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Abstract. The structure of non-singular projective surfaces admitting non-isomorphic surjective separable endomorphisms is studied in the positive characteristic case. The case of characteristic zero is treated in [2], [16] (cf. [3]). Many similar classification results are obtained also in this case; on the other hand, some examples peculiar to the positive characteristic are given explicitly.

1. Introduction

We work in the category of algebraic $\Bbbk$-schemes for an algebraically closed field $\Bbbk$ of characteristic $p > 0$. The main purpose of this article is to prove Theorems 1.1 and 1.2 below on the classification of non-singular projective surfaces X which admit non-isomorphic surjective *separable* endomorphisms $f\colon X \to X$. Here, f is a finite surjective morphism of $\deg f > 1$ and the field extension $\Bbbk(X)/f^*\Bbbk(X)$ is separable. In the case of characteristic zero, the non-singular projective surfaces admitting non-isomorphic surjective endomorphisms are classified by [2], [16] (cf. [3]) as follows:

- A toric surface.
- A $\mathbb{P}^1$-bundle over an elliptic curve.
- A $\mathbb{P}^1$-bundle over a curve of genus ≥ 2 which is trivialized after a finite étale base change.
- An abelian surface.
- A hyperelliptic surface.
- An elliptic surface with Kodaira dimension one and Euler number zero.

Here, any elliptic surface in the last case admits an étale covering from the product of an elliptic curve and a curve of genus ≥ 2. Even in the positive characteristic case, the arguments in the papers above are effective for the classification. However, there are

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strange phenomena not covered by the arguments. For example, there is a non-toric non-singular rational surface admitting non-isomorphic surjective separable endomorphisms (cf. Example 4.5 below).

Theorems 1.1 and 1.2 below almost correspond to the classification in characteristic zero. For their proofs, we apply results and arguments in the classification theory of algebraic surfaces of characteristic $p > 0$, mainly those by Bombieri and Mumford in [14], [1].

Theorem 1.1. *Let X be a non-singular projective surface admitting a non-isomorphic surjective separable endomorphism $f\colon X \to X$. Assume that the Kodaira dimension $\kappa(X) = -\infty$. Then, the following hold for the irregularity $q(X) = \dim \mathrm{Alb}(X)$:*

1. *If $q(X) = 0$, then X is a rational surface having at most finitely many negative curves and $-K_X$ is* big *(cf. Convention 3.2). If $p \nmid \deg f$ or f is* tame *(cf. Definition 2.1), in addition, then X is a toric surface.*
2. *If $q(X) = 1$, then X is a $\mathbb{P}^1$-bundle over an elliptic curve.*
3. *If $q(X) \geq 2$, then X is a $\mathbb{P}^1$-bundle over a non-singular projective curve T of genus $q(X)$, X has no negative curves, and the relative anti-canonical divisor $-K_{X/T}$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. If $p \nmid \deg f$ or f is tame, in addition, then $-K_{X/T}$ is semi-ample, and there is a finite surjective morphism $T' \to T$ from a non-singular projective curve T' such that $X \times_T T' \simeq \mathbb{P}^1 \times T'$ over T'.*

Here, a *negative curve* means a prime divisor on X with negative self-intersection number (cf. Section 3).

Theorem 1.2. *Let X be a non-singular projective surface of Kodaira dimension $\kappa(X) \geq 0$. Then, any surjective separable endomorphism of X is étale. Moreover, X admits a non-isomorphic surjective separable endomorphism if and only if one of the following conditions is satisfied:*

1. *X is a minimal surface with $\kappa(X) = 0$ and $\chi(X, \mathcal{O}_X) = 0$; in other words, X is an abelian surface, a hyperelliptic surface, or a quasi-hyperelliptic surface (cf. Fact 6.3 below).*
2. *X is a minimal elliptic surface with $\kappa(X) = 1$ and $\chi(X, \mathcal{O}_X) = 0$.*

There are two remarks on the theorems. First, the converse direction in Theorem 1.1 is not known, i.e., it is not clear whether a surface listed in Theorem 1.1 really admits non-isomorphic surjective separable endomorphisms. So, the classification is not complete in the case of $\kappa(X) = -\infty$. Second, similarly to the case of characteristic zero, we have few information on the structure of non-isomorphic surjective separable endomorphisms of a given surface.

Two peculiar examples related to Theorem 1.1 are given. One is the example mentioned above, which is a non-toric rational surface admitting non-isomorphic surjective separable endomorphisms. This is given in Example 4.5. The other is an example of a $\mathbb{P}^1$-bundle over a curve of genus ≥ 2 such that it admits a non-isomorphic surjective endomorphism but the $\mathbb{P}^1$-bundle structure is not trivialized after any finite base change whose degree is not divisible by p. This is given by Proposition 5.5 (cf. Remark 5.6). Both examples are related to Artin–Schreier coverings.

This article is organized as follows. In Section 2, we recall some basic properties of "separable coverings" and "fibrations." In Section 3, we study the set of negative curves, which is the key object in the classification in the case of negative Kodaira dimension. The case of rational surfaces, the case of irrational ruled surfaces, and the case of non-negative Kodaira dimension are treated separately in the remaining sections. The proof of Theorem 1.1 is given at the end of Section 5, and that of Theorem 1.2 is at the end of Section 6.

Notation and conventions. We fix an algebraically closed field $\Bbbk$ of characteristic $p > 0$ as a ground field. We use standard notation of algebraic geometry (cf. Table 1). By a *variety*, we mean an integral separated $\Bbbk$-scheme of finite type. Note that, since $\Bbbk$ is algebraically closed, a variety is non-singular if and only if it is smooth over $\operatorname{Spec} \Bbbk$. A *curve* (resp. *surface*) means a variety of dimension one (resp. two). Additional notation and conventions etc. are given later (cf. Section 2, Convention 3.2, Definitions 3.4, 3.6).

Remark. The following formulas are well-known for non-singular projective surfaces X (cf. [1]):

$$\dim \mathrm{H}^2(X, \mathcal{O}_X) \geq \dim \mathrm{H}^1(X, \mathcal{O}_X) - q(X) \geq 0,$$

$$12\chi(X, \mathcal{O}_X) = K_X^2 + e(X) \qquad \text{(so-called "Noether's formula")},$$

$$b_1(X) = b_3(X) = 2q(X), \qquad 1 \leq \rho(X) \leq b_2(X).$$

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2. Preliminaries

In this section, we recall some basic results on separable coverings and fibrations of normal varieties.

Let us begin with discussion on separable coverings. Let $\varphi\colon V_1 \to V_2$ be a finite surjective morphism of varieties. If the function field $\Bbbk(V_1)$ is a separable extension of $\varphi^*\Bbbk(V_2)$, then φ is called *separable*. In this case, $\Omega^1_{V_1/V_2}$ is zero at the generic point of V_1. If V_1 and V_2 are normal, then the finite surjective morphism φ is called a *covering* (or a finite covering). If further $\Bbbk(V_1)$ is a Galois extension of $\varphi^*\Bbbk(V_2)$, then the action of the Galois group on V_1 is regular and the quotient variety is isomorphic to V_2. In this case, φ is called a *Galois covering*.

Table 1. List of notation

$\kappa(X)$:	The Kodaira dimension of X.
K_X :	The canonical divisor of X.
$\mathrm{Alb}(X)$:	The Albanese variety of X.
$\Bbbk(X)$:	The function field of X.
$\mathbb{P}_X(\mathcal{E})$:	The projective bundle associated with a locally free sheaf $\mathcal{E}$ on X.
$b_i(X)$:	The i-th Betti number: $\operatorname{rank} \mathrm{H}^i(X_{\text{ét}}, \mathbb{Z}_l)$, where $p \nmid l$.
$e(X)$:	The Euler number: $\sum_{i \geq 0}(-1)^i b_i(X)$.
$q(X)$:	The irregularity: $\dim \mathrm{Alb}(X) = (1/2)b_1(X)$.
$\mathsf{N}(X)$:	The real vector space $\mathrm{NS}(X) \otimes \mathbb{R}$, where $\mathrm{NS}(X)$ is the Néron–Severi group.
$\mathrm{Nef}(X)$:	The nef cone.
$\overline{\mathrm{NE}}(X)$:	The pseudo-effective cone.
$\rho(X)$:	The Picard number: $\dim \mathrm{NS}(X)$.
$D_1 D_2$:	The intersection number of two divisors D_1, D_2.
D^2 :	The self-intersection number: DD.
$\sim$:	The linear equivalence relation of divisors.
$\approx$:	The numerical equivalence relation of divisors.
$\mathrm{cl}(D)$:	The numerical equivalence class ($\in \mathsf{N}(X)$) of a divisor D.
$p_a(D)$:	The arithmetic genus of a complete connected reduced curve D ($= \dim \mathrm{H}^1(D, \mathcal{O}_D) = 1 - \chi(D, \mathcal{O}_D)$).
f^k :	The k-times composite $f \circ \cdots \circ f$ of an endomorphism $f\colon X \to X$.

Suppose that V_1 and V_2 are non-singular and that φ is separable. Then, φ is flat, and the canonical homomorphism $\varphi^*\Omega^1_{V_2} \to \Omega^1_{V_1}$ induced from the pullback of differential one-forms is injective. The determinant of the homomorphism is an injection $\varphi^*\Omega^n_{V_2} \to \Omega^n_{V_1}$ of invertible sheaves, where $n = \dim V_1 = \dim V_2$. Hence, $\Omega^n_{V_1} \simeq \varphi^*\Omega^n_{V_2} \otimes \mathcal{O}_{V_1}(R_\varphi)$ for an effective divisor R_φ, equivalently, $K_{V_1} = \varphi^*(K_{V_2}) + R_\varphi$, where K_V denotes the canonical divisor of V. The homomorphism $\varphi^*\Omega^1_{V_2} \to \Omega^1_{V_1}$ is an isomorphism outside $\operatorname{Supp} R_\varphi$. Thus, φ is étale on $V_1 \setminus \operatorname{Supp} R_\varphi$. The effective divisor R_φ is called the *ramification divisor*.

Definition 2.1 (cf. [10], Section 2.1)**.** Let $\varphi\colon V_1 \to V_2$ be a finite surjective separable morphism of normal varieties. It is called *tame* over a prime divisor Θ on V_2 if the following conditions are satisfied for any prime divisor Γ on V_1 with $\varphi(\Gamma) = \Theta$:

(1) The ramification index of φ along Γ is not divisible by p, where the ramification index is the multiplicity of the divisor $\varphi^*(\Theta)$ along Γ.
(2) The induced finite surjective morphism $\varphi|_\Gamma\colon \Gamma \to \Theta$ is separable.

If φ is tame over any prime divisor on V_2, then φ is called tame.

Note that if φ is étale, then φ is tame. As a version of Abhyankar's lemma (cf. [9], Exp. X, Lemma 3.6, Exp. XIII, Section 5, and [10], Section 2.3), we have the following:

Lemma 2.2. *Let $\varphi\colon V_1 \to V_2$ be a finite surjective morphism of normal varieties. Suppose that V_2 is non-singular, φ is étale outside a non-singular divisor Θ on V_2 (i.e., $V_1 \setminus \varphi^{-1}(\Theta) \to V_2 \setminus \Theta$ is étale), and that φ is tame. Then, for any point $P \in \varphi^{-1}(\Theta)$, there exist an étale neighborhood $U_1 \to V_1$ of P, an affine étale neighborhood $U_2 = \operatorname{Spec} A \to V_2$ of $\varphi(P)$, and an étale morphism $U_1 \to U_2(m,a)$ over V_2 for the affine scheme*

$$U_2(m,a) = \operatorname{Spec} A[\mathsf{T}]/(\mathsf{T}^m - a),$$

where m is a positive integer not divisible by p and the zero subscheme of $a \in A$ is the non-singular divisor $\Theta \times_{V_2} U_2$. In particular, V_1 and $\varphi^{-1}(\Theta)$ are non-singular, and $\varphi^{-1}(\Theta)$ is étale over Θ.

