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Abstract

The competition graph of a digraph D is a graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if there exists a vertex v in D such that (x, v) and (y, v) are arcs of D. For any graph G, G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of a graph G is defined to be the smallest number of such isolated vertices. In general, it is hard to compute the competition number $k(G)$ for a graph G and characterizing a graph by its competition number has been one of important research problems in the study of competition graphs.

The Johnson graph $J(n, d)$ has the vertex set $\{v_X \mid X \in \binom{[n]}{d}\}$, where $\binom{[n]}{d}$ denotes the set of all d-subsets of an n-set $[n] = \{1, \ldots, n\}$, and two vertices v_{X_1} and v_{X_2} are adjacent if and only if $|X_1 \cap X_2| = d - 1$. In this paper, we study the edge clique number and the competition number of $J(n, d)$. Especially we give the exact competition numbers of $J(n, 2)$ and $J(n, 3)$.

Keywords: competition graph; competition number; edge clique cover; Johnson graph

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1 Introduction

The *competition graph* $C(D)$ of a digraph D is a simple undirected graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if there is a vertex v in D such that (x, v) and (y, v) are arcs of D. The notion of a competition graph was introduced by Cohen [3] as a means of determining the smallest dimension of ecological phase space (see also [4]). Since then, various variations have been defined and studied by many authors (see [11, 15] for surveys and [1, 2, 7, 8, 9, 10, 12, 14, 19, 20] for some recent results). Besides an application to ecology, the concept of competition graph can be applied to a variety of fields, as summarized in [17].

Roberts [18] observed that, for a graph G, G together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* $k(G)$ of a graph G to be the smallest number k such that G together with k isolated vertices is the competition graph of an acyclic digraph.

A subset S of the vertex set of a graph G is called a *clique* of G if the subgraph of G induced by S is a complete graph. For a clique S of a graph G and an edge e of G, we say e *is covered by* S if both of the endpoints of e are contained in S. An *edge clique cover* of a graph G is a family of cliques such that each edge of G is covered by some clique in the family. The *edge clique cover number* $\theta_E(G)$ of a graph G is the minimum size of an edge clique cover of G. We call an edge clique cover of G with the minimum size $\theta_E(G)$ a *minimum edge clique cover* of G. A *vertex clique cover* of a graph G is a family of cliques such that each vertex of G is contained in some clique in the family. The *vertex clique cover number* $\theta_V(G)$ of a graph G is the minimum size of a vertex clique cover of G. Dutton and Brigham [5] characterized the competition graphs of acyclic digraphs using edge clique covers of graphs.

Roberts [18] observed that the characterization of competition graphs is equivalent to the computation of competition numbers. It does not seem to be easy in general to compute $k(G)$ for a graph G, as Opsut [16] showed that the computation of the competition number of a graph is an NP-hard problem (see [11, 13] for graphs whose competition numbers are known). For some special graph families, we have explicit formulae for computing competition numbers. For example, if G is a choral graph without isolated vertices then $k(G) = 1$, and if G is a nontrivial triangle-free connected graph then $k(G) = |E(G)| - |V(G)| + 2$ (see [18]).

In this paper, we study the competition numbers of Johnson graphs. We denote an n-set $\{1, \dots, n\}$ by $[n]$ and the set of all d-subsets of an n-set by $\binom{[n]}{d}$. The *Johnson graph* $J(n, d)$ has the vertex set $\{v_X \mid X \in \binom{[n]}{d}\}$, and two vertices v_{X_1} and v_{X_2} are adjacent if and only if $|X_1 \cap X_2| = d - 1$ (for reference, see [6]). For example, the Johnson graph $J(5, 2)$ is given in Figure 1. As it is known that $J(n, d) \cong J(n, n - d)$, we assume that $n \geq 2d$. Our main results are the following.

