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INVARIANTS OF 3-MANIFOLDS DERIVED FROM COVERING PRESENTATIONS

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Abstract. By a covering presentation of a 3-manifold, we mean a labeled link (i.e., a link with a monodromy representation), which presents the 3-manifold as the simple 4-fold covering space of the 3-sphere branched along the link with the given monodromy. It is known that two labeled links present a homeomorphic 3-manifold if and only if they are related by a finite sequence of some local moves. This paper presents a recipe for constructing topological invariants of 3-manifolds based on their covering presentations. The proof of the topological invariance is shown by verifying the invariance under the local moves. As an example of such invariants, we present the Dijkgraaf-Witten invariant of 3-manifolds.

1. Introduction

Since late 1980's, many new invariants of 3-manifolds have been discovered, called *quantum invariants.* They were originally proposed by Witten [18] as a partition function given by a path integral, based on the quantum field theory whose Lagrangian is the Chern-Simons functional from the view point of mathematical physics. By the operator formalism of the path integral, quantum invariants can be formulated by a TQFT, and it enables us to compute the invariants by cut-and-paste method of 3-manifolds. Reshetikhin-Turaev [14] first gave a rigorous mathematical construction of quantum invariants based on the surgery presentation of 3-manifolds, where the surgery presentation is a presentation of a 3-manifold by a framed link such that the 3-manifold is obtained from the 3-sphere by surgery along the framed link. Since it is known that two framed links give homeomorphic 3-manifolds if and only if they are related by Kirby moves, we can show the topological invariance of an invariant given by framed links by verifying that it is invariant under Kirby moves. Further, Kohno [8] formulated quantum invariants based on Heegaard splitting of 3-manifolds. It is a presentation of a 3-manifold as a union of two handlebodies, and it is known that two Heegaard splittings give homeomorphic 3-manifolds if and only if they are related by moves of the Reidemeister-Singer theorem. Kohno showed the topological invariance by verifying the invariance under such moves. Furthermore, Turaev-Viro [16] formulated quantum invariants based on triangulations of 3-manifolds. It is known that two triangulations give homeomorphic 3-manifolds if and only if they are related by Pachner moves. Turaev-Viro showed the topological invariance by verifying the invariance under Pachner moves.

In this paper, we propose a new recipe for constructing topological invariants of 3-manifolds based on covering presentations of 3-manifolds (Theorem 3.8). By a covering presentation of a closed oriented 3-manifold, we mean a labeled link (i.e., a link with a monodromy representation), of the branch set in the base space of a

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simple branched covering from the 3-manifold to the 3-sphere S^3, with the given monodromy. It is known [2] that two labeled links give homeomorphic 3-manifolds if and only if they are related by the moves MI and MII shown in Figure 2. We show the topological invariance of our recipe by verifying the invariance under the moves MI and MII. Compared with the above mentioned presentations of 3-manifolds, covering presentation dose not arise from cut-and-paste method, and we expect that we obtain other kind of topological invariants of 3-manifolds than quantum invariants by our recipe. In our recipe we introduce coloring on diagrams of labeled links in finite groups. It is an assignment of an element in a finite group to each arc of a diagram of a labeled link subject to compatibility. We will see that the number of colorings on a diagram of a labeled link is an invariant of the 3-manifold, and then this invariant is completely determined by the number of representations of the fundamental group in the group. Making our recipe for a finer invariants than the number of representations, we propose establishing state sum invariants in Theorem 3.8, defined by using diagrams of labeled links and colorings on it. In the theorem, an invariant of 3-manifolds will be obtained for a pair of maps satisfying some conditions explicitly formulated, and this is the procedure of our recipe.

The Dijkgraaf-Witten invariant [4] of a closed oriented 3-manifold is a state sum invariant defined with its triangulation, and depends only on a group and its 3-cocycle. Following our recipe, we reconstruct the Dijkgraaf-Witten invariant as an example of invariants derived from covering presentations. It gives another proof of topological invariance of the Dijkgraaf-Witten invariant, and makes the computation that was based on triangulations easier, since it is now possible to use planar diagrams of the branch links. Moreover, we give an algorithm to obtain an useful triangulation of a 3-manifold from its covering presentation, which provides a correspondence between the set of tetrahedra of the triangulation and the set of crossings of the covering presentation.

It is a problem to construct a new topological invariant of 3-manifolds, by finding a pair of maps satisfying the conditions in Theorem 3.8. This problem is still unsolved, except for the Dijkgraaf-Witten invariant at present.

This paper is organized as follows. The next section is for an introduction of covering presentations of 3-manifolds. Using covering presentations, we propose a recipe for invariants in Section 3. Then we review the definition of the Dijkgraaf-Witten invariant in Section 4.1, and give an example of invariants constructed along the recipe in Section 4.2, which will be identified with the Dijkgraaf-Witten invariant in Section 6. Making use of this example, we give some calculations of the Dijkgraaf-Witten invariant in Section 5. We then provide an algorithm to obtain a triangulation of a 3-manifold from its covering presentation in Section 6, which leads to the identification of the example in Section 4 with the Dijkgraaf-Witten invariant.

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2. Covering presentations

In this section, we review covering presentations of 3-manifolds.

We consider a d-fold branched covering $p : M \to S^3$ from a closed connected oriented 3-manifold M to S^3 branched along a link L. Associated to such a branched covering, we have the monodromy representation $\pi_1(S^3 - L) \to \mathfrak{S}_d$ into the symmetric group of degree d (see, for example, [13]).

A *labeled link of degree* d is a link L with a representation $\pi_1(S^3 - L) \to \mathfrak{S}_d$. When we present a labeled link by a link diagram, we present such a representation by showing the image of a meridian of each over-arc of the diagram.

A branched covering $p : M \to S^3$ is said to be *simple* if its monodromy representation $\pi_1(S^3 - L) \to \mathfrak{S}_d$ maps each meridian to a transposition in $\mathfrak{S}_d$. It is known [7, 10] that any closed, connected and oriented 3-manifold can be represented by a simple branched covering of S^3 of degree 3 branched along a link. This implies, by stabilizing such a covering, that the same is true for degree ≥ 3. Here, a *stabilization* is a move adding an unknotted component with a label $(i, d+1) \in \mathfrak{S}_{d+1}$; see Figure 1. This move change the degree by 1, but does not change the topological type of the covering space since the move makes the connected sum of the original covering space and the 3-sphere which is the double cover of S^3 branched along an unknot.

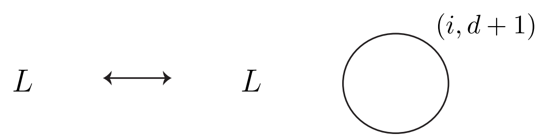


Figure 1. The stabilization of a labeled link L of degree d

It is known [11] that the moves MI and MII on labeled links, shown in Figure 2, does not change the topological type of the covering space. Here, the pictures of the two sides of each move mean two labeled links which are identical except for a 3-ball, where they differ as shown in the pictures. Since the 3-fold covering space of a 3-ball branched along tangles of the two sides of the move MI are homeomorphic to a 3-ball, the move MI does not change the topological type of the covering space. Further, when we relate the two sides of the move MII by a homotopy, it does not change the topological type of the covering space, since the branch loci in the covering space do not intersect by such a homotopy, and hence the move MII does not change the topological type of the covering space. Bobtcheva and Piergallini [2] showed that these moves are sufficient to relate any two homeomorphic branched covering spaces.

Theorem 2.1 (special case of [2, Theorem 3])**.** *Two* 4*-fold connected simple branched coverings of* S^3 *branched over links represent homeomorphic oriented 3-manifolds if and only if their labeled links can be related by a finite sequence of moves MI, MII and isotopy of* S^3.

In this paper we deal with labeled links of degree 4 particularly, based on the identity,

$$\{\text{closed oriented 3-manifolds}\}/\text{orientation preserving homeomorphism}$$
$$= \{\text{labeled links of degree 4}\}/\text{MI, MII and isotopy of } S^3.$$

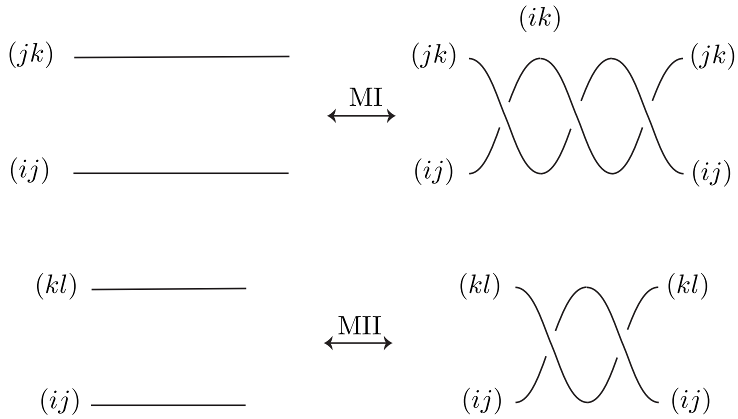


FIGURE 2. The moves MI and MII

We call a labeled link of degree 4 a *covering presentation* of the 3-manifold. The mirror image of a covering presentation gives the same 3-manifold with the opposite orientation. Some examples of covering presentations are shown in Figures 20, 21, and 22. We remark that the conjugate labelings give a homeomorphic 3-manifold, and such a labeling change can be realized by isotopy; see Figure 3.

Actually, we can define covering presentations of degree d for any $d \geq 3$. However, when $d = 3$, we need a global move to relate labeled links of degree 3 [12], and our recipe of constructing invariants would become complicated in this case. When $d \geq 5$, the argument is essentially the same with the one of degree 4. These are the reason why we choose the degree 4 to explain our recipe.

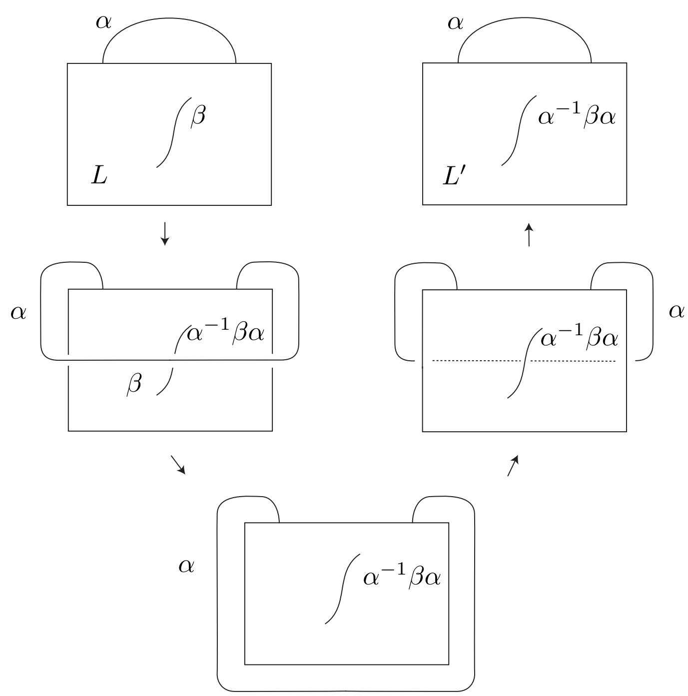


FIGURE 3. Conjugate labelings are related by isotopy

3. Recipe

In this section first we review a method to give a presentation of the fundamental group of a 3-manifold from its covering presentation in Section 3.1. Then we define coloring on diagrams of labeled links in finite groups in Section 3.2. We will see that the number of colorings on a diagram of a labeled link gives an invariant of the 3-manifold which the labeled link presents in Proposition 3.4. Furthermore, we give a correspondence between the set of colorings on a diagram of a labeled link and the set of representations of the fundamental group of the 3-manifold. To show the correspondence we use the presentation of the fundamental group of a 3-manifold given in Section 3.1. In Section 3.3, another coloring will be defined on the complementary regions of colored diagrams of labeled links. By using these two colorings, a recipe for constructing invariants of 3-manifolds will be stated in Theorem 3.8.

3.1. **A presentation of the fundamental groups of 3-manifolds.** In this subsection we review a method to give a presentation of the fundamental group of a 3-manifold from its covering presentation, which will be used in the proof of Proposition 3.5 in the next subsection.

Let $p : M \to S^3$ be a 4-fold simple branched covering from a closed oriented 3-manifold M, and L a labeled link in S^3 of degree 4 which is a covering presentation of M. We denote a planer diagram of L by D. By replacing each crossing of D with a 4-valent vertex, we obtain a graph Γ in $\mathbb{R}^2 \subset S^3$ as shown in Figure 4.

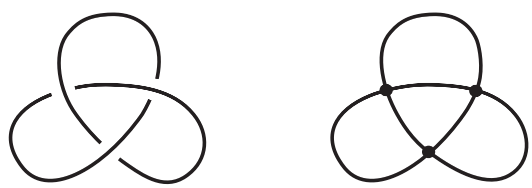


Figure 4. A diagram D of a labeled link (left) and its graph Γ (right).

We have a presentation of $\pi_1(S^3 - \Gamma)$ by simply observing this graph on a plane. Suppose that the graph Γ has e edges and v vertices. Let x_m $(m \in \{1, 2, \cdots e\})$ be the homotopy class of a loop based at a base point above the plane and running around an edge of Γ once, as depicted in Figure 5. Let r_n $(n \in \{1, 2, \cdots v\})$ stand for the loop generated by the loops $\{x_m\}$ and bounding the disk around an vertex, as indicated in the left hand side of Figure 6. Then the presentation is as follows;

$$\pi_1(S^3 - \Gamma) = \langle\ x_m \mid r_n\ \rangle_{m\in E,\ n\in V}, \tag{1}$$

where $E = \{1, 2, \cdots, e\}$ and $V = \{1, 2, \cdots, v\}$. On the other hand, the complement space $S^3 - \Gamma$ is homotopy equivalent to a bouquet of some circles. Therefore $\pi_1(S^3 - \Gamma)$ is a free group, and we have another presentation such as

$$\pi_1(S^3 - \Gamma) = \langle\ x_m \mid\ \rangle_{m\in E'}. \tag{2}$$

Here the subscript set E' can be regarded as a subset of E.

