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Abstract

In [S1] we showed that the growth function $P_M(t)$ for an Artin monoid of finite type M is a rational function of the form $1/N_M(t)$ where $N_M(t)$ is a polynomial[1], and gave three conjectures on the denominator polynomial $N_M(t)$. In the present note, we remove this assumption on M by showing the result for any type M. Then we give renewed three conjectures on the denominator poynomial $N_M(t)$ for an indecomposable affine type M. The new conjectures are verified for all types $\tilde{A}_2, \tilde{A}_3, \tilde{B}_3, \tilde{C}_2, \tilde{C}_3$ and $\tilde{G}_2$ of rank 2, 3 and type $\tilde{D}_4$.

1 Growth function for an Artin monoid

We recall the definition of a *growth function* for an Artin monoid.

Let $M=(m_{ij})_{i,j\in I}$ be a Coxeter matrix ([B]). The Artin monoid G_M^+ ([B-S, §1.2]) associated with M (or, of type M) is a monoid generated by the letters a_i, $i \in I$ which are subordinate to the relation generated by

$$(1.1) \qquad a_i a_j a_i \cdots \ = \ a_j a_i a_j \cdots \qquad i, j \in I,$$

where both hand sides of (1.1) are words of alternating sequences of letters a_i and a_j of the same length $m_{ij} = m_{ji}$ with the initials a_i and a_j, respectively. More precisely, G_M^+ is the quotient of the free monoid generated by the letters a_i $(i \in I)$ where two words U and V in these letters are equivalent if there exists a sequence $U_0 := U, U_1, \cdots, U_m := V$ such that the word U_k $(k = 1, \cdots, m)$ is obtained by replacing a phrase in U_{k-1} of the form on LHS of (1.1) by RHS of (1.1) for some $i, j \in I$. We write by $U \underset{\bullet}{=} V$ if U and V are equivalent. The equivalence class (i.e. an element of G_M^+) of a word W is denoted by the same notation W. By definition equivalent words have the same length. We therefore define the degree homomorphism

$$(1.2) \qquad \deg \ : \ G_M^+ \longrightarrow \mathbb{Z}_{\geq 0}$$

by assigning to each equivalence class the length of any representative word.

The growth function $P_{G_M^+,I}(t)$ for the Artin monoid G_M^+ is defined by

$$(1.3) \qquad P_{G_M^+,I}(t) \ := \sum_{n\in\mathbb{Z}_{\geq 0}} \#\{W \in G_M^+ \mid \deg(W) \leq n\}\ t^n.$$

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[1] Here M is a Coxeter matrix [B]. We shall refer M as the type of the Coxeter group, Artin group, Artin monoid, growth function,..., etc. associated with M. In the present note, we shall call a Coxeter matrix M is of finite (resp. affine) type if the associated Coxeter group $\overline{G}_M$ ([B, Ch.IV §1]) is finite (resp. affine), that is, the associated bilinear form B_M [B] is positive (resp. semi-positive with rank 1 kernel), but which may or may not be indecomposable.

2 Spherical growth function $\dot{P}_{G_M^+,I}(t)$

The *spherical growth function of the monoid* G_M^+ of type M is defined by

$$\dot{P}_{G_M^+,I}(t) \ := \sum_{n\in\mathbb{Z}_{\geq 0}} \#(\deg^{-1}(n))\ t^n \tag{2.1}$$

so that one has the obvious equality $P_{G_M^+,I}(t) = \dot{P}_{G_M^+,I}(t)/(1-t)$.

The goal of this section is the following.

Theorem 2.1. *Let G_M^+ be the Artin monoid of any type M. Then, the spherical growth function of the monoid is given by the Taylor expansion of the rational function of the form*

$$\dot{P}_{G_M^+,I}(t) \ = \ \frac{1}{N_M(t)}, \tag{2.2}$$

where $N_M(t)$, called the denominator polynomial, is a polynomial given by

$$N_M(t) \ := \sum_{J\subset I} (-1)^{\#(J)}\ t^{\deg(\Delta_J)}. \tag{2.3}$$

Here the summation index J runs over subsets of I such that the restricted Coxeter matrix[2] *$M|_J$ is of finite type, and Δ_J is the fundamental element in G_M^+ associated with J (*[B-S, §5 Definition]*. See also* Lemma-Definition 1 *and* Remark 2.3 *of the present note).*

Proof. The proof is achieved by a recursion formula (2.9) on the coefficients of the growth function. For the proof of the formula, we use the method used to solve the word problem for the Artin monoid ([B-S, §6.1]), which we recall below. We first recall the fact that an Artin monoid is cancelative in the following sense ([B-S, Prop.2.3]).