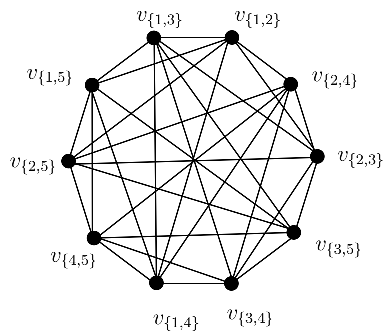

Figure 1: The Johnson graph $J(5, 2)$

Theorem 1. *For $n \geq 4$, we have $k(J(n, 2)) = 2$.*

Theorem 2. *For $n \geq 6$, we have $k(J(n, 3)) = 4$.*

We use the following notation and terminology in this paper. For a digraph D, an ordering $v_1, v_2, \ldots, v_n$ of the vertices of D is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i < j$. It is well-known that a digraph D is acyclic if and only if there exists an acyclic ordering of D. For a digraph D and a vertex v of D, the *out-neighborhood* of v in D is the set $\{w \in V(D) \mid (v, w) \in A(D)\}$. A vertex in the out-neighborhood of a vertex v in a digraph D is called a *prey* of v in D. For simplicity, we denote the out-neighborhood of a vertex v in a digraph D by $P_D(v)$ instead of usual notation $N_D^+(v)$. For a graph G and a vertex v of G, we define the *(open) neighborhood* $N_G(v)$ of v in G to be the set $\{u \in V(G) \mid uv \in E(G)\}$. We sometimes also use $N_G(v)$ to stand for the subgraph induced by its vertices.

2 A lower bound for the competition number of $J(n, d)$

In this section, we give lower bounds for the competition number of the Johnson graph $J(n, d)$.

Lemma 3. *Let n and d be positive integers with $n \geq 2d$. For any vertex x of the Johnson graph $J(n, d)$, we have $\theta_V(N_{J(n,d)}(x)) = d$.*

Proof. If $d = 1$, then $J(n, d)$ is a complete graph and the lemma is trivially true. Assume that $d \geq 2$. Take any vertex x in $J(n, d)$. Then $x = v_A$ for some $A \in \binom{[n]}{d}$. For any vertex v_A in $J(n, d)$, the set

$$S_i(v_A) := \{v_B \mid B = (A \setminus \{i\}) \cup \{j\} \text{ for some } j \in [n] \setminus A\}$$

forms a clique of $J(n, d)$ for each $i \in A$. To see why, take two distinct vertices v_B and v_C in $S_i(v_A)$. Then $B = (A \setminus \{i\}) \cup \{j\}$ and $C = (A \setminus \{i\}) \cup \{k\}$ for some distinct $j, k \in [n] \setminus A$. Clearly $|B \cap C| = d - 1$, and so v_B and v_C are adjacent in $J(n, d)$.

Take a vertex v_B in $N_{J(n,d)}(v_A)$. Then $B = (A \setminus \{i\}) \cup \{j\}$ for some $i \in A$ and $j \in [n] \setminus A$ and so $v_B \in S_i(v_A)$. Thus $\{S_i(v_A) \mid i = 1, \ldots, d\}$ is a vertex clique cover of $N_{J(n,d)}(v_A)$. Thus $\theta_V(N_{J(n,d)}(v_A)) \leq d$. On the other hand,

$$|((A \setminus \{i\}) \cup \{j\}) \cap ((A \setminus \{i'\}) \cup \{j'\})| = d - 2$$

if $i, i' \in A$ and $j, j' \in [n] \setminus A$ satisfy $i \neq i'$ and $j \neq j'$ (such i, i', j, j' exist since $n \geq 2d \geq 4$). This implies that $\theta_V(N_{J(n,d)}(v_A)) \geq d$. Hence $\theta_V(N_{J(n,d)}(v_A)) = d$. □

Opsut [16] gave a lower bound for the competition number of a graph G as follows:

$$k(G) \geq \min\{\theta_V(N_G(v)) \mid v \in V(G)\}.$$

Together with Lemma 3, we have $k(J(n,d)) \geq d$ for positive integers n and d satisfying $n \geq 2d$. The following theorem gives a better lower bound for $k(J(n,d))$ if $d \geq 2$.

