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By

Jung Yeun LEE, Suh-Ryung KIM, Seog-Jin KIM,
and Yoshio SANO

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

The competition number of a graph in the aspect of the number of holes

JUNG YEUN LEE* and SUH-RYUNG KIM*†
Department of Mathematics Education, Seoul National University
Seoul 151-742, Korea

SEOG-JIN KIM
Department of Mathematics Education, Konkuk University,
Seoul 143-701, Korea

YOSHIO SANO
Research Institute for Mathematical Sciences, Kyoto University,
Kyoto 606-8502, Japan

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Abstract

Let D be an acyclic digraph. The competition graph of D is a graph which has the same vertex set as D and has an edge between u and v if and only if there exists a vertex x in D such that (u,x) and (v,x) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of G is the smallest number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph G and it has been one of important research problems in the study of competition graphs to characterize a graph by its competition number.

A hole of a graph is a cycle of length at least 4 as an induced subgraph. Kim [2005] conjectured that the competition number of a graph with h holes is at most $h+1$. In this paper, we show that the conjecture is true for a graph all of whose holes are mutually edge-disjoint.

Key words and phrases: competition graphs, competition numbers, hole-edge-disjoint graphs

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†Corresponding author. *email address*: srkim@snu.ac.kr

1 Introduction

Suppose D is an acyclic digraph (for all undefined graph-theoretical terms, see [1] and [17]). The *competition graph* of D, denoted by $C(D)$, has the same set of vertices as D and an edge between vertices u and v if and only if there is a vertex x in D such that (u, x) and (v, x) are arcs of D. Roberts [16] observed that if G is any graph, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* $k(G)$ of a graph G to be the smallest number k such that G together with k isolated vertices added is the competition graph of an acyclic digraph.

The notion of competition graph was introduced by Cohen [4] as a means of determining the smallest dimension of ecological phase space. Since then, various variations have been defined and studied by many authors (see, for example, [2, 8, 10, 12, 13, 18]). Besides an application to ecology, the concept of competition graph can be applied to the study of communication over noisy channel (see Roberts [16] and Shannon [19]) and to problem of assigning channels to radio or television transmitters (see Cozzens and Roberts [5], Hale [7], or Opsut and Roberts [15]).

Roberts [16] observed that characterization of competition graph is equivalent to computation of competition number. It does not seem to be easy in general to compute $k(G)$ for all graphs G, as Opsut [14] showed that the computation of the competition number of a graph is an NP-hard problem (see [10, 12] for graphs whose competition numbers are known). It has been one of important research problems in the study of competition graphs to characterize a graph by its competition number.

We call a cycle of a graph G a *chordless cycle* of G if it is an induced subgraph of G. A chordless cycle of length at least 4 of a graph is called a *hole* of the graph and a graph without holes is called a *chordal graph*.

Cho and Kim [3] studied the competition number of a graph with exactly one hole and showed that the competition number of a graph with exactly one hole is at most 2. Kim [11] observed that the graph given in Figure 1 with h holes has competition number $h + 1$ and conjectures that $h + 1$ is the largest competition number that can be achieved by a graph with h holes.

In this paper, we show that the competition number of a graph all of whose holes are mutually edge-disjoint is at most $h + 1$ where h is the number of holes. From this result, it immediately follows that the competition number of a graph all of whose holes are mutually vertex-disjoint is at most $h + 1$ where h is the number of holes.

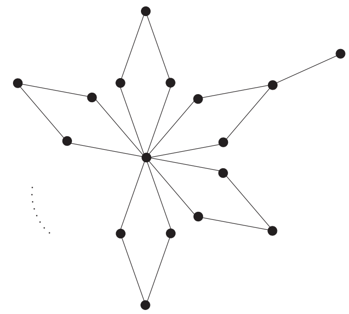

Figure 1: A graph G with h holes and $k(G) = h + 1$.

2 Preliminaries

Given a graph G and a hole C of G, we denote by X_C the set of vertices that are adjacent to every vertex of C. Given a graph G and a hole C of G, we call a walk (resp. path) W a *C-avoiding walk* (resp. *C-avoiding path*) if none of the internal vertices of W are on C or in X_C.

A set S of vertices of a graph G is called a *vertex cut* of G if the number of components of $G - S$ is greater than that of G.

Throughout this paper, we assume that all subscripts of vertices on a cycle are reduced to modular the length of the cycle.

Lemma 2.1 ([3]). *Suppose that a graph G has exactly one hole C. If there exists a C-avoiding (u, v)-path for some consecutive vertices u, v on C, then $X_C \cup \{u, v\}$ is a vertex cut.*

Theorem 2.2 ([3]). *If a graph G has exactly one hole, then $k(G) \leq 2$.*

Cho and Kim [3] showed that for a chordal graph G, we may construct an acyclic digraph D with the vertices of indegree 0 as many as the number of a clique so that the competition graph of D is G with one more isolated vertex:

Lemma 2.3 ([3]). *If K is a clique of a chordal graph G, then there exists an acyclic digraph D such that $C(D) = G \cup I_1$, and the vertices of K have only outgoing arcs in D.*

This lemma is useful when we construct an acyclic graph D whose competition graph has a nontrivial chordal component.

Theorem 2.4. *Suppose that a graph G has two subgraphs G_1 and G_2, and a clique X satisfying the following property: $E(G_1) \cup E(G_2) = E(G)$, $V(G_1) \cap V(G_2) = X$, and G_2 is a chordal graph where X is a clique of G_2. Then if $k(G_1) \leq k$, then $k(G) \leq k + 1$.*

Proof. Let $k(G_1) \leq k$, there exists an acyclic digraph D_1 such that $C(D_1) = G_1 \cup \{a_1, ..., a_k\}$ where $a_1, \ldots, a_k$ are isolated vertices not in $V(G)$.

