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By

Boram PARK and Yoshio SANO

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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BORAM PARK*

Department of Mathematics Education,
Seoul National University, Seoul 151-742, Korea.
kawa22@snu.ac.kr

YOSHIO SANO†

Research Institute for Mathematical Sciences,
Kyoto University, Kyoto 606-8502, Japan.
sano@kurims.kyoto-u.ac.jp

Abstract

The competition graph of a digraph D is a graph which has the same vertex set as D and has an edge between x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of a graph G is defined to be the smallest number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph G and it has been one of important research problems in the study of competition graphs to characterize a graph by its competition number. In this paper, we compute the competition numbers of Hamming graphs.

Keywords: competition graph; competition number; edge clique cover; Hamming graph

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1. Introduction

The notion of a competition graph was introduced by Cohen [2] as a means of determining the smallest dimension of ecological phase space (see also [3]). The *competition graph* $C(D)$ of a digraph D is a (simple undirected) graph which has the same vertex set as D and an edge between vertices u and v if and only if there is a vertex x in D such that (u, x) and (v, x) are arcs of D. Roberts [15] observed that if G is any graph, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* $k(G)$ of a graph G to be the smallest number k such that G together with k isolated vertices added is the competition graph of an acyclic digraph.

Roberts [15] observed that the characterization of competition graphs is equivalent to the computation of competition numbers. It does not seem to be easy in general to compute $k(G)$ for all graphs G, as Opsut [12] showed that the computation of the competition number of a graph is an NP-hard problem (see [6], [7], [9] for graphs whose competition numbers are known). It has been one of important research problems in the study of competition graphs to characterize a graph by its competition number (see [1], [8], [10], [11], [13], [14] for recent research).

For some special graph families, we have explicit formulae for computing competition numbers. For example, if G is a choral graph without isolated vertices then $k(G) = 1$, and if G is a nontrivial triangle-free connected graph then $k(G) = |E(G)| - |V(G)| + 2$ (see [15]).

In this paper, we study the competition numbers of Hamming graphs. For a positive integer q, we denote a q-set $\{1, 2 \ldots, q\}$ by $[q]$. Also we denote the set of n-tuple over $[q]$ by $[q]^n$. For positive integers n and q, the *Hamming graph* $H(n, q)$ is a graph which has the vertex set $[q]^n$, and two vertices x and y are adjacent if $d_H(x, y) = 1$, where $d_H : [q]^n \times [q]^n \to \mathbb{Z}$ is the *Hamming distance* defined by $d_H(x, y) := |\{i \in [n] \mid x_i \neq y_i\}|$. Note that the diameter of the Hamming graph $H(n, q)$ is equal to n.

Since the Hamming graph $H(n, q)$ is $n(q-1)$-regular and the number of vertices of $H(n, d)$ is q^n, it follows that the number of edges of the Hamming graph $H(n, q)$ is equal to $\frac{1}{2}n(q-1)q^n$.

If $n = 1$, then $H(1, q)$ is the complete graph K_q with q elements and thus $k(H(1, q)) = 1$. If $q = 1$, then $H(n, 1)$ is K_1 and thus $k(H(n, 1)) = 1$. If $q = 2$ then $H(n, 2)$ is triangle-free and connected, so we have

$$\begin{aligned} k(H(n,2)) &= |E(H(n,2))| - |V(H(n,2))| + 2 \\ &= n2^{n-1} - 2^n + 2 \\ &= (n-2)2^{n-1} + 2. \end{aligned}$$

However, in general, it is difficult to compute $k(H(n, q))$.

In this paper, we give a lower bound for $k(H(n,q))$ and also give the exact values of $k(H(2,q))$ and $k(H(3,q))$.

We use the following notation and terminology in this paper. For a digraph D, a sequence $v_1, v_2, \ldots, v_n$ of the vertices of D is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i > j$. It is well-known that a digraph D is acyclic if and only if there exists an acyclic ordering of D. For a digraph D and a vertex v of D, we define the *out-neighborhood* $N^+_D(v)$ of v in D to be the set $\{w \in V(D) \mid (v,w) \in A(D)\}$, and the *in-neighborhood* $N^-_D(v)$ of v in D to be the set $\{w \in V(D) \mid (w,v) \in A(D)\}$. A vertex in the out-neighborhood $N^+_D(v)$ of a vertex v in a digraph D is called a *prey* of v in D. For a graph G and a vertex v of G, we define the *open neighborhood* $N_G(v)$ of v in G to be the set $\{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* $N_G[v]$ of v in G to be the set $N_G(v) \cup \{v\}$. We denote the subgraph of G induced by $N_G(v)$ (resp. $N_G[v]$) by the same symbol $N_G(v)$ (resp. $N_G[v]$).

