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Free-fall orbits and heteroclinic orbits to triple collisions, and shadowing in the isosceles three-body problem

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Abstract

We investigate free-fall orbits in the isosceles three-body problem on $\mathbb{R}^2$ where the particles m_1 and m_2 have equal masses and the particle m_3 may have a different mass, and show that there exists a countable family of orbits which converge to triple collisions in the forward and backward time evolution. In the McGehee coordinates, the orbits correspond to topologically transeverse heteroclinic orbits between the fixed points of the extended equation on the collision manifold (Theorem 1.1).

Then, by applying the "window theory", we show that there exists an orbit of the isosceles three-body problem that shadows the prescribed path on the graph consisting of these heteroclinic orbits and already known ones (Theorem 1.2). In particular, if we choose the infinite paths appropriately, we obtain a variety of new oscillatory orbits (Corollary 1.1).

1 Introduction and Main Results

This paper is concerned with the Newtonian three-body problem given by the following set of ODEs:

$$m_\ell \frac{d^2\mathbf{x}_\ell}{dt^2} = \frac{\partial U}{\partial \mathbf{x}_\ell}, \quad \mathbf{x}_\ell = (x_\ell, y_\ell, z_\ell) \in \mathbb{R}^3, \quad \ell = 1, 2, 3, \quad m_\ell > 0 \tag{1}$$

where

$$U(\mathbf{x}) = \sum_{i<j} \frac{m_i m_j}{\|\mathbf{x}_i - \mathbf{x}_j\|}.$$

We consider the isosceles three-body problem; assume $m_1 = m_2$, and consider motions for which m_3 remains on the z-axis in $\mathbb{R}^3$, while m_1 and m_2 remain symmetric with respect to this axis (figure 1). We fix the center of masses

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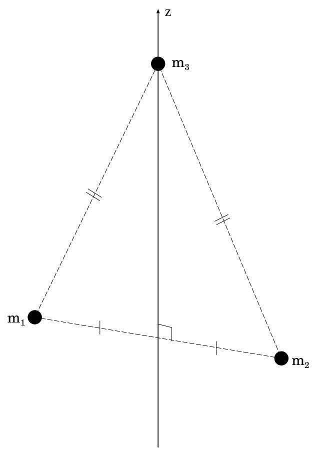

Figure 1: Isosceles three-body problem

$m_1\mathbf{x}_1+m_2\mathbf{x}_2+m_3\mathbf{x}_3$ at the origin without lack of generality. Then the isosceles three-body problem is given by the Hamiltonian system with Hamiltonian

$$H=\frac{1}{4}(p_x^2+p_y^2)+\frac{\alpha+2}{4\alpha}p_z^2-\frac{1}{2\alpha\sqrt{x^2+y^2}}-\frac{2}{\sqrt{x^2+y^2+z^2}}, \tag{2}$$

where

$$\mathbf{x}_1=\left(x,y,-\frac{\alpha}{\alpha+2}z\right),\mathbf{x}_2=\left(-x,-y,-\frac{\alpha}{\alpha+2}z\right),\mathbf{x}_3=\left(0,0,\frac{2}{\alpha+2}z\right)$$

and (p_x,p_y,p_z) is the momentum corresponding to (x,y,z). Throughout this paper assume that all double collisions are regularized: by a change of variables the singularities of the double collisions transform to regular points (see section 2). The behavior of the double collisions physically corresponds to an elastic bounce. By examining orbits with $p_z=0$, which we call free-fall orbit, we show that for each $k\in\{0,1,2,3,\dots\}$ there exists an orbit which has the triple collision in the future and past time evolution and which experiences the double collisions of m_1 and m_2 exactly k times between the triple collisions.

In order to study behaviors at and near triple collisions, it is convenient to use Devaney's coodinates [1] which we introduce in section 3. In the coordinates the triple collision singularity is blown-up to a two-dimensional invariant manifold called the *collision manifold* (figure 3). There are six equilibrium points $E_\pm, E_\pm^*, C, C^*$ on the collision manifold. Orbits of isosceles three-body problem ending (starting) at the triple collision correspond to ones converging to C^* or $E_\pm^*$ as $t\to\infty$ (C or $E_\pm$ as $t\to-\infty$) in the Devaney's coordinate. The limiting configurations are collinear, equilateral and opposite orientational triangles in the case of converging to C^* and $E_\pm^*$ (C and $E_\pm$) respectively. Therefore using the Devaney's coordinates, the orbits stated above correspond to heteroclinic orbits from E_+ to E_+^* (or from E_- to E_-^*). Note that the heteroclinic orbit lies outside the collision manifold. More precisely, we have:

