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Abstract

The notion of a competition graph was introduced by J. E. Cohen in 1968. The *competition graph* $C(D)$ of a digraph D is a (simple undirected) graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. In 1978, F. S. Roberts defined the *competition number* $k(G)$ of a graph G as the minimum number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph G and it has been one of important research problems in the study of competition graphs to characterize a graph by its competition number. In 1982, R. J. Opsut gave two lower bounds for the competition number of a graph. In this paper, we give a generalization of both of the Opsut's lower bounds for the competition numbers of graphs.

Keywords: competition graph; competition number; vertex clique cover number; edge clique cover number

1. Introduction

Throughout this paper, all graphs G are simple and undirected. The notion of a competition graph was introduced by J. E. Cohen [4] in connection with a problem in ecology (see also [5]). The *competition graph* $C(D)$ of a digraph D is a graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. From this observation, F. S. Roberts [16] defined the *competition number* $k(G)$ of a graph G to be the minimum number k such that G together with k isolated vertices is the competition graph of an acyclic digraph:

$$k(G) := \min\{k \in \mathbb{Z}_{\geq 0} \mid G \cup I_k = C(D) \text{ for some acyclic digraph } D\}, \tag{1.1}$$

where I_k denotes a set of k isolated vertices.

For a digraph D, an ordering $v_1, v_2, \ldots, v_n$ of the vertices of D is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i < j$. It is well-known that a digraph D is acyclic if and only if there exists an acyclic ordering of D.

A subset $S \subseteq V(G)$ of the vertex set of a graph G is called a *clique* of G if the subgraph $G[S]$ of G induced by S is a complete graph. For a clique S of a graph G and an edge e of G, we say e *is covered by* S if both of the endpoints of e are contained in S. An *edge clique cover* of a graph G is a family of cliques such that each edge of G is covered by some clique in the family (see [17] for applications of edge clique covers). The *edge clique cover number* $\theta_E(G)$ of a graph G is the minimum size of an edge clique cover of G. A *vertex clique cover* of a graph G is a family of cliques such that each vertex of G is contained in some clique in the family. The *vertex clique cover number* $\theta_V(G)$ of a graph G is the minimum size of a vertex clique cover of G.

R. D. Dutton and R. C. Brigham [6] characterized the competition graph of an acyclic digraph in terms of an edge clique cover as follows (see also [14], [18]).

Theorem 1.1 (Dutton and Brigham [6], Theorem 2)**.** *A graph G is the competition graph of an acyclic digraph if and only if there exist an ordering $v_1, \ldots, v_n$ of the vertices of G and an edge clique cover $\{S_1, ..., S_n\}$ of G such that $v_i \in S_j$ implies $i < j$.*

The above theorem characterizes graphs whose competition numbers are equal to 0. But R. J. Opsut [15] showed that the problem of determining whether a graph is the competition graph of an acyclic digraph or not is NP-complete. It follows that the computation of the competition number of a graph is an NP-hard problem, and thus it does not seem to be easy in general to compute $k(G)$ for an arbitrary graphs G (see [9], [10], [12] for graphs whose competition numbers are known). It has been one of important research problems in the study of competition graphs to characterize a graph by its competition number (see [1], [2], [3], [7], [8], [11], [13], [19] for recent research).

R. J. Opsut gave the following two lower bounds for the competition number of a graph.

Theorem 1.2 (Opsut [15], Proposition 5)**.** *For any graph G,*

$$k(G) \geq \theta_E(G) - |V(G)| + 2. \tag{1.2}$$

Theorem 1.3 (Opsut [15], Proposition 7)**.** *For any graph G,*

$$k(G) \geq \min\{\theta_V(N_G(v)) \mid v \in V(G)\}, \tag{1.3}$$

where $N_G(v) := \{u \in V(G) \mid uv \in E(G)\}$ is the open neighborhood of a vertex v in the graph G.

It should be noted that the above two are the only sharp lower bounds known to us by today which hold for the competition numbers of any graphs.

In this paper, we give a generalization of the Opsut's lower bounds, which also holds for the competition numbers of any graphs. In particular, our main result contains both lower bounds given in Theorems 1.2 and 1.3 as special cases. The proof of our main result is elementary, but the new lower bound given in this paper would be a strong tool in the study of the competition number of a graph.

2. Main Results

Let G be a graph and $F \subseteq E(G)$ be a subset of the edge set of G. An *edge clique cover* of F in G is a family of cliques of G such that each edge in F is covered by some clique in the family. We define the *edge clique cover number* $\theta_E(F; G)$ of $F \subseteq E(G)$ in G as the minimum size of an edge clique cover of F in G:

$$\theta_E(F; G) := \min\{|\mathcal{S}| \mid \mathcal{S} \text{ is an edge clique cover of } F \text{ in } G\}. \tag{2.1}$$

By definition, it follows that the edge clique cover number $\theta_E(E(G); G)$ of $E(G)$ in a graph G is equal to the edge clique cover number $\theta_E(G)$ of the graph G.

Let G be a graph and $U \subseteq V(G)$ be a subset of the vertex set of G. We define

$$N_G[U] := \{v \in V(G) \mid v \text{ is adjacent to a vertex in } U\} \cup U, \tag{2.2}$$
$$E_G[U] := \{e \in E(G) \mid e \text{ has an endvertex in } U\}. \tag{2.3}$$

We denote by the same symbol $N_G[U]$ the subgraph of G induced by $N_G[U]$. Note that $E_G[U]$ is contained in the edge set of the subgraph $N_G[U]$. We denote by $\binom{V}{m}$ the set of all m-subsets of a set V.