Let us consider a presentation of $\pi_1(S^3 - L)$ by using the presentation (1) of $\pi_1(S^3 - \Gamma)$. Let $N(\Gamma)$ be the neighborhood of Γ in S^3 as shown in the left hand

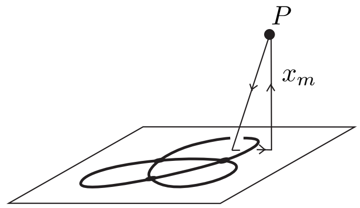
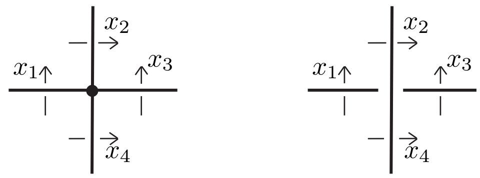


FIGURE 5. A generator x_m of $\pi_1(S^3 - \Gamma)$



$r_n = x_1 x_2 x_3^{-1} x_4^{-1}$ $\quad\quad$ $s_n = x_2 x_4^{-1}$

FIGURE 6. The relator r_n in $\pi_1(S^3 - \Gamma)$ at a vertex (left), and the relator s_n in $\pi_1(S^3 - L)$ at the corresponding crossing of D

side of Figure 7. The complement $S^3 - L$ is homotopy equivalent to the union of the closure of the complement $\mathrm{cl}(S^3 - N(\Gamma))$ and the disks bounded in $N(\Gamma)$ at the vertices of Γ as shown in the right hand side of Figure 7. Such a disk, which will be denoted by s_n $(n \in V)$, determines which is the over- or under-arc around the corresponding crossing of the diagram D. If the arcs around a crossing are as shown in the right hand side of Figure 6, then the disk s_n will be expressed by the generators $\{x_m\}$ as follows,

$$s_n = x_2 {x_4}^{-1}.$$

By the Van Kampen Theorem, we have the equation that

$$\pi_1(S^3 - L) = \pi_1(S^3 - \Gamma)/\{s_n = 1\}_{n \in V}.$$

By using the presentation (1) of $\pi_1(S^3 - \Gamma)$, we have a presentation of $\pi_1(S^3 - L)$,

$$\pi_1(S^3 - L) = \langle\ x_m \mid r_n,\ s_n\ \rangle_{m \in E,\ n \in V}. \tag{3}$$

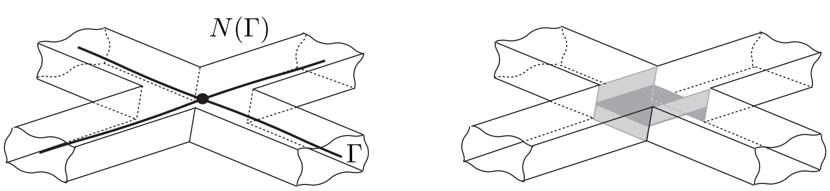


FIGURE 7. A neighborhood $N(\Gamma)$ of the graph Γ (left) and a disk at a vertex (right)

Let $\widetilde{L}$ be the link $p^{-1}(L)$ in the covering space M. To obtain a presentation of $\pi_1(M - \widetilde{L})$, we consider about the covering space of $S^3 - \Gamma$. Let $\phi : \pi_1(S^3 - L) \to \mathfrak{S}_4$ be the monodromy representation of the branched covering p. We define the "unbranched" 4-fold covering $p' : \widetilde{S^3 - \Gamma} \to S^3 - \Gamma$ by its monodromy representation $\phi' : \pi_1(S^3 - \Gamma) \to \mathfrak{S}_4$ such that each generator x_m $(m \in E)$ is mapped to $\phi(x_m) \in \mathfrak{S}_4$ for the corresponding meridian x_m of $L \subset S^3$, denoted by the same symbol.

Let us give a presentation of $\pi_1(\widetilde{S^3-\Gamma})$ by using the Reidemeister-Schreier method. For a detail about the Reidemeister-Schreier method, refer to [6], for example. Let P be the base point in $S^3-\Gamma$, and give an order to its preimages by p'^{-1} as P_1, P_2, P_3 and P_4. Each generator x_m $(m \in E)$ of $\pi_1(S^3-\Gamma)$ lifts to four paths by p'^{-1}. If its monodromy is the transposition (12), for example, then the paths run from P_1 to P_2, from P_2 to P_1, from P_3 to P_3 and from P_4 to P_4. We name these oriented paths $\widetilde{x}_{m1}$, $\widetilde{x}_{m2}$, $\widetilde{x}_{m3}$ and $\widetilde{x}_{m4}$ respectively by putting subscripts of their initial points. In the same manner, $\widetilde{x}_{md}$ $(m \in E,\ d \in \{1,2,3,4\})$ will denote the lifted path of the generator x_m running from P_d to $P_{\phi'(x_m)(d)}$. Furthermore, we fix three paths $w_2, w_3, w_4 \in \langle \widetilde{x}_{md} | \ \rangle_{m\in E, d\in\{1,2,3,4\}}$ running from P_1 to P_2, from P_1 to P_3 and from P_1 to P_4 respectively, and take the set $\{w_2, w_3, w_4\}$ as a Schreier tree. Choose P_1 for the base point of $\widetilde{S^3-\Gamma}$, and put

$$\widehat{x}_{md} = w_d\ \widetilde{x}_{md}\ {w_{\phi'(x_m)(d)}}^{-1} \qquad \text{for } m \in E,\ d \in \{1,2,3,4\},$$

where $w_1 = 1$, to make each lifted path a loop based at P_1, as shown in Figure 8.

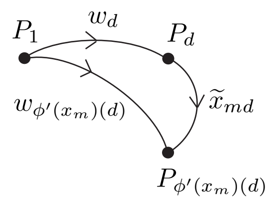


FIGURE 8. A generator $\widehat{x}_{md} = w_d\ \widetilde{x}_{md}\ {w_{\phi'(x_m)(d)}}^{-1}$ of $\pi_1(\widetilde{S^3-\Gamma})$.

Lemma 3.1. $\pi_1(\widetilde{S^3-\Gamma}) = \langle\ \widehat{x}_{md}\ |\ \rangle_{m\in E',\ d\in\{1,2,3,4\}}$.

Proof. It follows from Reidemeister-Schreier method with respect to the presentation (2) of $\pi_1(S^3-\Gamma)$. □

Let us consider the relation between the fundamental groups $\pi_1(M-\widetilde{L})$ and $\pi_1(\widetilde{S^3-\Gamma})$. By the argument of the base spaces above, the complement S^3-L would be regarded to differ (up to homotopy equivalence) from $S^3-\Gamma$ only on the disks attached at crossings of the diagram D of L. Such a disk was expressed by the relator s_n $(n \in V)$, and it lifts to four disks by $p_{|M-\widetilde{L}}$, which will be denoted by $\widetilde{s}_{nd}$ $(n \in V, d \in \{1,2,3,4\})$. Therefore $M-\widetilde{L}$ is homotopy equivalent to the union of $\widetilde{S^3-\Gamma}$ and the lifted disks $\{\widetilde{s}_{nd}\}_{n\in V,\ d\in\{1,2,3,4\}}$. By the Van Kampen Theorem, we have the equation that

$$\pi_1(M-\widetilde{L}) = \pi_1(\widetilde{S^3-\Gamma})/\{\widetilde{s}_{nd} = 1\}_{n\in V,\ d\in\{1,2,3,4\}}.$$

Now we give a presentation of $\pi_1(M-\widetilde{L})$ by using the presentation (3) of $\pi_1(S^3-L)$. Let us observe the lifts of disks expressed by the relators r_n and s_n in detail. Such disks are bounded around crossings of the diagram D. On the other hand, crossings of labeled diagrams are classified into 3 types (named type 1, type 2 and type 3) according to the labels on the arcs around, as shown in Figure 9. If a crossing is of type 1, then the disks r_n and s_n lifts to four disks $\{\widetilde{r}_{nd}\}_{d\in\{1,2,3,4\}}$ and $\{\widetilde{s}_{nd}\}_{d\in\{1,2,3,4\}}$ respectively, as indicated in Figure 10 in the case of $i = 1$ and $j = 2$, for example. If a crossing is of type 2, such lifts are as shown in Figure 11 in the case of $i = 1$, $j = 2$ and $k = 3$. At last, if a crossing is of type 3, such lifts are as shown in Figure 12 in the case of $i = 1$, $j = 2$, $k = 3$ and $l = 4$. For $\widetilde{r}_{nd}$ and $\widetilde{s}_{nd}$, replace each consisting element $\widetilde{x}_{md}$ with $\widehat{x}_{md}$ in order to express them by the generators of $\pi_1(\widetilde{S^3 - \Gamma})$, and name them $\widehat{r}_{nd}$ and $\widehat{s}_{nd}$, respectively. Thus we have a presentation of $\pi_1(M - \widetilde{L})$ as follows;

$$\pi_1(M - \widetilde{L}) = \langle\ \widehat{x}_{md} \mid \widehat{r}_{nd},\ \widehat{s}_{nd}\ \rangle_{m \in E,\ n \in V,\ d \in \{1,2,3,4\}}. \tag{4}$$

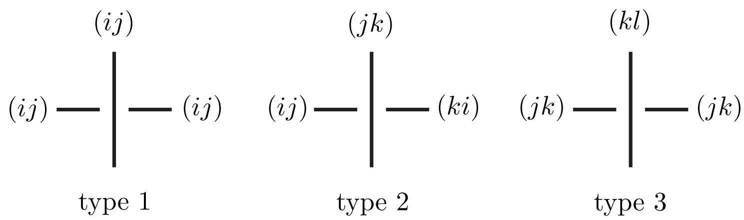


FIGURE 9. Three types of crossings of labeled diagrams. At each crossing, i, j, k and l are all distinct

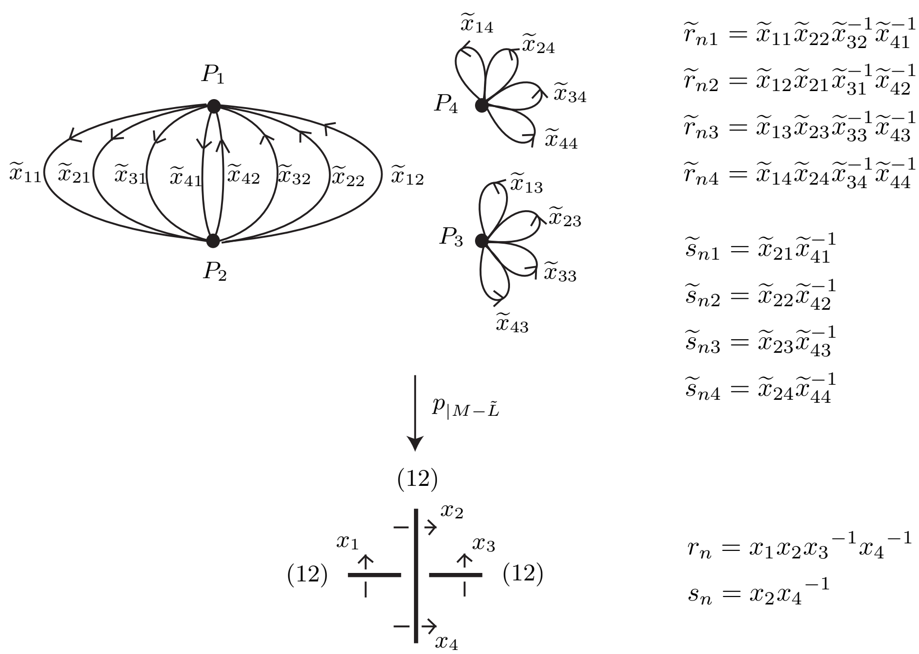
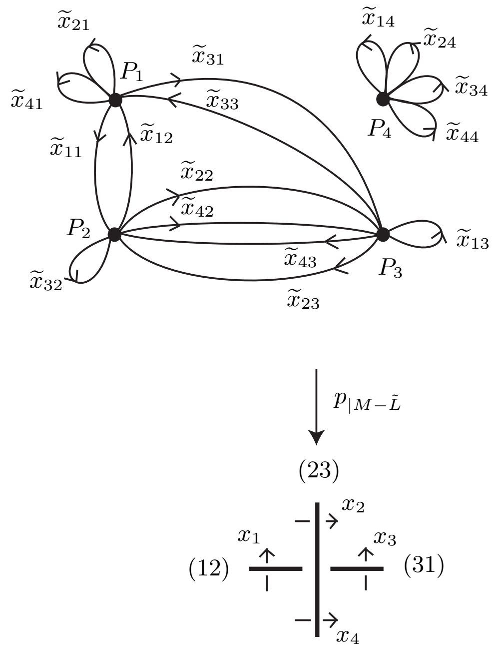


FIGURE 10. The lifts $\{\widetilde{r}_{nd}\}_{d \in \{1,2,3,4\}}$ and $\{\widetilde{s}_{nd}\}_{d \in \{1,2,3,4\}}$ of the relators r_n and s_n at a crossing of type 1

Let us give a presentation of $\pi_1(M)$ at last, from the presentation (4) of $\pi_1(M - \widetilde{L})$. To do this, we consider the meridian disks of $\widetilde{L}$. A meridian disk of L in S^3 whose bounding loop is homotopic to x_m lifts to three meridian disks of $\widetilde{L}$ in M, and we denote such lifted disks $\{\widetilde{t}_{mc}\}_{c \in \{1,2,3\}}$. If the monodromy $\phi(x_m) = (ij)$, then the disks are represented such as $\widetilde{t}_{m1} = \widetilde{x}_{mi}\widetilde{x}_{mj}$, $\widetilde{t}_{m2} = \widetilde{x}_{mk}$ and $\widetilde{t}_{m3} = \widetilde{x}_{ml}$, where $k, l \notin \{i, j\} \subset \{1, 2, 3, 4\}$. Again by the Van Kampen Theorem, we have the identity that

$$\pi_1(M) = \pi_1(M - \widetilde{L}) / \{\widetilde{t}_{mc} = 1\}_{m \in E, c \in \{1,2,3\}}.$$