Since X is a clique in G_2 which is a chordal graph by the hypothesis, there exists an acyclic digraph D_2 such that $C(D_2) = G_2 \cup \{a_{k+1}\}$ where a_{k+1} is an isolated vertex not in $V(G) \cup \{a_1, \ldots, a_k\}$ and the vertices in X have only outgoing arcs in D_2 by Lemma 2.3.

Now we define a digraph D as follows: $V(D) = V(D_1) \cup V(D_2)$ and $A(D) = A(D_1) \cup A(D_2)$. Firstly, note that $V(G_1) \cap V(G_2) = X$. Suppose that there is an edge in $E(C(D))$ but not in $E(C(D_1)) \cup E(C(D_2))$. Then there exist an arc (u, x) in D_1 and an arc (v, x) in D_2 for some $x \in X$. However, this is impossible since every vertex in X has indegree 0 in D_2. Thus $E(C(D)) \subset E(C(D_1)) \cup E(C(D_2))$. It is obvious that $E(C(D)) \supset E(C(D_1)) \cup E(C(D_2))$ since $E(C(D)) \supset E(C(D_i))$ for $i = 1, 2$. Thus

$$E(C(D)) = E(C(D_1)) \cup E(C(D_2)) = E(G_1) \cup E(G_2) = E(G).$$

Moreover, since D_1 and D_2 are acyclic, $V(G_1) \cap V(G_2) = X$, and each vertex in X has only outgoing arcs in D_2, it is true that D is acyclic. Hence $C(D) = G \cup \{a_1, \ldots, a_k, a_{k+1}\}$ and so $k(G) \leq k + 1$. □

Given a walk W of a graph G, we denote by W^{-1} the walk represented by the reverse of vertex sequence of W. We also denote the length of W by $|W|$.

Lemma 2.5. *Let C be a hole of a graph G. Suppose that v is a vertex not on C that is adjacent to two non-adjacent vertices x and y of C. Then exactly one of the following is true:*

(1) v is adjacent to all the vertices of C;

(2) v is on a hole C^* different from C such that there are at least two common edges of C and C^* and all the common edges are contained in exactly one of the (x, y)-sections of C.

Proof. Suppose that (1) is not true. Then there exists a vertex z on C that is not adjacent to v. Let P be the (x, y)-section of C that contains z. Let w (resp. u) be the first vertex right after z along P (resp. P^{-1}) that is adjacent to v. Such a vertex exists since v is adjacent to y (resp. x). Then the (u, w)-section of C containing z and uvw form a hole satisfying the property of C^* given in (2). □

Lemma 2.6. *Let $C = v_0 v_1 \cdots v_{n-1} v_0$ be a hole of a graph G. Suppose that there exists a vertex v satisfying the following properties:*

- *v is not on any hole of G.*

- *v is adjacent to v_i for some $i \in \{v_0, \ldots, v_{n-1}\}$.*
- *There is a C-avoiding path from v to a vertex on C other that v_i.*

Let v_j be a vertex with the smallest $|i - j|$ such that there is a C-avoiding (v, v_j)-path and P be the shortest among C-avoiding (v, v_j)-paths. Then v_i is adjacent to every internal vertex on P. Moreover, if none of internal vertices on P belongs to any hole, then $j = i - 1$ or $i + 1$.

Proof. Let Q be the shorter (v_i, v_j)-section of C. Firstly, consider the case where $|P| = 1$. If $j \neq i - 1$ or $i + 1$, then the hypothesis of Lemma 2.5 is satisfied. However, none of (1), (2) holds, which is a contradiction. Thus, $j \in \{i - 1, i + 1\}$ and we are done.

Now suppose that $|P| \geq 2$. Then $v_i P Q^{-1}$ is a cycle of length at least 4. Since v is not on any hole on G, it cannot be a hole and has a chord. Take an internal vertex w on P. If w is adjacent to a vertex v_k for some k, $1 \leq |i - k| \leq |i - j|$, then v_i, the (v, w)-section of P, and v_k form a C-avoiding path, which contradicts the choice of v_j. Thus no internal vertex of P is adjacent to any vertex on the shorter (v_i, v_j)-section of C except v_i. Thus v_i is adjacent to an internal vertex of P. Let x be the first internal vertex on P and P' be the (v, x)-section of P. Then $v_i P' v_i$ is a hole or a triangle. However, the former cannot happen by the condition on v. Thus x immediately follows v on P. By repeating this argument, we can show that v_i is adjacent to every internal vertex on P.

Now assume that none of internal vertices on P does not belong to any hole. Let y be the vertex immediately preceding v_j on P. Then $v_i y Q^{-1}$ is a hole or a triangle. By our assumption, the former does not hold. Thus Q is a path of length 1, that is, v_i and v_j are adjacent. Hence $j = i - 1$ or $j = i + 1$. □

3 Properties of hole-edge-disjoint graphs

We call a graph G a *hole-edge-disjoint graph* if all the holes of G are mutually edge-disjoint.

