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Abstract

In [W] and [TW] the exact WKB theoretic structure of the most degenerate third Painlevé equation of type ($D8$) was investigated. In this paper we announce a result that this degenerate third Painlevé equation of type ($D8$) plays a special role of the canonical equation near a simple pole of Painlevé equations. We also explain an outline of its proof after reviewing several relevant results of [W] and [TW].

1 Introduction

In this paper we consider the degenerate third Painlevé equation from the viewpoint of exact WKB analysis. Since the work of Sakai [S] on geometrical classification of the space of initial conditions of Painlevé equations, it is considered to be natural to distinguish the degenerate third Painlevé equations of type ($D7$) and ($D8$) from the generic third Painlevé equation (P_{III}). Several important properties such as τ-functions, irreducibility etc. of these degenerate third Painlevé equations were studied in [OKSO] and the asymptotics of their solutions was also investigated in [KV]. The purpose of

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this paper is to discuss the most degenerate third Painlevé equation of type $(D8)$ with a large parameter η of the form

$$(P_{\mathrm{III}'(D8)}) \qquad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\left(\frac{d\lambda}{dt}\right) + \eta^2\left(\frac{\lambda^2}{t^2} - \frac{1}{t}\right),$$

from the viewpoint of exact WKB analysis and clarify its role in exact WKB analysis.

Let us explain the background of this paper. In exact WKB analysis the most important role is played by turning points. In the case of second order linear ordinary differential equations, every equation can be transformed to the Airy equation near a simple turning point and the local behavior of its Borel resummed WKB solution is described by that of the Airy equation (see, e.g., [KT3, Chapter 2]). Furthermore, Koike showed in [K] that a simple pole, i.e., a degenerate regular singular point, should be considered also as a kind of turning points and that the canonical equation near a simple pole is given by the Whittaker equation. On the other hand, in the case of Painlevé equations, similar transformation theory near a simple turning point was established in [KT1], [AKT], [KT2]; the main result of this series of papers is that every formal Painlevé transcendent (a formal 2-parameter instanton-type solution of a Painlevé equation) is transformed to that of the first Painlevé equation (P_{I}) near its simple turning point. Then the following question naturally arises: *What can we say about a simple pole of Painlevé equations?* The answer to this question is a simple and impressive one: *The above equation* $(P_{\mathrm{III}'(D8)})$ *gives the canonical equation near a simple pole of Painlevé equations.* In this paper we explain the core part of the transformation theory to $(P_{\mathrm{III}'(D8)})$ near a simple pole of Painlevé equations after reviewing the exact WKB theoretic structure of $(P_{\mathrm{III}'(D8)})$ investigated in [W] and [TW].

The plan of the paper is as follows: In Section 2 we recall several relevant results of [W] and [TW] on the exact WKB theoretic structure of $(P_{\mathrm{III}'(D8)})$. The most important ones are the relationship between the Stokes geometry of $(P_{\mathrm{III}'(D8)})$ and that of the underlying linear differential equation (Proposition 2.2) and the connection formula for its instanton-type solutions (Corollary 2.4). Then in Section 3 we explain the core part of the transformation theory to $(P_{\mathrm{III}'(D8)})$ near a simple pole of Painlevé equations. The details of the proof of main theorems will be discussed in our forthcoming paper.

2 Exact WKB theoretic structure of $(P_{\mathrm{III}'(D8)})$

Exact WKB theoretic structure of $(P_{\mathrm{III}'(D8)})$ was investigated in [W] and [TW]. In this section we review the results of [W] and [TW].

First we introduce two classes of formal solutions of $(P_{\mathrm{III}'(D8)})$. It can be readily confirmed that $(P_{\mathrm{III}'(D8)})$, or the Hamiltonian system

$$(H_{\mathrm{III}'(D8)}) \qquad \begin{cases} \dfrac{d\lambda}{dt} = \eta \dfrac{\partial K}{\partial \nu} \\ \dfrac{d\nu}{dt} = -\eta \dfrac{\partial K}{\partial \lambda} \end{cases}$$

with the Hamiltonian

$$K = \frac{1}{t}\left[\lambda^2\nu^2 - \left(\frac{\lambda}{2} + \frac{t}{2\lambda}\right) - \eta^{-1}\lambda\nu\right] \tag{2.1}$$

which is equivalent to $(P_{\mathrm{III}'(D8)})$, has the following particular solution

$$(\lambda,\ \nu) = \left(\sqrt{t},\ \eta^{-1}\frac{3}{4\sqrt{t}}\right). \tag{2.2}$$