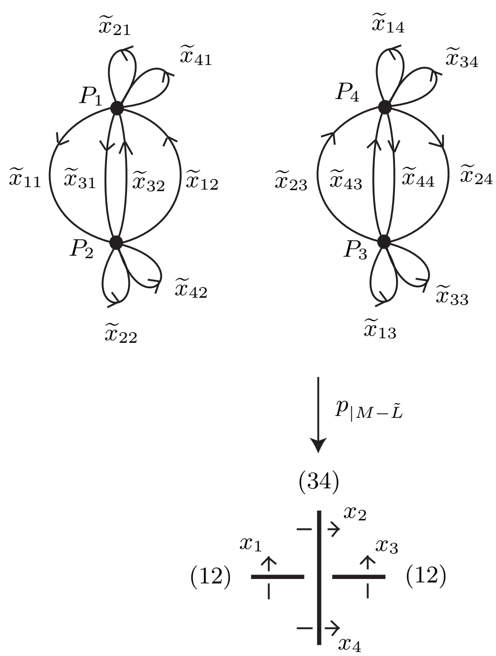


$$\widetilde{r}_{n1} = \widetilde{x}_{11}\widetilde{x}_{22}\widetilde{x}_{31}^{-1}\widetilde{x}_{41}^{-1}$$
$$\widetilde{r}_{n2} = \widetilde{x}_{12}\widetilde{x}_{21}\widetilde{x}_{33}^{-1}\widetilde{x}_{42}^{-1}$$
$$\widetilde{r}_{n3} = \widetilde{x}_{13}\widetilde{x}_{23}\widetilde{x}_{32}^{-1}\widetilde{x}_{43}^{-1}$$
$$\widetilde{r}_{n4} = \widetilde{x}_{14}\widetilde{x}_{24}\widetilde{x}_{34}^{-1}\widetilde{x}_{44}^{-1}$$

$$\widetilde{s}_{n1} = \widetilde{x}_{21}\widetilde{x}_{41}^{-1}$$
$$\widetilde{s}_{n2} = \widetilde{x}_{22}\widetilde{x}_{42}^{-1}$$
$$\widetilde{s}_{n3} = \widetilde{x}_{23}\widetilde{x}_{43}^{-1}$$
$$\widetilde{s}_{n4} = \widetilde{x}_{24}\widetilde{x}_{44}^{-1}$$

$$r_n = x_1 x_2 {x_3}^{-1} {x_4}^{-1}$$
$$s_n = x_2 {x_4}^{-1}$$

FIGURE 11. The lifts $\{\widetilde{r}_{nd}\}_{d\in\{1,2,3,4\}}$ and $\{\widetilde{s}_{nd}\}_{d\in\{1,2,3,4\}}$ at a crossing of type 2



$$\widetilde{r}_{n1} = \widetilde{x}_{11}\widetilde{x}_{22}\widetilde{x}_{31}^{-1}\widetilde{x}_{41}^{-1}$$
$$\widetilde{r}_{n2} = \widetilde{x}_{12}\widetilde{x}_{21}\widetilde{x}_{32}^{-1}\widetilde{x}_{42}^{-1}$$
$$\widetilde{r}_{n3} = \widetilde{x}_{13}\widetilde{x}_{23}\widetilde{x}_{34}^{-1}\widetilde{x}_{43}^{-1}$$
$$\widetilde{r}_{n4} = \widetilde{x}_{14}\widetilde{x}_{24}\widetilde{x}_{33}^{-1}\widetilde{x}_{44}^{-1}$$

$$\widetilde{s}_{n1} = \widetilde{x}_{21}\widetilde{x}_{41}^{-1}$$
$$\widetilde{s}_{n2} = \widetilde{x}_{22}\widetilde{x}_{42}^{-1}$$
$$\widetilde{s}_{n3} = \widetilde{x}_{23}\widetilde{x}_{43}^{-1}$$
$$\widetilde{s}_{n4} = \widetilde{x}_{24}\widetilde{x}_{44}^{-1}$$

$$r_n = x_1 x_2 {x_3}^{-1} {x_4}^{-1}$$
$$s_n = x_2 {x_4}^{-1}$$

FIGURE 12. The lifts $\{\widetilde{r}_{nd}\}_{d\in\{1,2,3,4\}}$ and $\{\widetilde{s}_{nd}\}_{d\in\{1,2,3,4\}}$ at a crossing of type 3

Replace each consisting element $\widetilde{x}_{md}$ of $\widetilde{t}_{mc}$ with $\widehat{x}_{md}$, and denote it $\widehat{t}_{mc}$. By using the presentation (4) of $\pi_1(M-\widetilde{L})$, we have a presentation of $\pi_1(M)$,

$$\pi_1(M) = \langle\ \widehat{x}_{md} \mid \widehat{r}_{nd},\ \widehat{s}_{nd},\ \widehat{t}_{mc}\ \rangle_{m\in E,\ n\in V,\ d\in\{1,2,3,4\},\ c\in\{1,2,3\}}. \tag{5}$$

From the presentation (5), we give another presentaiton of $\pi_1(M)$ which is equivalent and is reduced in the number of generators by using the relators. First of all, we use the relators $\{\widehat{t}_{mc}\}_{m\in E, c\in\{1,2,3\}}$. For x_m $(m \in E)$, assume that $\phi(x_m) = (ij)$ with $i < j$. Then we have the equations that

$$\widehat{x}_{mj} = \widehat{x}_{mi}^{-1},\ \widehat{x}_{mk} = 1 \text{ and } \widehat{x}_{ml} = 1,$$

where $k,\ l \notin \{i, j\} \subset \{1, 2, 3, 4\}$. From these equations, the set of generators of $\pi_1(M)$ is reduced to the set $\{\widehat{x}_{mi}\}_{m\in E}$, where $i \in \{1, 2, 3\}$ satisfies $\phi(x_m) = (ij)$ with $i < j$. Therefore the number of generators of $\pi_1(M)$ can be reduced to the number of edges of Γ. We rename the generator $\widehat{x}_{mi}$ with the symbol $\widehat{x}_m$ for the simplicity of notations. Secondly, we use the relators $\{\widehat{s}_{nd}\}_{n\in V,\ d\in\{1,2,3,4\}}$. Let us describe $\{\widehat{s}_{nd}\}_{d\in\{1,2,3,4\}}$ corresponding to a crossing of the diagram D with the reduced generators $\{\widehat{x}_m\}_{m\in E}$. If a crossing is of type 1 as shown in Figure 10, then

$$\widehat{s}_{n1} = \widehat{x}_2\widehat{x}_4^{-1},\ \widehat{s}_{n2} = \widehat{x}_2^{-1}\widehat{x}_4,\ \widehat{s}_{n3} = 1 \text{ and } \widehat{s}_{n4} = 1.$$

If a crossing is of type 2 as shown in Figure 11, then

$$\widehat{s}_{n1} = 1,\ \widehat{s}_{n2} = \widehat{x}_2\widehat{x}_4^{-1},\ \widehat{s}_{n3} = \widehat{x}_2^{-1}\widehat{x}_4 \text{ and } \widehat{s}_{n4} = 1.$$

If a crossing is of type 3 as shown in Figure 12, then

$$\widehat{s}_{n1} = 1,\ \widehat{s}_{n2} = 1,\ \widehat{s}_{n3} = \widehat{x}_2\widehat{x}_4^{-1} \text{ and } \widehat{s}_{n4} = \widehat{x}_2^{-1}\widehat{x}_4.$$

Therefore we have the equation $\widehat{x}_2 = \widehat{x}_4$ at a crossing of D in these figures, meaning that these generators come from the same meridian of an over-arc of D in the base space. The number of generators $\widehat{x}_m$ $(m \in E)$ can be reduced to the number of over-arcs of D. Let a be the number of crossings of D, and denote such a reduced set of generators $\{\widehat{x}_m\}_{m\in A}$, where $A = \{1, 2, \cdots, a\} \subset E$. Finally, we describe the relators $\{\widehat{r}_{nd}\}_{n\in V,\ d\in\{1,2,3,4\}}$ with the reduced generators $\{\widehat{x}_m\}_{m\in A}$. For a crossing of type 1 as shown in Figure 10, the relators are rewritten such as

$$\widehat{r}_{n1} = \widehat{x}_1\widehat{x}_2^{-1}\widehat{x}_3\widehat{x}_2^{-1},\ \widehat{r}_{n2} = \widehat{x}_1^{-1}\widehat{x}_2\widehat{x}_3^{-1}\widehat{x}_2,\ \widehat{r}_{n3} = 1 \text{ and } \widehat{r}_{n4} = 1,$$

where we identified $\widehat{x}_4$ with $\widehat{x}_2$. We note that $\widehat{r}_{n2}$ is found to be equal to $\widehat{r}_{n1}$ in $\pi_1(M)$. For a crossing of type 2 as shown in Figure 11,

$$\widehat{r}_{n1} = \widehat{x}_1\widehat{x}_2\widehat{x}_3^{-1},\ \widehat{r}_{n2} = \widehat{x}_1^{-1}\widehat{x}_3\widehat{x}_2^{-1},\ \widehat{r}_{n3} = 1 \text{ and } \widehat{r}_{n4} = 1,$$

noting that $\widehat{r}_{n2}$ is equal to $\widehat{r}_{n1}$. For a crossing of type 3 as shown in Figure 12,

$$\widehat{r}_{n1} = \widehat{x}_1\widehat{x}_3^{-1},\ \widehat{r}_{n2} = \widehat{x}_1^{-1}\widehat{x}_3,\ \widehat{r}_{n3} = 1 \text{ and } \widehat{r}_{n4} = 1,$$

noting that $\widehat{r}_{n2}$ is equal to $\widehat{r}_{n1}$. Hence we found that for each $n \in V$, two of $\{\widehat{r}_{nd}\}_{d\in\{1,2,3,4\}}$ are equivalent, and the other two are equal to 1 in $\pi_1(M)$. We choose one of the former two equivalent relators, and rename it $\widehat{r}_n$. At last, we obtained the following presentation of $\pi_1(M)$.

Proposition 3.2. *Let M be a closed oriented 3-manifold, and L be a labeled link in S^3 which is a covering presentation of M. Let D be a diagram of L with a over-arcs and v crossings. Then we have the following presentation of $\pi_1(M)$,*

$$\pi_1(M) = \langle\ \widehat{x}_m \mid \widehat{r}_n\ \rangle_{m\in\{1,2,\cdots,a\},\ n\in\{1,2,\cdots,v\}}. \tag{6}$$

The symbol $\widehat{x}_m$ represent a loop in M corresponding to an over-arc of D, and the symbol $\widehat{r}_n$ represent a disk in M corresponding to a crossing of D, as expressed in the argument above.

We remark that $\pi_1(M - \widetilde{L})$ can also be presented as a subgroup of $\pi_1(S^3 - L)$. The restriction $p_| : M - \widetilde{L} \to S^3 - L$ induces an injective homomorphism p_* between their fundamental groups. For the monodromy representation $\phi : \pi_1(S^3 - L) \to \mathfrak{S}_4$, we have an isomorphism that

$$p_*(\pi_1(M - \widetilde{L}, P_1)) \cong \phi^{-1}(\{\sigma \in \mathfrak{S}_4 | \sigma(1) = 1\})).$$

3.2. **Colorings.** In this subsection we first define coloring on diagrams of labeled links in finite groups. Then we show that the number of colorings on a diagram of a labeled link in a group gives an invariant of the 3-manifold which the labeled link present in Proposition 3.4. Moreover, in Proposition 3.5, this invariant turns out to be the number of representations of the fundamental group in the finite group up to constant multiple.

For a finite group G, we define a set $\widetilde{G}$ with a binary operation $*$. Let $\langle ij \rangle$ be an ordered transposition $(ij) \in \mathfrak{S}_4$, that is, we distinguish $\langle ji \rangle$ from $\langle ij \rangle$ derived from the same transposition (ij). Put $T = \{\langle ij \rangle \mid (ij) \in \mathfrak{S}_4\}$ be the set of such ordered transpositions, consisting of 12 elements. We define $\widetilde{G} = T \times G$ as the direct product set, and we give a rule in the notations of elements in $\widetilde{G}$ such that

$$(\langle ij \rangle, g) = (\langle ji \rangle, g^{-1}).$$

Furthermore we define a binary operation $* : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ in Table 1.

(t, g)	(t', g')	$(t, g) * (t', g')$
$(\langle ij \rangle, g)$	$(\langle ij \rangle, g')$	$(\langle ij \rangle, g' g^{-1} g')$
$(\langle ij \rangle, g)$	$(\langle jk \rangle, g')$	$(\langle ik \rangle, g g')$
$(\langle ij \rangle, g)$	$(\langle kl \rangle, g')$	$(\langle ij \rangle, g)$

TABLE 1. The binary operation $*$ in $\widetilde{G}$. In each line i, j, k, l are all distinct

Definition 3.3. Let $L \subset S^3$ be a labeled link of degree 4, and $\phi : \pi_1(S^3 - L) \to \mathfrak{S}_4$ its monodromy representation. Let D be a diagram of L with a arcs $\{\bar{x}_m\}_{m \in \{1,2,\cdots,a\}}$. Here each arc $\bar{x}_m$ is in a correspondence with a Wirtinger generator x_m of $\pi_1(S^3 - L)$ in the previous subsection. A *coloring* on D in a finite group G is defined to be a map

$$C : \{\bar{x}_m\}_{m \in \{1,2,\cdots,a\}} \to \widetilde{G},$$

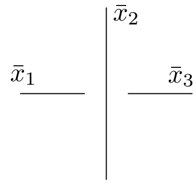
satisfying the following conditions.

(i) If $\phi(x_m) = (ij)$, then $C(\bar{x}_m) = (\langle ij \rangle, g)$.
(ii) At each crossing of D with two under-arcs $\bar{x}_1$, $\bar{x}_3$ and an over-arc $\bar{x}_2$, we have the equation that

$$C(\bar{x}_3) = C(\bar{x}_1) * C(\bar{x}_2), \tag{7}$$

as shown in Figure 13.

We call the element $C(\bar{x}_m)$ the *color* on the arc $\bar{x}_m$.



$$C(\bar{x}_3) = C(\bar{x}_1) * C(\bar{x}_2)$$

Figure 13. Coloring condition at a crossing

The coloring condition (ii) in Definition 3.3 is well defined, meaning that the equation does not depend on the choice of two under-arcs. Actually, if $C(\bar{x}_3) = C(\bar{x}_1) * C(\bar{x}_2)$, then $C(\bar{x}_3) * C(\bar{x}_2) = (C(\bar{x}_1) * C(\bar{x}_2)) * C(\bar{x}_2)$, and we can verify that it turns out to be $C(\bar{x}_1)$ in all cases of ordered transpositions in $C(\bar{x}_1)$ and $C(\bar{x}_2)$ according to Table 1.

Proposition 3.4. *Let $L \subset S^3$ be a labeled link of degree 4, and D its diagram. For a finite group G, the number of colorings on D in G is a topological invariant of the 3-manifold which L present.*

Proof. To show the invariance, first we have to see that the number of colorings does not depend on the choice of diagrams of a labeled link. Since any two diagrams of a link can be related by a finite sequence of Reidemeister moves, we will verify the invariance under the Reidemeister moves. Next we will see that the number of colorings is also preserved under the moves MI and MII, which shows the topological invariance for the 3-manifold which the labeled link presents.

