Thom polynomials and Schur functions: the singularities $A_3(-)$

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To the memory of Stanisław Balcerzyk

Abstract

Combining the "method of restriction equations" of Rimányi et al. with the techniques of symmetric functions, we establish the Schur function expansions of the Thom polynomials for the Morin singularities $A_3 : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ for any nonnegative integer k.

1 Introduction

The global behavior of singularities of maps is governed by their *Thom polynomials* (see [25], [11], [1], [9], [22]). Knowing the Thom polynomial of a singularity η, denoted $\mathcal{T}^\eta$, one can compute the cohomology class represented by the η-points of a map. In particular, if $f : X \to Y$ is a general map of complex analytic manifolds, where X is compact and $\dim(X)$ equals the codimension of the singularity η, then the degree $\int_X \mathcal{T}^\eta$ evaluates the number of points of X at which f has the singularity η.

In the present paper, following the "method of restriction equations" from a series of papers by Rimányi et al. [23], [22], [7], [2], we study the Thom polynomials for the singularities A_3 associated with maps $(\mathbf{C}^\bullet, 0) \to$

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$(\mathbf{C}^{\bullet+k}, 0)$ with parameter $k \geq 0$. We give the Schur function expansions of these Thom polynomials. This is the content of our main Theorem 8 and its proof in Section 4.

The way of obtaining the Thom polynomial is through the solution of a system of linear equations (see Theorem 1). This is fine when we want to find one concrete Thom polynomial, say, for a fixed k. However, if we want to find the Thom polynomials for a series of singularities, associated with maps $(\mathbf{C}^{\bullet}, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ with k as a parameter, we have to solve *simultaneously* a countable family of systems of linear equations. This cannot be done by computer, and must be done conceptually.

Thom polynomials are symmetric functions in the universal Chern roots. Instead of giving their expressions in terms of these variables, we use *Schur function expansions*. This puts a more transparent structure on computations of Thom polynomials (see [17], and also [6] for some second order Thom-Boardman singularities). In particular, in the Schur basis one can see some *recurrences* which are difficult or even impossible to notice in other bases (see [18]).

Another feature of using the Schur function expansions for Thom polynomials is that all the coefficients are *nonnegative*. This has been recently proved by A. Weber and the second author in [21].

To be more precise, we use here (the specializations of) *supersymmetric* Schur functions also called "Schur functions in difference of alphabets" together with their three basic properties: *vanishing*, *cancellation* and *factorization*, (see [24], [4], [16], [20], [13], and [12]). These functions contain resultants among themselves. They play a fundamental role in the study of *$\mathcal{P}$-ideals of singularities* Σ^i (see [18, end of Sect. 2 and Theorem 11] and Proposition 6) which is based on the enumerative geometry of degeneracy loci of [15].

Since the singularity A_3 is in the closure of the orbit of the singularity Σ^1, we have by Proposition 6 that all partitions in the Schur expansion of $\mathcal{T}^{A_3}$ (any k) contain the single row-partition $(k+1)$.

In [19] the decomposition of the Thom polynomial of the singularity A_i into *h-parts* was defined (see also the end of Section 3). In particular, the 1-part of the Thom polynomial of the Morin singularity A_i (any i, k) was computed. In the present paper, we work out the case of the singularities A_3 (any k), and we find the 2-part of this Thom polynomial (the h-parts, where $h \geq 3$, are equal to zero for these singularities).

In our calculations, we use extensively the functorial λ-ring approach to symmetric functions from [12] (e.g. we shall need to handle symmetric functions in $2x_1, 2x_2, x_1 + x_2$[1] at the same time as symmetric functions in x_1, x_2).

[1]Strictly speaking: symmetric functions in $\boxed{2x_1}$, $\boxed{2x_2}$, $\boxed{x_1 + x_2}$ after simplification, see Section 3.

The main results of the present paper were announced in [17].

Bérczi, Fehér and R. Rimányi gave without proof in [2] an expression for this Thom polynomial, but in terms of the monomial basis in Chern classes. We prompt the authors of [2] to publish their proof.

2 Reminder on Thom polynomials

Our main reference for this section is [22]. We start with recalling what we shall mean by a "singularity". Let $k \geq 0$ be a fixed integer. By a *singularity* we shall mean an equivalence class of stable germs $(\mathbf{C}^{\bullet}, 0) \to (\mathbf{C}^{\bullet+k}, 0)$, where $\bullet \in \mathbf{N}$, under the equivalence generated by right-left equivalence (i.e. analytic reparametrizations of the source and target) and suspension.

We recall[2] that the *Thom polynomial* $\mathcal{T}^{\eta}$ of a singularity η is a polynomial in the formal variables $c_1, c_2, \ldots$ which after the substitution of c_i to

$$c_i(f^*TY - TX) = [c(f^*TY)/c(TX)]_i \,, \tag{1}$$

for a general map $f : X \to Y$ between complex analytic manifolds, evaluates the Poincaré dual of $[V^{\eta}(f)]$, where $V^{\eta}(f)$ is the cycle carried by the closure of the set

$$\{x \in X : \text{the singularity of } f \text{ at } x \text{ is } \eta\} \,. \tag{2}$$

By *codimension of a singularity* η, $\operatorname{codim}(\eta)$, we shall mean $\operatorname{codim}(V^{\eta}(f), X)$ for such an f. The concept of the polynomial $\mathcal{T}^{\eta}$ comes from Thom's fundamental paper [25]. For a detailed discussion of the *existence* of Thom polynomials, see, e.g., [1]. Thom polynomials associated with group actions were studied by Kazarian in [9] and [10].