As this solution contains no free parameters, it is called a "0-parameter solution". In what follows we use the notation $(\widehat{\lambda}, \widehat{\nu})$ to denote the 0-parameter solution (2.2). In addition to $(\widehat{\lambda}, \widehat{\nu})$, by employing the multiple-scale analysis similarly to the case of (P_J) ([AKT, Section 1]), we can construct the following formal solution $\lambda(t;\alpha,\beta)$ of $(P_{\mathrm{III}'(D8)})$ with 2 free parameters (α,β) called a "(2-parameter) instanton-type solution":

$$\lambda(t;\alpha,\beta) = \sqrt{t} + \eta^{-1/2}\sum_{n=0}^{\infty}\eta^{-n/2}L_{n/2}(t,\eta), \tag{2.3}$$

where $L_0 = L_0(t,\eta)$ is given by

$$L_0 = 2^{1/4}t^{3/8}\left(\alpha(2^7t^{1/2}\eta^2)^{\alpha\beta}e^{\phi\eta} + \beta(2^7t^{1/2}\eta^2)^{-\alpha\beta}e^{-\phi\eta}\right) \tag{2.4}$$

with $\phi = 4\sqrt{2}t^{1/4}$ and $L_{n/2} = L_{n/2}(t,\eta)$ $(n \geq 1)$ is of the form

$$L_{n/2} = t^{(3-n)/8}\sum_{k=0}^{n+1} c^{(n/2)}_{n+1-2k}\left((2^7t^{1/2}\eta^2)^{\alpha\beta}e^{\phi\eta}\right)^{n+1-2k} \tag{2.5}$$

with $c_l^{(n/2)}$ being constants depending only on (α, β). As we will see below, the Stokes phenomenon that occurs on a Stokes curve of $(P_{\mathrm{III}'(D8)})$ is explicitly described in terms of these instanton-type solutions.

Remark 1. In this paper, for the sake of convenience in developing the transformation theory in Section 3 below, we use a pair of canonical variables (λ, ν) that is different from the variables (q, p) used in [W] and [TW]. They are related by the formula $(\lambda, \nu) = (q, p + \eta^{-1}q^{-1})$. As its consequence, the expressions (2.1) and (2.2) of the Hamiltonian and the 0-parameter solution become slightly different from those in [W] and [TW] in their appearance, but they are essentially the same. For the same reason we adopt a new scaling of the free parameters (α, β) in this paper and consequently the expression (2.3) $\sim$ (2.5) of an instanton-type solution becomes different from that in [W] and [TW], although they exactly coincide.

Next we define a turning point and a Stokes curve of $(P_{\mathrm{III}'(D8)})$. In the same manner as in the case of the usual six kinds of Painlevé equations (P_J) $(J = \mathrm{I}, \ldots, \mathrm{VI})$ they are defined by considering the Frechét derivative (or the linearized equation) of $(P_{\mathrm{III}'(D8)})$ at its 0-parameter solution $(\widehat{\lambda}, \widehat{\nu})$ denoted by $(\Delta P_{\mathrm{III}'(D8)})$:

$$(\Delta P_{\mathrm{III}'(D8)}) \qquad \frac{d^2}{dt^2}(\Delta\lambda) = \eta^2\left(\frac{2}{t^{3/2}} - \eta^{-2}\frac{1}{4t^2}\right)\Delta\lambda.$$

Definition 2.1. A turning point (resp., a Stokes curve) of $(P_{\mathrm{III}'(D8)})$ is, by definition, a turning point (resp., a Stokes curve) of $(\Delta P_{\mathrm{III}'(D8)})$. To be more specific, a turning point of $(P_{\mathrm{III}'(D8)})$ consists of one point $t = 0$ and a Stokes curve of $(P_{\mathrm{III}'(D8)})$, which is defined by the relation

$$\mathrm{Im}\int_0^t \sqrt{\frac{2}{t^{3/2}}}dt = \mathrm{Im}\ (4\sqrt{2}t^{1/4}) = 0, \tag{2.6}$$

is explicitly given by

$$\{t \in \mathbb{C} \mid \arg\sqrt{t} = 2n\pi\ (n \in \mathbb{Z})\} \tag{2.7}$$

(i.e., the positive real axis).

Note that a change of variables $(t, \Delta\lambda) = (\tilde{t}^2, \tilde{t}^{1/2}\widetilde{\Delta\lambda})$ transforms the Frechét derivative $(\Delta P_{\mathrm{III}'(\mathrm{D8})})$ into

$$\frac{d^2}{d\tilde{t}^2}\widetilde{\Delta\lambda} = \eta^2\left(\frac{8}{\tilde{t}} - \eta^{-2}\frac{1}{4\tilde{t}^2}\right)\widetilde{\Delta\lambda}, \tag{2.8}$$

which has a simple pole at $\tilde{t} = 0$ in the sense of [K]. Thus $\tilde{t} = 0$, i.e., the turning point $t = 0$ of $(P_{\mathrm{III}'(D8)})$ can be considered as a nonlinear analogue of a simple pole of second-order linear ordinary differential equations discussed in [K].

