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By

Boram PARK, Suh-Ryung KIM, and Yoshio SANO

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

The competition numbers of complete multipartite graphs and orthogonal families of Latin squares

BORAM PARK *, SUH-RYUNG KIM * ,

Department of Mathematics Education,
Seoul National University, Seoul 151-742, Korea.

YOSHIO SANO †‡

Research Institute for Mathematical Sciences,
Kyoto University, Kyoto 606-8502, Japan.

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Abstract

The competition graph of a digraph D is a graph which has the same vertex set as D and has an edge between u and v if and only if there exists a vertex x in D such that (u, x) and (v, x) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of a graph G is defined to be the smallest number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph G and it has been one of important research problems in the study of competition graphs to characterize a graph by its competition number.

In this paper, we give new upper and lower bounds for the competition number of a complete multipartite graph K_n^m on m partite sets of the same size n by using orthogonal Latin squares. Furthermore, we give better bounds for the competition number of the complete tetrapartite graph K_p^4 for a prime number p.

Keywords: competition graph, competition number, edge clique cover number, complete multipartite graph, orthogonal Latin squares

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‡Corresponding author: sano@kurims.kyoto-u.ac.jp ; y.sano.math@gmail.com

1 Introduction

The notion of a competition graph was introduced by Cohen [1] as a means of determining the smallest dimension of ecological phase space (see also [2]). The *competition graph* $C(D)$ of a digraph D is a graph which has the same vertex set as D and an edge between vertices u and v if and only if there is a vertex x in D such that (u, x) and (v, x) are arcs of D. Roberts [9] observed that if G is any graph, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* $k(G)$ of a graph G to be the smallest number k such that G together with k isolated vertices added is the competition graph of an acyclic digraph.

For a digraph D, an ordering $v_1, v_2, \ldots, v_n$ of the vertices of D is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i < j$. It is well-known that a digraph D is acyclic if and only if there exists an acyclic ordering of D.

For a clique S of a graph G and an edge e of G, we say e *is covered by* S if both of the endpoints of e are contained in S. An *edge clique cover* of a graph G is a family of cliques such that each edge of G is covered by some clique in the family. The *edge clique cover number* $\theta_e(G)$ of a graph G is the minimum size of an edge clique cover of G. Dutton and Brigham [3] characterized the competition graphs of acyclic digraphs in terms of an edge clique cover as follows.

Theorem 1.1 ([3]). *A graph G is the competition graph of an acyclic digraph if and only if there exist an ordering $v_1, \ldots, v_n$ of the vertices of G and an edge clique cover $\{S_1, ..., S_n\}$ of G such that $v_i \in S_j$ implies $i < j$.*

Roberts [9] observed that the characterization of competition graphs is equivalent to the computation of competition numbers. It does not seem to be easy in general to compute $k(G)$ for all graphs G, as Opsut [7] showed that the computation of the competition number of a graph is an NP-hard problem (see [4], [5] for graphs whose competition numbers are known). It has been one of important research problems in the study of competition graphs to characterize a graph by its competition number.

For some special graph families, we have explicit formulae for computing competition numbers. For example, if G is a choral graph without isolated vertices then $k(G) = 1$, and if G is a triangle-free connected graph then $k(G) = |E(G)| - |V(G)| + 2$ (see [7]).

We denote by K_n^m the complete multipartite graph on m partite sets of the same size n, and denote an n-set $\{1, ..., n\}$ by $[n]$. From the above formulae, it follows that for a complete graph $K_1^m = K_m$ we have $k(K_1^m) = 1$, and for a complete bipartite graph K_n^2 we have $k(K_n^2) = n^2 - 2n + 2$. For a graph $K_n^1 = I_n$ without edges, we have $k(K_n^1) = 0$. However, for general m and n, it is so hard to compute $k(K_n^m)$ since K_n^m has many cycles and many triangles.

Recently, Kim and Sano [6] gave the exact competition number of a complete tripartite graph K_n^3.

Theorem 1.2 ([6], Theorem 1)**.** *For* $n \geq 2$, $k(K_n^3) = n^2 - 3n + 4$.