According to Mather's classification, singularities are in one-to-one correspondence with finite dimensional $\mathbf{C}$-algebras. We shall use the following notation:

– A_i (of Thom-Boardman type Σ^{1_i}) will stand for the stable germs with local algebra $\mathbf{C}[[x]]/(x^{i+1})$, $i \geq 0$;

– $III_{2,2}$ (of Thom-Boardman type Σ^2) for stable germs with local algebra $\mathbf{C}[[x, y]]/(xy, x^2, y^2)$ (here $k \geq 1$).

In the present article, the computations of Thom polynomials shall use the method which stems from a sequence of papers by Rimányi et al. [23], [22], [7], [2]. We sketch briefly this approach, refering the interested reader for more details to these papers.

[2] This statement is usually called the Thom-Damon theorem [25], [5].

Let $k \geq 0$ be a fixed integer, and let $\eta : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ be a stable singularity with a prototype $\kappa : (\mathbf{C}^n, 0) \to (\mathbf{C}^{n+k}, 0)$. The *maximal compact subgroup of the right-left symmetry group*

$$\operatorname{Aut}\kappa = \{(\varphi, \psi) \in \operatorname{Diff}(\mathbf{C}^n, 0) \times \operatorname{Diff}(\mathbf{C}^{n+k}, 0) : \psi \circ \kappa \circ \varphi^{-1} = \kappa\} \qquad (3)$$

of κ will be denoted by G_η. Even if $\operatorname{Aut}\kappa$ is much too large to be a finite dimensional Lie group, the concept of its maximal compact subgroup (up to conjugacy) can be defined in a sensible way. In fact, G_η can be chosen so that the images of its projections to the factors $\operatorname{Diff}(\mathbf{C}^n, 0)$ and $\operatorname{Diff}(\mathbf{C}^{n+k}, 0)$ are linear. Its representations via the projections on the source $\mathbf{C}^n$ and the target $\mathbf{C}^{n+k}$ will be denoted by $\lambda_1(\eta)$ and $\lambda_2(\eta)$. The vector bundles associated with the universal principal G_η-bundle $EG_\eta \to BG_\eta$ using the representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$ will be called E'_η and E_η. The *total Chern class of the singularity* η is defined in $H^*(BG_\eta, \mathbf{Z})$ by

$$c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)}\,. \qquad (4)$$

The *Euler class* of η is defined in $H^{2\operatorname{codim}(\eta)}(BG_\eta, \mathbf{Z})$ by

$$e(\eta) := e(E'_\eta)\,. \qquad (5)$$

In the following theorem, we collect information from [22], Theorem 2.4 and [7], Theorem 3.5, needed for the calculations in the present paper.

Theorem 1 *Suppose, for a singularity η, that the Euler classes of all singularities of smaller codimension than* $\operatorname{codim}(\eta)$*, are not zero-divisors* [3]. *Then we have*

1. *if $\xi \neq \eta$ and* $\operatorname{codim}(\xi) \leq \operatorname{codim}(\eta)$*, then $\mathcal{T}^\eta(c(\xi)) = 0$;*
2. *$\mathcal{T}^\eta(c(\eta)) = e(\eta)$.*

This system of equations (taken for all such ξ's) determines the Thom polynomial $\mathcal{T}^\eta$ in a unique way[4].

To use this method of determining the Thom polynomials for singularities, one needs their classification, see, e.g., [14].

We record the following lemma (see [22] and [2]).

[3] This is the so-called "Euler condition" (*loc.cit.*). It holds for A_3.

[4] To make it precise, we need one more condition that the number of singularities (=contact orbits) of smaller codimension is finite: we may assume that η is a *simple* singularity type, i.e., there is no moduli adjacent to η.

Lemma 2 *(i) For the singularity of type A_i: $(\mathbf{C}^\bullet,0)\to(\mathbf{C}^{\bullet+k},0)$, we have $G_{A_i}=U(1)\times U(k)$. Moreover, denoting by x and $y_1,\dots,y_k$ the Chern roots of the tautological vector bundles on $BU(1)$ and $BU(k)$, we have*

$$c(A_i)=\frac{1+(i+1)x}{1+x}\prod_{j=1}^{k}(1+y_j) \tag{6}$$

and

$$e(A_3)=6\ x^3\ \prod_{j=1}^{k}(y_j-3x)(y_j-2x)(y_j-x)\,. \tag{7}$$

(ii) For the singularity $III_{2,2}:(\mathbf{C}^\bullet,0)\to(\mathbf{C}^{\bullet+k},0)$, where $k>0$, we have $G_\eta=U(2)\times U(k-1)$. Moreover, denoting by x_1,x_2 (resp. $y_1,\dots,y_{k-1}$) the Chern roots of the tautological vector bundle on $BU(2)$ (resp. $BU(k-1)$), we have

$$c(III_{2,2})=\frac{(1+2x_1)(1+2x_2)(1+x_1+x_2)}{(1+x_1)(1+x_2)}\prod_{j=1}^{k-1}(1+y_j)\,. \tag{8}$$

3 Reminder on Schur functions

In this section, we collect needed notions related to symmetric functions. We adopt a functorial λ-ring point of view of [12].

For $m\in\mathbf{N}$, by an alphabet $\mathbb{A}$ we shall mean a finite set of indeterminates $\mathbb{A}=\{a_1,\dots,a_m\}$.

We shall often identify an alphabet $\mathbb{A}=\{a_1,\dots,a_m\}$ with the sum $a_1+\dots+a_m$.