For a clique S of a graph G and an edge e of G, we say e *is covered by* S if both of the endpoints of e are contained in S. An *edge clique cover* of a graph G is a family of cliques of G such that each edge of G is covered by some clique in the family. The *edge clique cover number* $\theta_E(G)$ of a graph G is the minimum size of an edge clique cover of G. An edge clique cover of G is called a *minimum edge clique cover* of G if its size is equal to $\theta_E(G)$. A *vertex clique cover* of a graph G is a family of cliques of G such that each vertex of G is contained in some clique in the family. The smallest size of a vertex clique cover of G is called the *vertex clique cover number*, and is denoted by $\theta_V(G)$.

We denote a path with n vertices by P_n, a cycle with n vertices by C_n, and a complete multipartite graph by $K_{n_1,\ldots,n_m}$.

2. Main Results

2.1. A lower bound for the competition number of $H(n,d)$

For $j \in [n]$ and $\mathbf{p} \in [q]^{n-1}$, we put

$$S_j(\mathbf{p}) := \{x \in [q]^n \mid \pi_j(x) = \mathbf{p}\}, \tag{2.1}$$

where $\pi_j : [q]^n \to [q]^{n-1}$ is a map defined by

$$(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

Note that $S_j(\mathbf{p})$ is a clique of $H(n,q)$ with size q. Put

$$\mathcal{F}(n,q) := \{S_j(\mathbf{p}) \mid j \in [n], \mathbf{p} \in [q]^{n-1}\}. \tag{2.2}$$

Then $\mathcal{F}(n,q)$ is the family of maximal cliques of $H(n,q)$.

In the computation of the competition number of a graph, usually it is not so easy to give a sharp lower bound. In this subsection, we give a sharp lower bound for the competition numbers of Hamming graphs.

Lemma 1. *Let $n \geq 2$ and $q \geq 2$. For any vertex x of $H(n,q)$, we have $\theta_V(N_{H(n,q)}(x)) = n$.*

Proof. Take any $x \in [q]^n$. Then the vertex x is adjacent to a vertex y such that $\pi_j(x) = \pi_j(y)$ for some $j \in [n]$. We can easily check from the definition of $H(n,q)$ that, for any $j \in [n]$, the set $S_j(\pi_j(x)) := \{y \in [q]^n \mid \pi_j(x) = \pi_j(y)\}$ forms a clique of $H(n,q)$. Since $N_{H(n,q)}(x) = \cup_{j \in [n]} S_j(\pi_j(x)) \setminus \{x\}$, the family $\{S_j(\pi_j(x)) \mid j \in [n]\}$ is a vertex clique cover of $N_{H(n,q)}(x)$ and so $\theta_V(N_{H(n,q)}(x)) \leq n$.

Moreover, note that $S_j(\pi_j(x)) \cap S_{j'}(\pi_{j'}(x)) = \{x\}$ for $j, j' \in [n]$ where $j \neq j'$. Take $y_j \in S_j(\pi_j(x)) \setminus \{x\}$ for each $j \in [n]$. Then $y_1, y_2, \ldots, y_n$ are n vertices of $N_{H(n,q)}(x)$ such that no two of them can be covered by a same clique and so $\theta_V(N_{H(n,q)}(x)) \geq n$. □

Opsut showed the following lower bound for the competition number of a graph.

Theorem 2 ([12])**.** *For a graph G, it holds that $k(G) \geq \min\{\theta_V(N_G(v)) \mid v \in V(G)\}$.*

By Lemma 1 and Theorem 2, we have the following.

Corollary 3. *Let $n \geq 2$ and $q \geq 2$. Then it holds that $k(H(n,q)) \geq n$.*

Lemma 4. *Let $n \geq 2$ and $q \geq 2$, and let K be a clique of $H(n,q)$ with size at least 2. Then there is a unique maximal clique $S \in \mathcal{F}(n,q)$ containing K.*