After then, Park *et al.* [8] gave the exact competition numbers of K_2^m and K_3^m.

Theorem 1.3 ([8])**.** *For* $m \geq 2$, $k(K_2^m) = 2$.

Theorem 1.4 ([8])**.** *For* $m \geq 3$, $k(K_3^m) = 4$.

So, now, we are interested in the case $m \geq 4$ and $n \geq 4$.

In this paper, we continue to study the competition numbers of complete multipartite graphs K_n^m on m partite sets of the same size n. We give new upper and lower bounds for $k(K_n^m)$ by using orthogonal Latin squares. Furthermore, we give a better upper bounds for the competition number of a complete tetrapartite graph K_p^4 of the same size p which is a prime number greater than 4 and also give a better lower bound for $k(K_n^4)$. This paper is organized as follows. In Section 2, we give some bounds for $k(K_n^m)$ by using orthogonal Latin squares. In Section 3, we focus on complete tetrapartite graphs K_p^4 with prime numbers p greater than 4. In Section 4, we make some remarks.

2 Bounds for the Competition Number of K_n^m

In this section, we compute the edge clique cover number of K_n^m with $3 \leq m \leq n+1$ when there exists a family $\mathcal{L}$ of mutually orthogonal Latin squares of order n such that $|\mathcal{L}| \geq m-2$ (see, for example, [10] for all undefined terms related to Latin squares). Then we give some bounds for $k(K_n^m)$ with $3 \leq m \leq n+1$ when there exists a family $\mathcal{L}$ of mutually orthogonal Latin squares of order n such that $|\mathcal{L}| \geq m-2$.

Suppose that there exists a family $\mathcal{L}$ of mutually orthogonal Latin squares of order n such that $|\mathcal{L}| \geq m-2$. We denote by v_j^l the jth vertex in the lth partite set for $l \in [m]$ and $j \in [n]$. By the hypothesis, there are $m-2$ Latin squares of order n which are orthogonal each other. Let $L_1, L_2, \ldots, L_{m-2}$ be such Latin squares, and we denote the (i,j)-element of L_l by $L_l(i,j)$. Then, we define a set S_{ij} of vertices for $i, j \in [n]$ as follows:

$$S_{ij} = \{v_i^1, v_j^2, v_{L_1(i,j)}^3, v_{L_2(i,j)}^4, \ldots, v_{L_{m-2}(i,j)}^m\}. \tag{2.1}$$

(See Figure 1 for illustration.) We denote by $\mathcal{S}$ the collection of those S_{ij}, that is,

$$\mathcal{S} := \{S_{ij} \mid i, j \in [n]\}. \tag{2.2}$$

Theorem 2.1. *Let m and n be positive integers such that $3 \leq m \leq n+1$. Suppose that there exists a family $\mathcal{L}$ of mutually orthogonal Latin squares of order n such that $|\mathcal{L}| \geq m-2$. Then the following are true:*

(1) $\mathcal{S}$ defined by (2.1) and (2.2) is an edge clique cover of K_n^m of minimum size.

$L_1 =$

1	2	3	4	5
2	3	4	**5**	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

$L_2 =$

1	2	3	4	5
3	4	5	**1**	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

$L_3 =$

1	2	3	4	5
4	5	1	**2**	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

$L_4 =$

1	2	3	4	5
5	1	2	**3**	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

Figure 1: For the orthogonal family of Latin squares $\{L_1, L_2, L_3, L_4\}$, $S_{24} = \{v_2^1, v_4^2, v_{\mathbf{5}}^3, v_{\mathbf{1}}^4, v_{\mathbf{2}}^5, v_{\mathbf{3}}^6\}$.

(2) $\theta_e(K_n^m) = n^2$.

Proof. Since any pair of two vertices in S_{ij} belongs to distinct partite sets of K_n^m, the set S_{ij} is a clique of K_n^m. Now take an edge e of K_n^m. Then $e = v_j^l v_{j'}^{l'}$ for some $l, l' \in [m]$ and $j, j' \in [n]$ with $l \neq l'$. By symmetry, we may assume that $l < l'$.