Definition 3 *Given two alphabets $\mathbb{A}$, $\mathbb{B}$, the complete functions $S_i(\mathbb{A}-\mathbb{B})$ are defined by the generating series (with z an extra variable):*

$$\sum S_i(\mathbb{A}-\mathbb{B})z^i=\prod_{b\in\mathbb{B}}(1-bz)/\prod_{a\in\mathbb{A}}(1-az)\,. \tag{9}$$

Definition 4 *Given a partition[5] $I=(0\le i_1\le i_2\le\ldots\le i_s)\in\mathbf{N}^s$, and alphabets $\mathbb{A}$ and $\mathbb{B}$, the Schur function $S_I(\mathbb{A}-\mathbb{B})$ is*

$$S_I(\mathbb{A}-\mathbb{B}):=\left|S_{i_p+p-q}(\mathbb{A}-\mathbb{B})\right|_{1\le p,q\le s}\,. \tag{10}$$

These functions are often called *supersymmetric Schur functions* or *Schur functions in difference of alphabets*. Their properties were studied, among others, in [4], [16], [20], [13], and [12].

[5] We identify partitions with their Young diagrams, as is customary.

We have the following *cancellation property*:

$$S_I((\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C})) = S_I(\mathbb{A}-\mathbb{B})\,. \tag{11}$$

We identify partitions with their Young diagrams, as is customary.

We record the following property (*loc.cit.*):

$$S_I(\mathbb{A}-\mathbb{B}) = (-1)^{|I|}S_J(\mathbb{B}-\mathbb{A}) = S_J(\mathbb{B}^*-\mathbb{A}^*)\,, \tag{12}$$

where J is the conjugate partition of I (i.e. the consecutive rows of the diagram of J are the transposed columns of the diagram of I), and $\mathbb{A}^*$ denotes the alphabet $\{-a_1, -a_2, \dots\}$.

In the present paper, by a *symmetric function* we shall mean a $\mathbf{Z}$-linear combination of the operators S_I.

Instead of introducing, in the argument of a symmetric function, formal variables which will be specialized, we write $\boxed{r}$ for a variable which will be specialized to r (r can be $2x_1$, x_1+x_2,...). For example,

$$S_2(x_1+x_2) = x_1^2+x_1x_2+x_2^2 \quad\text{but}\quad S_2\big(\boxed{x_1+x_2}\big) = (x_1+x_2)^2 = x_1^2+2x_1x_2+x_2^2\,.$$

Definition 5 *Given two alphabets* $\mathbb{A},\mathbb{B}$*, we define their resultant:*

$$R(\mathbb{A},\mathbb{B}) := \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (a-b)\,. \tag{13}$$

For example, we have the following identity:

$$-6x^3\prod_{j=1}^{k}(3x-y_j)(2x-y_j)(x-y_j) = R\big(x+\boxed{2x}+\boxed{3x}, \mathbb{Y}+\boxed{4x}\big)\,, \tag{14}$$

where $\mathbb{Y} = \{y_1,\dots,y_k\}$.

We record the following *factorization property* ([12, Proposition 1.4.3]). Suppose that cardinality of B is n. Then for partitions $I=(i_1,\dots,i_m)$ and $J=(j_1,\dots,j_s)$, we have

$$S_{(j_1,\dots,j_s,i_1+n,\dots,i_m+n)}(\mathbb{A}-\mathbb{B}) = S_I(\mathbb{A})\ R(\mathbb{A},\mathbb{B})\ S_J(-\mathbb{B})\,. \tag{15}$$

In the present paper, it will be more handy to use, instead of k, a shifted parameter

$$r := k+1\,. \tag{16}$$

Sometimes, we shall write $\eta(r)$ for the singularity $\eta : (\mathbf{C}^\bullet,0)\to(\mathbf{C}^{\bullet+r-1},0)$, and denote the Thom polynomial of $\eta(r)$ by $\mathcal{T}_r^\eta$ – to emphasize the dependence of both items on r.

Let $f : X \to Y$ be a map of complex analytic manifolds, where $\dim(X) = m$ and $\dim(Y) = n$. Given a partition I, we define

$$S_I(T^*X - f^*(T^*Y))$$

to be the effect of the following specialization of $S_I(\mathbb{A} - \mathbb{B})$: we set the indeterminates of $\mathbb{A}$ to the Chern roots of T^*X, and the indeterminates of $\mathbb{B}$ to the Chern roots of $f^*(T^*Y)$.

Similarly to [17], [18], and [19], we shall write the Poincaré dual of $[V^\eta(f)]$, for a singularity η and a general map $f : X \to Y$, in the form

$$\sum_I \alpha_I S_I(T^*X - f^*(T^*Y))$$

with integer coefficients α_I. Accordingly, we shall write

$$\mathcal{T}^\eta = \sum_I \alpha_I S_I \,, \tag{17}$$

where S_I is identified with $S_I(\mathbb{A} - \mathbb{B})$ for the universal Chern roots $\mathbb{A}$ and $\mathbb{B}$.

Note that in this notation, the Thom polynomial of the singularity $A_1(r)$ for $r \geq 1$, is: $\mathcal{T}_r^{A_1} = S_r$. Another example is the Thom polynomial of $A_2(1)$. In [22], it is written as $c_1^2 + c_2$, whereas in the present notation it is written as $S_{11} + 2S_2$.

The arguments of the proof of [18, Theorem 11] give the following result[6].

Proposition 6 *Suppose that a singularity η is in the closure of the orbit of the singularity Σ^j. Then all summands in the Schur function expansion of $\mathcal{T}_r^\eta$ are indexed by partitions containing the rectangle partition $(r + j - 1)^j$.*

Recall (from [19]) that the *h-part* of $\mathcal{T}_r^{A_i}$ is the sum of all Schur functions appearing nontrivially in $\mathcal{T}_r^{A_i}$ (multiplied by their coefficients) such that the corresponding partitions satisfy the following condition: I contains the rectangle partition $\big((r{+}h{-}1)^h\big)$, but it does not contain the larger Young diagram $\big((r{+}h)^{h+1}\big)$. The polynomial $\mathcal{T}_r^{A_i}$ is a sum of its h-parts, $h = 1, 2, \ldots$.

In one instance (the proof of Proposition 14), we shall also use *multi-Schur functions*. For their definition and properties, we refer the reader to [12].