Lemma 2.3. *Let $\varphi\colon V_1 \to V_2$ be a finite surjective separable morphism of non-singular varieties. Let Γ be a prime divisor on V_1 and m the ramification index of φ along Γ. Then,* $\operatorname{mult}_\Gamma(R_\varphi) \geq m - 1$*. If $p \mid m$, then* $\operatorname{mult}_\Gamma(R_\varphi) \geq m$*. If φ is tame over $\varphi(\Gamma)$, then* $\operatorname{mult}_\Gamma(R_\varphi) = m - 1$*.*

Proof. Let x be a general point of Γ such that Γ is non-singular at x and $\varphi(\Gamma)$ is non-singular at $\varphi(x)$. Then, there exist local coordinate systems $(t_1, \ldots, t_n)$ of V_1 at x and

$(s_1, \ldots, s_n)$ of V_2 at $\varphi(x)$ such that t_1 is a local equation of Γ and s_1 is a local equation of $\varphi(\Gamma)$. Then, $\varphi^*(s_1) = ut_1^m$ for a regular function u not vanishing at x. Since

$$\varphi^*(\mathrm{d}s_1) = t_1^{m-1}(mu\,\mathrm{d}t_1 + t_1\,\mathrm{d}u),$$

there is a regular function v such that

$$\varphi^*(\mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_n) = vt_1^{m-1}(\mathrm{d}t_1 \wedge \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_n).$$

Hence, $\mathrm{mult}_\Gamma(R_\varphi) \geq m - 1$. If $p \mid m$, then $\varphi^*(\mathrm{d}s_1) = t_1^m\,\mathrm{d}u$; hence, $\mathrm{mult}_\Gamma(R_\varphi) \geq m$ by the same argument above. Suppose that φ is tame over $\varphi(\Gamma)$. By Lemma 2.2, étale locally on V_1 and V_2, φ is regarded as a cyclic covering $(t_1, t_2, \ldots, t_n) \mapsto (s_1, s_2, \ldots, s_n) = (t_1^m, t_2, \ldots, t_n)$. Hence,

$$\varphi^*(\mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_n) = mt_1^{m-1}(\mathrm{d}t_1 \wedge \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_n).$$

Therefore, $\mathrm{mult}_\Gamma(R_\varphi) = m - 1$. □

Corollary 2.4. *Let $\varphi\colon V_1 \to V_2$ be a finite surjective separable morphism between non-singular varieties. Let D_2 be a reduced divisor on V_2 such that φ is tame over D_2, and let D_1 be the reduced divisor $\varphi^{-1}(D_2) = \varphi^*(D_2)_{\mathrm{red}}$. Then, $\Delta = R_\varphi - \varphi^*(D_2) + D_1$ is an effective divisor having no common irreducible components with D_1.*

Proof. Since R_φ is effective, so is Δ at least on $V_1 \setminus D_1$. If Γ is a prime component of D_1, then $\mathrm{mult}_\Gamma(\Delta) = 0$ by Lemma 2.3. Thus, we are done. □

Note that the divisor Δ in Corollary 2.4 satisfies $K_{V_1} + D_1 = \varphi^*(K_{V_2} + D_2) + \Delta$.

Corollary 2.5. *Let $\varphi\colon C \to \mathbb{P}^1$ be a finite covering from a non-singular curve C such that $p \nmid \deg\varphi$. Assume that, for a point $P \in \mathbb{P}^1$, $\varphi^{-1}(P)$ is a point, and φ is étale on $C \setminus \varphi^{-1}(P)$. Then, φ is an isomorphism.*

Proof. We set $Q := \varphi^{-1}(P)$. Then, $\varphi^*(P) = mQ$ for $m = \deg\varphi$. Hence, φ is tame. By Corollary 2.4, we have $K_C + Q = \varphi^*(K_{\mathbb{P}^1} + P) + \Delta$ for an effective divisor Δ on C with $Q \notin \mathrm{Supp}\,\Delta$. Since φ is étale outside Q, we have $\Delta = 0$. Hence,

$$2g - 2 < 2g - 2 + 1 = \deg(K_C + Q) = -\deg\varphi < 0$$

for the genus g of C. Thus, $g = 0$ and $\deg\varphi = 1$. Therefore, φ is an isomorphism. □

The following is a typical example of separable surjective morphisms $\varphi\colon C \to \mathbb{P}^1$ with $p = \deg\varphi$ which is étale over $\mathbb{P}^1 \setminus \{P\}$ for a point P.

Example 2.6. Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the Artin–Schreier morphism defined by $(\mathtt{x} : \mathtt{y}) \mapsto (\mathtt{x}^p - \mathtt{x}\mathtt{y}^{p-1} : \mathtt{y}^p)$ for a homogeneous coordinate $(\mathtt{x}, \mathtt{y})$ of $\mathbb{P}^1$. Then, for the infinity point $P = (1 : 0)$, $f^{-1}(P) = \{P\}$ and f is étale on $\mathbb{P}^1 \setminus \{P\}$. In particular, f is a separable finite covering of degree p. Here, f is not tame over P, since the ramification index at P is p. Moreover, the ramification divisor R_f is calculated as $(2p-2)P$, since $\deg R_f = (1 - \deg f) \deg K_{\mathbb{P}^1} = 2p - 2$ by the ramification formula.

Next, we shall discuss on fibrations. First, we recall the notion of fibration of normal varieties.

Definition 2.7. Let $\pi \colon V \to W$ be a proper surjective morphism of normal varieties. If the canonical homomorphism $\mathcal{O}_W \to \pi_* \mathcal{O}_V$ is isomorphic, then π is called a *fibration* (or a fiber space).

For a proper surjective morphism $\pi \colon V \to W$ of normal varieties, it is known that π is a fibration if and only if the function field $\Bbbk(W)$ is algebraically closed in $\Bbbk(V)$ via $\pi^* \colon \Bbbk(W) \to \Bbbk(V)$. Moreover, if π is a fibration, then any fiber of π is connected (cf. [6], Théorème 4.3.1) and a general fiber of π is geometrically irreducible (cf. [7], Proposition 4.5.9, [8], Théorème 9.7.7). However, even if π is a fibration, a general fiber is not necessarily reduced.

Example 2.8. For $n \geq 2$, let V be the hypersurface of $\mathbb{P}^n \times \mathbb{P}^n$ defined by $\sum_{i=0}^{n} \mathtt{x}_i \mathtt{y}_i^p = 0$, where $(\mathtt{x}_0 : \cdots : \mathtt{x}_n)$ and $(\mathtt{y}_0 : \cdots : \mathtt{y}_n)$ are homogeneous coordinates of $\mathbb{P}^n$. Then the projection $V \to \mathbb{P}^n$ to the second factor is a $\mathbb{P}^{n-1}$-bundle, while the projection $V \to \mathbb{P}^n$ to the first factor is a fibration whose closed fibers are all non-reduced.

For a fibration $\pi \colon V \to W$, a general fiber is reduced if and only if the geometric general fiber is reduced (cf. [8], Théorème 9.7.7); this is also equivalent to the condition that $\Bbbk(V)$ is a separable over $\Bbbk(W)$ via π^*, i.e., $\Bbbk(V) \otimes_{\Bbbk(W)} L$ is reduced for any field L over $\Bbbk(W)$ (cf. [7], Proposition 4.6.1). Fortunately, if $\dim W = 1$, then a general fiber of a fibration $\pi \colon V \to W$ is always reduced by [11], Theorem 2 (cf. [20]).

For a fibration from a surface to a curve, we have the following well-known result in the classification theory of surfaces, which is mentioned in [14] without proof.

Proposition 2.9. *Let $\pi \colon X \to T$ be a fibration from a non-singular surface X to a non-singular curve T such that $K_X C = 0$ for any closed curve $C \subset X$ contained in a fiber of π. Then, a general fiber F of π is an irreducible and reduced curve of arithmetic genus one. Moreover, if $p > 3$, then F is an elliptic curve, and if $p \leq 3$, then F is an elliptic curve or a cuspidal cubic curve.*

Proof. As has been mentioned, $\Bbbk(X)$ is separable over $\pi^* \Bbbk(T)$ and F is irreducible and reduced. Hence, $p_a(F) = 1$ by $(K_X + F)F = 0$, and consequently, F is isomorphic to a plane cubic curve. Thus, it is enough to prove the last assertion. In many articles, the proof of this part is done by referring to [21], Theorem 2. Here, we shall present another proof. Assume that the general fiber F is not an elliptic curve. Then F is a rational curve with a unique singular point P, where P is a node or a cusp of type $(2,3)$; more precisely, the completion $\widehat{\mathcal{O}}$ of the local ring $\mathcal{O}_{F,P}$ is isomorphic to either $\Bbbk[\![\mathtt{u},\mathtt{v}]\!]/(\mathtt{uv})$ or $\Bbbk[\![\mathtt{u},\mathtt{v}]\!]/(\mathtt{u}^2-\mathtt{v}^3)$. Let $(\mathtt{x},\mathtt{y})$ be a local coordinate of $\mathbb{P}^2$ at P and let $\phi=\phi(\mathtt{x},\mathtt{y})$ be a local defining equation of F. From the natural exact sequence

$$0\to\mathcal{O}_F(-F)\to\Omega^1_{\mathbb{P}^2/\Bbbk}\to\Omega^1_{F/\Bbbk}\to 0,$$

we have an isomorphism

$$\mathcal{E}xt^1_{\mathcal{O}_F}(\Omega^1_{F/\Bbbk},\mathcal{O}_F)_P\simeq\mathcal{O}_{F,P}/(\phi,\partial\phi/\partial\mathtt{x},\partial\phi/\partial\mathtt{y})\simeq\operatorname{Ext}^1_{\widehat{\mathcal{O}}}\left(\Omega^1_{\widehat{\mathcal{O}}/\Bbbk},\widehat{\mathcal{O}}\right).$$

Therefore, in order to calculate the dimension of Ext^1 above, we may assume $(\mathtt{x},\mathtt{y})=(\mathtt{u},\mathtt{v})$, and $\phi=\mathtt{uv}$ or $\phi=\mathtt{u}^2-\mathtt{v}^3$. As a consequence, we infer that the dimension of the Ext^1 is 1, 2, 3, and 4 according as the conditions: (i) P is a node, (ii) P is a cusp and $p>3$, (iii) P is a cusp and $p=3$, and (iv) P is a cusp and $p=2$.

By the separability of $\Bbbk(X)/\pi^*\Bbbk(T)$, the natural sequence

$$0\to\pi^*\Omega^1_T\to\Omega^1_{X/\Bbbk}\to\Omega^1_{X/T}\to 0$$

is exact. Hence, there is a coherent $\mathcal{O}_X$-ideal $\mathcal{I}$ such that

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_{X/T},\mathcal{O}_X)\otimes\pi^*\Omega^1_T\simeq\mathcal{O}_X/\mathcal{I},$$
$$\mathcal{E}xt^1_{\mathcal{O}_F}(\Omega^1_{F/\Bbbk},\mathcal{O}_F)\simeq(\mathcal{O}_X/\mathcal{I})\otimes\mathcal{O}_F\simeq\mathcal{O}_F/\mathcal{I}\mathcal{O}_F.$$

Let $S\subset X$ be the reduced closed subscheme identified with the support of $\mathcal{O}_X/\mathcal{I}$. Then, $S\cap F=\{P\}$. In particular, $S\subset X\to T$ is a dominant purely inseparable morphism. If $S\to T$ is isomorphic, then $\pi\colon X\to T$ is smooth along S, since X is non-singular; this is a contradiction. Therefore, $\deg S/T\geq p$. On the other hand, $\deg S/T\leq\dim_{\Bbbk}(\mathcal{O}_F/\mathcal{I}\mathcal{O}_F)_P\leq 2$ if (i) P is a node or if (ii) P is a cusp and $p>3$, by the calculation of the dimension of the Ext^1 above. Hence, $p\leq 3$ and P is a cusp. □

Definition 2.10. Let $\pi\colon X\to T$ be a fibration from a normal surface X to a non-singular curve T. If a general fiber of π is an elliptic curve (resp. a cuspidal cubic curve), then π is called an *elliptic fibration* (resp. a *quasi-elliptic fibration*). In this case, X is called an *elliptic surface* (resp. a *quasi-elliptic surface*).