If $l = 1$ and $l' = 2$, then e is covered by $S_{jj'}$. If $l = 1$ and $l' \geq 3$, then, by the definition of a Latin square, there exists $j^* \in [n]$ such that $L_{l'-2}(j, j^*) = l$. Then e is covered by S_{jj^*}.

If $l = 2$, then, by the definition of a Latin square again, there exists $j^* \in [n]$ such that $L_{l'-2}(j^*, j) = l$. Then e is covered by S_{j^*j}.

Now suppose that $l \geq 3$. By the orthogonality of Latin squares, there exists $i^*, j^* \in [n]$ such that $L_{l-2}(i^*, j^*) = j$ and $L_{l'-2}(i^*, j^*) = j'$. Then e is covered by $S_{i^*j^*}$. Therefore $\mathcal{S} := \{S_{ij} \mid i, j \in [n]\}$ is an edge clique cover of K_n^m.

It is easy to see that $\mathcal{S}$ has size n^2. Thus $\theta_e(K_n^m) \leq n^2$.

Since any of edges joining a vertex in the 1st partite set and a vertex in the 2nd partite set belongs to distinct cliques, it follows that $\theta_e(K_n^m) \geq n^2$. Hence we have $\theta_e(K_n^m) = n^2$ and $\mathcal{S}$ is an edge clique number minimum size.

The statement (2) is an immediate consequence of (1). □

It is a well-known theorem that for any $n = p^r$, where p is a prime number and r is a positive integer, there exists a complete orthogonal family of Latin squares of order n. Since the size of a complete orthogonal family of Latin squares of order n is $n - 1$, we have a family $\mathcal{L}$ of mutually orthogonal Latin squares of order n with $|\mathcal{L}| \geq m - 2$ if $3 \leq m \leq n + 1$. Therefore the following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.2. *If n is a prime power and $3 \leq m \leq n + 1$, then $\theta_e(K_n^m) = n^2$.*

For distinct cliques S and S' of a graph G, we say S and S' are *edge-disjoint* if $|S \cap S'| \leq 1$.

Corollary 2.3. *Let m and n be positive integers such that $3 \leq m \leq n+1$. Suppose that there exists a family $\mathcal{L}$ of mutually orthogonal Latin squares of order n such that $|\mathcal{L}| \geq m-2$. Let $\mathcal{E}$ be an edge clique cover of K_n^m of minimum size. Then $\mathcal{E}$ consists of exactly n^2 cliques of size m which are edge disjoint each other.*

Proof. Let $\mathcal{E}$ be an edge clique cover of K_n^m of minimum size. By Theorem 2.1, we have $\theta_e(K_n^m) = n^2$. So we put $\mathcal{E} = \{S_1, ..., S_{n^2}\}$.

Suppose that there exist cliques $S_i, S_j \in \mathcal{E}$ such that $|S_i \cap S_j| \geq 2$ and $S_i \neq S_j$. Any maximal clique of K_n^m has size m. Now we count the number of edges which are covered by $\mathcal{E}$. The two cliques S_i and S_j cover at most $2 \cdot \binom{m}{2} - 1$ edges since $|S_i \cap S_j| \geq 2$, and $\mathcal{E} \setminus \{S_i, S_j\}$ covers at most $\binom{m}{2}(n^2-2)$ edges. Thus the family $\mathcal{E}$ covers at most $2 \cdot \binom{m}{2} - 1 + \binom{m}{2}(n^2-2) = \binom{m}{2}n^2 - 1$ edges of K_n^m. On the other hand, we know that $|E(K_n^m)| = \binom{m}{2}n^2$. This contradicts the hypothesis that $\mathcal{E}$ is an edge clique cover of K_n^m. Therefore any two distinct cliques in $\mathcal{E}$ are edge disjoint.

