RIMS-1669

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April 2009

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# The competition number of a graph with exactly two holes

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April 2009

**Abstract**

Let $D$ be an acyclic digraph. The competition graph of $D$ is a graph which has the same vertex set as $D$ and has an edge between $x$ and $y$ if and only if there exists a vertex $v$ in $D$ such that $(x, v)$ and $(y, v)$ are arcs of $D$. For any graph $G$, $G$ together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. The competition number $k(G)$ of $G$ is the smallest number of such isolated vertices.

A hole of a graph is a cycle of length at least 4 as an induced subgraph. In 2005, Kim [5] conjectured that the competition number of a graph with $h$ holes is at most $h + 1$. Though Kim *et al.* [7] and Li and Chang [8] showed that her conjecture is true when the holes do not overlap much, it still remains open for the case where the holes share edges in an arbitrary way. In order to share an edge, a graph must have at least two holes and so it is natural to start with a graph with exactly two holes. In this paper, the conjecture is true for such a graph.

**Keywords:** competition graph; competition number; hole

*This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2007-313-C00012).

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## 1 Introduction

Suppose $D$ is an acyclic digraph (for all undefined graph-theoretical terms, see [1] and [13]). The *competition graph* of $D$, denoted by $C(D)$, has the same vertex set as $D$ and has an edge between vertices $x$ and $y$ if and only if there exists a vertex $v$ in $D$ such that $(x, v)$ and $(y, v)$ are arcs of $D$. Roberts [12] observed that, for any graph $G$, $G$ together with sufficiently many isolated vertices is the competition graph of an acyclic digraph. Then he defined the *competition number* $k(G)$ of a graph $G$ to be the smallest number $k$ such that $G$ together with $k$ isolated vertices added is the competition graph of an acyclic digraph.

The notion of competition graph was introduced by Cohen [3] as a means of determining the smallest dimension of ecological phase space. Since then, various variations have been defined and studied by many authors (see [4, 9] for surveys). Besides an application to ecology, the concept of competition graph can be applied to a variety of fields, as summarized in [11].

Roberts [12] observed that characterization of competition graph is equivalent to computation of competition number. It does not seem to be easy in general to compute $k(G)$ for a graph $G$, as Opsut [10] showed that the computation of the competition number of a graph is an NP-hard problem (see [4, 6] for graphs whose competition numbers are known). It has been one of important research problems in the study of competition graphs to characterize a graph by its competition number. Cho and Kim [2] and Kim [5] studied the competition number of a graph with exactly one hole. A cycle of length at least 4 of a graph as an induced subgraph is called a *hole* of the graph and a graph without holes is called a *chordal graph*. As Roberts [12] showed that the competition number of a chordal graph is at most 1, the competition number of a graph with $0$ hole is at most $1$. Cho and Kim [2] showed that the competition number of a graph with exactly $1$ hole is at most $2$.

**Theorem 1.1** (Cho and Kim [2]). *Let $G$ be a graph with exactly one hole. Then the competition number of $G$ is at most $2$.*

Kim [5] conjectured that the competition number of a graph with $h$ holes is at most $h + 1$ from these results. Recently, Li and Chang [8] showed that her conjecture is true for graphs, all of whose holes are independent. In a graph $G$, a hole $C$ is *independent* if the following two conditions hold for any other hole $C'$ of $G$,

1. $C$ and $C'$ have at most two common vertices.

2. If $C$ and $C'$ have two common vertices, then they have one common edge and $C$ is of length at least $5$.

**Theorem 1.2** (Li and Chang [8]). *Suppose that $G$ is a graph with exactly $h$ holes, all of which are independent. Then $k(G) \leq h + 1$.*

After then, Kim, Lee, and Sano [7] generalized the above theorem to the following theorem.

**Theorem 1.3** (Kim *et al.* [7])**.** *Let $C_1, \dots, C_h$ be the holes of a graph $G$. Suppose that*

(a) *any pairs of $C_1, \dots, C_h$ shares at most one edge, and*

(b) *if $C_i$ and $C_j$ share an edge, then both $C_i$ and $C_j$ have length at least $5$.*

*Then $k(G) \leq h + 1$.*

Thus, it is natural to ask if the bound holds when the holes share the arbitrary many edges. In this paper, we show that the answer is yes for a graph $G$ with exactly two holes. Our main theorem is as follows.