A fibration π is called *minimal* if X is non-singular and any fiber of π contains no (-1)-curves. Here, a (-1)-curve is by definition a non-singular rational curve $C\subset X$ with $C^2=-1$; this is also called an *exceptional curve of the first kind.*

Remark 2.11. If an elliptic surface (resp. a quasi-elliptic surface) $\pi\colon X\to T$ is minimal, then $K_XC=0$ for any closed curve contained in a fiber. In fact, if $K_XC\neq 0$, then there is an irreducible component Γ in the same fiber such that $K_X\Gamma < 0$, since $K_X\pi^*(t) = 0$ for any $t \in T$. Here, if $\Gamma^2 < 0$, then Γ is a (-1)-curve by $2p_a(\Gamma) - 2 = (K_X + \Gamma)\Gamma < 0$; if $\Gamma^2 \geq 0$, then the fiber $\pi^*(t)$ is a multiple of Γ, since $\pi^*(t)$ is connected and $\pi^*(t)\Gamma = 0$; thus $K_X\Gamma = 0$, a contradiction.

Lemma 2.12. *Let $\pi\colon X \to T$ be an elliptic fibration from a normal surface X to a non-singular curve T. Assume that any fiber of π does not contain rational curves. Then, X is non-singular, π is minimal, and the support of every fiber is an elliptic curve.*

Proof. Let $\mu\colon Z \to X$ be a resolution of singularities. Contracting (-1)-curves contained in fibers of $\pi \circ \mu\colon Z \to T$, we have a proper birational morphism $\nu\colon Z \to Y$ to a minimal elliptic surface Y over T. Let $\varpi\colon Y \to T$ be the induced elliptic fibration. Let Γ be the proper transform in Y of an irreducible component of a fiber of π. Then, Γ is also an irrational irreducible component of a fiber F of ϖ. Hence, $0 \leq 2p_a(\Gamma) - 2 = (K_Y + \Gamma)\Gamma$. If F is reducible, then $\Gamma^2 < 0$. But $K_YF = 0$ and $K_Y\Gamma > 0$ imply that $\Gamma_1^2 < 0$ and $K_Y\Gamma_1 < 0$ for some other irreducible component Γ_1 of F; hence Γ_1 is a (-1)-curve. Therefore, F is irreducible, and hence $F = m\Gamma$ for some $m \geq 1$. Since $p_a(\Gamma) = 1$ by $K_Y\Gamma = \Gamma^2 = 0$ and since Γ is irrational, Γ is an elliptic curve. Therefore, the support of any fiber of ϖ is an elliptic curve. In particular, the rational curves contained in fibers of $\pi \circ \mu\colon Z \to T$ are all exceptional for both μ and ν. Hence, $X \simeq Y$ over T. Thus, we are done. □

Theorem 6.1 below explains in detail the structure of the elliptic fibration $\pi\colon X \to T$ in Lemma 2.12 above. The following result seems to be well-known.

Lemma 2.13. *Let $\pi\colon X \to T$ be an elliptic fibration from a non-singular projective surface X. If $\chi(X, \mathcal{O}_X) = e(X) = 0$, then the support of any fiber of π is an elliptic curve.*

Proof. We have $K_X^2 = 12\chi(X, \mathcal{O}_X) - e(X) = 0$. If π is not minimal, then we have a birational morphism $\mu\colon X \to Y$ for a minimal elliptic surface Y over T, where $0 = K_X^2 < K_Y^2$. However, since $K_YF = 0$ for a general fiber F of π, we have $K_Y^2 \leq 0$ by the Hodge index theorem. Therefore, π is a minimal elliptic fibration, and hence $K_XC = 0$ for any closed curve C contained in a fiber of π (cf. Remark 2.11). Let $U \subset T$ be a non-empty open subset such that $\pi|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to U$ is smooth. Then $\Omega^1_{X/T}$ is locally free of rank one over $\pi^{-1}(U)$. Thus, we have a surjection $\Omega^1_{X/T} \to \mathcal{J}\mathcal{M}$ for an invertible sheaf $\mathcal{M}$ on X and an ideal sheaf $\mathcal{J}$ on X such that the kernel is a torsion sheaf on X and $\operatorname{Supp} \mathcal{O}_X/\mathcal{J}$ is a finite subset of $X \setminus \pi^{-1}(U)$. Therefore, we have an effective divisor B on X with $\operatorname{Supp} B \cap \pi^{-1}(U) = \emptyset$ and an exact sequence

$$0 \to \pi^*\Omega^1_T \otimes \mathcal{O}_X(B) \to \Omega^1_X \to \mathcal{J}\mathcal{M} \to 0.$$

Considering the Chern classes of Ω^1_X, we have $\mathcal{M} \simeq \mathcal{O}_X(K_X - \pi^*(K_T) - B)$ and

$$\begin{aligned} c_2(\Omega^1_X) &= c_1(\pi^*\Omega^1_T \otimes \mathcal{O}_X(B))c_1(\mathcal{M}) + c_2(\mathcal{J}\mathcal{M}) \\ &= (\pi^*(K_T) + B)(K_X - \pi^*(K_T) - B) + \mathrm{length}(\mathcal{O}_X/\mathcal{J}) \\ &= -B^2 + \mathrm{length}(\mathcal{O}_X/\mathcal{J}) \geq \mathrm{length}(\mathcal{O}_X/\mathcal{J}) \geq 0, \end{aligned}$$

since $K_X\pi^*(K_T) = K_X B = 0$ and $B^2 \leq 0$. Note that $B^2 = 0$ if and only if $kB = \pi^*(\Theta)$ for a positive integer k and an effective divisor Θ on T (cf. Lemma in the proof of [1], Theorem 2). Thus, from the assumption $e(X) = c_2(\Omega^1_X) = 0$, we have $\mathcal{J} = \mathcal{O}_X$ and $B^2 = 0$. In particular, we have an exact sequence

$$0 \to \mathcal{O}_X(\pi^*(K_T) + B) \to \Omega^1_X \to \mathcal{O}_X(K_X - \pi^*(K_T) - B) \to 0, \tag{2.1}$$

where $BC = 0$ for any closed curve C contained in fibers of π.

Let C be an irreducible component of the fiber $\pi^*(t)$ over a point $t \in T \setminus U$. Then, we have a natural exact sequence

$$0 \to \mathcal{O}_C(-C) \to \Omega^1_X|_C \to \Omega^1_C \to 0. \tag{2.2}$$

By (2.1), we have a homomorphism

$$\varphi_C \colon \mathcal{O}_C(-C) \to \mathcal{O}_X(K_X - \pi^*(K_T) - B)|_C$$

to an invertible sheaf on C of degree zero. Suppose that φ_C is not zero. Then, $C^2 = 0$, and φ_C is an isomorphism. In this case, (2.2) is split and $\Omega^1_C \simeq \mathcal{O}_X(\pi^*(K_T) + B)|_C$ is locally free. Therefore, C is an elliptic curve, and $\pi^*(t) = mC$ for some $m > 0$. Suppose next that φ_C is zero. Then, we have an injection

$$\psi_C \colon \mathcal{O}_C(-C) \to \mathcal{O}_X(\pi^*(K_T) + B)|_C$$

to an invertible sheaf on C of degree zero. Hence, $C^2 = 0$ and ψ_C is an isomorphism. Then, Ω^1_C is isomorphic to the locally free sheaf $\mathcal{O}_X(K_X - \pi^*(K_T) - B)|_C$. Thus, C is an elliptic curve, and $\pi^*(t) = mC$ for some $m > 0$. Therefore, the support of any fiber $\pi^*(t)$ is an elliptic curve. □

3. Negative curves and endomorphisms

Let X be a non-singular projective surface. A prime divisor Γ on X is called a *negative curve* if the self-intersection number Γ^2 is negative. Let $\mathrm{Neg}(X)$ denote the set of negative curves on X. In this section, we shall give basic properties on the negative curves related to endomorphisms. In particular, we shall show that $\mathrm{Neg}(X)$ is finite if X admits a non-isomorphic surjective separable endomorphism. This is known in the case of characteristic zero by [16].

Let $\mathsf{N}(X)$ be the real vector space $\mathrm{NS}(X)\otimes\mathbb{R}$ for the Néron–Severi group $\mathrm{NS}(X)$ of X. Here, $\dim \mathsf{N}(X)$ equals the Picard number $\rho(X)$. Let $f\colon X \to Y$ be a surjective morphism of non-singular projective surfaces. For the pull-back and push-forward of divisors, we have

$$f^*(D)E = Df_*(E) \quad \text{and} \quad (\deg f)D = f_*(f^*(D))$$

for divisors D on Y and E on X. These are known as the projection formula. The maps $D \mapsto f^*(D)$ and $E \mapsto f_*(E)$ induce the homomorphisms $f^*\colon \mathsf{N}(Y) \to \mathsf{N}(X)$ and $f_*\colon \mathsf{N}(X) \to \mathsf{N}(Y)$, respectively. Here, $f_* \circ f^*$ is the multiplication map by $\deg f$. In particular, f^* is injective and f_* is surjective. Note that if $g\colon Y \to Z$ is a surjective morphism to another non-singular projective surface Z, then $f^* \circ g^* = (g \circ f)^*$ and $g_* \circ f_* = (g \circ f)_*$.

Remark (cf. [2], Lemma 2.3, (1)). Let $f\colon X \to X$ be a surjective endomorphism of a non-singular projective surface X. Then, $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ and $f_*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ are isomorphisms. In particular, f is a finite morphism, since no curve is contracted by f.

The following is proved in [16] in characteristic zero, and the same proof works in this case:

Lemma 3.1. *Let $f\colon X \to X$ be a non-isomorphic separable surjective endomorphism. Then,* $\mathrm{Neg}(X)$ *is a finite set, and there is a positive integer k such that $(f^k)^*\Gamma = (\deg f)^{k/2}\Gamma$ for any $\Gamma \in \mathrm{Neg}(X)$, where f^k stands for the k-times composite $f \circ \cdots \circ f$.*

Since this is a key lemma for our study of endomorphisms of surfaces, we write the proof.

Proof. Step 1 (cf. [16], Lemma 9). We shall show that the mapping $\Gamma \mapsto f(\Gamma)$ induces an injection $\psi\colon \mathrm{Neg}(X) \to \mathrm{Neg}(X)$. Let Γ be a negative curve on X. Assume that $f(\Gamma) = f(\Gamma')$ for some prime divisor Γ'. Then, $f_*(\Gamma') = \alpha f_*(\Gamma)$ for some rational number $\alpha > 0$, since $f_*(\Gamma) = d_\Gamma f(\Gamma)$ for the mapping degree d_Γ of $\Gamma \to f(\Gamma)$. Hence, the class of $\Gamma' - \alpha\Gamma$ in $\mathsf{N}(X)$ is zero by the injectivity of f_*. In particular, $\Gamma\Gamma' = \alpha\Gamma^2 < 0$. Therefore, $\Gamma = \Gamma'$. As a consequence, we have $f^*(f(\Gamma)) = m\Gamma$ for some $m \geq 1$. Here, $f(\Gamma)$ is a negative curve by

$$(\deg f)f(\Gamma)^2 = f^*(f(\Gamma)) \cdot f^*(f(\Gamma)) = m^2\Gamma^2 < 0.$$

Thus, $\Gamma \mapsto f(\Gamma)$ induces an injection $\psi\colon \mathrm{Neg}(X) \to \mathrm{Neg}(X)$.

