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On the WKB theoretic structure of a Schrödinger operator with a merging pair of a simple pole and a simple turning point

Dedicated to Professor A. Voros on the occasion of his sixtieth birthday

By

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On the WKB theoretic structure of a Schrödinger operator with a merging pair of a simple pole and a simple turning point

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0 Introduction

The principal aim of this paper is to form a basis for the exact WKB analysis of a Schrödinger equation

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x,\eta)\right)\psi = 0 \quad (\eta : \text{a large parameter}) \tag{0.1}$$

with one simple turning point and with one simple pole in the potential Q. As [Ko1] and [Ko2] emphasize, the Borel transform of a WKB solution of (0.1) displays, near the simple pole singularity, behavior similar to that near a simple turning point. Hence it is natural to expect that such an equation plays an important role in the exact WKB analysis in the large. Such an expectation has recently been enhanced by the discovery ([Ko4]) that the Voros coefficient of a WKB solution of (0.1) with

$$Q = \frac{1}{4} + \frac{\alpha}{x} + \eta^{-2}\frac{\gamma}{x^2} \quad (\alpha, \gamma : \text{fixed complex numbers}) \tag{0.2}$$

can be explicitly written down with the help of the Bernoulli numbers. The potential Q given by (0.2) will play an important role in Section 2; the Schrödinger equation with the potential Q of the form (0.2), that is, the Whittaker equation with a large parameter η, gives us a WKB theoretic canonical form of a Schrödinger equation with one simple turning point and with one simple pole in its potential. We note that the parameter α contained in the Whittaker equation in Section 2 is an infinite series $\alpha(\eta) = \sum_{k\geq 0} \alpha_k \eta^{-k}$ (α_k: a constant), and we call such an equation the ∞-Whittaker equation when we want to emphasize that α is not a genuine constant but an infinite series as above.

In order to make a semi-global study of a Schrödinger equation with one simple turning point and with a simple pole in its potential, we let the simple pole singular point merge with the turning point and observe what kind of equation appears. For example, what if we let α tend to 0 in (0.2) with γ being kept intact? Interestingly enough, the resulting equation is what we call a ghost equation ([Ko3]); we have been worrying where we should place the class of ghost equations in regard to the whole WKB analysis. A ghost equation has no turning point by its definition (cf. Remark 1.1 in Section 1); still a WKB solution of a ghost equation displays singularity similar to that which a WKB solution normally has near a turning point. The singularity is due to the singularities contained in the coefficients of η^{-k} ($k \geq 1$) in the potential Q. (See [Ko3] for details; there a ghost (point) is tentatively called a"new" turning point.) In view of the above observation, we regard a Schrödinger equation with one simple turning point and with one simple pole in its potential as an equation obtained through perturbation of a ghost equation by a simple pole term $aq(x,a)/x$, where a is a complex parameter and $q(x,a)$ is a holomorphic function defined on a neighborhood of $(x,a) = (0,0)$. An equation obtained by such a procedure is called an equation with a merging pair of a simple pole and a simple turning point, or, for short, an MPPT equation. Precisely speaking, we call a Schrödinger equation (0.1) an MPPT equation if its potential Q depends also on an auxiliary parameter a and has the following form

$$Q = \frac{Q_0(x,a)}{x} + \eta^{-1}\frac{Q_1(x,a)}{x} + \eta^{-2}\frac{Q_2(x,a)}{x^2}, \tag{0.3}$$

where $Q_j(x,a)$ ($j = 0, 1, 2$) are holomorphic near $(x,a) = (0,0)$ and $Q_0(x,a)$ satisfies the following conditions (0.4) and (0.5):

(0.4) $\quad Q_0(0,a) \neq 0$ if $a \neq 0$,

(0.5) $\quad Q_0(x,0) = c_0^{(0)}x + O(x^2)$ holds with $c_0^{(0)}$ being a constant different from 0.

Clearly we find a ghost equation at $a = 0$; furthermore the implicit function theorem together with the assumption (0.5) guarantees the existence of a unique holomorphic function $x(a)$ that satisfies

$$Q_0(x(a), a) = 0. \tag{0.6}$$

The assumption (0.4) entails

(0.7) $x(a) \neq 0$ if $a \neq 0$,

and the assumption (0.5) guarantees that, for a sufficiently small $a(\neq 0)$, $x = x(a)$ is a simple turning point of the operator in question.

As the above naming "an MPPT equation" indicates, it is a counterpart of an MTP equation in our context. An MTP equation, i.e., a merging-turning-points equation introduced in [AKT4] contains, by definition, two simple turning points that merge into one double turning point as the parameter t tends to 0, whereas, in an MPPT equation, a simple pole and a simple turning point merge into a ghost point where neither zero nor singularity is observed in the highest degree (i.e., degree 0) in η part of the potential. The parallelism of these two notions is not a superficial one. The reduction of an MPPT equation to a canonical one is achieved in Sections 1 and 2 below in a way parallel to that used in the reduction of MTP equation to a canonical one; first, in Section 1 we construct a WKB theoretic transformation that brings an MPPT equation with the parameter a being 0 to a particular ∞-Whittaker equation, that is, the ∞-Whittaker equation with the top degree part of the parameter $\alpha(\eta)$ being 0 (i.e., $\alpha(\eta) = \sum_{k\geq 1} \alpha_k \eta^{-k}$), and then in Section 2 we construct the transformation of a generic (i.e., $a \neq 0$) MPPT equation to the ∞-Whittaker equation in the form of a perturbation series in a, starting with the transformation constructed in Section 1. In Sections 1 and 2 we focus our attention on the formal aspect of the problem, and the estimation of the growth order of the coefficients that appear in several formal series is given separately in Appendices A and

B. One important implication of the estimates given in Appendix B is that they endow the formal transformation with an analytic meaning as a microdifferential operator through the Borel transformation. Furthermore, as is shown in Theorem 1.7 and Theorem 2.7, the action of the resulting microdifferential operator upon multi-valued analytic functions such as Borel transformed WKB solutions, is described in terms of an integro-differential operator of particular type; its kernel function contains a differential operator of infinite order in x-variable. Thus it is of local character in x-variable, whereas it is suited for the global study related to the resurgence phenomena in y-variable. (See e.g. [SKK] and [K] for the notion of a differential operator of infinite order. See also [AKT4] that has first used a differential operator of infinite order in exact WKB analysis.) As the domain of definition of the integro-differential operator may be chosen to be uniform with respect to the parameter a (Remark 2.3), our results in Section 2 are of semi-global character, as is noted in Remark 4.1. This uniformity is one of the most important advantages in introducing the notion of an MPPT operator. It is worth emphasizing that the uniformity becomes clearly visible through the Borel transformation. In order to use the results obtained in Section 2 for the detailed study of the structure of Borel transformed WKB solutions of an MPPT equation, we first study in Section 3 analytic properties of Borel transformed WKB solutions of the Whittaker equation, and then in Section 4 we analyze Borel transformed WKB solutions of the ∞-Whittaker equation using the results obtained in Section 3. The basis of the study in Section 3 is a recent result of Koike ([Ko4]), and the analysis in Section 4 makes essential use of the estimate (B.3) of the coefficients $\{\alpha_k(a)\}_{k\geq 0}$ of the parameter $\alpha(a,\eta) = \sum_{k\geq 0}\alpha_k(a)\eta^{-k}$; the effect of this infinite series that appears in the ∞-Whittaker equation is grasped as a microdifferential operator acting on Borel transformed WKB solutions of the Whittaker equation. Combining all the results obtained in Sections 2 and 4 we summarize in Section 5 basic properties of Borel transformed WKB solutions of an MPPT equation with $a \neq 0$.

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1 Construction of the transformation to the canonical form, I. — the case where $a = 0$

The purpose of this section is to show how to construct the Borel transformable series

(1.1) $$x^{(0)}(\tilde{x}, \eta) = \sum_{k \geq 0} x_k^{(0)}(\tilde{x}) \eta^{-k}$$

and

(1.2) $$\alpha^{(0)}(\eta) = \sum_{k \geq 0} \alpha_k^{(0)} \eta^{-k}$$

with $\alpha_0^{(0)}$ being 0, i.e.,

(1.2′) $$\alpha^{(0)}(\eta) = \sum_{k \geq 1} \alpha_k^{(0)} \eta^{-k}$$

so that the Schrödinger equation

(1.3)
$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 \Big(\frac{\tilde{Q}_0(\tilde{x}, 0)}{\tilde{x}} + \eta^{-1} \frac{\tilde{Q}_1(\tilde{x}, 0)}{\tilde{x}} + \eta^{-2} \frac{\tilde{Q}_2(\tilde{x}, 0)}{\tilde{x}^2} \Big) \right) \tilde{\psi}(\tilde{x}, \eta) = 0$$

with $\tilde{Q}_j(\tilde{x}, 0)$ $(j = 0, 1, 2)$ being holomorphic functions near the origin that satisfy (1.5) below may be brought to a particular ∞-Whittaker equation

$$(1.4)\qquad \left(\frac{d^2}{dx^2}-\eta^2\Big(\frac{1}{4}+\frac{\alpha^{(0)}(\eta)}{x}+\eta^{-2}\,\frac{\tilde{Q}_2(0,0)}{x^2}\Big)\right)\psi(x,\eta)=0.$$

Here the adjective "particular" refers to the vanishing of $\alpha_0^{(0)}$. The Borel transformability of $x^{(0)}$ and $\alpha^{(0)}$, i.e., the growth order conditions on their coefficients will be separately discussed in Appendix B. Thus the first task is to establish Theorem 1.1 below, which relates the potentials in (1.3) and (1.4); the relation (1.6) enables us to relate (1.3) and (1.4) in an appropriate way, as we will expound after proving Theorem 1.1.

Theorem 1.1. *Let $\tilde{Q}_j(\tilde{x},a)$ $(j=0,1,2)$ be holomorphic functions defined on a neighborhood of $(\tilde{x},a)=(0,0)$, and suppose that the following condition is satisfied:*

(1.5) $\tilde{Q}_0(\tilde{x},0)=c_0^{(0)}\tilde{x}+O(\tilde{x}^2)$ *with $c_0^{(0)}$ being a constant different from* 0.

Then there exist Borel transformable series $x^{(0)}(\tilde{x},\eta)$ and $\alpha^{(0)}(\eta)$ respectively given in (1.1) *and* (1.2$'$) *so that the following relations* (1.6) $\sim$ (1.9) *hold on an open neighborhood U of the origin $\tilde{x}=0$:*

$$(1.6)\qquad \tilde{x}^{-1}\tilde{Q}_0(\tilde{x},0)+\eta^{-1}\tilde{x}^{-1}\tilde{Q}_1(\tilde{x},0)+\eta^{-2}\tilde{x}^{-2}\tilde{Q}_2(\tilde{x},0)$$
$$=\Big(\frac{dx^{(0)}(\tilde{x},\eta)}{d\tilde{x}}\Big)^2\left(\frac{1}{4}+\frac{\alpha^{(0)}(\eta)}{x^{(0)}(\tilde{x},\eta)}+\eta^{-2}\frac{\tilde{Q}_2(0,0)}{x^{(0)}(\tilde{x},\eta)^2}\right)$$
$$-\frac{1}{2}\eta^{-2}\{x^{(0)}(\tilde{x},\eta);\tilde{x}\},$$

(1.7) $x_k^{(0)}(\tilde{x})$ $(k=0,1,2,\cdots)$ *is holomorphic on U,* (1.8) $$x_k^{(0)}(0) = 0 \quad (k = 0, 1, 2, \cdots),$$

(1.9) $$\bigl(dx_0^{(0)}/d\tilde{x}\bigr)(0) \neq 0.$$

Here $\{x^{(0)}(\tilde{x}, \eta); \tilde{x}\}$ *stands for the Schwarzian derivative, i.e.,*

(1.10) $$\frac{d^3x^{(0)}/d\tilde{x}^3}{dx^{(0)}/d\tilde{x}} - \frac{3}{2}\left(\frac{d^2x^{(0)}/d\tilde{x}^2}{dx^{(0)}/d\tilde{x}}\right)^2.$$

Remark 1.1. The assumption (1.5) entails that $\tilde{x}^{-1}\tilde{Q}_0(\tilde{x}, 0)$ is holomorphic near $\tilde{x} = 0$ and that it does not vanish there. Thus MPPT operator restricted to $\{a = 0\}$ is exactly of the form of a ghost operator ([Ko3]). Hence the content of Theorem 1.1 is essentially the same as [Ko3, Proposition 2.1].