Now we show that $|S_i| = m$ holds for any $i = 1, ..., n^2$. Since the size of a maximal clique of K_n^m is m, we have $|S_i| \leq m$ for any i. Suppose that there exists $S_j \in \mathcal{E}$ such that $|S_j| \leq m-1$. Then the number of edges which is covered by $\mathcal{E}$ is at most $\binom{m}{2}(n^2-1) + \binom{m-1}{2} = \binom{m}{2}n^2 - (m-1)$ which is less than $|E(K_n^m)|$. But it contradicts the hypothesis that $\mathcal{E}$ is an edge clique cover of K_n^m. Therefore any cliques in $\mathcal{E}$ have the size m. □

Theorem 2.4. *Let m and n be positive integers such that $3 \leq m \leq n+1$. Suppose that there exists a family $\mathcal{L}$ of mutually orthogonal Latin squares of order n such that $|\mathcal{L}| \geq m-2$. Then*

$$k(K_n^m) \leq n^2 - n + 1.$$

Proof. Take $\mathcal{S}$ given in (2.2) which is an edge clique cover of K_n^m by Theorem 2.1. Then we define a digraph D as follows:

$$\begin{aligned} V(D) &= V(K_n^m) \cup \{z_{ij} \mid i,j \in [n], i \neq n\} \cup \{z_{nn}\}, \\ A(D) &= \bigcup_{i=1}^{n-1}\bigcup_{j=1}^{n}\{(v, z_{ij}) \mid v \in S_{ij}\} \cup \bigcup_{j=1}^{n-1}\{(v, v_j^1) \mid v \in S_{nj}\} \\ &\quad \cup \{(v, z_{nn}) \mid v \in S_{nn}\}. \end{aligned}$$

Once we note that v_n^1 is the only vertex in the 1st partite set that is contained in $\bigcup_{j=1}^n S_{nj}$, it is not difficult to see that D is acyclic. It is obvious that

$$C(D) = K_n^m \cup \{z_{ij} \mid i,j \in [n], i \neq n\} \cup \{z_{nn}\}.$$

Hence we have shown that $k(K_n^m) \leq n^2 - n + 1$. □

As we mentioned above, there exists a family $\mathcal{L}$ of mutually orthogonal Latin squares of order n with $|\mathcal{L}| \geq m-2$ if n is a prime power and $3 \leq m \leq n+1$. Therefore the following corollary is an immediate consequence of Theorem 2.4.

Corollary 2.5. *If n is a prime power and $3 \leq m \leq n+1$, then $k(K_n^m) \leq n^2 - n + 1$.*

The following theorem gives a lower bound for $k(K_n^m)$ with $3 \leq m \leq n+1$ when there exists a family $\mathcal{L}$ of orthogonal Latin squares of order n such that $|\mathcal{L}| \geq m-2$.

Theorem 2.6. *Let m and n be positive integers such that $3 \leq m \leq n+1$. Suppose that there exists a family $\mathcal{L}$ of orthogonal Latin squares of order n such that $|\mathcal{L}| \geq m-2$. Then*

$$k(K_n^m) \geq n^2 - mn + m + 1.$$

Proof. By the definition of competition number, there exists an acyclic digraph D such that $C(D) = K_n^m \cup I_k$, where $k = k(K_n^m)$. By Theorem 1.1, there exist an ordering $v_1, \ldots, v_{mn+k}$ of the vertices of $K_{m\times n} \cup I_k$ and an edge clique cover $\mathcal{F} = \{S_1, ..., S_{mn+k}\}$ of $K_n^m \cup I_k$ such that $v_i \in S_j \Rightarrow i < j$. Note that $\mathcal{F}$ is also an edge clique cover of K_n^m and that $S_1 = \emptyset$, $S_2 \subseteq \{v_1\}, \ldots, S_j \subseteq \{v_1, \ldots, v_{j-1}\}$.

Consider the first m vertices $v_1, \ldots, v_m$. Then there are two cases: (1) any pair of the vertices $v_1, ..., v_m$ belongs to different partite sets; (2) there are at most $m-1$ partite sets that contain one of v_1, ..., v_m.