Step 2 (cf. [16], Lemma 10). Let Γ be a negative curve. We shall show that $f^k(\Gamma) \subset \operatorname{Supp} R_f$ for some $k \geq 0$. For an integer $k \geq 0$, we define m_k by $f^*(f^{k+1}(\Gamma)) = m_k f^k(\Gamma)$. It is enough to show that $m_k > 1$ for some k. Assume the contrary. Then, $(\deg f) f^{k+1}(\Gamma)^2 = (f^k(\Gamma))^2$ for any k. We have a contradiction by

$$\Gamma^2 = (\deg f) f(\Gamma)^2 = \cdots = (\deg f)^k f^k(\Gamma)^2 \in \bigcap\nolimits_{k=0}^{\infty} (\deg f)^k \mathbb{Z} = 0.$$

Step 3 (cf. [16], Proposition 11). Let $\mathrm{Neg}(X)_\circ$ be the set of negative curves Γ such that $\Gamma \subset \operatorname{Supp} R_f$. This is a finite set, and

$$\mathrm{Neg}(X) = \bigcup\nolimits_{k \geq 0} (\psi^k)^{-1}(\mathrm{Neg}(X)_\circ)$$

by *Step* 2. Since ψ is injective, $\mathrm{Neg}(X)$ is a finite set by [3], Lemma 3.4 (cf. The proof of [16], Proposition 11). Let k be the order of the permutation $\psi \colon \Gamma \mapsto f(\Gamma)$ of the finite set $\mathrm{Neg}(X)$. Then, $(f^k)^*(\Gamma) = n_{k,\Gamma}\Gamma$ for some positive integer $n_{k,\Gamma}$ for any $\Gamma \in \mathrm{Neg}(X)$. By calculation

$$(\deg f)^k \Gamma^2 = \left((f^k)^*(\Gamma)\right)^2 = n_{k,\Gamma}^2 \Gamma^2,$$

we have $n_{k,\Gamma} = (\deg f)^{k/2}$. Thus, we are done. □

Convention 3.2. An element of $\mathsf{N}(X)$ is regarded as the numerical equivalence class $\mathrm{cl}(D)$ of an $\mathbb{R}$-*divisor* D on X, where $\mathbb{R}$-divisor means a formal $\mathbb{R}$-linear combination of finitely many prime divisors. The numerical equivalence relation is denoted by $\approx$. Note that $D \approx 0$ if and only of $DC = 0$ for any closed curve C on X. An $\mathbb{R}$-divisor D is called *nef* if $DC \geq 0$ for any closed curve C on X. The *nef cone* $\mathrm{Nef}(X)$ is by definition the set of $\mathrm{cl}(D)$ for all the nef $\mathbb{R}$-divisors D on X. An *effective* $\mathbb{R}$-divisor is by definition a divisor of the form $\sum a_i \Gamma_i$, where Γ_i is a prime divisor and all $a_i \geq 0$. The *pseudo-effective cone* $\overline{\mathrm{NE}}(X)$ is the closure of the cone $\mathrm{NE}(X)$ consisting of $\mathrm{cl}(D)$ for all the effective $\mathbb{R}$-divisors D on X. An $\mathbb{R}$-divisor D is called *pseudo-effective* (resp. *big*) if $\mathrm{cl}(D) \in \overline{\mathrm{NE}}(X)$ (resp. $\mathrm{cl}(D)$ is in the interior of $\overline{\mathrm{NE}}(X)$).

Let $f \colon X \to Y$ be a surjective morphism of non-singular projective surfaces. Then, $f^* \mathrm{Nef}(Y) = \mathrm{Nef}(X) \cap f^* \mathsf{N}(Y)$ and $f_* \overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(Y)$ for the homomorphisms $f^* \colon \mathsf{N}(Y) \to \mathsf{N}(X)$ and $f_* \colon \mathsf{N}(X) \to \mathsf{N}(Y)$. The following is shown in [18], Section 4.4.

Proposition 3.3. *Let $f \colon X \to X$ be a non-isomorphic surjective endomorphism of a non-singular projective rational surface X. Let $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ be the pullback homomorphism. If $X \not\simeq \mathbb{P}^1 \times \mathbb{P}^1$, then some power $(f^*)^k = f^* \circ \cdots \circ f^*$ is a scalar map.*

Proof. In the proof, we may replace f with a composite $f^k = f \circ \cdots \circ f$, freely. Hence, by Lemma 3.1, we may assume that $f^*(\Gamma) = d\Gamma$ for any $\Gamma \in \mathrm{Neg}(X)$, where d is the positive integer equal to $(\deg f)^{1/2}$. If $\rho(X) = 1$, then $\mathsf{N}(X)$ is one-dimensional, so f^* is a scalar map. Suppose that $\rho(X) = 2$ and $X \not\simeq \mathbb{P}^1 \times \mathbb{P}^1$. Then, X is a Hirzebruch surface having a negative section Γ. Thus $f^*(\Gamma) = d\Gamma$. Let F be a fiber of the $\mathbb{P}^1$-bundle structure on

X. Then, $f^*(F) \approx mF$ for some $m > 0$, since $\overline{\mathrm{NE}}(X)$ is spanned by $\mathrm{cl}(F)$ and $\mathrm{cl}(\Gamma)$ and since $f^* \overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)$. Here, $m = d$ by $d^2 = \deg f = f^*(F) f^*(\Gamma) = md$. Thus, $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ is a scalar map.

Suppose that $\rho(X) \geq 3$. Then, there is a (-1)-curve C on X (cf. [15], Theorem (2.1)). Here, $f^*(C) = dC$. Let $\mu \colon X \to Y$ be the blowing down of C. Then, $\mathsf{N}(X) = \mu^* \mathsf{N}(Y) \oplus \mathbb{R}\,\mathrm{cl}(C)$. Since C is the unique curve contracted by $\mu \circ f \colon X \to Y$, as the Stein factorization of $\mu \circ f$, we have an endomorphism $f_Y \colon Y \to Y$ such that $f_Y \circ \mu = \mu \circ f$. Thus, $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ is isomorphic to the direct sum of $f_Y^* \colon \mathsf{N}(Y) \to \mathsf{N}(Y)$ and $d\times \colon \mathbb{R}\,\mathrm{cl}(C) \to \mathbb{R}\,\mathrm{cl}(C)$. Therefore, we can reduce to the case $\rho(X) = 3$. In this case, $\rho(Y) = 2$. If $Y \not\simeq \mathbb{P}^1 \times \mathbb{P}^1$, then f^* is a scalar map, since f_Y^* is the multiplication map by d. Hence, we may assume that $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$. For $i = 1, 2$, let F_i be the fiber of the i-th projection $Y \to \mathbb{P}^1$ which contains the point $\mu(C)$. Then, the proper transform F_i' of F_i in X is also a negative curve. Hence, $f^* F_i' = dF_i'$, which induces $f_Y^*(F_i) = dF_i$. Thus, $f_Y^* \colon \mathsf{N}(Y) \to \mathsf{N}(Y)$ is the multiplication map by d, since $\mathsf{N}(Y)$ is generated by $\mathrm{cl}(F_1)$ and $\mathrm{cl}(F_2)$. Therefore, $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ is a scalar map. Thus we are done. □

Definition 3.4. Let X be a non-singular projective surface. We define

$$N_X := \sum\nolimits_{\Gamma \in \mathrm{Neg}(X)} \Gamma$$

when $\mathrm{Neg}(X)$ is finite.

The following result is proved in [3], Lemma 3.7 in the case of characteristic zero. The same proof almost works in the positive characteristic case, but we need to modify some arguments by applying Lemma 2.2 and Corollaries 2.4 and 2.5.

Lemma 3.5. *Assume that X admits a non-isomorphic surjective endomorphism $f \colon X \to X$ with $p \nmid \deg f$. Then, a connected component of N_X is one of the following*:

(1) *An elliptic curve.*
(2) *A cyclic chain of rational curves.*
(3) *A straight chain of rational curves.*

Here, "cyclic chains of rational curves" and "straight chains of rational curves" are defined as follows:

Definition 3.6. Let D be a reduced and connected divisor on a non-singular projective surface. If D is expressed as $\sum_{i=1}^{k} C_i$ for mutually distinct non-singular rational curves C_i such that

$$C_i C_j = \begin{cases} 1 & \text{if } |i - j| = 1; \\ 0 & \text{if } |i - j| > 1, \end{cases}$$

then D is called a *straight chain of rational curves*. If D is expressed as $\sum_{i=1}^{k} C_i$ for irreducible components C_i satisfying the following conditions, then D is called a *cyclic chain of rational curves*:

(1) If $k = 1$, then $D = C_1$ is a nodal cubic curve.
(2) If $k \geq 2$, then, for any i, $C_i \simeq \mathbb{P}^1$, $(D - C_i)C_i = 2$, and $(D - C_i) \cap C_i$ consists of two points.

Proof of Lemma 3.5. By replacing f with a power f^k, we may assume that $f^*(\Gamma) = d\Gamma$ for any negative curve Γ, by Lemma 3.1, where $d^2 = \deg f$. Hence, the degree of $f|_\Gamma \colon \Gamma \to \Gamma$ is d, and $f|_\Gamma$ is separable. Thus, f is tame over Γ. By Corollary 2.4, we have an effective divisor Δ on X such that Δ has no common irreducible component with N_X and

$$K_X + N_X = f^*(K_X + N_X) + \Delta. \tag{3.1}$$

In particular, any irreducible component of Δ is nef. Let D be a connected component of N_X. Then, by Lemma 2.2, $f|_D \colon D \to D$ is étale over $D \setminus (\operatorname{Sing} D \cup \operatorname{Supp} \Delta)$. We set $D^\circ := D \setminus (\operatorname{Sing} D)$. Since we have

$$K_D = (f|_D)^* K_D + \Delta|_D$$

from (3.1), the ramification divisor of $f|_D$ over D° is just $\Delta|_{D^\circ}$. If Γ is an irreducible component of D, then, by (3.1), we have

$$\deg K_\Gamma + (D - \Gamma)\Gamma = (K_X + N_X)\Gamma = -\frac{1}{d-1}\Delta\Gamma \leq 0. \tag{3.2}$$

Summing up for all the Γ, we have

$$2p_a(D) - 2 = \deg K_D = (K_X + N_X)D = -\frac{1}{d-1}\Delta D \leq 0. \tag{3.3}$$

In particular, $p_a(D) \leq 1$.

Assume that $p_a(D) = 1$. Then, $\omega_D = \mathcal{O}_X(K_X + D) \otimes \mathcal{O}_D \simeq \mathcal{O}_D$, and $\Delta \cap D = \emptyset$ by (3.3). Hence, $f|_D$ is étale over D°. Suppose that D is irreducible. Then D is an elliptic curve or a cubic curve with a node or a cusp of type $(2, 3)$. However, the cusp case does not occur. For, otherwise, $f|_D$ is étale over $D^\circ \simeq \mathbb{A}^1$; this is impossible by Corollary 2.5. Suppose next that D is reducible. Let Γ be an irreducible component of D. Then $(D - \Gamma) \cap \Gamma \neq \emptyset$. By (3.2), we have $\Gamma \simeq \mathbb{P}^1$ and $(D - \Gamma)\Gamma = 2$. The finite surjective morphism $f|_\Gamma \colon \Gamma \to \Gamma$ is of degree d and is étale outside $(D - \Gamma) \cap \Gamma$. Hence, $(D - \Gamma) \cap \Gamma$ consists of two points by Corollary 2.5. Since the property holds for any irreducible component Γ of D, we infer that D is a cyclic chain of rational curves.