Proof. We construct $x_k^{(0)}$ inductively, and to facilitate the required computation we introduce a series $z^{(0)}(\tilde{x}, \eta)$ given by

(1.11) $$\tilde{x}^{-1}x^{(0)}(\tilde{x}, \eta).$$

By setting

(1.12) $$\gamma = \tilde{Q}_2(0, 0),$$

we define $\tilde{R}_2 = \tilde{R}_2(\tilde{x})$ by

(1.13) $$\tilde{x}^{-1}(\tilde{Q}_2(\tilde{x}, 0) - \gamma).$$

Then we find

(1.14) $$\begin{aligned} &\tilde{x}^{-2}\tilde{Q}_2(\tilde{x}, 0) - \gamma(dx^{(0)}/d\tilde{x})^2(x^{(0)})^{-2} \\ &\quad = \tilde{x}^{-1}\Bigl[\tilde{R}_2 - 2\gamma(dz^{(0)}/d\tilde{x})(z^{(0)})^{-1} - \gamma\tilde{x}(dz^{(0)}/d\tilde{x})^2(z^{(0)})^{-2}\Bigr]. \end{aligned}$$

Hence our task is to construct series $x^{(0)}(\tilde{x}, \eta)$ and $\alpha^{(0)}(\eta)$ so that they satisfy

(1.15)

$$\tilde{Q}_0(\tilde{x},0)+\eta^{-1}\tilde{Q}_1(\tilde{x},0)$$

$$=\left(\frac{dx^{(0)}}{d\tilde{x}}\right)^2\left(\frac{\tilde{x}}{4}+\frac{\alpha^{(0)}}{z^{(0)}}\right)+\eta^{-2}\Big[-\tilde{R}_2(\tilde{x})+2\gamma(dz^{(0)}/d\tilde{x})(z^{(0)})^{-1}$$

$$+\gamma\tilde{x}(dz^{(0)}/d\tilde{x})^2(z^{(0)})^{-2}-\frac{1}{2}\tilde{x}\{x^{(0)};\tilde{x}\}\Big].$$

Since we will choose $z_0^{(0)}(\tilde{x})$ so that it does not vanish at the origin the following relations (1.16) and (1.17) guarantee that the right-hand side of (1.15) is well-defined on a sufficiently small neighborhood U of the origin:

(1.16)

$$(z^{(0)})^{-1}$$

$$=\frac{1}{z_0^{(0)}(\tilde{x})}\left(1-\frac{z_1^{(0)}(\tilde{x})}{z_0^{(0)}(\tilde{x})}\eta^{-1}+\frac{z_1^{(0)}(\tilde{x})^2-z_0^{(0)}(\tilde{x})z_2^{(0)}(\tilde{x})}{z_0^{(0)}(\tilde{x})^2}\eta^{-2}+\cdots\right),$$

(1.17)

$$\left(\frac{dx^{(0)}}{d\tilde{x}}\right)^{-1}$$

$$=\frac{1}{z_0^{(0)}(\tilde{x})+\tilde{x}dz_0^{(0)}/d\tilde{x}}\left(1-\frac{z_1^{(0)}(\tilde{x})+\tilde{x}dz_1^{(0)}/d\tilde{x}}{z_0^{(0)}(\tilde{x})+\tilde{x}dz_0^{(0)}/d\tilde{x}}\eta^{-1}+\cdots\right).$$

Let us now compare the coefficients of η^0 in (1.15). Then we find

$$\tilde{Q}_0(\tilde{x},0)=\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2\left(\frac{\tilde{x}}{4}+\frac{\alpha_0^{(0)}}{z_0^{(0)}}\right), \tag{1.18}$$

and hence we choose

$$\alpha_0^{(0)} = 0 \tag{1.19}$$

and

$$x_0^{(0)}(\tilde{x}) = 2\int_0^{\tilde{x}} \sqrt{\tilde{x}^{-1}\tilde{Q}_0(\tilde{x},0)}\,d\tilde{x}. \tag{1.20}$$

It then follows from (1.5) that

$$z_0^{(0)}(0) = 2\sqrt{c_0^{(0)}} \neq 0. \tag{1.21}$$

Next, using (1.19) we obtain the following relation (1.22) by comparing the coefficients of η^{-1} in (1.15):

$$\tilde{Q}_1(\tilde{x},0) = 2\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_1^{(0)}}{d\tilde{x}}\frac{\tilde{x}}{4} + \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2 \left(\frac{\alpha_1^{(0)}}{z_0^{(0)}}\right). \tag{1.22}$$

Setting $\tilde{x} = 0$ in (1.22) we find that $\alpha_1^{(0)}$ should satisfy

$$\alpha_1^{(0)} = \tilde{Q}_1(0,0)/z_0^{(0)}(0). \tag{1.23}$$

Then we can find a holomorphic function $f_1(\tilde{x})$ which satisfies

$$\tilde{Q}_1(\tilde{x},0) - \left(\frac{dx_0^{(0)}(\tilde{x})}{d\tilde{x}}\right)^2 \frac{\alpha_1^{(0)}}{z_0^{(0)}(\tilde{x})} = \tilde{x}f_1(\tilde{x}). \tag{1.24}$$

Thus it suffices to solve

$$\frac{dx_1^{(0)}}{d\tilde{x}} = 2\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^{-1} f_1(\tilde{x}) \tag{1.25}$$

to find $x_1^{(0)}$ that satisfies (1.22). If we solve (1.25) with the initial condition at $\tilde{x} = 0$ being 0 on a sufficiently small disc U centered at the origin, we obtain $x_1^{(0)}(\tilde{x})$ that also satisfies the condition (1.8). The construction of $x_k^{(0)}$ and $\alpha_k^{(0)}$ $(k \geq 2)$ can be inductively done on the same disc U in a similar manner. For example, the comparison of the coefficients of η^{-2} in (1.15) results in the following:

$$0 = \left(2\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_2^{(0)}}{d\tilde{x}} + \Big(\frac{dx_1^{(0)}}{d\tilde{x}}\Big)^2\right)\frac{\tilde{x}}{4} + 2\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_1^{(0)}}{d\tilde{x}}\frac{\alpha_1^{(0)}}{z_0^{(0)}} \tag{1.26}$$

$$+ \Big(\frac{dx_0^{(0)}}{d\tilde{x}}\Big)^2\left(\frac{\alpha_2^{(0)}}{z_0^{(0)}} - \frac{\alpha_1^{(0)}z_1^{(0)}}{z_0^{(0)2}}\right) - \tilde{R}_2(\tilde{x}) + \frac{2\gamma\frac{dz_0^{(0)}}{d\tilde{x}}}{z_0^{(0)}}$$

$$+ \gamma\tilde{x}\left(\frac{\frac{dz_0^{(0)}}{d\tilde{x}}}{z_0^{(0)}}\right)^2 - \frac{1}{2}\tilde{x}\{x_0^{(0)};\tilde{x}\}.$$

Then we set $\tilde{x} = 0$ in (1.26) to find

$$\alpha_2^{(0)} = (z_0^{(0)}(0))^{-1}\left[\alpha_1^{(0)}\big(z_1^{(0)}(0) - 2z_1^{(0)}(0)\big) + \tilde{R}_2(0) - \frac{2\gamma\frac{dz_0^{(0)}}{d\tilde{x}}(0)}{z_0^{(0)}(0)}\right]. \tag{1.27}$$

After choosing $\alpha_2^{(0)}$ as in (1.27) we can divide (1.26) by $\tilde{x}$ to find a differential equation of the form

$$\frac{dx_2^{(0)}}{d\tilde{x}} = f_2(\tilde{x}), \tag{1.28}$$

where $f_2(\tilde{x})$ is holomorphic on U. Thus we can find the required $x_2^{(0)}(\tilde{x})$ by solving (1.28) with the initial condition $x_2^{(0)}(0) = 0$. The construction of $\alpha_k^{(0)}$ and $x_k^{(0)}(\tilde{x})$ can be performed in exactly the same manner: first compute the coefficients of η^{-k} in (1.15), set $\tilde{x}$ to be 0 to find $\alpha_k^{(0)}$ so that we may divide the sum of the coefficients by $\tilde{x}$ to find a first order equation of normal form for $x_k^{(0)}(\tilde{x})$ with holomorphic coefficients on U, and finally solve the differential equation with the initial condition $x_k^{(0)}(0) = 0$.

Q.E.D.

As is well known in the exact WKB analysis (e.g. [KT2, Theorem 2.16 and Corollary 2.18]), the relation (1.6) between potentials enables us to clarify the structure of WKB solutions of a general MPPT equation restricted to $\{a = 0\}$ in terms of WKB solutions of a particular (i.e., $\alpha_0^{(0)} = 0$) ∞-Whittaker equation; the concrete statements are as follows:

Theorem 1.2. *In the situation considered in Theorem 1.1, the infinite series $x^{(0)}(\tilde{x}, \eta)$ and $\alpha^{(0)}(\eta)$ satisfy*

$$\begin{aligned}\tilde{S}(\tilde{x}, \eta) = &\Big(\frac{dx^{(0)}}{d\tilde{x}}\Big) S(x^{(0)}(\tilde{x}, \eta), \alpha^{(0)}(\eta), \eta) \\ &- \frac{1}{2}\Big(\frac{d^2 x^{(0)}(\tilde{x}, \eta)}{d\tilde{x}^2}\Big) \Big/ \Big(\frac{dx^{(0)}(\tilde{x}, \eta)}{d\tilde{x}}\Big),\end{aligned} \tag{1.29}$$

where $\tilde{S}$ and S are formal series in η^{-1} respectively beginning with $\tilde{S}_{-1}(x)\eta$ and $S_{-1}(x)\eta$ which solve the Riccati equations

$$\tilde{S}^2 + \frac{d\tilde{S}}{dx} = \eta^2\Big(\frac{\tilde{Q}_0(\tilde{x}, 0)}{\tilde{x}} + \eta^{-1}\frac{\tilde{Q}_1(\tilde{x}, 0)}{\tilde{x}} + \eta^{-2}\frac{\tilde{Q}_2(\tilde{x}, 0)}{\tilde{x}^2}\Big) \tag{1.30}$$

and

$$S^2 + \frac{dS}{dx} = \eta^2\Big(\frac{1}{4} + \frac{\alpha^{(0)}(\eta)}{x} + \eta^{-2}\frac{\tilde{Q}_2(0, 0)}{x^2}\Big), \tag{1.31}$$

and for which

$$\arg \tilde{S}_{-1}(\tilde{x}) = \arg\Big(\frac{dx_0^{(0)}}{d\tilde{x}} S_{-1}\big(x_0^{(0)}(\tilde{x})\big)\Big) \tag{1.32}$$

holds (and hence $\tilde{S}_{-1}(\tilde{x})$ *and* $\big(dx_0^{(0)}/d\tilde{x}\big)\ S_{-1}\big(x_0^{(0)}(\tilde{x})\big)$ *themselves coincide.)*

Theorem 1.3. *Let us consider the situation assumed in Theorem 1.1, and let* ψ *be a WKB solution of the* ∞*-Whittaker equation*

$$\left(\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{4} + \frac{\alpha^{(0)}(\eta)}{x} + \eta^{-2}\,\frac{\tilde{Q}_2(0,0)}{x^2}\Big)\right)\psi = 0, \tag{1.33}$$

where $\alpha^{(0)}(\eta)$ *is the infinite series constructed there; in particular*

$$\alpha_0^{(0)} = 0. \tag{1.34}$$

Then for the infinite series $x^{(0)}(\tilde{x},\eta)$ *constructed there we find*

$$\tilde{\psi}(\tilde{x},\eta) = \Big(\frac{dx^{(0)}(\tilde{x},\eta)}{d\tilde{x}}\Big)^{-1/2}\psi\big(x^{(0)}(\tilde{x},\eta),\eta\big) \tag{1.35}$$

satisfies the following MPPT equation restricted to $\{a=0\}$*:*

$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2\Big(\frac{\tilde{Q}_0(\tilde{x},0)}{\tilde{x}} + \eta^{-1}\,\frac{\tilde{Q}_1(\tilde{x},0)}{\tilde{x}} + \eta^{-2}\,\frac{\tilde{Q}_2(\tilde{x},0)}{\tilde{x}^2}\Big)\right)\tilde{\psi}(\tilde{x},\eta) = 0. \tag{1.36}$$

See [KT2, Section 2] for the derivation of Theorems 1.2 and 1.3 from Theorem 1.1; although the situation considered in [KT2] is a much simpler one (the situation where only one simple turning point is relevant) the logical structure of the derivation is exactly the same.

The analytic meaning of Theorem 1.3 becomes much more transparent if we apply the Borel transformation to all the relevant functions and equations; for example, the Borel transformed ∞-Whittaker equation turns out to be a microdifferential equation

$$\left(\frac{\partial^2}{\partial x^2} - \Big(\frac{1}{4} + \frac{1}{x}\,\alpha^{(0)}\big(\frac{\partial}{\partial y}\big)\Big)\frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_2(0,0)}{x^2}\right)\psi_B(x,y) = 0, \tag{1.37}$$

thanks to the estimate (B.3) in Appendix B of the growth order of $\alpha_k^{(0)}$ $(k \geq 1)$. Before embarking on the analytic study of the Borel transformed relations, we present an important relation between the infinite series $\alpha^{(0)}(\eta)$ and $\tilde{S}(\tilde{x},\eta)$ in Theorem 1.2. For that purpose we recall the definition of the odd part S_{odd} of a solution S of the Riccati equation with η-dependent potential.