Let us see the Reidemeister moves in Figure 14. In this figure the boxed characters should be ignored here. For the Reidemeister move I, assume that the color on the arc in the left hand side of the figure is $(\langle ij\rangle, g)$. Then in the right hand side of the figure, if we put the color $(\langle ij\rangle, g)$ on the left bottom arc, then the color on the right bottom arc is

$$(\langle ij\rangle, g) * (\langle ij\rangle, g) = (\langle ij\rangle, gg^{-1}g) = (\langle ij\rangle, g)$$

from the coloring condition and Table 1. Hence we found that colorings correspond in 1-to-1 before and after the Reidemeister move I. Next for the Reidemeister II, assume the colors on the left and the right arcs in the left hand side of the figure (t, g) and (t', g'), respectively. Then in the right hand side of the figure, putting the colors on the left top and the right top arcs (t, g) and (t', g'), we have the color $((t, g) * (t', g')) * (t', g')$ on the left bottom arc. As we saw in the well definedness of the coloring condition (ii) above, we know it turns out to be (t, g) for each case of the ordered transpositions t and t'. Thus we have a coloring correspondence in the Reidemeister move II. At last for the Reidemeister move III, assume that the colors on the left top, the middle top and the right top arcs on both sides of the figure (t, g), (t', g') and (t'', g''), respectively. Looking at the colors on the bottom arcs, we know that it is sufficient to show that

$$((t, g) * (t', g')) * (t'', g'') = ((t, g) * (t'', g'')) * ((t', g') * (t'', g'')). \tag{8}$$

We show the equation in some cases of ordered transpositions, for example. Let us consider in the case of $t = t' = t'' = \langle ij \rangle$. For the left hand side,

$$\begin{aligned}((\langle ij\rangle, g) * (\langle ij\rangle, g')) * (\langle ij\rangle, g'') &= (\langle ij\rangle, g'g^{-1}g') * (\langle ij\rangle, g'') \\ &= (\langle ij\rangle, g''g'^{-1}gg'^{-1}g'').\end{aligned}$$

On the other hand in the right hand side,

$$\begin{aligned}((\langle ij\rangle, g) * (\langle ij\rangle, g'')) * ((\langle ij\rangle, g') * (\langle ij\rangle, g'')) &= (\langle ij\rangle, g''g^{-1}g'') * (\langle ij\rangle, g''g'^{-1}g'') \\ &= (\langle ij\rangle, g''g'^{-1}gg'^{-1}g'').\end{aligned}$$

Thus we have the equation (8). Moreover in the case of $t = \langle ij \rangle$, $t' = \langle jk \rangle$ and $t'' = \langle kl \rangle$, we have the identity that

$$\begin{aligned}((\langle ij\rangle, g) * (\langle jk\rangle, g')) * (\langle kl\rangle, g'') &= (\langle ik\rangle, gg') * (\langle kl\rangle, g'') \\ &= (\langle il\rangle, gg'g'').\end{aligned}$$

On the other hand,

$$\begin{aligned}((\langle ij\rangle, g) * (\langle kl\rangle, g'')) * ((\langle jk\rangle, g') * (\langle kl\rangle, g'')) &= (\langle ij\rangle, g) * (\langle jl\rangle, g'g'') \\ &= (\langle il\rangle, gg'g''),\end{aligned}$$

and we have the equation (8). Since we can show this equation in other cases in the same manner, we omit the redundant computation.

Now let us consider the invariance of the number of the colorings in the moves MI and MII in Figure 15. For the move MI, assume that the colors on the bottom and the top arcs in the left hand side of the figure $(\langle ij\rangle, g)$ and $(\langle jk\rangle, g')$, respectively. Then in the right hand side of the figure, putting the colors $(\langle ij\rangle, g)$ and $(\langle jk\rangle, g')$ on the left bottom and left top arcs respectively, we obtain the color on the right bottom arc such as

$$\begin{aligned}(\langle jk\rangle, g') * ((\langle ij\rangle, g) * (\langle jk\rangle, g')) &= (\langle jk\rangle, g') * (\langle ik\rangle, gg') \\ &= (\langle ij\rangle, g).\end{aligned}$$

The color on the right top arc is

$$(\langle ik\rangle, gg') * (\langle ij\rangle, g) = (\langle jk\rangle, g'),$$

and hence we have a 1-to 1 correspondence in coloring in the move MI. Finally, for the move MII, assume that the colors on the bottom and the top arcs in the left hand side of the figure $(\langle ij\rangle, g)$ and $(\langle kl\rangle, g')$, respectively. Then in the right hand side of the figure, put the colors $(\langle ij\rangle, g)$ and $(\langle kl\rangle, g')$ on the left bottom and left top arcs respectively. The color on the right bottom arc is

$$(\langle ij\rangle, g) * (\langle kl\rangle, g') = (\langle ij\rangle, g),$$

and the color on the right top arc is

$$(\langle kl\rangle, g') * (\langle ij\rangle, g) = (\langle kl\rangle, g').$$

Therefore we have a 1-to 1 correspondence in coloring in the move MII, and complete the proof.

□

We remark that for a finite group G the set $\widetilde{G}$ forms a quandle with the operation $*$.

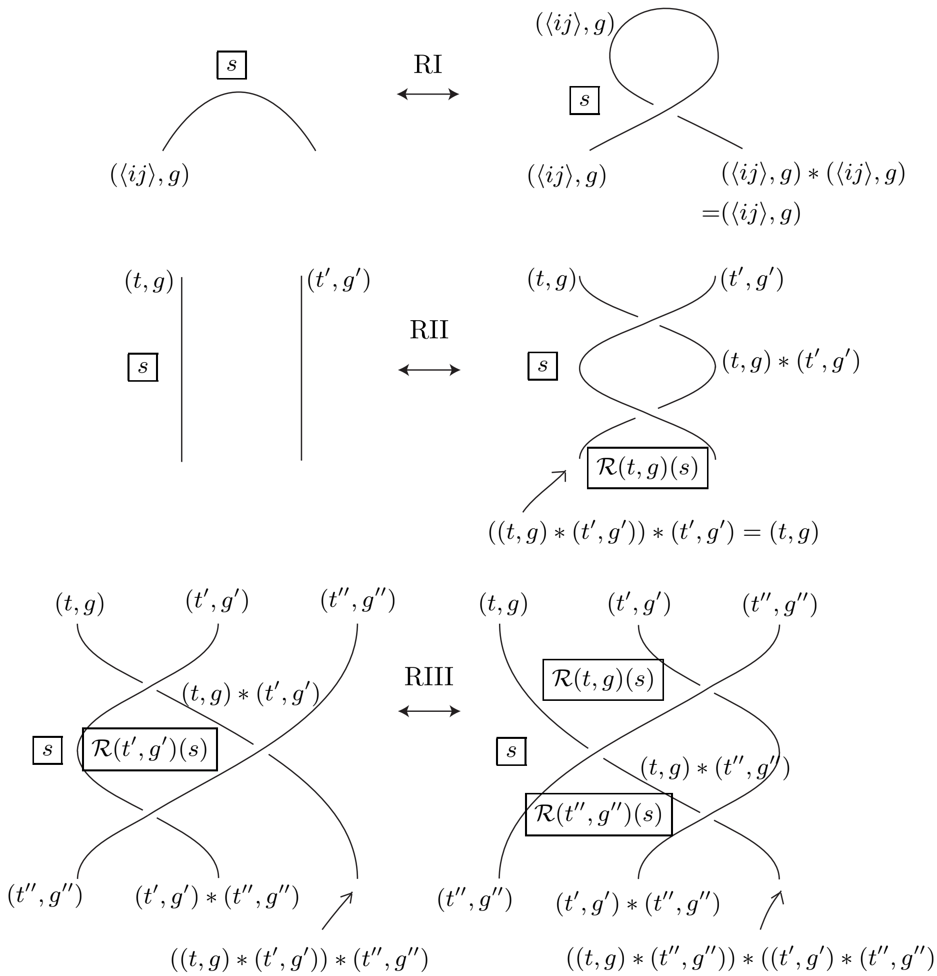


FIGURE 14. The correspondences of colorings in Reidemeister moves RI, RII and RIII

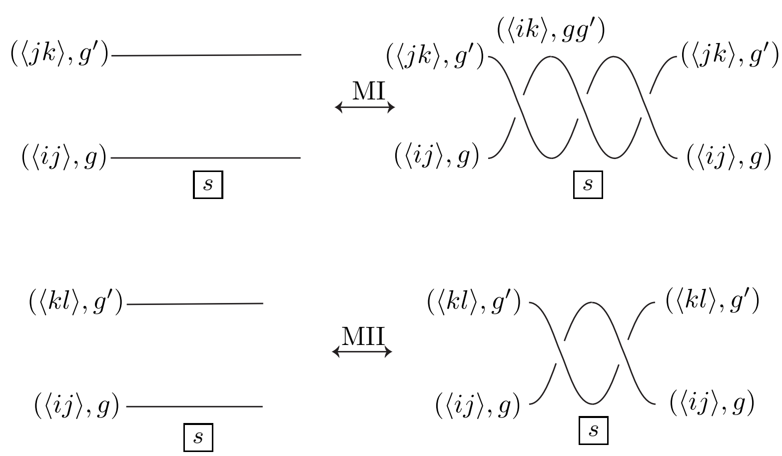


FIGURE 15. The correspondence of colorings in the moves MI and MII

Proposition 3.5. *Let M be a closed oriented 3-manifold, and G a finite group. For a diagram D of a labeled link L which presents M, we have the following equation,*

$$\sharp\{\textit{colorings on } D \textit{ in } G\} = |G|^3 \cdot \sharp\{\textit{representations } \pi_1(M) \to G\}.$$

Proof. To prove this proposition, first we see that one coloring gives one representation. Then conversely, we see that one representation corresponds $|G|^3$ colorings.

Let C be a coloring on D in G. For an arc $\bar{x}_m$ of D, we have the corresponding Wirtinger generator x_m in $\pi_1(S^3 - L)$, the lifted paths $\widetilde{x}_{md}$ ($d \in \{1,2,3,4\}$) by $p|_{M-\widetilde{L}}$, and a generator $\widehat{x}_m$ of a presentation (6) of $\pi_1(M)$ as stated in the previous subsection. We consider a homomorphism

$$\widetilde{C} : \langle \widetilde{x}_{md} | \ \rangle_{m\in\{1,2,\cdots a\},\ d\in\{1,2,3,4\}} \to G$$

defined by the generators such as if $C(\bar{x}_m) = (\langle ij\rangle, g)$ ($i < j$), then $\widetilde{C}(\widetilde{x}_{mi}) = g$, $\widetilde{C}(\widetilde{x}_{mj}) = g^{-1}$ and $\widetilde{C}(\widetilde{x}_{mk}) = 1$ ($k \notin \{i,j\}$). Using this homomorphism, we define a representation $\rho_C : \pi_1(M) \to G$ by giving a mapping of each generator $\widehat{x}_m$ of the presentation (6) as follows. If $\widehat{x}_m = w_i \widetilde{x}_{mi} w_j^{-1}$, where w_i and w_j are the Schreier trees in $\langle \widetilde{x}_{md} | \ \rangle$, then

$$\rho_C(\widehat{x}_m) = \widetilde{C}(w_i \widetilde{x}_{mi} w_j^{-1}).$$

To see that this map ρ_C is well defined, we have to check that $\rho_C(\widehat{r}_n) = 1$ for each relator $\widehat{r}_n$ in the presentation (6), which corresponds to a crossing of D. Let us consider in the case of the relator $\widehat{r}_n = \widehat{x}_1\widehat{x}_2^{-1}\widehat{x}_3\widehat{x}_2^{-1}$ corresponding to a crossing of type 1 with the notations in Figure 10, for example. Assume that $C(\bar{x}_1) = (\langle 12\rangle, g_1)$, $C(\bar{x}_2) = (\langle 12\rangle, g_2)$ and $C(\bar{x}_3) = (\langle 12\rangle, g_3)$, then $C(\bar{x}_3) = C(\bar{x}_2) * C(\bar{x}_1)$ from the coloring condition, and hence we have a relation that $g_3 = g_2 g_1^{-1} g_2$. Using this relation, we have the identity that

$$\begin{aligned}
\rho_C(\widehat{x}_1\widehat{x}_2^{-1}\widehat{x}_3\widehat{x}_2^{-1}) &= \widetilde{C}(w_1\widetilde{x}_{11}w_2^{-1} \cdot (w_1\widetilde{x}_{21}w_2^{-1})^{-1} \cdot w_1\widetilde{x}_{31}w_2^{-1} \cdot (w_1\widetilde{x}_{21}w_2^{-1})^{-1}) \\
&= \widetilde{C}(w_1\widetilde{x}_{11}\widetilde{x}_{21}^{-1}\widetilde{x}_{31}\widetilde{x}_{21}^{-1}w_1^{-1}) \\
&= g_1 g_2^{-1} g_3 g_2^{-1} \\
&= g_1 g_2^{-1} \cdot g_2 g_1^{-1} g_2 \cdot g_2^{-1} \\
&= 1.
\end{aligned}$$

We can verify that $\rho_C(\widehat{r}_n) = 1$ for the relators corresponding to crossings of other types in the same manner as above, and therefore we see that ρ_C is well defined.

Conversely, we give a coloring C_ρ from a representation $\rho : \pi_1(M) \to G$. First we give arbitrary elements in G for the Schreier trees w_2, w_3 and w_4, and denote them by $C_\rho(w_2)$, $C_\rho(w_3)$ and $C_\rho(w_4) \in G$, for the convenience of the notation. We put $C_\rho(w_1) = 1$. We define the color $C_\rho(\bar{x}_m)$ as follows. If $\phi(x_m) = (ij)$ ($i < j$), then

$$C_\rho(\bar{x}_m) = (\langle ij\rangle,\ C_\rho(w_i)^{-1}\rho(\widehat{x}_m)C_\rho(w_j)).$$

We can verify that C_ρ satisfies the coloring condition (ii) in the same manner as above, and that if we give a representation ρ_{C_ρ} from this coloring C_ρ, then it turns out to be the original representation ρ. ∎

In fact, Proposition 3.5 gives another proof for Proposition 3.4, the invariance of the number of colorings, and even independence of the orientations of 3-manifolds. But, our formulation would have its own right since, for instance, it gives a new combinatorially and algorithmically refined method to compute a known invariant, the number of representations of a fundamental group. Moreover, our formulation enables us to give a recipe for constructing finer invariants derived from covering presentations, which depends on the orientation of manifolds.

Let us consider coloring by using labeled links directly without link diagrams. Let G be a finite group. We consider the direct product set $\mathfrak{S}_4 \times G^4$, and define a binary operation in it as follows;

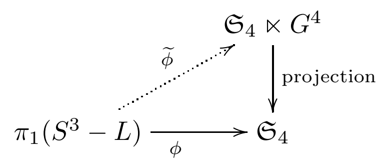
$$(\sigma, g_1, g_2, g_3, g_4) \cdot (\sigma', {g_1}', {g_2}', {g_3}', {g_4}') = (\sigma\sigma', g_1 {g_{\sigma(1)}}', g_2 {g_{\sigma(2)}}', g_3 {g_{\sigma(3)}}', g_4 {g_{\sigma(4)}}'),$$

where σ, $\sigma' \in \mathfrak{S}_4$, and g_i, ${g_i}' \in G$ $(i \in \{1, 2, 3, 4\})$. With the binary operation, the set $\mathfrak{S}_4 \times G^4$ forms a semidirect group with a normal subgroup G^4, and we denote it $\mathfrak{S}_4 \ltimes G^4$. We define a coloring on a labeled link $L \subset S^3$ by a homomorphism

$$\widetilde{\phi} : \pi_1(S^3 - L) \to \mathfrak{S}_4 \ltimes G^4,$$

satisfying the folowing conditions;

(i) The projection of $\widetilde{\phi}$ on $\mathfrak{S}_4$ coincides with the monodromy $\phi : \pi_1(S^3 - L) \to \mathfrak{S}_4$ of the labeled link L.



(ii) For each meridian x in $\pi_1(S^3 - L)$, if $\widetilde{\phi}(x) = ((ij), g_1, g_2, g_3, g_4)$, then $g_i = {g_j}^{-1}$ and $g_k = 1$ $(k \notin \{i, j\})$.

A coloring on a diagram D of L is obtained from a coloring $\widetilde{\phi}$ on L as follows. For each arc $\bar{x}_m$ of D, if $\phi(x_m) = (ij)$ $(i < j)$ and the i-th element of $\widetilde{\phi}(x_m)$ in G^4 is g, then put the color $(\langle ij \rangle, g)$ on $\bar{x}_m$.