Lemma 2.2. *Let $A, B, X, Y \in G_M^+$. If $AXB \underset{\bullet}{=} AYB$. Then $X \underset{\bullet}{=} Y$.*

A word U is said to be divisible (from the left) by a word V, and we write $V|U$, if there exists a word W such that $U \underset{\bullet}{=} VW$. Since $V \underset{\bullet}{=} V'$, $U \underset{\bullet}{=} U'$ and $V|U$ implies $V'|U'$, we use the notation "$|$" of divisibility also between elements of the monoid G_M^+. We have the following basic concepts ([B-S, §5 Definition and §6.1]).

Lemma-Definition 1. *Let M be a Coxeter matrix of any type, and let $J \subset I$ be a subset of I such that $M|_J$ is of finite type (though not necessarily indecomposable). Then, there exists a unique element $\Delta_J \in G_M^+$, called the* fundamental element*, such that i) $a_i|\Delta_J$ for all $i \in J$, and ii) if $W \in G_M^+$ and $a_i|W$ for all $i \in J$, then $\Delta_J|W$.*

2. *To an element $W \in G_M^+$, we associate the subset*

$$I(W) \ := \ \{i \in I \ \mid \ a_i|W\}. \tag{2.4}$$

of I. Then the restricted Coxeter matrix $M|_{I(W)}$ is of finite type for any $W \in G_M^+$.

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[2] For a Coxeter matrix $M = (m_{ij})_{i,j\in I}$ and a subset J of I, we define the restricted Coxeter matrix by $M|_J := (m_{ij})_{i,j\in J}$ which, obviously, is also a Coxeter matrix.

Proof. **2.** This follows from the fact that the existence of Δ_J holds under an assumption weaker than $M|_J$ being of finite type, namely that there exists a common multiple of a_j for $j \in J$ in G_M^+ (see [B-S, Prop. (4.1)]). $\Box$

By definition (2.4), one has $\Delta_{I(W)}|W$ and if $\Delta_J|W$ then $J \subset I(W)$.

We return to the proof of Theorem.
For $n \in \mathbb{Z}_{\geq 0}$ and for any subset $J \subset I$, put

$$G_n^+ \ := \ \{W \in G_M^+ \mid \deg(W) = n\} \tag{2.5}$$

$$G_{n,J}^+ \ := \ \{W \in G_n^+ \mid I(W) = J\}. \tag{2.6}$$

We note that $G_{n,J}^+ = \emptyset$ if $M|_J$ is not of finite type. By definition, we have the disjoint decomposition

$$G_n^+ \ = \ \amalg_{J \subset I} \ G_{n,J}^+, \tag{2.7}$$

where J runs over all subsets of I. Note that $G_{n,\emptyset}^+ = \emptyset$ if $n > 0$ but $G_{0,\emptyset}^+ = \{\emptyset\} \neq \emptyset$. For any subset J of I the union $\amalg_{J \subset K \subset I} G_{n,K}^+$, where the index K runs over all subsets of I containing J, is equal to the subset of G_n^+ consisting of elements divisible by a_j for $j \in J$. That is, one has

$$\amalg_{J \subset K \subset I} G_{n,K}^+ \ = \ \begin{cases} \Delta_J \cdot G_{n-\deg(\Delta_J)}^+ & \text{if } M|_J \text{ is of finite type,} \\ \emptyset & \text{if } M|_J \text{ is not of finite type.} \end{cases}$$

Thus, if $M|_J$ is of finite type, the cancelativity Lemma 2.3 implies the multiplication map of Δ_J is injective and we find a bijection $G_{n-\deg(\Delta_J)}^+ \simeq \amalg_{J \subset K \subset I} G_{n,K}^+$. This implies the equality

$$\#(G_{n-\deg(\Delta_J)}^+) \ = \ \textstyle\sum_{J \subset K \subset I} \#(G_{n,K}^+). \tag{2.8}$$