Proof. Take any $x, y \in K$ with $x \neq y$ and let $x = (x_1, x_2 \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Since x and y are adjacent, there is a unique integer $j \in [n]$ such that $\pi_j(x) = \pi_j(y)$. Then

$$x_j \neq y_j. \tag{2.3}$$

Take any $z \in K \setminus \{x, y\}$ and let $z = (z_1, z_2, \ldots, z_n)$. We will show that $\pi_j(z) = \pi_j(x)$ by contradiction. Suppose that $\pi_j(z) \neq \pi_j(x)$. Since x and z are adjacent, there is $j_1 \in [n]$ with $j_1 \neq j$ such that $\pi_{j_1}(x) = \pi_{j_1}(z)$, and thus $x_j = z_j$. Since y and z are adjacent, there is $j_2 \in [n]$ with $j_2 \neq j$ such that $\pi_{j_2}(y) = \pi_{j_2}(z)$, and thus $y_j = z_j$. Thus we have $x_j = z_j = y_j$, which is a contradiction to (2.3). Therefore, $\pi_j(z) = \pi_j(x)$. It implies that there is an integer $j \in [n]$ such that $\pi_j(x) = \pi_j(z)$ for all $z \in K$. Hence K is contained in $S := S_j(\mathbf{p}) \in \mathcal{F}(n,q)$ with $\mathbf{p} = \pi_j(x) \in [q]^{n-1}$. From the uniqueness of $j \in [n]$ and the fact that $\mathbf{p} = \pi_j(x)$ does not depend on the choice of $x \in K$, it follows that S is the unique maximal clique containing K. □

Lemma 5. *Let $n \geq 2$ and $q \geq 2$. Let D be an acyclic digraph such that $C(D) = H(n,q) \cup I_k$ with $I_k = \{z_1, z_2, \ldots, z_k\}$. Let $z_1, z_2, \ldots, z_k, v_1, v_2, \ldots, v_{q^n}$ be an acyclic ordering of D. Let $U_i := \{v_1, \ldots, v_i\}$. Then*

$$|\{S \in \mathcal{F}(n,q) \mid S \cap U_i \neq \emptyset\}| \leq k + i - 1.$$

Proof. For a digraph D, we define $\mathcal{N}^-(D) := \{N^-_D(x) \mid x \in V(D), |N^-_D(x)| \geq 2\}$. We will show that

$$|\{S \in \mathcal{F}(n,q) \mid S \cap U_i \neq \emptyset\}| \leq |\{K \in \mathcal{N}^-(D) \mid K \cap U_i \neq \emptyset\}|. \tag{2.4}$$

For convenience, let $\mathcal{A} := \{S \in \mathcal{F}(n,q) \mid S \cap U_i \neq \emptyset\}$ and $\mathcal{B} := \{K \in \mathcal{N}^-(D) \mid K \cap U_i \neq \emptyset\}$. By Lemma 4, each element K in $\mathcal{B}$ is contained in exactly one element S in $\mathcal{F}(n,d)$. From the fact that $(K \cap U_i) \subseteq (S \cap U_i)$ and $K \cap U_i \neq \emptyset$, we have $S \cap U_i \neq \emptyset$ and so $S \in \mathcal{A}$. Therefore, to show (2.4), it is sufficient to show that each element of $\mathcal{A}$ contains an element of $\mathcal{B}$.

Take an element $S \in \mathcal{A}$. Then there is a vertex $x \in S$ such that $x \in S \cap U_i$. Since $n \geq 2$, there is another vertex $y \in S \setminus \{x\}$. By Lemma 4, S is the unique clique in $\mathcal{F}(n,q)$ containing x and y.

Since $C(D) = H(n,q) \cup I_k$ and the vertices x and y are adjacent, there is a common prey u of x and y in D. Then $x \in N^-_D(u) \cap U_i$ and so $N^-_D(u) \in \mathcal{B}$. Then, by Lemma 4, there is the unique clique S' in $\mathcal{F}(n,q)$ containing $N^-_D(u)$. Since $N^-_D(u)$ contains x and y, S' is the unique clique containing x and y. Therefore $S' = S$ and so S contains an element $N^-_D(u) \in \mathcal{B}$.

Let $\mathcal{S}_i := \{N^-_D(x) \mid x \in U_{i-1} \cup I_k, |N^-_D(x)| \geq 2\}$. Then $|\mathcal{S}_i| \leq k + i - 1$. From the acyclicity of D, it holds that $\{K \in \mathcal{N}^-(D) \mid K \cap U_i \neq \emptyset\} = \{K \in \mathcal{S}_i \mid K \cap U_i \neq \emptyset\}$. Therefore it follows that

$$|\{K \in \mathcal{S}_i \mid K \cap U_i \neq \emptyset\}| \leq |\mathcal{S}_i| \leq k + i - 1, \tag{2.5}$$

Hence, from (2.4) and (2.5), the theorem holds. □

Theorem 6. *For $n \geq 3$ and $q \geq 3$, we have $k(H(n,q)) \geq 3n - 4$.*

Proof. Let D be an acyclic digraph such that $C(D) = H(n,q) \cup I_k$ with $I_k = \{z_1, z_2, \ldots, z_k\}$, where $k := k(H(n,q))$. Let $z_1, z_2, \ldots, z_k, v_1, v_2, \ldots, v_{q^n}$ be an acyclic ordering of D. Let $U_3 := \{v_1, v_2, v_3\}$. By Lemma 5, it holds that

$$|\{S \in \mathcal{F}(n,q) \mid S \cap U_3 \neq \emptyset\}| \leq k + 2. \tag{2.6}$$

In addition, it holds that $|\{S \in \mathcal{F}(n,q) \mid S \cap U_3 \neq \emptyset\}| \geq 3n - 2$ whose proof will be shown in next paragraph. Therefore, we have $3n - 2 \leq k + 2$, or $k \geq 3n - 4$.