Theorem 1.1. *For any $k \in \{0, 1, 2, 3 \dots\}$, there exists a topologically transverse heteroclinic orbit $\gamma_\pm^k$ on the energy surface (which we will denote $\mathscr{M}_+$ in section 3) from $E_\pm$ to $E_\pm^*$ such that $\gamma_\pm^k$ has double collisions of m_1 and m_2 k times.*

We can easily obtain three heteroclinic orbits κ_C and $\kappa_{E_\pm}$ from C to C^* and from $E_\pm$ to $E_\pm^*$ respectively. These behave homothetically while the particles hold collinear and equilateral configurations respectively. Moeckel [3] proved $\kappa_{E_\pm}$ are transverse. Heteroclinic orbits $\gamma_\pm^0$ in theorem 1.1 are consistence with $\kappa_{E_\pm}$. As we state after, Moeckel [3] proved the existence of multiple heteroclinic, but they are different from $\gamma_\pm^k (k = 1, 2, 3 \dots)$. Theorem 1.1 is closely related to [8]. McGehee [2] showed that the set of parabolic orbits ($z_3 \to \pm\infty$ and $\dot{z}_3 \to 0$ as $t \to +\infty$) is a two-dimensional analytic manifold $\mathcal{P}_\pm$ called parabolic manifold. Simó and Martínez [8] studied dynamics near $\mathcal{P}_\pm$ and proved the existence of infinite many heteroclinic orbits from $E_\pm$ to $E_\pm^*$ passing near $\mathcal{P}_\pm$. The heteroclinic orbits experience double collision many times between triple collisions. We focus free-fall orbits to obtain heteroclinic orbits $\gamma_\pm^k$. The heteroclinic orbit $\gamma_\pm^k$ may be consistent with ones obtained by Simó and Martínez for sufficiently large k but $\gamma_\pm^k$ is new heteroclinic orbit for not so large $k \geqq 1$.

The parabolic manifolds $\mathcal{P}_\pm$ can be viewed as the stable manifolds of virtual hyperbolic periodic orbits $\beta_\pm$ where m_3 stays at infinity point ($z \to \pm\infty$), and m_1 and m_2 behave according to the Kepler motion. As before, by using the results of Moeckel [3], there are heteroclinc orbits related to $\beta_\pm$ and the equilibrium points in the collision manifold. Namely, for any mass ratio, there are

- a heteroclinic orbit η_+ from E_+ to β_+ (and η_- from E_- to β_-),
- a heteroclinic orbit θ_+ from β_+ to E_+^* (and θ_- from β_- to E_-^*),

for large $m_3 > 0$ and in the collision manifold, there are

- Infinitely many heteroclinic orbits $\{\rho_+^k\}_{k\in\mathbb{N}}$ from E_+ to E_+^* (and $\{\rho_-^k\}_{k\in\mathbb{N}}$ from E_- to E_-^*),
- Infinitely many heteroclinic orbits $\{\lambda_+^k\}_{k\in\mathbb{N}}$ from E_+ to E_-^* (and $\{\lambda_-^k\}_{k\in\mathbb{N}}$ from E_- to E_+^*),
- a heteroclinic orbit δ_+ from E_+^* to E_- (and δ_- from E_-^* to E_+);

for small $m_3 > 0$ and in the collision manifold, there are

- a heteroclinic orbit α_+ from E_+^* to E_+ (and α_- from E_-^* to E_-).

Figure 2 stands for heteroclinic orbits between $E_\pm$ and $E_\pm^*$. All above heteroclinic orbits except κ_C are topologically transverse on an invariant set $\mathscr{M}$, that we will define in section 3. The energy surface $\mathscr{M}_+$ is open subset of $\mathscr{M}$. The heteroclinic orbits $\rho_\pm^k$ and $\lambda_\pm^k$ pass near the heteroclinic orbit κ_C and has

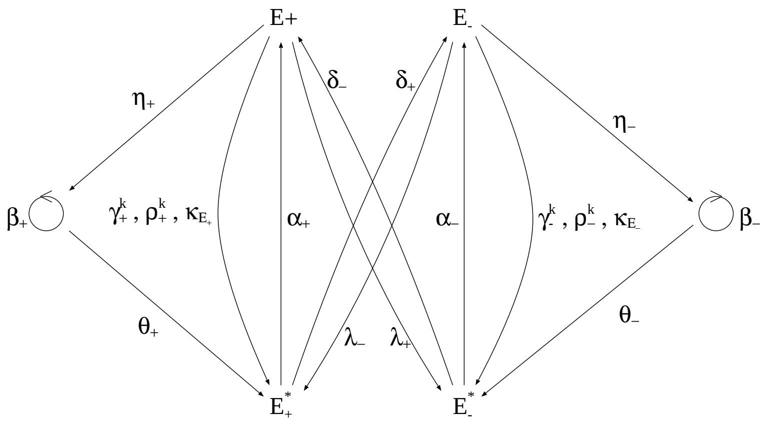

Figure 2: Topologically transverse heteroclinic orbits. The arrow $A \xrightarrow{\xi} B$ means that there exist a topologically transverse heteroclinic orbit ξ from A to B.