Lemma 3.1. *Given a hole-edge-disjoint graph G, let C be a hole of G. If $v \notin V(C)$ is a vertex adjacent to two non-adjacent vertices of C, then v is adjacent to all the vertices of C.*

Proof. Since G is a hole-edge-disjoint graph G, (2) of Lemma 2.5 cannot happen. Thus the lemma immediately follows. □

Lemma 3.2. *Let G be a hole-edge-disjoint graph and C be a hole of G. Then there is no C-avoiding path joining two nonconsecutive vertices of C.*

Proof. By contradiction. Suppose that there is a C-avoiding (v_i, v_j)-path P for some i, $j \in \{0, \ldots, m-1\}$ satisfying $|i-j| \geq 2$ where $C = v_0v_1 \cdots v_{m-1}v_0$. Let P be the shortest among the C-avoiding (v_i, v_j)-paths. Then there is no edge joining two nonconsecutive vertices on P. Let P_1 and P_2 be the two (v_i, v_j)-sections of C containing v_{i-1} and v_{i+1}, respectively. Then P and P_1 form a cycle in G and so do P and P_2. By the hypothesis, these cycles cannot be holes. Then, by the choice of P, an internal vertex of P is adjacent to an internal vertex on P_1. Let u be the first internal vertex on P that is adjacent to an internal vertex on P_1. Then let v be the first internal vertex on P_1 that is adjacent to u. Then the (v_i, u)-section of P, the edge uv, the (v, v_i)-section of P_1^{-1} form a triangle or a hole. Since it shares an edge with C, it must form a triangle and so u is the vertex immediately following v_i on P and $v = v_{i-1}$. By applying a similar argument for P_2, we can show that u is adjacent to v_{i+1}. Therefore, by Lemma 3.1, u belongs to X_C. However, since P is a C-avoiding path, u does not belong to X_C and we reach a contradiction. □

Corollary 3.3. *Let G be a hole-edge-disjoint graph and C be a hole of G. Given a vertex v of C, joining v and every other vertex on C by a new edge reduces the number of holes of G.*

Proof. It is obvious that C is not a hole in the resulting graph. Thus it is sufficient to show that no new hole has been created. We show it by contradiction. Suppose that a new hole is created. Then there exists a vertex w on C such that w is not adjacent to v and there is a C-avoiding (v, w)-path P in G. This contradicts Lemma 3.2. □

Lemma 3.4. *Let G be a hole-edge-disjoint graph and C be a hole of G. Suppose that G has a C-avoiding (u, v)-path for some consecutive vertices u, v on C. Then $X_C \cup \{u, v\}$ is a vertex cut.*

Proof. We prove by induction on the number h of holes of a graph. If a graph has exactly one hole, then it immediately follows from Lemma 2.1. Suppose that the lemma holds for any hole-edge-disjoint graph with at most $h-1$ holes for $h \geq 2$. Now take a hole-edge-disjoint graph G with h holes. Suppose that G has a C-avoiding (u, v)-path for some hole C of G and some consecutive vertices u, v on C. Since $h \geq 2$, there exists another hole C'. Take a vertex w of C' and join w and every other vertex on C' by a new edge. Then by Corollary 3.3, the resulting graph G' has less than h holes. It is easy to see that G' is still a hole-edge-disjoint graph and that C is a hole of G'. By the induction hypothesis, $X_C \cup \{u, v\}$ is a vertex cut of G'. Since G' is obtained by adding edges to G, it is true that $X_C \cup \{u, v\}$ is a vertex cut of G. □

We denote by K_2^m a complete multipartite with m parts each of which has size 2, which is called a 'cocktail party graph'. We say that a graph is

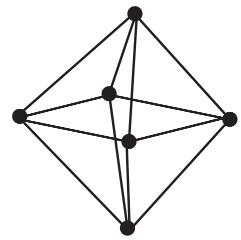

Figure 2: K_2^3. Note that K_2^3 is induced by the edges of 3 edge-disjoint holes of length 4

K_2^3*-free* if it does not contain a complete tripartite graph K_2^3 as an induced subgraph (see Figure 2).

The following lemma shows that the subgraph induced by X_C is a clique if G is an K_2^3-free hole-edge-disjoint graph:

Lemma 3.5. *If a graph G is a K_2^3-free hole-edge-disjoint graph and C is a hole of G, then X_C is a clique.*

Proof. Suppose that there are two nonadjacent vertices u and w in X_C. If $|C| = 4$, then $V(C) \cup \{u, w\}$ induces K_2^3. Thus, $|C| \geq 5$. Let $C = v_0 v_1 ... v_{m-1} v_1$ for $m \geq 4$. Then uv_0wv_2u and uv_0wv_3 are holes sharing the edge uv_0, which is a contradiction. □

Lemma 3.6. *Let G be a K_2^3-free hole-edge-disjoint graph with exactly h holes and $C = v_0 \cdots v_{m-1}v_0$ be a hole of G. Suppose that G has no C-avoiding path between v_i and v_{i+1} for some $i \in \{0, 1, \ldots, m-1\}$. Then $G - v_iv_{i+1}$ has at most $h - 1$ holes.*

Proof. Suppose that $G - v_iv_{i+1}$ has more than $h - 1$ holes. Then, since C is not a hole in $G - v_iv_{i+1}$, there is a hole C' in $G - v_iv_{i+1}$ that is not a hole in G. Obviously v_iv_{i+1} is a chord of C' in G. Since C' is a hole in $G - v_iv_{i+1}$, it is true that v_iv_{i+1} is the only chord of C'.

Now consider the two distinct (v_i, v_{i+1})-sections P_1 and P_2 of C'. If $|P_1| \geq 3$ or $|P_2| \geq 3$, then $P_1v_iv_{i+1}$ or $P_2v_iv_{i+1}$ is a hole in G that shares an edge with C, which contradicts the hypothesis that G has only edge-disjoint holes. Thus $|P_1| = 2$ and $|P_2| = 2$. We denote $P_1 = v_iuv_{i+1}$ and $P_2 = v_iu'v_{i+1}$. Since G does not contain a C-avoiding path between v_i and v_{i+1} by the hypothesis, it is true that $\{u, u'\} \subset X_C \cup V(C)$. However, if $u \in V(C)$, then at least one of v_iu, $v_{i+1}u$ is a chord of C, which is a contradiction. If $u \in X_C$, then u and u' are adjacent by Lemma 3.5. Then the edge uu' is a chord C', which is a contradiction again. Therefore $G - v_iv_{i+1}$ has at most $h - 1$ holes. □

In the following, we present some results on structures of hole-edge-disjoint graphs having K_2^3 as an induced subgraph.