[6]This justifies the remark in [19] p. 166, lines 28–31.

4 Main result and its proof

Since the singularities $\neq A_3$, whose codimension is $\leq \mathrm{codim}(A_3)$ are: A_0, A_1, A_2 and, for $r \geq 2$, $III_{2,2}$ (see [14]), Theorem 1 yields the following equations (in T), characterizing the Thom polynomial $\mathcal{T}_r^{A_3}$:

$$T(-\mathbb{B}_{r-1}) = T(x - \mathbb{B}_{r-1} - \boxed{2x}) = T(x - \mathbb{B}_{r-1} - \boxed{3x}) = 0\,, \tag{18}$$

$$T(x - \mathbb{B}_{r-1} - \boxed{4x}) = R(x + \boxed{2x} + \boxed{3x}, \mathbb{B}_{r-1} + \boxed{4x}) \tag{19}$$

$$T(x_1 + x_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0\,. \tag{20}$$

Here,

$$\mathbb{D} = \boxed{2x_1} + \boxed{2x_2} + \boxed{x_1 + x_2}\,.$$

We assume that x, x_1, x_2, and $b_1, \ldots, b_n$ are variables. Note that these variables, in the following, will be specialized to the Chern roots of the *cotangent* bundles.

By [19], we know that $\mathcal{T}_r^{A_3}$ must contain (as its 1-part) the following combination of Schur functions, denoted by $F_r^{(3)}$ in [19]:

$$F_r := \sum_{j_1 \leq j_2 \leq r} S_{j_1,j_2}(\boxed{2} + \boxed{3}) S_{r-j_2, r-j_1, r+j_1+j_2}\,. \tag{21}$$

By [19, Corollary 11], Eqs. (18) and (19) are satisfied by the function F_r. For $r = 1$, this means that

$$F_1 = S_{111} + 5S_{12} + 6S_3 \tag{22}$$

is the Thom polynomial for $A_3(1)$.

However, for $r \geq 2$, F_r does not satisfy the last vanishing, imposed by $III_{2,2}$. In the following we shall modify F_r in order to obtain the Thom polynomial for A_3. In fact, our goal is to give an expresion for the Thom polynomial for A_3 (any r) as a $\mathbf{Z}$-linear combination of Schur functions. For $r = 2$, the Thom polynomial is

$$S_{222} + 5S_{123} + 6S_{114} + 19S_{24} + 30S_{15} + 36S_6 + 5S_{33}\,, \tag{23}$$

and it differs from its 1-part F_2 by $5S_{33}$ which is the "correction" 2-part in this case (see [19]).

Define integers $e_{i,j}$, for $i \geq 2$ and $j \geq 0$ in the following way. First, $e_{20}, e_{30}, e_{40}, \ldots$ are the coefficients $5, 24, 89, \ldots$ in the Taylor expansion of

$$\frac{5 - 6z}{(1-z)(1-2z)(1-3z)} = 5 + 24z + 89z^2 + 300z^3 + 965z^4 + 3024z^5 + 9329z^6 + \ldots\,.$$

Moreover, we set $e_{2,j} = e_{3,j} = 0$ for $j \geq 1$, $e_{4,j} = e_{5,j} = 0$ for $j \geq 2$, $e_{6,j} = e_{7,j} = 0$ for $j \geq 3$ etc. To define the remaining $e_{i,j}$'s, we use the recursive formula

$$e_{i+1,j} = e_{i,j-1} + e_{i,j} \,. \tag{24}$$

We obtain the following matrix $[e_{i,j}]_{i\geq 2, j\geq 0}$:

$$\begin{matrix} e_{20} & 0 & 0 & 0 & 0 & \dots \\ e_{30} & 0 & 0 & 0 & 0 & \dots \\ e_{40} & e_{41} & 0 & 0 & 0 & \dots \\ e_{50} & e_{51} & 0 & 0 & 0 & \dots \\ e_{60} & e_{61} & e_{62} & 0 & 0 & \dots \\ e_{70} & e_{71} & e_{72} & 0 & 0 & \dots \\ e_{80} & e_{81} & e_{82} & e_{83} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix} \quad = \quad \begin{matrix} 5 & 0 & 0 & 0 & 0 & \dots \\ 24 & 0 & 0 & 0 & 0 & \dots \\ 89 & 24 & 0 & 0 & 0 & \dots \\ 300 & 113 & 0 & 0 & 0 & \dots \\ 965 & 413 & 113 & 0 & 0 & \dots \\ 3024 & 1378 & 526 & 0 & 0 & \dots \\ 9329 & 4402 & 1904 & 526 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

Remark 7 Note that arguing similarly as in the proof of Proposition 19 in [18], we get the following closed formula for $e_{i,j}$. For $i \geq 2$ and $j \geq 0$, we have

$$e_{i,j} = \frac{1}{2^{j+1}}\Big[(3^{i+1}-3^{2(j+1)})-(2^{i+j+2}-2^{3(j+1)}) - \sum_{s=1}^{j} 2^s\big(3^{2(j-s+1)}-2^{3(j-s+1)}\big)\Big(\binom{i-2j-2s+1}{s}-\binom{2s-2}{s}\Big)\Big] \,.$$

For example, we have

$$e_{i,2} = \frac{1}{2^3}\Big[(3^{i+1}-3^6)-(2^{i+4}-2^9)-2(3^4-2^6)(i-5)-2^2\Big(\binom{i-3}{2}-1\Big)\Big] \,.$$

Consider the following matrix whose elements are two row partitions (the symbol "$\emptyset$" denotes the empty partition):

$$\begin{matrix} 33 & \emptyset & \emptyset & \emptyset & \emptyset & \dots \\ 45 & \emptyset & \emptyset & \emptyset & \emptyset & \dots \\ 57 & 66 & \emptyset & \emptyset & \emptyset & \dots \\ 69 & 78 & \emptyset & \emptyset & \emptyset & \dots \\ 7,11 & 8,10 & 9,9 & \emptyset & \emptyset & \dots \\ 8,13 & 9,12 & 10,11 & \emptyset & \emptyset & \dots \\ 9,15 & 10,14 & 11,13 & 12,12 & \emptyset & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

We use for this matrix the same "matrix coordinates" as for the previous one. Denote by $I(i,j)$ the partition occupying the (i,j)th place in this matrix. So, e.g., $I(i,0) = (i+1, 2i-1)$ for $i \geq 2$.