Assume next that $p_a(D) = 0$. Then, $\mathrm{H}^1(D, \mathcal{O}_D) = 0$. Hence, every irreducible component of D is $\mathbb{P}^1$. There is an irreducible component Γ_1 such that $\Delta\Gamma_1 > 0$ by (3.3). Then, $(D - \Gamma_1)\Gamma_1 = 1$ and $\Delta\Gamma = d - 1$ by (3.2). Thus, there is another irreducible component

Γ_2 of D such that $\Delta\Gamma_2 > 0$ by (3.3). If Γ is an irreducible component different from Γ_1 and Γ_2, then $\Gamma \cap \Delta = \emptyset$ and $(D - \Gamma)\Gamma = 2$ by (3.3) and (3.2). In this case, since $f|_\Gamma$ is étale outside $(D - \Gamma) \cap \Gamma$, by Corollary 2.5, $(D - \Gamma) \cap \Gamma$ consists of two points. Then these properties imply that D is a straight chain of rational curves. □

In the case of characteristic zero, the following result on N_X is proved by Proposition 14, (3), and Theorem 17 in [16]. Since the same arguments in their proofs work in the positive characteristic case, we omit the proof.

Proposition 3.7. *Let X be a non-singular rational surface with* $\mathrm{Neg}(X)$ *finite. Then, any negative curve on X is a non-singular rational curve. Assume further that any connected component of N_X is either a cyclic chain of rational curves or a straight chain of rational curves. Then, X is a toric surface.*

4. Rational surfaces

If X is a non-singular projective rational surface with $\rho(X) \leq 2$, then $-K_X$ is big and $\mathrm{Neg}(X)$ consists of at most one curve. For the case $\rho(X) \geq 3$, we have the following:

Theorem 4.1. *Let X be a non-singular projective rational surface with $\rho(X) > 2$. Suppose that $-K_X$ is pseudo-effective and that, for any negative curve Γ, $-m_\Gamma K_X - \Gamma$ is pseudo-effective for some $m_\Gamma > 0$. Then, $-K_X$ is big,* $\mathrm{Neg}(X)$ *is finite, and* $\overline{\mathrm{NE}}(X)$ *is a polyhedral cone generated by the classes of negative curves.*

This is a generalization of [17], Proposition 3.3. For the proof, we need:

Lemma 4.2. *Let X be a non-singular projective surface such that $-K_X$ is pseudo-effective. Let P be the positive part of the Zariski decomposition of $-K_X$. Suppose that $P \not\approx 0$ $P^2 = 0$, and $P\Gamma = 0$ for any $\Gamma \in$* $\mathrm{Neg}(X)$*. Then, X is a $\mathbb{P}^1$-bundle over an elliptic curve and* $\mathrm{Neg}(X) = \emptyset$.

Proof. Let $-K_X = P + N$ be the Zariski decomposition (cf. [22], [5]). Then, $(-K_X)^2 = P^2 + N^2 \leq 0$. There is a birational morphism $\mu\colon X \to Y$ to a non-singular projective surface Y without (-1)-curves. It is well-known that Y is a $\mathbb{P}^1$-bundle over a curve or $Y \simeq \mathbb{P}^2$ (cf. [15], Theorem (2.1)). Moreover, $K_Y^2 = 8(1 - g)$ if Y is a $\mathbb{P}^1$-bundle over a curve of genus g. Since $P\Gamma = 0$ for any μ-exceptional curve Γ, there is a nef $\mathbb{Q}$-divisor P_0 on Y such that $P = \mu^* P_0$. If γ is a negative curve on Y or an irreducible component of $\mu_*(N)$, then $\gamma = \mu_*(\Gamma)$ for a negative curve Γ on X, and hence $P_0\gamma = P\Gamma = 0$. Let $\mu_* N = P_1 + N_1$ be the Zariski decomposition, where P_1 is the positive part. Then, $P_0 P_1 = 0$. Since $P_0 \not\approx 0$, by the Hodge index theorem, $P_1 \approx rP_0$ for some rational number $r \geq 0$. For the Zariski decomposition $-K_Y = P_Y + N_Y$ of $-K_Y = \mu_*(-K_X) = P_0 + \mu_*(N)$, the positive part P_Y equals $P_0 + P_1 \approx (r+1)P_0$, and $N_Y = N_1$ by the uniqueness of the Zariski decomposition. Hence, the Zariski decomposition of $-K_Y$ satisfies the same condition as $-K_X$, i.e., $P_Y \not\approx 0$, $P_Y^2 = 0$, $P_Y\gamma = 0$ for any negative curve γ on Y. In particular, $K_Y^2 \leq 0$. Therefore, Y is not rational, and Y is a $\mathbb{P}^1$-bundle over a curve T of genus $g \geq 1$. Let F be a fiber of π. Then, $P_Y \not\approx \alpha F$ for any $\alpha \in \mathbb{R}$; for, otherwise, we have $2\alpha = (-K_Y)P_Y = P_Y^2 + P_Y N_Y = 0$. Thus, $P_Y F > 0$. Since $\mathsf{N}(Y)$ is two-dimensional, $\overline{\mathrm{NE}}(Y)$ is fan-shaped; thus $\overline{\mathrm{NE}}(Y)$ is generated by $\mathrm{cl}(P_Y)$ and $\mathrm{cl}(F)$. In particular, $\overline{\mathrm{NE}}(Y) = \mathrm{Nef}(Y)$ and $\mathrm{Neg}(Y) = \emptyset$. Hence, $N_Y = 0$ and $-8(g-1) = K_Y^2 = P_Y^2 = 0$. Thus, T is an elliptic curve. If $\mu\colon X \to Y$ is not an isomorphism, then there is a reducible fiber of $X \to Y \to \mathbb{P}^1$, which consists of negative curves; hence $P\pi^*(F) = P_Y F = 0$, a contradiction. Therefore, $X \simeq Y$. This completes the proof. □

Proof of Theorem 4.1. *Step* 1. First of all, we shall show that $\mathrm{Neg}(X)$ is finite. If $-K_X$ is big, then the finiteness of $\mathrm{Neg}(X)$ is proved by [19], Proposition 4.4 (cf. The first half of the proof of [17], Proposition 3.3). Thus, we may assume that $-K_X$ is not big. Let $-K_X = P + N$ be the Zariski decomposition of $-K_X$, where P is the positive part. Then, $P^2 = 0$. Moreover, $P\Gamma = 0$ for any negative curve Γ, since $P\Gamma \leq -m_\Gamma K_X P = m_\Gamma P^2 = 0$; hence, $P \approx 0$ by Lemma 4.2. For a negative curve Γ, $-K_X - r\Gamma$ is pseudo-effective for $r := m_\Gamma^{-1}$; let $-K_X - r\Gamma = P_1 + N_1$ be the Zariski decomposition, where P_1 is the positive part. Since $P + N = P_1 + N_1 + r\Gamma$, we have $P \geq P_1$, equivalently, $N_1 + r\Gamma \geq N$. Since $P \approx 0$, we have $P_1 = P \approx 0$ and $N = N_1 + r\Gamma$. In particular, $N \geq r\Gamma$. This implies that $\mathrm{Supp}\, N$ contains all the negative curves. Consequently, $\mathrm{Neg}(X)$ is finite.

Step 2. Let Λ be the polyhedral cone in $\mathsf{N}(X)$ generated by the classes of negative curves on X. Then, $\Lambda \subset \overline{\mathrm{NE}}(X)$. We shall show that if $\Lambda = \overline{\mathrm{NE}}(X)$, then $-K_X$ is big. Assume the contrary. Then, $\mathrm{cl}(-K_X)$ is contained in a face of $\Lambda = \overline{\mathrm{NE}}(X)$, thus $-K_X D = 0$ for a nef divisor $D \not\approx 0$. However, in this situation, $D\Gamma = 0$ for any negative curve Γ by $0 \leq D\Gamma \leq -m_\Gamma K_X D = 0$; hence $D \approx 0$ by $\Lambda = \overline{\mathrm{NE}}(X)$, a contradiction. Therefore, we have only to prove: $\Lambda = \overline{\mathrm{NE}}(X)$.

Step 3. For $z \in \overline{\mathrm{NE}}(X)$, we define a closed convex set

$$\Lambda_{\leq z} := \{y \in \Lambda \mid z - y \in \overline{\mathrm{NE}}(X)\}.$$

We shall show that $\Lambda_{\leq z} \neq \{0\}$ if $z \neq 0$. Assume the contrary. Then, there is an $\mathbb{R}$-divisor D such that $D \not\approx 0$ and $\Lambda_{\leq \mathrm{cl}(D)} = \{0\}$. Clearly, $\mathrm{cl}(D) \notin \Lambda$. By considering the Zariski decomposition of D, we infer that D is nef. Since $\rho(X) > 2$, every K_X-negative extremal ray of $\overline{\mathrm{NE}}(X)$ is generated by the class of a (-1)-curve, by [15], Theorem (2.1). Hence, $K_X D \geq 0$ by the cone theorem ([15], Theorem (1.4)). This implies that $K_X D = 0$

and $D\Gamma = 0$ for any negative curve Γ, since $0 \le D\Gamma \le -m_\Gamma K_X D \le 0$. Since X is a rational surface, we have a birational morphism $\mu\colon X \to Y$ to a non-singular rational surface Y with $\rho(Y) \le 2$. Then, $D = \mu^* D_0$ for a nef $\mathbb{R}$-divisor D_0 on Y and $-K_Y D_0 = \mu_*(-K_X)D_0 = -K_X D = 0$. Here, we have $D_0 \approx 0$ by the Hodge index theorem, since $-K_Y$ is big. This contradicts $D = \mu^*(D_0) \not\approx 0$.

Step 4. There is a linear form $\chi\colon \mathsf{N}(X) \to \mathbb{R}$ such that $\chi > 0$ on $\overline{\mathrm{NE}}(X) \setminus \{0\}$. For $z \in \overline{\mathrm{NE}}(X)$ and $y \in \Lambda_{\le z}$, we have $\chi(y) \le \chi(z)$. Hence, the closed convex set $\Lambda_{\le z}$ is compact for any $z \in \overline{\mathrm{NE}}(X)$. We set $c(z) = \max\{\chi(y) \mid y \in \Lambda_{\le z}\}$. Then, $c(z+y) \ge c(z) + \chi(y)$ for any $y \in \Lambda$ and $z \in \overline{\mathrm{NE}}(X)$. Assume that $z \notin \Lambda$. Then, there is a vector $y_0 \in \Lambda_{\le z}$ such that $\chi(y_0) = c(z)$. Since $z - y_0 \in \overline{\mathrm{NE}}(X) \setminus \Lambda$, by *Step* 3, we have $0 < c(z - y_0) \le c(z) - \chi(y_0) = 0$. This is a contradiction. Therefore, $\overline{\mathrm{NE}}(X) = \Lambda$. Thus, the proof of Theorem 4.1 has been completed. □

Corollary 4.3. *Let X be a non-singular projective rational surface admitting a non-isomorphic surjective separable endomorphism. Then $-K_X$ is big.*

Proof. We may assume that $\rho(X) > 2$: Indeed, if $\rho(X) \le 2$, then $-K_X$ is big. Thus, $\mathrm{Neg}(X) \ne \emptyset$. Note that $\mathrm{Neg}(X)$ is finite by Lemma 3.1. Let $f\colon X \to X$ be the non-isomorphic surjective separable endomorphism. By replacing f with a power f^k, we may assume that f satisfies the following conditions by Lemma 3.1 and Proposition 3.3:

(1) $d = (\deg f)^{1/2}$ is a positive integer.
(2) $f^*(\Gamma) = d\Gamma$ for any $\Gamma \in \mathrm{Neg}(X)$.
(3) $f^*(D) \approx dD$ for any divisor D on X.