no double collision and then ones obtained at theorem 1.1 are different from them.

Since orbits with a triple collision have zero angular momentum ($xp_y - yp_x = 0$), so do all above heteroclinic orbits. Moeckel [3] studied the window theory to show the existence of an orbit with small angular momentum shadowing a path of heteroclinic orbits. Let $\mathcal{H}$ be the set of topologically transverse heteroclinic orbits on $\mathscr{M}$ and periodic orbits $\beta_\pm$. We apply the window theory to the isosceles three-body problem to obtain orbits shadowing the sequence of the heteroclinic and periodic orbits. That is,

Theorem 1.2 ([3]). *Fix $h < 0$. For a given finite, path-connected subgraph $\mathcal{B}$ of $\mathcal{H}$, there is a positive number $\omega(\mathcal{B})$ such that for $0 < |\omega| < \omega(\mathcal{B})$, any infinite path in $\mathcal{B}$ is shadowed by at least one solution of the isosceles three-body problem with the angular momentum ω and with total energy $H = h$. In particular the shadowing orbits converge to the infinite path of heteroclinic orbits as $\omega \to 0$.*

Adding heteroclinic orbits of theorem 1.1 to known ones, we know more multiple orbits experiencing near-triple collision. In particular we obtain new oscillatory orbits as below.

Definition 1.1. If an orbit $\mathbf{x} = (\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t))$ of a three-body problem 1 satisfies

$$\liminf_{t\to\infty} \max_{i<j} ||\mathbf{x}_i(t) - \mathbf{x}_j(t)|| < \infty \text{ and } \limsup_{t\to\infty} \max_{i<j} ||\mathbf{x}_i(t) - \mathbf{x}_j(t)|| = \infty,$$

we call $\mathbf{x}$ an *oscillatory orbit.*

Corollary 1.1. *In Theorem* 1.2 *if we choose a path which contains subsequences* $\{\eta_\pm, \underbrace{\beta_\pm, \ldots, \beta_\pm}_{j_i}, \theta_\pm\}_{i\in\mathbb{N}}$ *satisfying* $j_i \to \infty$ *as* $i \to \infty$, *the orbit is an oscillatory orbit.*

By choosing a sequence with $\gamma_\pm^k$ and satisfying the condition in the Corollary 1.1, we obtain a variety of new oscillatory orbits.

In Section 2 we consider free-fall orbits and show the existence of orbits having triple collisions in the forward and backward time evolution. In Section 3, in order to investigate the behavior at and near the triple collision, we introduce Devaney's coordinates. We show the orbits obtained in Section 4 are topologically transverse heteroclinic orbits in the Devaney's coordinates.

2 Free-fall orbits

In this section we only consider orbits the subsystem $\{y = p_y = 0\}$ of the isosceles three-body problem. The problem is Hamiltonian system with respect to the Hamiltonian

$$H(x, z, p_x, p_z) = \frac{1}{4}p_x^2 + \frac{\alpha+2}{4\alpha}p_z^2 - \frac{1}{2\alpha|x|} - \frac{2}{\sqrt{x^2+z^2}}.$$

Here we will blow-up double collision singularity $\{x = 0\}$ by so-called Levi-Civita coordinates. We fix a constant $h < 0$ and consider orbits with $H = h$. We define canonical transformation by

$$x = \frac{1}{2}\xi^2, p_x = \frac{p_\xi}{\xi}, \tag{3}$$

and change the time by

$$dt = \xi^2 d\tau.$$

Then the equations become

$$\frac{d\xi}{d\tau} = \frac{1}{2}p_\xi \tag{4}$$

$$\frac{dz}{d\tau} = \frac{\alpha+2}{2\alpha}\xi^2 p_z \tag{5}$$

$$\frac{dp_\xi}{d\tau} = \frac{\alpha+2}{2\alpha}\xi p_z^2 - \frac{32\xi z^2}{(\xi^4+4z^2)^{3/2}} - 2h\xi \tag{6}$$