Definition 1.1. ([AKT3, Definition 2.1]) *Consider the following Riccati equation with η-dependent potential:*

$$S(x,\eta) + \frac{dS}{dx}(x,\eta) = \eta^2\Big(\sum_{k\geq 0} Q_k(x)\eta^{-k}\Big). \tag{1.38}$$

Let $S^{(\pm)}$ respectively denote the solution of (1.38) *that begins with $\pm\eta\sqrt{Q_0(x)}$. Then the odd part S_{odd} of S is, by definition, given by*

$$S_{\mathrm{odd}} = \frac{1}{2}(S^{(+)} - S^{(-)}). \tag{1.39}$$

With the help of Definition 1.1, Theorem 1.2 immediately entails the following

Corollary 1.4. *For S and $\tilde{S}$ in Theorem 1.2 their odd parts satisfy the following relation*

$$\tilde{S}_{\mathrm{odd}}(\tilde{x},\eta) = \Big(\frac{dx^{(0)}}{d\tilde{x}}\Big) S_{\mathrm{odd}}(x^{(0)}(\tilde{x},\eta), \alpha^{(0)}(\eta), \eta), \tag{1.40}$$

if the branches of $\tilde{S}_{-1}$ and S_{-1} are chosen so that (1.32) *is satisfied.*

Using this result we find the following

Proposition 1.5. ([Ko3, Proposition 2.1]) *Let $\tilde{S}_{\mathrm{odd}}$ denote the odd part of $\tilde{S}$ in Theorem 1.2. Then we find*

$$\operatorname*{Res}_{\tilde{x}=0} \tilde{S}_{\mathrm{odd}} = \eta\alpha^{(0)}. \tag{1.41}$$

Proof. In view of the relation (1.40) it suffices to prove (1.41) for S in Theorem 1.2. To verify (1.41) for S_{odd}, we study the concrete form of solutions $S^{(+)}$ and $S^{(-)}$ of (1.31) whose top degree (i.e., degree 1 in η) parts are respectively given by $+\eta/2$ and $-\eta/2$. One can then immediately see

$$S_0^{(\pm)} = \pm\frac{\alpha_1^{(0)}}{x}. \tag{1.42}$$

Here, and in what follows, the sign $\pm$ is chosen correspondingly in each formula. Next

$$2S_{-1}^{(\pm)}S_1^{(\pm)} + \left(S_0^{(\pm)}\right)^2 + \frac{d}{dx}S_0^{(\pm)} = \frac{\alpha_2^{(0)}}{x} + \frac{\tilde{Q}_2(0,0)}{x^2} \tag{1.43}$$

entails

$$\pm S_1^{(\pm)} = \frac{\alpha_2^{(0)}}{x} + \frac{\beta_1^{(\pm)}}{x^2} \tag{1.44}$$

with constants $\beta_1^{(\pm)}$. Similarly the computation of the coefficients of η^{-l} ($l \geq 1$) in (1.31) entails

$$\pm S_{l+1}^{(\pm)} + \sum_{\substack{j+k=l \\ j,k\geq 0}} S_j^{(\pm)}S_k^{(\pm)} + \frac{d}{dx}S_l^{(\pm)} = \frac{\alpha_{l+2}^{(0)}}{x}. \tag{1.45}$$

Since each $S_j^{(\pm)}$ ($j \geq 0$) is a sum of pole terms, (1.45) implies

$$\pm S_{l+1}^{(\pm)} = \frac{\alpha_{l+2}^{(0)}}{x} + (\text{multiple pole terms}). \tag{1.46}$$

Thus the residue of $S_{\mathrm{odd}} = \frac{1}{2}(S^{(+)} - S^{(-)})$ at the origin is $\alpha^{(0)}$, as is expected. This completes the proof of the proposition.

Q.E.D.

We have so far studied the formal aspect of the problem; the growth order conditions (B.3) and (B.4) (with $a = 0$) that $\{x_k^{(0)}(\tilde{x})\}_{k\geq 0}$ and

$\{\alpha_k^{(0)}\}_{k\geq 0}$ respectively satisfy enable us to obtain much deeper analytic results. Applying the Borel transformation ([KT2]) to (1.35), we find that $\tilde{\psi}_B(\tilde{x}, y)$, the Borel transform of $\tilde{\psi}(\tilde{x}, \eta)$, and $\psi_B(x_0^{(0)}(\tilde{x}), y)$, the Borel transform of $\psi(x_0^{(0)}(\tilde{x}), \eta)$, are related by a microdifferential operator. This is one of the most important observations made in [AKT1, Section 2], where a simple turning point problem was studied. Following the presentation of [AY] and [AKT4], we formulate this fact in Theorem 1.6 below as the existence of intertwining operators of a Borel transformed MPPT operator with $a = 0$ and the Borel transformed particular (i.e., $\alpha_0^{(0)} = 0$) ∞-Whittaker operator; furthermore the intertwining operators enjoy beautiful expressions which are most amenable to the study of the exact WKB analysis. (Theorem 1.7.)

To state Theorem 1.6 and Theorem 1.7 we make some notational preparations. First we let $g(x)$ denote the inverse function of

$$x = x_0^{(0)}(\tilde{x}), \tag{1.47}$$

where $x_0^{(0)}(\tilde{x})$ is the function given by (1.20), that is,

$$x = x_0^{(0)}(g(x)), \quad \tilde{x} = g(x_0^{(0)}(\tilde{x})). \tag{1.48}$$

The existence of $g(x)$ is guaranteed by the condition (1.9). Then, by rewriting the Borel transform $\tilde{A}$ of an MPPT operator restricted to $\{a = 0\}$, i.e.,

$$\tilde{A} = \frac{\partial^2}{\partial \tilde{x}^2} - \frac{\tilde{Q}_0(\tilde{x}, 0)}{\tilde{x}} \frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_1(\tilde{x}, 0)}{\tilde{x}} \frac{\partial}{\partial y} - \frac{\tilde{Q}_2(\tilde{x}, 0)}{\tilde{x}^2}, \tag{1.49}$$

in (x, y)-coordinate, we find by (1.18) and (1.19)

(1.50)

$$\tilde{A}\big|_{\tilde{x}=g(x)} = \Big(\frac{dg}{dx}\Big)^{-2} \left[\frac{\partial^2}{\partial x^2} - \Big(\frac{d^2g/dx^2}{dg/dx}\Big)\frac{\partial}{\partial x}\right]$$

$$- \frac{\tilde{Q}_0(g(x),0)}{g(x)} \frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_1(g(x),0)}{g(x)} \frac{\partial}{\partial y} - \frac{\tilde{Q}_2(g(x),0)}{g(x)^2}$$

$$= \big(\frac{dg}{dx}\big)^{-2} \Bigg[\frac{\partial^2}{\partial x^2} - \Big(\frac{d^2g/dx^2}{dg/dx} \Big) \frac{\partial}{\partial x} - \frac{1}{4} \frac{\partial^2}{\partial y^2}$$

$$- \frac{(dg/dx)^2}{g(x)} \tilde{Q}_1(g(x),0) \frac{\partial}{\partial y} - \frac{(dg/dx)^2}{g(x)^2} \tilde{Q}_2(g(x),0) \Bigg].$$

We now define microdifferential operators L and M respectively by

(1.51)

$$L = \frac{\partial^2}{\partial x^2} - \Big(\frac{d^2g/dx^2}{dg/dx} \Big) \frac{\partial}{\partial x}$$

$$- \frac{1}{4} \frac{\partial^2}{\partial y^2} - \frac{(dg/dx)^2}{g(x)} \tilde{Q}_1(g(x),0) \frac{\partial}{\partial y} - \frac{(dg/dx)^2}{g(x)^2} \tilde{Q}_2(g(x),0)$$

and

(1.52) $$M = \frac{\partial^2}{\partial x^2} - \Big(\frac{1}{4} + \frac{\alpha^{(0)}(\partial/\partial y)}{x} \Big) \frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_2(0,0)}{x^2}.$$

Then we have the following

Theorem 1.6. *Let ω_0 be an open neighborhood of $x = 0$, and set*

(1.53) $$\Omega_0 = \{(x, y; \xi, \eta) \in T^*\mathbb{C}^2_{(x,y)}; x \in \omega_0, \eta \neq 0\}$$

and

(1.54) $$\Omega_0^* = \{(x, y; \xi, \eta) \in \Omega_0; x \neq 0\}.$$

Then there exist microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ defined on Ω_0 that satisfy

(1.55) $$L\mathcal{X} = \mathcal{Y}M$$

on Ω_0^* and that are invertible on Ω_0.

Proof. In this proof, and in what follows, we follow [A] in the usage of terminologies and ideograms in symbol calculus; for example, for a microdifferential operator $\mathcal{X}$, $\sigma(\mathcal{X})$ stands for its symbol, and for a symbol $s(x,y,\xi,\eta)$, $: s(x,y,\xi,\eta) :$ designates the corresponding normal ordered product operator, and so on. As was first emphasized by [AKT1],

(1.56)

$$\psi\bigl(x^{(0)}(\tilde{x},\eta),\eta\bigr) = \psi\bigl(x_0^{(0)}(\tilde{x}) + x_1^{(0)}(\tilde{x})\eta^{-1} + x_2^{(0)}(\tilde{x})\eta^{-2} + \cdots, \eta\bigr)$$

that appears in the right-hand side of (1.35) can be formally rewritten as

$$\sum_{n\geq 0} \frac{1}{n!}\Bigl(\sum_{k\geq 1} x_k^{(0)}(\tilde{x})\eta^{-k}\Bigr)^n \Bigl(\frac{\partial^n}{\partial x^n}\psi(x,\eta)\Bigr)\Big|_{x=x_0^{(0)}(\tilde{x})}, \tag{1.57}$$

and hence its Borel transform is expressed in (x,y)-coordinate as

$$\begin{aligned}\Biggl(\sum_{n\geq 0} \frac{1}{n!}\Bigl(\sum_{k\geq 1} x_k^{(0)}(g(x))\bigl(\frac{\partial}{\partial y}\bigr)^{-k}\Bigr)^n \frac{\partial^n}{\partial x^n}\Biggr)\psi_B(x,y) \\ = : \exp\Bigl(\Bigl(\sum_{k\geq 1} x_k^{(0)}(g(x))\eta^{-k}\Bigr)\xi\Bigr) : \psi_B(x,y).\end{aligned} \tag{1.58}$$

Having this expression in mind, we try to find operators $\mathcal{X}$ and $\mathcal{Y}$ in the following form:

$$\mathcal{X} = : C(x,\eta)\exp(r(x,\eta)\xi) :, \tag{1.59}$$

$$\mathcal{Y} = : C^*(x,\eta)\exp(r(x,\eta)\xi) :, \tag{1.60}$$

where $C(x,\eta)$, $C^*(x,\eta)$ and $r(x,\eta)$ are symbols of microdifferential operators respectively of order $0, 0$ and -1. As the notation indicates we suppose they are free from (y,ξ). Let $r_k(x)$ denote the coefficient of η^{-k} in r; that is,

$$r(x,\eta) = \sum_{k\geq 1} r_k(x)\eta^{-k}. \tag{1.61}$$

Then, by the symbol calculus of the composition of operators, we find

$$\sigma(L\mathcal{X}) = \sigma(L)\sigma(\mathcal{X}) + \sigma_\xi(L)\sigma_x(\mathcal{X}) + \frac{1}{2!}\sigma_{\xi\xi}(L)\sigma_{xx}(\mathcal{X}). \tag{1.62}$$

Note that $\mathcal{X}$ is free from y and that

$$\frac{\partial^p}{\partial\xi^p}\sigma(L) = 0 \quad \text{if} \quad p \geq 3. \tag{1.63}$$

Here and in what follows we use the subscripts x (resp., ξ) to designate the differentiation by x (resp., ξ): $r_x = dr/dx$, $r_{xx} = d^2r/dx^2$, etc. We also use the letter E as an abbreviation of $\exp(r(x,\eta)\xi)$. Under these conventions we find

(1.64)

$$\begin{aligned}
&\sigma(L\mathcal{X})\\
&=\Big[\xi^2 - \frac{1}{4}\eta^2 - \frac{g_{xx}}{g_x}\xi - \frac{(g_x)^2}{g}\tilde{Q}_1(g(x),0)\eta - \frac{(g_x)^2}{g^2}\tilde{Q}_2(g(x),0)\Big]CE\\
&\quad + \Big(2\xi - \frac{g_{xx}}{g_x}\Big)\Big(C_xE + r_x\xi CE\Big)\\
&\quad + \frac{1}{2!}(2)\Big(C_{xx}E + 2C_xr_x\xi E + Cr_{xx}\xi E + C(r_x\xi)^2E\Big)\\
&= (1+r_x)^2C\xi^2E + \Big[2(1+r_x)C_x - \frac{g_{xx}}{g_x}(1+r_x)C + r_{xx}C\Big]\xi E\\
&\quad + \Big[\Big(-\frac{1}{4}\eta^2 - \frac{(g_x)^2}{g}\tilde{Q}_1(g(x),0)\eta - \frac{(g_x)^2}{g^2}\tilde{Q}_2(g(x),0)\Big)C\\
&\quad - \frac{g_{xx}}{g_x}C_x + C_{xx}\Big]E.
\end{aligned}$$