Then, $\mathrm{mult}_\Gamma(R_f) \ge d - 1$ for the ramification divisor R_f by Lemma 2.3. In particular, there is an effective divisor Δ such that $K_X + N_X = f^*(K_X + N_X) + \Delta$, where $N_X = \sum_{\Gamma \in \mathrm{Neg}(X)} \Gamma$. Since $f^*(K_X + N_X) \approx d(K_X + N_X)$, $-(K_X + N_X) \approx (d-1)^{-1}\Delta$ is pseudo-effective. Then, $-K_X$ is big by Theorem 4.1. □

Proposition 4.4. *Let X be a non-singular projective rational surface admitting a non-isomorphic surjective separable endomorphism $f\colon X \to X$. If f is tame or $p \nmid \deg f$, then X is toric.*

Proof. We may assume that $\rho(X) > 2$, since any non-singular projective rational surface with Picard number ≤ 2 is always toric. Moreover, as in the proof of Corollary 4.3, we may assume that $f^*(\Gamma) = d\Gamma$ for any $\Gamma \in \mathrm{Neg}(X)$ and for the positive integer $d = (\deg f)^{1/2}$. Hence, f is tame over any $\Gamma \in \mathrm{Neg}(X)$ even if $p \nmid \deg f$. If $p \mid \deg f$, then $p \mid d$ and f is not tame along any $\Gamma \in \mathrm{Neg}(X)$. Thus, $p \nmid \deg f$. Then, by Lemma 3.5, any connected component of N_X is an elliptic curve, a cyclic chain of rational curves, or a straight chain of rational curves. Therefore, X is a toric surface by Proposition 3.7. □

Theorem 1.1, (1) is derived from Lemma 3.1, Corollary 4.3 and Proposition 4.4. The following is an example of non-toric rational surfaces admitting non-isomorphic surjective separable endomorphisms.

Example 4.5. Let $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ be the endomorphism defined by

$$\mathbb{P}^2 \ni (\mathtt{X} : \mathtt{Y} : \mathtt{Z}) \mapsto (\mathtt{X}^p - \mathtt{X}\mathtt{Z}^{p-1} : \mathtt{Y}^p - \mathtt{Y}\mathtt{Z}^{p-1} : \mathtt{Z}^p).$$

Then $f^*(H) = pH$ for the line $H = \{\mathtt{Z} = 0\}$ and the restriction $\mathbb{P}^2 \setminus H \to \mathbb{P}^2 \setminus H$ of f is étale. Let $\mathcal{S}$ be the set of points $P \in \mathbb{P}^2$ such that $f^{-1}(P) = P$. Then, $\mathcal{S} \subset H$. Since $f|_H\colon H \to H$ is just the endomorphism given by $(\mathtt{X} : \mathtt{Y} : 0) \mapsto (\mathtt{X}^p : \mathtt{Y}^p : 0)$, we infer that $\mathcal{S} = \{P_0, P_1, \ldots, P_{p-1}, P_\infty\}$, where $P_i := (1 : i : 0)$ for $0 \le i \le p-1$ and $P_\infty := (0 : 1 : 0)$. Let $\psi_i\colon \mathbb{P}^2 \cdots\to \mathbb{P}^1$ be the projection from P_i for $0 \le i \le p-1$ or $i = \infty$. Here, ψ_i is given explicitly by

$$(\mathtt{X} : \mathtt{Y} : \mathtt{Z}) \mapsto \begin{cases} (-i\mathtt{X} + \mathtt{Y} : \mathtt{Z}), & \text{for } 0 \le i \le p-1; \\ (\mathtt{X} : \mathtt{Z}), & \text{for } i = \infty. \end{cases}$$

Let $h\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the endomorphism defined by

$$\mathbb{P}^1 \ni (\mathtt{u} : \mathtt{v}) \mapsto (\mathtt{u}^p - \mathtt{u}\mathtt{v}^{p-1} : \mathtt{v}).$$

Then, $\psi_i \circ f = h \circ \psi_i$. In fact, this follows directly in case $i = \infty$, and in the other cases, this follows from the calculation

$$\begin{aligned} -i(\mathtt{X}^p - \mathtt{X}\mathtt{Z}^{p-1}) + (\mathtt{Y}^p - \mathtt{Y}\mathtt{Z}^{p-1}) &= -i\mathtt{X}^p + \mathtt{Y}^p - (-i\mathtt{X} + \mathtt{Y})\mathtt{Z}^{p-1} \\ &= (-i\mathtt{X} + \mathtt{Y})^p - (-i\mathtt{X} + \mathtt{Y})\mathtt{Z}^{p-1} \end{aligned}$$

for $0 \le i \le p-1$, where we use $i^p = i$. Therefore, the endomorphism f lifts to an endomorphism $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$ of the blown up surface $\widetilde{X}$ of $\mathbb{P}^2$ along $\mathcal{S}$. The proper transform of H is a curve with self-intersection number $1 - (p+1) = -p < 0$ and intersects all the exceptional curves for $\widetilde{X} \to \mathbb{P}^2$. Since the number of the exceptional curves is $p + 1 \ge 3$, the surface $\widetilde{X}$ is not toric. In fact, for a non-singular projective toric surface, a negative curve is contained in the complement of the open torus, and the complement is a cyclic chain of rational curves; hence every negative curve on the toric surface intersects at most two other negative curves.

5. Irrational ruled surfaces

Let X be an irrational and ruled surface, i.e., $\kappa(X) = -\infty$ and $q(X) > 0$. Then, we have a ruling $\pi\colon X \to T$ to a non-singular projective irrational curve T uniquely up to isomorphism. Here, a general fiber of π is $\mathbb{P}^1$, π is given by the Albanese map, and the genus of T is $q(X)$. We shall study the structure of the surface X when it admits a non-isomorphic surjective separable endomorphism. Let $f\colon X \to X$ be such an endomorphism.

Lemma 5.1. *There is an étale endomorphism $h\colon T \to T$ such that $\pi \circ f = h \circ \pi$. If $q(X) > 1$, then h is an automorphism of finite order.*

Proof. By the universality of the Albanese map, we have an endomorphism $h\colon T \to T$ with $\pi \circ f = h \circ \pi$. Since f is separable, so is h. By the ramification formula $K_T = h^*(K_T) + R_h$, we have

$$\deg K_T = \deg h^*(K_T) + \deg R_h \geq (\deg h) \deg K_T \geq 0.$$

Therefore, $R_h = 0$, and h is étale. If $q(X) > 1$, then $\deg K_T = 2q(X) - 2 > 0$ and $\deg h = 1$; thus h is an automorphism. Moreover, in this case, h is of finite order, since $\mathrm{Aut}(T)$ is finite when $\deg K_T > 0$. □

Lemma 5.2 ([16], Proposition 14, (1))**.** *The ruling $\pi\colon X \to T$ is a $\mathbb{P}^1$-bundle.*

Proof. Assume the contrary. Then, there is a reducible fiber $F = \pi^*(t)$. Let Γ be an irreducible component of F. Then, Γ is a negative curve. By Lemma 3.1, by replacing f with a suitable power f^k, we may assume that $f^*(\Gamma) = d\Gamma$ and $d^2 = \deg f > 1$. Then, $h^{-1}(t) = \{t\}$. This implies that h is an automorphism of T, since h is étale by Lemma 5.1. We have $f^*F = \pi^*h^*(t) = F$. In particular, $f^*\Gamma = \Gamma$. This contradicts $f^*(\Gamma) = d\Gamma$ with $d > 1$. □

Remark. Every $\mathbb{P}^1$-bundle over an elliptic curve seems to admit a non-isomorphic surjective separable endomorphism. In fact, this is true in the case of characteristic zero (cf. [16], Proposition 5). This is also true in the case where the $\mathbb{P}^1$-bundle has a negative section, which is proved by the same argument as in the proof of [16], Proposition 5, (1).

By [12], Theorem 3.1, we can prove the following result on $\mathbb{P}^1$-bundles over curves, which is not related to the existence of endomorphisms. In the case of characteristic zero, this is proved in [16], Theorem 8.

Proposition 5.3. *Let $\pi\colon X \to T$ be a $\mathbb{P}^1$-bundle over a non-singular projective curve T. Then the following three conditions are mutually equivalent:*

(1) *$-K_{X/T}$ is semi-ample.*
(2) *There exist at least three distinct closed curves C on X such that $\pi(C) = T$ and $C^2 = 0$.*
(3) *There is a finite surjective morphism $T' \to T$ from a non-singular projective curve T' such that $X \times_T T'$ is a trivial bundle over T'.*

Proof. The implications (1) ⇒ (2) and (2) ⇒ (3) are proved by the same argument as in the proof of [16], Theorem 8. Hence, it is enough to prove (3) ⇒ (1). For the pullback $X' := X \times_T T' \to T'$ of the $\mathbb{P}^1$-bundle π, let $\nu\colon X' = X \times_T T' \to X$ be the first projection. Then, $-K_{X'/T'} \sim \nu^*(-K_{X/T})$. Let D_1, D_2 be two distinct fibers of the projection $X' \simeq \mathbb{P}^1 \times T' \to \mathbb{P}^1$. Then, $D_1 \sim D_2$ and $-K_{X'/T'} \sim 2D_i$ for $i = 1, 2$. Therefore,

$$2\nu_*(D_1) \sim 2\nu_*(D_2) \sim \nu_*(-K_{X'/T'}) \sim m(-K_{X/T}),$$

where $m := \deg \nu = \deg(T'/T)$. Since $X' \to \mathbb{P}^1$ has infinitely many fibers, we may assume that $\nu(D_1) \neq \nu(D_2)$. Then, $\nu(D_1)\nu(D_2) = 0$ by $(-K_{X/T})^2 = 0$; hence $\nu(D_1) \cap \nu(D_2) = \emptyset$. Therefore, $|-mK_{X/T}|$ is base point free. Thus, we are done. □

Proposition 5.4. *Let X be a $\mathbb{P}^1$-bundle over a non-singular projective curve T of genus at least two and let $f\colon X \to X$ be a non-isomorphic surjective separable endomorphism. Then, X contains no negative curves, and $-K_{X/T}$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. If $p \nmid \deg f$ or if f is tame, then there is a finite surjective morphism $T' \to T$ from another non-singular projective curve T' such that $X \times_T T'$ is a trivial $\mathbb{P}^1$-bundle over T'.*

Proof. Let $\pi\colon X \to T$ be the $\mathbb{P}^1$-bundle. Then, $\pi \circ f = h \circ \pi$ for an automorphism of T of finite order by Lemma 5.1. By replacing f with a power f^k, we may assume that h is the identity map.

Assume that X contains a negative curve Γ. We may assume that $f^*(\Gamma) = d\Gamma$ for the integer $d = (\deg f)^{1/2} > 1$ by Lemma 3.1. For a fiber $F = \pi^*(t)$, we have $f^*(F) = F$, and $F\Gamma = 0$ by

$$d^2F\Gamma = (\deg f)F\Gamma = f^*(F)f^*(\Gamma) = dF\Gamma.$$

Hence, Γ is contained in a fiber of π, but there is no negative curve in any fiber, since π is a $\mathbb{P}^1$-bundle. Therefore, X contains no negative curves.

Consequently, by [12], Theorem 3.1, $-K_{X/T}$ is nef and $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X)$ is spanned by $\mathrm{cl}(F)$ and $\mathrm{cl}(-K_{X/T})$. The pullback homomorphism $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is an automorphism preserving $\mathrm{Nef}(X)$. Since $f^*(F) = F$ for a fiber F, there is a rational number $r > 0$ such that $f^*(-K_{X/T}) \approx r(-K_{X/T})$. Here, we have $r = \deg f$ by

$$2 \deg f = f^*(-K_{X/T})f^*F = r(-K_{X/T})F = 2r.$$

Therefore, $-(\deg f - 1)K_{X/T} \approx R_f$ by the ramification formula $K_X = f^*(K_X) + R_f$. Since R_f is effective, the first assertion has been proved.

In the rest of the proof, we assume either that $p \nmid \deg f$ or that f is tame. Let $\mathcal{S}$ be the set of closed curves C on X such that $C^2 = 0$ and $\pi(C) = T$, equivalently, $\mathrm{cl}(C)$ is contained in the ray $\mathbb{R}_{\geq 0}\,\mathrm{cl}(-K_{X/T})$. Any irreducible component C of R_f belongs to $\mathcal{S}$.