$$\frac{dp_z}{d\tau} = -\frac{16\xi^2 z}{(\xi^4+4z^2)^{3/2}} \tag{7}$$

which is the Hamiltonian system with Hamiltonian

$$\begin{aligned}\Gamma(p_\xi, p_z, \xi, z) &= H\left(\frac{p_\xi}{\xi}, p_z, \frac{1}{2}\xi^2, z\right)\xi^2 - h\xi^2 \\ &= \frac{1}{4}p_\xi^2 + \frac{\alpha+2}{4\alpha}\xi^2 p_z^2 - \frac{1}{\alpha} - \frac{4\xi^2}{\sqrt{\xi^4+4z^2}} - h\xi^2\end{aligned}$$

We should consider orbits with $\Gamma = 0$ since we just consider orbits with $H = h$, and we should identify (ξ, z, p_ξ, p_z) with $(-\xi, z, -p_\xi, p_z)$ since the transformation (3) is 2-to-1 except $\xi = 0$. We consider orbits with $p_\xi = p_z = 0$ at $\tau = 0$, that is, free-fall orbits. It is east to show that for any $\sqrt{-\frac{1}{h\alpha}} < \xi < \sqrt{-\frac{4\alpha+1}{h\alpha}}$ there is a unique $z > 0$ such that $\Gamma(0, 0, \xi, z) = 0$, so we use ξ as a parameter of the free-fall orbits.

Such free-fall orbits may have double collisions of m_1 and m_2 for the future time evolution before m_3 reaches the origin. Let $N(\xi)$ be the number of these double collisions of m_1 and m_2 (including a double collision at $t = 0$ but not counting a triple collision and letting $N = 0$ if the orbit has no double collision).

Proposition 2.1. (i) *The function $N(\xi)$ is well-defined for all $\xi \in \left(\sqrt{-\frac{1}{h\alpha}}, \sqrt{-\frac{4\alpha+1}{h\alpha}}\right)$.*
(ii) $N\left(\sqrt{-\frac{2\alpha+1}{h\alpha}}\right) = 0$
(iii) $N(\xi) \to \infty$ *as* $\xi \to \sqrt{-\frac{1}{h\alpha}} + 0$.
(iv) *For each $k \in \mathbb{N}$, there is a discontinuous point $\xi_k \in \left(\sqrt{-\frac{1}{h\alpha}}, \sqrt{-\frac{4\alpha+1}{h\alpha}}\right)$ of N such that $N(\xi_k) = k$, $\lim_{\xi \to \xi_k - 0} N(\xi) = k$ and that $\lim_{\xi \to \xi_k + 0} N(\xi) = k + 1$.*
(v) *The free-fall orbit corresponding to ξ_k has a triple collision after the double collisions k times.*

Proof. (i) Assume a free-fall orbit $(\xi(\tau), z(\tau), p_\xi(\tau), p_z(\tau))$ is defined for $\tau < \tau_0$ where $\tau_0 \in (0, \infty]$ is maximal. If $\tau_0 < \infty$, the orbit has a triple collision as $\tau \to \tau_0 - 0$, since singularities in the three body problem are only collision singularity (see [5]). Hence N is determined as the number of double collisions before the triple collision. So we consider the case that $\tau_0 = \infty$ and that $z > 0$ for all $\tau > 0$. From $p_z(0) = 0$ and (7), $p_z < 0$ for $\tau > 0$. z and p_z are monotonically decreasing from (5) and (7). Therefore there is the limit $\lim_{\tau \to \infty} z(\tau) \geqq 0$ of $z(t)$ as $\tau \to \infty$. From $\frac{dz}{d\tau} \to 0(\ \tau \to \infty)$ and (5), $\xi \to 0$ or $p_z \to 0$ $(\tau \to \infty)$. It is impossible that $p_z \to 0$ because $p_z(0) = 0$ and $p_z(\tau)$ is monotonically decreasing. If $\xi \to 0$, $p_\xi = 2\frac{d\xi}{d\tau} \to 0$ from (4). It contradicts $\Gamma = 0$. Consequently for any $\xi \in \left(\sqrt{-\frac{1}{h\alpha}}, \sqrt{-\frac{4\alpha+1}{h\alpha}}\right)$, m_3 reach the origin in finite time, and hence $N(\xi)$ can be defined.
(ii) In the case of $\xi = \sqrt{-\frac{2\alpha+1}{h\alpha}}$, the corresponding initial point is $(p_\xi, p_z, \xi, z) = \left(0, 0, \sqrt{-\frac{2\alpha+1}{h\alpha}}, -\frac{\sqrt{3}(2\alpha+1)}{2h\alpha}\right)$. The initial configuration is the equilateral triangle and the initial velocity of each particle is zero. It is known that such a solution is the homothetic solution. The solution has a triple collision without double collision and hence $N = 0$.
(iii) With respect to the initial points of the free-fall orbits $z \to \infty$ as $\xi \to \sqrt{-\frac{1}{h\alpha}}$. The limit $p_z = 0, z \to \infty$ of Γ is