In parallel with (1.64), by setting

$$\beta(\eta) = \eta\alpha^{(0)}(\eta) = \sum_{k\geq 1}\alpha_k^{(0)}\eta^{-k+1} \tag{1.65}$$

and

$$\gamma = \tilde{Q}_2(0,0), \tag{1.66}$$

we find

(1.67)

$$\begin{aligned}
&\sigma(\mathcal{Y}M)\\
&= \sum_{n\geq 0} \frac{1}{n!}\Big(\frac{\partial^n}{\partial \xi^n}\sigma(\mathcal{Y})\Big)\Big(\frac{\partial^n}{\partial x^n}\sigma(M)\Big)\\
&= (C^*E)\Big(\xi^2 - \frac{1}{4}\eta^2 - \frac{\beta(\eta)\eta}{x} - \frac{\gamma}{x^2}\Big)\\
&+ \sum_{n\geq 1} \frac{1}{n!}\big(r^n C^* E\big)\Big(\frac{(-1)^{n+1} n!\beta(\eta)\eta}{x^{n+1}} + \frac{(-1)^{n+1}(n+1)!\gamma}{x^{n+2}}\Big)\\
&= (C^*E)\big(\xi^2 - \frac{1}{4}\eta^2\big)\\
&\quad -(C^*E)\Big[\sum_{n\geq 0}\frac{\beta(\eta)\eta}{x}\big(\frac{-r}{x}\big)^n + \sum_{n\geq 0}\frac{(n+1)\gamma}{x^2}\big(\frac{-r}{x}\big)^n\Big]\\
&= (C^*E)\big(\xi^2 - \frac{1}{4}\eta^2\big) - (C^*E)\Big[\frac{\beta(\eta)\eta}{x}\big(1+\frac{r}{x}\big)^{-1} + \frac{\gamma}{x^2}\big(1+\frac{r}{x}\big)^{-2}\Big]\\
&= (C^*E)\Big(\xi^2 - \frac{1}{4}\eta^2 - \frac{\beta(\eta)\eta}{x+r} - \frac{\gamma}{(x+r)^2}\Big).
\end{aligned}$$

Hence we obtain the following relations by comparing the coefficients of $\xi^l E$ $(l = 2,1,0)$ in (1.64) and (1.67):

$$(1+r_x)^2 C = C^* \tag{1.68}$$

$$(1+r_x)\Big(2C_x - \frac{g_{xx}}{g_x}C\Big) + r_{xx}C = 0 \tag{1.69}$$

(1.70)

$$\Big[-\frac{1}{4}\eta^2-\frac{(g_x)^2}{g}\tilde{Q}_1(g(x),0)\eta-\frac{(g_x)^2}{g^2}\tilde{Q}_2(g(x),0)\Big]C-\frac{g_{xx}}{g_x}C_x+C_{xx}$$

$$=C^*\Big(-\frac{1}{4}\eta^2-\frac{\beta(\eta)\eta}{x+r}-\frac{\gamma}{(x+r)^2}\Big).$$

If we set

$$s(x,\eta)=x+r(x,\eta), \tag{1.71}$$

(1.69) is rewritten as follows:

$$\frac{C_x}{C}=\frac{1}{2}\Big(\frac{g_{xx}}{g_x}-\frac{s_{xx}}{s_x}\Big). \tag{1.72}$$

Hence C is fixed by g and s aside from a constant multiple Γ:

$$C=\Gamma(g_x)^{1/2}(s_x)^{-1/2}. \tag{1.73}$$

As the arbitrariness of Γ is absorbed by the freedom in choosing the constant multiple of C^* if we define it by (1.68), i.e.,

$$C^*=s_x^2C. \tag{1.74}$$

Thus we may choose $\Gamma=1$ in (1.73) without loss of generality. Substituting (1.74) into (1.70), we obtain

$$\frac{1}{4}\eta^2+\frac{(g_x)^2}{g(x)}\tilde{Q}_1(g(x),0)\eta+\frac{(g_x)^2}{g(x)^2}\tilde{Q}_2(g(x),0) \tag{1.75}$$

$$=s_x^2\Big(\frac{1}{4}\eta^2+\frac{\beta(\eta)\eta}{s}+\frac{\gamma}{s^2}\Big)-C^{-1}\Big(\frac{g_{xx}}{g_x}C_x-C_{xx}\Big).$$

Further (1.18) entails

$$\frac{\tilde{Q}_0(\tilde{x},0)}{\tilde{x}}\Big|_{\tilde{x}=g(x)}=\frac{1}{4}\Big(\frac{dx_0^{(0)}}{d\tilde{x}}\Big)^2\Big|_{\tilde{x}=g(x)}=\frac{1}{4}g_x(x)^{-2}. \tag{1.76}$$

Hence we may rewrite (1.75) as

$$\frac{\tilde{Q}_0(g(x),0)}{g(x)}\eta^2 + \frac{\tilde{Q}_1(g(x),0)}{g(x)}\eta + \frac{\tilde{Q}_2(g(x),0)}{g(x)^2} \tag{1.77}$$

$$= g_x^{-2} s_x^2\Big(\frac{1}{4}\eta^2 + \frac{\beta(\eta)\eta}{s} + \frac{\gamma}{s^2}\Big) - D(x,\eta)$$

where

$$D(x,\eta) = g_x(x)^{-2} C(x,\eta)^{-1}\Big(\frac{g_{xx}(x)}{g_x(x)} C_x(x,\eta) - C_{xx}(x,\eta)\Big). \tag{1.78}$$

Thus our task is to find the series $s(x,\eta)$ that satisfies (1.77), and we want to find the required series in terms of $x^{(0)}(\tilde{x},\eta)$ constructed in the proof of Theorem 1.1, by somehow relating (1.77) with (1.6). In order to relate (1.77) with (1.6), we substitute $x = x_0^{(0)}(\tilde{x})$ into (1.77) so that the relation is described in terms of the $\tilde{x}$-variable. To facilitate the description of (1.77) in $\tilde{x}$-coordinate, we introduce

$$\tilde{s}(\tilde{x},\eta) = s(x_0^{(0)}(\tilde{x}),\eta) \tag{1.79}$$

and

$$\tilde{C}(\tilde{x},\eta) = C(x_0^{(0)}(\tilde{x}),\eta). \tag{1.80}$$

Then we find

$$\frac{d\tilde{s}}{d\tilde{x}} = \Big(\frac{ds}{dx}\Big|_{x=x_0^{(0)}(\tilde{x})}\Big)\frac{dx_0^{(0)}}{d\tilde{x}} = \Big(\frac{ds}{dx}\Big|_{x=x_0^{(0)}(\tilde{x})}\Big)\Big(\Big(\frac{dg}{dx}\Big)^{-1}\Big|_{x=x_0^{(0)}(\tilde{x})}\Big), \tag{1.81}$$

and hence by (1.73) with $\Gamma = 1$

$$\tilde{C}(\tilde{x},\eta) = \Big(\frac{d\tilde{s}}{d\tilde{x}}\Big)^{-1/2}. \tag{1.82}$$

On the other hand it follows from the definition (1.80) of $\tilde{C}(\tilde{x},\eta)$ that

$$C(x,\eta) = \tilde{C}(g(x),\eta), \tag{1.83}$$

(1.84)
$$C_x(x,\eta) = \left(\frac{d\tilde{C}}{d\tilde{x}}\Big|_{\tilde{x}=g(x)} \right) \frac{dg}{dx},$$

(1.85)
$$C_{xx}(x,\eta) = \left(\frac{d^2\tilde{C}}{d\tilde{x}^2}\Big|_{\tilde{x}=g(x)} \right) \left(\frac{dg}{dx}\right)^2 + \left(\frac{d\tilde{C}}{d\tilde{x}}\Big|_{\tilde{x}=g(x)} \right) \frac{d^2g}{dx^2}.$$

Thus the substitution of (1.84) and (1.85) into (1.78) shows

(1.86)
$$\begin{aligned} D(x,\eta) &= g_x^{-2} C(x,\eta)^{-1} \left(-\frac{d^2\tilde{C}}{d\tilde{x}^2}\Big|_{\tilde{x}=g(x)} \right) g_x^2 \\ &= -C(x,\eta)^{-1} \left(\frac{d^2\tilde{C}}{d\tilde{x}^2}\Big|_{\tilde{x}=g(x)} \right). \end{aligned}$$

We now use (1.82) to compute $\tilde{C}_{\tilde{x}\tilde{x}}$ $(= d^2\tilde{C}/d\tilde{x}^2)$:

(1.87)
$$\frac{d^2\tilde{C}}{d\tilde{x}^2} = -\frac{1}{2}\left(\frac{d\tilde{s}}{d\tilde{x}}\right)^{-1/2} \left(\frac{\tilde{s}_{\tilde{x}\tilde{x}\tilde{x}}}{\tilde{s}_{\tilde{x}}} - \frac{3}{2}\left(\frac{\tilde{s}_{\tilde{x}\tilde{x}}}{\tilde{s}_{\tilde{x}}}\right)^2 \right).$$

Then the substitution of $x = x_0^{(0)}(\tilde{x})$ into (1.86) entails

(1.88)
$$D\big(x_0^{(0)}(\tilde{x}),\eta\big) = \frac{1}{2}\tilde{C}(\tilde{x},\eta)^{-1}\left(\frac{d\tilde{s}}{d\tilde{x}}\right)^{-1/2} \left(\frac{\tilde{s}_{\tilde{x}\tilde{x}\tilde{x}}}{\tilde{s}_{\tilde{x}}} - \frac{3}{2}\left(\frac{\tilde{s}_{\tilde{x}\tilde{x}}}{\tilde{s}_{\tilde{x}}}\right)^2 \right) = \frac{1}{2}\{\tilde{s};\tilde{x}\}.$$

Now we substitute $x = x_0^{(0)}(\tilde{x})$ into (1.77) and use (1.81) and (1.88) to obtain

(1.89)
$$\begin{aligned} &\frac{\tilde{Q}_0(\tilde{x},0)}{\tilde{x}}\eta^2 + \frac{\tilde{Q}_1(\tilde{x},0)}{\tilde{x}}\eta + \frac{\tilde{Q}_2(\tilde{x},0)}{\tilde{x}^2} \\ &= \left(\frac{d\tilde{s}}{d\tilde{x}}\right)^2 \left(\frac{1}{4}\eta^2 + \frac{\beta(\eta)\eta}{\tilde{s}(\tilde{x},\eta)} + \frac{\gamma}{\tilde{s}(\tilde{x},\eta)^2} \right) - \frac{1}{2}\{\tilde{s};\tilde{x}\}. \end{aligned}$$

Comparing (1.89) with (1.6) we find by (1.65) and (1.66) that the series $x^{(0)}(\tilde{x},\eta)$ constructed in the proof of Theorem 1.1 gives us the series

$\tilde{s}(\tilde{x}, \eta)$ that satisfies (1.89). Furthermore the growth order condition (B.4) in Appendix B guarantees that $\tilde{s}(\tilde{x}, \eta)$ is the symbol of a microdifferential operator of order 0. Therefore we obtain the required symbol $s(x, \eta)$ by setting

$$s(x, \eta) = \tilde{s}(g(x), \eta). \tag{1.90}$$

Note that the top degree part of $s(x, \eta)$, i.e., $s_0(x)$ is, by its definition, $x_0^{(0)}(g(x)) = x$. Hence the series s given by (1.90) has the form (1.71). Hence $r(x, \eta)$ is the symbol of a microdifferential operator of order -1. Furthermore the fact that $s_0(x) = x$ together with (1.73) and (1.74) entails that the highest degree in η parts, i.e., degree 0 parts of C and C^* are both $(g_x)^{1/2}$, which never vanishes on a sufficiently small neighborhood ω_0 of the origin. This implies that C and C^* are invertible on Ω_0, and hence $\mathcal{X} = CE$ and $\mathcal{Y} = C^*E$ are also invertible there. Since

$$\sigma(L\mathcal{X}) = \sigma(\mathcal{Y}M) \tag{1.91}$$

holds on Ω_0^* by the way of constructing $\mathcal{X}$ and $\mathcal{Y}$, we find

$$L\mathcal{X} = \mathcal{Y}M \tag{1.92}$$

on Ω_0^*. This completes the proof of the theorem.

Q.E.D.

Remark 1.2. As is evident from the above proof of Theorem 1.6, Theorem 1.6 may be understood as a Borel-transformed version of Theorem 1.3. Actually it follows from (1.59), (1.81) and (1.73) with Γ being 1 that, if we write down the Borel transform of $(dx^{(0)}(\tilde{x}, \eta)/d\tilde{x})^{-1/2}$ $\psi(x^{(0)}(\tilde{x}, \eta), \eta)$ in (x, y)-coordinate (not in $(\tilde{x}, y)$-coordinate) for a WKB solution of (1.33), we then find $\mathcal{X}\psi_B(x, y)$ for the operator $\mathcal{X}$ in Theorem 1.6.