If $M|_J$ is not of finite type, still the formula (2.8) holds formally, by putting $\deg(\Delta_J) := \infty$ and $G_{-\infty}^+ := \emptyset$, i.e. $\#(G_{n-\deg(\Delta_J)}^+) := 0$. Then, for $n > 0$, using (2.8) we get the recursion relation

$$\sum_{J \subset I} (-1)^{\#(J)} \#(G_{n-\deg(\Delta_J)}^+) \ = \ 0, \tag{2.9}$$

where the index J may run either over all subsets of I, or over only those subsets J such that the restricted Coxeter matrix $M|_J$ is of finite type. Together with $\#(G_0^+) = 1$ for $n = 0$, this is equivalent to the formula:

$$\dot{P}_{G_M^+, I}(t) N_M(t) \ = \ 1. \tag{2.10}$$

This completes the proof of Theorem 2.1. $\Box$

Remark 2.3. We have the equality ([B-S, §5.7]):
$\deg(\Delta_J) = \#\{\text{reflections in } \overline{G}_{M|_J}\} = \text{the length of the longest element of } \overline{G}_{M|_J}$.

By the definition (2.3) of the denominator polynomial, one has

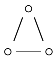
$$N_M(1) = \textstyle\sum_{\substack{J \subset I, M|_J \text{is} \\ \text{of finite type}}} (-1)^{\#J}$$

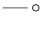
This, in particular, implies

i) $N_M(t)$ has the factor $1-t$ if M contains a component of finite type,

ii) $N_M(1) = (-1)^l$ if M is of indecomposable affine type of rank l (that is, by definition, M is indecomposable and affine such that $\#(I)=l+1$).[3]
We refer to [S1] and http://www.kurims.kyoto-u.ac.jp/~saito/FFST/ for examples of finite type. (The author express his gratitude to S. Tsuchioka for preparing this page). Here we give a few more examples of affine type.

Example. There are three types of indecomposable affine Coxeter matrices of rank 2. In the following, for each type, we associate the Coxeter diagram Γ_M and the denominator polynomial $N_M(t)$.

1. $\tilde{A}_2$ $\Gamma_{\tilde{A}_2} =$  . $N_{\tilde{A}_2}(t) = 1 - 3t + 3t^3$.
2. $\tilde{C}_2$ $\Gamma_{\tilde{C}_2} =$  . $N_{\tilde{C}_2}(t) = 1 - 3t + t^2 + 2t^4$.

3. $\tilde{G}_2$. $\Gamma_{\tilde{G}_2} =$  . $N_{\tilde{G}_2}(t) = 1 - 3t + t^2 + t^3 + t^6$.

3 A bound on the zeroes of the denominator polynomial $N_M(t)$ of affine type

The following lemma gives a numerical bound on the zeroes of the denominator polynomials for indecomposable affine type.

Lemma 3.1. *Let M be a Coxeter matrix of indecomposable affine type of rank l. Then, all the roots of $N_M(t) = 0$ are contained in the open disc of radius r centered at the origin, where r is give by*

$$r := \Big(\frac{2^{l+1} - s - 1}{s}\Big)^{1/(\deg(\Delta_{M|_{I\setminus\{v\}}})-d)}, \tag{3.1}$$

where $\deg(\Delta_{M|_{I\setminus\{v\}}}), d, s$ *are invariants of M explained in the proof.*

Proof. In the affine Coxeter graph Γ_M (whose vertex set is identified with I, and hence $\#(\Gamma_M) = \#(I) = l+1$) there is a vertex v, called *special* [B, p.87], such that $\Gamma_M \setminus \{v\}$ is the Coxeter graph of the finite Coxeter group isomorphic to the radical quotient of the affine Coxeter group $\overline{G}_M$. The number, denoted s, of special vertices in Γ_M of types $\tilde{A}_l, \tilde{B}_l, \tilde{C}_l, \tilde{D}_l, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2$ are $l+1$, 2, 2, 4, 3, 2, 1, 1, 1, respectively.