**Theorem 1.4.** *Let $G$ be a graph with exactly two holes. Then the competition number of $G$ is at most $3$.*

This paper is organized as follows. In Section 2, we investigate some properties of graphs with holes. In Section 3, we give a proof of Theorem 1.4.

## 2 Preliminaries

A set $S$ of vertices of a graph $G$ is called a *clique* of $G$ if the subgraph of $G$ induced by $S$ is a complete graph. A set $S$ of vertices of a graph $G$ is called a *vertex cut* of $G$ if the number of connected components of $G - S$ is greater than that of $G$.

Cho and Kim [2] showed that for a chordal graph $G$, we can construct an acyclic digraph $D$ with the vertices of indegree $0$ as many as the number of the vertices of a clique so that the competition graph of $D$ is $G$ with one more isolated vertex:

**Lemma 2.1** ([2])**.** *If $X$ is a clique of a chordal graph $G$, then there exists an acyclic digraph $D$ such that $C(D) = G \cup I_1$, and the vertices of $X$ have only outgoing arcs in $D$.*

**Theorem 2.2.** *Let $G$ be a graph and $k$ be a nonnegative integer. Suppose that $G$ has a subgraph $G_1$ with $k(G_1) \leq k$ and a chordal subgraph $G_2$ such that $E(G_1) \cup E(G_2) = E(G)$ and $X := V(G_1) \cap V(G_2)$ is a clique of $G_2$. Then $k(G) \leq k + 1$.*

*Proof.* Since $k(G_1) \leq k$, there exists an acyclic digraph $D_1$ such that $C(D_1) = G_1 \cup I_k$ where $I_k$ is a set of $k$ isolated vertices with $I_k \cap V(G) = \emptyset$. Since $X$ is a clique of a chordal graph $G_2$, there exists an acyclic digraph $D_2$ such that $C(D_2) = G_2 \cup \{a\}$ where $a$ is an isolated vertex not in $V(G) \cup I_k$ and the vertices in $X$ have only outgoing arcs in $D_2$ by Lemma 2.1. Now we define a digraph $D$ as follows: $V(D) = V(D_1) \cup V(D_2)$ and $A(D) = A(D_1) \cup A(D_2)$.

Suppose that there is an edge in $E(C(D))$ but not in $E(C(D_1)) \cup E(C(D_2))$. Then there exist an arc $(u, x)$ in $D_1$ and an arc $(v, x)$ in $D_2$ for some $x \in X$. However, this is impossible since every vertex in $X$ has indegree 0 in $D_2$. Thus $E(C(D)) \subset E(C(D_1)) \cup E(C(D_2))$. It is obvious that $E(C(D)) \supset E(C(D_1)) \cup E(C(D_2))$ since $E(C(D)) \supset E(C(D_i))$ for $i = 1, 2$. Thus $E(C(D)) = E(C(D_1)) \cup E(C(D_2)) = E(G_1) \cup E(G_2) = E(G)$.

Moreover, since $D_1$ and $D_2$ are acyclic, $V(G_1) \cap V(G_2) = X$, and each vertex in $X$ has only outgoing arcs in $D_2$, it is true that $D$ is also acyclic. Hence $C(D) = G \cup I_k \cup \{a\}$ and so $k(G) \leq k + 1$. □

**Lemma 2.3** ([7]). *Let $G$ be a graph and $C$ be a hole of $G$. Suppose that $v$ is a vertex not on $C$ that is adjacent to two non-adjacent vertices $x$ and $y$ of $C$. Then exactly one of the following is true:*

(1) *$v$ is adjacent to all the vertices of $C$;*

(2) *$v$ is on a hole $C^*$ different from $C$ such that there are at least two common edges of $C$ and $C^*$ and all the common edges are contained in exactly one of the $(x, y)$-sections of $C$.*

For a graph $G$ and a hole $C$ of $G$, we denote by $X_C$ the set of vertices which are adjacent to all vetices of $C$. Given a walk $W$ of a graph $G$, we denote by $W^{-1}$ the walk represented by the reverse of vertex sequence of $W$ and denote the length of $W$ by $|W|$. For a graph $G$ and a hole $C$ of $G$, we call a walk (resp. path) $W$ a *$C$-avoiding walk* (resp. *$C$-avoiding path*) if one of the following hold:

- $|W| \geq 2$ and none of the internal vertices of $W$ are in $V(C) \cup X_C$;
- $|W| = 1$ and one of the two vertices of $W$ is not in $V(C) \cup X_C$.