$$\frac{1}{4}p_\xi^2 - \frac{1}{\alpha} - 4 - h\xi^2,$$

which is the Hamiltonian of a harmonic oscillator. The solution on Γ is a periodic orbit with a double collision, which corresponds to β_+. The initial point of free-fall orbits $z \to \infty$ as $\xi \to \sqrt{-\frac{1}{h\alpha}} + 0$ converges to β_+. The set of the double collisions is the codimension-one subspace $\{\xi = 0\}$. Therefore $N(\xi) = \infty$ as $\xi \to \sqrt{-\frac{1}{h\alpha}} + 0$.
(iv) It follow from (i), (ii) and (iii).
(v) It is sufficient to show that the free-fall orbit corresponding to ξ_k has a triple collision when m_3 reach the origin first time. Assume that the free-fall orbit does not have a triple collision. By the continuity of orbits with respect to initial points, $N(\xi) = N(\xi_k)$ for ξ near ξ_k, since the orbits do not have collision when m_3 arrives at the origin, and hence ξ_k is continuous point. □

Let γ_+^{2k} be the free-fall orbit corresponding to ξ_k. We define γ_+^{2k-1} and γ_-^l later. From the time reversibility, γ_+^{2k} has a triple collision before k double collisions in the past time evolution. Therefore the orbit has $2k$ double collisions between the triple collisions.

Next we investigate orbits with odd number of double collisions between triple collisions. Consider the orbit with $(p_\xi, p_z, \xi, z) = (\frac{2}{\sqrt{\alpha}}, 0, 0, z)$ at $\tau = 0$ and $M(z)$ as the number of double collisions (including at $\tau = 0$) before m_3 reaches the origin. The function M has similar property as N:

Proposition 2.2. (i) *The function M is well-defined for all $z > 0$.*
(ii) *There is $c > 0$ such that $M(c) = 1$.*
(iii) *$M(z) \to \infty$ as $z \to \infty$.*
(iv) *For each $k \in \mathbb{N}$, there is a discontinuous point $z_k \in (c, \infty)$ of M such that $M(z_k) = k$, $\lim_{z \to z_k + 0} N(z) = k$ and that $\lim_{z \to z_k - 0} N(z) = k + 1$.*
(v) *The free-fall orbit corresponding to z_k has a triple collision after the double collisions k times including one at $\tau = 0$.*

Proof. As the proof of proposition 2.1, we can similarly prove this proposition except (ii). In [6], by using the variational methods we proved the existence of a periodic orbit which alternately experiences double collisions $(p_\xi, p_z, \xi, z) = (\frac{2}{\sqrt{\alpha}}, 0, 0, \pm c)$ and has no collision between these double collisions. Hence c satisfies $N(c) = 1$ and this completes the proof of (ii). □

Let γ_+^{2k-1} be the free-fall orbit corresponding to z_k. Similarly γ_+^{2k} has a triple collision before k double collisions (include the double collision at $\tau = 0$) in the past time evolution. From the construction γ_+^l has l double collisions and satisfies $z > 0$ between triple collisions. If $(p_\xi(\tau), p_z(\tau), \xi(\tau), z(\tau))$ is a solution, so is the reflection $(p_\xi(\tau), -p_z(\tau), \xi(\tau), -z(\tau))$. Define γ_-^l be the reflected solution of γ_+^l. Similarly γ_-^l has l double collisions and satisfies $z < 0$ between triple collisions.

3 Collision manifold

In order to investigate the behavior of orbits with a triple collision, we introduce a blow-up technic of triple collision singularity by [1]. Let

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{2\alpha}{\alpha+2} \end{pmatrix}$$

and define the canonical transformation

$$\zeta = M^{1/2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \eta = M^{-1/2} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}.$$

In the new coordinates Hamiltonian is $H = \frac{1}{2}|\eta|^2 - U(\zeta)$ where

$$U(\zeta) = \frac{1}{\sqrt{2}} \left[\alpha^{-1}(\zeta_1^2 + \zeta_2^2)^{-1/2} + 4(\zeta_1^2 + \zeta_2^2 + (1 + 2\alpha^{-1})\zeta_3^2)^{-1/2} \right].$$