In stating Theorem 1.6 we have considered the relation (1.55) only on Ω_0^*. This is just because operators L and M contain singularities at $x = 0$. As is clear from the above construction, operators $\mathcal{X}$ and $\mathcal{Y}$ are well-defined on Ω_0. Furthermore, as we will show in Appendix C, Proposition C.1 and Theorem B.1 in Appendix B entail Theorem 1.7 below. In stating the theorem, we let U (resp., S_j $(j = 1, 2, \cdots, N)$) denote an open set (resp., an analytic hypersurface) given by the following:

$$U = \{(x, y) \in \mathbb{C}^2; |x|, |y| < \delta\} \tag{1.93}$$

and

$$S_j = \{(x, y) \in U; y = s_j(x)\}, \tag{1.94}$$

where δ is a sufficiently small positive number. We also define

$$U^* = U - \Big(\{(x, y) \in U; x = 0\} \cup \big(\bigcup_{j=1}^{N} S_j\big)\Big). \tag{1.95}$$

Theorem 1.7. *Let $\mathcal{X}$ be the microdifferential operator given by* (1.59)*. Then its action upon a multi-valued analytic function $\varphi(x, y)$ defined on U^* is represented as an integro-differential operator of the form*

$$\mathcal{X}\varphi(x, y) = \int_{y_0}^{y} K(x, y - y', \partial/\partial x)\varphi(x, y')dy', \tag{1.96}$$

where $K(x, y, \partial/\partial x)$ is a differential operator of infinite order that is defined on $\{(x, y) \in \mathbb{C}^2; |x| < C$ and $|y| < C'$ for some positive constants C and $C'\}$, and y_0 is a constant that fixes the action of $(\partial/\partial y)^{-1}$ as an integral operator. (See Figure 1.1 below.) The operator $\mathcal{Y}$ given by (1.60) *also enjoys a similar expression.*

Remark 1.3. When the operand φ is a Borel transformed WKB solution of a particular (i.e., $\alpha_0^{(0)} = 0$) ∞-Whittaker equation, the relevant

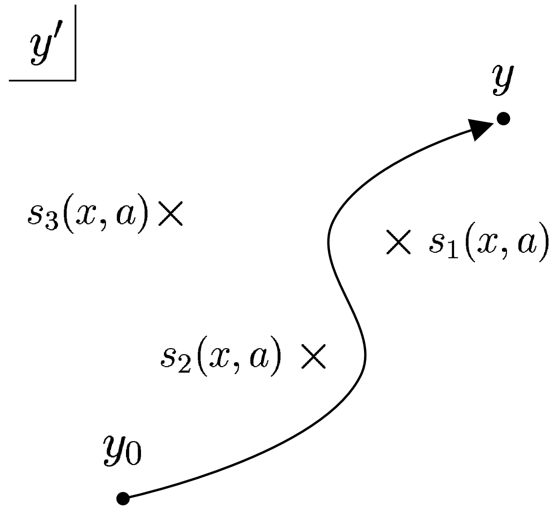

Figure 1.1.

singular points are only $y = s_1(x) = x/2$ and $y = s_2(x) = -x/2$ ([Ko3]); that is, no fixed singularities are observed in this case. (See [KT2, Future Directions and Problems] for the notion and importance of fixed singularities (versus movable ones like the pair $(s_1(x), s_2(x))$) as above.) On the other hand, the power of the expression (1.96) is most manifest when we study the structure of a Borel trasformed WKB solution near its fixed singular points, as we will do in Section 5. Hence we do not discuss the action of operators upon Borel transformed WKB solutions of an MPPT equation with $a = 0$ any more. One more reason to avoid here the further discussion of WKB solutions of an MPPT equation with $a = 0$, i.e., a ghost equation, is that we have not yet been able to find a universal and canonical way (like that to be used in Theorem 2.2 in the next section) of normalizing WKB solutions applicable to all ghost equations. This is mainly due to the existence of infinitely many simple poles in S_{odd}, as is shown in Corollary 1.4, and it stands in total contrast to the situation of MPPT equation with $a \neq 0$, which we will discuss in Section 2 and Section 5.

2 Construction of the transformation to the canonical form, II. — the case where $a \neq 0$

The purpose of this section is to find a canonical form of an MPPT equation, i.e., a Schrödinger equation obtained by the addition of a term $aq(x,a)/x$ to the potential of the ghost equation; to begin with we present the following

Theorem 2.1. *Let $\tilde{Q}_j(\tilde{x},a)$ $(j=0,1,2)$ be holomorphic functions defined on a neighborhood of $(\tilde{x},a)=(0,0)$, and suppose that*

(2.1) $\quad \tilde{Q}_0(0,a) \neq 0$ *if* $a \neq 0$,

and

(2.2) $\quad \tilde{Q}_0(\tilde{x},0) = c_0^{(0)}\tilde{x} + O(\tilde{x}^2)$ *holds with* $c_0^{(0)}$ *being a constant different from* 0.

Then there exist an open neighborhood U of $\tilde{x}=0$, an open neighborhood V of $a=0$, holomorphic functions $x_k^{(j)}(\tilde{x})$ $(j,k \geq 0)$ defined on U and constants $\alpha_k^{(j)}$ for which the following conditions (2.3) $\sim$ (2.8) are satisfied:

$$(2.3) \qquad \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)(0) \neq 0,$$

$$(2.4) \qquad x_k^{(j)}(0) = 0 \quad \textit{for every } j \textit{ and } k,$$

$$(2.5) \qquad x_k(\tilde{x},a) = \sum_{j\geq 0} x_k^{(j)}(\tilde{x})a^j \quad \textit{is holomorphic on } U \times V,$$

$$(2.6) \qquad \alpha_k(a) = \sum_{j\geq 0} \alpha_k^{(j)} a^j \quad \textit{is holomorphic on } V,$$

(2.7) $x(\tilde{x},a,\eta)=\sum_{k\geq 0}x_k(\tilde{x},a)\eta^{-k}$ *and*
$\alpha(a,\eta)=\sum_{k\geq 0}\alpha_k(a)\eta^{-k}$ *are Borel transformable series,*

(2.8)

$$\tilde{x}^{-1}\tilde{Q}_0(\tilde{x},a)+\eta^{-1}\tilde{x}^{-1}\tilde{Q}_1(\tilde{x},a)+\eta^{-2}\tilde{x}^{-2}\tilde{Q}_2(\tilde{x},a)$$

$$=\left(\frac{\partial x(\tilde{x},a,\eta)}{\partial\tilde{x}}\right)^2\left(\frac{1}{4}+\frac{\alpha(a,\eta)}{x(\tilde{x},a,\eta)}+\eta^{-2}\frac{\tilde{Q}_2(0,a)}{x(\tilde{x},a,\eta)^2}\right)-\frac{1}{2}\eta^{-2}\{x;\tilde{x}\}.$$

In this section we only describe how to construct $x_k^{(j)}(\tilde{x})$ and $\alpha_k^{(j)}$ so that they formally satisfy (2.8); (2.5), (2.6) and (2.7) are proved in Appendix B (Theorem B.1).

The construction of $\{x_k^{(j)}\}$ and $\{\alpha_k^{(j)}\}$ makes use of the perturbation in powers of a, starting with $x^{(0)}(\tilde{x},\eta)$ and $\alpha^{(0)}(\eta)$ constructed in the preceding section. We introduce $z(\tilde{x},a,\eta)$ given by

$$\tilde{x}^{-1}x(\tilde{x},a,\eta) \tag{2.9}$$

to find (2.10) below in parallel with (1.15):

(2.10)

$$\tilde{Q}_0(\tilde{x},a)+\eta^{-1}\tilde{Q}_1(\tilde{x},a)=\left(\frac{dx}{d\tilde{x}}\right)^2\left(\frac{\tilde{x}}{4}+\frac{\alpha(a,\eta)}{z}\right)$$

$$+\eta^{-2}\left(-\tilde{R}_2(\tilde{x},a)+2\tilde{Q}_2(0,a)\frac{z_{\tilde{x}}}{z}+\tilde{Q}_2(0,a)\tilde{x}\left(\frac{z_{\tilde{x}}}{z}\right)^2-\frac{1}{2}\tilde{x}\{x;\tilde{x}\}\right),$$

where

$$\tilde{R}_2(\tilde{x},a)=\big(\tilde{Q}_2(\tilde{x},a)-\tilde{Q}_2(0,a)\big)/\tilde{x}. \tag{2.11}$$

As (1.16) shows, $(z^{(0)})^{-1}$ is a well-defined (formal) series in η^{-1} thanks to (1.21); hence z^{-1} is a well-defined formal power series of a:

$$z^{-1}=\big(z^{(0)}+az^{(1)}+a^2z^{(2)}+\cdots\big)^{-1} \tag{2.12}$$

$$= \left(z^{(0)}\right)^{-1}\Bigg(1 - a\left(\frac{z^{(1)}}{z^{(0)}} + a\frac{z^{(2)}}{z^{(0)}} + \cdots\right)$$
$$+ a^2\left(\frac{z^{(1)}}{z^{(0)}} + a\frac{z^{(2)}}{z^{(0)}} + \cdots\right)^2 + \cdots\Bigg).$$

Thus, if we let R denote the coefficient of η^{-2} in the right-hand side of (2.10), we find it can be formally expanded as a power series of a:

(2.13) $$R = R^{(0)} + aR^{(1)} + a^2R^{(2)} + \cdots,$$

where

(2.14) $R^{(N)}$ is free from a and expressed in terms of $z^{(j_0)}$, $z^{(j_1)}_{\tilde{x}}$, $z^{(j_2)}_{\tilde{x}\tilde{x}}$, $z^{(j_3)}_{\tilde{x}\tilde{x}\tilde{x}}$ $(0 \leq j_0, j_1, j_2, j_3 \leq N)$ and $\tilde{x}$;

furthermore (2.14) entails

(2.15) the coefficient $R^{(N)}_l$ of η^{-l} in $R^{(N)}$ is expressed in terms of $\tilde{x}$ and $z^{(j)}_k$ and its derivatives with $0 \leq j \leq N$ and $0 \leq k \leq l-2$.

Here $z^{(j)}_k$ stands for the coefficient of η^{-k} of $z^{(j)}$.

Theorem 1.1 shows that $x^{(0)}$ and $z^{(0)} = \tilde{x}^{-1}x^{(0)}$ satisfy (2.10) with $a = 0$. The comparison of coefficients of a^1 in (2.10) leads to

(2.16) $$\frac{\partial}{\partial a}\left(\tilde{Q}_0(\tilde{x}, a) + \eta^{-1}\tilde{Q}_1(\tilde{x}, a)\right)\Big|_{a=0}$$
$$= \frac{\tilde{x}}{2}\left(x^{(0)}_{\tilde{x}}x^{(1)}_{\tilde{x}}\right) + \frac{2\alpha^{(0)}}{z^{(0)}}\left(x^{(0)}_{\tilde{x}}x^{(1)}_{\tilde{x}}\right) + \left(x^{(0)}_{\tilde{x}}\right)^2\frac{\alpha^{(1)}}{z^{(0)}}$$
$$- \left(x^{(0)}_{\tilde{x}}\right)^2\frac{\alpha^{(0)}z^{(1)}}{z^{(0)2}} + \eta^{-2}R^{(1)}.$$

In what follows we let $\tilde{Q}^{(j)}_k(\tilde{x})$ $(k = 0, 1)$ denote the following:

(2.17) $$\frac{1}{j!}\frac{\partial^j}{\partial a^j}\tilde{Q}_k(\tilde{x}, a)\Big|_{a=0}.$$

Let us first pick up every coefficient of η^0 in (2.16), including some terms which actually vanish:

$$(2.16.0) \qquad \tilde{Q}_0^{(1)}(\tilde{x}) = \frac{\tilde{x}}{2}\left(\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_0^{(1)}}{d\tilde{x}}\right) + \frac{2\alpha_0^{(0)}}{z_0^{(0)}}\left(\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_0^{(1)}}{d\tilde{x}}\right) + \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2 \frac{\alpha_0^{(1)}}{z_0^{(0)}} - \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2 \frac{\alpha_0^{(0)} z_0^{(1)}}{z_0^{(0)2}}.$$

In the right-hand side of (2.16.0) the second term and the fourth term vanish because $\alpha_0^{(0)}$ vanishes by (1.19). Hence, by setting $\tilde{x} = 0$ in (2.16.0), we obtain

$$(2.18) \qquad \tilde{Q}_0^{(1)}(0) = \alpha_0^{(1)} z_0^{(0)}(0).$$

Choosing $\alpha_0^{(1)}$ as above, we find a holomorphic function $h(\tilde{x})$ that satisfies

$$(2.19) \qquad \tilde{Q}_0^{(1)}(0) - \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2 \frac{\alpha_0^{(1)}}{z_0^{(0)}} = \tilde{x} h(\tilde{x}).$$

Hence, by dividing (2.16.0) by $\tilde{x}$, we arrive at

$$(2.20) \qquad \frac{1}{2}\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_0^{(1)}}{d\tilde{x}} = h(\tilde{x}).$$

Then we solve (2.20) with the initial condition

$$(2.21) \qquad x_0^{(1)}(0) = 0.$$

Thus we find a solution $x_0^{(1)}$ such that $z_0^{(1)} = \tilde{x}^{-1} x_0^{(1)}$ is holomorphic near $\tilde{x} = 0$ and that satisfies (2.16.0).