**Lemma 2.4.** *Let $G$ be a graph and $C = v_0v_1 \cdots v_{m-1}v_0$ be a hole of $G$. Suppose that there exists a vertex $v$ satisfying the following properties:*

- *$v$ is not on any hole of $G$.*
- *$v$ is adjacent to $v_i$ for some $i \in \{0, \ldots, m-1\}$.*
- *There is a $C$-avoiding path from $v$ to a vertex on $C$ other than $v_i$.*

*Let $v_j$ be a vertex with the smallest $|i - j|$ such that there is a $C$-avoiding $(v, v_j)$-path and $P$ be the shortest among $C$-avoiding $(v, v_j)$-paths. Then $v_i$ is adjacent to every internal vertex on $P$. Moreover, if none of internal vertices on $P$ belongs to any hole, then $j = i-1$ or $i + 1$.*

*Proof.* Let $Q$ be the shorter $(v_i, v_j)$-section of $C$. Firstly, consider the case where $|E(P)| = 1$. If $j \neq i-1$ or $i+1$, then the hypothesis of Lemma 2.3 is satisfied. However, none of (1), (2) holds, which is a contradiction. Thus, $j \in \{i-1, i+1\}$ and we are done.

Now suppose that $|E(P)| \geq 2$. Then $v_i P Q^{-1}$ is a cycle of length at least 4. Since $v$ is not on any hole on $G$, it cannot be a hole and has a chord. Take an internal vertex $w$ on $P$. If $w$ is adjacent to a vertex $v_k$ for some $k$, $1 \leq |i-k| \leq |i-j|$, then $v_i$, the $(v, w)$-section of $P$, and $v_k$ form a $C$-avoiding $(v_i, v_k)$-path, which contradicts the choice of $v_j$. Thus no internal vertex of $P$ is adjacent to any vertex on the shorter $(v_i, v_j)$-section of $C$ except $v_i$. Thus $v_i$ is adjacent to an internal vertex of $P$. Let $x$ be the first internal vertex on $P$ and $P'$ be the $(v, x)$-section of $P$. Then $v_i P' v_i$ is a hole or a triangle. However, the former cannot happen by the condition on $v$. Thus $x$ immediately follows $v$ on $P$. By repeating this argument, we can show that $v_i$ is adjacent to every internal vertex on $P$.

Now assume that none of internal vertices on $P$ does not belong to any hole. Let $y$ be the vertex immediately preceding $v_j$ on $P$. Then $v_i y Q^{-1}$ is a hole or a triangle. By our assumption, the former does not hold. Thus $Q$ is a path of length 1, that is, $v_i$ and $v_j$ are adjacent. Hence $j = i-1$ or $j = i+1$. □

## 3 Proof of Theorem 1.4

In this section, we shall show that the competition number of a graph with exactly two holes cannot exceed 3.

Let $G$ be a graph with exactly two holes $C_1$ and $C_2$. We denote the holes of $G$ by

$$C_1 : v_0 v_1 \cdots v_{m-1} v_0, \quad C_2 : w_0 w_1 \cdots w_{m'-1} w_0,$$

where $m$ and $m'$ are the length of the holes $C_1$ and $C_2$, respectively. In the following, we assume that all subscripts of vertices on a cycle are considered in modulo the length of the cycle.

Without loss of generality, we may assume that $m \geq m' \geq 4$. For $t \in \{1, 2\}$, let

$$X_t := X_{C_t} = \{x \in V(G) \mid xv \in E(G) \text{ for all } v \in V(C_t)\}.$$

In the following, we deal with the case that the two holes have a common edge since Theorem 1.3 covers the case that the two holes are edge disjoint.