Now an important problem in exact WKB analysis of $(P_{\mathrm{III}'(D8)})$ is to seek for an explicit connection formula that describes the Stokes phenomenon for instanton-type solutions on its Stokes curve defined above, i.e., the positive real axis. For that purpose, as in the case of the first Painlevé equation (P_{I}) (cf. [T2]), we make full use of the well-known relationship of $(P_{\mathrm{III}'(D8)})$ with the theory of isomonodromic deformations of linear differential equations (cf. [OKSO, Section 3]) formulated as follows: Let $(SL_{\mathrm{III}'(D8)})$ and $(D_{\mathrm{III}'(D8)})$ denote the following linear differential equations, respectively.

$$(SL_{\mathrm{III}'(D8)}) \qquad \left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q\right)\psi = 0,$$

$$(D_{\mathrm{III}'(D8)}) \qquad \frac{\partial \psi}{\partial t} = A\frac{\partial \psi}{\partial x} - \frac{1}{2}\frac{\partial A}{\partial x}\psi,$$

where

$$Q = \frac{tK}{x^2} + \frac{1}{2x} + \frac{t}{2x^3} - \eta^{-1}\frac{\lambda\nu}{x(x-\lambda)} + \eta^{-2}\frac{3}{4(x-\lambda)^2}, \tag{2.9}$$

$$A = \frac{x\lambda}{t(x-\lambda)}, \tag{2.10}$$

and K appearing in (2.9) is the Hamiltonian defined by (2.1). Then the compatibility condition of $(SL_{\mathrm{III}'(D8)})$ and $(D_{\mathrm{III}'(D8)})$ is exactly described by the Hamiltonian system $(H_{\mathrm{III}'(D8)})$. Otherwise stated, the Hamiltonian system $(H_{\mathrm{III}'(D8)})$ or the Painlevé equation $(P_{\mathrm{III}'(D8)})$ governs the isomonodromic deformation of $(SL_{\mathrm{III}'(D8)})$ in the sense of [JMU]. In particular, if a solution of $(H_{\mathrm{III}'(D8)})$ or $(P_{\mathrm{III}'(D8)})$ is substituted into the coefficients of $(SL_{\mathrm{III}'(D8)})$, the monodromy data of $(SL_{\mathrm{III}'(D8)})$ are preserved (i.e., they are not depending on the deformation parameter t). To seek for the connection formula of $(P_{\mathrm{III}'(D8)})$, we substitute an instanton-type solution (2.3) $\sim$ (2.5) into the coefficients of $(SL_{\mathrm{III}'(D8)})$ and compute its monodromy data by applying the exact WKB analysis to $(SL_{\mathrm{III}'(D8)})$. A key proposition in determining the connection formula is the following

Proposition 2.2. *Suppose that an instanton-type solution of* $(H_{\mathrm{III}'(D8)})$ *is substituted into the coefficients of* $(SL_{\mathrm{III}'(D8)})$*. Then the following hold:*
(i) *The top order term (with respect to* η^{-1}*)* Q_0 *of the potential* Q *of* $(SL_{\mathrm{III}'(D8)})$ *can be factorized as*

$$Q_0 = \frac{(x-\sqrt{t})^2}{2x^3}. \tag{2.11}$$

Hence $(SL_{\mathrm{III}'(D8)})$ *has a unique double turning point at* $x=\sqrt{t}$*.*
(ii) *When and only when* t *lies on a Stokes curve* (2.7) *of* $(P_{\mathrm{III}'(D8)})$*, there exists a Stokes curve of* $(SL_{\mathrm{III}'(D8)})$ *that starts from* $\sqrt{t}$*, encircles* $x=0$ *and returns to* $\sqrt{t}$ *(cf.* Fig.1, (ii)*).*

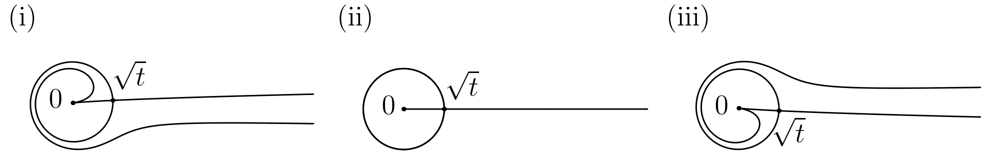

Figure 1: Stokes curves of $(SL_{\mathrm{III}'(D8)})$ in the case of (i) $\arg\sqrt{t} > 0$, (ii) $\arg\sqrt{t} = 0$, and (iii) $\arg\sqrt{t} < 0$.

