$ [which is ramified, of degree 2], $X \xrightarrow{\sim} Y_0 \xrightarrow{\sim} Y_1 \to Y_2$ [which is unramified, of some degree d] yield equations

$$d \cdot \chi_2 = \chi = 2 \cdot \chi_2 + 1$$

[where we write $\chi \stackrel{\mathrm{def}}{=} 2g - 2 + r$, $\chi_2 \stackrel{\mathrm{def}}{=} 2g_2 - 2 + r_2$] — which imply [since d, χ, χ_2 are positive integers] that $d - 2 = \chi_2 = 1$, hence that $d = 3$, $\chi_2 = 1$, $\chi = 3$. In particular, by choosing X so that χ is > 3 [e.g., X such that $g \geq 3$], we obtain a *contradiction.*

(ii) *The case of* **type-chain** $\circledcirc, \curlyvee$: Let X be a *proper hyperbolic orbicurve* over k; $X \to C$ the *coarse space* associated to the algebraic stack X. Let us assume further that C is a *[proper] hyperbolic curve* over k; that the morphism $X \to C$ is a *non-isomorphism* which restrict to an *isomorphism* away from some point $c \in C(k)$; and that there exists a *finite étale covering* $\epsilon : C \to D$ of *degree* 2 [so D is also a proper hyperbolic curve over k, which is *not* isomorphic to C]. [It is easy to construct such objects by *starting from* D and then constructing C, X.] Now we take $X_0 = X \rightsquigarrow X_1 \stackrel{\mathrm{def}}{=} C$ to be the elementary operation of type $\circledcirc$ determined by the unique point of $x \in X(k)$ lying over $c \in C(k)$; $C = X_1 \rightsquigarrow X_2 \stackrel{\mathrm{def}}{=} D$ to be the elementary operation of type $\curlyvee$ determined by the finite étale covering $\epsilon : C \to D$. On the other hand, let us suppose that $Y_0 \rightsquigarrow Y_1 \rightsquigarrow Y_2$ is a chain of length 2 with associated type-chain $\curlyvee, \circledcirc$ such that $X_2 \xrightarrow{\sim} Y_2$ over k. Then since $D = X_2 \xrightarrow{\sim} Y_2$ is a *scheme*, it follows that the hyperbolic orbicurve Y_1 admits a point $y_1 \in Y_1(k)$ such that Y_1 is a *scheme* away from y_1. Note that if Y_1 is a scheme, then the finite étale covering $X = Y_0$ of Y_1 is as well, a *contradiction.* Thus, we conclude that Y_1 is *not* a scheme at y_1. Next, let us observe that if the finite étale morphism $Y_0 \to Y_1$ is *not* an isomorphism [i.e., of degree ≥ 2], then Y_0 *fails* to be a scheme at some k-étale divisor of Y_0 [namely, the inverse image of y_1] of degree ≥ 2; thus, since $Y_0 = X$ in fact fails to be a scheme only at the unique point $x \in X(k)$, we thus conclude that this finite étale covering is, in fact, an *isomorphism* $X = Y_0 \xrightarrow{\sim} Y_1$. But this implies that Y_2 is isomorphic to the coarse space associated to X, i.e., we have an isomorphism $Y_2 \xrightarrow{\sim} C$, hence an isomorphism $D = X_2 \xrightarrow{\sim} Y_2 \xrightarrow{\sim} C$, a *contradiction.*

Next, we recall the *group-theoretic characterization of the cuspidal decomposition groups* of a hyperbolic [orbi]curve given in [Mzk13].

Lemma 4.5. **(Cuspidal Decomposition Groups)** *Let G be a* **slim** *profinite group;*

$$1 \to \Delta \to \Pi \to G \to 1$$

an **extension of GSAFG-type** *that admits* **base-prime** *[cf. Definition 2.1, (iv)] partial construction data* $(k, \widetilde{k}, X, \Sigma)$*, where* X *is a* **hyperbolic orbicurve***;* $\alpha : \pi_1^{\mathrm{tame}}(X) \twoheadrightarrow \Pi$ *a* **scheme-theoretic envelope***;* $l \in \Sigma$ *a prime such that the* **cyclotomic character** $\chi_G^{\mathrm{cyclo}} : G \to \mathbb{Z}_l^\times$ *[i.e., the character whose composite with* α *is the usual cyclotomic character* $\pi_1^{\mathrm{tame}}(X) \twoheadrightarrow \mathrm{Gal}(\widetilde{k}/k) \to \mathbb{Z}_l^\times$*] has* **open image** *[i.e., in the terminology of [Mzk13], "the outer action of* G *on* Δ *is* **l-cyclotomically full***"]. We recall from [Mzk13] that a character* $\chi : G \to \mathbb{Z}_l^\times$ *is called* $\mathbb{Q}$**-cyclotomic** **[of weight** $w \in \mathbb{Q}$**]** *if there exist integers* a, b*, where* $b > 0$*, such that* $\chi^b = (\chi_G^{\mathrm{cyclo}})^a$*,* $w = 2a/b$ *[cf. [Mzk13], Definition 2.3, (i), (ii)]. Then:*

(i) X *is* **non-proper** *if and only if every torsion-free pro-*Σ *open subgroup of* Δ *is* **free pro-**Σ*.*

(ii) Let M *be a finite-dimensional* $\mathbb{Q}_l$*-vector space equipped with a continuous* G*-action. Then we shall say that this action is* **quasi-trivial** *if it factors through a finite quotient of* G *[cf. [Mzk13], Definition 2.3, (i)]. We shall write* $\tau(M)$ *for the* $\mathbb{Q}_l$*-dimension of the* **maximal quasi-trivial** $\mathbb{Q}_l$**-subspace** *of* M*. If* $\chi : G \to \mathbb{Z}_l^\times$ *is a* **character***, then we shall write*

$$d_\chi(M) \stackrel{\mathrm{def}}{=} \tau(M(\chi^{-1})) - \tau(\mathrm{Hom}_{\mathbb{Q}_l}(M, \mathbb{Q}_l))$$

[where "$M(\chi^{-1})$*" denotes the result of "twisting"* M *by the character* χ^{-1}*]. We shall say that two characters* $G \to \mathbb{Z}_l^\times$ *are* **power-equivalent** *if there exists a positive integer* n *such that the* n*-th powers of the two characters coincide. Then* $d_\chi(M)$*, regarded as a* **function of** χ*, depends only on the* **power-equivalence class** *of* χ*.*

(iii) Suppose that X *is* **not proper** *[cf. (i)]. Then the character* $G \to \mathbb{Z}_l^\times$ *arising from the* **determinant** *of the* G*-module* $H^{\mathrm{ab}} \otimes \mathbb{Q}_l$*, where* $H \subseteq \Delta$ *is a characteristic open subgroup such that* $H^{\mathrm{ab}} \otimes \mathbb{Q}_l \neq 0$*, is* $\mathbb{Q}$**-cyclotomic of positive weight***. Moreover, for every* **sufficiently small** *characteristic open subgroup* $H \subseteq \Delta$*, the power-equivalence class of the* **cyclotomic character** χ_G^{cyclo} *may be characterized as the unique power-equivalence class of characters* $\chi : G \to \mathbb{Z}_l^\times$ *of the form* $\chi = \chi^* \cdot \chi_*$*, where* $\chi^* : G \to \mathbb{Z}_l^\times$ *(respectively,* $\chi_* : G \to \mathbb{Z}_l^\times$*) is a* $\mathbb{Q}$*-cyclotomic character of* **maximal** *(respectively,* **minimal***) weight such that* $\tau(M(\chi^{-1})) \neq 0$ *for some subquotient* G*-module* M *of* $(H^{\mathrm{ab}} \otimes \mathbb{Q}_l) \oplus \mathbb{Q}_l$ *[where the final direct summand* $\mathbb{Q}_l$ *is equipped with the trivial* G*-action]. Moreover, in this situation, if* $\chi = \chi_G^{\mathrm{cyclo}}$*, then the* **divisor of cusps** *of the covering of* $X \times_k \widetilde{k}$ *determined by* H *is a disjoint union of* $d_\chi(H^{\mathrm{ab}} \otimes \mathbb{Q}_l) + 1$ *copies of* $\mathrm{Spec}(\widetilde{k})$*.*

(iv) Suppose that X *is* **not proper** *[cf. (i)]. Let* $H \subseteq \Delta$ *be a torsion-free pro-*Σ *characteristic open subgroup;* $H \twoheadrightarrow H^*$ *the maximal pro-*l *quotient of* H*. Then the* **decomposition groups of cusps** $\subseteq H^*$ *may be characterized [***"group-theoretically"***] as the* **maximal** *closed subgroups* $I \subseteq H^*$ *isomorphic to* $\mathbb{Z}_l$ *which satisfy the following condition: We have*

$$d_{\chi_G^{\mathrm{cyclo}}}(J^{\mathrm{ab}} \otimes \mathbb{Q}_l) + 1 = [I \cdot J : J] \cdot d_{\chi_G^{\mathrm{cyclo}}}((I \cdot J)^{\mathrm{ab}} \otimes \mathbb{Q}_l) + 1$$

for every characteristic open subgroup $J \subseteq H^*$.

(v) Let X, H, H^* *be as in (iv). Then the set of cusps of the covering of* $X \times_k \widetilde{k}$ *determined by* H *is in natural* **bijective** *correspondence with the set of conjugacy classes in* H^* *of decomposition groups of cusps [as described in (iv)]. Moreover, this correspondence is* **functorial** *in* H *and* **compatible** *with the natural actions by* Π *on both sides. In particular, by allowing* H *to* **vary**, *this yields a [***"group-theoretic"***] characterization of the* **decomposition groups of cusps** *in* Π.

(vi) Let $I \subseteq \Pi$ *be a* **decomposition group of a cusp**. *Then* $I = C_\Pi(I \bigcap \Delta)$ *[cf. §0].*

Proof. Assertion (i) may be reduced to the case of *hyperbolic curves* via Lemma 4.1, (iv), in which case it is well-known [cf., e.g., [Mzk13], Remark 1.1.3]. Assertion (ii) makes sense in light of our assumption of *"l-cyclotomic fullness"* on χ_G^{cyclo}, and its content is immediate from the definitions. Assertion (iii) follows immediately from [Mzk13], Proposition 2.4, (iv), (vii); the proof of [Mzk13], Corollary 2.7, (i). Assertion (iv) is [in light of assertion (iii)] precisely a summary of the argument of [Mzk13], Theorem 1.6, (i). Finally, assertions (v), (vi) follow immediately from the *commensurable terminality* of [Mzk13], Proposition 1.2, (ii). ◯

Definition 4.6.

(i) Let $\mathbb{V}$ (respectively, $\mathbb{F}$; $\mathbb{S}$) be a *set of isomorphism classes of algebraic stacks* (respectively, *set of isomorphism classes of fields*; *set of nonempty subsets of* $\mathfrak{Primes}$);

$$\mathbb{D} \subseteq \mathbb{V} \times \mathbb{F} \times \mathbb{S}$$

a subset of the direct product set $\mathbb{V} \times \mathbb{F} \times \mathbb{S}$, which we shall think of as a set of collections of *partial construction data*. In the following discussion, we shall use "$[-]$" to denote the *isomorphism class* of "$-$". We shall say that $\mathbb{D}$ is *chain-full* if for every *extension* $1 \to \Delta \to \Pi \to G \to 1$ *of GSAFG-type*, where G is *slim*, that admits *base-prime* partial construction data (X, k, Σ) such that $([X], [k], \Sigma) \in \mathbb{D}$ [cf. Definition 4.2], it follows that every "X_j, k_j" [cf. Definition 4.2, (i)] appearing in an $\widetilde{X}/X$-chain [where $\widetilde{X} \to X$ is the pro-finite étale covering of X determined by some scheme-theoretic envelope for Π] determines an element $([X_j], [k_j], \Sigma) \in \mathbb{D}$.

(ii) Let $\mathbb{D}$ be as in (i); suppose that $\mathbb{D}$ is *chain-full*. Then we shall say that *rel-isom-*$\mathbb{D}$*GC holds* [i.e., "the relative isomorphism version of the Grothendieck Conjecture for $\mathbb{D}$ holds"] (respectively, *rel-hom-*$\mathbb{D}$*GC holds* [i.e., "the relative homomorphism version of the Grothendieck Conjecture for $\mathbb{D}$ holds"]), or that, *rel-isom-GC holds for* $\mathbb{D}$ (respectively, *rel-hom-GC holds for* $\mathbb{D}$) if the following condition is satisfied: For $i = 1, 2$, let

$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$

be an *extension of GSAFG-type*, where G_i is *slim*, that admits *base-prime* partial construction data (k_i, X_i, Σ_i) such that $([X_i], [k_i], \Sigma_i) \in \mathbb{D}$; $\alpha_i : \pi_1^{\text{tame}}(X_i) \twoheadrightarrow \Pi_i$ a *scheme-theoretic envelope*; $\zeta_k : k_1 \xrightarrow{\sim} k_2$ an *isomorphism of fields* that induces, via the α_i, an outer isomorphism $\zeta_G : G_1 \xrightarrow{\sim} G_2$. Then the *natural map*

$$\mathrm{Isom}_{k_1,k_2}(X_1, X_2) \to \mathrm{Isom}^{\mathrm{out}}_{G_1,G_2}(\Pi_1, \Pi_2)$$

$$(\textit{respectively}, \ \mathrm{Hom}^{\mathrm{dom}}_{k_1,k_2}(X_1, X_2) \to \mathrm{Hom}^{\text{out-open}}_{G_1,G_2}(\Pi_1, \Pi_2))$$

determined by the α_i from the set of isomorphisms of schemes $X_1 \xrightarrow{\sim} X_2$ lying over $\zeta_k : k_1 \xrightarrow{\sim} k_2$ (respectively, the set of dominant morphisms of schemes $X_1 \to X_2$ lying over $\zeta_k : k_1 \xrightarrow{\sim} k_2$) to the set of outer isomorphisms of profinite groups $\Pi_1 \xrightarrow{\sim} \Pi_2$ lying over $\zeta_G : G_1 \xrightarrow{\sim} G_2$ (respectively, the set of open outer homomorphisms of profinite groups $\Pi_1 \to \Pi_2$ lying over $\zeta_G : G_1 \xrightarrow{\sim} G_2$) is a *bijection*.

Remark 4.6.1. Of course, in a similar vein, one may also formulate the notions that "the *absolute isomorphism* version of the Grothendieck Conjecture holds for $\mathbb{D}$", "the *absolute homomorphism* version of the Grothendieck Conjecture holds for $\mathbb{D}$", "the *semi-absolute isomorphism* version of the Grothendieck Conjecture holds for $\mathbb{D}$", "the *semi-absolute homomorphism* version of the Grothendieck Conjecture holds for $\mathbb{D}$", etc. Since we shall not use these versions in the discussion to follow, we leave the routine details of their formulation to the interested reader.