**Lemma 3.1.** *If a graph $G$ has exactly two holes $C_1$ and $C_2$, then both $X_1$ and $X_2$ are cliques.*

*Proof.* Suppose that two distinct vertices $u_1$ and $u_2$ in $X_1$ are not adjacent. Then $u_1 v_0 u_2 v_2 u_1$ and $u_1 v_1 u_2 v_3 u_1$ are two holes other than $C_1$. That is, $G$ has at least three holes, which is a contradiction. □

**Lemma 3.2.** *Let $G$ be a connected graph having exactly two holes $C_1$ and $C_2$. If $C_1$ and $C_2$ have a common edge, then $G[E(C_1) \cap E(C_2)]$ is a path.*

*Proof.* Suppose that $G[E(C_1) \cap E(C_2)]$ is not a path. Without loss of generality, we may assume that $v_0v_1$ is a common edge but $v_1v_2$ is not common. Let $v_i$ is the first vertex on $C$ after $v_1$ common to $C_1$ and $C_2$. Then $i \in \{2, \dots, m-2\}$. Other than $v_0$, let $w$ be the vertex on $C_2$ that is adjacent to $v_1$. In addition, let $Z$ be the $(w, v_i)$-section of $C_2$ not containing $v_0$.

Now, consider a $(w, v_{m-1})$-walk $W = Zv_{i+1} \cdots v_{m-1}$. Then $G[W]$ contains a $(w, v_{m-1})$-path. Let $P$ be a shortest path among such paths. We shall claim that $C = v_0v_1Pv_0$ is a hole. Since neither $v_0$ nor $v_1$ is on $W$, none of $v_0$, $v_1$ is on $P$. Thus $C$ is a cycle. By the definition of $P$, there is no chord between any pair of vertices on $P$. Since $C_1$ is a hole, $v_0$ is not adjacent to any of $v_{i+1}, \dots, v_{m-2}$. On the other hand, $V(Z) \subset V(C_2)$ and $v_0 \in V(C_2)$. Thus $v_0$ is not adjacent to any vertex on $Z$. Thus $v_0$ is not adjacent to any vertex on $P$. By a similar argument, we can claim that $v_1$ is not adjacent to any vertex on $P$ except $u$. Hence we have shown that $C$ is a hole of $G$. Since $C$ does not contain the edge $v_1v_2$, $C$ is not $C_1$ and so $C = C_2$.

Suppose that $v_j$ is adjacent to a vertex $v$ on $Z$ for some $j \in \{i+1, \dots, m-1\}$. Then $v_jv$ is shorter than any $(v, v_j)$-path containing $v_i$ in $G[W]$ and so $P$ does not contain $v_i$. Thus $v_i \notin V(C)$. Hence $C$ does not contain $v_i$ and so $C$ is distinct from $C_2$, which is a contradiction. Therefore, $v_j$ is not adjacent to any vertex on $Z$ for any $j \in \{i+1, \dots, m-1\}$. If $v_j$ is on $Z$, then $v_j$ is adjacent to a vertex on $Z$, which is impossible. This implies that no vertex on $W$ repeats and that no two nonconsecutive vertices in $W$ are adjacent. Thus $W = P$. Then $G[E(C_1) \cap E(C_2)] = v_iv_{i+1} \cdots v_{m-1}v_0v_1$ is a path and we reach a contradiction. □

**Lemma 3.3.** *Let $G$ be a connected graph having exactly two holes $C_1$ and $C_2$. If $|E(C_1) \cap E(C_2)| \geq 2$, then $X_1 = X_2$.*

*Proof.* By Lemma 3.2, $G[E(C_1) \cap E(C_2)] = w_iw_{i+1} \cdots w_j$ where $|j - i| \geq 2$. We take a vertex $v \in X_1$. Note that $v \notin \{w_i, \dots, w_j\}$. Then $v$ must be contained in $V(C_2)$ or $X_2$ by the Lemma 2.3 since $v$ is adjacent to non-adjacent vertices $w_i$ and $w_j$ in $V(C_2)$. If $v \in V(C_2)$, then $C_2$ has a chord $vw_{i+1}$, which is a contradiction. Therefore, $v \in X_2$. Thus, $X_1 \subseteq X_2$. Similarly, it can be shown that $X_2 \subseteq X_1$. □