Now it remains to show that $|\{S \in \mathcal{F}(n,q) \mid S \cap U_3 \neq \emptyset\}| \geq 3n-2$. Consider the subgraph of $H(n,q)$ induced by U_3, say H. Then H is isomorphic to one of following:

$$(1)K_3 \qquad (2)P_3 \qquad (3)P_2 \cup I_1 \qquad (4)I_3.$$

<u>(1) $H \cong K_3$</u>. By Lemma 4, U_3 is contained exactly one maximal clique. We may assume that U_3 is contained in $S_1((\underbrace{1,\ldots,1}_{n-1}))$, and so may assume that

$$U_3 = \{(\underbrace{1,1,\ldots,1}_{n}),(2,\underbrace{1,\ldots,1}_{n-1}),(3,\underbrace{1,\ldots,1}_{n-1})\}.$$

Then $\{S \in \mathcal{F}(n,q) \mid S \cap U_3 \neq \emptyset\}$ consists of $3n-2$ elements, that is,

$$\{S_1((\underbrace{1,\ldots,1}_{n-1}))\} \cup \left(\bigcup_{i=2}^{n}\{S_i((\underbrace{1,\ldots,1}_{n-1})),S_i((2,\underbrace{1,\ldots,1}_{n-2})),S_i((3,\underbrace{1,\ldots,1}_{n-2}))\}\right).$$

<u>(2) $H \cong P_3$</u>. We may assume that

$$U_3 = \{(\underbrace{1,\ldots,1}_{n}),(2,\underbrace{1,\ldots,1}_{n-1}),(1,2,\underbrace{1,\ldots,1}_{n-2})\}.$$

Then $\{S \in \mathcal{F}(n,q) \mid S \cap U_3 \neq \emptyset\}$ consists of $3n-2$ elements, that is,

$$\begin{aligned}&\{S_2((2,\underbrace{1,\ldots,1}_{n-2})),S_1((1,2,\underbrace{1,\ldots,1}_{n-3}))\}\\ &\cup\left(\bigcup_{i=1}^{q}\{S_i((\underbrace{1,\ldots,1}_{n-1}))\}\right)\cup\left(\bigcup_{i=3}^{q}\{S_i((2,\underbrace{1,\ldots,1}_{n-2})),S_i((1,2,\underbrace{1,\ldots,1}_{n-3}))\}\right).\end{aligned}$$

<u>(3) $H \cong P_2 \cup I_1$ or (4) $H \cong I_3$</u>. Then H has an isolated vertex, say v. Since the n cliques of $\mathcal{F}(n,d)$ containing v does not contain the other vertices of U_3, it is sufficient to show that $\{S \in \mathcal{F}(n,q) \mid S \cap (U_3 \setminus \{v\}) \neq \emptyset\}$ has at least $2n-2$ elements. Since, for each vertex $u \in U_3 \setminus \{v\}$, there are n cliques of $\mathcal{F}(n,d)$ containing u and there is at most one clique of $\mathcal{F}(n,d)$ containing the two vertices of $U_3 \setminus \{v\}$. Thus we can conclude that $\{S \in \mathcal{F}(n,q) \mid (S \cap U_3) \setminus \{v\} \neq \emptyset\}$ has at least $2n-1$ elements. We complete the proof. □

In subsection 2.4, we can see that the bound in Theorem 6 is sharp.

2.2. An edge clique cover of $H(n,q)$

Lemma 7. *The following hold.*

(a) *The family $\mathcal{F}(n,q)$ defined by (2.10) and (2.2) is an edge clique cover of $H(n,q)$.*

(b) $\theta_E(H(n,q)) = nq^{n-1}$.

(c) *Any minimum edge clique cover of $H(n,q)$ consists of edge disjoint maximum cliques.*

Proof. Take any edge xy of $H(n,q)$. Then $d_H(x,y) = 1$. Let $j \in [n]$ be the index such that $x_j \neq y_j$ and let $p := \pi_j(x) = \pi_j(y)$. Then both of x and y are contained in the clique $S_j(p) \in \mathcal{F}(n,q)$. Thus $\mathcal{F}(n,q)$ is an edge clique cover of $H(n,q)$.