Theorem 4. *For $n \geq 2d \geq 4$, we have $k(J(n,d)) \geq 2d - 2$.*

Proof. Put $k := k(J(n,d))$. Then there exists an acyclic digraph D such that $C(D) = J(n,d) \cup I_k$, where $I_k = \{z_1, z_2, \ldots, z_k\}$ is a set of isolated vertices. Let x_1, x_2, …, $x_{\binom{n}{d}}$, z_1, z_2, …, z_k be an acyclic ordering of D. Let $v_1 := x_{\binom{n}{d}}$ and $v_2 := x_{\binom{n}{d}-1}$. By Lemma 3, we have $\theta_V(N_{J(n,d)}(x_i)) = d$ for $i = 1, \ldots, \binom{n}{d}$. Thus v_i has at least d distinct prey in D, that is,

$$|P_D(v_i)| \geq d. \tag{2.1}$$

Since $x_1, x_2, \ldots, x_{\binom{n}{d}}, z_1, z_2, \ldots, z_k$ is an acyclic ordering of D, we have

$$P_D(v_1) \cup P_D(v_2) \subset I_k \cup \{v_1\}. \tag{2.2}$$

Moreover, we may claim the following:

Claim. For any two adjacent vertices v_{X_1} and v_{X_2} of $J(n,d)$, we have $|P_D(v_{X_1}) \setminus P_D(v_{X_2})| \geq d - 1$.

Proof of Claim. Suppose that v_{X_1} and v_{X_2} are adjacent in $J(n,d)$. Then $|X_1 \cap X_2| = d - 1$ and

$$|[n] \setminus (X_1 \cup X_2)| \geq 2d - |X_1| - |X_2| + |X_1 \cap X_2| = d - 1.$$

We take $d-1$ elements from $[n] \setminus (X_1 \cup X_2)$, say $z_1, z_2, \ldots, z_{d-1}$, and put $X_1 \cap X_2 := \{y_1, y_2, \ldots, y_{d-1}\}$.

For each $1 \leq j \leq d - 1$, we put $Z_j := X_1 \cup \{z_j\} \setminus \{y_j\}$. Then $|Z_j| = d$ and so v_{Z_j} is a vertex in $J(n,d)$. Note that $|Z_j \cap X_1| = d - 1$ and $|Z_j \cap X_2| = d - 2$. Thus v_{Z_j} is adjacent to v_{X_1} while it is not adjacent to v_{X_2}. Therefore

$$P_D(v_{X_1}) \cap P_D(v_{Z_j}) \neq \emptyset \quad \text{and} \quad P_D(v_{X_2}) \cap P_D(v_{Z_j}) = \emptyset.$$

Now we show that

$$P_D(v_{X_1}) \setminus P_D(v_{X_2}) \supseteq \bigcup_{j=1}^{d-1} \left(P_D(v_{X_1}) \cap P_D(v_{Z_j})\right). \tag{2.3}$$

Take a vertex x in $\bigcup_{i=1}^{d-1} \left(P_D(v_{X_1}) \cap P_D(v_{Z_j})\right)$. Then $x \in P_D(v_{X_1})$ and $x \in P_D(v_{Z_j})$ for some $j \in \{1, \ldots d-1\}$. Since $P_D(v_{X_2}) \cap P_D(v_{Z_j}) = \emptyset$, $x \notin P_D(v_{X_2})$ and so $x \in P_D(v_{X_1}) \setminus P_D(v_{X_2})$. Thus (2.3) follows.