**Lemma 3.4.** *Let $G$ be a connected graph having exactly two holes $C_1$ and $C_2$. If there is no $C_t$-avoiding $(u, v)$-path for consecutive vertices $u$, $v$ on $C_t$ for $t \in \{1, 2\}$, then $G - uv$ has at most one hole.*

*Proof.* First consider the case where $uv$ is not a common edge of $C_1$ and $C_2$. We may assume that $uv$ is an edge of $C_1$. Suppose that $G - uv$ has at least two holes. Let $C^*$ be a hole of $G - uv$ distinct from $C_2$. Now $C^* + uv$ are two cycles $C_1$ and $C'$ sharing exactly one edge that is $uv$. Since $uv$ does not belong to $C_2$, $C'$ cannot be $C_2$. If the length of $C'$ is greater than 3, then $C'$ is a hole, contradicting the hypothesis. Thus, $C' - uv$ is a path of length 2. Let $x$ be the internal vertex of $C' - uv$. Since there is no $C_1$-avoiding $(u, v)$-path, it is true that $x \in X_1$. However, this implies that $C^*$ has a chord joining $x$ and every vertex in $V(C_1) \setminus \{u, v\}$, which is a contradiction.

Now we consider the case where $uv$ is a common edge of $C_1$ and $C_2$. Then $G - uv$ contains neither $C_1$ nor $C_2$. Moreover, $X_1 = X_2$. For, if there exist a vertex $x \in X_1 \setminus X_2$ (resp. $x \in X_2 \setminus X_1$), $uxv$ is a $C_2$-avoiding (resp. $C_1$-avoiding) path, which is a contradiction. We let $X = X_1 = X_2$.

Now we show that $G - uv$ does not contain a hole. Suppose that $G - uv$ contains a hole $H$. We will reach a contradiction. Since $H$ is not a hole of $G$, $uv$ is a chord of $H$ in $G$. In fact, $uv$ is the only chord of $H$ in $G$ and so $|H| = 4$. Let $H = uxvyu$. Then since there is no $C_1$-avoiding $(u, v)$-path in $G$, both $x$ and $y$ belong to $X$. Then $x$ and $y$ are adjacent by Lemma 3.1 and so $xy$ is a chord of $H$ in $G - uv$, which contradicts the assumption that $H$ is a hole of $G - uv$ □

**Lemma 3.5.** *Let $G$ be a graph with exactly two holes $C_1$ and $C_2$. Suppose that there exists $i \in \{0, 1, \ldots, m-1\}$ such that $v_i v v_{i+1}$ is a $C_1$-avoiding $(v_i, v_{i+1})$-path for some $v$ not in $V(C_1) \cup V(C_2)$. In addition, suppose that $v_i v_{i+1}$ is an edge of $C_2$ if $C_1$ and $C_2$ share at least two edges. Then $X_1 \cup \{v_i, v_{i+1}\}$ is a vertex cut of $G$.*