Let $\mathcal{E}$ be a minimum edge clique cover for $H(n,q)$, that is, $\theta_E(H(n,q)) = |\mathcal{E}|$. Since $\mathcal{F}(n,q)$ is an edge clique cover with $|\mathcal{F}(n,q)| = nq^{n-1}$, we have $|\mathcal{E}| \leq nq^{n-1}$.

Now we will show that $|\mathcal{E}| \geq nq^{n-1}$. Since the maximum size of a clique of $H(n,q)$ is q, we have $|E(S)| \leq \binom{q}{2}$ for each $S \in \mathcal{E}$, where $E(S) := \binom{S}{2}$. Therefore,

$$|E(H(n,q))| \;\leq\; \sum_{S\in\mathcal{E}} |E(S)| \;\leq\; \binom{q}{2} \times |\mathcal{E}|, \tag{2.7}$$

and the first equality holds if and only if none of two distinct cliques in $\mathcal{E}$ have a common edge, and the second equality holds if and only if any element of $\mathcal{E}$ is a maximum clique in $H(n,q)$.

Note that $|E(H(n,q))| = \frac{1}{2}n(q-1)q^n$. From the equality that

$$|E(H(n,q))| = \frac{1}{2}n(q-1) \times q^n = n \times \frac{1}{2}(q-1)q \times q^{n-1} = \binom{q}{2} \times nq^{n-1},$$

it follows that

$$\binom{q}{2} \times nq^{n-1} = |E(H(n,q))| \leq \sum_{S\in\mathcal{E}} |E(S)| \leq \binom{q}{2} \times |\mathcal{E}|.$$

Thus we have $nq^{n-1} \leq |\mathcal{E}|$, or $nq^{n-1} = |\mathcal{E}|$. Moreover, since two equalities of (2.7) hold, we can conclude that any minimum edge clique cover of $H(n,q)$ consists of edge disjoint maximum cliques. □

Corollary 8. *The family $\mathcal{F}(n,q)$ defined by (2.10) and (2.2) is a minimum edge clique cover of $H(n,q)$.*

Proof. It follows from the fact that $|\mathcal{F}(n,q)| = nq^{n-1}$ and Lemma 7. □

2.3. The competition number of $H(2, q)$

In this subsection, we give the competition number of a Hamming graph $H(2, q)$ with $q \geq 2$.

First, we define a total order $\prec$ on the set $[q]^n$ as follows. Take two distinct elements x and y in $[q]^n$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Then we define $x \prec y$ if there exists $t \in [n]$ such that $x_s = y_s$ for $s \leq t - 1$ and $x_t < y_t$.

The *lexicographic ordering* of $V(H(n, q))$ is the ordering $v_1, v_2, \ldots, v_{q^n}$ such that $v_1 \prec v_2 \prec \ldots \prec v_{q^n}$.

Theorem 9. *For $q \geq 2$, we have $k(H(2, q)) = 2$.*

Proof. By Corollary 3, it follows that $k(H(2, q)) \geq 2$. Now we show that $k(H(2, q)) \leq 2$. We define a digraph D as follows.

$$\begin{aligned} V(D) &= V(H(2, q)) \cup \{z_1, z_2\}, \\ A(D) &= \left(\bigcup_{i=2}^{q} \{(x, (1, i-1)) \mid x \in S_1(i)\}\right) \cup \left(\bigcup_{i=2}^{q} \{(x, (i-1, q)) \mid x \in S_2(i)\}\right) \\ &\quad \cup \{(x, z_1) \mid x \in S_1(1)\} \cup \{(x, z_2) \mid x \in S_2(1)\}, \end{aligned}$$

where $S_j(i)$ with $j \in \{1, 2\}$ and $i \in [q]$ is the clique of $H(2, q)$ defined by (2.10). Since $\{N_D^-(v) \mid v \in V(D), |N_D^-(v)| \geq 2\} = \mathcal{F}(2, q)$, it is easy to check that $C(D) = H(2, q) \cup \{z_1, z_2\}$. In addition, the ordering obtained by adding z_1, z_2 on the head of the lexicographic ordering of $V(H(2, q))$ is an acyclic ordering of D. To see why, take an arc $(x, y) \in A(D)$. If $x \in S_1(1)$ or $x \in S_2(1)$ then y is either z_1 or z_2. If $x \in S_1(i)$ or $x \in S_2(i)$ for some $2 \leq i \in [q]$, then $x = (l, i)$ or $x = (i, l)$ for some $l \in [q]$ Since $y = (1, i-1)$ or $y = (i-1, q)$, we have $y \prec x$. Therefore D is acyclic. Hence we have $k(H(2, q)) \leq 2$. □

2.4. The competition number of $H(3, q)$

In this subsection, we give the competition number of a Hamming graph $H(3, q)$ with $q \geq 3$.