Firstly we consider the case in which any pair of vertices $v_1, ..., v_m$ belongs to different partite sets. Then $S' = \{v_1, ..., v_m\}$ is a clique of K_n^m by the definition of K_n^m and S' contains each of $S_1, ..., S_{m+1}$ since $S_j \subseteq \{v_1, ..., v_{j-1}\}$ for $j = 1, ..., m+1$. Therefore $\mathcal{F}' := \mathcal{F} \cup \{S'\} \setminus \{S_1, ..., S_{m+1}\}$ is also an edge clique cover of K_n^m. Now consider S_{m+2}. We know that $S_{m+2} \subseteq \{v_1, ..., v_{m+1}\}$.

If $|S' \cap S_{m+2}| \geq 2$, then S' and S_{m+2} are not edge-disjoint and, by Corollary 2.3, $\mathcal{F}'$ is not an edge clique cover of K_n^m of minimum size. If $|S' \cap S_{m+2}| \leq 1$, then S_{m+2} contains at most one of $v_1, \ldots, v_m$ and so $|S_{m+2}| \leq 2 < m$. Thus $\mathcal{F}'$ is not an edge clique cover of K_n^m of minimum size by Corollary 2.3. Thus, in both cases, we have

$$\theta_e(K_n^m) < |\mathcal{F}'| = mn + k - (m+1) + 1.$$

By Theorem 2.4, we have $\theta_e(K_n^m) = n^2$. Hence we have $n^2 < mn + k - (m+1) + 1$, that is,

$$k(K_n^m) \geq n^2 - mn + m + 1.$$

Now consider the case where there are at most $m-1$ partite sets that contain one of v_1, ..., v_m. That is, there exists a partite set, say P, that does not contain any of $v_1, ..., v_m$. To cover all the edges which have an endpoint in P, we need at least n^2 cliques. Since

$S_j \cap P = \emptyset$ for $j = 1, ..., m+1$, there are at least $n^2 + m + 1$ distinct cliques in $\mathcal{S}$ and so $n^2 + m + 1 \leq mn + k$. Therefore we have

$$k(K_n^m) \geq n^2 - mn + m + 1.$$

Hence we can conclude that $k(K_n^m) \geq n^2 - mn + m + 1$. $\square$

Corollary 2.7. *If n is a prime power and $3 \leq m \leq n+1$, then $k(K_n^m) \geq n^2 - mn + m + 1$.*

3 The Competition Numbers of Complete Tetrapartite Graphs

In this section, for a prime number p, we give sharper bounds for the competition numbers of complete tetrapartite graphs K_p^4 than the upper bound $p^2 - p + 1$ and the lower bound $p^2 - 4p + 5$ obtained in the previous section. Let $\mathbb{F}_p$ denote the finite field with p elements.

For $p = 2$ and $p = 3$, we have $k(K_2^4) = 2$ and $k(K_3^4) = 4$ by Theorem 1.3 and Theorem 1.4. In the following, we consider the case $p \geq 5$.

Theorem 3.1. *Let K_p^4 be a complete tetrapartite graph whose partite sets have the same size p which is a prime number greater than or equal to 5. Then we have*

$$k(K_p^4) \leq p^2 - 4p + 8.$$

Proof. Let $\{a_i \mid i \in \mathbb{F}_p\}$, $\{b_i \mid i \in \mathbb{F}_p\}$, $\{c_i \mid i \in \mathbb{F}_p\}$, and $\{d_i \mid i \in \mathbb{F}_p\}$ denote the 4 partite sets of K_p^4. Note that since p is prime and $p \geq 5$, there exists a pair of orthogonal Latin squares of order p. Let

$$\mathcal{S} = \{\{a_i, b_j, c_{j-i+1}, d_{j-2i+2}\} \mid i, j \in \mathbb{F}_p\},$$

which is an edge clique cover of K_p^4 obtained from a pair of orthogonal Latin squares of order p as given in (2.2). Note that $|\mathcal{S}| = p^2$ and any two of cliques in $\mathcal{S}$ are edge-disjoint by Corollary 2.3.