Note that for any $j \in \{1, \ldots d-1\}$, since $P_D(v_{X_1}) \cap P_D(v_{Z_j}) \neq \emptyset$,

$$|P_D(v_{X_1}) \cap P_D(v_{Z_j})| \geq 1. \tag{2.4}$$

Moreover, $P_D(v_{X_1}) \cap P_D(v_{Z_i})$ and $P_D(v_{X_1}) \cap P_D(v_{Z_j})$ are mutually disjoint for $i \neq j$. To see why, note that $|Z_j \cap Z_i| = d-2$ for $i \neq j$. Therefore v_{Z_i} and v_{Z_j} are not adjacent and so $P_D(v_{Z_i}) \cap P_D(v_{Z_j}) = \emptyset$. Thus

$$(P_D(v_{X_1}) \cap P_D(v_{Z_i})) \cap \left(P_D(v_{X_1}) \cap P_D(v_{Z_j})\right) = \emptyset. \tag{2.5}$$

From (2.3), (2.4), and (2.5), it follows that

$$|P_D(v_{X_1}) \setminus P_D(v_{X_2})| \geq \sum_{i=1}^{d-1} |P_D(v_{X_1}) \cap P_D(v_{Z_j})| \geq d-1.$$

This completes the proof of the claim. □

Now suppose that v_1 and v_2 are not adjacent in $J(n,d)$. Then v_1 and v_2 do not have a common prey in D, that is,

$$P_D(v_1) \cap P_D(v_2) = \emptyset. \tag{2.6}$$

By (2.1), (2.2) and (2.6), we have

$$k+1 \geq |P_D(v_1) \cup P_D(v_2)| = |P_D(v_1)| + |P_D(v_2)| \geq 2d.$$

Hence $k \geq 2d-1 > 2d-2$.

Next suppose that v_1 and v_2 are adjacent in $J(n,d)$. Then v_1 and v_2 have at least one common prey in D, that is,

$$|P_D(v_1) \cap P_D(v_2)| \geq 1. \tag{2.7}$$

By the above claim,

$$|P_D(v_1) \setminus P_D(v_2)| \geq d-1 \quad \text{and} \quad |P_D(v_2) \setminus P_D(v_1)| \geq d-1. \tag{2.8}$$

Then

$$\begin{aligned} k+1 &\geq |P_D(v_1) \cup P_D(v_2)| \qquad \text{(by (2.2))} \\ &= |P_D(v_1) \setminus P_D(v_2)| + |P_D(v_2) \setminus P_D(v_1)| + |P_D(v_1) \cap P_D(v_2)| \\ &\geq (d-1) + (d-1) + 1 \qquad \text{(by (2.7) and (2.8))} \\ &= 2d-1. \end{aligned}$$

Hence it holds that $k \geq 2d-2$. □

3 An edge clique cover of $J(n, d)$

In this section, we build a minimum edge clique cover of $J(n, d)$.

Given a Johnson graph $J(n, d)$, we define a family $\mathcal{F}_d^n$ of cliques of $J(n, d)$ as follows. For each $Y \in \binom{[n]}{d-1}$, we put

$$S_Y := \{v_X \mid X = Y \cup \{j\} \text{ for } j \in [n] - Y\}$$

Note that S_Y is a clique of $J(n, d)$ with size $n - d + 1$. We let

$$\mathcal{F}_d^n := \{S_Y \mid Y \in \binom{[n]}{d-1}\}. \tag{3.1}$$

Then it is not difficult to show that $\mathcal{F}_d^n$ is the collection of cliques of maximum size. Moreover the family $\mathcal{F}_d^n$ is an edge clique cover of $J(n, d)$. To wee why, take any edge $v_{X_1}v_{X_2}$ of $J(n, d)$. Then $|X_1 \cap X_2| = d - 1$ and both of v_{X_1} and v_{X_2} belong to the clique $S_{X_1 \cap X_2} \in \mathcal{F}_d^n$. Thus $\mathcal{F}_d^n$ is an edge clique cover of $J(n, d)$.

We will show that $\mathcal{F}_d^n$ is a minimum edge clique cover of $J(n, d)$. Prior to that, we present the following theorem. For two distinct cliques S and S' of a graph G, we say S and S' are *edge disjoint* if $|S \cap S'| \leq 1$.