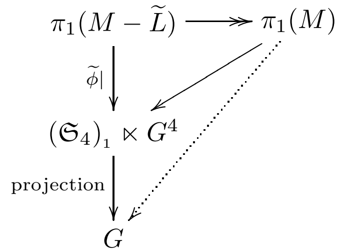
Similar to Proposition 3.5, we have a correspondence between colorings on labeled links and representations $\pi_1(M) \to G$. As we saw in the last of the previous subsection, $\pi_1(M - \widetilde{L})$ can be regarded as a subgroup of $\pi_1(S^3 - L)$ by the isomorphism $\pi_1(M - \widetilde{L}) \cong \phi^{-1}(\{\sigma \in \mathfrak{S}_4 | \sigma(1) = 1\})$. Considering a restriction of $\widetilde{\phi}$ to $\pi_1(M - \widetilde{L})$, we have a homomorphism

$$\widetilde{\phi}| : \pi_1(M - \widetilde{L}) \to (\mathfrak{S}_4)_1 \ltimes G^4,$$

where $(\mathfrak{S}_4)_1 = \{\sigma \in \mathfrak{S}_4 | \sigma(1) = 1\}$. Moreover, since $\pi_1(M)$ can be presented as a quotient group of $\pi_1(M - \widetilde{L})$ by the Van Kampen Theorem, and since the normal subgroup generated by the homotopy classes of meridians of $\widetilde{L}$ is mapped to the unit in $(\mathfrak{S}_4)_1 \ltimes G^4$ by the condition (ii) of coloring on labeled links, the map $\widetilde{\phi}$ induces a homomorphism

$$\pi_1(M) \to (\mathfrak{S}_4)_1 \ltimes G^4.$$

Taking the projection on the first element in G^4 of this map, we have a representation $\pi_1(M) \to G$. Conversely by the reverse procedure, from a representation $\pi_1(M) \to G$ we have three colorings on the labeled link L, meaning that we have free choices on three elements in $\pi_1(S^3 - L)$ labeled (12), (13) and (14), respectively, in giving a coloring from the representation. Therefore we have 3-to-1 correspondence in the numbers of colorings and representations.



3.3. **Invariants of 3-manifolds.** To make finer invariants than the number of colorings, we give a recipe of invariants by defining another coloring on the complementary regions of diagrams of labeled links in a set. For the pair of a coloring on a diagram and a coloring on the complementary regions of the diagram, we assign an element of an Abelian group. Summing up such elements for all the pairs of the colorings with respect to a diagram of a labeled link, we obtain a state sum invariant like the quandle cocycle invariants introduced in [3] for knots and surface-knots.

Definition 3.6. Let G be a finite group, and S a set. A *switching map* in S associated with G is defined to be a map

$$\mathcal{R} : \widetilde{G} \to \mathrm{Aut}(S),$$

satisfying the following conditions. Here, $\widetilde{G}$ is the direct product set $T \times G$ with a binary operation $*$ defined in Table 1 in the previous subsection.

(R1) $\mathcal{R}((t, g))^2 = \mathrm{id}_S$, and

(R2) $\mathcal{R}((t, g)) \circ \mathcal{R}((t', g')) \circ \mathcal{R}((t, g) * (t', g')) \circ \mathcal{R}((t', g')) = \mathrm{id}_S$,

for any (t, g), $(t', g') \in \widetilde{G}$.

Definition 3.7. Let D be a diagram of a labeled link L of degree 4. We assume that D has a coloring in a finite group G. Let $\mathcal{R} : \widetilde{G} \to \mathrm{Aut}(S)$ be a switching map in a set S associated with G. A *coloring on the regions* of D in S associated with $\mathcal{R}$ will be a map

$$R : \{\text{regions of } \mathbb{R}^2 \setminus D\} \to S$$

such that the two colors on the adjacent regions separated by an arc with a color $(t, g) \in \widetilde{G}$ are expressed as $s \in S$ and $\mathcal{R}(t, g)(s) \in S$, as depicted in Figure 16.

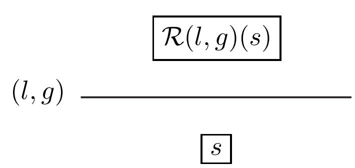


Figure 16. The condition of coloring on the regions

We remark that the coloring on the regions is compatible around any crossing by the condition (R2) in Definition 3.6, as depicted in Figure 17. Besides, with the condition (R1), the color on the unbounded region determines the colors on the other regions uniquely. This means that the number of colorings on the regions of a colored diagram with a fixed switching map $\mathcal{R}$ is $|S|$.

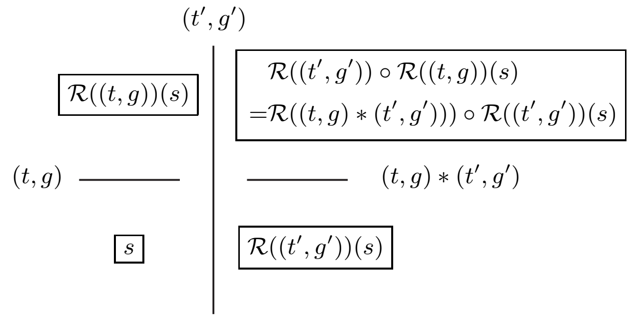


FIGURE 17. Coloring on the regions is compatible around any crossing

We define a weight at a crossing. Let A be an Abelian group written multiplicatively, and

$$\mathcal{X} : S \times \widetilde{G} \times \widetilde{G} \to A$$

be a map. For a diagram D of a labeled link L, fix a coloring C in a finite group G and a coloring R on the regions in S associated with a switching map $\mathcal{R} : \widetilde{G} \to \mathrm{Aut}(S)$. The *weight* at a crossing x of D, with colors as depicted in Figure 18, is defined to be $\mathcal{X}(s, (t, g), (t', g')) \in A$. Here s is the color on one of the two regions around x with the under-arc left towards the crossing, and (t, g), (t', g') are the colors on the under- and over-arcs touching the region, respectively. Such a weight at a crossing x will be denoted by $X(x; C, R)$.

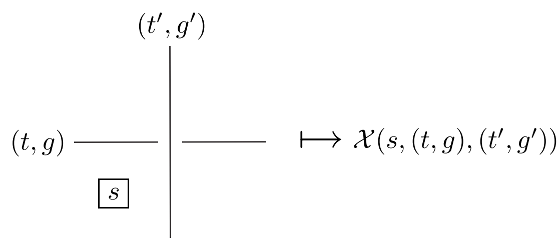


FIGURE 18. The weight $X(x; C, R)$ at a crossing x

Theorem 3.8. *Let M be a closed oriented 3-manifold. Let D be a diagram of a labeled ink L which presents M as a covering presentation. Moreover, we let C be a coloring on D in a finite group G, R a coloring on the regions in a set S associated with a switching map $\mathcal{R} : \widetilde{G} \to Aut(S)$, where $\widetilde{G} = T \times G$, and $X(x; C, R)$ the weight at a crossing x of D given by a map $\mathcal{X} : S \times \widetilde{G} \times \widetilde{G} \to A$ in an Abelian group A. If the maps $\mathcal{R}$ and $\mathcal{X}$ satisfy the following conditions (X1)–(X6), then the state sum*

$$I(D) = \sum_{C} \sum_{R} \prod_{x} X(x; C, R) \in \mathbb{Z}[A]$$

is an topological invariant of M. In the expression we take the product over all the crossings of D, the inner sum over all the colorings on the regions in S and the outer sum over all the colorings on D in G.

(X1) $\mathcal{X}(s, (t, g), (t', g')) = \mathcal{X}(\mathcal{R}((t', g')) \circ \mathcal{R}((t, g))(s), (t, g) * (t', g'), (t', g'))$,
(X2) $\mathcal{X}(s, (t, g), (t, g)) = 1_A$,
(X3) $\mathcal{X}(s, (t, g), (t', g')) \cdot \mathcal{X}(\mathcal{R}((t, g))(s), (t, g), (t', g')) = 1_A$,

(X4) $\mathcal{X}(s,(t,g),(t',g'))\cdot\mathcal{X}(\mathcal{R}((t',g'))(s),(t,g)*(t',g'),(t'',g''))\cdot\mathcal{X}(s,(t',g'),(t'',g''))$
$= \mathcal{X}(\mathcal{R}((t,g))(s),(t',g'),(t'',g''))\cdot\mathcal{X}(s,(t,g),(t'',g''))$
$\cdot\,\mathcal{X}(\mathcal{R}((t'',g''))(s),(t,g)*(t'',g''),(t',g')*(t'',g''))$,

(X5) $\mathcal{X}(s,(\langle ij\rangle,g),(\langle jk\rangle,g'))\cdot\mathcal{X}(s,(\langle jk\rangle,g'),(\langle ik\rangle,gg'))\cdot\mathcal{X}(s,(\langle ik\rangle,gg'),(\langle ij\rangle,g))$
$= 1_A$,

(X6) $\mathcal{X}(s,(\langle ij\rangle,g),(\langle kl\rangle,g'))\cdot\mathcal{X}(s,(\langle kl\rangle,g'),(\langle ij\rangle,g)) = 1_A$

for any $s\in S$, (t,g), (t',g'), $(t'',g'')\in\widetilde{G}$, *where* i,j,k,l *in the ordered transpositions are all distinct.*

Proof. To show the topological invariance, we show that the definition of the weight $X(x;C,R)$ is well defined, meaning that it does not depend on the choice of the colors on the two regions around any crossing x. Then we show that the product $\prod_x X(x;C,R)$ does not depend on the choice of diagrams of a link, and the choice of labeled links presenting homeomorphic 3-manifolds.

The condition (X1) leads the invariance of $X(x;C,R)$ in the choice of the two regions around a crossing, as illustrated in Figure 17.

To show the invariance of the product $\prod_x X(x;C,R)$ in the choice of diagrams of a link, it is enough to show that it is invariant under the Reidemeister moves. The conditions (X2), (X3) and (X4) leads the invariance under the Reidemeister moves I, II and III respectively, as depicted in Figure 14.

In the same manner, the conditions (X5) and (X6) leads the invariance of the product $\prod_x X(x;C,R)$ under the moves MI and MII in Figure 15 respectively, and this means that it does not depend on the choice of labeled links presenting homeomorphic 3-manifolds. □

We denote this state sum invariant $I(M)$. We call a map $\mathcal{X}$ satisfying all the conditions (X1) – (X6) for a switching map $\mathcal{R}$, a *weight function.* In Section 4.2 we give an example of such weight functions.

Since the state sum $I(M)$ is defined on a diagram of a labeled link, and it is invariant under Reidemeister moves as shown in the proof above, we have the following corollary.

Corollary 3.9. *Let* D *be a diagram of a labeled link* L. *If the maps* $\mathcal{R}$ *and* $\mathcal{X}$ *in Theorem 3.8 satisfy the conditions (X1)–(X4), then the state sum*

$$I(D) = \sum_C\sum_R\prod_x X(x;C,R)\in\mathbb{Z}[A]$$

is an invariant of labeled links.

In particular, if we consider only labeled links such that each Wirtinger generator of $\pi_1(S^3-L)$ *is labeled* (12), *then the state sum* $I(D)$ *gives an invariant of unoriented links.*

4. The Dijkgraaf-Witten invariant

We review the Dijkgraaf-Witten invariant of closed oriented 3-manifolds in Section 4.1. Then in Section 4.2, we give an example of invariants based on the recipe introduced in Theorem 3.8, and state that the example is essentially the Dijkgraaf-Witten invariant.

4.1. **Definition of the Dijkgraaf-Witten invariant.** Let M be a closed oriented 3-manifold with a triangulation T. A *coloring* on T in a finite group G is defined to be a map

$$C : \{\text{oriented edges of } T\} \to G,$$

satisfying the following conditions.

(i) For any 'oriented' 2-simplex F, we have

$$C(\partial F) = 1$$

as depicted in Figure 19. Here the symbol ∂F stands for the image of F under the boundary operator ∂, when we regard F as a generator of the chain group $C_2(M;\mathbb{Z})$.

(ii) For any oriented edge E, we have

$$C(-E) = C(E)^{-1},$$

where $-E$ is the edge E with the opposite orientation.

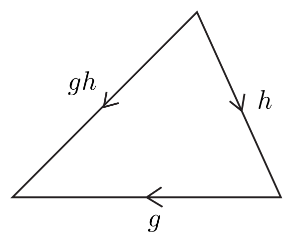


Figure 19. The coloring condition on triangulations

A map $\theta : G \times G \times G \to A$, where A is an Abelian group, is a *3-cocycle* with value in A if, by definition, it satisfies the identity

$$(\sharp) \qquad \theta(y,z,w)\cdot\theta(xy,z,w)^{-1}\cdot\theta(x,yz,w)\cdot\theta(x,y,zw)^{-1}\cdot\theta(x,y,z) = 1$$

for any $x, y, z, w \in G$. For any group G, we denote its classifying space by BG. Here we use the semi-simplicial method by Milnor [9] to construct BG. Taking the unitary group $U(1)(\cong \mathbb{R}/\mathbb{Z})$ as an Abelian group A, we define a map

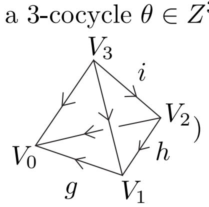
$$\psi : \mathrm{Hom}(C_n(BG,\mathbb{Z}), U(1)) \to C^n(G, U(1))$$

by

$$\psi(\theta)(g_1, \cdots, g_n) = \theta([g_1|\cdots|g_n]),$$

where $[g_1|\cdots|g_n]$ is the n-cell defined by $g_1, \cdots, g_n$. We see that the map ψ is a cochain map and induces an isomorphism from $H^n(BG, U(1))$ to $H^n(G, U(1))$.

We give an ordering to the vertices of T. Let us give the orientation of each tetrahedron in the ascending order. The *weight* $W_\theta(t;C)$ of a tetrahedron t with a coloring C associated with a 3-cocycle $\theta \in Z^3(BG, U(1))$ is defined such as

$W_\theta($  $) = \theta([g|h|i])^{\epsilon(t)}.$

Here the order of the vertices is $V_0 < V_1 < V_2 < V_3$. The elements $g, h, i \in G$ are the colors of edges $\langle V_0, V_1\rangle, \langle V_1, V_2\rangle, \langle V_2, V_0\rangle$ respectively, and $[g|h|i]$ is the 3-cell of BG defined by g, h and $i \in G$. The sign $\epsilon(t) \in \{\pm 1\}$ is 1 if the orientation of the tetrahedron t is compatible with that of M, and -1 otherwise. The *Dijkgraaf-Witten invariant* [4] is defined by

$$Z_\theta(M) = \frac{1}{|G|^N} \sum_C \prod_t W_\theta(t; C) \in \mathbb{C},$$

where N is the number of the vertices of T. The product is taken over all the tetrahedra of T and the sum is taken over all the colorings on T. Wakui [17] showed that $Z_\theta(M)$ does not depend on the choice of orderings of the vertices and the triangulations on T, and depends only on the cohomology class θ if $\partial M = \emptyset$. Further he showed that this definition can be extended for 3-manifolds with boundaries, and the construction gives an example of the topological quantum field theory. Note that this invariant $Z_\theta(M)$ is also expressed by

$$Z_\theta(M) = \frac{1}{|G|} \sum_{\gamma \in \mathrm{Hom}(\pi_1(M), G)} \langle \gamma^*[\theta], [M] \rangle.$$

Here $[\theta]$ is the cohomology class of θ and $[M]$ is the fundamental class of M. γ^* is the map $H^3(BG, U(1)) \to H^3(M, U(1))$ induced by the classifying map $M \to BG$ corresponding to a representation $\gamma : \pi_1(M) \to G$. We remark that

$$Z_\theta(-M) = \overline{Z_\theta(M)}$$

for the 3-manifold with the opposite orientation.