Lemma 3.7. *Suppose that a hole-edge-disjoint graph G has K_2^3 as an induced subgraph. Let m be the maximum integer such that K_2^m is an induced subgraph of G. If X is the set of vertices of G each of which is adjacent to every vertex of K_2^m, then X is a clique.*

Proof. By contradiction. Suppose that there exist two nonadjacent vertices u and v in X. Then $V(K_2^m) \cup \{u, v\}$ induces K_2^{m+1}, which contradicts the choice of m. □

Lemma 3.8. *Suppose that a hole-edge-disjoint graph G has K_2^3 as an induced subgraph. Let m be the maximum integer such that K_2^m is an induced subgraph of G and X be the set of vertices of G each of which is adjacent to every vertex of K_2^m. Then $N(u) \cap N(v) \subset X \cup V(K_2^m)$ for any nonadjacent vertices u, v in $V(K_2^m)$.*

Proof. Take a vertex $w \in N(u) \cap N(v)$ that is not in $V(K_2^m)$. Then take a vertex x in $V(K_2^m) \setminus \{u, v\}$. By the definition of K_2^m, x is adjacent to both u and v. If w is not adjacent to x, then $uwvxu$ and $uyvxu$ are holes where y is a vertex of K_2^m that belongs to the same partite set as x. This contradicts the hypothesis that G is a hole-edge-disjoint graph. Thus w is adjacent to x. Since x is chosen arbitrarily from $V(K_2^m)$, it is true that $w \in X$. □

4 The competition number of a hole-edge-disjoint graph

In this section, we shall show that the competition number of a hole-edge-disjoint graph G does not exceed $h + 1$ where h is the number of holes of G. This result partially answers the conjecture given by Kim [11]. In order to do so, we need the following notations: Let G be a hole-edge-disjoint graph with exactly h holes C_1, C_2, $\ldots$, C_h. For each $t = 1, \ldots, h$, we let

$$C_t = v_{t,0}v_{t,1}...v_{t,m_t-1}v_{t,0},$$

where m_t is the length of the hole C_t. We denote X_{C_t} by X_t for short.

If there exists a C_t-avoiding $(v_{t,i}, v_{t,i+1})$-path for some $t \in [h]$, where $[h]$ denotes the set $\{1, \ldots, h\}$, and for some $i \in \{0, \ldots, m_t - 1\}$, then the set

$$\{v \in V(G) \mid v_{t,i}vv_{t,i+1} \text{ is a } C_t\text{-avoiding path}\}$$

is not empty by Lemma 2.6. We denote it by $A_{t,i}$. By Lemma 3.4, $\{v_{t,i}, v_{t,i+1}\} \cup X_t$ is a vertex cut. For simplicity, we denote $\{v_{t,i}, v_{t,i+1}\} \cup X_t$ by $X_{t,i}$. Let $Q_{t,i}$ be the component of $G - \{v_{t,i}, v_{t,i+1}\} - X_t$ containing $V(C_t) \setminus \{v_{t,i}, v_{t,i+1}\}$. Among the components of $G - \{v_{t,i}, v_{t,i+1}\} - X_t$ other than $Q_{t,i}$, we take the components each of which contains a vertex in $A_{t,i}$. Then we denote the union of such components by $G_{t,i}$. It is easy to see that $A_{t,i} \subset V(G_{t,i})$.