Theorem 4.7. (Semi-absoluteness of Chains of Elementary Operations) *Let* $\mathbb{D}$ *be a* **chain-full** *set of collections of* **partial construction data** *[cf. Definition 4.6, (i)] such that the* **rel-isom-$\mathbb{D}$GC** *holds [cf. Definition 4.6, (ii)]. For* $i = 1, 2$, *let* G_i *be a* **slim** *profinite group;*

$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$

an **extension of GSAFG-type** *that admits* **base-prime** *[cf. Definition 2.1, (iv)] partial construction data* $(k_i, \widetilde{k}_i, X_i, \Sigma_i)$ *such that* $([X_i], [k_i], \Sigma_i) \in \mathbb{D}$; $\alpha_i : \pi_1^{\mathrm{tame}}(X_i) \twoheadrightarrow \Pi_i$ *a* **scheme-theoretic envelope***. Also, let us suppose further that the following conditions are satisfied:*

(a) if **either** X_1 *or* X_2 *is a* **hyperbolic orbicurve***, then* **both** X_1 *and* X_2 *are hyperbolic orbicurves;*

(b) if either X_1 *or* X_2 *is a* **non-proper hyperbolic orbicurve***, then there exists a prime number* $l \in \Sigma_1 \bigcap \Sigma_2$ *such that for* $i = 1, 2$, *the* **cyclotomic character** $G_i \to \mathbb{Z}_l^\times$ *[i.e., the character whose composite with* α_i *is the usual cyclotomic character* $\pi_1^{\mathrm{tame}}(X_i) \twoheadrightarrow \mathrm{Gal}(\widetilde{k}/k) \to \mathbb{Z}_l^\times$*] has* **open image**.

Let

$$\phi : \Pi_1 \xrightarrow{\sim} \Pi_2$$

be an **isomorphism of profinite groups** *that induces isomorphisms* $\phi_\Delta : \Delta_1 \xrightarrow{\sim} \Delta_2$, $\phi_G : G_1 \xrightarrow{\sim} G_2$. *Then:*

(i) The **natural functors** *[cf. Remark 4.2.1]*

$$\mathrm{Chain}(\widetilde{X}_i/X_i) \to \mathrm{Chain}(\Pi_i); \quad \mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}_i/X_i) \to \mathrm{Chain}^{\text{iso-trm}}(\Pi_i)$$
$$\text{ÉtLoc}(\widetilde{X}_i/X_i) \to \text{ÉtLoc}(\Pi_i)$$

are **equivalences of categories** *that are compatible with passing to* **type-chains**.

(ii) The isomorphism ϕ induces **equivalences of categories**

$$\mathrm{Chain}(\Pi_1) \xrightarrow{\sim} \mathrm{Chain}(\Pi_2); \quad \mathrm{Chain}^{\text{iso-trm}}(\Pi_1) \xrightarrow{\sim} \mathrm{Chain}^{\text{iso-trm}}(\Pi_2)$$
$$\text{ÉtLoc}(\Pi_1) \xrightarrow{\sim} \text{ÉtLoc}(\Pi_2)$$

that are compatible with passing to **type-chains** *and* **functorial** *in ϕ.*

(iii) Suppose further that the **rel-hom-$\mathbb{D}$GC** *holds [cf. Definition 4.6, (ii)], and that for $i = 1, 2$, X_i is a* **hyperbolic orbicurve**. *Then the* **natural functors** *[cf. Remark 4.2.1]*

$$\mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}_i/X_i) \to \mathrm{Chain}^{\mathrm{trm}}(\Pi_i); \quad \mathrm{DLoc}(\widetilde{X}_i/X_i) \to \mathrm{DLoc}(\Pi_i)$$

are **equivalences of categories** *that are compatible with passing to* **type-chains**.

(iv) In the situation of (iii), the isomorphism ϕ induces **equivalences of categories**

$$\mathrm{Chain}^{\mathrm{trm}}(\Pi_1) \xrightarrow{\sim} \mathrm{Chain}^{\mathrm{trm}}(\Pi_2); \quad \mathrm{DLoc}(\Pi_1) \xrightarrow{\sim} \mathrm{DLoc}(\Pi_2)$$

that are compatible with passing to **type-chains** *and* **functorial** *in ϕ.*

Proof. First, we consider the natural functor

$$\mathrm{Chain}(\widetilde{X}_i/X_i) \to \mathrm{Chain}(\Pi_i)$$

of Remark 4.2.1. To conclude that this functor is an *equivalence of categories*, it follows immediately from the definitions of the categories involved that it suffices to verify that the $\widetilde{X}/X$-chain and Π-chain versions of the four types of elementary operations $\curlywedge$, $\curlyvee$, $\bullet$, $\circledcirc$ described in Definition 4.2, (i), (iii), *correspond bijectively* to one another. This is immediate from the definitions (respectively, the cuspidal portion of Lemma 4.1, (i), (v); the "closed point of X" portion of Lemma 4.1, (iii), (v)) for $\curlywedge$ (respectively, $\bullet$; $\circledcirc$). [Here, we note that in the case of $\bullet$, $\circledcirc$, the *"k_{j+1}-rationality"* of the cusp or non-scheme-like point in question follows immediately from Lemma 4.1, (vi) [by taking "x_B" to be the various Galois conjugates of this point].] Finally, the desired correspondence for $\curlyvee$ follows from our assumption that the *rel-isom-$\mathbb{D}GC$* holds by applying this "rel-isom-$\mathbb{D}$GC" as was done in the proofs of [Mzk7], Theorem 2.4; [Mzk9], Theorem 2.3, (i). This completes the proof that the natural functor $\mathrm{Chain}(\widetilde{X}_i/X_i) \to \mathrm{Chain}(\Pi_i)$ is an *equivalence*. A similar application of the *"rel-isom-$\mathbb{D}GC$"* then yields the *equivalences* $\mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}_i/X_i) \xrightarrow{\sim} \mathrm{Chain}^{\text{iso-trm}}(\Pi_i)$, $\text{ÉtLoc}(\widetilde{X}_i/X_i) \xrightarrow{\sim} \text{ÉtLoc}(\Pi_i)$. In a similar vein, the *"rel-hom-$\mathbb{D}GC$"* [cf. assertion (iii)] implies the *equivalences* $\mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X) \xrightarrow{\sim} \mathrm{Chain}^{\mathrm{trm}}(\Pi)$, $\mathrm{DLoc}(\widetilde{X}/X) \xrightarrow{\sim} \mathrm{DLoc}(\Pi)$. This completes the proof of assertions (i), (iii).

Finally, to obtain the equivalences of assertions (ii), (iv), it suffices to observe that the definitions of the various categories involved are entirely *"group-theoretic"*. Here, we note that the "group-theoreticity" of the elementary operations of type $\curlywedge$, $\curlyvee$, $\circledcirc$ is immediate; the "group-theoreticity" of the elementary operations of type $\bullet$ follows immediately from Lemma 4.5, (v) [in light of our assumptions (a), (b)]. Also, we observe that Σ_i may be recovered "group-theoretically" from Δ_i [i.e., as the unique minimal subset $\Sigma' \subseteq \mathfrak{Primes}$ such that Δ_i is *almost pro*-Σ']. This completes the proof of assertions (ii), (iv). $\bigcirc$

Remark 4.7.1. The portion of Theorem 4.7 concerning the categories "ÉtLoc(−)" [cf. also Example 4.8 below; Corollary 2.8, (ii)] and "DLoc(−)" allows one to relate the theory of the present §4 to the theory of [Mzk9], §2 [cf., especially, [Mzk9], Theorem 2.3].

Example 4.8. Hyperbolic Orbicurves. Let p be a *prime number*; $\mathbb{S}$ the set of subsets of $\mathfrak{Primes}$ containing p; $\mathbb{V}$ the set of isomorphism classes of *hyperbolic orbicurves* over fields of cardinality $\leq$ the cardinality of $\mathbb{Q}_p$.

(i) Let $\mathbb{F}$ be the set of isomorphism classes of *generalized sub-p-adic fields* [i.e., subfields of finitely generated extensions of the quotient field of the ring of Witt vectors with coefficients in an algebraic closure of $\mathbb{F}_p$ — cf. [Mzk5], Definition 4.11]; $\mathbb{D} = \mathbb{V} \times \mathbb{F} \times \mathbb{S}$. Then let us observe that:

The hypotheses of Theorem 4.7, (i), (ii), are satisfied relative to this $\mathbb{D}$.

Indeed, it is immediate that $\mathbb{D}$ is *chain-full*; the *rel-isom-$\mathbb{D}GC$* follows from [Mzk5], Theorem 4.12; the prime p clearly serves as a prime "l" as in the statement of Theorem 4.7. Moreover, we recall that from [Mzk5], Lemma 4.14, that the *absolute Galois group of a generalized sub-p-adic field* is always *slim*.

(ii) Let $\mathbb{F}$ be the set of isomorphism classes of *sub-p-adic fields* [i.e., subfields of finitely generated extensions of $\mathbb{Q}_p$ — cf. [Mzk3], Definition 15.4, (i)]; $\mathbb{D} = \mathbb{V} \times \mathbb{F} \times \mathbb{S}$. Then let us observe that:

The hypotheses of Theorem 4.7, (iii), (iv), are satisfied relative to this $\mathbb{D}$.

Indeed, it is immediate that $\mathbb{D}$ is *chain-full*; the *rel-hom-$\mathbb{D}GC$* follows from [Mzk3], Theorem A; the prime p clearly serves as a prime "l" as in the statement of Theorem 4.7. Moreover, we recall that from [Mzk3], Lemma 15.8, that the *absolute Galois group of a sub-p-adic field* is always *slim*.

Example 4.9. Iso-poly-hyperbolic Orbisurfaces.

(i) Let k be a *field of characteristic zero*. Then we recall from [Mzk3], Definition a2.1, that a smooth k-scheme X is called a *hyperbolically fibred surface* if it is admits the structure of a *family of hyperbolic curves* [cf. §0] over a hyperbolic curve Y over k. If X is a smooth, generically scheme-like, geometrically connected algebraic stack over k, then we shall say that X is an *iso-poly-hyperbolic orbisurface* [cf. the term "poly-hyperbolic" as it is defined in [Mzk4], Definition 4.6] if X admits a finite étale covering which is a hyperbolically fibred surface over some finite extension of k.

(ii) Let p be a *prime number*; $\mathbb{S} \stackrel{\text{def}}{=} \{\mathfrak{Primes}\}$ [where we regard $\mathfrak{Primes}$ as the unique non-proper subset of $\mathfrak{Primes}$]; $\mathbb{F}$ the set of isomorphism classes of *sub-p-adic fields*; $\mathbb{V}$ the set of isomorphism classes of *iso-poly-hyperbolic orbisurfaces* [cf. (i)] over sub-p-adic fields; $\mathbb{D} = \mathbb{V} \times \mathbb{F} \times \mathbb{S}$. Then let us observe that:

The hypotheses of Theorem 4.7, (i), (ii), are satisfied relative to this $\mathbb{D}$.

Indeed, it is immediate that $\mathbb{D}$ is *chain-full*; the *rel-isom-*$\mathbb{D}$*GC* follows from [Mzk3], Theorem D. Moreover, we recall that from [Mzk3], Lemma 15.8, that the *absolute Galois group of a sub-p-adic field* is always *slim*.

(iii) Let k be a *sub-p-adic field*; X the *moduli stack of hyperbolic curves* of type $(0,5)$ [i.e., the moduli stack of smooth curves of genus 0 with 5 distinct, unordered points] over k; $\widetilde{X} \to X$ a *"universal"* pro-finite étale covering of X; $\overline{k}$ the algebraic closure of k determined by $\widetilde{X} \to X$. Then one verifies immediately that X is an *iso-poly-hyperbolic orbisurface* over k. Write $1 \to \Delta \to \Pi \to G \to 1$ for the *GSAFG-extension* defined by the natural surjection $\pi_1(X) = \mathrm{Gal}(\widetilde{X}/X) \twoheadrightarrow \mathrm{Gal}(\overline{k}/k)$ [which we regard as equipped with the tautological scheme-theoretic envelope given by the identity]. Then we have an *equivalence of categories*

$$\text{ÉtLoc}(\widetilde{X}/X) \xrightarrow{\sim} \text{ÉtLoc}(\Pi)$$

[cf. (ii); Theorem 4.7, (i)]; the object of these categories determined by X, Π [i.e., by the unique chain of length 0] is *terminal* [cf. [Mzk2], Theorem C] — i.e., a *"core"* [cf. the terminology of [Mzk7], §2; [Mzk8], §2].

Finally, we observe that the theory of the present §4 admits a *"tempered version"*, in the case of hyperbolic orbicurves over MLF's. We begin by recalling basic facts concerning *tempered fundamental groups*. Let k be an *MLF* of *residue characteristic p*; $\overline{k}$ an algebraic closure of k; X a *hyperbolic orbicurve* over k. We shall use a subscript $\overline{k}$ to denote the result of a *base-change* from k to $\overline{k}$. Write

$$\pi_1^{\text{tp}}(X); \quad \pi_1^{\text{tp}}(X_{\overline{k}})$$

for the *tempered fundamental groups* of X, $X_{\overline{k}}$ [cf. [André], §4; [Mzk10], Examples 3.10, 5.6]. Thus, the *profinite completion* of $\pi_1^{\text{tp}}(X)$ (respectively, $\pi_1^{\text{tp}}(X_{\overline{k}})$)

is naturally isomorphic to the usual étale fundamental group $\pi_1(X)$ (respectively, $\pi_1(X_{\overline{k}})$). If $H \subseteq \pi_1^{\mathrm{tp}}(X_{\overline{k}})$ is an open subgroup of finite index, then recall that the *minimal co-free subgroup* of H

$$H^{\text{co-fr}} \subseteq H$$

[cf. §0] is precisely the subgroup of H with the property that the quotient $H \twoheadrightarrow H/H^{\text{co-fr}}$ corresponds to the tempered covering of $X_{\overline{k}}$ determined by the *universal covering* of the dual graph of the special fiber of a stable model of $X_{\overline{k}}$ — cf. [André], proof of Lemma 6.1.1.