For $r \geq 2$, we set

$$\overline{H}_r := \sum_{j\geq 0} e_{r,j} \ S_{I(r,j)} \,. \qquad (25)$$

We have

$$\begin{array}{lllll}
\overline{H}_2 & = 5S_{33} & & & \\
\overline{H}_3 & = 24S_{45} & & & \\
\overline{H}_4 & = 89S_{57} & +24S_{66} & & \\
\overline{H}_5 & = 300S_{69} & +113S_{78} & & \\
\overline{H}_6 & = 965S_{7,11} & +413S_{8,10} & +113S_{99} & \\
\overline{H}_7 & = 3024S_{8,13} & +1378S_{9,12} & +526S_{10,11} & \\
\overline{H}_8 & = 9329S_{9,15} & +4402S_{10,14} & +1904S_{11,13} & +526S_{12,12} \,.
\end{array}$$

Denote now by Φ the linear endomorphism on the free $\mathbf{Z}$-module spanned by Schur functions indexed by partitions of length ≤ 3, that sends a Schur function S_{i_1,i_2,i_3} to S_{i_1+1,i_2+1,i_3+1}. We define

$$H_r := \overline{H}_r + \Phi(H_{r-1}) \,, \qquad (26)$$

or equivalently, by iteration

$$H_r = \overline{H}_r + \Phi(\overline{H}_{r-1}) + \Phi^2(\overline{H}_{r-2}) + \cdots + \Phi^{r-2}(\overline{H}_2) \,. \qquad (27)$$

We have the following values of $H_2, H_3 = \Phi(H_2)+\overline{H_3}, \ldots, H_7 = \Phi(H_6)+\overline{H_7}$:

$$\begin{aligned}
H_2 &= 5S_{33} \\
H_3 &= 5S_{144}+24S_{45} \\
H_4 &= 5S_{255}+24S_{156}+24S_{66}+89S_{57} \\
H_5 &= 5S_{366}+24S_{267}+24S_{177}+89S_{168}+113S_{78}+300S_{69}, \\
H_6 &= 5S_{477}+24S_{378}+24S_{288}+89S_{279}+113S_{189}+300S_{1,7,10}+113S_{99}+413S_{8,10} \\
&\quad +965S_{7,11} \\
H_7 &= 5S_{588}+24S_{489}+24S_{399}+89S_{3,8,10}+113S_{2,9,10}+300S_{2,8,11}+113S_{1,10,10} \\
&\quad +413S_{1,9,11}+965S_{1,8,12}+526S_{10,11}+1378S_{9,12}+3024S_{8,13} \,.
\end{aligned}$$

Alternatively,

$$H_r = \sum_{i=0}^{r-2} \ \sum_{\{j\geq 0:\ i+2j\leq r-2\}} e_{r-i,j} \ S_{i,r+j+1,2r-i-j-1} \,. \qquad (28)$$

We now state the main result of this paper.

Theorem 8 *For $r \geq 1$, the Thom polynomial of $A_3(r)$ is equal to $F_r + H_r$.*

In other words, the function H_r is the 2-part of $\mathcal{T}_r^{A_3}$, and its h-parts are zero for $h \geq 3$.

In the proof of the theorem, we shall need several properties of the functions H_r and F_r.

The next result says that the addition of H_r to F_r is "irrelevant" for what concerns the conditions (18) and (19) imposed by the singularities A_i, $i = 0, 1, 2, 3$.

Lemma 9 *The function H_r satisfies Eqs. (18), and the equation*

$$H_r(x - \mathbb{B}_{r-1} - \boxed{4x}) = 0\,. \tag{29}$$

Proof. According to (15), each Schur function of index (i_1, i_2, i_3) with $i_2, i_3 \geq r{+}1$ vanishes when evaluated in $x - \mathbb{B}_{r-1} - y$, y any indeterminate. Therefore H_r satisfies the required nullities, which correspond to taking $y = 0, x, \boxed{2x}, \boxed{3x}$ or $\boxed{4x}$. $\square$

Thanks to the lemma, in order to prove the theorem, it suffices to show the equality

$$(F_r + H_r)(x_1 + x_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0\,, \tag{30}$$

which is equivalent to the vanishing of $\mathcal{T}_r^{A_3}$ at the Chern class $c(III_{2,2}(r))$.

Set $\mathbb{X}_2 = (x_1, x_2)$. Due to (15), each Schur function occuring in the expansion of H_r is such that

$$S_{c,r+1+a,r+1+b}(\mathbb{X}_2{-}\mathbb{D}{-}\mathbb{B}_{r-2}) = R(\mathbb{X}_2, \mathbb{D}{+}\mathbb{B}_{r-2}) \cdot S_c(-\mathbb{D}{-}\mathbb{B}_{r-2}) \cdot S_{a,b}(\mathbb{X}_2)\,,$$

We set

$$V_r(\mathbb{X}_2; \mathbb{B}_{r-2}) = \frac{H_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2})}{R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2})}\,, \tag{31}$$

so that

$$V_r(\mathbb{X}_2; \mathbb{B}_{r-2}) = \sum_{i=0}^{r-2} \sum_{\{j\geq 0:\ i+2j\leq r-2\}} e_{r-i,j}\ S_i(-\mathbb{D} - \mathbb{B}_{r-2})\ S_{j,r-i-j-2}(\mathbb{X}_2)\,. \tag{32}$$

We have the following recursive relation which follows from the observation that the coefficient of b_{r-2} in $V_r(\mathbb{X}_2; \mathbb{B}_{r-2})$ is equal to $-V_{r-1}(\mathbb{X}_2; \mathbb{B}_{r-3})$.