Proposition 2.2, (ii) implies that the configuration of Stokes curves of $(SL_{\mathrm{III}'(D8)})$ when $\arg\sqrt{t} > 0$ is different from the configuration when $\arg\sqrt{t} < 0$ (cf. Fig.1). Hence, if we apply the exact WKB analysis for linear ordinary differential equations to $(SL_{\mathrm{III}'(D8)})$ to compute its monodromy data, the concrete expressions of monodromy data thus obtained become different (as functions of the parameters (α,β)) according as t belongs to the region $\Omega_+ = \{t \mid \arg\sqrt{t} > 0\}$ or $\Omega_- = \{t \mid \arg\sqrt{t} < 0\}$ since the computation of monodromy data through the exact WKB analysis heavily depends on the configuration of Stokes curves. On the other hand, the isomonodromic property requires the monodromy data to be unchanged. Consequently, after crossing the Stokes curve $\arg\sqrt{t} = 0$, an instanton-type solution $\lambda(t;\alpha,\beta)$ of $(P_{\mathrm{III}'(D8)})$ with given parameters (α,β) in the region Ω_+ should be analytically continued to another instanton-type solution $\lambda(t;\alpha',\beta')$ with different parameters (α',β') in Ω_- so that the monodromy data of $(SL_{\mathrm{III}'(D8)})$ in Ω_- may coincide with those in Ω_+. This is the Stokes phenomenon for instanton-type solutions of $(P_{\mathrm{III}'(D8)})$ that occurs on its Stokes curve.

In the case of $(P_{\mathrm{III}'(D8)})$, the underlying linear equation $(SL_{\mathrm{III}'(D8)})$ has irregular singularities at $x = 0$ and $x = \infty$ with Poincaré rank $1/2$ and its monodromy data consist of the following three matrices:

$$S_0 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad S_\infty = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \quad M = \begin{pmatrix} c & d \\ e & f \end{pmatrix}, \tag{2.12}$$

where S_0 (resp., S_∞) is the Stokes matrix around the irregular singular point $x = 0$ (resp., $x = \infty$) and M designates the connection matrix between $x = 0$ and $x = \infty$. Note that among these monodromy data we can take, for example, $(m_1, m_2) = (a, d)$ as a basis for them.

The computation of the monodromy data (m_1, m_2) through the application of exact WKB analysis was explicitly done in [W] and [TW]. The result is given by the following:

Theorem 2.3. *Suppose that an instanton-type solution of $(H_{\mathrm{III}'(D8)})$ is substituted into the coefficients of $(SL_{\mathrm{III}'(D8)})$. Then the monodromy data (m_1, m_2) of $(SL_{\mathrm{III}'(D8)})$ can be explicitly computed as follows:*

(*In the region* Ω_+, *i.e., when* $\arg \sqrt{t} > 0$)

$$\begin{aligned} m_1 &= -2\sqrt{\pi}\left(2^{-E/4}\frac{i\beta}{\Gamma(-\frac{E}{4}+1)} + 2^{E/4}e^{-i\pi E/4}\frac{\alpha}{\Gamma(\frac{E}{4}+1)}\right), \\ m_2 &= 2\sqrt{\pi}\ 2^{E/4}\frac{\alpha}{\Gamma(\frac{E}{4}+1)}. \end{aligned} \tag{2.13}$$

(*In the region* Ω_-, *i.e., when* $\arg \sqrt{t} < 0$)

$$\begin{aligned} m_1 &= -2\sqrt{\pi}\left(2^{-E/4}\frac{i\beta}{\Gamma(-\frac{E}{4}+1)} - 2^{E/4}e^{i\pi E/4}\frac{\alpha}{\Gamma(\frac{E}{4}+1)}\right), \\ m_2 &= 2\sqrt{\pi}\ 2^{E/4}\frac{\alpha}{\Gamma(\frac{E}{4}+1)}. \end{aligned} \tag{2.14}$$

Here E designates $-8\alpha\beta$.

For the details of the computation see [TW, Section 5]. (Note that the above formulas for the monodromy data are slightly different from those of [TW] due to the adoption of a new scaling of the parameters (α, β). Cf. Remark 1.)

As an corollary we thus obtain the following connection formula for instanton-type solutions of $(P_{\mathrm{III}'(D8)})$ on its Stokes curve $\arg \sqrt{t} = 0$.