Lemma 10. *For $q \geq 3$, we have $k(H(3, q)) \geq 6$.*

Proof. Let $G = H(3, q)$. By Theorem 6, we have $k(G) \geq 5$. Suppose that $k(G) = 5$. Then there exists an acyclic digraph D such that $C(D) = G \cup I_5$ with $I_5 = \{z_1, z_2, \ldots, z_5\}$. Let $z_1, z_2, \ldots, z_5, v_1, v_2, \ldots, v_{q^3}$ be an acyclic ordering of D. Let $U_i := \{v_1, \ldots, v_i\}$. For convenience, let

$$\begin{aligned} A_1 : &= \{S \in \mathcal{F}(3, q) \mid S \cap U_4 \neq \emptyset\}, \\ A_2 : &= \{S \in \mathcal{F}(3, q) \mid S \cap \{v_5\} \neq \emptyset\}. \end{aligned}$$

Now we consider the subgraph $G[U_4]$ of G induced by U_4. Any graph on 4 vertices is isomorphic to one of the following.

(1) K_4	(2) $K_{1,1,2}$	(3) $K_4 - E(P_3)$	(4) C_4
(5) P_4	(6) $K_{1,3}$	(7) $K_3 \cup I_1$	(8) $K_2 \cup K_2$
(9) $P_3 \cup I_1$	(10) $K_2 \cup I_2$	(11) I_4	

Since $H(3, q)$ does not contain an induced subgraph isomorphic to $K_{1,1,2}$ by Lemma 4, $G[U_4]$ is one of the above graphs except (2). For each cases, the number $|A_1|$ is given as follows.

(1) 9	(2) −	(3) 9	(4) 8
(5) 9	(6) 9	(7) 10	(8) 10
(9) 10	(10) 11	(11) 12	

By Lemma 5, we have $|A_1| \leq 8$. Therefore $G[U_4] \cong C_4$ and so $|A_1| = 8$.

Since $|A_1| = 8$ and $|A_2| = 3$, $|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2| = 11 - |A_1 \cup A_2|$. From the fact that

$$A_1 \cup A_2 = \{S \in \mathcal{F}(3, q) \mid S \cap U_5 \neq \emptyset\},$$

it holds that $|A_1 \cup A_2| \leq 9$ by Lemma 5 Therefore, $|A_1 \cap A_2| \geq 2$.

Take two distinct cliques $S, S' \in A_1 \cap A_2$. Then $S \cap U_4 \neq \emptyset$, $S' \cap U_4 \neq \emptyset$ and so take $x \in S \cap U_4$ and $y \in S' \cap U_4$. If $x = y$ or x and y are adjacent, then $S = S'$ by Lemma 4. Therefore x and y are not adjacent. Then together fact that $G[U_4] \cong C_4$, without loss of generality, we may assume that

$$\begin{aligned} U_4 &:= \{(1,1,1), (1,1,2), (1,2,2), (1,2,1)\}, \\ x &:= (1,1,1), \\ y &:= (1,2,2). \end{aligned}$$

Since x and v_5 are adjacent, one of following holds:

$$\pi_1(v_5) = \pi_1(x) = (1,1), \quad \pi_2(v_5) = \pi_2(x) = (1,1), \quad \pi_3(v_5) = \pi_3(x) = (1,1). \tag{2.8}$$

Since y and v_5 are adjacent, one of following holds:

$$\pi_1(v_5) = \pi_1(y) = (2,2), \quad \pi_2(v_5) = \pi_2(y) = (1,2), \quad \pi_3(v_5) = \pi_3(y) = (1,2). \tag{2.9}$$

However, it is impossible that v_5 satisfies both one of (2.8) and one of (2.9). We reach a contradiction. Hence we conclude $k(H(3, q)) \geq 6$. □

In the following, we will show the following theorem.

Theorem 11. *For $q \geq 3$, we have $k(H(3, q)) = 6$.*

For $q_1, q_2, q_3 \geq 2$, we denote by $H_3(q_1, q_2, q_3)$ a graph with $V(H_3(q_1, q_2, q_3)) = [q_1] \times [q_2] \times [q_3]$ such that two vertices x and y are adjacent if $d_H(x, y) = 1$.