Lemma 10 *For $r \geq 2$, we have*

$$V_r(\mathbb{X}_2; \mathbb{B}_{r-2}) = \sum_{i=0}^{r-2} V_{r-i}(\mathbb{X}_2; 0)\ S_i(-\mathbb{B}_{r-2})\,. \tag{33}$$

Thus it is sufficient to compute $V_r(\mathbb{X}_2; 0)$.

Proposition 11 *For $r \geq 2$, we have*

$$V_r(\mathbb{X}_2;0) = 3^{r-2}\Big(3S_{r-2}(\mathbb{X}_2) - 2S_{1,r-3}(\mathbb{X}_2)\Big)\,. \tag{34}$$

(In particular, $V_2(\mathbb{X}_2;0) = 5$ and $V_3(\mathbb{X}_2;0) = 9S_1(\mathbb{X}_2)$.)
The proof of the proposition is given in the Appendix.

We now determine the specialization $F_r(\mathbb{X}_2-\mathbb{D}-\mathbb{B}_{r-2})$.

Lemma 12 *The resultant $R(\mathbb{X}_2, \mathbb{D}+\mathbb{B}_{r-2})$ divides $F_r(\mathbb{X}_2-\mathbb{D}-\mathbb{B}_{r-2})$.*

Proof. By [19, Proposition 10], we have

$$F_r(x-\mathbb{B}_r) = R(x+\boxed{2x}+\boxed{3x}, \mathbb{B}_r)\,,$$

and making into $F_r(\mathbb{X}_2-\mathbb{D}-\mathbb{B}_{r-2})$ the substitutions: $x_1 = 0$ and $x_1 = 2x_2$, we get

$$F_r(-\boxed{2x_2}-\mathbb{B}_{r-2}) = R(0+0+0, \boxed{2x_2}+\mathbb{B}_{r-2}+0) = 0\,,$$

and

$$\begin{aligned} F_r(x_2-\boxed{2x_1}-\boxed{x_1+x_2}-\mathbb{B}_{r-2}) &= R(x_2+\boxed{2x_2}+\boxed{3x_2}, \boxed{2x_1}+\boxed{x_1+x_2}+\mathbb{B}_{r-2}) \\ &= R(x_2+\boxed{2x_2}+\boxed{3x_2}, \boxed{2x_1}+\boxed{3x_2}+\mathbb{B}_{r-2}) = 0\,. \end{aligned}$$

Moreover, if $x_1 \in \mathbb{B}_{r-2}$ and $\mathbb{B}_{r-3} := \mathbb{B}_{r-2}-x_1$, then $F_r(\mathbb{X}_2-\mathbb{D}-\mathbb{B}_{r-2})$ becomes

$$\begin{aligned} &F_r(x_2-\boxed{2x_1}-\boxed{2x_2}-\boxed{x_1+x_2}-\mathbb{B}_{r-3}) \\ &\qquad = R(x_2+\boxed{2x_2}+\boxed{3x_2}, \boxed{2x_1}+\boxed{2x_2}+\boxed{x_1+x_2}+\mathbb{B}_{r-3}) = 0\,. \end{aligned}$$

These vanishings imply the assertion of the lemma. □

We set

$$U_r(\mathbb{X}_2;\mathbb{B}_{r-2}) = \frac{F_r(\mathbb{X}_2-\mathbb{D}-\mathbb{B}_{r-2})}{R(\mathbb{X}_2, \mathbb{D}+\mathbb{B}_{r-2})}\,. \tag{35}$$

Note that each variable $b \in \mathbb{B}_{r-2}$ appears at most with degree 3 in $F_r(\mathbb{X}_2-\mathbb{D}-\mathbb{B}_{r-2})$, and hence at most with degree 1 in $U_r(\mathbb{X}_2;\mathbb{B}_{r-2})$. We have the following precise recursive relation which follows from the observation that the coefficient of b_{r-2}^3 in $F_r(\mathbb{X}_2-\mathbb{D}-\mathbb{B}_{r-2})$ is equal to $F_{r-1}(\mathbb{X}_2-\mathbb{D}-\mathbb{B}_{r-3})$.

Lemma 13 *For $r \geq 2$, we have*

$$U_r(\mathbb{X}_2;\mathbb{B}_{r-2}) = \sum_{i=0}^{r-2} U_{r-i}(\mathbb{X}_2;0)\ S_i(-\mathbb{B}_{r-2})\,. \tag{36}$$

Let π be the endomorphism of the $\mathbf{C}$-vector space of functions of x_1, x_2, defined by

$$\pi\big(f(x_1,x_2)\big) = \frac{x_1 f(x_1,x_2) - x_2 f(x_2,x_1)}{x_1 - x_2}\,.$$

For any $i, j \in \mathbb{N}$, we have

$$\pi(x_1^j x_2^i) = S_{i,j}(\mathbb{X}_2)\,. \tag{37}$$

Proposition 14 *The following identity holds for $r \geq 2$,*

$$F_r(\mathbb{X}_2 - \mathbb{D}) = -3^{r-2} R(X_2,\mathbb{D})(x_1x_2)^{r-2}\big(3S_{r-2}(\mathbb{X}_2) - 2S_{1,r-3}(\mathbb{X}_2)\big)\,. \tag{38}$$