Corollary 2.4. *Let $\lambda(t;\alpha,\beta)$ and $\lambda(t;\alpha',\beta')$ be instanton-type solutions of $(P_{\mathrm{III}'(D8)})$ in the region Ω_+ and Ω_-, respectively. If $\lambda(t;\alpha',\beta')$ is the analytic continuation of $\lambda(t;\alpha,\beta)$ across the Stokes curve $\arg\sqrt{t}=0$, then the following connection formula should hold:*

$$
\begin{aligned}
2^{-E/4}\frac{i\beta}{\Gamma(-\frac{E}{4}+1)} &+ 2^{E/4}e^{-i\pi E/4}\frac{\alpha}{\Gamma(\frac{E}{4}+1)} \\
&= 2^{-E'/4}\frac{i\beta'}{\Gamma(-\frac{E'}{4}+1)} - 2^{E'/4}e^{i\pi E'/4}\frac{\alpha'}{\Gamma(\frac{E'}{4}+1)}, \\
2^{E/4}\frac{\alpha}{\Gamma(\frac{E}{4}+1)} &= 2^{E'/4}\frac{\alpha'}{\Gamma(\frac{E'}{4}+1)},
\end{aligned}
\tag{2.15}
$$

where $E=-8\alpha\beta$ and $E'=-8\alpha'\beta'$.

3 Transformation theory to $(P_{\mathrm{III}'(D8)})$ near a simple pole

As was discussed in the preceding section, the connection formula for instanton-type solutions on a Stokes curve can be explicitly written down for the most degenerate third Painlevé equation $(P_{\mathrm{III}'(D8)})$. More interesting is that this equation $(P_{\mathrm{III}'(D8)})$ plays a special role of the canonical equation near a simple pole of Painlevé equations, that is, every 2-parameter instanton-type solution of a Painlevé equation (P_J) can be transformed to an appropriately chosen 2-parameter instanton-type solution of $(P_{\mathrm{III}'(D8)})$ near a simple pole. In this section we explain the transformation theory to $(P_{\mathrm{III}'(D8)})$ near a simple pole of Painlevé equations.

First of all, we list up Painlevé equations (P_J) ($J=$ I, II, III′, III′$(D7)$, IV, V, VI) for the reference of the reader.

$$(P_{\mathrm{I}})\quad \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2+t),$$

$$(P_{\mathrm{II}})\quad \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3+t\lambda+c),$$

$$(P_{\mathrm{III}'})\quad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \eta^2\left[\frac{c_\infty\lambda^3}{t^2} - \frac{c'_\infty\lambda^2}{t^2} + \frac{c'_0}{t} - \frac{c_0}{\lambda}\right],$$

$$(P_{\mathrm{III}'(D7)}) \quad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} - \eta^2\left[\frac{2\lambda^2}{t^2} + \frac{c}{t} + \frac{1}{\lambda}\right],$$

$$(P_{\mathrm{IV}}) \quad \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + \eta^2\left[\frac{3}{2}\lambda^3 + 4t\lambda^2 + \left(2t^2 + c_1\right)\lambda - \frac{c_0}{\lambda}\right],$$

$$(P_{\mathrm{V}}) \quad \frac{d^2\lambda}{dt^2} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda - \frac{1}{2\lambda}\right) + \eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[c_\infty - \frac{c_0}{\lambda^2} - \frac{c_2 t}{(\lambda-1)^2} - \frac{c_1 t^2(\lambda+1)}{(\lambda-1)^3}\right],$$

$$(P_{\mathrm{VI}}) \quad \frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} + \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2} + \eta^2\left\{c_\infty - \frac{c_0 t}{\lambda^2} + \frac{c_1(t-1)}{(\lambda-1)^2} - \frac{c_t t(t-1)}{(\lambda-t)^2}\right\}\right],$$

where c, c_* and c'_* $(* = 0, 1, 2, t, \infty)$ denote complex constants. (Instead of the ordinary (P_{III}), here and in what follows, we deal with its equivalent form $(P_{\mathrm{III}'})$ for the sake of convenience in discussing the behavior of solutions near its simple pole.) In view of this list we readily find that Painlevé equations (P_J) have the following singular points:

$$(3.1) \qquad \begin{array}{lcl} (P_{\mathrm{I}}),\ (P_{\mathrm{II}}),\ (P_{\mathrm{IV}}) & : & \{\infty\}, \\ (P_{\mathrm{III}'}),\ (P_{\mathrm{III}'(D7)}),\ (P_{\mathrm{V}}) & : & \{0, \infty\}, \\ (P_{\mathrm{VI}}) & : & \{0, 1, \infty\}. \end{array}$$