Proposition 4.10. (Basic Properties of Tempered Fundamental Groups) *In the notation of the above discussion, suppose further that $\phi : X \to Y$ is a morphism of hyperbolic orbicurves over k. For $Z = X, Y$, let us write*

$$\Pi_Z^{\mathrm{tp}} \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{tp}}(Z); \quad \Delta_Z^{\mathrm{tp}} \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{tp}}(Z_{\overline{k}})$$

and denote the **profinite completions** *of Π_Z^{tp}, Δ_Z^{tp} by $\widehat{\Pi}_Z^{\mathrm{tp}}$, $\widehat{\Delta}_Z^{\mathrm{tp}}$, respectively; in the following, all "co-free completions" [cf. §0] of open subgroups of finite index in Π_X^{tp} (respectively, Δ_X^{tp}) will be with respect to the subgroup $\Delta_X^{\mathrm{tp}} \subseteq \Pi_X^{\mathrm{tp}}$ (respectively, $\Delta_X^{\mathrm{tp}} \subseteq \Delta_X^{\mathrm{tp}}$). Then:*

(i) The natural homomorphism $\Pi_X^{\mathrm{tp}} \to \widehat{\Pi}_X^{\mathrm{tp}} \xrightarrow{\sim} \pi_1(X)$ (respectively, $\Delta_X^{\mathrm{tp}} \to \widehat{\Delta}_X^{\mathrm{tp}} \xrightarrow{\sim} \pi_1(X_{\overline{k}})$) is **injective**. *In fact, if $H \subseteq \Delta_X^{\mathrm{tp}}$ is any characteristic open subgroup of finite index, then $\Pi_X^{\mathrm{tp}}/H^{\text{co-fr}}$, $\Delta^{\mathrm{tp}}/H^{\text{co-fr}}$* **inject** *into their respective profinite completions. In particular, $\pi_1^{\mathrm{tp}}(X)$ (respectively, $\pi_1^{\mathrm{tp}}(X_{\overline{k}})$) is naturally isomorphic to its $\pi_1(X)$*-**co-free completion** *(respectively, $\pi_1(X_{\overline{k}})$*-**co-free completion**) *[cf. §0].*

(ii) Π_X^{tp} (respectively, Δ_X^{tp}) is **normally terminal** *in $\widehat{\Pi}_X^{\mathrm{tp}}$ (respectively, $\widehat{\Delta}_X^{\mathrm{tp}}$).*

(iii) Suppose that ϕ is either a **de-cuspidalization** *morphism [i.e., an open immersion whose image is the complement of a single k-valued point of Y — cf. Definition 4.2, (i), (c)] or a* **de-orbification** *morphism [i.e., a partial coarsification morphism which is an isomorphism over the complement of a single k-valued point of Y — cf. Definition 4.2, (i), (d)]. Then the natural homomorphism $\Pi_X^{\mathrm{tp}} \to \Pi_Y^{\mathrm{tp}}$ (respectively, $\Delta_X^{\mathrm{tp}} \to \Delta_Y^{\mathrm{tp}}$) may be reconstructed —* **"group-theoretically"** *— from its profinite completion $\widehat{\Pi}_X^{\mathrm{tp}} \twoheadrightarrow \widehat{\Pi}_Y^{\mathrm{tp}}$ (respectively, $\widehat{\Delta}_X^{\mathrm{tp}} \twoheadrightarrow \widehat{\Delta}_Y^{\mathrm{tp}}$) as the natural morphism from Π_X^{tp} (respectively, Δ_X^{tp}) to the* **co-free completion** *of Π_X^{tp} with respect to $\widehat{\Pi}_Y^{\mathrm{tp}}$ (respectively, $\widehat{\Delta}_Y^{\mathrm{tp}}$) [cf. §0].*

(iv) Let $l \in \mathfrak{Primes}$. If $J \subseteq \Delta_X^{\mathrm{tp}}$ is an open subgroup of finite index, write $J \to J^{[l]}$ for the **co-free completion** *of J with respect to the* **maximal pro-l quotient** *of the profinite completion of J. Let $H \subseteq \Delta_X^{\mathrm{tp}}$ be an open subgroup of finite index. Suppose that $l \neq p$. Then the dual graph Γ_H of the special fiber of a stable model of the covering of $X_{\overline{k}}$ corresponding to H determines* **verticial** *and* **edge-like subgroups** *of $H^{[l]}$ [i.e., decomposition groups of the vertices and edges*

of Γ_H *— cf. [Mzk10], Theorem 3.7, (i), (iii)]. The verticial (respectively, edge-like) subgroups of* $H^{[l]}$ *may be characterized —* **"group-theoretically"** *— as the* **maximal compact subgroups** *(respectively,* **nontrivial intersections of two distinct maximal compact subgroups***) of* $H^{[l]}$*. In particular, the graph* Γ_H *may be reconstructed —* **"group-theoretically"** *— from the verticial and edge-like subgroups of* $H^{[l]}$*, together with their various mutual inclusion relations.*

(v) The **prime number** $\boldsymbol{p}$ *may be characterized —* **"group-theoretically"** *— as the unique prime number* l *such that their exist open subgroups* $H, J \subseteq \Delta_X^{\mathrm{tp}}$ *of finite index, together with distinct prime numbers* l_1*,* l_2*, satisfying the following properties: (a)* H *is a normal subgroup of* J *of index* l*; (b) for* $i = 1, 2$*, the outer action of* J *on* $H^{[l_i]}$ *[cf. (iv)]* **fixes** *[the conjugacy class in* $H^{[l_i]}$ *of] and* **induces the trivial outer action** *on some* **maximal compact subgroup** *of* $H^{[l_i]}$ *[cf. (iv)].*

(vi) Let l *be a prime number* $\neq p$*;* $H \subseteq \Delta_X^{\mathrm{tp}}$ *an open subgroup of finite index. Then the set of* **cusps** *of the covering of* $X_{\overline{k}}$ *corresponding to* H *may be characterized —* **"group-theoretically"** *— as the set of conjugacy classes in* $H^{[l]}$ *of the commensurators in* $H^{[l]}$ *of the* **images** *in* $H^{[l]}$ *of* **edge-like subgroups** *of* $J^{[l]}$ *[cf. (iv)], where* $J \subseteq H$ *is an open subgroup of finite index, which are* **not contained** *in edge-like subgroups of* $H^{[l]}$*. In particular, by allowing* H *to* **vary***, this yields a [***"group-theoretic"***] characterization of the* **decomposition groups of cusps** *in* Δ_X^{tp}*,* Π_X^{tp} *[i.e., a* **"tempered version"** *of Lemma 4.5, (v)].*

Proof. Assertion (i) follows immediately from the discussion at the beginning of [Mzk10], §6 [cf. also the discussion of [André], §4.5]. Assertion (ii) is the content of [Mzk10], Lemma 6.1, (ii), (iii) [cf. also [André], Corollary 6.2.2]. Assertion (iii) follows immediately from assertion (i). Assertion (iv) follows immediately from [Mzk10], Theorem 3.7, (iv); [Mzk10], Corollary 3.9 [cf. also the proof of [Mzk10], Corollary 3.11]. Assertions (v), (vi) amount to summaries of the relevant portions of the proof of [Mzk10], Corollary 3.11. Here, in assertion (v), we observe that *at least one* of the l_i is $\neq p$; thus, for this choice of l_i, the action of J *fixes* and *induces the trivial outer action* on some *verticial subgroup* of $H^{[l_i]}$. ○

Remark 4.10.1. It is not clear to the author at the time of writing how to prove a version of Proposition 4.10, (vi), for *decomposition of closed points which are not cusps* [i.e., a "tempered version" of Lemma 4.1, (iii)].

Remark 4.10.2. A certain fact applied in the portion of the proof of [Mzk10], Corollary 3.11 summarized in Proposition 4.10, (vi), is only given a somewhat sketchy proof in *loc. cit.* A more detailed treatment of this fact is given in [Mzk15], Corollary 2.11.

Now we are ready to state the *tempered version* of Definition 4.2.

Definition 4.11. In the notation of the above discussion, let

$$1 \to \Delta \to \Pi \to G \to 1$$

be an *extension of topological groups* that is isomorphic to the natural extension $1 \to \pi_1^{\mathrm{tp}}(X_{\overline{k}}) \to \pi_1^{\mathrm{tp}}(X) \to \mathrm{Gal}(\overline{k}/k) \to 1$ via some isomorphism $\alpha : \pi_1^{\mathrm{tp}}(X) \xrightarrow{\sim} \Pi$, which we shall refer to as a *scheme-theoretic envelope*. Write $\widehat{\Pi}$ for the *profinite completion* of Π, $\widetilde{X} \to X$ for the *pro-finite étale covering* of X determined by the completion of α [so $\widehat{\Pi} = \mathrm{Gal}(\widetilde{X}/X)$]; $\widetilde{k}$ for the resulting field extension of k. In a similar vein, we shall write $\widetilde{\Pi}$ for the *projective system of topological groups* determined by the open subgroups of finite index of Π [cf. Definition 4.2]. Then:

(i) We shall refer to as an *[*Π*-]chain [of length* n*]* [where $n \geq 0$ is an integer] any finite sequence

$$\Pi_0 \rightsquigarrow \Pi_1 \rightsquigarrow \ldots \rightsquigarrow \Pi_{n-1} \rightsquigarrow \Pi_n$$

of topological groups Π_j [for $j = 0, \ldots, n$] with *slim* profinite completions $\widehat{\Pi}_j$, each equipped with a *"rigidifying homomorphism"* $\rho_j : \widetilde{\Pi} \to \Pi_j$ which is *of DOF-type* [i.e., which maps some member of the projective system $\widetilde{\Pi}$ onto a dense subgroup of an open subgroup of finite index of Π_j — cf. §0] satisfying the following conditions:

(0_{tp}) $\Pi_0 = \Pi$ [equipped with its natural rigidifying homomorphism $\widetilde{\Pi} \to \Pi$].

(1_{tp}) There exists a [uniquely determined] surjection $\Pi_j \twoheadrightarrow G_j$ compatible with ρ_j, where $G_j \subseteq G$ is an open subgroup.

(2_{tp}) Each kernel

$$\Delta_j \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Pi_j \twoheadrightarrow G_j \hookrightarrow G)$$

has a *slim*, *nontrivial* profinite completion $\widehat{\Delta}_j$.

(3_{tp}) The topological groups Π_j, Δ_j are *residually finite*. We shall refer to as a *cuspidal decomposition group* in $\widehat{\Delta}_j$ any $\widehat{\Delta}_j$-conjugate of the *commensurator* in $\widehat{\Delta}_j$ of the image via ρ_j of the inverse image in $\widetilde{\Pi}$ of the decomposition group in Δ [determined by α] of a *cusp* of X.

(4_{tp}) Each "$\Pi_j \rightsquigarrow \Pi_{j+1}$" [for $j = 0, \ldots, n-1$] is an *"elementary operation"*, as defined below.

Here, an *elementary operation* "$\Pi_j \rightsquigarrow \Pi_{j+1}$" is defined to consist of the datum of an *"operation homomorphism"* ϕ of DOF-type either from Π_j to Π_{j+1} or from Π_{j+1} to Π_j which is *compatible* with ρ_j, ρ_{j+1}, and, moreover, is of one of the following four types:

(a) *Type* $\curlywedge$: In this case, the elementary operation $\Pi_j \rightsquigarrow \Pi_{j+1}$ consists of an *immersion of OF-type* [cf. §0] $\phi : \Pi_{j+1} \hookrightarrow \Pi_j$.

(b) *Type* $\curlyvee$: In this case, the elementary operation $\Pi_j \rightsquigarrow \Pi_{j+1}$ consists of an *immersion of OF-type* [cf. §0] $\phi : \Pi_j \hookrightarrow \Pi_{j+1}$.

(c) *Type* $\bullet$: In this case, the elementary operation $\Pi_j \rightsquigarrow \Pi_{j+1}$ consists of a *dense homomorphism* $\phi : \Pi_j \to \Pi_{j+1}$ which is isomorphic to the *co-free completion* of Π_j with respect to the induced profinite quotient $\widehat{\phi} : \widehat{\Pi}_j \twoheadrightarrow \widehat{\Pi}_{j+1}$ [and the subgroup Δ_j], such that $\mathrm{Ker}(\widehat{\phi})$ is topologically normally generated by a *cuspidal decomposition group* C in $\widehat{\Delta}_j$ such that C is contained in some normal open *torsion-free* subgroup of $\widehat{\Delta}_j$.

(d) *Type* $\circledcirc$: In this case, the elementary operation $\Pi_j \rightsquigarrow \Pi_{j+1}$ consists of a *dense homomorphism* $\phi : \Pi_j \to \Pi_{j+1}$ which is isomorphic to the *co-free completion* of Π_j with respect to the induced profinite quotient $\widehat{\phi} : \widehat{\Pi}_j \twoheadrightarrow \widehat{\Pi}_{j+1}$ [and the subgroup Δ_j], such that $\mathrm{Ker}(\widehat{\phi})$ is topologically normally generated by a *finite closed subgroup* of $\widehat{\Delta}_j$.

Thus, any Π-chain determines a *sequence of symbols* $\in \{\curlywedge, \curlyvee, \bullet, \circledcirc\}$ [corresponding to the types of elementary operations in the Π-chain], which we shall refer to as the *type-chain* associated to the Π-chain.

(ii) An *isomorphism* between two Π-chains with *identical type-chains* [hence of the same length]

$$(\Pi_0 \rightsquigarrow \ldots \rightsquigarrow \Pi_n) \xrightarrow{\sim} (\Psi_0 \rightsquigarrow \ldots \rightsquigarrow \Psi_n)$$

is defined to be a collection of isomorphisms of topological groups $\Pi_j \xrightarrow{\sim} \Psi_j$ [for $j = 0, \ldots, n$] that are compatible with the rigidifying homomorphisms. [Here, we note that the condition of *compatibility* with the rigidifying homomorphisms implies [since all of the topological groups involved are *residually finite* with *slim* profinite completions] that every *automorphism* of a Π-chain is given by the *identity*, and that every isomorphism of Π-chains of the same length is *compatible* with the respective *operation homomorphisms*.] Thus, one obtains a *category*

$$\mathrm{Chain}(\Pi)$$

whose *objects* are the Π-chains [with arbitrary associated type-chain], and whose *morphisms* are the isomorphisms between Π-chains [with identical type-chains]. A *terminal homomorphism* between two Π-chains [with arbitrary associated type-chain]

$$(\Pi_0 \rightsquigarrow \ldots \rightsquigarrow \Pi_n) \to (\Psi_0 \rightsquigarrow \ldots \rightsquigarrow \Psi_m)$$

is defined to be an outer homomorphism of DOF-type [cf. [Mzk10], Theorem 6.4] $\Pi_n \to \Psi_m$ that is *compatible* [up to composition with an inner automorphism] with the open homomorphisms $\Pi_n \to G$, $\Psi_m \to G$. Thus, one obtains a *category*

$$\mathrm{Chain}^{\mathrm{trm}}(\Pi)$$

whose *objects* are the Π-chains [with arbitrary associated type-chain], and whose *morphisms* are the *terminal homomorphisms* between Π-chains; write

$$\mathrm{Chain}^{\text{iso-trm}}(\Pi) \subseteq \mathrm{Chain}^{\mathrm{trm}}(\Pi)$$

for the subcategory determined by the *terminal isomorphisms* [i.e., the isomorphisms of $\mathrm{Chain}^{\mathrm{trm}}(\Pi)$]. Thus, it follows immediately from the definitions that we obtain *natural functors* $\mathrm{Chain}(\Pi) \to \mathrm{Chain}^{\text{iso-trm}}(\Pi) \to \mathrm{Chain}^{\mathrm{trm}}(\Pi)$. Finally, we obtain (sub)categories

$$\mathrm{Chain}^{\text{iso-trm}}(\Pi)\{-\} \subseteq \mathrm{Chain}^{\text{iso-trm}}(\Pi); \quad \mathrm{Chain}^{\mathrm{trm}}(\Pi)\{-\} \subseteq \mathrm{Chain}^{\mathrm{trm}}(\Pi)$$

$$\mathrm{DLoc}(\Pi) \stackrel{\mathrm{def}}{=} \mathrm{Chain}^{\mathrm{trm}}(\Pi)\{\curlywedge, \bullet\}; \quad \text{ÉtLoc}(\Pi) \stackrel{\mathrm{def}}{=} \mathrm{Chain}^{\text{iso-trm}}(\Pi)\{\curlywedge, \curlyvee\}$$

[cf. Definition 4.2, (v)].