Proof. The identity is true for $r = 2$. To prove the assertion for $r \geq 3$, we compute in two different ways the action of π on the multi-Schur function (see [12, 1.4.7] p. 9):

$$S_{r,r;r}(\mathbb{X}_2 + \boxed{2x_1} + \boxed{3x_1} - \mathbb{D}; x_1 - \mathbb{D})\,. \tag{39}$$

Firstly, expanding (39), we have

$$\begin{aligned}
&\pi\big(S_{r,r;r}(\mathbb{X}_2 + \boxed{2x_1} + \boxed{3x_1} - \mathbb{D}; x_1 - \mathbb{D})\big)\\
&= \pi\Big(\sum_{j_1 \leq j_2 \leq r} S_{j_1,j_2}(\boxed{2x_1} + \boxed{3x_1})\; S_{r-j_2,r-j_1,r}(\mathbb{X}_2 - \mathbb{D}; x_1 - \mathbb{D})\Big)\\
&= \pi\Big(\sum_{j_1 \leq j_2 \leq r} S_{j_1,j_2}(\boxed{2} + \boxed{3})\; S_{r-j_2,r-j_1,r+j_1+j_2}(\mathbb{X}_2 - \mathbb{D}; x_1 - \mathbb{D})\Big)\\
&= \sum_{j_1 \leq j_2 \leq r} S_{j_1,j_2}(\boxed{2} + \boxed{3})\; S_{r-j_2,r-j_1,r+j_1+j_2}(\mathbb{X}_2 - \mathbb{D})\\
&= F_r(\mathbb{X}_2 - \mathbb{D})\,.
\end{aligned}$$

Secondly, we subtract x_1 from the arguments in the first two rows of (39) without changing the determinant (see [12, Transformation Lemma 1.4.1]):

$$\begin{aligned}
&S_{r,r;r}(\mathbb{X}_2 + \boxed{2x_1} + \boxed{3x_1} - \mathbb{D}; x_1 - \mathbb{D})\\
&\quad = S_{r,r;r}(\mathbb{X}_2 + \boxed{3x_1} - \boxed{2x_2} - \boxed{x_1 + x_2}; x_1 - \mathbb{D})\,.
\end{aligned} \tag{40}$$

Then the elements in the first two rows of the last column become zero, and we get the following factorization of the latter determinant in (40):

$$S_{r,r}(x_2 + \boxed{3x_1} - \boxed{2x_2} - \boxed{x_1 + x_2}) \cdot S_r(x_1 - \mathbb{D})\,.$$

Using the following two factorizations:

$$S_{r,r}(x_2 + \boxed{3x_1} - \boxed{2x_2} - \boxed{x_1 + x_2}) = -3^{r-2}(x_2 - 2x_1)(x_1x_2)^{r-1}(3x_1 - 2x_2)\,,$$

and

$$S_r(x_1 - \mathbb{D}) = x_1^{r-2} x_2 (x_1 - 2x_2)\,,$$

we infer that

$$S_{r,r;r}(\mathbb{X}_2+\boxed{2x_1}+\boxed{3x_1}-\mathbb{D};x_1-\mathbb{D}) = -3^{r-2}R(\mathbb{X}_2,\mathbb{D})(x_1x_2)^{r-2}x_1^{r-3}(3x_1-2x_2)\,. \tag{41}$$

By (37), the result of applying π to (41) is

$$-3^{r-2}R(X_2,\mathbb{D})(x_1x_2)^{r-2}\big(3S_{r-2}(\mathbb{X}_2)-2S_{1,r-3}(\mathbb{X}_2)\big)\,.$$

Comparison of both these computations of π applied to (39) yields the proposition. □

In terms of U_r, we rewrite Proposition 14 into

Corollary 15 *For $r \geq 2$,*

$$U_r(\mathbb{X}_2;0) = -3^{r-2}\big(3S_{r-2}(\mathbb{X}_2)-2S_{1,r-3}(\mathbb{X}_2)\big)\,. \tag{42}$$

Lemmas 10, 13, Proposition 11, and Corollary 15 imply Eq. (30), and this finishes the proof of Theorem 8.

5 Appendix: The Pascal starcaise

We shall use the following variant of the Pascal triangle. Consider an infinite matrix $P=[p_{s,t}]$ with rows and columns numbered by $s,t=1,2,\ldots$.

We assume that $p_{1,t}=p_{2,t}=0$ for $t\geq 2$, $p_{3,t}=p_{4,t}=0$ for $t\geq 3$, $p_{5,t}=p_{6,t}=0$ for $t\geq 4$ etc. (Speaking less formally, P is filled with 0's above the diagram of the infinite partition $(0,0,1,1,2,2,3,3,\ldots)$.)

The first column is an arbitrary sequence $v=(v_1,v_2,\ldots)$. In the case when this sequence is the sequence of coefficients of the Taylor expansion of a function $f(z)$, we write P_f for the corresponding P.

To define the remaining $p_{s,t}$'s, we use the recursive formula

$$p_{s+1,t}=p_{s,t-1}+p_{s,t}. \tag{43}$$

We visualize this definition by

$$\begin{matrix} a & b \\ & \square \end{matrix} \quad \Rightarrow \quad \begin{matrix} a & b \\ & a+b \end{matrix}$$

We thus get the following *Pascal staircase* $P=[p_{i,j}]_{i,j=1,2,\ldots}$:

$$\begin{matrix}
v_1 & 0 & 0 & 0 & 0 & \ldots \\
v_2 & 0 & 0 & 0 & 0 & \ldots \\
v_3 & v_2 & 0 & 0 & 0 & \ldots \\
v_4 & v_3{+}v_2 & 0 & 0 & 0 & \ldots \\
v_5 & v_4{+}v_3{+}v_2 & v_3{+}v_2 & 0 & 0 & \ldots \\
v_6 & v_5{+}v_4{+}v_3{+}v_2 & v_4{+}2v_3{+}2v_2 & 0 & 0 & \ldots \\
v_7 & v_6{+}v_5{+}v_4{+}v_3{+}v_2 & v_5{+}2v_4{+}3v_3{+}3v_2 & v_4{+}2v_3{+}2v_2 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{matrix}$$