Among them $t = 0$ for $(P_{\mathrm{III}'})$, $t = 0$ for $(P_{\mathrm{III}'(D7)})$, $t = 0$ for (P_{V}) and $t = 0, 1, \infty$ for (P_{VI}) are of the first kind (or "regular singular type"). At these singular points of the first kind, in addition to double pole type 0-parameter solutions, there exist simple pole type 0-parameter solutions. For example, let us consider the sixth Painlevé equation (P_{VI}) near its singular point $t = 0$ of the first kind. Since the top order part $\lambda_0(t)$ of a 0-parameter solution $\widehat{\lambda} = \lambda_0(t) + \eta^{-1}\lambda_1(t) + \cdots$ of (P_{VI}) satisfies an algebraic equation

(3.2)
$$F_{\mathrm{VI}}(\lambda_0, t) \stackrel{\mathrm{def}}{=} \frac{2\lambda_0(\lambda_0-1)(\lambda_0-t)}{t^2(t-1)^2}\left\{c_\infty - \frac{c_0 t}{\lambda_0^2} + \frac{c_1(t-1)}{(\lambda_0-1)^2} - \frac{c_t t(t-1)}{(\lambda_0-t)^2}\right\} = 0,$$

$(P_{\rm VI})$ has six 0-parameter solutions. As was observed in [T1], their local behavior near $t=0$ can be classified into the following three groups:

Group (A): The case where $\lambda_0(t)$ has an expansion $\lambda_0(t) = a_0 + a_1 t + \cdots$ with $a_0 \neq 0$. Here a_0 is a root of $(a_0-1)^2 = c_1/c_\infty$. In this case we have

$$\frac{\partial F_{\rm VI}}{\partial \lambda_0}(\lambda_0(t), t) = \frac{4}{t^2}\,(\sqrt{c_\infty} \pm \sqrt{c_1})^2 + O(t^{-1}). \tag{3.3}$$

Group (B): The case where $\lambda_0(t)$ can be expanded as $\lambda_0(t) = a_1 t + \cdots$, that is, $\lambda_0(t)$ has a Taylor expansion with vanishing constant term at $t=0$. Here a_1 is a root of $((1/a_1)-1)^2 = c_t/c_0$ and we have

$$\frac{\partial F_{\rm VI}}{\partial \lambda_0}(\lambda_0(t), t) = \frac{4}{t^2}\,(\sqrt{c_0} \pm \sqrt{c_t})^2 + O(t^{-1}). \tag{3.4}$$

Group (C): The case where $\lambda_0(t)$ can be expanded as $\lambda_0(t) = a_{1/2}t^{1/2} + a_1 t + \cdots$ with respect to $t^{1/2}$. Here $a_{1/2}$ is a root of $a_{1/2}^2 = (c_t - c_0)/(c_1 - c_\infty)$. In this case we have

$$\frac{\partial F_{\rm VI}}{\partial \lambda_0}(\lambda_0(t), t) = \pm\frac{4}{t^{3/2}}\sqrt{(c_t-c_0)(c_1-c_\infty)} + O(t^{-1}). \tag{3.5}$$

In parallel to the case of $(P_{{\rm III}'(D8)})$, through the study of the behavior of the Frechét derivative of $(P_{\rm VI})$ at these 0-parameter solutions near the origin after the change of variables $(t, \Delta\lambda) = (\tilde{t}^2, \tilde{t}^{1/2}\widetilde{\Delta\lambda})$, we find that a 0-parameter solution belonging to the group (C) is of simple pole type (while a 0-parameter solution belonging to the other two groups is of double pole type). In a similar manner we can confirm that for the following pairs

$$\begin{aligned} &((P_{{\rm III}'}), 0),\ \ ((P_{{\rm III}'(D7)}), 0),\ \ ((P_{\rm V}), 0),\\ &((P_{\rm VI}), 0),\ \ ((P_{\rm VI}), 1),\ \ ((P_{\rm VI}), \infty) \end{aligned} \tag{3.6}$$

of a Painlevé equation and a singular point of it, there exist simple pole type 0-parameter solutions whose top order part $\lambda_0(t)$ satisfies

$$\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t) = O((t-t_*)^{-3/2}) \quad \text{as} \quad t \to t_*, \tag{3.7}$$

where $F_J(\lambda, t)$ denotes the coefficient of η^2 in the right-hand side of the explicit formula of (P_J) given in the above list and t_* designates a simple pole type singular point for a 0-parameter solution in question.