Now we label all the cliques in $\mathcal{S}$ as follows. For $1 \leq i \leq 7$, we put S_i as

$$\begin{array}{lll} S_1 = \{a_1, b_1, c_1, d_1\}, & S_2 = \{a_1, b_2, c_2, d_2\}, & \\ S_3 = \{a_2, b_3, c_2, d_1\}, & S_4 = \{a_1, b_3, c_3, d_3\}, & \\ S_5 = \{a_2, b_2, c_1, d_p\}, & S_6 = \{a_2, b_4, c_3, d_2\}, & S_7 = \{a_3, b_4, c_2, d_p\}. \end{array}$$

For $8 \leq i \leq 3p - 2$, we put S_i as

$$\begin{array}{lll} S_8 = \{a_3, b_3, c_1, d_{p-1}\}, & S_9 = \{a_2, b_1, c_p, d_{p-1}\}, & S_{10} = \{a_1, b_p, c_p, d_p\}, \\ S_{11} = \{a_3, b_2, c_p, d_{p-2}\}, & S_{12} = \{a_2, b_p, c_{p-1}, d_{p-2}\}, & S_{13} = \{a_1, b_{p-1}, c_{p-1}, d_{p-1}\}, \\ S_{14} = \{a_3, b_1, c_{p-1}, d_{p-3}\}, & S_{15} = \{a_2, b_{p-1}, c_{p-2}, d_{p-3}\}, & S_{16} = \{a_1, b_{p-2}, c_{p-2}, d_{p-2}\}, \\ S_{17} = \{a_3, b_p, c_{p-2}, d_{p-4}\}, & S_{18} = \{a_2, b_{p-2}, c_{p-3}, d_{p-4}\}, & S_{19} = \{a_1, b_{p-3}, c_{p-3}, d_{p-3}\}, \\ & \vdots & \\ S_{3p-7} = \{a_3, b_8, c_6, d_4\}, & S_{3p-6} = \{a_2, b_6, c_5, d_2\}, & S_{3p-5} = \{a_1, b_5, c_5, d_5\}, \\ S_{3p-4} = \{a_3, b_7, c_5, d_3\}, & S_{3p-3} = \{a_2, b_5, c_4, d_3\}, & S_{3p-2} = \{a_1, b_4, c_4, d_4\}. \end{array}$$

More precisely, we put

$$\begin{aligned} S_{3t-1} &= \{a_3, b_{p+6-t}, c_{p+4-t}, d_{p+2-t}\}, \\ S_{3t} &= \{a_2, b_{p+4-t}, c_{p+3-t}, d_{d+2-t}\}, \\ S_{3t+1} &= \{a_1, b_{p+3-t}, c_{p+3-t}, d_{p+3-t}\} \end{aligned}$$

for $3 \leq t \leq p-1$, where all the indices are reduced to modulo p.
Furthermore, if $p \geq 7$, then we put S_i for $3p-1 \leq i \leq 4p-8$ as

$$S_i = \{a_4, b_{i+2}, c_{i-1}, d_{i-4}\}.$$

Then there are p^2-4p+8 cliques in $\mathcal{S} \setminus \{S_1, ..., S_{4p-8}\}$ and we label them as $T_1, ..., T_{p^2-4p+8}$ arbitrarily. (Note that, in the case $p=5$, $4p-8=12=3p-3$.)

Now we label the vertices of K_p^4 in the following way. We label a_1, b_1, c_1, d_1 in S_1 as v_1, v_2, v_3, v_4. Then label the vertices b_2, c_2, d_2 in $S_2 \setminus S_1$ as v_5, v_6, v_7. Inductively label the vertices of $S_i \setminus \bigcup_{t=1}^{i-1} S_t$ in alphabetical order as $v_{j+1}, \ldots, v_{j+\ell}$ where $j = \left|\bigcup_{t=1}^{i-1} S_t\right|$ and $\ell = \left|S_i \setminus \bigcup_{t=1}^{i-1} S_t\right|$. That is, we label the vertices of K_p^4