Remark 4.11.1. Just as in the profinite case [i.e., Remark 4.2.1], we have *natural functors*

$$\mathrm{Chain}(\widetilde{X}/X) \to \mathrm{Chain}(\Pi) \to \mathrm{Chain}(\widehat{\Pi})$$

$$\mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X) \to \mathrm{Chain}^{\text{iso-trm}}(\Pi) \to \mathrm{Chain}^{\text{iso-trm}}(\widehat{\Pi})$$

$$\mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X) \to \mathrm{Chain}^{\mathrm{trm}}(\Pi) \to \mathrm{Chain}^{\mathrm{trm}}(\widehat{\Pi})$$

— where the second arrow in each line is the natural functor obtained by *profinite completion*; the various *composite functors* of the two functors in each line are the natural functors of Remark 4.2.1.

Remark 4.11.2. A similar remark to Remark 4.2.2 applies in the present tempered case.

Theorem 4.12. (Tempered Chains of Elementary Operations) *For* $i = 1, 2$, *let* k_i *be an* **MLF** *of* **residue characteristic** p_i; $\overline{k}_i$ *an algebraic closure of* k_i; X_i *a* **hyperbolic orbicurve** *over* k_i;

$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$

an **extension of topological groups** *that is isomorphic to the natural extension* $1 \to \pi_1^{\mathrm{tp}}((X_i)_{\overline{k}_i}) \to \pi_1^{\mathrm{tp}}(X_i) \to \mathrm{Gal}(\overline{k}_i/k_i) \to 1$ *via some* **scheme-theoretic envelope** $\alpha_i : \pi_1^{\mathrm{tp}}(X_i) \stackrel{\sim}{\to} \Pi_i$. *Let*

$$\phi : \Pi_1 \stackrel{\sim}{\to} \Pi_2$$

be an **isomorphism of topological groups**. *Then:*

(i) The **natural functors** *[cf. Remark 4.2.1]*

$$\mathrm{Chain}(\widetilde{X}_i/X_i) \to \mathrm{Chain}(\Pi_i); \quad \mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}_i/X_i) \to \mathrm{Chain}^{\text{iso-trm}}(\Pi_i)$$

$$\text{ÉtLoc}(\widetilde{X}_i/X_i) \to \text{ÉtLoc}(\Pi_i)$$

$$\mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}_i/X_i) \to \mathrm{Chain}^{\mathrm{trm}}(\Pi_i); \quad \mathrm{DLoc}(\widetilde{X}_i/X_i) \to \mathrm{DLoc}(\Pi_i)$$

are **equivalences of categories** *that are compatible with passing to* **type-chains***.*

(ii) We have $p_1 = p_2$*; the isomorphism* ϕ *induces isomorphisms* $\phi_\Delta : \Delta_1 \xrightarrow{\sim} \Delta_2$*,* $\phi_G : G_1 \xrightarrow{\sim} G_2$*, as well as* **equivalences of categories**

$$\mathrm{Chain}(\Pi_1) \xrightarrow{\sim} \mathrm{Chain}(\Pi_2); \quad \mathrm{Chain}^{\text{iso-trm}}(\Pi_1) \xrightarrow{\sim} \mathrm{Chain}^{\text{iso-trm}}(\Pi_2)$$

$$\text{ÉtLoc}(\Pi_1) \xrightarrow{\sim} \text{ÉtLoc}(\Pi_2)$$

$$\mathrm{Chain}^{\mathrm{trm}}(\Pi_1) \xrightarrow{\sim} \mathrm{Chain}^{\mathrm{trm}}(\Pi_2); \quad \mathrm{DLoc}(\Pi_1) \xrightarrow{\sim} \mathrm{DLoc}(\Pi_2)$$

that are compatible with passing to **type-chains** *and* **functorial** *in* ϕ*.*

Proof. In light of Proposition 4.10, (iii), together with the *"tempered anabelian theorem"* of [Mzk10], Theorem 6.4, the proof of Theorem 4.12 is entirely similar to the proof of Theorem 4.7. [Here, we note that in the case of *de-cuspidalization* operations, instead of applying the de-cuspidalization portion of Proposition 4.10, (iii), one may instead apply the *"group-theoretic" characterization* of Proposition 4.10, (vi).] Also, we recall that the portion of assertion (ii) concerning, "$p_1 = p_2$", "ϕ_Δ", "ϕ_G" follows immediately [by considering the *profinite completion* of ϕ] from Theorem 2.14, (i). ○

Remark 4.12.1. A similar remark to Remark 4.7.1 applies in the present tempered case [cf. [Mzk10], Theorem 6.8].

Appendix: The Theory of Albanese Varieties

In the present Appendix, we review the basic theory of *Albanese varieties* [cf., e.g., [NS], [Serre1], [Chev], [BS], [SS]], as it will be applied in the present paper. One of our aims here is to present the theory in *modern scheme-theoretic language* [i.e., as opposed to [NS], [Serre1], [Chev]], but without resorting to the introduction of *motives and derived categories*, as in [BS], [SS]. Put another way, although there is no doubt that the content of the present Appendix is *implicit* in the literature, the lack of an appropriate reference that discusses this material *explicitly* seemed to the author to constitute sufficient justification for the inclusion of a detailed discussion of this material in the present paper.

In the following discussion, we fix a *perfect field* k, together with an *algebraic closure* $\overline{k}$ of k. The result of base-change [of k-schemes and morphisms of k-schemes] from k to $\overline{k}$ will be denoted by means of a subscript "$\overline{k}$". Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the *absolute Galois group* of k.

We will apply basic well-known properties of *commutative group schemes of finite type over* k without further explanation. In particular, we recall the following:

(I) The category of such group schemes is *abelian* [cf., e.g., [SGA3-1], $\mathrm{VI_A}$, 5.4]; subgroup schemes are always *closed* [cf., e.g., [SGA3-1], $\mathrm{VI_B}$, 1.4.2]; reduced group schemes over k are *k-smooth* [cf., e.g., [SGA3-1], $\mathrm{VI_A}$, 1.3.1].

(II) Every *connected reduced subquotient* of a semi-abelian variety over k [i.e., an extension of an *abelian variety* by a *torus*] is itself a *semi-abelian variety* over k. [Indeed, this may be verified easily by reducing to the corresponding fact for tori [cf., e.g., [SGA3-2], IX, 8.1] and abelian varieties.]

(III) Let $\phi : B \to A$ be a *connected finite étale Galois covering* of a semi-abelian variety A over k, with identity element $0_A \in A(k)$, such that $(\phi^{-1}(0_A))(k) \neq \emptyset$, and the degree of ϕ is *prime* to the characteristic of k. Then each element of $b \in (\phi^{-1}(0_A))(k)$ determines on B a *unique* structure of *semi-abelian variety* over k on B such that b is the identity element of the group $B(k)$, and ϕ is a homomorphism of group schemes over k. [Indeed this may be verified easily by reducing to the corresponding fact for tori and abelian varieties.] Note, moreover, that in this situation, if $k = \overline{k}$, then we obtain an inclusion $\mathrm{Gal}(B/A) \hookrightarrow B(k)$, which implies, in particular, that the covering ϕ is *abelian*, and, moreover, appears as a *subcovering* of a covering $A \to A$ given by multiplication by some n invertible in k.

Definition A.1.

(i) A *variety over* k, or *k-variety*, is defined to be a geometrically integral separated scheme of finite type over k. A k-variety will be called *complete* if it is proper over k. We shall refer to a pair (V, v), where V is a k-variety and $v \in V(k)$, as a *pointed variety over* k; a *morphism of pointed varieties over* k, or *pointed k-morphism*, $(V, v) \to (W, w)$ [which we shall often simply write $V \to W$, when v, w

are *fixed*] is a morphism of k-varieties that maps $v \mapsto w$. Any *reduced group scheme* G over k has a natural structure of pointed variety over k determined by the *identity element* $0_G \in G(k)$. If G, H are group schemes over k, then we shall refer to a k-morphism $G \to H$ as a *[k-]trans-homomorphism* if it factors as the composite of a homomorphism of group schemes $G \to H$ over k with an automorphism of H given by *translation* by an element of $H(k)$. If V is a *k-variety*, then we shall use the notation $\pi_1(V)$ to denote the *étale fundamental group* [relative to an appropriate choice of basepoint] of V. Thus, we have a natural exact sequence of fundamental groups $1 \to \pi_1(V \otimes_k \overline{k}) \to \pi_1(V) \to G_k \to 1$. Let $\Sigma_k \subseteq \mathfrak{Primes}$ [cf. §0] be the set of primes *invertible* in k; use the superscript "(Σ_k)" to denote the *maximal pro-Σ_k quotient* of a profinite group; if V is a *k-variety*, then we shall write

$$\Delta_V \stackrel{\mathrm{def}}{=} \pi_1(V_{\overline{k}})^{(\Sigma_k)}; \qquad \Pi_V \stackrel{\mathrm{def}}{=} \pi_1(V)/\mathrm{Ker}(\pi_1(V_{\overline{k}}) \twoheadrightarrow \pi_1(V_{\overline{k}})^{(\Sigma_k)})$$

for the resulting *geometrically pro-Σ_k fundamental groups*, so we have a natural exact sequence of fundamental groups $1 \to \Delta_V \to \Pi_V \to G_k \to 1$.

(ii) Let $\mathcal{C}$ be a *class of commutative group schemes of finite type* over k. If A is a group scheme over k that belongs to the class $\mathcal{C}$, then we shall write $A \in \mathcal{C}$. If (V, v) is a pointed k-variety, then we shall refer to a morphism of pointed k-varieties

$$\phi : V \to A$$

as a *$\mathcal{C}$-Albanese morphism* if $A \in \mathcal{C}$ [so A is equipped with a point $0_A \in A(k)$, as discussed in (i)], and, moreover, for any pointed k-morphism $\phi' : V \to A'$, where $A' \in \mathcal{C}$, there exists a unique homomorphism $\psi : A \to A'$ of group schemes over k such that $\phi' = \psi \circ \phi$. In this situation, A will also be referred to as the *$\mathcal{C}$-Albanese variety* of V. We shall write $\mathcal{C}_k^{\mathrm{ab}}$ for the class of *abelian varieties* over k and $\mathcal{C}_k^{\text{s-ab}}$ for the class of *semi-abelian varieties* over k. When $\mathcal{C} = \mathcal{C}_k^{\text{s-ab}}$, the term "$\mathcal{C}$-Albanese", will often be abbreviated *"Albanese"*.

(iii) If X is a k-variety (respectively, noetherian scheme) which admits a log structure such that the resulting log scheme $X^{\log}$ is *log smooth* over k [where we regard $\mathrm{Spec}(k)$ as equipped with the trivial log structure] (respectively, *log regular* [cf. [Kato]]), then we shall refer to X as *k-toric* (respectively, *absolutely toric*) and to $X^{\log}$ as a *torifier*, or *torifying log scheme*, for X. [Thus, "k-toric" implies "absolutely toric".]

(iv) If k is of *positive characteristic*, then, for any k-scheme X and integer $n \geq 1$, we shall write X^{F^n} for the result of base-changing X by the n-th iterate of the Frobenius morphism on k; thus, we obtain a k-linear *relative Frobenius morphism* $\Phi_X^n : X \to X^{F^n}$. If k is of *characteristic zero*, then we set $X^{F^n} \stackrel{\mathrm{def}}{=} X$, $\Phi_X^n \stackrel{\mathrm{def}}{=} \mathrm{id}_X$, for integers $n \geq 1$. If $\phi : X \to Y$ is a morphism of k-schemes, then we shall refer to ϕ as a *sub-Frobenius morphism* if, for some integer $n \geq 1$, there exists a k-morphism $\psi : Y \to X^{F^n}$ such that $\psi \circ \phi = \Phi_X^n$, $\phi^{F^n} \circ \psi = \Phi_Y^n$. [Thus, in characteristic zero, a sub-Frobenius morphism is simply an automorphism.]

Remark A.1.1. As is well-known, if V is a k-variety, then Φ_V^n induces an *isomorphism* $\Pi_V \xrightarrow{\sim} \Pi_{V^{F^n}}$, for all integers $n \geq 1$. Note that this implies that every *sub-Frobenius morphism* $V \to W$ of k-varieties induces *isomorphisms* $\Pi_V \xrightarrow{\sim} \Pi_W$, $\Delta_V \xrightarrow{\sim} \Delta_W$.

Before proceeding, we review the following well-known result.