Using the top order part $\lambda_0(t)$ of these 0-parameter solutions, we can also construct a simple pole type 2-parameter instanton-type solution $\lambda_J(t;\alpha,\beta)$ for each pair $((P_J), t_*)$ of a Painlevé equation and a singular point of it listed in (3.6). The problem we want to discuss is then the analysis of the Stokes phenomena for these instanton-type solutions of Painlevé equations that occur on their Stokes curves emanating from the corresponding simple pole type singular points $t = t_*$. (Note that, as in Definition 2.1 for $(P_{\mathrm{III}'(D8)})$, every simple pole type singular point $t = t_*$ of (P_J) should be regarded as a turning point and that a Stokes curve of (P_J) emanating from t_* is defined as a Stokes curve (emanating from t_*) of the Frechét derivative of (P_J) at the corresponding 0-parameter solution of simple pole type in question. Thanks to (3.7) we readily find that only one Stokes curve emanates from a simple pole t_*.) In order to attack this problem, we develop a transformation theory near simple poles. As a matter of fact, generalizing the transformation theory near simple turning points established in our series of papers ([KT1], [AKT], [KT2]), we can prove the following theorem which claims that every 2-parameter instanton-type solution of simple pole type of a Painlevé equation can be transformed to that of $(P_{\mathrm{III}'(D8)})$ near its simple pole. (In the statement of the theorem we add a lower suffix "III′($D8$)" and put $\tilde{}$ as $\tilde{\lambda}_{\mathrm{III}'(D8)}(\tilde{t};\tilde{\alpha},\tilde{\beta})$ to the variables relevant to $(P_{\mathrm{III}'(D8)})$ to distinguish them from those relevant to other Painlevé equations (P_J).)

Theorem 3.1. *Let $\lambda_J(t;\alpha,\beta)$ be a 2-parameter instanton-type solution of simple pole type for one of the pairs $((P_J), t_*)$ of a Painlevé equation and a singular point of it listed in* (3.6)*. Let σ be any point on a Stokes curve emanating from the corresponding simple pole type singular point $t = t_*$. Then we can find a neighborhood V of σ and a 2-parameter instanton-type solution $\tilde{\lambda}_{\mathrm{III}'(D8)}(\tilde{t};\tilde{\alpha},\tilde{\beta})$ of $(P_{\mathrm{III}'(D8)})$ so that $\lambda_J(t;\alpha,\beta)$ is formally transformed to $\tilde{\lambda}_{\mathrm{III}'(D8)}(\tilde{t};\tilde{\alpha},\tilde{\beta})$ in V. To be more specific, there exist a formal transformation $\tilde{t} = \tilde{t}(t,\eta)$ of an independent variable and a formal transformation $\tilde{x} = \tilde{x}(x,t,\eta)$ of an unknown function of the form*

$$\tilde{t}(t,\eta) = \sum_{j\geq 0} \tilde{t}_{j/2}(t,\eta)\eta^{-j/2}, \tag{3.8}$$

$$\tilde{x}(x,t,\eta) = \sum_{j\geq 0} \tilde{x}_{j/2}(x,t,\eta)\eta^{-j/2}, \tag{3.9}$$

where $\tilde{t}_{j/2}$ *and* $\tilde{x}_{j/2}$ *are holomorphic in both* x *and* t*, that satisfy the following relation:*

$$\tilde{x}(\lambda_J(t;\alpha,\beta),t,\eta) = \tilde{\lambda}_{\mathrm{III}'(D8)}(\tilde{t}(t,\eta);\tilde{\alpha},\tilde{\beta}). \tag{3.10}$$

Thus the most degenerate third Painlevé equation ($P_{\mathrm{III}'(D8)}$) gives the canonical equation of Painlevé equations near a simple pole. In particular, it can be expected that the connection formula similar to that for ($P_{\mathrm{III}'(D8)}$) described in the preceding section (Corollary 2.4) should hold also for an instanton-type solution of simple pole type for a pair $((P_J), t_*)$ listed in (3.6) on its Stokes curve emanating from $t = t_*$.

In what follows we explain an outline of the construction of the transformations $\tilde{t}(t,\eta)$ and $\tilde{x}(x,t,\eta)$. It is done in a way parallel to the transformation theory near a simple turning point; we again make full use of the relationship between Painlevé equations and isomonodromic deformations of linear differential equations. That is, we use the fact that a Painlevé equation (P_J) is equivalent to the compatibility condition of a system of the following linear differential equations:

$$\left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_J\right)\psi = 0, \tag{SL_J}$$

$$\frac{\partial \psi}{\partial t} = A_J \frac{\partial \psi}{\partial x} - \frac{1}{2}\frac{\partial A_J}{\partial x}\psi. \tag{D_J}$$

(For the concrete form of Q_J and A_J see, e.g., [KT1] or [KT3, Chapter 4].) If we substitute a 2-parameter instanton-type solution $\lambda_J(t;\alpha,\beta)$ of simple pole type in question into the coefficients of (SL_J), we then find that the following proposition, which is a generalization of Proposition 2.2 to any general instanton-type solution of simple pole type, holds between the Stokes geometry of (P_J) and that of (SL_J).