*Proof.* Suppose that there is a vertex in $V(C_1) \setminus \{v_i, v_{i+1}\}$ that is reachable from $v$ by a $C_1$-avoiding path. Let $v_j$ be the vertex in $V(C_1) \setminus \{v_i, v_{i+1}\}$ of the smallest index that is reachable from $v$ by a $C_1$-avoiding path. Let $P$ be a shortest $C_1$-avoiding $(v, v_j)$-path. By Lemma 2.4, $j = i + 2$ and $v_{i+1}$ is adjacent to all the vertices of $P$. If $|P| = 1$, then $v$ is adjacent to two nonadjacent vertices on $C_1$. Since $v \notin X_1$, $v$ is contained in a hole $C$ by Lemma 2.3. Since $\{v_i, v, v_{i+1}\}$ forms a clique, $C$ contains at most one of $v_i$, $v_{i+1}$. Thus, $C$ is different from $C_1$. Since $v \notin V(C_2)$, $C$ is different from $C_2$ and we reach a contradiction. Hence $|P| \geq 2$. Let $P = vw_1w_2 \cdots w_l v_{i+2}$ for $l \geq 2$. We first consider the cycle $C' = v_2 P v_2$. Then $|C'| \geq 4$. Now we consider the cycle $C'' = v_i P v_{i+3} v_{i+1} \cdots v_{i-1} v_i$. Then $|C''| \geq 4$. If $C''$ is a hole, then $C''$ is a hole distinct from $C_1$ and $C_2$ since it contains $v$, which is a contradiction. Thus $C''$ has a chord. Note that any two nonconsecutive vertices on $P$ cannot be adjacent and that any two nonconsecutive vertices in $V(C'') \setminus V(P)$ cannot be adjacent. Thus a vertex $u \in V(P) \setminus \{v_i, v_{i+2}\}$ is adjacent to a vertex $v_k$ on the $(v_{i+3}, v_{i-1})$-section of $C''$ not containing $v$. Therefore $u$ is adjacent to two nonconsecutive vertex $v_{i+1}$ and $v_k$ on $C_1$. If $C_1$ and $C_2$ share at most one edge, then we reach a contradiction by Lemma 2.3. Consider the case where $C_1$ and $C_2$ share at least two edges. Then $u$ is contained in a hole which has at least two common edges with $C_1$. Since $C_2$ is the only hole other than $C_1$, $u$ is on $C_2$. Then $v_{i+1}u$ is a chord of $C_2$, which is a contradiction. □

**Lemma 3.6.** *Let G be a graph with exactly two holes $C_1$ and $C_2$ sharing exactly one edge $v_iv_{i+1}$. Suppose that there exists a $C_1$-avoiding $(v_{i+2}, v_{i+3})$-path. Then $X_1 \cup \{v_{i+2}, v_{i+3}\}$ is a vertex cut.*

*Proof.* Without loss of generality, we may assume that $v_i = w_0$ and $v_{i+1} = w_1$. Then $v_{i+2}v_{i+3}$ in $E(C_1) \setminus E(C_2)$. If a shortest $C_1$-avoiding $(v_{i+2}, v_{i+3})$-path has length greater than 2, then the path together with $v_{i+2}v_{i+3}$ form a hole sharing an edge. Since $v_{i+2}v_{i+3}$ is not an edge of $C_2$, this hole is distinct from $C_2$, which is a contradiction. Thus a shortest $C_1$-avoiding $(v_{i+2}, v_{i+3})$-path is $v_{i+2}vv_{i+3}$ for some $v \in V(G) \setminus V(C_1)$. If $v \in V(C_2)$, then $v = w_{n-1}$ by Theorem 2.3. Then $w_{n-1}$ is adjacent to $v_{i+1}$ and $v_{i+3}$, which contradicts Lemma 2.3 as $w_{n-1} \notin X_1$ by the definition of a $C_1$-avoiding path. Thus $v \notin V(C_2)$. By Lemma 3.5, $X_1 \cup \{v_{i+2}, v_{i+3}\}$ is a vertex cut. □

**Lemma 3.7.** *Let G be a graph with exactly two holes $C_1$ and $C_2$ sharing at least two edges. Suppose that for any $i \in \{0, 1 \dots, m-1\}$, there exists a $C_1$-avoiding $(v_i, v_{i+1})$-path. Then there exists a vertex cut consisting of $X_1$ together with two consecutive vertices on both $C_1$ and $C_2$.*

*Proof.* By Lemma 3.2, $G[E(C_1) \cap E(C_2)]$ is a path. Without loss of generality, we may assume that $G[E(C_1) \cap E(C_2)] = v_1v_2 \dots v_k$ for some integer $k \geq 2$. We shall claim that $X_1 \cup \{v_1, v_2\}$ is a vertex cut as follows. If a shortest $C_1$-avoiding $(v_1, v_2)$-path has length greater than 2, then the path together with $v_1v_2$ form a hole sharing exactly one edge, which is a contradiction. Thus a shortest $C_1$-avoiding $(v_1, v_2)$-path is $v_1vv_2$ for some $v \in V(G) \setminus V(C_1)$. From the fact that $C_1$ and $C_2$ are holes and $v_1v_2 \in E(C_1) \cap E(C_2)$, it follows that $v$ is not on $C_2$. Thus, by Lemma 3.5, $X_1 \cup \{v_1, v_2\}$ is a vertex cut. □

Now, we prove our main theorem.