Lemma A.2. **(Morphisms to Abelian and Semi-abelian Schemes)** *Let S be a noetherian scheme; X an S-scheme whose underlying scheme is* **absolutely toric***; A an* **abelian scheme** *over S (respectively, a* **semi-abelian scheme** *over S which is an extension of an abelian scheme $B \to S$ by a torus $T \to S$); $V \subseteq X$ an open subscheme whose complement in X is of* **codimension** ≥ 1 *(respectively, ≥ 2) in X. Then any morphism of S-schemes $V \to A$* **extends uniquely** *to X.*

Proof. First, we consider the case where A is an *abelian scheme*. If X is *regular*, then Lemma A.2 follows from [BLR], §8.4, Corollary 6. When X is an *arbitrary* absolutely toric scheme with *torifier* $X^{\log}$, we reduce immediately to the case where X is *strictly henselian*, hence admits a *resolution of singularities* [cf., e.g., [Mzk4], §2]

$$Y^{\log} \to X^{\log}$$

— i.e., a log étale morphism of log schemes which induces an *isomorphism* $U_Y \xrightarrow{\sim} U_X$ *between the respective interiors* such that $Y^{\log}$ arises from a divisor with normal crossings in a *regular* scheme Y. Since the "regular case" has already been settled, we may assume that $U_X \subseteq V$; also, it follows that the restriction $U_Y \to A$ to U_Y of the resulting morphism $U_X \to A$ *extends uniquely* to a morphism $Y \to A$. The image of this morphism determines a closed subscheme $Z \subseteq A_Y \stackrel{\text{def}}{=} A \times_S Y$. Moreover, by considering the *image* of Z under the morphism $A_Y \to A_X \stackrel{\text{def}}{=} A \times_S X$ of proper X-schemes, we conclude from *"Zariski's main theorem"* [since X is *normal*] that to obtain the [manifestly *unique*, since V is *schematically dense* in X] desired *extension* $X \to A$, it suffices to show that the fibers of $Y \to X$ map to points of A. On the other hand, as is observed in the discussion of [Mzk4], §2, each irreducible component of the fiber of $Y \to X$ at a point $x \in X$ is a *rational variety* over the residue field $k(x)$ at x, hence maps to a point in the abelian variety $A_x \stackrel{\text{def}}{=} A \times_S k(x)$ [cf., e.g., [BLR], §10.3, Theorem 1, (b), (c)]. This completes the proof of Lemma A.2 in the non-resp'd case. Thus, to complete the proof of Lemma A.2 in the resp'd case, we may assume that $A = T$ is a *torus* over S. In fact, by étale descent, we may even assume that T is a *split torus* over S. Then it suffices to show that if $\mathcal{L}$ is any line bundle on X that admits a generating section $s_V \in \Gamma(V, \mathcal{L})$, then it follows that s_V extends to a *generating section* of $\mathcal{L}$ over X. But since X is *normal*, this follows immediately from [SGA2], XI, 3.4; [SGA2], XI, 3.11. ◯

Proposition A.3. **(Basic Properties of Albanese Varieties)** *Let $\mathcal{C} \in \{\mathcal{C}_k^{\text{ab}}, \mathcal{C}_k^{\text{s-ab}}\}$; $\phi_V : V \to A$, $\phi_W : W \to B$ $\mathcal{C}$-***Albanese morphisms***. Then:*

(i) **(Base-change)** *Let k' be an* **algebraic field extension** *of k; denote the result of base-change [of k-schemes and morphisms of k-schemes] from k to k' by*

means of a subscript "k'". If $\mathcal{C} = \mathcal{C}_k^{\mathrm{ab}}$ (respectively, $\mathcal{C} = \mathcal{C}_k^{\text{s-ab}}$), then set $\mathcal{C}' = \mathcal{C}_{k'}^{\mathrm{ab}}$ (respectively, $\mathcal{C}' = \mathcal{C}_{k'}^{\text{s-ab}}$). Then $(\phi_V)_{k'} : V_{k'} \to A_{k'}$ is a $\mathcal{C}'$-**Albanese morphism**.

(ii) **(Functoriality)** *Given any k-morphism $\beta_V : V \to W$, there* **exists a unique** *k-trans-homomorphism $\beta_A : A \to B$ such that $\phi_W \circ \beta_V = \beta_A \circ \phi_V$. If, moreover, β_V is pointed, then β_A is a homomorphism.*

(iii) **(Relative Frobenius Morphisms)** *For any integer $n \geq 1$, $\phi_V^{F^n} : V^{F^n} \to A^{F^n}$ is a* $\mathcal{C}$-**Albanese morphism**. *If, moreover, in (ii), $\phi_W = \phi_V^{F^n}$, $\beta_V = \Phi_V^n$, then $\beta_A = \Phi_A^n$.*

(iv) **(Sub-Frobenius Morphisms)** *If, in (ii), β_V is a* **sub-Frobenius morphism**, *then so is β_A.*

(v) **(Toric Open Immersions)** *Suppose, in (ii), that β_V is an* **open immersion**, *that W is* ***k*-toric**, *and that if $\mathcal{C} = \mathcal{C}_k^{\mathrm{ab}}$ (respectively, $\mathcal{C} = \mathcal{C}_k^{\text{s-ab}}$), then the* **codimension** *of the complement of the image of β_V in W is ≥ 1 (respectively, ≥ 2). Then β_A is an* **isomorphism**.

(vi) **(Dominant Quotients)** *If, in (ii), β_V is* **dominant**, *then β_A is* **surjective**.

(vii) **(Surjectivity of Fundamental Groups)** *The [outer] homomorphisms $\Pi_{\phi_V} : \Pi_V \to \Pi_A$, $\Delta_{\phi_V} : \Delta_V \to \Delta_A$ induced by ϕ_V are* **surjective**.

(viii) **(Semi-abelian versus Abelian Albanese Morphisms)** *Suppose that $\mathcal{C} = \mathcal{C}_k^{\text{s-ab}}$. Write $A \twoheadrightarrow A^{\mathrm{ab}}$ for the maximal quotient of group schemes over k such that $A^{\mathrm{ab}} \in \mathcal{C}_k^{\mathrm{ab}}$. Then the composite morphism $V \to A \twoheadrightarrow A^{\mathrm{ab}}$ is a* $\mathcal{C}_k^{\mathrm{ab}}$-**Albanese morphism**.

(ix) **(Group Law Generation)** *For integers $n \geq 1$, write*

$$\zeta_n : V \times_k \ldots \times_k V \to A$$
$$(v_1, \ldots, v_n) \mapsto \textstyle\sum_{j=1}^n \; v_j$$

for the morphism from the product over k of n copies of V to A given by adding the images under ϕ_V of the points in the n factors. Then there exists an integer N such that ζ_n is **surjective** *for all $n \geq N$. In particular, if V is* **proper**, *then so is A.*

Proof. To verify assertion (i), we may assume that k' is a *finite* [hence necessarily *étale*, since k is *perfect*] extension of k. Then assertion (i) follows immediately by considering the *Weil restriction* functor $W_{k'/k}(-)$ from k' to k. That is to say, it is immediate that $W_{k'/k}(-)$ takes objects in $\mathcal{C}'$ to objects in $\mathcal{C}$. Thus, to give a k'-morphism $V_{k'} \to A'$ (respectively, $A_{k'} \to A'$) is equivalent to giving a k-morphism $V \to W_{k'/k}(A')$ (respectively, $A \to W_{k'/k}(A')$). This completes the proof of assertion (i). Assertions (ii), (iii) follow immediately from the definition of a *"$\mathcal{C}$-Albanese morphism"*; assertion (iv) follows immediately from assertion (iii).

Assertion (v) follows immediately from the definition of a *"$\mathcal{C}$-Albanese morphism"*, in light of Lemma A.2.

Assertion (vi) follows from the definition of a *"$\mathcal{C}$-Albanese morphism"*, by arguing as follows: First, we observe that β_V is an *epimorphism* in the category of schemes. Also, we may assume without loss of generality that β_V is *pointed*. Now consider the composite $\beta \circ \phi_W : W \to B/C$ of $\phi_W : W \to B$ with the natural quotient morphism $\beta : B \twoheadrightarrow B/C$, where we write $C \stackrel{\text{def}}{=} \text{Im}(\beta_A) \subseteq B$ [so $C \in \mathcal{C}$]. Since $\beta \circ \phi_W$ has the same restriction [via β_V] to V as the constant pointed morphism $W \to B/C$, we thus conclude that $\beta \circ \phi_W$ is *constant*, i.e., that $\text{Im}(\phi_W) \subseteq C$. But, by the definition of a *"$\mathcal{C}$-Albanese morphism"*, this implies the existence of a *section* $B \to C$ of the natural inclusion $C \hookrightarrow B$, hence that $B = C$, as desired. In a similar vein, assertion (vii) follows from the definition of a *"$\mathcal{C}$-Albanese morphism"*, by observing that if $\Pi_{\phi_V} : \Pi_V \to \Pi_A$ *fails* to surject, then [after possibly replacing k by a finite extension of k, which is possible, by assertion (i)] it follows that $\phi_V : V \to A$ factors $V \to C \to A$, where the morphism $C \to A$ is a *nontrivial finite étale Galois covering*, with C *geometrically connected* over k, so $C \in \mathcal{C}$. But this implies, by the definition of a *"$\mathcal{C}$-Albanese morphism"*, the existence of a *section* $A \to C$ of the surjection $C \twoheadrightarrow A$, hence that this surjection is an isomorphism $C \xrightarrow{\sim} A$, a contradiction.

Next, we observe that assertion (viii) follows immediately from the definitions, in light of the well-known fact that any homomorphism $G \to H$ of group schemes over k, where G is a *torus* and H is an *abelian variety*, is *trivial* [cf., e.g., [BLR], §10.3, Theorem 1, (b), (c)].

Finally, we consider assertion (ix). First, let us observe that we may assume without loss of generality that $k = \overline{k}$. Next, let us observe that since the image of ϕ_V contains $0_A \in A(k)$, it follows that for $n \geq m$, the image $I_n \subseteq A(k)$ of ζ_n contains the image I_m of ζ_m. Write $F_n \subseteq A$ for the [reduced closed subscheme given by the] *closure* of I_n. Since the domain of ζ_n is *irreducible*, it follows immediately that F_n is *irreducible*. Thus, the ascending sequence $\ldots \subseteq F_m \subseteq \ldots \subseteq F_n \subseteq \ldots$ *terminates*, i.e., we have $F_n = F_m$ for all $n, m \geq N'$, for some N'; write $F \stackrel{\text{def}}{=} F_{N'}$. Since $I_{N'}$ is *constructible*, it follows that $I_{N'}$ contains a nonempty open subset U of [the underlying topological space of] F; let $u \in U(k)$. Now let us write I'_n for the union of the translates of U by elements of I_n; thus, one verifies immediately that I'_n is *open* in F, that $I'_n \subseteq I_{n+N'}$, and that $u + I_n \subseteq I'_n$. Since F is *noetherian*, it thus follows that the ascending sequence $\ldots \subseteq I'_m \subseteq \ldots \subseteq I'_n \subseteq \ldots$ *terminates*, i.e., that for some $N'' > N'$, we have $I'_n = I'_m$ for all $n, m \geq N''$; write $I \subseteq F$ for the resulting open subscheme. Thus, for $n \geq N''$, $u + I \subseteq u + I_{n+N'} \subseteq I$. On the other hand, again since F is *noetherian*, it follows that the ascending sequence $I \subseteq I - u \subseteq I - 2u \subseteq \ldots$ *terminates*, hence that $u + I = I$. In particular, for some $N''' > N''$, we have $I_n = I$, for $n \geq N'''$. Next, let us observe that for any $j \in I(k)$, it follows from the definition of the I_n that $j + I \subseteq I$, hence [as in the case where $j = u$], we have $j + I = I$. Since $0_A \in I$, it thus follows that I is *closed* under the group operation on A, as well as taking inverses in A. Thus, it follows that I is a *subgroup scheme* of A, hence that I is a *closed subscheme* of A [so $I = F$]. But this implies, by the definition of a *"$\mathcal{C}$-Albanese morphism"*, the existence of a *homomorphism* $A \to I$ whose composite with the inclusion $I \hookrightarrow A$ is the identity on A. Thus, we conclude that the inclusion $I \hookrightarrow A$ is a *surjection*, i.e., that $I = A$, as desired. ○

A proof of the following result may be found, in essence, in [NS] [albeit in somewhat *archaic* language], as well as in [FGA], 236, Théorème 2.1, (ii) [albeit in somewhat *sketchy* form]. Various other approaches [e.g., via *Weil divisors*] to this result are discussed in [Klei], Theorem 5.4 and the discussion following [Klei], Theorem 5.4.

Theorem A.4. (Properness of the Identity Component of the Picard Scheme) *The identity component of the Picard scheme*

$$\mathrm{Pic}^0_{V/k}$$

[cf., e.g., [BLR], §8.2, Theorem 3; [BLR], §8.4] associated to a **complete normal variety** V *over a field* k *is* **proper**.

Proof. Write G for the *reduced group scheme* $(\mathrm{Pic}^0_{V/k})_{\mathrm{red}}$ over k. Then by a well-known *theorem of Chevalley* [cf., e.g., [Con], for a treatment of this result in modern language], it follows that to show that G [hence also $\mathrm{Pic}^0_{V/k}$] is *proper*, it suffices to show that G does not contain any copies of the multiplicative group $(\mathbb{G}_{\mathrm{m}})_k$ or the additive group $(\mathbb{G}_{\mathrm{a}})_k$. On the other hand, since $(\mathbb{G}_{\mathrm{m}})_k$, $(\mathbb{G}_{\mathrm{a}})_k$ may be thought of as open subschemes of the affine line $\mathbb{A}^1_k$, this follows immediately from Lemma A.5 below [i.e., by applying the *functorial interpretation* of $\mathrm{Pic}^0_{V/k}$ — cf., e.g., [BLR], §8.1, Proposition 4]. ○

Lemma A.5. (Rational Families of Line Bundles) *Let* V *be a* **normal variety** *over* k*;* $U \subseteq \mathbb{A}^1_k$ *a nonempty open subscheme of the* **affine line** $\mathbb{A}^1_k$*. Then every line bundle* $\mathcal{L}_U$ *on* $V \times_k U$ *arises via* **pull-back** *from a line bundle* $\mathcal{L}_k$ *on* V.

Proof. In the following, let us regard $\mathbb{A}^1_k$ as an open subscheme $\mathbb{A}^1_k \subseteq \mathbb{P}^1_k$ of the projective line [obtained in the standard way by removing the point at infinity $\infty_k \in \mathbb{P}^1_k(k)$]. First, let us verify Lemma A.5 under the further hypothesis that V is *smooth* over k. Then it follows immediately that $V \times_k \mathbb{P}^1_k$ is *smooth* over k, hence *locally factorial* [cf., e.g., [SGA2], XI, 3.13, (i)]. Thus, $\mathcal{L}_U$ *extends* to a line bundle $\mathcal{L}_P$ on $P \stackrel{\mathrm{def}}{=} V \times_k \mathbb{P}^1_k$ $(\supseteq V \times_k \mathbb{A}^1_k \supseteq V \times_k U)$. Moreover, by tensoring with line bundles associated to multiples of the divisor on P arising from ∞_k, we may assume that the degree of $\mathcal{L}_P$ on the fibers of the trivial projective bundle $f : P \to V$ is *zero*. Thus, the natural morphism $f^* f_* \mathcal{L}_P \to \mathcal{L}_P$ is an *isomorphism*, which exhibits $\mathcal{L}_P$, hence also $\mathcal{L}_U$, as a line bundle $\mathcal{L}_k$ pulled back from V.