Proposition 3.2. *Suppose that an instanton-type solution* $\lambda_J(t;\alpha,\beta)$ *of simple pole type of* (P_J) *is substituted into the coefficients of* (SL_J)*. Then the following hold:*
(i) *The top order term* (*with respect to* η^{-1}) Q_0 *of the potential* Q *of* (SL_J) *has a double zero at* $x = \lambda_0(t)$*. Hence* (SL_J) *has a double turning point at* $x = \lambda_0(t)$*.*
(ii) *When and only when* t *lies on a Stokes curve of* (P_J) *emanating from a*

simple pole type singular point $t = t_*$ in question, there exists a Stokes curve of (SL_J) that starts from $\lambda_0(t)$ and returns to $\lambda_0(t)$ after encircling several singular points and turning points of (SL_J).

This Proposition 3.2 of geometric character again plays a key role in the proof of Theorem 3.1 in the following manner. Let $t = \sigma$ be a point on a Stokes curve of (P_J) emanating from $t = t_*$ and let γ denote a Stokes curve of (SL_J) that starts from $\lambda_0(t)$ and returns to $\lambda_0(t)$ at $t = \sigma$ whose existence is guaranteed by Proposition 3.2, (ii). Then we can construct an invertible formal transformation $(\tilde{x}(x,t,\eta), \tilde{t}(t,\eta))$ which brings the simultaneous equations (SL_J) and (D_J) into $(SL_{\mathrm{III}'(D8)})$ and $(D_{\mathrm{III}'(D8)})$ in a neighborhood of $\gamma \times \{\sigma\}$. That is, we have

Theorem 3.3. *Under the above geometric situation there exist a neighborhood U of the Stokes curve γ, a neighborhood V of σ, and a formal coordinate transformation*

$$\tilde{x} = \tilde{x}(x,t,\eta) = \sum_{j \geq 0} \tilde{x}_{j/2}(x,t,\eta)\eta^{-j/2}, \tag{3.11}$$

$$\tilde{t} = \tilde{t}(t,\eta) = \sum_{j \geq 0} \tilde{t}_{j/2}(t,\eta)\eta^{-j/2} \tag{3.12}$$

with $\tilde{x}_{j/2}(x,t,\eta)$ and $\tilde{t}_{j/2}(t,\eta)$ being holomorphic on $U \times V$ and V, respectively, for which the following conditions (i) $\sim$ (v) *are satisfied:*

(i) *The function $\tilde{x}_0(x,t,\eta)$ is independent of η and $\partial\tilde{x}_0/\partial x$ never vanishes on $U \times V$.*

(ii) *The function $\tilde{t}_0(t,\eta)$ is also independent of η and $d\tilde{t}_0/dt$ never vanishes on V.*

(iii) *$\tilde{x}_0(x,t)$ and $\tilde{t}_0(t)$ satisfy*

$$\tilde{x}_0(\lambda_0(t),t) = \sqrt{\tilde{t}_0(t)}. \tag{3.13}$$

(iv) *$\tilde{x}_{1/2}$ and $\tilde{t}_{1/2}$ identically vanish.*

(v) *If $\tilde{\psi}(\tilde{x},\tilde{t},\eta)$ is a WKB solution of $(SL_{\mathrm{III}'(D8)})$ that satisfies $(D_{\mathrm{III}'(D8)})$ also, then $\psi(x,t,\eta)$ defined by*

$$\psi(x,t,\eta) = \left(\frac{\partial\tilde{x}(x,t,\eta)}{\partial x}\right)^{-1/2} \tilde{\psi}(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta),\eta) \tag{3.14}$$

satisfies both (SL_J) and (D_J) near $x = \lambda_0(\sigma)$.

Furthermore we can verify that this semi-global transformation $(\tilde{x}(x,t,\eta),\tilde{t}(t,\eta))$ that brings (SL_J) and (D_J) into $(SL_{\mathrm{III}'(D8)})$ and $(D_{\mathrm{III}'(D8)})$ provides a local equivalence (3.10) between $\lambda_J(t;\alpha,\beta)$ and $\tilde{\lambda}_{\mathrm{III}'(D8)}(\tilde{t};\tilde{\alpha},\tilde{\beta})$. Otherwise stated, by considering a transformation for the underlying system (SL_J) and (D_J) of linear differential equations, we can find a transformation of an independent variable and a transformation of unknown functions for a solution $\lambda_J(t;\alpha,\beta)$ of the nonlinear equation (P_J) in question.

This is a sketch of the proof of Theorem 3.1. The details will be discussed in our forthcoming paper.

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