4.2. **An example of invariants.** Following the recipe in Section 3.3, we practically give a switching map $\mathcal{R}$ and a weight function $\mathcal{X}$, satisfying all the conditions in Theorem 3.8. Then we state that the invariant of 3-manifolds derived from the recipe in the theorem turns out to be the the Dijkgraaf-Witten invariant.

Letting the direct product G^4 of a finite group G be the set S, we define a map

$$\mathcal{R} : \widetilde{G} \to \mathrm{Aut}(G^4)$$

as follows.

$$\begin{aligned}
&\mathcal{R}((\langle 12 \rangle, g))(s_1, s_2, s_3, s_4) = (gs_2, g^{-1}s_1, s_3, s_4),\\
&\mathcal{R}((\langle 13 \rangle, g))(s_1, s_2, s_3, s_4) = (gs_3, s_2, g^{-1}s_1, s_4),\\
&\mathcal{R}((\langle 14 \rangle, g))(s_1, s_2, s_3, s_4) = (gs_4, s_2, s_3, g^{-1}s_1),\\
&\mathcal{R}((\langle 23 \rangle, g))(s_1, s_2, s_3, s_4) = (s_1, gs_3, g^{-1}s_2, s_4),\\
&\mathcal{R}((\langle 24 \rangle, g))(s_1, s_2, s_3, s_4) = (s_1, gs_4, s_3, g^{-1}s_2),\\
&\mathcal{R}((\langle 34 \rangle, g))(s_1, s_2, s_3, s_4) = (s_1, s_2, gs_4, g^{-1}s_3),
\end{aligned}$$

for any g and $s_i \in G$ ($i \in \{1, 2, 3, 4\}$). Since this map $\mathcal{R}$ satisfies the conditions (R1) and (R2) in Definition 3.6, it is a switching map.

Let $\theta \in Z^3(G, U(1))$ be a 3-cocycle. We define a map $\mathcal{X}$ using θ as follows, assuming that i, j, k and l are all distinct.

$$\begin{aligned}
&\mathcal{X}((s_1, s_2, s_3, s_4), (\langle ij \rangle, g), (\langle ij \rangle, g'))\\
&= \theta(g, g^{-1}g', g'^{-1}gs_j) \cdot \theta(g', g'^{-1}g, g^{-1}s_i) \cdot \theta(g'g^{-1}g', g'^{-1}g, s_j) \cdot \theta(g', g^{-1}g', g'^{-1}s_i)\\
&\mathcal{X}((s_1, s_2, s_3, s_4), (\langle ij \rangle, g), (\langle jk \rangle, g')) = \theta(g'^{-1}, g^{-1}, s_i)^{-1} \cdot \theta(g'^{-1}, g^{-1}, gs_j)\\
&\mathcal{X}((s_1, s_2, s_3, s_4), (\langle ij \rangle, g), (\langle kl \rangle, g')) = 1
\end{aligned}$$

Proposition 4.1. *If a 3-cocycle $\theta \in Z^3(G, U(1))$ satisfies the two conditions,*

(Coc1) $\theta(1, x, y) = \theta(x, 1, y) = \theta(x, y, 1) = 1$ *and*
(Coc2) $\theta(x, x^{-1}, y) = \theta(x, y, y^{-1}) = 1$,

then the maps $\mathcal{R}$ and $\mathcal{X}$ defined above satisfy the conditions (X1) – (X6) in Theorem 3.8.

Proof. The weight conditions (X1) – (X6) can be checked by hand calculations in each case, using the cocycle conditions. Here we check some of these cases for instance, because other cases can be similarly verified.

(X1) In the case that $t = t' = \langle 12\rangle$, beginning with the right hand side, we have

$$\begin{aligned}&\mathcal{X}(\mathcal{R}((\langle 12\rangle, g')) \circ \mathcal{R}((\langle 12\rangle, g))(s_1, s_2, s_3, s_4), (\langle 12\rangle, g) * (\langle 12\rangle, g'), (\langle 12\rangle, g'))\\ =&\mathcal{X}((g'g^{-1}s_1, g'^{-1}gs_2, s_3, s_4), (\langle 12\rangle, g'g^{-1}g'), (\langle 12\rangle, g'))\\ =&\theta(g'g^{-1}g', g'^{-1}g, s_2) \cdot \theta(g', g^{-1}g', g'^{-1}s_1) \cdot \theta(g, g^{-1}g', g'^{-1}gs_2) \cdot \theta(g', g'^{-1}g, g^{-1}s_1)\\ =&\mathcal{X}((s_1, s_2, s_3, s_4), (\langle 12\rangle, g), (\langle 12\rangle, g')).\end{aligned}$$

(X2) Putting $\langle ij\rangle$ to t, we have

$$\begin{aligned}&\mathcal{X}((s_1, s_2, s_3, s_4), (\langle ij\rangle, g), (\langle ij\rangle, g))\\ =&\theta(g, g^{-1}g, g^{-1}gs_j) \cdot \theta(g, g^{-1}g, g^{-1}s_i) \cdot \theta(gg^{-1}g, g^{-1}g, s_j) \cdot \theta(g, g^{-1}g, g^{-1}s_i)\\ =&\theta(g, 1, s_j) \cdot \theta(g, 1, g^{-1}s_i) \cdot \theta(g, 1, s_j) \cdot \theta(g, 1, g^{-1}s_i)\\ =&1,\end{aligned}$$

because of the condition (Coc1) of 3-cocycles.

(X3) In the case that $t = t' = \langle 12\rangle$, we have

$$\begin{aligned}&\mathcal{X}((s_1, s_2, s_3, s_4), (\langle 12\rangle, g), (\langle 12\rangle, g')) \cdot \mathcal{X}(\mathcal{R}((\langle 12\rangle, g'))(s_1, s_2, s_3, s_4), (\langle 12\rangle, g), (\langle 12\rangle, g'))\\ &= \theta(g, g^{-1}g', g'^{-1}gs_2) \cdot \theta(g', g'^{-1}g, g^{-1}s_1) \cdot \theta(g'g^{-1}g', g'^{-1}g, s_2) \cdot \theta(g', g^{-1}g', g'^{-1}s_1)\\ &\cdot \theta(g, g^{-1}g', g'^{-1}s_1) \cdot \theta(g', g'^{-1}g, s_2) \cdot \theta(g'g^{-1}g', g'^{-1}g, g^{-1}s_1) \cdot \theta(g', g^{-1}g', g'^{-1}gs_2)\\ &= 1.\end{aligned}$$

Here we used the property

$$\theta(x^{-1}, z, w) = \theta(x, x^{-1}z, w)^{-1}$$

for any x, z and $w \in G$ by the cocycle conditions ($\sharp$), (Coc1) and (Coc2).

(X4) We can verify that the product

$$(\text{LHS}) \cdot (\text{RHS})^{-1} = 1,$$

in each situation using the cocycle conditions repeatedly, and so omit the redundant computations.

(X5) Beginning with the left hand side, we have

$$\begin{aligned}&\mathcal{X}((s_1, s_2, s_3, s_4), (\langle ij\rangle, g), (\langle jk\rangle, g')) \cdot \mathcal{X}((s_1, s_2, s_3, s_4), (\langle jk\rangle, g'), (\langle ik\rangle, gg'))\\ &\cdot \mathcal{X}((s_1, s_2, s_3, s_4), (\langle ik\rangle, gg'), (\langle ij\rangle, g))\\ =&\theta(g'^{-1}, g^{-1}, s_i)^{-1} \cdot \theta(g'^{-1}, g^{-1}, gs_j) \cdot \theta(gg', g'^{-1}, s_j)^{-1} \cdot \theta(gg', g'^{-1}, g's_k)\\ &\cdot \theta(g^{-1}, gg', s_k)^{-1} \cdot \theta(g^{-1}, gg', g'^{-1}g^{-1}s_i)\\ =&1,\end{aligned}$$

by the property

$$\theta(y, y^{-1}x^{-1}, w) \cdot \theta(x, y, y^{-1}x^{-1}w)^{-1} = 1$$

obtained by putting $y^{-1}x^{-1}$ to z in the equation ($\sharp$).

(X6) From the definition of the map $\mathcal{X}$ above,

$$\mathcal{X}((s_1,s_2,s_3,s_4),(\langle ij\rangle,g),(\langle kl\rangle,g'))\cdot\mathcal{X}((s_1,s_2,s_3,s_4),(\langle kl\rangle,g'),(\langle ij\rangle,g))=1.$$

□

We remark that a large number of cohomology classes are realized by 3-cocycles having the properties (Coc1) and (Coc2), though we do not have a nontrivial 3-cocycle with these properties in $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

Therefore, by Theorem 3.8, we obtain an invariant of a 3-manifold M from a diagram D of a labeled link L presenting M,

$$I_\theta(M)=\sum_C\sum_R\prod_x X_\theta(x;C,R)\in\mathbb{C},$$

where X_θ denotes the weight at crossings given by the weight function $\mathcal{X}$ defined above. Moreover, by Corollary 3.9, $I_\theta(M)$ is also an invariant $I_\theta(L)$ of the unoriented link L presented by D.

Theorem 4.2. *For a closed oriented 3-manifold M, we have the identity between the state sum invariant $I_\theta(M)$ and the Dijkgraaf-Witten invariant $Z_\theta(M)$ associated with a finite group G and a 3-cocycle $\theta\in Z^3(G,U(1))$ which satisfies the conditions (Coc1) and (Coc2) in Proposition 4.1,*

$$I_\theta(M)=|G|^8\cdot Z_\theta(M).$$

The proof of this theorem will be given in Section 6, after explaining how we obtain such an example of the pair of a switching map $\mathcal{R}$ and a weight function $\mathcal{X}$.

5. Examples

Theorem 4.2 gives a combinatorial way to compute the Dijkgraaf-Witten invariant using covering presentations of 3-manifolds. In this section we give some calculations.

5.1. **Lens spaces L(5,1) and L(5,2).** Let the finite group G be $\mathbb{Z}_5=\{0,1,2,3,4\}$ written additively. We have a 3-cocycle $\theta_5\in Z(\mathbb{Z}_5,U(1))$ satisfying the conditions (Coc1) and (Coc2) in Theorem 3.8, for example, as the following map.

$$\begin{aligned}\theta_5=&t_{(1,1,1)}\cdot t^4_{(1,1,2)}\cdot t_{(1,2,1)}\cdot t^4_{(1,2,4)}\cdot t_{(1,3,3)}\cdot t^4_{(1,3,4)}\\ &\cdot t^4_{(2,1,1)}\cdot t_{(2,1,3)}\cdot t^4_{(2,2,2)}\cdot t_{(2,2,4)}\cdot t^4_{(2,4,2)}\cdot t_{(2,4,3)}\\ &\cdot t_{(3,1,2)}\cdot t^4_{(3,1,3)}\cdot t_{(3,3,1)}\cdot t^4_{(3,3,3)}\cdot t_{(3,4,2)}\cdot t^4_{(3,4,4)}\\ &\cdot t^4_{(4,2,1)}\cdot t_{(4,2,2)}\cdot t^4_{(4,3,1)}\cdot t_{(4,3,4)}\cdot t^4_{(4,4,3)}\cdot t_{(4,4,4)},\end{aligned}$$

where

$$t_{(x,y,z)}(x',y',z')=\begin{cases}e^{\frac{2\pi\sqrt{-1}}{5}} & \text{if } (x,y,z)=(x',y',z')\\ \\ 1 & \text{otherwise.}\end{cases}$$

Computing the state sum invariant I_{θ_5} with this 3-cocycle θ_5 for the lens spaces $L(5,1)$ and $L(5,2)$, where their diagrams of labeled links are depicted in Figure 20 and 21 respectively, we have

$$I_{\theta_5}(L(5,1))=5^7+2\cdot5^7e^{\frac{4\pi\sqrt{-1}}{5}}+2\cdot5^7e^{\frac{6\pi\sqrt{-1}}{5}}=-5^7\sqrt{5}$$

and

$$I_{\theta_5}(L(5,2)) = 5^7 + 2\cdot 5^7 e^{\frac{2\pi\sqrt{-1}}{5}} + 2\cdot 5^7 e^{\frac{8\pi\sqrt{-1}}{5}} = 5^7\sqrt{5}.$$

Hence on their Dijkgraaf-Witten invariant,

$$Z_{\theta_5}(L(5,1)) = -\frac{1}{\sqrt{5}} \quad \text{and} \quad Z_{\theta_5}(L(5,2)) = \frac{1}{\sqrt{5}},$$

by Theorem 4.2.

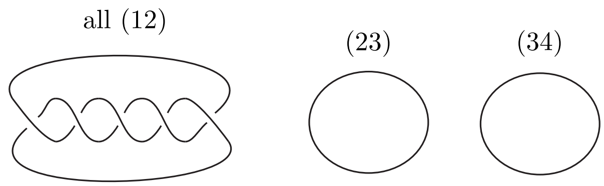


FIGURE 20. A diagram of a labeled link presenting the lens space $L(5,1)$.

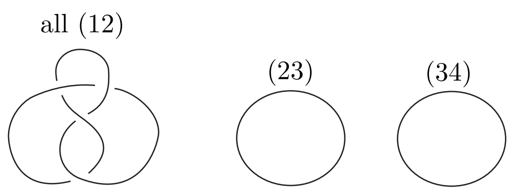


FIGURE 21. A diagram of a labeled link presenting the lens space $L(5,2)$.

5.2. **Seifert manifolds.** The Seifert manifold $M = (S^2; 5, 5, \cdots, 5)$ with n fibers has a covering presentation of the pretzel link as illustrated in Figure 22. It has $5^{(n+2)}$ colorings in $\mathbb{Z}_5$, and with the same 3-cocycle θ_5 defined in the previous subsection, we have

$$I_{\theta_5}(M) = 5^7 \sum_{x_1,\cdots,x_{n-1}} \exp(\frac{2\pi\sqrt{-1}}{5}((\sum_{i=1}^{n-1} f(x_i)) + f(-x_1 - \cdots - x_{n-1})),$$

where the outer sum is taken over all the combinations of $(x_1, \cdots, x_{n-1}) \in (\mathbb{Z}_5)^{n-1}$. The map $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ is defined by $f(0) = 0$, $f(1) = f(4) = 2$ and $f(2) = f(3) = 3$. Then the Dijkgraaf-Witten invariant of M is

$$Z_{\theta_5}(M) = \frac{1}{5} \sum_{x_1,\cdots,x_{n-1}} \exp(\frac{2\pi\sqrt{-1}}{5}((\sum_{i=1}^{n-1} f(x_i)) + f(-x_1 - \cdots - x_{n-1})).$$

6. PROOF OF THEOREM 4.2

The aim of this section is to prove Theorem 4.2. We give a proof in Section 6.4, and before it, we introduce the idea of obtaining the pair of the switching map $\mathcal{R}$ and the weight function $\mathcal{X}$ introduced in Section 4.2. It comes from an attempt to compute the Dijkgraaf-Witten invariant of a 3-manifold using a diagram of a labeled link presenting the 3-manifold, and the process is divided into the following three steps.