Now we return to the case of an *arbitrary normal variety* V. As is well-known, V contains a dense open subscheme $W \subseteq V$ which is *smooth* over k and such that the closed subscheme $F \stackrel{\mathrm{def}}{=} V \backslash W$ [where we equip F with the reduced induced structure] is of *codimension* ≥ 2 in V [cf., e.g., [SGA2], XI, 3.11, applied to the geometric fiber of $V \to \text{Spec}(k)$]. Thus, by the argument given in the smooth case, we conclude that $\mathcal{M}_U \stackrel{\text{def}}{=} \mathcal{L}_U|_{W \times_k U}$ arises from a line bundle $\mathcal{M}_k$ on W. Next, let us write $\iota_k : W \hookrightarrow V$, $\iota_U : W \times_k U \hookrightarrow V \times_k U$ for the natural open immersions. Since U is *k-flat*, it follows immediately that we have a *natural isomorphism*

$$((\iota_k)_* \mathcal{M}_k)|_{V \times_k U} \xrightarrow{\sim} (\iota_U)_* \mathcal{M}_U$$

[arising, for instance, by computing the right-hand side by means of an affine covering of $W \times_k U$ obtained by taking the product over k with U of an affine covering of W]. On the other hand, since $V \times_k U$ is *normal* and $F \times_k U \subseteq V \times_k U$ is a closed subscheme of *codimension* ≥ 2, it follows from the definition of $\mathcal{M}_U$ that $(\iota_U)_* \mathcal{M}_U \xrightarrow{\sim} \mathcal{L}_U$ [cf., e.g., [SGA2], XI, 3.4; [SGA2], XI, 3.11], i.e., that $((\iota_k)_* \mathcal{M}_k)|_{V \times_k U}$ is a *line bundle* on $V \times_k U$. On the other hand, since the morphism $U \to \text{Spec}(k)$, hence also the projection morphism $V \times_k U \to V$, is *faithfully flat*, we thus conclude that $\mathcal{L}_k \stackrel{\text{def}}{=} (\iota_k)_* \mathcal{M}_k$ is a line bundle on V whose pull-back to $V \times_k U$ is isomorphic to $\mathcal{L}_U$, as desired. ◯

Proposition A.6. (Duals of Picard Varieties as Albanese Varieties) *Let V be a* **complete normal variety** *over k;* $\text{Pic}^0_{V/k}$ *the identity component of the associated* **Picard scheme***; A the* **dual abelian variety** *to* $G \stackrel{\text{def}}{=} (\text{Pic}^0_{V/k})_{\text{red}}$ *[which is an abelian variety by Theorem A.4]; $v \in V(k)$. Then the* **universal line bundle** $\mathcal{P}_V$ *[cf., e.g., [BLR], §8.1, Proposition 4] on $V \times_k G$ relative to the rigidification determined by v [i.e., such that $\mathcal{P}_V|_{\{v\} \times G}$ is* **trivial***] determines [by the definition of A] a morphism of pointed k-varieties*

$$\phi : V \to A$$

such that the pull-back of the Poincaré bundle $\mathcal{P}_A$ on $A \times_k G$ via $\phi \times_k G : V \times_k G \to A \times_k G$ is isomorphic to $\mathcal{P}_V$ [in a fashion compatible with the respective rigidifications]. Moreover:

(i) The morphism ϕ is a $\mathcal{C}^{\text{ab}}_k$**-Albanese morphism***.*

(ii) Suppose, in the situation of Proposition A.3, (ii), that W is also **complete** *and* **normal***, and that β_V is pointed and* **birational***. Then the* **dual morphism** *$\beta_G : H \to G$ to $\beta_A : A \to B$ is a* **closed immersion***. In particular, β_A is an* **isomorphism** *if and only if* $\dim_k(A) \leq \dim_k(B)$.

(iii) The morphism ϕ induces an **injection** $H^1(A, \mathcal{O}_A) \hookrightarrow H^1(V, \mathcal{O}_V)$.

(iv) The morphism ϕ induces an **isomorphism** $\Delta_V^{\text{ab-t}} \xrightarrow{\sim} \Delta_A$ *[where we refer to §0 for the notation "*ab-t*"].*

Proof. First, we consider assertion (i). Let $\psi_V : V \to C$ be a morphism of pointed k-varieties, where $C \in \mathcal{C}^{\text{ab}}_k$. Now by the *functoriality* of "$\text{Pic}^0_{(-)/k}$", ψ_V induces a morphism $D \stackrel{\mathrm{def}}{=} \mathrm{Pic}^0_{C/k} \to \mathrm{Pic}^0_{V/k}$ [so D is the dual abelian variety to C], hence a morphism $\psi_D : D \to G$, whose dual gives a morphism $\psi_A : A \to C$. The fact that $\psi_V = \psi_A \circ \phi : V \to C$ follows by thinking of morphisms as *classifying morphisms* for line bundles and considering the following [*a priori*, not necessarily commutative] *diagram* of morphisms between varieties equipped with [isomorphism classes of] line bundles:

$$\begin{array}{ccccc} (V \times_k D, \mathcal{L}) & \stackrel{\mathrm{id}}{\longrightarrow} & (V \times_k D, \mathcal{L}) & \stackrel{V \times_k \psi_D}{\longrightarrow} & (V \times_k G, \mathcal{P}_V) \\ \downarrow{\scriptstyle \psi_V \times_k D} & & \downarrow{\scriptstyle \phi \times_k D} & & \downarrow{\scriptstyle \phi \times_k G} \\ (C \times_k D, \mathcal{P}_C) & \stackrel{\psi_A \times_k D}{\longleftarrow} & (A \times_k D, \mathcal{M}) & \stackrel{A \times_k \psi_D}{\longrightarrow} & (A \times_k G, \mathcal{P}_A) \end{array}$$

— where we write $\mathcal{L} \stackrel{\mathrm{def}}{=} (\psi_V \times_k D)^* \mathcal{P}_C$; $\mathcal{M} \stackrel{\mathrm{def}}{=} (\psi_A \times_k D)^* \mathcal{P}_C \cong (A \times_k \psi_D)^* \mathcal{P}_A$. That is to say, the desired commutativity of the left-hand square follows by computing:

$$\begin{aligned} (\phi \times_k D)^* (\psi_A \times_k D)^* \mathcal{P}_C &\cong (\phi \times_k D)^* (A \times_k \psi_D)^* \mathcal{P}_A \\ &\cong (V \times_k \psi_D)^* (\phi \times_k G)^* \mathcal{P}_A \\ &\cong (V \times_k \psi_D)^* \mathcal{P}_V \\ &\cong (\psi_V \times_k D)^* \mathcal{P}_C \end{aligned}$$

— which implies that $\psi_V = \psi_A \circ \phi$. Finally, the *uniqueness* of such a "ψ_A" follows immediately by applying "$\mathrm{Pic}^0_{(-)/k}$" to the condition "$\psi_V = \psi_A \circ \phi : V \to A \to C$". This completes the proof of assertion (i).

Next, we consider assertion (ii). First, observe that there exists a *k-smooth* open subscheme $U \subseteq W$ such that $W \backslash U$ has codimension ≥ 2 in W [cf., e.g., [SGA2], XI, 3.11, as it was applied in the proof of Lemma A.5], and, moreover, $\beta_V : V \to W$ admits a *section* $\sigma : U \to V$ over U. Note, moreover, that if S is any local artinian finite k-scheme, and we write $\iota_S : U_S \stackrel{\mathrm{def}}{=} U \times_k S \hookrightarrow W_S \stackrel{\mathrm{def}}{=} W \times_S k$ for the natural inclusion, then for any line bundle $\mathcal{L}$ on W_S, we have a *natural isomorphism* $(\iota_S)_*(\iota_S^* \mathcal{L}) \stackrel{\sim}{\to} \mathcal{L}$ [cf., e.g., [SGA2], XI, 3.4; [SGA2], XI, 3.11]. Thus, by applying this natural isomorphism, together with the section σ, we conclude that the map $\mathrm{Pic}^0_{W/k}(S) \to \mathrm{Pic}^0_{V/k}(S)$ [induced by β_V] is an *injection*, which implies that the kernel group scheme of $\beta_G : H \to G$ is *trivial*, hence that β_G is a *closed immersion*, as desired. This completes the proof of assertion (ii).

Next, we consider assertion (iii). The morphism $H^1(A, \mathcal{O}_A) \to H^1(V, \mathcal{O}_V)$ in question may be interpreted as the *morphism induced by ϕ on tangent spaces to the Picard scheme*, i.e., as the morphism

$$G(k[\epsilon]/(\epsilon^2)) = \mathrm{Pic}^0_{A/k}(k[\epsilon]/(\epsilon^2)) \to \mathrm{Pic}^0_{V/k}(k[\epsilon]/(\epsilon^2))$$

[cf., e.g., [BLR], §8.4, Theorem 1, (a)]. But, by the definition of G, this morphism arises from a closed immersion $G \hookrightarrow \mathrm{Pic}^0_{V/k}$, hence is an *injection*, as desired.

Finally, we consider assertion (iv). The *surjectivity* portion of assertion (iv) follows immediately from Proposition A.3, (vii). To verify the fact that the surjection $\Delta_V^{\text{ab-t}} \twoheadrightarrow \Delta_A$ is an *isomorphism*, we reason as follows: First, we recall that a line bundle $\mathcal{L}$ on V such that $\mathcal{L}^{\otimes n}$ [where $n \geq 1$ is an integer *invertible* in k] is trivial may be interpreted [via the *Kummer exact sequence* in étale cohomology] as a continuous homomorphism $\Delta_V \to (\mathbb{Z}/n\mathbb{Z})(1)$ [where the "(1)" denotes a *"Tate twist"*]. On the other hand, by [BLR], §8.4, Theorem 7, there exists an integer $m \geq 1$ such that for every integer $n \geq 1$, the *cokernel* of the inclusion ${}_nG(\overline{k}) \hookrightarrow {}_n\mathrm{Pic}_{V/k}(\overline{k})$ [where the "${}_n$" preceding an abelian group denotes the kernel of multiplication by n] is *annihilated by* m. In light of the *functorial interpretation* of the inclusion $G \hookrightarrow \mathrm{Pic}^0_{V/k} \subseteq \mathrm{Pic}_{V/k}$, this implies that the *cokernel* of the homomorphism $\mathrm{Hom}(\Delta_A, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(\Delta_V, \mathbb{Q}/\mathbb{Z})$ is *annihilated by* m. But, by applying $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$, this implies that the induced homomorphism $\Delta_V^{\mathrm{ab}} \to \Delta_A$ has *finite kernel*, hence [in light of the *surjectivity* already verified] induces an *isomorphism* upon passing to "ab-t". ◯

Remark A.6.1. The content of Proposition A.6, (i), is discussed in [FGA], 236, Théorème 3.3, (iii).

Remark A.6.2. Suppose that we are in the situation of Proposition A.6, (ii). Then it is *not* necessarily the case that the induced morphism β_A is an *isomorphism*. This phenomenon already appears in the work of *Chevalley* — cf. [Chev]; the discussion of [Klei], p. 248; Example A.7 below.

Example A.7. Albanese Varieties and Resolution of Singularities. For simplicity, suppose that $k = \overline{k}$. Write $\mathbb{P}^2_k = \mathrm{Proj}(k[x_1, x_2, x_3])$ [i.e., where we consider $k[x_1, x_2, x_3]$ as a *graded ring*, in which x_1, x_2, x_3 are of degree 1]. Let $f \in k[x_1, x_2, x_3]$ be a homogeneous polynomial that defines a *smooth plane curve* $X \subseteq \mathbb{P}^2_k$ of genus ≥ 1. Thus, any $x \in X(k)$ determines an embedding $X \hookrightarrow J$, where J is the *Jacobian variety* of X. Set $Y \stackrel{\mathrm{def}}{=} \mathrm{Spec}(k[x_1, x_2, x_3]/(f))$; write $y \in Y(k)$ for the origin, $U_Y \stackrel{\mathrm{def}}{=} Y \backslash \{y\}$. Thus, we have a natural morphism $Y \supseteq U_Y \to X$; $U_Y \to X$ is a $\mathbb{G}_{\mathrm{m}}$-*torsor* over X. In particular, U_Y is k-*smooth*. Thus, since Y is clearly a *local complete intersection* [hence, in particular, Cohen-Macaulay], it follows from *Serre's criterion of normality* [cf., e.g., [SGA2], XI, 3.11] that Y is *normal*. Let $Z \to Y$ be the *blow-up* of Y at the origin y. Thus, we obtain an *isomorphism* $U_Z \stackrel{\mathrm{def}}{=} Z \times_Y U_Y \stackrel{\sim}{\to} U_Y$. Moreover, one verifies immediately that the morphism $U_Z \stackrel{\sim}{\to} U_Y \to X$ extends to a morphism $Z \to X$ which has the structure of an $\mathbb{A}^1$-*bundle*, in which $E \stackrel{\mathrm{def}}{=} Z \times_Y \{y\} \subseteq Z$ forms a *"zero section"* [so $E \stackrel{\sim}{\to} X$]. Thus, Z admits a natural compactification $Z \hookrightarrow Z^*$ to a $\mathbb{P}^1$-*bundle* $Z^* \to X$. Moreover, by gluing $Z^* \backslash E$ to Y along $Z \backslash E = U_Z \stackrel{\sim}{\to} U_Y \subseteq Y$, we obtain a compactification $Y \hookrightarrow Y^*$ such that the blow-up morphism extends to a morphism $Z^* \to Y^*$ [which may be thought of as the blow-up of Y^* at $y \in Y(k) \subseteq Y^*(k)$]. On the other hand, note that the composite $Z^* \to X \hookrightarrow J$ determines a *closed immersion* $Z^* \supseteq E \stackrel{\sim}{\to} X \hookrightarrow J$. Thus, the restriction $U_Y \stackrel{\sim}{\to} U_Z \to J$ of this morphism $Z^* \to J$ to $U_Y \stackrel{\sim}{\to} U_Z$ *does not extend* to Y or Y^*. In particular, it follows that if we write $Y^* \to A_Y$, $Z^* \to A_Z$ for the $\mathcal{C}_k^{\mathrm{ab}}$-*Albanese varieties* of Proposition A.6, (i), then the surjection $A_Y \twoheadrightarrow A_Z$ induced by $Y^* \to Z^*$ [cf. Proposition A.6, (ii)] *is not an isomorphism*.