To study orbits near triple collision it is convenient to introduce new variables

$$r = |\zeta|, \boldsymbol{s} = r^{-1}\zeta, \boldsymbol{z} = r^{1/2}\eta$$

and multiply the resulting vector field by $r^{3/2}$. The result is

$$\begin{aligned} \dot{r} &= (\boldsymbol{s} \cdot \boldsymbol{z})r, \\ \dot{\boldsymbol{s}} &= \boldsymbol{z} - (\boldsymbol{s} \cdot \boldsymbol{z})\boldsymbol{s}, \\ \dot{\boldsymbol{z}} &= \frac{\partial U}{\partial \zeta}(\boldsymbol{s}) + \frac{1}{2}(\boldsymbol{s} \cdot \boldsymbol{z})\boldsymbol{z}. \end{aligned} \tag{8}$$

We use spherical coordinates $\boldsymbol{s} = (s_1, s_2, s_3) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$. Now the vectors

$$\begin{aligned} \boldsymbol{u}_1 &= \boldsymbol{s} \\ \boldsymbol{u}_2 &= \frac{\partial \boldsymbol{s}}{\partial \theta} = (-\sin\theta\cos\phi, \cos\theta\cos\phi, 0) \\ \boldsymbol{u}_3 &= \frac{\partial \boldsymbol{s}}{\partial \phi} = (-\cos\theta\sin\phi, -\sin\theta\sin\phi, \cos\phi) \end{aligned}$$

form an orthogonal basis for $\mathbb{R}^3$. If we write $\boldsymbol{z} = v\boldsymbol{u}_1 + w_2\boldsymbol{u}_2 + w_3\boldsymbol{u}_3$ and use (8) to find equations for the variables $(r, \theta, \phi, v, w_2, w_3)$, we get:

$$\begin{aligned} \dot{r} &= vr \\ \dot{\theta} &= w_2 \\ \dot{\phi} &= w_3 \\ \dot{v} &= \frac{1}{2}v^2 + w_2^2\cos^2\phi + w_3^2 - U(\phi) \\ \dot{w}_2 &= -\frac{1}{2}vw_2 + 2w_2w_3\tan\phi \\ \dot{w}_3 &= U'(\phi) - \frac{1}{2}vw_3 - w_2^2\cos^2\phi\tan\phi. \end{aligned}$$

where $U(\phi) = \frac{1}{\sqrt{2}}\alpha^{-1}\left[\sec\phi + 4\alpha^{3/2}(\alpha + 2\sin^2\phi)^{-1/2}\right]$. One can easily show that $\omega = r^{1/2}w_2\cos^2\phi$ is a constant of the motion which corresponds to the angular momentum.

We will now make some familiar regularizing transformations to eliminate the singularities at $\phi = \pm\pi/2$. Then replace w_3 by $w = w_3\cos\phi$ and multiply the resulting vector field by $\cos\phi$. We define T as the new time ($dt = r^{3/2}cos(\phi)dT$). The equations become

$$
\begin{aligned}
\frac{dr}{dT} &= vr\cos\phi, \quad \frac{d\phi}{dT} = w,\\
\frac{dv}{dT} &= \left(U(\phi) - \frac{1}{2}v^2 + 2rh\right)\cos\phi,\\
\frac{dw}{dT} &= \frac{dU}{d\phi}(\phi)\cos^2\phi - \frac{1}{2}vw\cos\phi - \left(2U(\phi) - v^2 + 2rh\right)\sin\phi\cos\phi
\end{aligned} \tag{9}
$$

with an energy relation

$$
\frac{1}{2}\left(v^2\cos^2\phi + w^2 + \frac{\omega^2}{r}\right) - U(\phi)\cos^2\phi = rh\cos^2\phi. \tag{10}
$$

In the coordinates the triple collision ($r = 0$) and the double collisions $\phi = \pm\frac{\pi}{2}$ singularities are blown-up. We call (r, ϕ, v, w) Devaney's coordinates. Mockel [3] defined the following invariant sets:

$$
\begin{aligned}
\mathscr{M}(\omega) &= \{(r, \phi, v, w) \mid (10) \text{ is satisfied for } \omega\}\\
\mathscr{M}_+ &= \{(r, \phi, v, w) \mid r \geq 0, v^2\cos^2\phi + w^2 - 2U(\phi)\cos^2\phi = 2rh\cos^2\phi\}\\
\mathscr{M}_0 &= \{(r, \phi, v, w) \mid r = 0, 2U(\phi)\cos^2\phi \geq v^2\cos^2\phi + w^2\}\\
\mathscr{C} &= \mathscr{M}_+ \cap \mathscr{M}_0 = \{(r, \phi, v, w) \mid r = 0, 2U(\phi)\cos^2\phi = v^2\cos^2\phi + w^2\}\\
\mathscr{M} &= \mathscr{M}_+ \cup \mathscr{M}_0.
\end{aligned}
$$