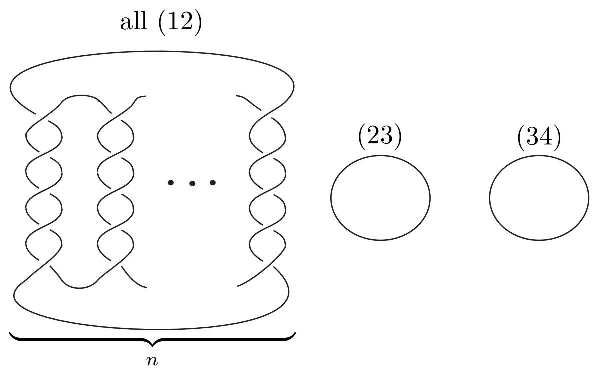


FIGURE 22. A diagram of a labeled link presenting the Seifert manifold $(S^2; 5, 5, \cdots, 5)$ with n fibers.

Step 1. Give a triangulation T of a 3-manifold M by using a diagram D of a labeled link presenting M. With the simple 4-fold branched covering $p : M \to S^3$, we make a correspondence between the set of tetrahedra of T and the set of crossings of D.

Step 2. Give a coloring C_T on the triangulation T obtained in Step 1, from a coloring C on D both in the same finite group G. To do this, we use a coloring on the regions associated with the switching map $\mathcal{R}$. This map $\mathcal{R}$ is defined so that the colors on the regions will indicate the colors on some edges of T directly for a coloring C on D, and it makes possible to see the colors on the tetrahedra corresponding to a crossing, only by looking the colors around the crossing.

Step 3. Compute the product $\prod W_\theta(t; C_T)$ of the weight of the Dijkgraaf-Witten invariant, for the tetrahedra corresponding to a crossing of each type using the coloring C_T which is obtained from a coloring C on D in Step 2. The weight function $\mathcal{X}$ is defined so that the weight $X_\theta(x; C, R)$ at a crossing x will be equal to this product $\prod W_\theta(t; C_T)$.

Therefore taking the product $\prod_x X_\theta(x; C, R)$ of the weights for all the crossings of D is nothing but computing the contribution $\prod_t W_\theta(t; C_T)$ of the Dijkgraaf-Witten invariant, and hence the state sum $\sum_C \sum_R \prod_x X_\theta(x; C, R)$ should be an invariant. We see these steps in detail by looking at some figures throghout the following three subsections.

6.1. **Step 1.** In this subsection we introduce an algorithm to give a triangulation of a 3-manifold from a diagram of a labeled link presenting the 3-manifold.

First we give a "banana division" of the base space S^3 with respect to a diagram D of a labeled link. Put D on the 2-sphere S^2. We can assume that the projection is connected on S^2 by a finite sequence of Reidemeister moves if necessary. Now taking the dual graph of D on this 2-sphere, we have a division of S^2 into a number of squares. Each square corresponds to a crossing of D as shown in Figure 23. Put the 2-sphere in the 3-sphere S^3, and take two points P and Q in the inner and the outer 3-balls of this 2-sphere respectively. Take a suspension of the dual graph with respect to P and Q, and we obtain a division of S^3 into thin 3-balls like bananas as illustrated in Figure 24. Each banana contains a crossing, and its boundary consists of four faces. This is the banana division of S^3 given by D.

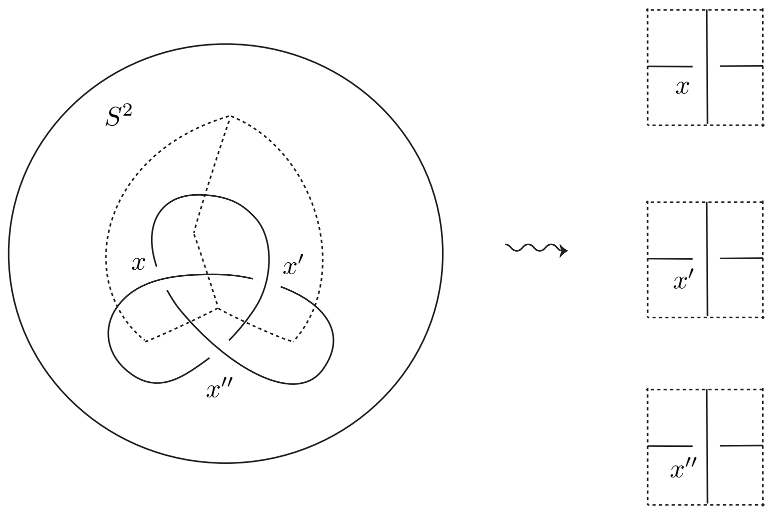


FIGURE 23. Divide S^2 into some squares by the dual graph of a link diagram

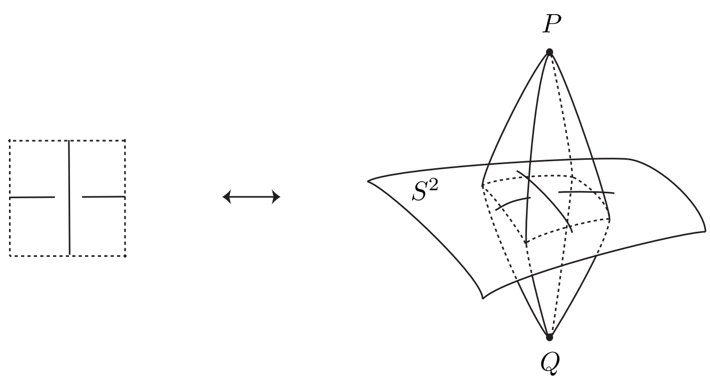


FIGURE 24. A banana contains a crossing of D

Let us see the branched covering spaces of such bananas branched along the two arcs with labels inside of each, in order to make a division of the branched covering space of the total base space S^3 branched along the whole diagram. The branched covering space of a banana containing a crossing of type 1 consists of a solid torus, and two 3-balls which are the copies of the base banana as depicted in Figure 25. On the other hand, the branched covering space of a banana of type 2 consists of a 3-ball and a copy of the base banana as illustrated in Figure 26, and the one of type 3 consists of two 3-balls, as illustrated in Figure 27. Since the base space S^3 is the union of such bananas, its branched covering space M is the union of the branched covering spaces of the bananas, that is, solid tori, two kinds of 3-balls and copies of the base bananas. They are pasted with their boundaries by the pasting rule of the boundary faces of the base bananas.

A triangulation on the branched covering space M will be obtained by giving triangulations on the branched covering spaces of the base bananas. Note that such triangulations should be compatible when we paste them along their boundaries. We give it in the following way. First we make a triangulation on a base banana

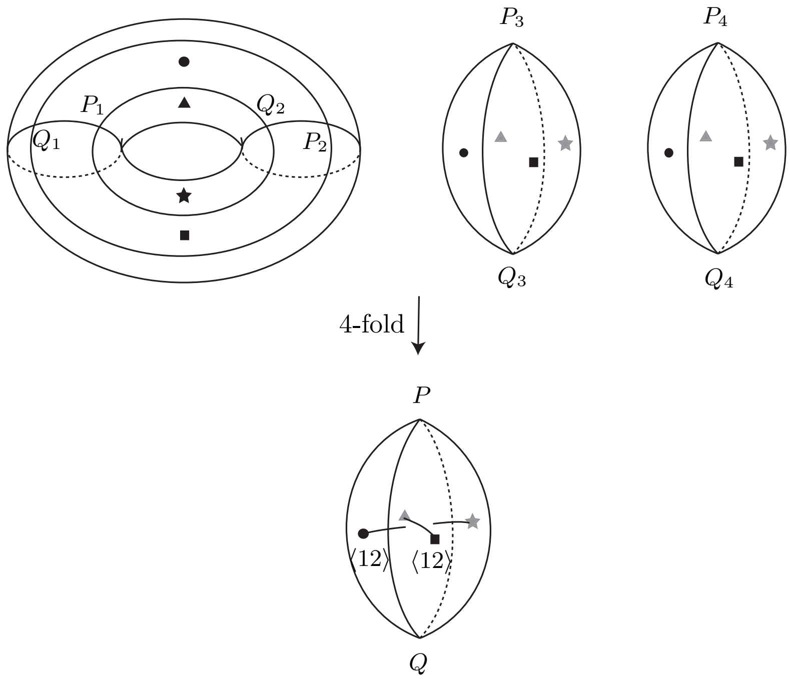


FIGURE 25. The branched covering space of a banana branched over a crossing of type 1 is a solid torus and two copies of the base banana

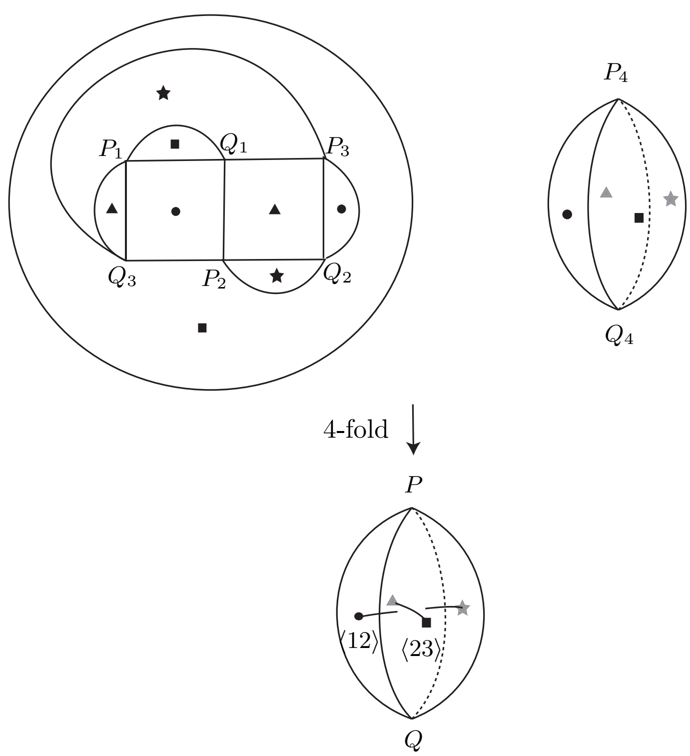


FIGURE 26. The branched covering space of a banana branched over a crossing of type 2 is a 3-ball and a copy of the base banana

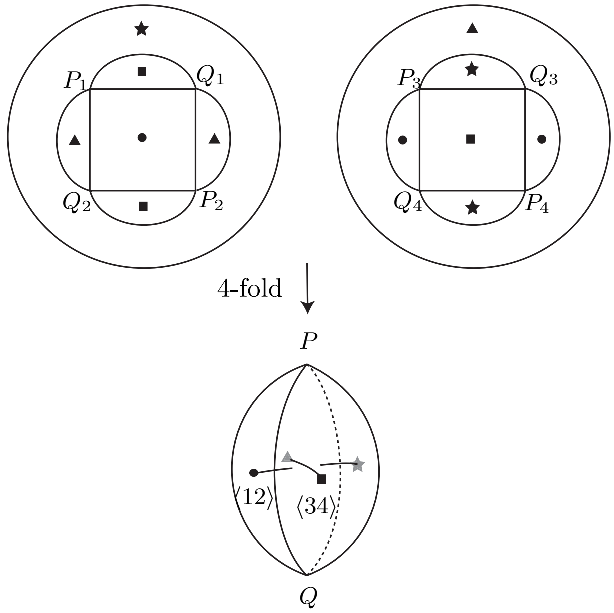


FIGURE 27. The branched covering space of a banana branched over a crossing of type 3 is two 3-balls

with nine tetrahedra as illustrated in Figure 28. Remark that the branch arcs are contained as edges of the triangulation. For a triangulation on a solid torus, which appears in the covering space of a base banana corresponding to a crossing of type 1, provide two copies of it, and give a pasting rule on the corresponding faces as illustrated in Figure 29. For triangulations on two kinds of 3-balls, which appear in the covering spaces at crossings of type 2 and 3, and a copy of banana upstairs, first we give triangulations on their boundaries using the one on the boundary of the base banana; see Figure 30. Then put a new interior vertex inside of each 3-balls, and take a cone to the boundary. Now we have a triangulation of the whole covering space M, of which 34 $(= 9\times 2+8\times 2)$ tetrahedra corresponds to a crossing of type 1, 32 $(= 24+8)$ tetrahedra corresponds to a crossing of type 2, and 32 $(= 16\times 2)$ tetrahedra corresponds to a crossing of type 3. We give an ordering to the vertices satisfying that $P_i < V < Q_j$ for all $i, j \in \{1,2,3,4\}$ and for any vertex V except for P_i and Q_j.

6.2. **Step 2.** In the previous subsection we obtained a triangulation T of a 3-manifold M from a diagram D of a labeled link presenting M, such that some tetrahedra of T corresponds to a crossing of D. In this subsection we give a coloring on this triangulation T of the Dijkgraaf-Witten invariant, from a coloring on D both in the same finite group G.