Now we are ready to state our main result. The following is our main theorem.

Theorem 2.1. *Let $G = (V, E)$ be a graph. Then*

$$k(G) \geq \max_{m\in\{1,\dots,|V|\}} \min_{U\in\binom{V}{m}} \Big(\theta_E(E_G[U]; N_G[U]) - |U| + 1\Big). \tag{2.4}$$

To prove our main theorem, we show the following lemma.

Lemma 2.2. *Let $G = (V, E)$ be a graph. Let m be an integer such that $1 \leq m \leq |V|$. Then*

$$k(G) \geq \min_{U\in\binom{V}{m}} \theta_E(E_G[U]; N_G[U]) - m + 1. \tag{2.5}$$

Proof. Let $k := k(G)$ for convenience. Fix an integer m such that $1 \leq m \leq |V|$. Let D be an acyclic digraph such that $C(D) = G \cup I_k$, where $I_k := \{z_1, ..., z_k\}$ is a set of k isolated vertices. Let $v_1, ..., v_n, z_1, ..., z_k$ be an acyclic ordering of D, and put $W := \{v_{n-m+1}, ..., v_n\}$. Note that $|W| = m$. Let

$$\mathcal{S} := \{N^-_D(w) \cap N_G[W] \mid w \in (W \cup I_k) \setminus \{v_{n-m+1}\}\},$$

where $N^-_D(w) := \{v \in V(D) \mid (v, w) \in A(D)\}$ is the *in-neighborhood* of a vertex w in the digraph D. For each $w \in (W \cup I_k) \setminus \{v_{n-m+1}\}$, since $N^-_D(w)$ forms a clique of the graph G, the set $N^-_D(w) \cap N_G[W]$ forms a clique of the induced subgraph $N_G[W]$ of G. Thus $\mathcal{S}$ is a family of cliques of $N_G[W]$.

Since $v_1, ..., v_n, z_1, ..., z_k$ is an acyclic ordering of D, it holds that the *out-neighborhood* $N^+_D(u) := \{v \in V(D) \mid (u, v) \in A(D)\}$ of a vertex u in the digraph D is contained in the set $(W \cup I_k) \setminus \{v_{n-m+1}\}$ for each vertex $u \in W$. Take any edge $e = uv \in E_G[W]$, where $u \in W$ and $v \in N_G(u)$. Since u and v are adjacent, there exists a common prey $w \in N^+_D(u) \cap N^+_D(v) \subseteq (W \cup I_k) \setminus \{v_{n-m+1}\}$. Then the edge e is covered by $N^-_D(w) \cap N_G[W] \in \mathcal{S}$.

Therefore the family $\mathcal{S}$ is an edge clique cover of of $E_G[W]$ in $N_G[W]$. So we have $\theta_E(E_G[W]; N_G[W]) \leq |\mathcal{S}| = m + k - 1$, that is, $\theta_E(E_G[W]; N_G[W]) - m + 1 \leq k$. Thus

$$\min_{U\in\binom{V}{m}} \theta_E(E_G[U]; N_G[U]) - m + 1 \leq \theta_E(E_G[W]; N_G[W]) - m + 1 \leq k(G).$$

Hence the lemma holds. □

Proof of Theorem 2.1. Since the inequality (2.5) holds for any $m \in \{1, ..., |V|\}$, it follows that the inequality (2.4) holds. □

Remark 2.3. Consider the case $m = 1$ in the inequality (2.5). Then we obtain

$$k(G) \geq \min_{v\in V(G)} \theta_E(E_G[v]; N_G[v]).$$

Since a family $\{S_1, ..., S_r\}$ of cliques is an edge clique cover of $E_G[v]$ in G if and only if $\{S_1 \cap N_G[v], ..., S_r \cap N_G[v]\}$ is an edge clique cover of $E_G[v]$ in $N_G[v]$, it holds that $\theta_E(E_G[v]; N_G[v]) = \theta_E(E_G[v]; G)$. Since a family $\{S_1, ..., S_r\}$ of cliques is an edge clique cover of $E_G[v]$ in G if and only if $\{S_1 \setminus \{v\}, ..., S_r \setminus \{v\}\}$ is a vertex clique cover of $N_G(v)$ in G, it holds that $\theta_E(E_G[v]; G) = \theta_V(N_G(v))$. Therefore we have $\theta_E(E_G[v]; N_G[v]) = \theta_V(N_G(v))$. Hence the above inequality coincides with the Opsut's lower bound (1.3) in Theorem 1.3.

Remark 2.4. Consider the case $m = |V| - 1$ in the inequality (2.5). Then we obtain

$$k(G) \geq \min_{v \in V} \theta_E(E_G[V \setminus \{v\}]; N_G[V \setminus \{v\}]) - |V| + 2.$$

Since $G = (V, E)$ has no loops, it holds that $E_G[V \setminus \{v\}] = E$. If the vertex v is not isolated in G, then we have $N_G[V \setminus \{v\}] = V$ and thus $\theta_E(E_G[V \setminus \{v\}]; N_G[V \setminus \{v\}]) = \theta_E(E; G) = \theta_E(G)$. If v is an isolated vertex, then we have $N_G[V \setminus \{v\}] = V \setminus \{v\}$ and thus $\theta_E(E_G[V \setminus \{v\}]; N_G[V \setminus \{v\}]) = \theta_E(E; G - \{v\}) = \theta_E(E; G) = \theta_E(G)$. Hence the above inequality coincides with the Opsut's lower bound (1.2) in Theorem 1.2.

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