We will refer to $\mathscr{M}$ as the limiting variety of $\mathscr{M}(\omega)$ as $\omega \to 0$. Topologically $\mathscr{M}_0 \cong D^3\backslash\{4 \text{ points}\}$, $\mathscr{C} = \partial\mathscr{M}_0 \cong S^2\backslash\{4 \text{ points}\}$. We call $\mathscr{C}$ collision manifold(figure 3). In the sets, 4 points stands for $(\phi, v, w) = (\pm\frac{\pi}{2}, +\infty, 0)$ and $(\pm\frac{\pi}{2}, -\infty, 0)$. There are six equilibrium points

$$
\begin{aligned}
C &= (0, \sqrt{2U(0)}, 0), & C^* &= (0, -\sqrt{2U(0)}, 0)\\
E_+ &= (\phi_+, \sqrt{2U(\phi_+)}, 0) & E_+^* &= (\phi_+, -\sqrt{2U(\phi_+)}, 0)\\
E_- &= (\phi_-, \sqrt{2U(\phi_-)}, 0) & E_-^* &= (\phi_-, -\sqrt{2U(\phi_-)}, 0)
\end{aligned}
$$

in the coordinates (ϕ, v, w), where $0, \phi_+(> 0)$ and $\phi_-(= -\phi_+)$ are the critical points of $U(\phi)$. If an orbit converges to a triple collision in the forward (resp. backward) time evolution in the original system, the corresponding orbits converges to E_+^*, E_-^* or C^* (resp. E_+, E_- or C) as $T \to \infty$ (resp. $T \to -\infty$) in the Devaney's coordinates (see [1]). The limit configuration $\phi = 0$ is the collinear one, while $\phi = \phi_+$ and $\phi = \phi_-$ represent equilateral configurations

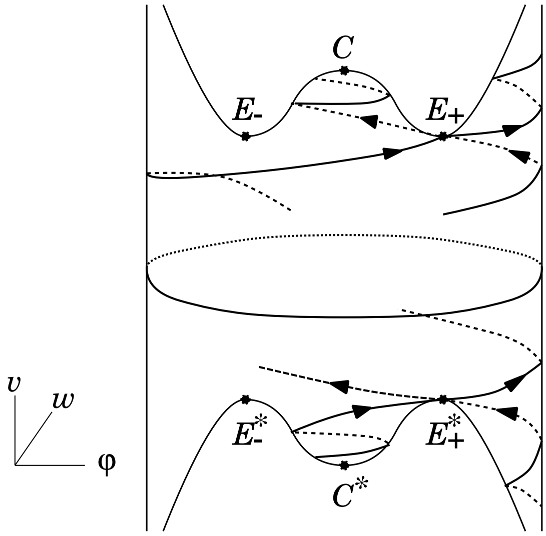

Figure 3: Collision manifold $\mathscr{C}$

with z positive and negative respectively. The all equilibrium points are saddle. Let D be one of the equilibrium points and define $W^s(D)$ ($W^u(D)$) as the stable(unstable) manifold of D. The dimension of the manifolds are following:

$$
\begin{aligned}
\dim W^s(C) &= \dim W^u(C^*) = 3,\\
\dim W^u(C) &= \dim W^s(C^*) = 1,\\
\dim W^s(E_\pm) &= \dim W^u(E^*_\pm) = 2,\\
\dim W^u(E_\pm) &= \dim W^s(E^*_\pm) = 2.
\end{aligned}
$$

Note that $W^s(D)$ and $W^u(D^*)$ ($W^u(D)$ and $W^s(D)$) are symmetric with respect to the reflection $(v, w) \mapsto (-v, -w)$. Let $D = C$ or $E_\pm$. The manifolds $W^s(D)$ and $W^u(D^*)$ are subsets of M_0, and $W^u(D)$ and $W^s(D^*)$ are subsets of M_+. See [1, 3] for more detail.

4 Proof of Theorem 1.1

Theorem 1.1 follows from the following two lemmata.