Next we collect terms of degree -1 in η in (2.16); this time we dispose of terms containing $\alpha_0^{(0)}$ as a factor. Then we find

(2.16.1)

$$\tilde{Q}_1^{(1)}(\tilde{x})$$

$$= \frac{\tilde{x}}{2}\left(\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_1^{(1)}}{d\tilde{x}} + \frac{dx_1^{(0)}}{d\tilde{x}}\frac{dx_0^{(1)}}{d\tilde{x}}\right) + 2\left(\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_0^{(1)}}{d\tilde{x}}\right)\frac{\alpha_1^{(0)}}{z_0^{(0)}}$$

$$+ 2\left(\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_1^{(0)}}{d\tilde{x}}\right)\frac{\alpha_0^{(1)}}{z_0^{(0)}} + \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2\left(\frac{\alpha_1^{(1)}}{z_0^{(0)}} - \frac{\alpha_0^{(1)}z_1^{(0)}}{z_0^{(0)2}}\right)$$

$$- \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2\frac{\alpha_1^{(0)}z_0^{(1)}}{z_0^{(0)2}}$$

$$= \left[\frac{\tilde{x}}{2}\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_1^{(1)}}{d\tilde{x}} + \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2\frac{\alpha_1^{(1)}}{z_0^{(0)}}\right]$$

$$+ \left[\frac{\tilde{x}}{2}\frac{dx_1^{(0)}}{d\tilde{x}}\frac{dx_0^{(1)}}{d\tilde{x}} + 2\left(\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_0^{(1)}}{d\tilde{x}}\right)\frac{\alpha_1^{(0)}}{z_0^{(0)}} + 2\left(\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_1^{(0)}}{d\tilde{x}}\right)\frac{\alpha_0^{(1)}}{z_0^{(0)}}\right.$$

$$\left.- \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2\frac{\alpha_0^{(1)}z_1^{(0)}}{z_0^{(0)2}} - \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2\frac{\alpha_1^{(0)}z_0^{(1)}}{z_0^{(0)2}}\right].$$

Hence (2.16.1) evaluated at $\tilde{x} = 0$ reads as follows:

(2.22)

$$\tilde{Q}_1^{(1)}(0)$$

$$= z_0^{(0)}(0)\alpha_1^{(1)} + 2z_0^{(1)}(0)\alpha_1^{(0)} + 2z_1^{(0)}(0)\alpha_0^{(1)} - \alpha_0^{(1)}z_1^{(0)}(0) - \alpha_1^{(0)}z_0^{(1)}(0)$$

$$= z_0^{(0)}(0)\alpha_1^{(1)} + z_0^{(1)}(0)\alpha_1^{(0)} + z_1^{(0)}(0)\alpha_0^{(1)}.$$

Since all terms in (2.22) are, except for $z_0^{(0)}(0)\alpha_1^{(1)}$, values of functions which have already been fixed, (2.22) fixes the constant $\alpha_1^{(1)}$. Furthermore this choice of $\alpha_1^{(1)}$ enables us to divide (2.16.1) by $\tilde{x}$ to find a differential equation of the form

$$\frac{dx_1^{(1)}(\tilde{x})}{d\tilde{x}} = f(\tilde{x}) \tag{2.23}$$

for a holomorphic function $f(\tilde{x})$ defined near the origin. We then solve (2.23) with the initial condition

$$x_1^{(1)}(0) = 0 \tag{2.24}$$

to obtain the required $x_1^{(1)}(\tilde{x})$. The treatment of terms of η^{-l} in (2.16) can be done in a similar way; we first find

(2.16.l) $\quad (l \geq 2)$

$$0 = \frac{\tilde{x}}{2}\left(\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_l^{(1)}}{d\tilde{x}} + F_l\right) + \left(\frac{2\alpha_0^{(0)}}{z_0^{(0)}}\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_l^{(1)}}{d\tilde{x}} + G_l\right)$$

$$+\left(\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2\frac{\alpha_l^{(1)}}{z_0^{(0)}} + H_l\right) - \left(\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2\frac{\alpha_0^{(0)}z_l^{(1)}}{(z_0^{(0)})^2} + K_l\right) + R_l^{(1)},$$

where F_l etc. are respectively collections of terms of degree l in η^{-1} that originate from $(x_{\tilde{x}}^{(0)}x_{\tilde{x}}^{(1)})$ etc. and that have been already fixed (like $(dx_j^{(0)}/d\tilde{x})\,(dx_k^{(1)}/d\tilde{x})$ $(j+k=l, 0 \leq k \leq l-1)$. In the above, in order to manifest the origin of G_l and K_l we have included terms which are actually 0, i.e., terms multiplied by $\alpha_0^{(0)}$. Thus (2.16.l) assumes the following form:

$$\frac{\tilde{x}}{2}\left(\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_l^{(1)}}{d\tilde{x}}\right) + \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2\frac{\alpha_l^{(1)}}{z_0^{(0)}} + L_l = 0, \tag{2.25}$$

where L_l is a sum of terms which have already been fixed. Thus we should, and really do, choose

$$\alpha_l^{(1)} = -\left(\frac{1}{z_0^{(0)}} L_l\right)\bigg|_{\tilde{x}=0}. \tag{2.26}$$

Then dividing (2.25) by $\tilde{x}$ we obtain

$$\left(\frac{1}{2}\frac{dx_0^{(0)}}{d\tilde{x}}\right)\frac{dx_l^{(1)}}{d\tilde{x}} = h(\tilde{x}) \tag{2.27}$$

with a holomorphic function h near the origin. Hence we can solve (2.27) with the initial condition $x_l^{(1)}(0) = 0$. Then the resulting function $x_l^{(1)}$ together with the constant $\alpha_l^{(1)}$ satisfies (2.16.l).

It is now evident that we can construct $\{\alpha_k^{(j)}, x_k^{(j)}\}$ for any (j, k) by the same procedure. Actually the comparison of the coefficients of a^N gives us an equation (E_N), and the computation of the coefficients of η^{-l} in (E_N) presents the equation (E_N, l) to be resolved. In the equation (E_N, l), $\{x_k^{(j)}, z_k^{(j)}, \alpha_k^{(j)}\}$ are regarded to be known objects if
(i) $\quad j \leq N-1$

or

(ii) $\quad j = N$, $k \leq l-1$.

The concrete form of (E_N, l) is as follows;

$$0 = \frac{\tilde{x}}{2}\frac{dx_0^{(0)}}{d\tilde{x}}\frac{dx_l^{(N)}}{d\tilde{x}} + \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2 \frac{\alpha_l^{(N)}}{z_0^{(0)}} + (\text{known functions}). \tag{2.28}$$

Here we note that $-\tilde{Q}_l^{(N)}$ is included among known functions when l is 0 or 1. Thus we first fix $\alpha_l^{(N)}$ so that the equation (2.28) is divisible by $\tilde{x}$, and then the equation for $x_l^{(N)}$ obtained by the division by $\tilde{x}$ assumes the normal form. Thus we can solve the equation with the initial condition $x_l^{(N)}(0) = 0$. Thus we can construct $x(\tilde{x}, a, \eta)$

$= \sum_{j,k\geq 0} x_k^{(j)}(\tilde{x}) a^j \eta^{-k}$ and $\alpha(a,\eta) = \sum_{j,k\geq 0} \alpha_k^{(j)} a^j \eta^{-k}$ that satisfy (2.8). The convergence of these series in a and their Borel transformability concerning η are assured by Theorem B.1 in Appendix B.

Q.E.D.

Remark 2.1. (i) It is worth emphasizing that the growth order properties of $\{x_k^{(j)}, \alpha_k^{(j)}\}$ as j tends to ∞ and those as k tends to ∞ are substantially different despite the fact that the construction of $\{x_k^{(j)}, \alpha_k^{(j)}\}$ can be done in a symmetric way with respect to indexes j and k; the equation for $x_l^{(N)}$ can be found by first writing down the equation $(\mathcal{E}_l)$ through the comparison of the coefficients of η^{-l} under the assumption that all coefficients of $\eta^{-l'}$ $(l' \leq l-1)$ are known and then finding out the required equation by the comparison of the coefficients of a^N in $(\mathcal{E}_l)$ under the assumption that all the coefficients of $a^{N'}$ $(N' \leq N-1)$ in $(\mathcal{E}_l)$ are known. The asymmetry of the growth order is tied up with the estimation of higher order derivatives contained in the seemingly ancillary term $\eta^{-2}\tilde{x}\{x;\tilde{x}\}/2$ in (2.10). (See Remark B.2 in Appendix B.)

(ii) It is also noteworthy that the convergence property (2.5) (with $k = 0$) automatically entails the following geometric result: it follows from (2.3) and (2.8) that the solution $\tilde{x} = \tilde{x}_0(a)$ of the equation

$$x_0(\tilde{x}, a) + 4\alpha_0(a) = 0, \tag{2.29}$$

whose existence is guaranteed again by (2.3) for $|a|$ sufficiently small, satisfies

$$\tilde{Q}_0(\tilde{x}_0(a), a) = 0. \tag{2.30}$$

Otherwise stated, the function $x = x_0(\tilde{x}, a)$ maps the simple turning point of the given MPPT equation to that of the ∞-Whittaker equation. Note that it should be difficult to image such a picture only by tracing the algebraic construction of $x(\tilde{x}, a, \eta)$ given above.

In parallel with the reasoning in Section 1, Theorem 2.1 gives us several results on the structure of WKB solutions of a generic (i.e., $a \neq 0$) MPPT equation. Among other things, we first note Theorem 2.2 below. To obtain Theorem 2.2 we make essential use of the simple turning point $\tilde{x} = \tilde{x}_0(a)$; it is known ([AKT2, Proposition 1.6]) that $\tilde{S}_{\text{odd}}$, the odd part of a solution $\tilde{S}$ of the associated Riccati equation, has singularities of square-root type near a simple turning point $\tilde{x} = t$ in general. Hence the integral

$$\int_t^{\tilde{x}} \tilde{S}_{\text{odd}} d\tilde{x} \tag{2.31}$$

is well-defined ([KT2, (2.24)]), and we use this integral to define a WKB solution $\tilde{\psi}_{\pm}$ of an MPPT equation that is normalized at the simple turning point in question, that is,

$$\tilde{\psi}_{\pm}(\tilde{x}, a, \eta) = \frac{1}{\sqrt{\tilde{S}_{\text{odd}}}} \exp\Big(\pm \int_{\tilde{x}_0(a)}^{\tilde{x}} \tilde{S}_{\text{odd}}(\tilde{x}, a, \eta) d\tilde{x} \Big). \tag{2.32}$$

As is shown in [KT2, Section 2], we can deduce Theorem 2.2 below from Theorem 2.1 using the above normalization of WKB solutions.

Theorem 2.2. *Let $\tilde{\psi}_+(\tilde{x}, a, \eta)$ be a WKB solution of an MPPT equation* (2.33) *below, and suppose that it is normalized at its simple turning point as above.*

$$\Big(\frac{d^2}{d\tilde{x}^2} - \eta^2 \tilde{Q}(\tilde{x}, a, \eta)\Big)\tilde{\psi}(\tilde{x}, a, \eta) = 0 \quad (a \neq 0), \tag{2.33}$$

where

$$\tilde{Q} = \frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}} + \eta^{-1}\frac{\tilde{Q}_1(\tilde{x}, a)}{\tilde{x}} + \eta^{-2}\frac{\tilde{Q}_2(\tilde{x}, a)}{\tilde{x}^2} \tag{2.34}$$

satisfies (2.1) *and* (2.2). *Then, for a sufficiently small* a ($\neq 0$), *we can find a WKB solution* $\psi_+(x, \eta; \alpha(a, \eta))$ *of the* ∞*-Whittaker equation*

(2.35)
$$\left(\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{4} + \frac{\alpha(a,\eta)}{x} + \eta^{-2}\frac{\tilde{Q}_2(0,a)}{x^2}\Big)\right)\psi\big(x, \eta; \alpha(a, \eta)\big) = 0$$

that is also normalized at its simple turning point $x = -4\alpha_0(a)$ *so that it satisfies the following relation:*

$$\tilde{\psi}_+(\tilde{x}, a, \eta) = \Big(\frac{\partial x(\tilde{x}, a, \eta)}{\partial \tilde{x}}\Big)^{-1/2} \psi_+\big(x(\tilde{x}, a, \eta), \eta; \alpha(a, \eta)\big), \tag{2.36}$$

where $x(\tilde{x}, a, \eta)$ *and* $\alpha(a, \eta)$ *are the series constructed in Theorem 2.1.*

The proof of Theorem 2.2 is essentially the same as that of Corollary 2.18 in [KT2], and we omit it here. We call the attention of the reader to the fact that normalization of the WKB solution $\tilde{\psi}(\tilde{x}, \eta)$ is not fixed in the corresponding result in Section 1, i.e., Theorem 1.3.

As there is no problem related to the normalization concerning solutions of the Riccati equation, we can obtain the results similar to Theorem 1.2 and Corollary 1.4 by using the series $x(\tilde{x}, a, \eta)$ and $\alpha(a, \eta)$ constructed in Theorem 2.1. For example we obtain the following Theorem 2.3 as a counterpart of Corollary 1.4.

Theorem 2.3. *Let* S *and* $\tilde{S}$ *respectively be a solution of*

$$S^2 + \frac{dS}{dx} = \eta^2\left(\frac{1}{4} + \frac{\alpha(a,\eta)}{x} + \eta^{-2}\frac{\tilde{Q}_2(0,a)}{x^2}\right) \tag{2.37}$$

and

$$\tilde{S}^2 + \frac{d\tilde{S}}{d\tilde{x}} = \eta^2 \tilde{Q}(\tilde{x}, a, \eta), \tag{2.38}$$

and suppose that

$$\arg \tilde{S}_{-1}(\tilde{x}, a) = \arg \left(\frac{dx_0(\tilde{x}, a)}{d\tilde{x}} S_{-1}\big(x_0(\tilde{x}, a), \alpha_0(a)\big) \right) \tag{2.39}$$

holds. Then they satisfy

$$\tilde{S}_{\mathrm{odd}}(\tilde{x}, a, \eta) = \left(\frac{dx(\tilde{x}, a, \eta)}{d\tilde{x}} \right) S_{\mathrm{odd}}\big(x(\tilde{x}, a, \eta), \alpha(a, \eta), \eta\big). \tag{2.40}$$

We refer the reader to [KT2, Section 2] for the proof.

Now we note the following important

Lemma 2.4. *Let S be a solution of* (2.37) *whose top degree part $S_{-1}(x, \alpha_0)$ is chosen so that it is positive for positive x and α_0. Then we find*

$$\oint_{\gamma(\alpha_0)} S_{\mathrm{odd}}\big(x, \alpha(a, \eta), \eta\big) dx = 2\pi i \alpha(a, \eta)\eta, \tag{2.41}$$

where $\gamma(\alpha_0)$ designates a closed curve in the cut plane shown in Figure 2.1 below.