*Proof of Theorem 1.4.* Suppose that there is no $C_1$-avoiding $(v_i, v_{i+1})$-path for some $i \in \{0, \dots, m-1\}$. Then $G_1 := G - v_iv_{i+1}$ has at most one hole by Lemma 3.4 and so $k(G_1) \leq 2$ by Theorem 1.1. Let $G_2 := v_iv_{i+1}$. Then $G_2$ is chordal, $E(G_1) \cup E(G_2) = E(G)$, and $V(G_1) \cap V(G_2) = \{v_i, v_{i+1}\}$ is a clique of $G_2$. By Theorem 2.2, we have $k(G) \leq 3$. Similarly, if there is no $C_2$-avoiding $(w_i, w_{i+1})$-path for some $i \in \{0, \dots, m'-1\}$, then we can show that $k(G) \leq 3$.

Now we suppose that there are a $C_1$-avoiding $(v_i, v_{i+1})$-path for any $i \in \{0, \dots, m-1\}$ and a $C_2$-avoiding $(w_i, w_{j+1})$-path for any $j \in \{0, \dots, m'-1\}$. We note that we deal with the case where $C_1$ and $C_2$ share an edge.

If $C_1$ and $C_2$ share exactly one edge, say $v_iv_{i+1}$ for some $i \in \{0, 1, \dots, m-1\}$, then $X_1 \cup \{v_{i+2}, v_{i+3}\}$ is a vertex cut of $G$ by Lemma 3.6. Let $Q$ be the components of $G - X_1 - \{v_{i+2}, v_{i+3}\}$ that contains $V(C_1) \setminus \{v_{i+2}, v_{i+3}\}$. Let $G_2$ be the subgraph of $G$ induced by the vertex set of remaining components and $X_1 \cup \{v_{i+2}, v_{i+3}\}$. Since $v_iv_{i+1}$ is contained in $Q$, $C_2$ is not contained in $G_2$ and so $G_2$ is chordal. Note that the subgraph induced by $V(Q) \cup X_1 \cup \{v_{i+2}, v_{i+3}\}$ contains no $C_1$-avoiding $(v_{i+2}, v_{i+3})$-path and so $G_1 := Q - v_{i+2}v_{i+3}$ has exactly one hole by Lemma 3.4. Thus $k(G_1) \leq 2$ by Theorem 1.1. Thus $k(G) \leq 3$ by Theorem 2.2.

Suppose that $C_1$ and $C_2$ share at least two edges. Then by Lemma 3.7, there exists $i \in \{0, 1, \ldots, m-1\}$ such that $X_1 \cup \{v_i, v_{i+1}\}$ is a vertex cut of $G$ and $v_iv_{i+1}$ is a common edge of $C_1$ and $C_2$. Let $Q^*$ be the connected component of $G - X_1 - \{v_i, v_{i+1}\}$ that contains $V(C_1) \setminus \{v_i, v_{i+1}\}$. If $C_1$ and $C_2$ share at least two edges, $C_2$. Let $G_2$ be the subgraph of $G$ induced by the vertices in $V(Q) \cup X_1 \cup \{v_i, v_{i+1}\}$. Then $C_2$ is not contained in $G_1$ since at least one edge other than $v_iv_{i+1}$ is contained in $Q^*$ by the assumption that $C_1$ and $C_2$ share at least two edges. Thus $G_2$ is chordal. Note that the subgraph induced by $V(Q) \cup X_1 \cup \{v_i, v_{i+1}\}$ contains no $C_1$-avoiding $(v_i, v_{i+1})$-path and so $G_1 := Q - v_iv_{i+1}$ has exactly one hole by Lemma 3.4. Thus $k(G_1) \leq 2$ by Theorem 1.1. Thus $k(G) \leq 3$ by Theorem 2.2. □

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