In fact, R_f contains no fiber $F = \pi^*(t)$, since $f^*(F) = F$; hence $\pi(C) = T$. We have $R_f^2 = 0$ by $R_f \approx (\deg f - 1)(-K_{X/T})$. Hence $C^2 = 0$, since $\mathrm{Neg}(X) = \emptyset$. In order to prove the remaining assertion, we may assume that $\mathcal{S}$ consists of at most two curves, by Proposition 5.3. Taking suitable base change, we may assume furthermore that the curves in $\mathcal{S}$ are sections of π. Let $\{C_1\}$ or $\{C_1, C_2\}$ be the set $\mathcal{S}$. Then an irreducible component of R_f is one of C_i. Any irreducible component of $f^*(C_i)$ belongs to $\mathcal{S}$, since $(f^*(C_i))^2 = 0$. Therefore, by replacing f with $f \circ f$ if necessary, we may assume that $f^{-1}(C_i) = C_i$ for any i. Here, we have $f^*(C_i) = (\deg f)C_i$, by $(\deg f)C_iF = f^*(C_i)f^*(F) = f^*(C_i)F$.

Assume that R_f is irreducible. Let C_1 be the irreducible component. Then, $X \setminus C_1 = f^{-1}(X \setminus C_1)$ is étale over $X \setminus C_1$ by f. In particular, for a fiber $F = \pi^{-1}(t) \simeq \mathbb{P}^1$, the restriction $f|_F \colon F \to F$ is étale outside the point $F \cap C_1$. By Lemma 2.5, $\deg f|_F = \deg f$ is divisible by p. But in this case, f is not tame, since $f^*(C_1) = (\deg f)C_1$. This contradicts our assumption.

Therefore, R_f is reducible and it has just two irreducible components C_1, C_2, where $C_1 \cap C_2 = \emptyset$ by $C_1C_2 = 0$. Then, there is a divisor L on T such that $C_2 \sim C_1 + \pi^*(L)$, and $X \simeq \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{O}_T(L))$. Since $f^*(C_i) = (\deg f)C_i$ for $i = 1$, 2, we have

$$(\deg f)\pi^*(L) \sim f^*\pi^*(L) = \pi^*(L).$$

Hence, $(\deg f - 1)L \sim 0$, i.e., $\mathcal{O}_T(L)$ is a torsion in $\mathrm{Pic}(T)$. We have a finite surjective morphism $\tau \colon T' \to T$ such that $\tau^*(L) \sim 0$. In fact, $\mathcal{S}pec$ over T of a suitable $\mathcal{O}_T$-algebra $\bigoplus_{i=0}^{b-1} \mathcal{O}_T(-iL)$ for the order b of $\mathcal{O}_T(L)$ produces such $T' \to T$. Thus, $X \times_T T' \simeq \mathbb{P}^1 \times T'$. □

Remark. In Proposition 5.4, the finite surjective morphism $T' \to T$ may not be separable. In the case of characteristic zero, we can find such a morphism $T' \to T$ as a finite étale covering (cf. [16], Theorem 15).

The following gives examples of $\pi \colon X \to T$ and f in Proposition 5.4 with $p = \deg f$.

Proposition 5.5. *Let T be a non-singular projective curve and let η be a non-zero element of $\mathrm{H}^1(T, \mathcal{O}_T)$ such that $\eta^p \in \Bbbk\eta$, where $\eta \mapsto \eta^p$ denotes the p-linear map $\mathrm{H}^1(T, \mathcal{O}_T) \to \mathrm{H}^1(T, \mathcal{O}_T)$ induced from the absolute Frobenius morphism of T. Let $\mathcal{E}$ be the locally free sheaf on T of rank two obtained as the extension of $\mathcal{O}_T$ by $\mathcal{O}_T$ corresponding to $\eta \in \mathrm{Ext}^1_T(\mathcal{O}_T, \mathcal{O}_T) \simeq \mathrm{H}^1(T, \mathcal{O}_T)$. Let $\pi \colon X = \mathbb{P}_T(\mathcal{E}) \to T$ be the $\mathbb{P}^1$-bundle associated with $\mathcal{E}$ and let $C \subset X$ be the section corresponding to the injection $\mathcal{O}_C \to \mathcal{E}$. Then, there is a non-isomorphic surjective separable endomorphism $f \colon X \to X$ of degree p over T such that $f^{-1}(C) = C$ and $f|_{X\setminus C} \colon X \setminus C \to X \setminus C$ is étale.*

Proof. By a scalar multiplication, we may assume that $\eta^p + (c-1)\eta = 0$ for some constant $c \in \Bbbk \setminus \{0\}$. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open affine covering of T. Then, η is represented by a Čech 1-cocycle $\{\eta_{i,j}\}$ of $\mathcal{O}_T$ with respect to $\mathcal{U}$, i.e., $\eta_{i,j} \in \mathrm{H}^0(U_i \cap U_j, \mathcal{O}_T)$ satisfy

$$\eta_{i,i} = 0, \quad \text{and} \quad \eta_{i,j} + \eta_{j,k} + \eta_{k,i} = 0 \text{ on } U_i \cap U_j \cap U_k$$

for i, j, $k \in I$. Let $\mathtt{u} \in \mathrm{H}^0(T, \mathcal{E})$ be the image of 1 under the injection $\mathcal{O}_T \to \mathcal{E}$. Then, we have sections $\mathtt{v}_i \in \mathrm{H}^0(U_i, \mathcal{E})$ for $i \in I$ such that

$$\mathcal{E}|_{U_i} = \mathcal{O}_{U_i}\mathtt{u} \oplus \mathcal{O}_{U_i}\mathtt{v}_i \quad \text{and} \quad \mathtt{v}_j = \mathtt{v}_i + \eta_{i,j}\mathtt{u} \text{ on } U_i \cap U_j$$

for any i, $j \in I$. Note that $\{\eta_{i,j}^p\}$ is also a Čech 1-cocycle of $\mathcal{O}_T$, and its cohomology class is just η^p. Since $\eta^p + (c-1)\eta = 0$, we have functions $a_i \in \mathrm{H}^0(U_i, \mathcal{O}_T)$ such that

$$\eta_{i,j}^p + (c-1)\eta_{i,j} = a_i|_{U_i \cap U_j} - a_j|_{U_i \cap U_j}$$

for any i, $j \in I$. We define a homomorphism $\Phi_i \colon \mathcal{E}|_{U_i} \to \mathrm{Sym}^p(\mathcal{E})|_{U_i}$ by

$$\Phi_i(\mathtt{u}) = \mathtt{u}^p \quad \text{and} \quad \Phi_i(\mathtt{v}_i) = \mathtt{v}_i^p + c\mathtt{u}^{p-1}\mathtt{v}_i + a_i\mathtt{u}^p,$$

where $\mathtt{u}^l\mathtt{v}_i^{p-l}$ for $0 \le l \le p$ are regarded as sections of $\mathrm{Sym}^p(\mathcal{E})$ over U_i and they form a free basis of $\mathrm{Sym}^p(\mathcal{E})|_{U_i}$ as an $\mathcal{O}_{U_i}$-module. Since $\mathtt{v}_j = \mathtt{v}_i + \eta_{i,j}\mathtt{u}$, we have

$$\begin{aligned}\Phi_i(\mathtt{v}_j) - \Phi_j(\mathtt{v}_j) &= \Phi_i(\mathtt{v}_i) + \eta_{i,j}\Phi_i(\mathtt{u}) - \Phi_j(\mathtt{v}_j) \\ &= (\mathtt{v}_i - \mathtt{v}_j)^p + c\mathtt{u}^{p-1}(\mathtt{v}_i - \mathtt{v}_j) + (\eta_{i,j} + a_i - a_j)\mathtt{u}^p \\ &= (-\eta_{i,j}^p - (c-1)\eta_{i,j} + a_i - a_j)\mathtt{u}^p = 0\end{aligned}$$

on $U_i \cap U_j$. Hence, $\{\Phi_i\}$ can be glued to a global homomorphism $\Phi \colon \mathcal{E} \to \mathrm{Sym}^p(\mathcal{E})$ on T. The natural surjection $\pi^*\mathcal{E} \simeq \pi^*\pi_*\mathcal{O}_X(C) \to \mathcal{O}_X(C)$ induces a surjection $\pi^*(\mathrm{Sym}^p(\mathcal{E})) \to \mathcal{O}_X(pC)$. The composite

$$\pi^*\mathcal{E} \xrightarrow{\Phi^*(\Phi)} \pi^*(\mathrm{Sym}^p(\mathcal{E})) \to \mathcal{O}_X(pC)$$

is surjective by the construction of Φ. Hence, Φ induces a surjective endomorphism $f \colon X \to \mathbb{P}_X(\mathcal{E}) = X$ of degree p over T such that $f^*\mathcal{O}_X(C) \simeq \mathcal{O}_X(pC)$. Moreover, $f^*(C) = pC$, since C is defined by $\mathtt{u}$ and pC is defined by $\Phi(\mathtt{u}) = \mathtt{u}^p$. For the fiber $F = \pi^{-1}(t)$ over a point $t \in U_i$, the induced endomorphism $f|_F$ of $F \simeq \mathbb{P}^1$ is isomorphic to

$$(\mathtt{x} : \mathtt{y}) \mapsto (\mathtt{x}^p : \mathtt{y}^p + c\mathtt{x}^{p-1}\mathtt{y} + a_i(t)\mathtt{x}^p),$$

which is a kind of Artin–Schreier morphism. Hence, $f|_{X \setminus C} \colon X \setminus C \to X \setminus C$ is étale, since $c \neq 0$. □

Remark 5.6. There is a non-singular projective curve T of genus at least two such that $\eta^p \in \Bbbk\eta$ for some non-zero element $\eta \in \mathrm{H}^1(T, \mathcal{O}_T)$. Let $\pi \colon X \to T$ be the $\mathbb{P}^1$-bundle constructed as in Proposition 5.5. Then, $X \times_T T' \not\simeq \mathbb{P}^1 \times T'$ for any finite surjective morphism $T' \to T$ with $p \nmid \deg(T'/T)$, since $\mathrm{H}^1(T, \mathcal{O}_T) \to \mathrm{H}^1(T', \mathcal{O}_{T'})$ is injective.

However, there is a finite surjective morphism $T' \to T$ such that $X \times_T T' \simeq \mathbb{P}^1 \times T'$. In fact, by considering the Albanese map $\alpha\colon T \to A := \mathrm{Alb}(T)$ and the multiplication map $\varpi\colon A \to A$ by p, we have a finite surjective morphism $\tau\colon T' \to T$ and a morphism $\beta\colon T' \to A$ such that $\alpha \circ \tau = \varpi \circ \beta$. Since $\alpha^*\colon \mathrm{H}^1(A, \mathcal{O}_A) \to \mathrm{H}^1(T, \mathcal{O}_T)$ is isomorphic and since $\varpi^*\colon \mathrm{H}^1(A, \mathcal{O}_A) \to \mathrm{H}^1(A, \mathcal{O}_A)$ is zero, $\tau^*(\eta) = 0$ in $\mathrm{H}^1(T', \mathcal{O}_{T'})$. Therefore, $X \times_T T' \simeq \mathbb{P}^1 \times T'$.

We close this section by proving Theorem 1.1.

Proof of Theorem 1.1. The first assertion (1) of Theorem 1.1 is derived from Lemma 3.1, Corollary 4.3, and Proposition 4.4. The remaining assertions (2) and (3) are derived from Lemma 5.2, Proposition 5.3, and Proposition 5.4. □

6. The case of non-negative Kodaira dimension

We shall prove Theorem 1.2 in this section. We begin with the following existence theorem of non-isomorphic surjective separable endomorphisms for certain elliptic surfaces.