Noting that $N(t) := (-1)^l s \cdot t^{\deg(\Delta_{M|_{I\setminus\{v\}}})}$, for v a special vertex, is the leading term of $N_M(t)$, one has $|N_M(t) - N(t)| \le (2^{l+1} - s - 1)|t|^d$ for $t \in \mathbb{C}$ with $|t| > 1$ (strict inequality holds except for the type $\tilde{A}_1$), where

$$d := \max\{\deg(\Delta_J) \mid J \subset I \ s.t. \ I \setminus J \text{ is not a single special vertex}\}.$$

Hence $|N_M(t) - N(t)|/|N(t)| \le \frac{2^{l+1}-s-1}{s}|t|^{d-\deg(\Delta_{M|_{I\setminus\{v\}}})}$. If $r \in \mathbb{R}_{>1}$ satisfies an inequality $\frac{2^{l+1}-s-1}{s}r^{d-\deg(\Delta_{M|_{I\setminus\{v\}}})} \le 1$ then by Rouché's theorem the number of roots of $N_M(t)=0$ in the disc of radius r is equal to that of $N(t)=0$, where $N(t) = 0$ has zeroes only at 0 of multiplicity $\deg(N(t)) = \deg(N_W(t))$. That is, all the roots of $N_M(t)=0$ are in the disc $\{|t| < r\}$ for the radius r given in (3.1). □

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[3] The discrepancy between the rank l and the number $\#(I) = l+1$ for a Coxeter matrix M of indecomposable affine type comes from the fact that the associated affine Coxeter group acts on a semi-positive lattice of signature $(l, 1, 0)$. Put $N_M^k(t) := \sum_{J \subset I, \#(J) \le k} (-1)^{\#(J)} t^{\deg(\Delta_J)}$ for $0 \le k \le l$. Then, $N_M(t) = N_M^l(t)$, and one has $N_M^k(1) = (-1)^k C_k^l$ for $0 \le k \le l$.

4 Conjectures on the zeroes of the denominator polynomial $N_M(t)$ of affine type

Some structures and examples discussed at the end of §2 lead us to the following three conjectures on the distribution of zeroes for the denominator polynomial $N_M(t)$ of indecomposable finite or affine type.

Conjecture 1. i) The polynomial $\tilde{N}_M(t) := N_M(t)/(1-t)$ is irreducible over $\mathbb{Z}$ for any indecomposable finite type M. ii) The polynomial $N_M(t)$ is irreducible over $\mathbb{Z}$ for any indecomposable affine type M.

Conjecture 2. i) There are $l-1$ pairwise distinct real roots of $N_M(t)=0$ on the interval $(0,1)$ for any indecomposable finite type M of rank l. ii) There are l pairwise distinct real roots of $N_M(t)=0$ on the interval $(0,1)$ for any indecomposable affine type M of rank l.

Conjecture 3. Let r_M be the smallest of the real roots on the interval $(0,1)$. Then, the absolute values of the other roots of $N_M(t)=0$ are strictly larger than r_W.

Conjectures on the denominator polynomials of finite type were already stated in [S1] and verified by direct computer calculations for the types A_l, B_l, C_l, D_l $(l \leq 30)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4 and $I_2(p)$ $(p \in \mathbb{Z}_{\geq 3})$ by M. Fuchiwaki, S. Tsuchioka and others (see http://www.kurims.kyoto-u.ac.jp/∼saito/FFST/). A theoretical work on these conjectures in this case is in progress by S. Yasuda.

Conjectures on the denominator polynomial of affine type are affirmative for the three types $\tilde{A}_2$, $\tilde{C}_2$ and $\tilde{G}_2$ from the explicit expressions given in §2 Example and for a few further types $\tilde{A}_3$, $\tilde{B}_3$, $\tilde{C}_3$ and $\tilde{D}_4$.

Remark 4.1. In Conjecture 3, the fact that r_M is at most the absolute value of any other root of $N_W(t)=0$ is trivially true, since r_M is equal to the radius of convergence of the series $P_M(t)$ of non-negative real coefficients. Therefore, the true question here is whether there are no other roots of $N_W(t)$=0 whose absolute value is equal to r_W. This is equivalent that whether the sequence $\{\#(G^+_{n-1})/\#(G^+_n)\}_{n\in\mathbb{Z}_{>0}}$ converges to the single value r_M. These questions were motivated by a study of the author on limit functions associated with the monoid G^+_M (see [S2, §11], [S1, §5]).

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