Proposition A.8. (Albanese Varieties of Complements of Divisors with Normal Crossings) *Let Z be a* **smooth projective variety** *over k; $D \subseteq Z$ a* **divisor with normal crossings**; *$Y \stackrel{\text{def}}{=} Z \backslash D \subseteq Z$; $y \in Y(k)$;*

$$D = \bigcup_{n=1}^{r} D_n$$

[for some integer $r \geq 1$] the decomposition of D into irreducible comonents; M the free $\mathbb{Z}$-module [of rank r] of **divisors** *supported on D; $P \subseteq M$ the submodule of divisors that determine a line bundle $\in \mathrm{Pic}^0_{Z/k}(k)$. Then:*

(i) (Y, y) admits an **Albanese morphism** *$Y \to A_Y$.*

(ii) Suppose that each of the D_n is **geometrically irreducible**. *Then the A_Y of (i) may be taken to be an* **extension** *of the* **abelian variety** *A_Z given by the* **dual** *to $G_Z \stackrel{\text{def}}{=} (\mathrm{Pic}^0_{Z/k})_{\text{red}}$ [cf. Propositions A.3, (viii); A.6, (i)] by a* **torus** *whose* **character group** *is naturally isomorphic to P.*

(iii) The morphism $Y \to A_Y$ of (i) induces an **isomorphism** *$\Delta_Y^{\text{ab-t}} \xrightarrow{\sim} \Delta_{A_Y}$.*

Proof. By étale descent [with respect to finite extensions of k], it follows immediately that to verify assertion (i), it suffices to verify assertion (ii). Next, we consider assertion (ii). Again, by étale descent, we may assume without loss of generality that $k = \overline{k}$. Note that the tautological homomorphism $P \to G_Z(k)$ determines an *extension*

$$0 \to T_Y \to A_Y \to A_Z \to 0$$

of A_Z by a split torus T_Y with *character group* P. Now the fact that A_Y serves as an *Albanese variety* for Y is essentially a *tautology*: Indeed, since any pointed morphism from Y to an abelian variety C extends [cf. Lemma A.2] to a pointed morphism $Z \to C$, and, moreover, we already know that A_Z is a $\mathcal{C}_k^{\text{ab}}$-*Albanese variety* for Z [cf. Proposition A.6, (i)], it follows that it suffices to consider pointed morphisms $Y \to B$, where B is an extension of A_Z by a [split] torus. In fact, for simplicity, we may even assume that this torus is simply $(\mathbb{G}_{\text{m}})_k$. Thus, it suffices to consider pointed morphisms $Y \to B$, where B is an extension of A_Z by $(\mathbb{G}_{\text{m}})_k$, determined by some extension class $\kappa_B \in G_Z(k)$. Then the datum of a morphism $Y \to B$ corresponds precisely to an *invertible section* of the restriction to Y of the line bundle $\mathcal{L}$ on Z given by pulling back the $\mathbb{G}_{\text{m}}$-torsor $B \to A_Z$ via the Albanese morphism $Z \to A_Z$. Note that such an invertible section of $\mathcal{L}|_Y$ may be thought of as the datum of an *isomorphism* $\mathcal{O}_Z(E) \xrightarrow{\sim} \mathcal{L}$ for some divisor E supported on D. That is to say, since the isomorphism class of $\mathcal{L}$ is precisely the class determined by the element $\kappa_B \in G_Z(k) \subseteq \mathrm{Pic}_{Z/k}(k)$, it thus follows that $E \in P$, and that κ_B is the image of $E \in P$ in $\mathrm{Pic}^0_{Z/k}(k) = G_Z(k)$. Thus, in summary, the datum of a pointed morphism $Y \to B$, where B is an extension of A_Z by a [split] torus, is equivalent [in a *functorial* way] to the datum of a homomorphism $A_Y \to B$ lying over the identity morphism of A_Z. In particular, the identity morphism $A_Y \to A_Y$ determines a morphism $Y \to A_Y$. This completes the proof of assertion (ii).

Finally, we consider assertion (iii). We may assume without loss of generality that $k = \overline{k}$. Let $F \subseteq D$ be a closed subscheme of codimension ≥ 1 in D such that $Z' \stackrel{\mathrm{def}}{=} Z \backslash F \subseteq Z$, $D' \stackrel{\mathrm{def}}{=} D \backslash F \subseteq D$ are *k-smooth*. Then one has the associated *Gysin sequence* in étale cohomology

$$0 \to H^1_{\text{ét}}(Z', \mathbb{Z}_l(1)) \to H^1_{\text{ét}}(Y, \mathbb{Z}_l(1)) \to M \otimes \mathbb{Z}_l \to H^2_{\text{ét}}(Z', \mathbb{Z}_l(1))$$

for $l \in \Sigma_k$ [cf. [Milne], p. 244, Remark 5.4, (b)]. Moreover, we have natural isomorphisms $H^j_{\text{ét}}(Z', \mathbb{Z}_l(1)) \xrightarrow{\sim} H^j_{\text{ét}}(Z, \mathbb{Z}_l(1))$, for $j = 1, 2$. [Indeed, by applying *noetherian induction*, it suffices to verify these isomorphisms in the case where F is *k-smooth*, in which case these isomorphisms follow from [Milne], p. 244, Remark 5.4, (b).] Note, moreover, that the morphism $M \otimes \mathbb{Z}_l \to H^2_{\text{ét}}(Z', \mathbb{Z}_l(1)) \xrightarrow{\sim} H^2_{\text{ét}}(Z, \mathbb{Z}_l(1))$ is precisely the *"fundamental class map"*, hence factors through the *natural inclusion*

$$\mathrm{Pic}_{Z/k}(k)^{\wedge} \hookrightarrow H^2_{\text{ét}}(Z, \mathbb{Z}_l(1))$$

[where the "$\wedge$" denotes the *pro-l completion*] arising from the *Kummer exact sequence* on Z. On the other hand, since $\mathrm{Pic}^0_{Z/k}(k)$ is *l-divisible*, and the quotient $\mathrm{Pic}_{Z/k}(k)/\mathrm{Pic}^0_{Z/k}(k)$ is *finitely generated* [cf. [BLR], §8.4, Theorem 7], it follows that we have an *isomorphism*

$$(\mathrm{Pic}_{Z/k}(k)/\mathrm{Pic}^0_{Z/k}(k)) \otimes \mathbb{Z}_l \xrightarrow{\sim} \mathrm{Pic}_{Z/k}(k)^{\wedge}$$

— i.e., that the kernel of the morphism $M \otimes \mathbb{Z}_l \to H^2_{\text{ét}}(Z', \mathbb{Z}_l(1))$ is precisely $P \otimes \mathbb{Z}_l$. In particular, the isomorphism $\Delta^{\text{ab-t}}_Z \xrightarrow{\sim} \Delta_{A_Z}$ of Proposition A.6, (iv), implies [in light of the above exact sequence] that $H^1_{\text{ét}}(Y, \mathbb{Z}_l(1))$ [i.e., $\mathrm{Hom}(\Delta^{\text{ab-t}}_Y, \mathbb{Z}_l(1))$], hence also $\Delta^{\text{ab-t}}_Y \otimes \mathbb{Z}_l$, is a *free* $\mathbb{Z}_l$*-module of rank* $\dim_k(A_Y)$. Thus, we conclude that the *surjection* $\Delta^{\text{ab-t}}_Y \twoheadrightarrow \Delta_{A_Y}$ of Proposition A.3, (vii), is an *isomorphism*, as desired. ○

Remark A.8.1. A sharper version [in the sense that it includes a computation of the torsion subgroup of Δ^{ab}_Y] of Proposition A.8, (iii), is given in [SS], Proposition 4.2. The discussion of [SS] involves the point of view of 1-*motives*. On the other hand, such a sharper version may also be obtained directly from the *Gysin sequence* argument of the above proof of Proposition A.8, (iii), by working with torsion coefficients.

The following result is elementary and well-known.

Lemma A.9. **(Descending Chains of Subgroup Schemes)** *Let G be a [not necessarily reduced]* **commutative group scheme** *of finite type over k;*

$$\ldots \subseteq G_n \subseteq \ldots \subseteq G_1 \subseteq G_0 = G$$

a **descending chain** *of [not necessarily reduced!] subgroup schemes of G, indexed by the nonnegative integers. Then there exists an integer N such that $G_n = G_m$ for all $n, m \geq N$.*

Proof. First, let us consider the case where all of the G_n, for $n \geq 0$, are *reduced* and *connected*. Then since all of the G_n are *closed irreducible subschemes* of G, it follows immediately that if we take any integer N such that $\dim_k(G_n) = \dim_k(G_m)$ for all $n, m \geq N$, then $G_n = G_m$ for all $n, m \geq N$. Now we return to the *general case*. By what we have done so far, we may assume without loss of generality that $(G_0)_{\text{red}} = (G_n)_{\text{red}}$ for all $n \geq 0$. Thus, by forming the quotient by $(G_0)_{\text{red}}$, we may assume that all of the G_n are *finite* over $\text{Spec}(k)$. Then Lemma A.9 follows immediately. ○

Before proceeding, we recall the following result of *de Jong*.

Lemma A.10. **(Equivariant Alterations)** *Suppose that $k = \overline{k}$; let V be a* **variety** *over k. Then there exists a* **smooth projective variety** *Z over k, a* **finite group** *Γ of automorphisms of Z over k, a* **divisor with normal crossings** *$D \subseteq Z$ fixed by Γ, and a Γ-equivariant [relative to the trivial action of Γ on V]* **dominant, proper, generically quasi-finite morphism**

$$Y \stackrel{\text{def}}{=} Z \backslash D \to V$$

such that if we write $k(Z)$, $k(V)$ for the respective function fields of Z, V, then the subfield of Γ-invariants $k(Z)^\Gamma \subseteq k(Z)$ forms a **purely inseparable extension** *of $k(V)$.*

Proof. This is the content of [deJong], Theorem 7.3. ○

We are now ready to prove the *main result* of the present Appendix, the first portion of which [i.e., Corollary A.11, (i)] is due to *Serre* [cf. [Serre1]].

Corollary A.11. **(Albanese Varieties of Arbitrary Varieties)**

(i) Every **pointed variety** *(V, v) over k admits an* **Albanese morphism** *$V \to A$.*

(ii) Let $\phi : V \to A$ be an **Albanese morphism**, *where (V, v) is a* ***k*-toric pointed variety**. *Then ϕ induces an* **isomorphism** *$\Delta_V^{\text{ab-t}} \xrightarrow{\sim} \Delta_A$.*

Proof. First, we consider assertion (i). By applying étale descent, we may assume without loss of generality that $k = \overline{k}$. Let $Z \supseteq Y \to V$ be as in Lemma A.10, $y \in Y(k)$ a point that maps to $v \in V(k)$ [where we observe that, as is easily verified, the existence of an Albanese morphism as desired is *independent* of the choice of v]. Then by Proposition A.8, (i), it follows that Y admits an *Albanese morphism* $Y \to B$. Thus, every pointed morphism $\nu : V \to C$, where $C \in \mathcal{C}_k^{\text{s-ab}}$, determines, by restriction to Y, a homomorphism $B \to C$, whose kernel is a *subgroup scheme* $H_\nu \subseteq B$. In particular, the pointed morphisms $\nu : V \to C$ determine a *projective system* of subgroup schemes $H_\nu \subseteq B$ which is *filtered* [a fact that is easily verified by considering product morphisms $V \to C_1 \times_k C_2$ of pointed morphisms $\nu_1 : V \to C_1$, $\nu_2 : V \to C_2$]. Moreover, by Lemma A.9, this projective system admits a cofinal subsystem which is *constant*, i.e., given by a single subgroup scheme $H \subseteq B$. Now it is a *tautology* that the composite morphism $Y \to B \twoheadrightarrow B/H$ *factors uniquely* [where we observe that uniqueness follows from the fact that $Y \to V$ is *dominant*] through a morphism $V \to B/H$ which serves as an *Albanese morphism* for V.

Next, we consider assertion (ii). First, let us observe that, by Proposition A.3, (i), we may assume without loss of generality that $k = \overline{k}$. Let $Z \supseteq Y \to V$, Γ be as in Lemma A.10; write $Y \to V' \to V$ for the factorization through the *normalization* $V' \to V$ of V in the *purely inseparable* extension $k(Z)^\Gamma$ of $k(V)$. Let $V \to A$, $V' \to A'$ be *Albanese morphisms* [which exist by assertion (i)]. Since V is *normal*, it follows immediately that $V' \to V$ is a *sub-Frobenius morphism*. Thus, by Proposition A.3, (iv) [cf. also Remark A.1.1], it follows that $V' \to V$ induces *isomorphisms* $\Delta^{\text{ab-t}}_{V'} \xrightarrow{\sim} \Delta^{\text{ab-t}}_{V}$, $\Delta_{A'} \xrightarrow{\sim} \Delta_A$. In particular, by replacing V by V', we may assume without loss of generality that $Y \to V$ is *generically étale*.

Next, let us observe that since V is *k-toric*, it follows that there exists a closed subscheme $F \subseteq V$ of *codimension* ≥ 2 such that $U \stackrel{\text{def}}{=} V \backslash F$ is *k-smooth* [cf., e.g., [SGA2], XI, 3.11]. Note, moreover, that the composite $U \to V \to A$ is an *Albanese morphism* for U [cf. Proposition A.3, (v)]. Thus, we have *surjections*

$$\Delta^{\text{ab-t}}_{U} \twoheadrightarrow \Delta^{\text{ab-t}}_{V} \twoheadrightarrow \Delta_A$$

[cf. Proposition A.3, (vii)]. In particular, if the surjection $\Delta^{\text{ab-t}}_{U} \twoheadrightarrow \Delta_A$ is an *isomorphism*, then so is the surjection $\Delta^{\text{ab-t}}_{V} \twoheadrightarrow \Delta_A$. Thus, we may assume without loss of generality that V is *k-smooth*.

Next, let $Y \to B$ be an *Albanese morphism* for B [cf. Proposition A.8, (i)]. Then, by Proposition A.3, (ii), the action of Γ on Y extends to a compatible action of Γ on B by *k-trans-homomorphisms*. This action of Γ on B may be thought of as the combination of an action of Γ on the *group scheme* B [i.e., via group scheme automorphisms], together with a *twisted homomorphism* $\chi : \Gamma \to B(k)$ [where Γ acts on $B(k)$ via the *group scheme action* of Γ on B]. Write $B \twoheadrightarrow C'$ for the *quotient semi-abelian scheme* of B by the *group scheme action* Γ, i.e., the quotient of B by the subgroup scheme generated by the images of the group scheme endomorphisms $(1 - \gamma) : B \to B$, for $\gamma \in \Gamma$. Thus, χ determines a *homomorphism* $\chi' : \Gamma \to C'(k)$; write $C' \to C$ for the *quotient semi-abelian scheme* of C by the finite subgroup scheme of C' determined by the image of χ'. Note that every trans-homomorphism of semi-abelian schemes $B \to D$ which is Γ-*equivariant* with respect to the trivial action of Γ on D and the trans-homomorphism action of Γ of B *factors uniquely* through $B \twoheadrightarrow C$. Note, moreover, that the composite $Y \to B \twoheadrightarrow C$ *factors uniquely* through V. [Indeed, this is clear generically; then since V is *normal*, one may extend such a factorization [uniquely] to points of height 1 of V by applying the *properness* of $Y \to V$; finally, since V is *smooth*, one may extend such a factorization [uniquely] to the entire scheme V by applying Lemma A.2.] Thus, it is a *tautology* that the resulting morphism $V \to C$ is an *Albanese morphism* for V. In particular, we may assume without loss of generality that $V \to A$ is $V \to C$. Also, let us observe that it follows immediately from the description of finite étale coverings of semi-abelian schemes reviewed at the beginning of the present Appendix that, for $l \in \Sigma_k$, the surjection $\Delta_B \otimes \mathbb{Q}_l \twoheadrightarrow \Delta_A \otimes \mathbb{Q}_l$ induces an *isomorphism*

$$(\Delta_B \otimes \mathbb{Q}_l)/\Gamma \xrightarrow{\sim} \Delta_A \otimes \mathbb{Q}_l$$

[where the "$/\Gamma$" denotes the *maximal quotient* on which Γ acts *trivially*].