In Proposition 3.5, we saw the correspondence

$$\{\text{colorings on } D \text{ in } G\} \overset{|G|^3:1}{\longleftrightarrow} \{\text{representations } \pi_1(M)\to G\}$$

for a finite group G. On the other hand, noting the two expressions of the Dijkgraaf-Witten invariant described in section 4.1, we also have the following correspondence,

$$\{\text{colorings on } T \text{ in } G\} \overset{|G|^{N-1}:1}{\longleftrightarrow} \{\text{representations } \pi_1(M)\to G\},$$

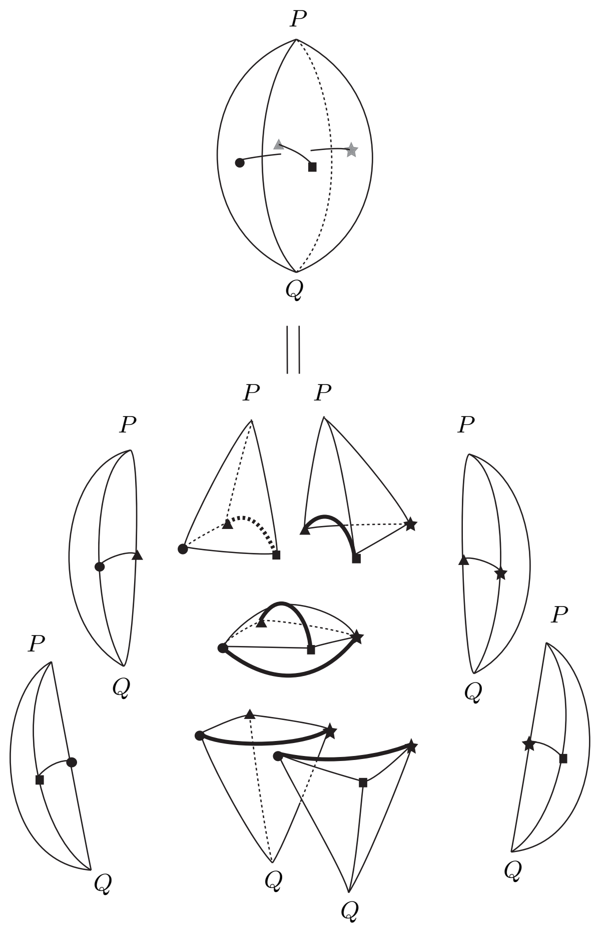


FIGURE 28. A triangulation of a base banana

where N is the number of the vertices of T. Hence a correspondence holds between the two sets of colorings,

$$\{\text{colorings on } D \text{ in } G\} \quad \overset{|G|^3:|G|^{N-1}}{\longleftrightarrow} \quad \{\text{colorings on } T \text{ in } G\},$$

coming from the same representations. We intend to give a coloring on T from a coloring on D in this correspondence, and in the process coloring on the regions associated with the switching map $\mathcal{R}$ defined in Section 4.2 is used effectively. A color on a region of D in G^4 stands for the colors on four edges of T in G, so as to make it possible to compute the product of the weights of the Dijkgraaf-Witten invariant for the tetrahedra corresponding to a crossing, from information only around the crossing.

We fix a coloring C on D in G, and a coloring R on the regions in G^4. Note that for a fixed coloring on D, a color in G^4 on a region determines the colors on the other regions uniquely. As we saw in Figure 24 in the previous subsection, a region corresponds to an edge of the banana division of the base space S^3, connecting vertices P and Q. This edge lifts to four oriented edges of the triangulation T, connecting P_i and Q_j for $i, j \in \{1, 2, 3, 4\}$; see Figures 25, 26 and 27. Take the i-th element of the color in G^4 on a region to be the color on the lifted edge of

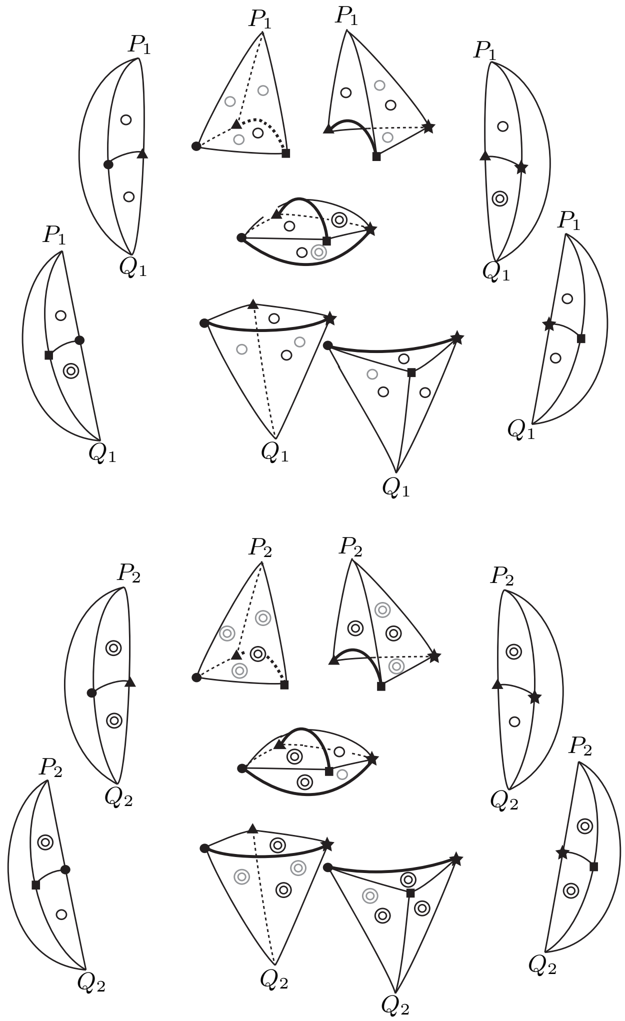


FIGURE 29. A triangulation of the solid torus in Figure 25. Provide two copies of the triangulation of the base banana in Figure 28. Then paste a one-circled face to the corresponding one-circled one, and a double-circled face to the double-circled one

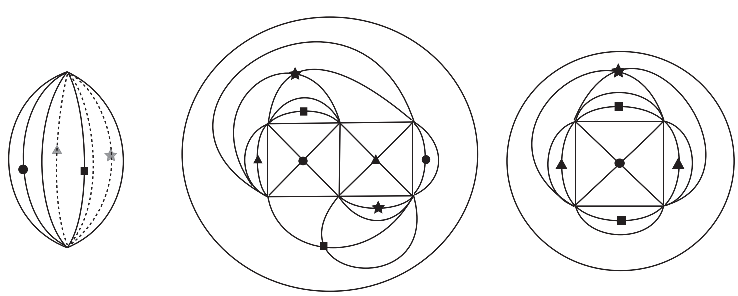


FIGURE 30. Triangulations of boundaries of a copy of banana upstairs (left), the 3-ball in Figure 26 (middle) and the 3-ball in Figure 27 (right). Put a new interior vertex inside of each, and take a cone to the boundary

T containing P_i. Except for these lifted edges, we have more edges of T to be colored. Among the uncolored edges, put the unit element $1_G \in G$ on the ones that connect the branch point labeled $\langle ij \rangle$ and P_* for $* \neq j$. Besides, choose an inner edge arbitrarily in each triangulation of two kinds of 3-balls and the copies of the bananas upstairs, and put also 1_G on them. Then the rest of the edges will be automatically colored by definition. Refer to Figure 31, 32 and 33 for the colors on edges of the covering spaces corresponding to a crossing of type 1, type 2 and type 3, respectively. We put 1_G on $N-8$ edges of T in all. Such a coloring is compatible in pasting of the boundaries of covering spaces of base bananas, and we can verify that it gives the same representation $\pi_1(M) \to G$ that C gives.

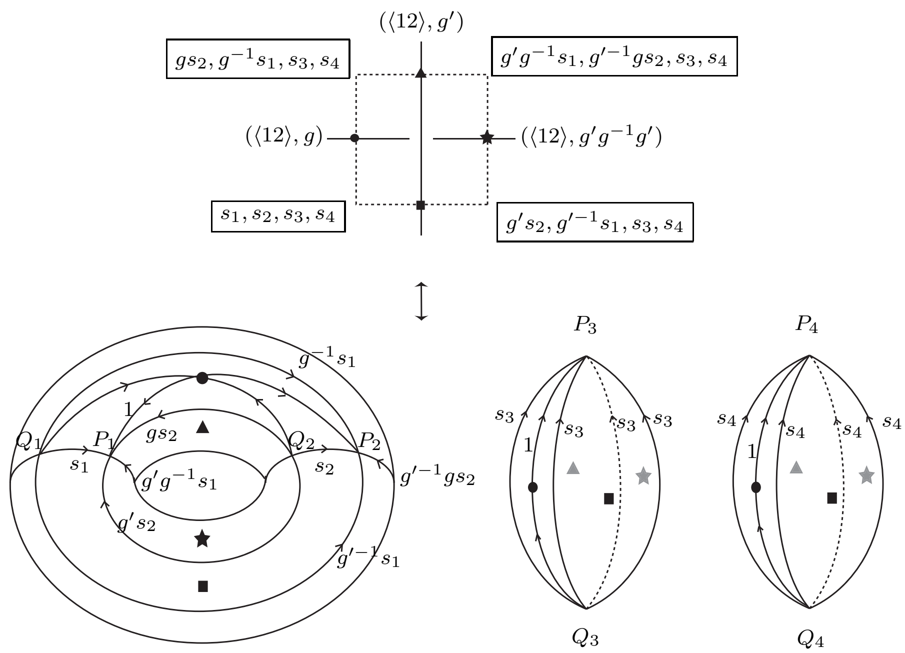


FIGURE 31. Colors on the regions around a crossing of type 1 (above) and the colors on the edges of the branched covering space (below). The other edges are automatically colored by putting 1_G on some edges

6.3. **Step 3.** The weight function $\mathcal{X}$, which is introduced in Section 4.2, is defined so that the identity holds,

$$X_\theta(x; C, R) = \prod_t W_\theta(t; C_T),$$

for a crossing x of D, a coloring C on D and a coloring on the regions R. Here X_θ denotes the weight at a crossing given by this weight function $\mathcal{X}$. The right hand side is concerning the weights of the Dijkgraaf-Witten invariant for the triangulation T given by D in Section 6.1. The product is taken over the tetrahedra of T corresponding to a crossing x, in each type of crossing, and C_T is the coloring on T given by using C and R as described in Section 6.2. The first formula in the definition of the weight function $\mathcal{X}$ in Section 4.2 expresses the product for the tetrahedra corresponding to a crossing of type 1, depicted in Figure 31. The

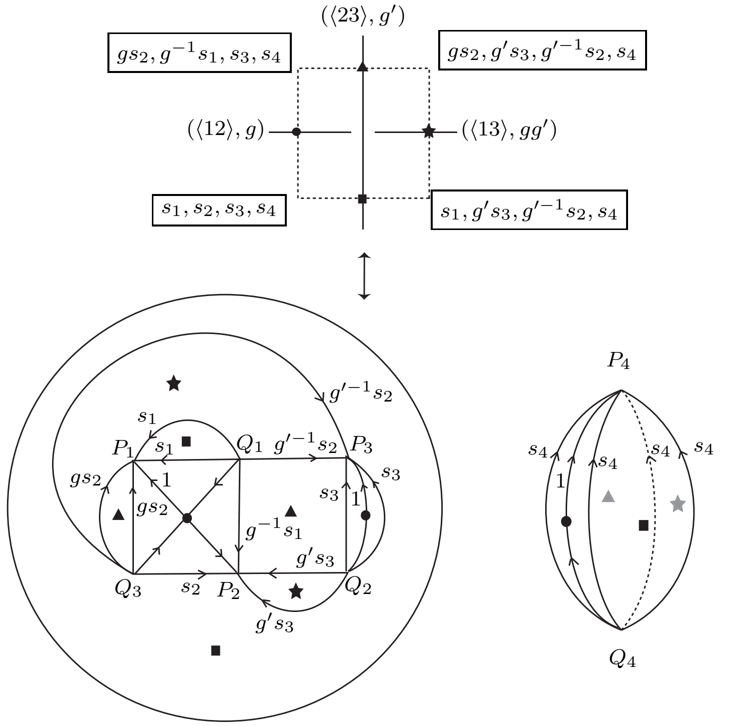


FIGURE 32. Colors around a crossing of type 2 (above), and the colors on the corresponding branched covering space (below)

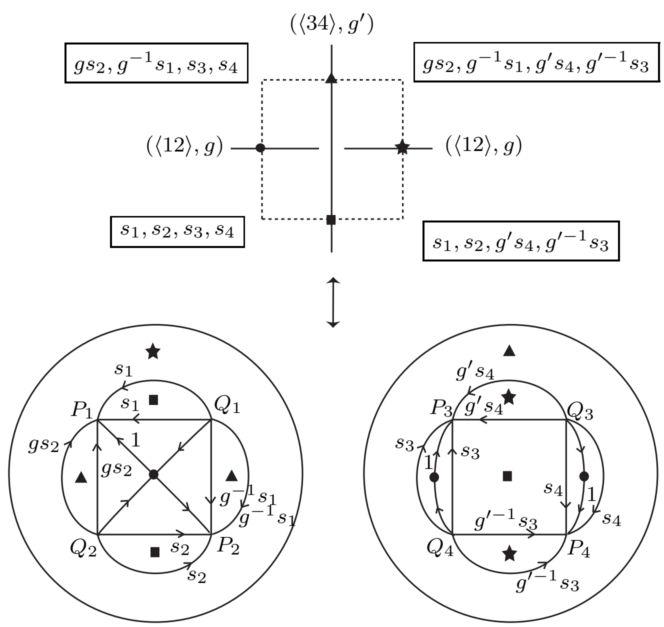


FIGURE 33. Colors around a crossing of type 3 (above), and the colors on the corresponding branched covering space (below).

next four formulas are the products for the tetrahedra corresponding to crossings of type 2, as depicted in Figure 32, in the four cases of the labels of the under- and over-arcs. The last formula is the product for the tetrahedra corresponding to a crossing of type 3, depicted in Figure 33. If we take the product of the weight $X_\theta(x; C, R)$ over all the crossings of D, then we have

$$\prod_{\text{all crossings of } D} X_\theta(x; C, R) = \prod_{\text{all tetrahedra of } T} W_\theta(t; C_T),$$

and it gives the contribution of the Dijkgraaf-Witten invariant for the coloring C_T.

6.4. **Proof of Theorem 4.2.** From the second expression of the Dijkgraaf-Witten invariant in Section 4.1, we see that two contributions $\prod_t W_\theta(t; C)$ and $\prod_t W_\theta(t; C')$, where both of the products are taken over all the tetrahedra of a triangulation of a 3-manifold, gives the same value if the two colorings C and C' in a finite group G correspond to the same representation $\pi_1(M) \to G$. Therefore by the equation in the last of the previous subsection, the contribution $\prod_x X_\theta(x; C, R)$, where the product is taken over all the crossings of a labeled diagram D, does not depend on the colorings on the regions R for a coloring C on D. We fix a coloring on the regions by giving $(1_G, 1_G, 1_G, 1_G) \in G^4$ on the unbounded region for each coloring C on D in G, and denote it by R_C. Now taking the state sum on a diagram D, the argument throughout the three steps stated from Section 6.1 to 6.3 leads the following identities,

$$\begin{aligned}
I_\theta(M) &= \sum_{C;\text{ all colorings}} \sum_{R;\text{ all colorings}} \prod_{x;\text{ all crossings}} X_\theta(x; C, R) \\
&= |G|^4 \cdot \sum_{C;\text{ all colorings}} \prod_{x;\text{ all crossings}} X_\theta(x; C, R_C) \\
&= |G|^4 \cdot \frac{|G|^3}{|G|^{N-1}} \cdot \sum_{C_T;\text{ all colorings}} \prod_{t;\text{ all tetrahedra}} W_\theta(t; C_T) \\
&= |G|^4 \cdot \frac{|G|^3}{|G|^{N-1}} \cdot |G|^N \cdot Z_\theta(M) \\
&= |G|^8 \cdot Z_\theta(M).
\end{aligned}$$

Problem 6.1. *Find a pair of a switching map $\mathcal{R}$ and a weight function $\mathcal{X}$, satisfying the conditions in Theorem 3.8, such that the consequent state sum $I(M)$ for a 3-manifold M gives a new topological invariant of M, except for the Dijkgraaf-Witten invariant.*

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