$$\begin{aligned} &a_1, b_1, c_1, d_1, b_2, c_2, d_2, a_2, b_3, c_3, d_3, d_p, b_4, a_3, \\ &d_{p-1}, c_p, b_p, d_{p-2}, c_{p-1}, b_{p-1}, \ldots, d_4, c_5, b_5, \\ &c_4, a_4, a_5, \ldots, a_{p-1}, a_p \end{aligned}$$

as $v_1, v_2, ..., v_{4p}$. Since $S_7 = \{c_2, d_p, b_4, a_3\} = \{v_6, v_{12}, v_{13}, v_{14}\}$ and $\left|S_i \setminus \bigcup_{t=1}^{i-1} S_t\right| = 1$ for $8 \leq i \leq 4p-8$, it holds that

$$S_i \subseteq \{v_1, v_2, \ldots, v_{i+7}\} \tag{3.1}$$

for $1 \leq i \leq 4p-8$. We define a digraph D as follows.

$$\begin{aligned} V(D) &= V(K_p^4) \cup \{z_1, z_2, \ldots, z_{p^2-4p+8}\}, \\ A(D) &= \bigcup_{i=1}^{4p-8} \{(x, v_{i+8}) \mid x \in S_i\} \cup \bigcup_{i=1}^{p^2-4p+8} \{(x, z_i) \mid x \in T_i\}. \end{aligned}$$

Then D is acyclic by (3.1). The following statements are equivalent:

- $uv \in E(C(D))$;
- There exits $w \in V(D)$ such that $(u, w) \in A(D)$ and $(v, w) \in A(D)$;
- There exists $w \in V(D)$ such that $\{u, v\} \subset S_i$ and $w = v_{i+8}$ for some $i \in \{1, \ldots, 4p-8\}$ or that $\{u, v\} \subset T_j$ and $w = z_j$ for some $j \in \{1, \ldots, p^2-4p+8\}$;

- $\{u,v\} \subset S_i$ for some $i \in \{1,\ldots,4p-8\}$, or $\{u,v\} \subset T_j$ for some $j \in \{1,\ldots,p^2-4p+8\}$;
- $uv \in E(K_p^4)$.

Thus $E(C(D)) = E(K_p^4)$ and so $C(D) = K_p^4 \cup \{z_1, z_2, \ldots, z_{p^2-4p+8}\}$. Hence we have $k(K_p^4) \leq p^2 - 4p + 8$. □

Next we give a lower bound for the competition numbers of complete tetrapartite graph. The following theorem does not require that the size of the partite sets of a complete tetrapartite graph is prime.

Theorem 3.2. *Let K_n^4 be a complete tetrapartite graph whose partite sets have the same size n with $n \geq 4$. Then we have*

$$k(K_n^4) \geq n^2 - 4n + 6.$$

Proof. We put $k = k(K_n^4)$ for convenience. Let D be an acyclic digraph such that $C(D) = K_n^4 \cup I_k$ and let $v_1, v_2, \ldots, v_{4n+k}$ be an acyclic ordering of the vertices of D. Let $\mathcal{F} = \{N_D^-(v) \mid v \in V(D)\}$ where $N_D^-(v)$ denotes the set $\{w \in V(D) \mid (w,v) \in A(D)\}$ of in-neighbors of a vertex v in a digraph D. By the definition, $N_D^-(v)$ forms a clique in $C(D)$ and so $\mathcal{F}$ is an edge clique cover of K_n^4. Then since $v_1, \ldots, v_{4n+k}$ is an acyclic ordering of the vertices of D, we have

$$N_D^-(v_i) \subseteq \{v_1, ..., v_{i-1}\}.$$

Let E_i be the set of edges of K_n^4 covered by the cliques $N_D^-(v_i)$. We define e_1 as the number of edges in E_1 and e_i $(i \geq 2)$ as the number of edges in $E_i \setminus \cup_{j=1}^{i-1} E_j$. Since $\mathcal{F}$ is an edge clique cover of K_n^4,

$$\sum_{i=1}^{4n+k} e_i = \left| \bigcup_{i=1}^{4n+k} E_i \right| = |E(K_n^4)| = 6n^2.$$

Let $U_7 = \{v_1, v_2, \ldots, v_7\}$ and $n_l = |P_l \cap U_7|$ for $l \in \{1,2,3,4\}$, where P_1, P_2, P_3, P_4 denote the 4 partite sets of K_n^4. Without loss of generality, we may assume that $n_1 \geq n_2 \geq n_3 \geq n_4$. Since $n_1 + n_2 + n_3 + n_4 = 7$, we have $n_4 = 0$ or $n_4 = 1$.