Theorem 6.1. *Let $\pi\colon X \to T$ be an fibration from a non-singular projective surface X to a non-singular projective curve T. Assume that the support of any fiber is an elliptic curve. Then, X admits a non-isomorphic surjective separable endomorphism $f\colon X \to X$ such that $\pi \circ f = \pi$.*

Proof. Step 1. There is a non-singular ample divisor C on X such that $C \subset X \to T$ is a separable finite surjective morphism. In fact, for an ample divisor H on X and for a smooth fiber F, $\mathrm{H}^0(X, \mathcal{O}_X(kH)) \to \mathrm{H}^0(F, \mathcal{O}_X(kH)|_F)$ is surjective for $k \gg 0$, hence, by Bertini's theorem, there is a non-singular ample divisor $C \in |kH|$ such that $C|_F$ is also non-singular. As a consequence, $C \to T$ is étale along $C \cap F$, and $C \to T$ is separable.

Step 2. Let $T' \to T$ be the Galois closure of $C \to T$, i.e., T' is the normalization of C in the Galois closure of $\Bbbk(C)/\Bbbk(T)$. We consider the base change of π by the Galois covering $T' \to T$. Let X' be the normalization of $X \times_T T'$ and let $\pi'\colon X' \to T'$ be the induced elliptic fibration. Then, any irreducible component of a fiber of π' is an irrational curve, since it dominates a fiber of π which is assumed to be an elliptic curve. Therefore, by Lemma 2.12, X' is non-singular and the support of any fiber of π' is also an elliptic curve. Now the natural morphism $T' \to C \times_T T'$ induces a section $e\colon T' \to X'$ of π'. Hence, any fiber of π' is reduced. As a consequence, all the fibers are non-singular and π' is a smooth morphism. Moreover, π' together with the section e is an abelian scheme by [13], Theorem 6.14.

Step 3. We regard the Galois group G of T'/T as an automorphism group of T'. For $\sigma \in G$, let $L_\sigma\colon X' \to X'$ be the automorphism induced from $\mathrm{id}_X \times \sigma\colon X \times_T T' \to X \times_T T'$.

Here, $\pi' \circ L_\sigma = \sigma \circ \pi'$. Thus, we have an action of G on X' such that $\pi' \colon X' \to T'$ is G-equivariant. In order to construct an endomorphism of X, we use the argument in the proof of [4], Theorem 2.26. The set S of sections of $\pi' \colon X' \to T'$ is an abelian group by the abelian group scheme structure, where e is the zero section. A section $b \in \mathsf{S}$ defines the translation morphism $\mathrm{tr}(b) \colon X' \to X'$ over T'. Then, L_σ is expressed uniquely as $\mathrm{tr}(b_\sigma) \circ \alpha_\sigma$ for a section $b_\sigma \in \mathsf{S}$ and for an automorphism $\alpha_\sigma \colon X' \to X'$ such that $\pi' \circ \alpha_\sigma = \sigma \circ \pi'$ and $\alpha_\sigma \circ e = e \circ \sigma$. Here, α_σ is regarded as a homomorphism between the abelian schemes $\sigma \circ \pi' \colon X' \to T'$ and $\pi' \colon X' \to T'$ over T' (cf. [13], Corollary 6.4). We define

$$\sigma \cdot b := \alpha_\sigma \circ b \circ \sigma^{-1}$$

for $\sigma \in G$ and $b \in \mathsf{S}$. Then, $\alpha_\sigma \circ \mathrm{tr}(b) \circ \alpha_\sigma^{-1} = \mathrm{tr}(\sigma \cdot b)$. Moreover, we have

$$\alpha_{\sigma_1\sigma_2} = \alpha_{\sigma_1} \circ \alpha_{\sigma_2} \quad \text{and} \quad b_{\sigma_1\sigma_2} = b_{\sigma_1} + \sigma_1 \cdot b_{\sigma_2}$$

for $\sigma_1, \sigma_2 \in G$, by $L_{\sigma_1\sigma_2} = L_{\sigma_1} \circ L_{\sigma_2}$. Thus, S has a left G-module structure by $(\sigma, b) \mapsto \sigma \cdot b$, and $\{b_\sigma\}$ is a 1-cocycle defining an element β of $\mathrm{H}^1(G, \mathsf{S})$. Then, $n_G\beta = 0$ for the order n_G of G, where $n_G = \deg T'/T$. Let n be the least common multiple of n_G and p. Then, we have a section $c \in \mathsf{S}$ such that $nb_\sigma = \sigma \cdot c - c$ for any $\sigma \in G$.

Let $\mu_{n+1} \colon X' \to X'$ be the multiplication map by $n+1$ with respect to the abelian scheme structure of $\pi' \colon X' \to T'$. We define $f' := \mathrm{tr}(c) \circ \mu_{n+1}$. Then, f' is a non-isomorphic étale endomorphism of X', since $p \nmid \deg f' = (n+1)^2 > 1$. For $\sigma \in G$, we have $L_\sigma \circ f' = f' \circ L_\sigma$ by

$$\begin{aligned} &\mathrm{tr}(b_\sigma) \circ \alpha_\sigma \circ \mathrm{tr}(c) \circ \mu_{n+1} = \mathrm{tr}(b_\sigma + \sigma \cdot c) \circ \alpha_\sigma \circ \mu_{n+1} \\ &= \mathrm{tr}(c + (n+1)b_\sigma) \circ \mu_{n+1} \circ \alpha_\sigma = \mathrm{tr}(c) \circ \mu_{n+1} \circ \mathrm{tr}(b) \circ \alpha_\sigma. \end{aligned}$$

Therefore, f' descends to a surjective separable endomorphism $f \colon X \to X$ such that $\pi \circ f = \pi$ and $\deg f = (n+1)^2 > 1$. Thus, we are done. □

Lemma 6.2 (cf. [2], Lemma 2.3)**.** *Let $f \colon X \to X$ be a surjective separable endomorphism of a non-singular projective surface X of $\kappa(X) \geq 0$. Then, f is étale. If $\deg f > 1$, then K_X is nef (i.e., X is minimal), X has no negative curves, $\kappa(X) \leq 1$, and $\chi(X, \mathcal{O}_X) = e(X) = 0$.*

Proof. By the ramification formula $K_X = f^*(K_X) + R_f$, if $R_f \neq 0$, then we have

$$K_X A = (f^k)^*(K_X)A + \left((f^{k-1})^*(R_f) + \cdots + R_f\right) A \geq k$$

for any ample divisor A and any positive integer k; this is a contradiction. Hence, $R_f = 0$, and f is étale. Assume that $\deg f > 1$. Then, X has no negative curve by Lemma 3.1.

Hence, K_X is nef, since $\kappa(X) \geq 0$. Since f is étale, we have

$$\chi(X, \mathcal{O}_X) = (\deg f)\chi(X, \mathcal{O}_X), \quad e(X) = (\deg f)e(X), \quad \text{and}$$
$$K_X^2 = f^*(K_X)f^*(K_X) = (\deg f)K_X^2.$$

Hence, $\chi(X, \mathcal{O}_X) = e(X) = K_X^2 = 0$. In particular, $\kappa(X) \leq 1$. □

The following is well-known as a part of the classification theory of non-singular projective surfaces by Bombieri and Mumford [1]:

Fact 6.3. The non-singular projective minimal surfaces X satisfying $\kappa(X) = \chi(X, \mathcal{O}_X) = 0$ are classified as follows: The irregularity $q(X) = 1$ or 2 for such a surface X. Here, $q(X) = 2$ if and only if X is abelian. Suppose that $q(X) = 1$. Then, $12K_X \sim 0$, $b_1(X) = b_2(X) = \rho(X) = 2$, and the Albanese map $\alpha\colon X \to \mathrm{Alb}(X)$ is a fibration to an elliptic curve. If α is an elliptic fibration, then X is called a *hyperelliptic* surface. If not, α is a quasi-elliptic fibration (cf. Definition 2.10), and X is called a *quasi-hyperelliptic* surface. The case of quasi-hyperelliptic surfaces occurs only when $p \leq 3$ (cf. Proposition 2.9). The following assertion is known by [1], Theorem 3 and its proof: *If X is a hyperelliptic surface or a quasi-hyperelliptic surface, then there is an elliptic fibration $\pi\colon X \to T \simeq \mathbb{P}^1$ such that the support of any fiber of π is an elliptic curve.*

Lemma 6.4. *Let $f\colon X \to X$ be a non-isomorphic surjective separable endomorphism of a non-singular projective surface X of $\kappa(X) \geq 0$. Suppose that there exist a fibration $\pi\colon X \to T$ to a non-singular projective curve T and an automorphism $h\colon T \to T$ satisfying $\pi \circ f = h \circ \pi$. Then, the support of every fiber of π is an elliptic curve.*

Proof. By Lemma 6.2, f is étale. Let F_t be the fiber $\pi^*(t)$ over a point $t \in T$. Then, $F_t = f^*(F_{h(t)})$. Hence, $K_X F_t = 0$ by

$$K_X F_t = f^*(K_X)f^*(F_{h(t)}) = (\deg f)K_X F_{h(t)} = (\deg f)K_X F_t.$$

Note that every fiber of π is irreducible, since X has no negative curve by Lemma 6.2. Thus, $F_t = m_t\Gamma_t$ for a prime divisor Γ_t and for some $m_t > 0$, in which $p_a(\Gamma_t) = 1$ by $K_X\Gamma_t = 0$. Restricting f to F_t, we have an étale morphism $F_t \to F_{h(t)}$ of degree $\deg f > 1$. Hence, $m_t = m_{h(t)}$, and the induced morphism $\Gamma_t \to \Gamma_{h(t)}$ is étale of the same degree. If $\Gamma_{h(t)}$ is rational, then the normalization of $\Gamma_{h(t)}$ produces a non-trivial étale covering over $\mathbb{P}^1$; this is impossible. Therefore, Γ_t is an elliptic curve for any $t \in T$. □

Finally, we shall prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that X admits a non-isomorphic surjective separable endomorphism $f\colon X \to X$. Then, X is a minimal surface, f is étale, $\chi(X, \mathcal{O}_X) = e(X) = 0$, and $\kappa(X) = 0$ or 1 by Lemma 6.2. Hence, if $\kappa(X) = 0$, then the condition (1) is satisfied.

Assume that $\kappa(X) = 1$. Then, by [14], we have the so-called "Iitaka fibration" $\pi\colon X \to T$ to a non-singular projective curve T such that $bK_X \sim \pi^*(H)$ for some $b > 0$ and a very ample divisor H on T. In order to check the condition (2), it is enough to prove that π is an elliptic fibration. By the uniqueness of the Iitaka fibration, considering the Stein factorization of $\pi \circ f$, we have an endomorphism $h\colon T \to T$ such that $\pi \circ f = h \circ \pi$. Here, we have $H \sim h^*(H)$ by

$$\pi^*(H) \sim bK_X \sim f^*(bK_X) \sim f^*\pi^*(H) = \pi^*h^*(H).$$

Thus, h is an automorphism, since we have $\deg h = 1$ from $\deg H = (\deg h)(\deg H) > 0$. Applying Lemma 6.4 to $\pi\colon X \to T$ and h, we infer that π is an elliptic fibration.

The rest of the proof of Theorem 1.2 is to construct a non-isomorphic surjective separable endomorphism of any surface X satisfying one of the conditions (1) and (2). We may assume that X is not abelian, since, for any abelian variety, the multiplication map by a positive integer not divisible by p is a non-isomorphic surjective étale endomorphism. Then, there is an elliptic fibration $\pi\colon X \to T$ such that the support of any fiber is an elliptic curve. In fact, if X satisfies (1), then X is a hyperelliptic surface or a quasi-hyperelliptic surface, and the existence of such π is known as in Fact 6.3. If X satisfies (2), then $K_X^2 = e(X) = 0$ by the minimality of X and Noether's formula. Hence, the support of any fiber of the elliptic surface X is an elliptic curve by Lemma 2.13. Therefore, X admits a non-isomorphic surjective separable endomorphism by Theorem 6.1. Thus, we are done. □

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