On the other hand, by Proposition A.8, (iii), it follows that we have a natural isomorphism $\Delta_Y^{\text{ab-t}} \xrightarrow{\sim} \Delta_B$, hence, in particular, a natural isomorphism

$$(\Delta_Y^{\text{ab-t}} \otimes \mathbb{Q}_l)/\Gamma \xrightarrow{\sim} (\Delta_B \otimes \mathbb{Q}_l)/\Gamma \xrightarrow{\sim} \Delta_A \otimes \mathbb{Q}_l$$

for $l \in \Sigma_k$. Moreover, since $Y \to V$ is *dominant*, it induces an open homomorphism $\Delta_Y \to \Delta_V$, hence a *surjection* $\Delta_Y^{\text{ab-t}} \otimes \mathbb{Q}_l \twoheadrightarrow \Delta_V^{\text{ab-t}} \otimes \mathbb{Q}_l$ which is Γ-*equivariant* [with respect to the trivial action of Γ on $\Delta_V^{\text{ab-t}} \otimes \mathbb{Q}_l$]. In particular, we obtain that the natural isomorphism $(\Delta_Y^{\text{ab-t}} \otimes \mathbb{Q}_l)/\Gamma \xrightarrow{\sim} \Delta_A \otimes \mathbb{Q}_l$ *factors* as the composite of surjections

$$(\Delta_Y^{\text{ab-t}} \otimes \mathbb{Q}_l)/\Gamma \twoheadrightarrow \Delta_V^{\text{ab-t}} \otimes \mathbb{Q}_l \twoheadrightarrow \Delta_A \otimes \mathbb{Q}_l$$

[cf. Proposition A.3, (vii)]. Thus, we conclude that these surjections are *isomorphisms*, hence that the surjection $\Delta_V^{\text{ab-t}} \twoheadrightarrow \Delta_A$ of Proposition A.3, (vii), is an *isomorphism*, as desired. ◯

Remark A.11.1. In fact, given any variety V over k, one may construct an *"Albanese morphism"* $V \to A$, where A is a *torsor* over a semi-abelian variety over k, by passing to a finite [separable] extension k' of k such that $V(k') \neq \emptyset$, applying Corollary A.11, (i), over k', and then descending back to k. This morphism $V \to A$ will then satisfy the *universal property* for morphisms $V \to A'$ to *torsors* A' over semi-abelian varieties over k [i.e., every such morphism $V \to A'$ admits a unique factorization $V \to A \to A'$, where the morphism $A \to A'$ is a k-morphism that base-changes to a trans-homomorphism over $\overline{k}$]. In the present Appendix, however, we always assumed the existence of rational points in order to simplify the discussion.

Remark A.11.2. One may further generalize Remark A.11.1, as follows. If V is a *geometrically integral separated algebraic stack of finite type* over k that is obtained by forming the quotient, in the sense of stacks, of some variety W over k by the action of a finite group of automorphisms $\Gamma \subseteq \text{Aut}(W)$, then, by applying Remark A.11.1 to W to obtain an Albanese morphism $W \to B$ for W, one may construct an *"Albanese morphism"*

$$V \to A$$

for V [i.e., which satisfies the *universal property* described in Remark A.11.1] by forming the quotient $B \to A$ of B as in the proof of Corollary A.11, (ii): That is to say, after reducing, via étale descent, to the case $k = \overline{k}$, the action of Γ on W induces an action of Γ by k-*trans-homomorphisms* on B, hence an action of Γ by *group scheme automorphisms* on B, together with a *twisted homomorphism*

$\chi : \Gamma \to B(k)$. Then we take $B \twoheadrightarrow A'$ to be the quotient by the images of the group scheme endomorphisms [arising from the group scheme action of Γ on B] $(1 - \gamma) : B \to B$, for $\gamma \in \Gamma$, and $A' \twoheadrightarrow A$ to be the quotient by the image of the homomorphism $\chi' : \Gamma \to A'(k)$ determined by χ. Moreover, just as in the proof of Corollary A.11, (ii), we obtain a *natural isomorphism*

$$\Delta_V^{\text{ab-t}} \xrightarrow{\sim} \Delta_A$$

[where we use the notation "$\Delta_{(-)}$" to denote the evident stack-theoretic generalization of this notation for varieties].

The content of more classical works [cf., e.g., [NS], [Chev]] written from the point of view of *birational geometry* may be recovered via the following result.

Corollary A.12. (Albanese Varieties and Birational Geometry)

(i) Let $\beta_V : V' \to V$ be a **proper birational** *morphism of* **normal varieties** *over k which restricts to an* **isomorphism** *$\beta_U : U' \stackrel{\text{def}}{=} V' \times_V U \xrightarrow{\sim} U$ over some nonempty open subscheme $U \subseteq V$; $\beta_A : A' \to A$ the induced morphism on* **Albanese varieties** *[cf. Corollary A.11, (i)]; $W \subseteq V$ a* ***k*-toric** *open subscheme. Then the composite morphism $U \bigcap W \hookrightarrow U \xrightarrow{\sim} U' \hookrightarrow V' \to A'$* **extends uniquely** *to a morphism $W \to A'$ which induces a* **surjection** *$\Delta_W \twoheadrightarrow \Delta_{A'}$.*

(ii) Let

$$\ldots \to V_n \to \ldots \to V_1 \to V_0 = V$$

be a sequence [indexed by the nonnegative integers] of **birational** *morphisms of* **complete normal varieties** *over k. Then there exists an integer N such that for all $n, m \geq N$, where $n \geq m$, the induced morphism on* **Albanese varieties** *$A_n \to A_m$ is an* **isomorphism**. *If V is* ***k*-toric**, *then one may take $N = 0$.*

Proof. First, we consider assertion (i). We may assume without loss of generality that $U \subseteq W$. Then since $V' \to V$ is *proper*, and W is *normal*, it follows that the morphism $U \xrightarrow{\sim} U' \hookrightarrow V'$ *extends uniquely* to an open subset $W \backslash F \subseteq W$, where F is a closed subscheme of codimension ≥ 2 in W. Thus, the fact that the resulting morphism $W \backslash F \to V' \to A'$ extends uniquely to W follows immediately from Lemma A.2. To verify the *surjectivity* of $\Delta_W \to \Delta_{A'}$, it suffices to verify the surjectivity of $\Delta_U \to \Delta_{A'}$, i.e., of $\Delta_{U'} \to \Delta_{V'} \to \Delta_{A'}$. On the other hand, this follows from the surjectivity of $\Delta_{V'} \to \Delta_{A'}$ [cf. Proposition A.3, (vii)], together with the surjectivity of $\Delta_{U'} \to \Delta_{V'}$ [cf. the fact that $U' \subseteq V'$ is a nonempty open subscheme of the normal variety V'].

Next, we consider assertion (ii). By Proposition A.6, (i), (ii) [cf. also Proposition A.3, (viii), (ix); Corollary A.11, (i)], each induced morphism on *Albanese varieties* $A_n \to A_m$, for $n \geq m$, is a *surjection of abelian varieties* which is an *isomorphism* if and only if $\dim_k(A_n) \leq \dim_k(A_m)$. On the other hand, if $W \subseteq V$ is *any nonempty k-toric* [e.g., k-smooth] open subscheme whose Albanese morphism [cf. Corollary A.11, (i)] we denote by $W \to A_W$, then assertion (i) yields a morphism $W \to A_n$ that induces a surjection $\Delta_W \twoheadrightarrow \Delta_{A_n}$, hence, in particular, a morphism $A_W \to A_n$ that induces a surjection $\Delta_{A_W} \twoheadrightarrow \Delta_{A_n}$. But this implies that $\dim_k(A_n) \leq \dim_k(A_W)$, hence that for some integer N, $\dim_k(A_n) = \dim_k(A_m)$, for all $n, m \geq N$. In particular, if $W = V$, then $\dim_k(A_n) \leq \dim_k(A_0)$, for all $n \geq 0$. ◯

Bibliography

[BS] L. Barbieri-Viale, V. Srinivas, Albanese and Picard 1-motives, *Mem. Soc. Math. Fr. (N.S.)* **87** (2001).

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **21**, Springer-Verlag (1990).

[Chev] C. Chevalley, Sur la théorie de la variété de Picard, *Amer. J. Math.* **82** (1960), pp. 435-490.

[Con] B. Conrad, A modern proof of Chevalley's theorem on algebraic groups, *J. Ramanujan Math. Soc.* **17** (2002), pp. 1-18.

[deJong] A. J. de Jong, Smoothness, Semi-stability, and Alterations, *IHES Publ. Math.* **83** (1996), pp. 51-93.

[DM] P. Deligne and D. Mumford, The Irreducibility of the Moduli Space of Curves of Given Genus, *IHES Publ. Math.* **36** (1969), pp. 75-109.

[FC] G. Faltings and C.-L. Chai, *Degenerations of Abelian Varieties*, Springer Verlag (1990).

[FGA] A. Grothendieck, *Fondements de la géométrie algébrique*, *Séminaire Bourbaki*, 1957-1962; also available as a PDF file from: `http://www.math.columbia.edu/~lipyan/FGA.pdf`.

[FJ] M. Fried and M. Jarden, *Field Arithmetic* (second edition), Springer-Verlag (2005).

[Kato] K. Kato, Toric Singularities, *Amer. J. Math.* **116** (1994), pp. 1073-1099.

[Klei] S. Kleiman, The Picard Scheme, in *Fundamental Algebraic Geometry: Grothendieck's FGA Explained*, B. Fantechi, L. Göttsche, L. Illusie, S. Kleiman, N. Nitsure, and A. Vistoli, *Mathematical Surveys and Monographs* **123** (2005); also available as arXiv e-print `math.AG/0504020` (April 2005).

[Milne] J. S. Milne, *Étale Cohomology*, Princeton Mathematical Series **33**, Princeton University Press (1980).

[Mzk1] S. Mochizuki, A Version of the Grothendieck Conjecture for p-adic Local Fields, *The International Journal of Math.* **8** (1997), pp. 499-506.

[Mzk2] S. Mochizuki, Correspondences on Hyperbolic Curves, *Journ. Pure Appl. Algebra* **131** (1998), pp. 227-244.

[Mzk3] S. Mochizuki, The Local Pro-p Anabelian Geometry of Curves, *Invent. Math.* **138** (1999), pp. 319-423.

[Mzk4] S. Mochizuki, Extending Families of Curves over Log Regular Schemes, *J. reine angew. Math.* **511** (1999), pp. 43-71.

[Mzk5] S. Mochizuki, Topics Surrounding the Anabelian Geometry of Hyperbolic Curves, *Galois Groups and Fundamental Groups*, *Mathematical Sciences Research Institute Publications* **41**, Cambridge University Press (2003), pp. 119-165.

[Mzk6] S. Mochizuki, The Absolute Anabelian Geometry of Hyperbolic Curves, *Galois Theory and Modular Forms*, Kluwer Academic Publishers (2003), pp. 77-122.

[Mzk7] S. Mochizuki, The Absolute Anabelian Geometry of Canonical Curves, *Kazuya Kato's fiftieth birthday, Doc. Math. 2003, Extra Vol.*, pp. 609-640.

[Mzk8] S. Mochizuki, The Geometry of Anabelioids, *Publ. Res. Inst. Math. Sci.* **40** (2004), pp. 819-881.

[Mzk9] S. Mochizuki, Galois Sections in Absolute Anabelian Geometry, *Nagoya Math. J.* **179** (2005), pp. 17-45.

[Mzk10] S. Mochizuki, Semi-graphs of Anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), pp. 221-322.

[Mzk11] S. Mochizuki, Global Solvably Closed Anabelian Geometry, *Math. J. Okayama Univ.* **48** (2006), pp. 57-71.

[Mzk12] S. Mochizuki, *The Étale Theta Function and its Frobenioid-theoretic Manifestations*, RIMS Preprint **1569** (September 2006); updated version available at the following web site:
`http://www.kurims.kyoto-u.ac.jp/~motizuki/papers-english.html`

[Mzk13] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math J.* **59** (2007), pp. 455-479.

[Mzk14] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, *J. Math. Kyoto Univ.* **47** (2007), pp. 451-539.

[Mzk15] S. Mochizuki, *Topics in Absolute Anabelian Geometry II: Decomposition Groups and Endomorphisms*, manuscript.

[MT] S. Mochizuki and A. Tamagawa, *The Algebraic and Anabelian Geometry of Configuration Spaces*, RIMS Preprint **1570** (September 2006); to appear in *Hokkaido Math. J.* **37** (2008).

[Mumf] D. Mumford, *Abelian Varieties*, Oxford Univ. Press (1974).

[NSW] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, *Grundlehren der Mathematischen Wissenschaften* **323**, Springer-Verlag (2000).

[NS] A. Néron and P. Samuel, La variété de Picard d'une variété normale, Ann. Inst. Fourier **4** (1952), pp. 1-30.

[RZ] L. Ribes and P. Zaleskii, *Profinite Groups*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **3**, Springer-Verlag (2000).

[Serre1] J.-P. Serre, Morphismes universels et variété d'Albanese, *Seminaire Claude Chevalley* **4** (1958-1959), Expose No. 10, 22 pp.

[Serre2] J.-P. Serre, Local Class Field Theory in *Algebraic Number Theory*, ed. J.W.S. Cassels and A. Fröhlich, Academic Press (1967).

[Serre3] J.-P. Serre, *Abelian l-adic Representations and Elliptic Curves*, W. A. Benjamin, Inc., 1968.

[SS] T. Szamuely and M. Spiess, On the Albanese map for smooth quasi-projective varieties, *Mathematische Annalen* **325** (2003), pp. 1-17.

[SGA2] A. Grothendieck et al., *Cohomologie locale des faisceaux cohérents et théorèmes de Lefshetz locaux et globaux*, North-Holland, Amsterdam (1968).

[SGA3-1] M. Demazure, A. Grothendieck, eds., *Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - Schémas en groupes - (SGA 3) - vol. 1*, Lecture notes in mathematics **151**, Springer-Verlag (1970).

[SGA3-2] M. Demazure, A. Grothendieck, eds., *Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - Schémas en groupes - (SGA 3) - vol. 2*, Lecture notes in mathematics **152**, Springer-Verlag (1970).

[Tama] A. Tamagawa, Resolution of nonsingularities of families of curves, *Publ. of RIMS* **40** (2004), pp. 1291-1336.

[Tate] J. Tate, *p-Divisible Groups*, Proceedings of a Conference on Local Fields, Driebergen, Springer-Verlag (1967), pp. 158-183.

[Yoshi] T. Yoshida, Finiteness theorems in the class field theory of varieties over local fields, *J. of Number Theory* **101** (2003), pp. 138-150.