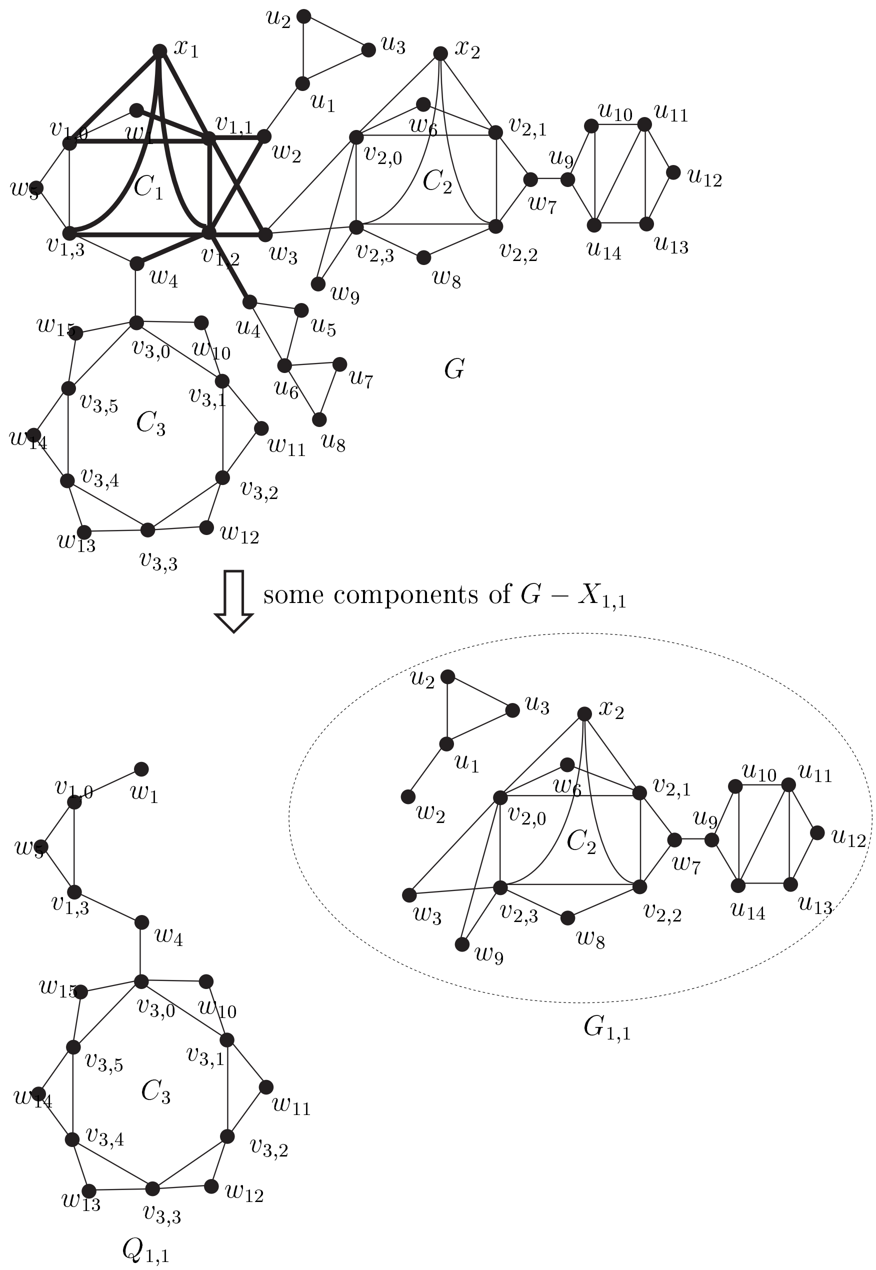

Figure 3: A hole-edge-disjoint graph G, $X_{1,1} = \{v_{1,1}, v_{1,2}\} \cup \{x_1\}$, $A_{1,1} = \{w_2, w_3\}$, $Q_{1,1}$ together with the components of $G - X_{1,1}$ containing w_2 or w_3.

Also note that $A_{t,i} \cap X_t = \emptyset$ for any $i \in \{0, \ldots, m_t - 1\}$. We summarize the notations introduced above in Table 1. See Figure 4 for illustration.

Now we are ready to present the following lemma:

Lemma 4.1. *Let G be a K_2^3-free hole-edge-disjoint graph with exactly h holes $C_1, C_2, \ldots, C_h$. Suppose that G has a C_t-avoiding $(v_{t,i}, v_{t,i+1})$-path for each $t \in [h]$ and for each $i \in \{0, \ldots, m_t - 1\}$. Suppose that for some $t^* \in [h]$ and $i^* \in \{0, \ldots, m_{t^*} - 1\}$, $G[V(G_{t^*,i^*}) \cup X_{t^*,i^*}]$ contains a hole. Then for any hole C_s in $G[V(G_{t^*,i^*}) \cup X_{t^*,i^*}]$, the following are true:*

(1) If there exists a vertex u in $A_{s,j}$ not belonging to $V(G_{t^*,i^*}) \cup X_{t^*,i^*}$ for some $j \in \{0, \ldots, m_s - 1\}$, then $v_{s,j}uv_{s,j+1}$ is a C_s-avoiding path and

$$\{v_{s,j}, v_{s,j+1}\} \subset X_{t^*,i^*};$$

(2) $|\{j \mid A_{s,j} \subset V(G_{t^*,i^*})\}| \geq m_s - 2$;

(3) For some $k \in \{j \mid A_{s,j} \subset V(G_{t^*,i^*})\}$, there is no C_s-avoiding path from any vertex in $A_{s,k}$ to any vertex in X_{t^*,i^*} in G.

(4) For some $k \in \{j \mid A_{s,j} \subset V(G_{t^*,i^*})\}$,

$$V(G_{s,k}) \cup X_{s,k} \subsetneq V(G_{t^*,i^*}) \cup X_{t^*,i^*}.$$

Proof. We show (1) as follows. Since u is in $A_{s,j}$, it is true that $v_{s,j}uv_{s,j+1}$ is a C_s-avoiding path. Suppose that one of $v_{s,j}$, $v_{s,j+1}$ is not contained in X_{t^*,i^*}. We may assume that $v_{s,j}$ is not in X_{t^*,i^*}. Then $v_{s,j}$ is contained in $V(G_{t^*,i^*})$ since C_s is contained in $V(G_{t^*,i^*}) \cup X_{t^*,i^*}$. Since $u \notin X_{t^*,i^*}$ by the hypothesis, u and $v_{s,j}$ are still adjacent in $G - X_{t^*,i^*}$. However, note that $u \notin V(G_{t^*,i^*})$ by the hypothesis while $v_{s,j} \in V(G_{t^*,i^*})$. This implies that they belong to distinct components in $G - X_{t^*,i^*}$ and so we reach a contradiction. Thus, $\{v_{s,j}, v_{s,j+1}\} \subset X_{t^*,i^*}$.

We show (2) by contradiction. Suppose that $|\{j \mid A_{s,j} \subset V(G_{t^*,i^*})\}| < m_s - 2$. Then $|\{j \mid A_{s,j} \not\subset V(G_{t^*,i^*})\}| > m_s - (m_s - 2)$ and so there exist p, $q \in \{0, \ldots, m_s - 1\}$ such that there exist vertices u_p and u_q of G such

$X_{t,i}$	$\{v_{t,i}, v_{t,i+1}\} \cup X_t$
$A_{t,i}$	$\{v \in V(G) \mid v_{t,i}vv_{t,i+1}$ is a C_t-avoiding path$\}$
$Q_{t,i}$	the component of $G - X_{t,i}$ containing $V(C_t) \setminus \{v_{t,i}, v_{t,i+1}\}$
$G_{t,i}$	the union of the components of $G - X_{t,i} - V(Q_{t,i})$ each of which contains a vertex in $A_{t,i}$