Suppose that $n_4 = 0$. Then we need n^2 cliques to cover all the edges incident to some vertex in P_4. Since $P_4 \cap N_D^-(v_i) = \emptyset$ for each $i \in \{1,\ldots,8\}$, it follows that $n^2 + 8 \leq |\mathcal{F}| = 4n + k$, which implies $k \geq n^2 - 4n + 8 > n^2 - 4n + 6$.

Now we suppose that $n_4 = 1$. Then there are three possibilities for (n_1, n_2, n_3, n_4), that is, $(4,1,1,1)$, $(3,2,1,1)$, and $(2,2,2,1)$. We show that $\sum_{i=1}^{8} e_i \leq 17$ in each case. Since $N_D^-(v_i) \subseteq U_7$ for any $i \in \{1,\ldots,8\}$, it follows that $E_1 \cup ... \cup E_8 \subseteq E(K_n^4[U_7])$ and thus $\sum_{i=1}^{8} e_i \leq |E(K_n^4[U_7])|$, where $K_n^4[U_7]$ denotes the subgraph of K_n^4 induced by U_7.

If $(n_1, n_2, n_3, n_4) = (4, 1, 1, 1)$, then $|E(K_n^4[U_7])| = 15$ and so $\sum_{i=1}^{8} e_i \leq 15$.
If $(n_1, n_2, n_3, n_4) = (3, 2, 1, 1)$, then $|E(K_n^4[U_7])| = 17$ and so $\sum_{i=1}^{8} e_i \leq 17$.
Suppose that $(n_1, n_2, n_3, n_4) = (2, 2, 2, 1)$. Then $|E(K_n^4[U_7])| = 18$. Since

$$k(K_n^4[U_7]) \geq \min\{\theta_v(N_{K_n^4[U_7]}(v)) \mid v \in U_7\} = 2$$

(see [7], Proposition 7, for this inequality), the set $\{N_D^-(v_i) \mid i \in \{1, \ldots, 8\}\}$ cannot cover all the edges in $K_n^4[U_7]$. Otherwise, we have $k(K_n^4[U_7]) \leq 1$, which is a contradiction. Therefore $\sum_{i=1}^{8} e_i \leq 18 - 1 = 17$.

Since the size of maximal cliques in K_n^4 is 4, we have $e_i \leq |E_i| \leq \binom{4}{2} = 6$ for each i. Therefore it holds that

$$6n^2 = \sum_{i=1}^{4n+k} e_i = \sum_{i=1}^{8} e_i + \sum_{i=9}^{4n+k} e_i \leq 17 + 6(4n + k - 8),$$

which implies $n^2 - 4n + 6 - \frac{5}{6} \leq k$. Since k is an integer, we have $n^2 - 4n + 6 \leq k$. □

Corollary 3.3. *If p is a prime number greater than or equal to 5, then*

$$p^2 - 4p + 6 \leq k(K_p^4) \leq p^2 - 4p + 8.$$

4 Concluding Remarks

In this paper, we gave upper and lower bounds for the competition number of a complete multipartite graph K_n^m with a prime power n and $3 \leq m \leq n + 1$. Furthermore we gave better bounds for the competition number of a complete tetrapartite graph.

We conclude this paper with leaving the following questions for further study.

- Give the exact value of the competition number of a complete tetrapartite graph K_p^4 with a prime number $p \geq 5$.
- Give the exact values or better bounds for the competition numbers of complete multipartite graphs K_n^m.

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