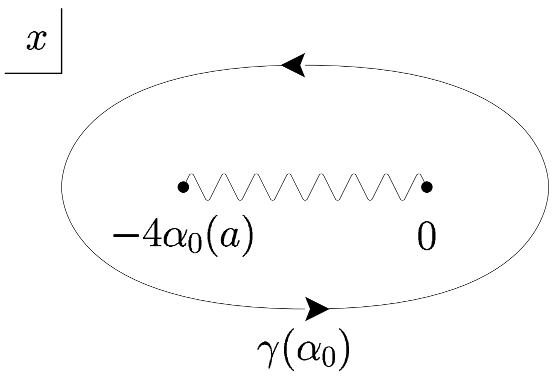

Figure 2.1.

Proof. By a straightforward computation we find

$$S_{-1}^{(\pm)} = \pm \frac{1}{2} \sqrt{\frac{x + 4\alpha_0}{x}}, \tag{2.42}$$

$$(2.43) \qquad S_0^{(\pm)} = \frac{\alpha_0}{x(x+4\alpha_0)} \pm \frac{\alpha_1}{\sqrt{x}\sqrt{x+4\alpha_0}}.$$

Then we can readily find the concrete form of $S_l^{(\pm)}$ $(l \geq 1)$ by the induction on l:

$$(2.44) \qquad S_l^{(\pm)} = \sum c_{p,q}^{(\pm)}(l) x^{-\frac{p}{2}}(x+4\alpha_0)^{-\frac{q}{2}},$$

where $c_{p,q}^{(\pm)}(l)$ are constants, p and q are integers that satisfy

$$(2.45) \qquad p+q = 2m, \quad m = l+1, l, \cdots, 1.$$

Furthermore we see that the surviving constant $c_{p,q}^{(\pm)}(l)$ with $p+q=2$ is only for $p=q=1$ and that

$$(2.46) \qquad c_{1,1}^{(\pm)}(l) = \alpha_{l+1}.$$

By computing the residue at ∞ of $x^{-p/2}(x+4\alpha_0)^{-q/2}$, we find

$$(2.47) \qquad \oint_{\gamma(\alpha_0)} \sqrt{\frac{x+4\alpha_0}{x}}\, dx = 4\pi i \alpha_0,$$

$$(2.48) \qquad \oint_{\gamma(\alpha_0)} \frac{dx}{\sqrt{x(x+4\alpha_0)}} = 2\pi i$$

and

$$(2.49) \qquad \oint_{\gamma(\alpha_0)} \frac{dx}{x^{p/2}(x+4\alpha_0)^{q/2}} = 0 \quad \text{if} \quad p+q = 2m \geq 4.$$

Therefore (2.43), (2.44) and (2.46) imply

$$(2.50) \qquad \oint_{\gamma(\alpha_0)} S_{\mathrm{odd}} dx = 2\pi i \alpha(\eta)\eta.$$

Q.E.D.

Combining Theorem 2.3 and Lemma 2.4 we obtain the following

Proposition 2.5. *Let $\tilde{S}$ be a solution of the Riccati equation* (2.38) *that is associated with a generic MPPT equation. Then with an appropriate choice of the branch of $\tilde{S}_{-1}$, we find*

$$\oint_{\tilde{\gamma}(a)} \tilde{S}_{\mathrm{odd}}(\tilde{x}, a, \eta) d\tilde{x} = 2\pi i \alpha(a, \eta)\eta, \tag{2.51}$$

where $\tilde{\gamma}(a)$ designates a closed curve in the cut plane shown in Figure 2.2.

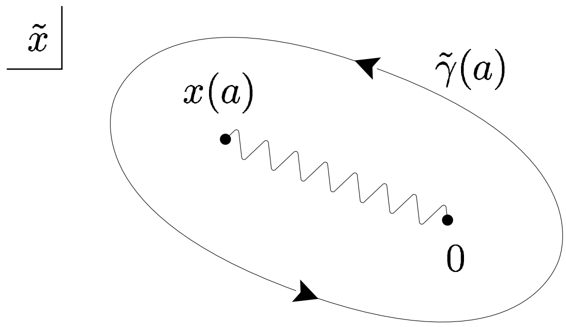

Figure 2.2.

In view of the logical structure of the discussions in Section 1, one naturally expects that some intertwining microdifferential operators between a generic MPPT operator and an ∞-Whittaker operator may be constructed with the help of the series $x(\tilde{x}, a, \eta)$ and $\alpha(a, \eta)$ constructed in Theorem 2.1. This expectation can be readily validated if we introduce a holomorphic function $g(x, a)$, instead of $g(x)$ given in (1.48), which satisfies

$$x = x_0\big(g(x, a), a\big), \quad \tilde{x} = g\big(x_0(\tilde{x}, a), a\big) \tag{2.52}$$

on a neighborhood of $(x, a) = (0, 0)$. The unique existence of such a holomorphic function is guaranteed by (2.3), and hence we find

$$g(x, 0) = g(x). \tag{2.53}$$

The proof of Theorems 2.6 and 2.7 below are essentially the same as that of Theorems 1.6 and 1.7. Here we only repeat the definitions of relevant operators for the convenience of the reader. First L designates a Borel transformed generic MPPT operator expressed in (x, a, y)-coordinate and then multiplied by $(\partial g/\partial x)^2$. That is,

$$L = \frac{\partial^2}{\partial x^2} - \left(\frac{\partial^2 g/\partial x^2}{\partial g/\partial x}\right)\frac{\partial}{\partial x} - \left(\frac{\partial g}{\partial x}\right)^2 \tilde{Q}\Big(g(x, a), a, \frac{\partial}{\partial y}\Big). \tag{2.54}$$

In parallel with (1.52) we designate by M the Borel transformed ∞-Whittaker equation, that is,

$$\frac{\partial^2}{\partial x^2} - \left(\frac{1}{4} + \frac{\alpha(a, \partial/\partial y)}{x}\right)\frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_2(0, a)}{x^2}. \tag{2.55}$$

Using the series $x(\tilde{x}, a, \eta) = \sum_{k\geq 0} x_k(\tilde{x}, a)\eta^{-k}$ constructed in Theorem 2.1, we define another series $r(x, a, \eta)$ by

$$\sum_{k\geq 1} x_k\big(g(x, a), a\big)\eta^{-k}. \tag{2.56}$$

Then, using the same reasoning as in the proof of Theorems 1.6, we obtain Theorem 2.6 below with the help of Theorem B.1 in Appendix B.

Theorem 2.6. *There exist invertible microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ with a holomorphic parameter a that satisfy*

$$L\mathcal{X} = \mathcal{Y}M \tag{2.57}$$

near $(x, a) = (0, 0)$ with the exception of $x\eta = 0$. The concrete form of operators $\mathcal{X}$ and $\mathcal{Y}$ are as follows:

$$\mathcal{X} =: \left(\frac{\partial g}{\partial x}\right)^{1/2}\left(1 + \frac{\partial r}{\partial x}\right)^{-1/2} \exp\big(r(x, a, \eta)\xi\big) :, \tag{2.58}$$

$$\mathcal{Y} =: \left(\frac{\partial g}{\partial x}\right)^{1/2}\left(1 + \frac{\partial r}{\partial x}\right)^{3/2} \exp\big(r(x, a, \eta)\xi\big) : . \tag{2.59}$$

Remark 2.2. In parallel with Remark 1.2, we see from (2.56) and (2.58) that Theorem 2.6 is a Borel-transformed version of Theorem 2.2; $\mathcal{X}\psi_{+,B}$ is the Borel transform of $(\partial x(\tilde{x}, a, \eta)/\partial \tilde{x})^{-1/2}\, \psi_+(x(\tilde{x}, a, \eta), \eta; \alpha(\alpha, \eta))$ written down in (x, y)-coordinate (not in $(\tilde{x}, y)$-coordinate), where ψ_+ is a WKB solution of the ∞-Whittaker equation (2.35).

Furthermore Theorem B.1 together with Proposition C.1 entails the following

Theorem 2.7. *The action of the microdifferential operator $\mathcal{X}$ upon the Borel transformed WKB solution $\psi_{+,B}$ of the ∞-Whittaker equation is expressed as an integro-differential operator of the following form:*

$$\mathcal{X}\psi_{+,B} = \int_{y_0}^{y} K(x, a, y - y', \partial/\partial x)\psi_{+,B}(x, a, y')dy', \tag{2.60}$$

where $K(x, a, y, \partial/\partial x)$ is a differential operator of infinite order that is defined on $\{(x, a, y) \in \mathbb{C}^3;\ (x, a) \in \omega$ for an open neighborhood ω of the origin and $|y| < C$ for some positive constant $C\}$, and y_0 is a constant that fixes the action of $(\partial/\partial y)^{-1}$ as an integral operator.

Remark 2.3. Since $\alpha_0(a)$ tends to 0 as a tends to 0, Theorem B.1 guarantees that we can choose ω to be of the form $\omega_0 \times D$, where

(2.61) $D = \{a \in \mathbb{C}; |a| < \delta$ for some positive constant $\delta\}$,

and

(2.62) ω_0 is a simply connected open set in $\mathbb{C}$ that contains the origin and the simple turning point of the ∞-Whittaker equation, i.e., $x = -4\alpha_0(a)$, for every a in D.

Then the integral operator in the right-hand side of (2.60) acts on any multi-valued analytic function defined on $\omega_0 \times D \times \{y \in \mathbb{C};\ |y - y_0| < C\}$.

3 Analytic properties of WKB solutions of the Whittaker equation with a large parameter

In order to analyze WKB solutions of the ∞-Whittaker equation, which plays a central role in subsequent sections as the canonical form of an MPPT equation for $a \neq 0$, we first recall several basic facts about WKB solutions of the Whittaker equation with a large parameter η, i.e., the equation:

$$\left(\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{4} + \frac{\alpha}{x} + \eta^{-2}\,\frac{\gamma(\gamma+1)}{x^2}\Big)\right)\psi = 0, \tag{3.1}$$

where $\alpha\,(\neq 0)$ and γ are complex numbers. We refer the reader to [KoT] for the details. As [Ko4] has recently found, the Voros coefficient $\phi(\alpha,\gamma;\eta)$ for (3.1) can be explicitly expressed in terms of the Bernoulli numbers and its Borel transform $\phi_B(\alpha,\gamma;y)$ is concretely written down by elementary functions. Here the Voros coefficient means, by definition,

$$\int_{-4\alpha}^{\infty} (S_{\mathrm{odd}} - \eta S_{-1})dx, \tag{3.2}$$

where S_{odd} designates the odd part of a solution S of the Riccati equation associated with (3.1), that is,

$$S^2 + \frac{dS}{dx} = \eta^2\Big(\frac{1}{4} + \frac{\alpha}{x} + \eta^{-2}\,\frac{\gamma(\gamma+1)}{x^2}\Big). \tag{3.3}$$

As we see in Theorem 3.1 below, the concrete form of $\phi_B(\alpha,\gamma;y)$ enables us to find the singularity structure of Borel transformed WKB solution of (3.1) through the relation

$$\psi_+(x,\eta) = (\exp(\phi(\alpha,\gamma;\eta)))\,\psi_+^{(\infty)}(x,\eta), \tag{3.4}$$

where $\psi_+(x,\eta)$ (resp., $\psi_+^{(\infty)}(x,\eta)$) designates the WKB solution of (3.1) that is normalized at the simple turning point $x = -4\alpha$ (resp., at infinity); that is,

$$\psi_+(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\Big(\int_{-4\alpha}^{x} S_{\mathrm{odd}}dx\Big) \tag{3.5}$$

and

(3.6)
$$\psi_+^{(\infty)}(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\Big(\int_{-4\alpha}^{x} \eta S_{-1}dx + \int_{\infty}^{x} \big(S_{\mathrm{odd}} - \eta S_{-1}\big)dx\Big).$$

An important property of $\psi_+^{(\infty)}(x,\eta)$ is that it is Borel summable on the condition that

(3.7) the path of integration from ∞ to x in the right-hand side of (3.6) never touches a Stokes curve of (3.1).

See [KoT] for the proof of the Borel summability of $\psi_+^{(\infty)}(x,\eta)$. See also [DDP1] and [DP] for the corresponding result for the Weber equation. Thus (3.4) implies that the computation of the alien derivative of $\psi_+(x,\eta)$ is reduced to that of $\exp\phi(\alpha,\gamma;\eta)$. In order to compute the latter one we first recall the concrete form of $\phi_B(\alpha,\gamma;y)$ and then employ the alien calculus ([P], [Sa]) to obtain the required result.

Now, the result in [Ko4] tells us the following:

(3.8)
$$\phi_B(\alpha,\gamma;y) = \frac{1}{2y}\left(\frac{\exp(y/\alpha)+1}{\exp(y/\alpha)-1}\right)\cosh\left(\frac{\gamma y}{\alpha}\right) - \frac{\alpha}{y^2} + \frac{1}{2y}\sinh\left(\frac{\gamma y}{\alpha}\right).$$

A straightforward computation shows that

$$\phi_B(\alpha,\gamma;y) = \frac{1}{2\alpha}\Big(\frac{1}{6} + \gamma + \gamma^2\Big) + O(y) \quad \text{near} \quad y = 0 \tag{3.9}$$

and that

(3.10) $\phi_B(\alpha,\gamma;y)$

$$= \left(\frac{\exp(2m\pi i\gamma) + \exp(-2m\pi i\gamma)}{4m\pi i} \right) \frac{1}{y - 2m\pi i\alpha} + O(1)$$

near $\quad y = 2m\pi i\alpha \quad (m$: a non-zero integer).