Table 1: Notations needed to prove Lemma 4.1 and Theorems 4.2 and 4.3.

that $u_p \in A_{s,p}$ and $u_q \in A_{s,q}$, but $u_p \notin V(G_{t^*,i^*})$ and $u_q \notin V(G_{t^*,i^*})$. Then $v_{s,p}u_pv_{s,p+1}$ and $v_{s,q}u_qv_{s,q+1}$ are C_s-avoiding paths. Since there are at least two vertices in $S = \{v_{s,p},\ v_{s,p+1},\ v_{s,q},\ v_{s,q+1}\}$ which are not adjacent, there exists a vertex in S not in X_{t^*,i^*}. Without loss of generality, we may assume that $v_{s,p} \notin X_{t^*,i^*}$. By (1), $A_{s,p} \subset V(G_{t^*,i^*}) \cup X_{t^*,i^*}$. Since $A_{s,p} \cap V(G_{t^*,i^*}) = \emptyset$, $A_{s,p} \subset X_{t^*,i^*}$ and so $u_p \in X_{t^*,i^*}$. Suppose that $u_q \in X_{t^*,i^*}$. Since $G[X_{t^*,i^*}]$ is a clique by Lemma 3.5, u_p and u_q are adjacent. Then there exist both a C_s-avoiding $(v_{s,p}, v_{s,q})$-path and a C_s-avoiding $(v_{s,p}, v_{s,q+1})$-path, contradicting Lemma 3.2. Thus $u_q \notin X_{t^*,i^*}$. Then $u_q \notin V(G_{t^*,i^*}) \cup X_{t^*,i^*}$. By (1), $\{v_{s,q}, v_{s,q+1}\} \subset V(G_{t^*,i^*}) \cup X_{t^*,i^*}$. This implies that $v_{s,p}u_pv_{s,q}$ and $v_{s,p}u_pv_{s,q+1}$ are C_s-avoiding paths, which contradicts Lemma 3.2.

Now we show (3) in the following. By (2), there exist two distinct integers k and l in $\{j \mid A_{s,j} \subset V(G_{t^*,i^*})\}$. That is, $A_{s,k} \subset V(G_{t^*,i^*})$ and $A_{s,l} \subset V(G_{t^*,i^*})$.

Suppose that there exist C_s-avoiding paths P and Q from w_k to a vertex X_{t^*,i^*} and from w_l to a vertex Y_{t^*,i^*} in G, respectively, for some $w_k \in A_{s,k}$ and $w_l \in A_{s,l}$. Then PQ^{-1} contains a C_s-avoiding (w_k, w_l)-path. However, this path extends to a C_s-avoiding $(v_{s,k}, v_{s,l+1})$-path, which contradicts Lemma 3.2. This argument implies that for at least one of $A_{s,k}$, $A_{s,l}$, there is no C_s-avoiding path from any of its vertices to any vertex belonging to X_{t^*,i^*} in G. Without loss of generality, we may assume that $A_{s,k}$ satisfies this property (for, otherwise, we can relabel the vertices on C_s so that the vertex v_l is labeled as v_k).

Finally we show (4). By (3), there exists $k \in \{j \mid A_{s,j} \subset V(G_{t^*,i^*})\}$ such that there is no C_s-avoiding path from any vertex in $A_{s,k}$ to any vertex in X_{t^*,i^*} in G. Since $V(C_s) \subset V(G_{t^*,i^*}) \cup X_{t^*,i^*}$ by the hypothesis, it holds that $\{v_{s,k}, v_{s,k+1}\} \subset V(G_{t^*,i^*}) \cup X_{t^*,i^*}$. Now take a vertex x in X_s. If $x \notin X_{t^*,i^*}$, then x is still adjacent to a vertex on C_s in $G - X_{t^*,i^*}$ and so $x \in V(G_{t^*,i^*})$. Thus $X_s \subset V(G_{t^*,i^*}) \cup X_{t^*,i^*}$ and therefore $X_{s,k} = X_s \cup \{v_{s,k}, v_{s,k+1}\} \subset V(G_{t^*,i^*}) \cup X_{t^*,i^*}$. Now it remains to show that $V(G_{s,k}) \subset V(G_{t^*,i^*}) \cup X_{t^*,i^*}$. Take a vertex y in $G_{s,k}$. Then y belongs to a component W of $G - X_{s,k}$. By the definition of $G_{s,k}$, $V(W) \cap A_{s,k} \neq \emptyset$. Take $z \in V(W) \cap A_{s,k}$. Then, since any vertex in W and z belong to a component of $G_{s,k}$, any vertex in W and z are connected by a C_s-avoiding path. Thus, by (3), $W \cap X_{t^*,i^*} = \emptyset$ and so W is a connected subgraph of $G - X_{t^*,i^*}$. Since $A_{s,k} \subset V(G_{t^*,i^*})$, it is true that $z \in V(G_{t^*,i^*})$. Therefore, $V(W) \subset V(G_{t^*,i^*})$ since z belongs to W, which is connected in $G - X_{t^*,i^*}$. Since $y \in V(W)$, it is true that $y \in V(G_{t^*,i^*})$. We have just shown that $V(G_{s,k}) \subset V(G_{t^*,i^*})$. Hence $V(G_{s,k}) \cup X_{s,k} \subset V(G_{t^*,i^*}) \cup X_{t^*,i^*}$.