Define

$$\begin{array}{ll} \pi_1 : [q_1] \times [q_2] \times [q_3] \to [q_2] \times [q_3], & (x_1, x_2, x_3) \mapsto (x_2, x_3), \\ \pi_2 : [q_1] \times [q_2] \times [q_3] \to [q_1] \times [q_3], & (x_1, x_2, x_3) \mapsto (x_1, x_3), \\ \pi_3 : [q_1] \times [q_2] \times [q_3] \to [q_1] \times [q_2], & (x_1, x_2, x_3) \mapsto (x_1, x_2). \end{array}$$

For given vectors $\mathbf{p}_1 \in [q_2] \times [q_3]$, $\mathbf{p}_2 \in [q_1] \times [q_3]$, $\mathbf{p}_3 \in [q_1] \times [q_2]$, let

$$\begin{array}{rcl} S_1(\mathbf{p}_1) & := & \{x \in [q_1] \times [q_2] \times [q_3] \mid \pi_1(x) = \mathbf{p}_1\}, \\ S_2(\mathbf{p}_2) & := & \{x \in [q_1] \times [q_2] \times [q_3] \mid \pi_2(x) = \mathbf{p}_2\}, \\ S_3(\mathbf{p}_3) & := & \{x \in [q_1] \times [q_2] \times [q_3] \mid \pi_3(x) = \mathbf{p}_3\}. \end{array}$$

Note that $S_1(\mathbf{p}_1)$, $S_2(\mathbf{p}_2)$, and $S_3(\mathbf{p}_3)$ are maximal cliques of $H_3(q_1, q_2, q_3)$. We denote the set of all maximal cliques $S_1(\mathbf{p}_1)$, $S_2(\mathbf{p}_2)$ and $S_3(\mathbf{p}_3)$ by $\mathcal{F}_3(q_1, q_2, q_3)$. Then $\mathcal{F}_3(q_1, q_2, q_3)$ is an edge clique cover of $H_3(q_1, q_2, q_3)$.

Theorem 12. *For $q_1, q_2, q_3 \geq 2$, we have $k(H_3(q_1, q_2, q_3)) \leq 6$ and there exists an acyclic digraph D such that $C(D) = H_3(q_1, q_2, q_3) \cup I_6$ and $\{N_D^-(v) \mid v \in V(D), |N_D^-(v)| \geq 2\} = \mathcal{F}_3(q_1, q_2, q_3)$.*

Proof. For any digraph D, we define $\mathcal{N}^-(D) := \{N_D^-(v) \mid v \in V(D), |N_D^-(v)| \geq 2\}$. We will show by induction on $m = q_1 + q_2 + q_3$. Note that $m \geq 6$. Suppose $m = 6$ or $q_1 = q_2 = q_3 = 2$, then $H_3(2, 2, 2) = H(3, 2)$. From the fact that $H(3, 2)$ is connected and triangle-free, $k(H(3, 2)) = |E(H(3, 2))| - |V(H(3, 2))| + 2 = 12 - 8 + 2 = 6$ and any acyclic digraph D such that $C(D) = H(3, 2) \cup I_6$ satisfies that $N_D^-(v)$ is either an empty set or a maximum clique for each vertex $v \in V(D)$.

Suppose that the statement is true for $m = q_1 + q_2 + q_3$ where $m \geq 6$. Take a graph $H_3(q_1, q_2, q_3)$ such that $m+1 = q_1+q_2+q_3$. Since $q_1+q_2+q_3 > 3$, one of q_1, q_2, or q_3 is greater than 2. Without loss of generality, we may assume that $q_1 > 2$. Consider a graph $H_3(q_1 - 1, q_2, q_3)$. Then by the induction hypothesis, we have $k(H_3(q_1 - 1, q_2, q_3)) \leq 6$ and there exists an acyclic digraph D_0 such that $C(D_0) = H_3(q_1 - 1, q_2, q_3) \cup I_6$ and

$$\mathcal{N}^-(D_0) = \mathcal{F}_3(q_1 - 1, q_2, q_3). \tag{2.10}$$

Let $v_1, v_2, \ldots, v_{q_1q_2q_3 - q_2q_3 + 6}$ be an acyclic ordering of D_0. For convenience, we put $w_1 := v_{q_1q_2q_3 - q_2q_3 + 5}$ and $w_2 := v_{q_1q_2q_3 - q_2q_3 + 6}$.