Theorem 5. $\theta_E(J(n, d)) = \binom{n}{d-1}$ *and any minimum edge clique cover of* $J(n, d)$ *consists of edge disjoint maximum cliques.*

Proof. Let $\mathcal{E}$ be a minimum edge clique cover for $J(n, d)$, that is, $\theta_E(J(n, d)) = |\mathcal{E}|$. Since $\mathcal{F}_d^n$ is an edge clique cover with $|\mathcal{F}_d^n| = \binom{n}{d-1}$, we have $\theta_E(J(n, d)) \leq \binom{n}{d-1}$.

Now we show that $|\mathcal{E}| \geq \binom{n}{d-1}$. Since the size of a maximum clique is $n - d + 1$, we have $|E(S)| \leq \binom{n-d+1}{2}$ for each $S \in \mathcal{E}$ where $E(S) = \binom{S}{2}$. Therefore,

$$|E(J(n, d))| \leq \sum_{S \in \mathcal{E}} |E(S)| \leq \binom{n-d+1}{2} \times |\mathcal{E}|, \tag{3.2}$$

and the first equality holds if and only if none of two distinct cliques in $\mathcal{E}$ have a common edge, and the second equality holds if and only if any element of $\mathcal{E}$ is a maximum clique in $J(n, d)$.

Since the Johnson graph $J(n, d)$ is a $d(n-d)$-regular graph and the number of vertices of $J(n, d)$ is $\binom{n}{d}$,

$$|E(J(n, d))| = \frac{1}{2} d(n - d) \times \binom{n}{d} = \binom{n-d+1}{2} \times \binom{n}{d-1}. \tag{3.3}$$

From (3.2) and (3.3), it follows that $\binom{n-d+1}{2} \times \binom{n}{d-1} \leq \binom{n-d+1}{2} \times |\mathcal{E}|$. Thus we have $\binom{n}{d-1} \leq |\mathcal{E}|$. Hence we can conclude that $\theta_E(J(n, d)) = \binom{n}{d-1}$.

Furthermore, two equalities in (3.2) must hold, and therefore any minimum edge clique cover of $J(n, d)$ consists of edge disjoint maximum cliques. □

Since $|\mathcal{F}_d^n| = \binom{n}{d-1}$, the following corollary is an immediate consequence of Theorem 5:

Corollary 6. *The edge clique cover $\mathcal{F}_d^n$ of $J(n,d)$ defined in (3.1) is a minimum edge clique cover of $J(n,d)$.*

4 Proofs of Theorems 1 and 2

First, we define an order $\prec$ on the set $\binom{[n]}{d}$ as follows. Take two distinct elements X_1 and X_2 in $\binom{[n]}{d}$. Let $X_1 = \{i_1, i_2, \ldots, i_d\}$ and $X_2 = \{j_1, j_2, \ldots, j_d\}$ where $i_1 < \ldots < i_d$ and $j_1 < \ldots < j_d$. Then we define $X_1 \prec X_2$ if there exists $t \in \{1, \ldots, d\}$ such that $i_s = j_s$ for $1 \leq s \leq t-1$ and $i_t < j_t$. It is easy to see that $\prec$ is a total order.

Now we prove Theorem 1.

Proof of Theorem 1. As $k(J(n,2)) \geq 2$ by Theorem 4, it remains to show $k(J(n,2)) \leq 2$. We define a digraph D as follows:

$$V(D) = V(J(n,2)) \cup I_2$$

where $I_2 = \{z_1, z_2\}$, and

$$A(D) = \bigcup_{i=1}^{n-2} \{(x, v_{\{i+1,i+2\}}) \mid x \in S_{\{i\}} \in \mathcal{F}_2^n\} \cup \bigcup_{i=1}^{2} \{(x, z_i) \mid x \in S_{\{n-2+i\}} \in \mathcal{F}_2^n\}.$$