Furthermore by (2), there is another $l \in \{0, \ldots, m_s - 1\}$ such that $A_{s,l} \subset V(G_{t^*,i^*})$. Now take w_l in $A_{s,l}$. Then $w_l \in V(G_{t^*,i^*}) \cup X_{t^*,i^*}$. However, $w_l \notin V(G_{s,k}) \cup X_{s,k}$ since w_l is still adjacent to at least one of $v_{s,l}$, $v_{s,l+1}$ in

$G - X_{s,k}$. Thus $V(G_{s,k}) \cup X_{s,k} \subsetneq V(G_{t^*,i^*}) \cup X_{t^*,i^*}$ and (4) follows. □

Given a K_2^3-free hole-edge-disjoint graph G, we say that G has the *chordal property* if there exist $t \in [h]$ and $i \in \{0, \ldots, m_t - 1\}$ such that $G[V(G_{t,i}) \cup X_t \cup \{v_{t,i}, v_{t,i+1}\}]$ is chordal.

Theorem 4.2. *Let G be a K_2^3-free hole-edge-disjoint graph with exactly h holes C_1, C_2, $\ldots$, C_h. Suppose that G has a C_t-avoiding $(v_{t,i}, v_{t,i+1})$-path for each $t \in [h]$ and for each $i \in \{0, \ldots, m_t - 1\}$. Then G has the chordal property.*

Proof. By contradiction. Suppose that G does not have the chordal property. Then given $t \in [h]$, and $i \in \{0, \ldots, m_t - 1\}$, $G[V(G_{t,i}) \cup X_{t,i}]$ contains a hole. We denote the set of such holes by $\mathcal{C}_{t,i}$. Take a hole $C_s \in \mathcal{C}_{t,k_1}$ where $k_1 = 1$. We denote t and s by t_1 and t_2, respectively. By Lemma 4.1 (4), there exists $k_2 \in \{j \mid A_{t_2,j} \subset V(G_{t_1,1})\}$ such that $V(G_{t_2,k_2}) \cup X_{t_2,k_2} \subsetneq V(G_{t_1,k_1}) \cup X_{t_1,k_1}$. Again, by our assumption that G does not have the chordal property, there exists a hole $C_{t_3} \in \mathcal{C}_{t_2,k_2}$. Then, by Lemma 4.1 (4), there exists $k_3 \in \{j \mid A_{t_3,j} \subset V(G_{t_2,k_2})\}$ such that $V(G_{t_3,k_2}) \cup X_{t_3,k_3} \subsetneq V(G_{t_2,k_2}) \cup X_{t_2,k_2}$.

Repeating this process, we have $t_1, t_2, \ldots, t_i, \ldots$ and $k_1, k_2 \ldots, k_i, \ldots$ such that

$$\cdots \subsetneq V(G_{t_i,k_i}) \cup X_{t_i,k_i} \subsetneq \cdots \subsetneq V(G_{t_2,k_2}) \cup X_{t_2,k_2} \subsetneq V(G_{t_1,k_1}) \cup X_{t_1,k_1},$$

which is impossible since $V(G_{t_1,k_1}) \cup X_{t_1,k_1}$ is finite. This completes the proof. □

Now we are ready to present our main theorem:

Theorem 4.3. *If G is a hole-edge-disjoint graph with exactly h holes, then $k(G) \leq h + 1$.*

Proof. We prove by induction on h. The case $h = 1$ corresponds to Theorem 2.2. Suppose that the statement holds for any hole-edge-disjoint graph with exactly $h - 1$ holes for $h \geq 2$. Let G be a hole-edge-disjoint graph with exactly h holes $C_1, ..., C_h$.

Firstly we suppose that G is K_2^3-free. Then assume that there exist t and i such that G has no C_t-avoiding $(v_{t,i}, v_{t,i+1})$-path. By Lemma 3.6, $G - v_{t,i}v_{t,i+1}$ has at most $h - 1$ holes. By induction hypothesis, there exists a digraph D' such that $C(D') = (G - v_{t,i}v_{t,i+1}) \cup I$ where $I = \{a_1, a_2, ..., a_h\}$ is the set of newly added isolated vertices. Then we construct an acyclic digraph D from D' as follows:

$$V(D) = V(D') \cup \{a_{h+1}\};$$

$$A(D) = A(D') \cup \{(v_{t,i}, a_{h+1}), (v_{t,i+1}, a_{h+1})\}.$$

Then it is easy to check that D is acyclic and that $C(D) = G \cup \{a_1, a_2, ..., a_{h+1}\}$.

Now suppose that G has a C_t-avoiding $(v_{t,i}, v_{t,i+1})$-path for any t in $[h]$ and for any i in $\{0, \ldots, m_t - 1\}$. Then, by Lemma 4.2, G has the chordal property. That is, there exist $t \in [n]$ and $i \in \{0, \ldots, m_t - 1\}$ such that $G[V(G_{t,i}) \cup X_t \cup \{v_{t,i}, v_{t,i+1}\}]$ is chordal.

Let $H_{t,i}$ be the subgraph of G induced by $V(G) \setminus V(G_{t,i})$. Then $H_{t,i}$ does not contain any C_t-avoiding path by the definition of $G_{t,i}$. Moreover, $H_{t,i}$ is K_2^3-free. Thus $H_{t,i} - v_{t,i}v_{t,i+1}$ contains at most $h-1$ holes by Lemma 3.6. We denote $H_{t,i} - v_{t,i}v_{t,i+1}$ by $H^*_{t,i}$. Then, by the induction hypothesis, we have $k(H^*_{t,i}) \leq h$. Denote $G[V(G_{t,i}) \cup X_t \cup \{v_{t,i}, v_{t,i+1}\}]$ by $G^*_{t,i}$. Then $G^*_{t,i}$ is a chordal graph and $X_t \cup \{v_{t,i}, v_{t,i+1}\}$ is a clique of $G^*_{t,i}$.