Let H^* be the subgraph of $H_3(q_1, q_2, q_3)$ induced by

$$\begin{array}{rcl} V^* & := & V(H_3(q_1, q_2, q_3)) - V(H_3(q_1 - 1, q_2, q_3)) \\ & = & \{(q_1, p_2, p_3) \mid p_2 \in [q_2], p_3 \in [q_3]\}. \end{array}$$

Now we will define a digraph D_1 as follows.

$$\begin{aligned} V(D_1) &= V^* \cup \{w_1, w_2\}, \\ A(D_1) &= \left(\bigcup_{i=2}^{q_3} \{(x, (q_1, 1, i-1)) \mid x \in S_2(q_1, i)\}\right) \\ &\cup \left(\bigcup_{i=2}^{q_2} \{(x, (q_1, i-1, q)) \mid x \in S_3(q_1, i)\}\right) \\ &\cup \{(x, w_1) \mid x \in S_2(q_1, 1)\} \cup \{(x, w_2) \mid x \in S_3(q_1, 1)\}. \end{aligned}$$

The ordering obtained by adding w_1, w_2 on the head of the lexicographic ordering of V^* is an acyclic ordering of D_1, and let $w_1, w_2, \ldots, w_{q_2q_3+2}$ be the ordering. In addition,

$$\mathcal{N}^-(D_1) = \{S_2((q_1, i)), S_3((q_1, i')) \mid i \in [q_3], i'[q_2]\}. \tag{2.11}$$

Therefore D_1 is an acyclic digraph such that $C(D_1) = H^* \cup \{w_1, w_2\}$.

Note that, for $\mathbf{p} = (p_2, p_3) \in [q_2] \times [q_3]$, the clique in $H_3(q_1, q_2, q_3)$ obtained by deleting the vertex (q_1, p_2, p_3) from an element $S_1(\mathbf{p})$ of $H_3(q_1, q_2, q_3)$ is a maximal clique of $H_3(q_1 - 1, q_2, q_3)$. Then by (2.10), for each $\mathbf{p} \in [q_2] \times [q_3]$, the set $\{v \in V(D_0) \mid N^-_{D_0}(v) = S_1(\mathbf{p}) \setminus \{(q_1, p_2, p_3)\}\}$ is not empty, and so we take an element $y_{\mathbf{p}}$ of this set.

Now we will define a digraph D as follows.

$$\begin{aligned} V(D) &= V(H_3(q_1, q_2, q_3)) \cup I_6 \\ A(D) &= A(D_0) \cup A(D_1) \\ &\cup \{((q_1, p_2, p_3), y_{\mathbf{p}}) \mid \mathbf{p} = (p_2, p_3) \in [q_2] \times [q_3]\} \end{aligned}$$

Note that since $N^-_{D_0}(w_1) = N^-_{D_0}(w_2) = \emptyset$,

$$\{N^-_D(w_1), N^-_D(w_2)\} = \{S_2(q_1, 1), S_2(q_1, 1)\}. \tag{2.12}$$

From the construction of D and the equations that (2.10), (2.11) and (2.12), we can conclude that $\mathcal{N}^-(D) = \mathcal{F}_3(q_1, q_2, q_3)$ and so $E(C(D)) = E(H_3(q_1, q_2, q_3))$. Thus, $C(D) = H_3(q_1, q_2, q_3) \cup I_6$.

Then the ordering

$$v_1, v_2, \ldots, v_{q_1q_2q_3 - q_2q_3 + 6}, w_3, w_4, \ldots, w_{q_2q_3+2} \tag{2.13}$$

is an acyclic ordering of D. To see why, take an arc $a = (x, y) \in A(D)$. If $a \in A(D_0) \cup A(D_1)$, then y is appeared in front to x in (2.13), since D_0 and D_1 is acyclic. If $a \notin A(D_0) \cup A(D_1)$ then $x \in \{w_3, w_4, \ldots, w_{q_2q_3+2}\}$ and $y \in \{v_1, v_2, \ldots, v_{q_1q_2q_3 - q_2q_3+6}\}$, thus y is appeared in proceed to x in (2.13). Thus D is an acyclic digraph.

Hence $k(H_3(q_1, q_2, q_3)) \leq 6$ and theorem holds. □

Proof of Theorem 11. By Lemma 10, we have $k(H(3, q)) \geq 6$. By Theorem 12, we obtain that $k(H(3, q)) = k(H_3(q, q, q)) \leq 6$. Hence Theorem 11 holds. □

3. Concluding Remarks

In this paper, we gave the exact values of the competition numbers of Hamming graphs with diameter 2 or 3.

We conclude this paper with leaving the following questions for further study.

- What is the competition number of a Hamming graph $H(4, q)$ with diameter 4?
- What is the competition number of a ternary Hamming graph $H(n, 3)$ for $n \geq 4$?
- Give the exact values or a good bound for the competition numbers of Hamming graphs $H(n, q)$.

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