Since the vertices of each clique in the edge clique cover $\mathcal{F}_2^n$ has a common prey in D, it holds that $C(D) = J(n,2) \cup I_2$. Each vertex in S_i is denoted by v_X for some $X \in \binom{[n]}{2}$ which contains i. Then by the definition of $\prec$, $v_X \prec v_{\{i+1,i+2\}}$ for $i = 1, \ldots, n-2$. Thus, there exists an arc from a vertex x to a vertex y in D if and only if either $x = v_X$ and $y = v_Y$ with $X \prec Y$, or $x = v_X$ and $y = z_i$ with $X \in S_{\{n-1\}} \cup S_{\{n\}}$ and $i \in \{1, 2\}$. Therefore D is acyclic. Thus we have $k(J(n,2)) \leq 2$ and this completes the proof. $\square$

Proof of Theorem 2. By Theorem 4, we have $k(J(n,3)) \geq 4$. It remains to show $k(J(n,3)) \leq 4$. We define a digraph D as follows:

$$V(D) = V(J(n,3)) \cup I_4$$

where $I_4 = \{z_1, z_2, z_3, z_4\}$, and

$$\begin{aligned} A(D) &= \bigcup_{i=1}^{n-3} \bigcup_{j=i+1}^{n-2} \left\{(x, v_{\{i,j+1,j+2\}}) \mid x \in S_{\{i,j\}} \in \mathcal{F}_3^n\right\} \\ &\cup \bigcup_{i=1}^{n-3} \left\{(x, v_{\{i+1,i+2,i+3\}}) \mid x \in S_{\{i,n-1\}} \in \mathcal{F}_3^n\right\} \\ &\cup \bigcup_{i=1}^{n-4} \left\{(x, v_{\{i+1,i+2,i+4\}}) \mid x \in S_{\{i,n\}} \in \mathcal{F}_3^n\right\} \\ &\cup \bigcup_{i=1}^{3} \{(x, z_i) \mid x \in S_{\{n-4+i,n\}} \in \mathcal{F}_3^n\} \\ &\cup \{(x, z_4) \mid x \in S_{\{n-2,n-1\}} \in \mathcal{F}_3^n\}. \end{aligned}$$

It is easy to check that

$$\begin{aligned} \mathcal{F}_3^n &= \{S_{\{i,j\}} \mid i = 1, \ldots, n-3; j = i+1, \ldots, n-2\} \cup \{S_{\{i,n-1\}} \mid i = 1, \ldots, n-3\} \\ &\cup \{S_{\{i,n\}} \mid i = 1, \ldots, n-4\} \cup \{S_{\{n-3,n\}}, S_{\{n-2,n\}}, S_{\{n-1,n\}}\} \cup \{S_{\{n-2,n-1\}}\}. \end{aligned}$$

Thus $C(D) = J(n, 3) \cup I_4$. Moreover, any vertex $x \in S_{\{i,j\}}$ is denoted by v_X for some $X \in \binom{[n]}{3}$ which contains i and j. By the definition of $\prec$, $X \prec \{i, j+1, j+2\}$. In a similar manner, for x in other cliques in $\mathcal{F}_3^n$, we may show that $(x, y) \in A(D)$ if and only if either $x = v_X$ and $y = v_Y$ with $X \prec Y$, or $x = v_X$ and $y = z_i$ with $X \in S_{\{n-3,n\}} \cup S_{\{n-2,n\}} \cup S_{\{n-1,n\}} \cup S_{\{n-2,n-1\}}$ and $i \in \{1, 2, 3, 4\}$. Thus D is acyclic. Hence $k(J(n, 3)) \leq 4$. □

5 Concluding Remarks

In this paper, we gave some lower bounds for the competition numbers of Johnson graphs, and computed the competition numbers of Johnson graphs $J(n, 2)$ and $J(n, 3)$. It would be natural to ask: What is the exact value of the competition number of a Johnson graph $J(n, 4)$ for $n \geq 8$? Eventually, what is the exact values of the competition numbers of Johnson graphs $J(n, q)$?

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