Moreover, $E(H^*_{t,i}) \cup E(G^*_{t,i}) = E(G)$, and $V(H^*_{t,i}) \cap V(G^*_{t,i}) = X_t \cup \{v_{t,i}, v_{t,i+1}\}$. Hence, by Theorem 2.4, $k(G) \leq h + 1$.

Now consider the case where G contains K_2^3 as an induced subgraph. Let m be the maximum integer such that G contains K_2^m as an induced subgraph. Then $m \geq 3$. Now take two vertices u and v in the same partite set in K_2^m. Then they are nonadjacent and we join them by adding a new edge e. We call the resulting graph G'. Lemma 3.2 assures that G' does not contain any new hole. In fact, u and v belong to at least two distinct holes of length 4 in K_2^m and these holes become 4-cycles with chord uv in G'. Thus G' has at most 2 holes less than G. Therefore, by induction hypothesis, there exists an acyclic digraph D' such that $C(D') = G' \cup I_{h-1}$.

In the following, we shall construct an acyclic digraph D such that $C(D) = G \cup I_\ell$ by using D' for some positive integer $\ell \leq h$. If $|N^+_{D'}(u) \cap N^+_{D'}(v)| = 1$, then we construct D as follows:

$$V(D) = V(D') \cup \{a\};$$

$$A(D) = A(D') \setminus \{(x, w) \mid (x, w) \in A(D')\} \cup \{(x, w) \mid x \in Y_1\} \cup \{(x, a) \mid x \in Y_2\}$$

where $N^+_{D'}(u) \cap N^+_{D'}(v) = \{w\}$, and Y_1, Y_2 are the two cliques resulting from deleting e from $N^-_{D'}(w)$ which forms a clique in G'. Since $Y_1 \subset N^-_{D'}(w)$, D is still acyclic. From the construction, it can easily be checked that $C(D) = G \cup I_h$.

Now assume that $|N^+_{D'}(u) \cap N^+_{D'}(v)| \geq 2$. Let $N^+_{D'}(u) \cap N^+_{D'}(v) = \{w_1, \ldots, w_p\}$ for some integer $p \geq 2$. For each $i \in \{1, \ldots, p\}$, $N^-_{D'}(w_i)$ forms a clique in G'. Thus the edges of the subgraph of G induced by $N^-_{D'}(w_i)$ are covered by exactly two cliques $N^-_{D'}(w_i) \setminus \{u\}$ and $N^-_{D'}(w_i) \setminus \{v\}$. For simplicity, we denote $N^-_{D'}(w_i) \setminus \{v\}$ by Y_i^u and $N^-_{D'}(w_i) \setminus \{u\}$ by Y_i^v. Note that $Y_i^u \setminus \{u\} = Y_i^v \setminus \{v\}$ and that $N^-_{D'}(w_i) = Y_i^v \cup \{u\}$.

Furthermore, as $N^-_{D'}(w_i)$ forms a clique containing u and v in G',

$$\bigcup_{i=1}^{p} Y_i^v \cup \{u\} = \bigcup_{i=1}^{p} N^-_{D'}(w_i) \subset N_G(u) \cap N_G(v) \cup \{u, v\}.$$

By Lemma 3.8, $N_G(u) \cap N_G(v) \subset X \cup V(K_2^m)$ where X is the clique each vertex of which is adjacent to every vertex of K_2^m in G. Thus

$$Y_i^v = N_G(v) \cap \bigcup_{i=1}^{p} N_{D'}^-(w_i) \subset N_G(v) \cap [X \cup V(K_2^m)].$$

The vertices in $N_G(v) \cap [X \cup V(K_2^m)]$ are covered by exactly two cliques. We denote those cliques by Z_1 and Z_2.

We define a digraph D as follows:

$$V(D) = V(D') \cup \{a, b\};$$

$$\begin{aligned} A(D) =& A(D') \setminus \bigcup_{i=1}^{p} N_{D'}^-(w_i) \cup \bigcup_{i=1}^{p} \{(x, w_i) \mid x \in Y_i^u\} \\ & \cup \{(x, a) \mid x \in Z_1\} \cup \{(x, b) \mid x \in Z_2\} \cup \{(v, a), (v, b)\}. \end{aligned}$$

Since $Y_i^u \subset N_{D'}^-(w_i)$ for each $i \in \{1, \ldots, p\}$, the acyclicity of D is guaranteed by that of D'.

It is easy to see that $E(C(D)) \subset E(G)$. To show that $E(C(D)) \supset E(G)$, take an edge $f = yz$ in G. If $\{y, z\} \not\subset N_{D'}^-(w_i)$ for any $i \in \{1, \ldots, p\}$, then clearly $f \in E(C(D))$. Now suppose that $\{y, z\} \subset N_{D'}^-(w_i)$ for some $i \in \{1, \ldots, p\}$. If $y \neq v$ and $z \neq v$, then $\{y, z\} \subset Y_i^u$ and so $(y, w_i) \in A(D)$ and $(z, w_i) \in A(D)$. Thus $f \in E(C(D))$. If $y = v$ or $z = v$, then we may assume that $y = v$ without loss of generality. Then $z \neq u$. Then $z \in Z_1$ or $z \in Z_2$. That is, $(z, a) \in A(D)$ or $(z, b) \in A(D)$. Since $(v, a) \in A(D)$ and $(v, b) \in A(D)$, it holds that $f \in E(C(D))$. Thus $C(D) = G \cup I_{h+1}$ and so $k(G) \leq h + 1$. □

The upper bound given in Theorem 4.3 is sharp as the graph given in Figure 1 has h holes and competition number $h + 1$.

5 Closing Remarks

In this paper, we have shown that the competition number of a hole-edge-disjoint graph with exactly h holes is at most $h + 1$, which strongly implies that Kim's conjecture might be true. It would be natural to see whether the conjecture is true for a graph with two holes.

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