Given an integer $n \geq 0$, and an alphabet $\mathbb{A}$, we define the function $W(n) = W(n, \mathbb{A})$ by

$$W(n, \mathbb{A}) = \sum_{i,j} p_{n+1-i,j+1}\, S_i(-\mathbb{A})\, S_{j,n-i-j}(\mathbb{X}_2). \tag{44}$$

The function $W(n, \mathbb{A})$ is linear in the elements of the first column of P. Therefore it is sufficient to restrict to the case $v = (1, y, y^2, \ldots)$, i.e. to take $P = P_{1/(1-zy)}$ to determine it.

Lemma 16 *If $P = P_{1/(1-zy)}$ and $\mathbb{A} = \boxed{x_1 + x_2}$, then $W(0) = 1$ and for $n \geq 1$*

$$W(n, \boxed{x_1 + x_2}) = (y-1)y^{n-1} S_n(\mathbb{X}_2). \tag{45}$$

Proof. The entries contributing to $S_{k,n-k}(\mathbb{X}_2)$, where $k > 0$ and $2k < n$ are, for some a, b,

$$\begin{array}{ll} -a(x_1+x_2)S_{k-1,n-k}(\mathbb{X}_2) & -b(x_1+x_2)S_{k,n-k-1}(\mathbb{X}_2) \\ & \quad (a+b)S_{k,n-k}(\mathbb{X}_2) \end{array}$$

and give $-aS_{k,n-k}(\mathbb{X}_2) - bS_{k,n-k}(\mathbb{X}_2) + (a+b)S_{k,n-k}(\mathbb{X}_2) = 0$.

The entries contributing to $S_{k,k}(\mathbb{X}_2)$, where $k > 0$ and $n = 2k$ are, for some a,

$$\begin{array}{lc} -a(x_1+x_2)S_{k,k}(\mathbb{X}_2) & 0 \\ & aS_{k,k}(\mathbb{X}_2) \end{array}$$

and give $-aS_{k,k}(\mathbb{X}_2) + aS_{k,k}(\mathbb{X}_2) = 0$.

Moreover, the first column contributes to $(y^n - y^{n-1})S_n(\mathbb{X}_2)$. □

Taking now $\mathbb{A} = \boxed{x_1 + x_2} + \mathbb{B}$ instead of $\boxed{x_1 + x_2}$, and using that

$$W(n, \mathbb{A}) = \sum_{i,j,k} p_{n+1-i-k,j+1}\, S_i\left(-\boxed{x_1 + x_2}\right) S_{j,n-i-j-k}(\mathbb{X}_2) S_k(-\mathbb{B})$$

$$= \sum_k W\left(n-k, \boxed{x_1 + x_2}\right) S_k(-\mathbb{B})$$

$$= (1 - y^{-1}) \sum_k y^{n-k} S_{n-k}(\mathbb{X}_2) S_k(-\mathbb{B}) = y^n((1 - y^{-1})S_n(\mathbb{X}_2 - y^{-1}\mathbb{B}),$$

we get the following corollary.

Corollary 17 *For $P = P_{1/(1-zy)}$, $\mathbb{B}$ an arbitrary alphabet, then (apart from initial values), we have*

$$W(n, \boxed{x_1 + x_2} + \mathbb{B}) = (y-1)y^{n-1} S_n(\mathbb{X}_2 - y^{-1}\mathbb{B}). \tag{46}$$

We apply the corollary with $\mathbb{B} = \boxed{2x_1} + \boxed{2x_2}$. Expanding

$$S_n\left(\mathbb{X}_2 - y^{-1}(\boxed{2x_1} + \boxed{2x_2})\right)$$
$$= S_n(\mathbb{X}_2) - \frac{2x_1 + 2x_2}{y} S_{n-1}(\mathbb{X}_2) + 4\frac{x_1 x_2}{y^2} S_{n-2}(\mathbb{X}_2)\,,$$

we get, for $n \geq 3$,

$$W(n, \mathbb{D}) = y^{n-2}(y-1)(y-2)S_n(\mathbb{X}_2)) - 2y^{n-3}(y-1)(y-2)S_{1,n-1}(\mathbb{X}_2) \quad (47)$$

and initial conditions

$$W(0) = 1,\ W(1) = (y-3)S_1(\mathbb{X}_2),\ W(2) = (y-1)(y-2)S_2(\mathbb{X}_2) - 2(y-3)S_{11}(\mathbb{X}_2)\,.$$

We come back to Proposition 11, and we take the Pascal staircase P_f associated with the function

$$f = \frac{5-6z}{(1-z)(1-2z)(1-3z)} = -\frac{1/2}{1-z} - \frac{8}{1-2z} + \frac{27/2}{1-3z}.$$

Then for $P = P_f$, and $n = r - 2$, the function $W(n, \mathbb{D})$ is the function $V_r(\mathbb{X}_2; 0)$.

We thus have to specialize y into $1, 2, 3$ successively. Apart from initial values, only $y = 3$ contributes, and we get, for $n \geq 3$,

$$W(n, \mathbb{D}) = 3^{n+1} S_n(\mathbb{X}_2) - 2 \cdot 3^n S_{1,n-1}(\mathbb{X}_2)\,.$$

This proves Proposition 11, checking the cases $r = 2, 3, 4$ directly.

Note As the referee of [19] points out, the Thom polynomials for Morin singularities have been recently also studied – using quite different methods – by Fehér and Rimányi in [8], and by Bérczi and Szenes in [3].

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