Lemma 4.1. *The orbit $\gamma^k_\pm$ is a heteroclinic orbit from $E_\pm$ to $E^*_\pm$.*

Proof. The orbit $\gamma^k_\pm$ converges to either C^*, E^*_+ or E^*_- in the forward time. The stable manifold $W^s(C^*)$ of C^* is included in the invariant set $\{\phi = w = 0\} \subset \mathscr{M}_+$, which corresponds to the collinear motions $\{y = z = p_y = p_z = 0\}$ in the original problem (2) (see [3]). Since the free-fall orbit is not a collinear motion, the limit is E^*_+ or E^*_-. Orbits converging to E^*_+ (resp. E^*_-) have an equilateral

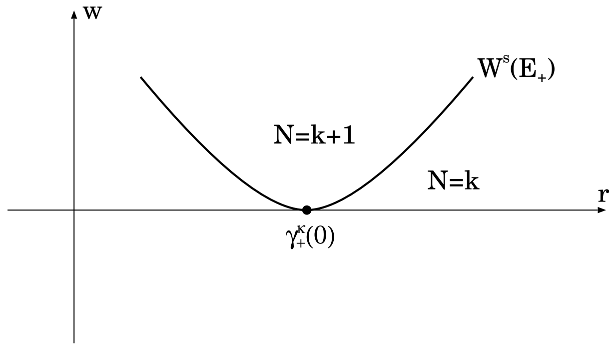

Figure 4:

triangular configuration with z positive (resp. negative) as converging to the limit. Since z-component on $\gamma_\pm^k$ is positive between the triple collisions, the limit must be $E_\pm^*$. Similarly the limit of $\gamma_\pm^k$ in the backward time is $E_\pm$. □

Lemma 4.2. *The heteroclinic orbit* $\gamma_\pm^k$ *is topologically transverse in* $\mathscr{M}_+$.

Proof. We can assume that on $\gamma_\pm^k$ the time T corresponding to $t=0$ in the orginal time is zero without lack of generality.

We first claim that $\gamma_\pm^k$ transversely crosses the hyperplane $\{v=0\}$ at $T=0$. From the construction the orbits $\gamma_\pm^k$ have zero angular momentum ($\omega=0$) and hence belongs to $\mathscr{M}_+$. In the case of odd k, since $p_z=0$ and $x=y=0$ at $t=0$ in the original coordinates, $(p_z=)\sqrt{\frac{2\alpha}{\alpha+2}}r^{-1/2}(v\sin\phi+w)=0$ and $\phi=\pm\frac{\pi}{2}$ at $T=0$ in the Deveney's coordinates. From $\phi=0$ and (10), $w=0$ and hence $v=0$ at $T=0$. It is easier to show that $v=w=0$ at $T=0$ also in the case of even k. From (9) and (10), $\dot{v}=-U(\phi)\cos\phi<0$ at $T=0$ (remark that $U(\phi)\cos\phi\neq 0$ even if $\phi=\pm\frac{\pi}{2}$) and hence $\gamma_\pm^k$ transversely cross the hyperplane $\{v=0\}$.

Next we will show the transversality of $\gamma_\pm^k$ on $\mathscr{M}_+$. We can use (r,v,w) as the coordinates on a neighborhood of $\gamma_\pm^k(0)$ in $\mathscr{M}_+$. On the hyperplane $\{v=0\}$, the stable manifold $W^s(E_\pm)$ topologically transversely crosses ϕ-axis at $\gamma_\pm^k(0)$. In fact if not, free-fall orbits ($w=0$) near $\gamma_\pm^k(0)$ belong to same domain split by $W^s(E_\pm)$ (see figure 4). Hence have same number of double collisions before m_3 arrives at the origin. This contradicts (iv) of proposition 2.1 or 2.2.

The stable manifold $W^s(E_\pm^*)$ of $E_\pm^*$ and the unstable manifold $W^u(E_\pm)$ of $E_\pm$ are two-dimensional analytic manifolds, and then these manifolds make analytic curves on $\{v=0\}$. Since the equations (9) are reversible, if $(r(T),v(T),w(T))$ is a solution, so is $(r(-T),-v(-T),-w(-T))$. Therefore $W^s(E_\pm^*)$ and $W^u(E_\pm)$ are symmetric with respect to ϕ-axis on $\{v=0\}$. If $W^s(E_\pm^*)$ and $W^u(E_\pm)$ do not cross topologically transversely, they coincide in the neighborhood of $\gamma_\pm^k$ on $\{v=0\}$. By the analycity of these manifolds, whole manifolds $W^s(E_\pm)^*$ and $W^u(E_\pm)$ coincide. But as stated in section 1, Mockel [3] proved that the heteroclinic orbits $\gamma^0_\pm = \kappa_{E_\pm}$ (homothetic orbits) are transverse. It follows the contradiction. □

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