Thus $\phi_B(\alpha, \gamma; y)$ is seen to be a single-valued analytic function with simple poles located at $y = 2m\pi i\alpha$ $(m \neq 0)$. The computation of the alien derivative $\Delta\phi$ of such a series, i.e., a series whose Borel transform is single-valued and only with simple poles, is exceptionally simple;

$$\Delta\phi = \sum_{m \geq 1} \Delta_{y=2m\pi i\alpha}\phi \tag{3.11}$$

with

$$\Delta_{y=2m\pi i\alpha}\phi = \frac{\exp(2m\pi i\gamma) + \exp(-2m\pi i\gamma)}{2m}. \tag{3.12}$$

(See [P] and [Sa].) Hence, by using the alien calculus, we find

$$\Delta_{y=2m\pi i\alpha}(\exp\phi) = \frac{\exp(2m\pi i\gamma) + \exp(-2m\pi i\gamma)}{2m} \exp\phi. \tag{3.13}$$

(See [P], [CNP] and [Sa].) For the convenience of the description of several formulae below we introduce

$$y_+(x) = \int_{-4\alpha}^{x} S_{-1}dx = \int_{-4\alpha}^{x} \sqrt{\frac{x + 4\alpha}{4x}}dx. \tag{3.14}$$

Then, on the condition that (3.7) is satisfied, we find

$$\Delta\Big(\exp(-y_+(x)\eta)\psi_+^{(\infty)}(x, \eta)\Big) = 0. \tag{3.15}$$

Hence we conclude that

(3.16)

$$\begin{aligned}
&\Delta_{y=-y_+(x)+2m\pi i\alpha}\Big(\exp(-y_+(x)\eta)\psi_+(x, \eta)\Big) \\
&= \Delta_{y=-y_+(x)+2m\pi i\alpha}\Big(\exp(-y_+(x)\eta) \exp(\phi(\alpha, \gamma; \eta))\psi_+^{(\infty)}(x, \eta)\Big)
\end{aligned}$$

$$= \frac{\exp(2m\pi i\gamma) + \exp(-2m\pi i\gamma)}{2m}$$
$$\times \big(\exp(-y_+(x)\eta)\exp(\phi(\alpha,\gamma;\eta))\psi_+^{(\infty)}(x,\eta)\big)$$
$$= \frac{\exp(2m\pi i\gamma) + \exp(-2m\pi i\gamma)}{2m}\big(\exp(-y_+(x)\eta)\psi_+(x,\eta)\big)$$

holds if x is chosen so that the condition (3.7) may be satisfied.

Summing up the obtained results, we find the following

Theorem 3.1. *Let $\psi_+(x,\eta)$ denote the WKB solution of the Whittaker equation that is normalized at the simple turning point $x = -4\alpha$ as in* (3.5). *Then its Borel transform $\psi_{+,B}(x,y)$ is singular at*

(3.17) $$y = -y_+(x) + 2m\pi i\alpha \quad (m = 0, \pm 1, \pm 2, \cdots),$$

where $y_+(x)$ is the function given by (3.14), *and its alien derivative there, i.e., $\Delta_{y=-y_+(x)+2m\pi i\alpha}\psi_+(x,\eta)$ satisfies the relation* (3.18) *below for x that can be connected with a point at infinity by a path that is contained in the interior of a Stokes region of the Whittaker equation.*

(3.18) $$\big(\Delta_{y=-y_+(x)+2m\pi i\alpha}\psi_+\big)_B(x,y)$$
$$= \frac{\exp(2m\pi i\gamma) + \exp(-2m\pi i\gamma)}{2m}\,\psi_{+,B}(x, y - 2m\pi i\alpha).$$

4 Structure of WKB solutions of the ∞-Whittaker equation

As Theorems 2.1, 2.2 and 2.7 show, the WKB-theoretic canonical form of an MPPT equation for $a \neq 0$ is the ∞-Whittaker equation

(4.1)
$$\left(\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{4} + \frac{\alpha(a,\eta)}{x} + \eta^{-2}\frac{c(a)}{x^2}\Big)\right)\tilde{\psi}\big(x,\eta;\alpha(a,\eta),c(a)\big) = 0,$$

where $\alpha(a,\eta)$ satisfies the condition (B.3) and $c(a)$ is $\tilde{Q}_2(0,a)$. Hence the study of singularity structure of Borel transformed WKB solutions of an MPPT equation for $a \neq 0$ is reduced to the study of the corresponding objects of the ∞-Whittaker equation. Thus the analysis of the ∞-Whittaker equation is our next target, and by relating (4.1) with the Whittaker equation

$$\left(\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{4} + \frac{\alpha}{x} + \eta^{-2}\frac{c}{x^2}\Big)\right)\psi(x,\eta;\alpha,c) = 0 \tag{4.2}$$

we achieve the target. A crucial idea in achieving it is the use of microdifferential operators, which becomes possible thanks to the estimate (B.3) of $\{\alpha_k^{(j)}\}$. (See also (B.32.k.j).)

In what follows, to avoid technical complexities, we assume the following condition:

$$\left(\frac{\partial \tilde{Q}_0}{\partial a}\right)(0,0) \neq 0. \tag{4.3}$$

This is a natural strengthening of the assumption (2.1); actually by using the Taylor expansion of $\tilde{Q}_0(\tilde{x},a)$, one immediately sees that the assumption (4.3) together with (2.2) entails (2.1). It is also clear from (2.18) that (4.3) entails

$$\alpha_0^{(1)} \neq 0, \tag{4.4}$$

and hence we find by using (2.6)

$$\frac{d\alpha_0(a)}{da}\Big|_{a=0} \neq 0. \tag{4.5}$$

Therefore we may employ α_0 as an independent variable in substitution for a; thus we regard $\alpha_j(a)$ $(j \geq 1)$ as functions of α_0 in what follows.

Now, in order to relate the Borel transformed WKB solution ψ_B of the Whittaker equation (3.1) and the Borel transformed WKB solution $\tilde{\psi}_B$ of the ∞-Whittaker equation, we rewrite a WKB solution $\tilde{\psi}\big(x, \eta; \alpha(\alpha_0, \eta), c(\alpha_0)\big)$ of (4.1) in the following manner:

$$\begin{aligned} &\tilde{\psi}\big(x, \eta; \alpha(\alpha_0, \eta), c(\alpha_0)\big) \\ &\quad = \left(\sum_{n\geq 0} \frac{(\alpha_1\eta^{-1} + \alpha_2\eta^{-2} + \cdots)^n}{n!} \frac{\partial^n}{\partial\alpha_0^n}\psi\big(x, \eta; \alpha_0, c\big)\right)\Big|_{c=c(\alpha_0)}, \end{aligned} \tag{4.6}$$

where $\psi\big(x, \eta; \alpha_0, c\big)$ designates a WKB solution of (4.2) with

$$\alpha = \alpha_0. \tag{4.7}$$

Then the estimate (B.3) that α_k's satisfy enables us to apply the Borel transformation to (4.6); we then find

$$\tilde{\psi}_B(x, y) = \left(\mathcal{A}\Big(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial\alpha_0}\Big)\psi_B\big(x, y; \alpha_0, c\big)\right)\Big|_{c=c(\alpha_0)}, \tag{4.8}$$

where

$$\mathcal{A}\left(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial\alpha_0}\right) = \sum_{n\geq 0} \frac{\big(\alpha_1(\partial/\partial y)^{-1} + \alpha_2(\partial/\partial y)^{-2} + \cdots\big)^n}{n!} \frac{\partial^n}{\partial\alpha_0^n} \tag{4.9}$$

is a well-defined microdifferential operator on

$$\{(y, \alpha_0; \eta, \theta) \in T^*\mathbb{C}^2; |\alpha_0| < \delta_0, \eta \neq 0\} \tag{4.10}$$

for some positive constant δ_0. In what follows we identify η and θ respectively with the symbol $\sigma(\partial/\partial y)$ and the symbol $\sigma(\partial/\partial\alpha_0)$; using these symbols we may write

$$(4.11) \qquad \mathcal{A} =: \sum_{n\geq 0} \frac{\left(\alpha_1\eta^{-1} + \alpha_2\eta^{-2} + \cdots\right)^n \theta^n}{n!} : .$$

In parallel with the above treatment of Borel transformed WKB solutions with the use of a microdifferential operator relevant to the parameter α, the Borel transform $V_B(y)$ of the exponential of the Voros coefficient of the ∞-Whittaker equation can be expressed in terms of the corresponding function of the Whittaker equation in the following manner:

$$(4.12) \quad V_B(y) = \left(\mathcal{A}(\alpha_0, \partial/\partial y, \partial/\partial\alpha_0)\left(\left(\exp\phi\left(\alpha_0, c, \eta\right)\right)_B\right)\right)\Big|_{c=c(\alpha_0)}.$$

Remark 4.1. Although the target variable is α_0, not x, we can use the same reasoning as in Section 2 to see the concrete expression of the operator $\mathcal{A}$ as an integro-differential operator; the right-hand side of (4.8) and (4.12) should be understood as a multi-valued analytic function acted upon by an integro-differential operator determined by the microdifferential operator $\mathcal{A}$. While the estimate (B.3) guarantees the existence of a common domain of definition of the operator as a tends to 0, the quantity $\alpha_0(a)$ tends to 0 as a tends to 0. On the other hand (3.17) means that a fixed singular point of $\psi_{+,B}(x,y)$ ("fixed" with respect to $y = -y_+(x)$) is located at $y = -y_+(x) + 2m\pi i\alpha$. Thus each individual fixed singular point of $\tilde{\psi}_{+,B}(x,y)$ is contained, for sufficiently small a, in the domain of definition of the integro-differential operator in question. Hence, in what follows, we do not worry about the existence of a sufficiently large domain of definition of the integro-differential operator; if necessary, we assume that a (or, equivalently α_0) is sufficiently close to 0.

Using the results obtained in the preceding section for the Whittaker equation we obtain the following

Theorem 4.1. *Let $\tilde{\psi}_+(x,\eta)$ and $\phi(\alpha(a),\gamma(a);\eta)$ respectively denote*

$$\frac{1}{\sqrt{\tilde{S}_{\mathrm{odd}}}}\exp\left(\int_{-4\alpha_0(a)}^{x}\tilde{S}_{\mathrm{odd}}dx\right) \tag{4.13}$$

and

$$\int_{-4\alpha_0(a)}^{\infty}\left(\tilde{S}_{\mathrm{odd}}-\eta\tilde{S}_{-1}\right)dx, \tag{4.14}$$

where $\tilde{S}_{\mathrm{odd}}$ designates the odd part of a solution $\tilde{S}$ of the following Riccati equation

$$\tilde{S}^2+\frac{d\tilde{S}}{dx}=\eta^2\left(\frac{1}{4}+\frac{\alpha(a)}{x}+\eta^{-2}\,\frac{\gamma(a)^2+\gamma(a)}{x^2}\right) \tag{4.15}$$

with

$$\gamma(a)^2+\gamma(a)=c(a). \tag{4.16}$$

Then the Borel transform $\tilde{\psi}_{+,B}(x,y)$ of $\tilde{\psi}_+(x,\eta)$ and the Borel transform V_B of the exponentiated Voros coefficient $V=\exp\big(\phi(\alpha(a),\gamma(a);\eta)\big)$ satisfy the following relations:

$$\begin{aligned}&\big(\Delta_{y=-y_+(x)+2m\pi i\alpha_0}\tilde{\psi}_+\big)_B(x,y)\\&=\frac{\exp(2m\pi i\gamma(\alpha_0))+\exp(-2m\pi i\gamma(\alpha_0))}{2m}\\&\times :\exp\big(-2m\pi i(\alpha_1+\alpha_2\eta^{-1}+\cdots)\big):\tilde{\psi}_{+,B}(x,y-2m\pi i\alpha_0),\end{aligned} \tag{4.17}$$

$$\begin{aligned}&\big(\Delta_{y=2m\pi i\alpha_0}V\big)_B(y)\\&=\frac{\exp(2m\pi i\gamma(\alpha_0))+\exp(-2m\pi i\gamma(\alpha_0))}{2m}\end{aligned} \tag{4.18}$$

$$\times : \exp\big(-2m\pi i(\alpha_1 + \alpha_2\eta^{-1} + \cdots)\big) : V_B(y - 2m\pi i\alpha_0),$$

where $m = 1, 2, 3, \cdots$, *and* $y_+(x)$ *denotes*

$$\int_{-4\alpha_0}^{x} \sqrt{\frac{x + 4\alpha_0}{4x}}\, dx. \tag{4.19}$$

Proof. For the notational convenience let $\mathcal{B}^{-1}\rho$ denote the inverse Borel transform of ρ. (This is just to avoid the use of the sign $\Delta\rho$ when ρ is the Borel transform of a formal series χ, although $\Delta\rho$ is sometimes used to mean $\Delta\chi$ in references in alien calculus.) Then it follows from (4.8) and the definition of the alien derivative that we obtain

(4.20)

$$\begin{aligned}
&\big(\Delta_{y=-y_+(x)+2m\pi i\alpha_0}\tilde{\psi}_+\big)_B(x, y) \\
=&\Big(\Delta_{y=-y_+(x)+2m\pi i\alpha_0}\mathcal{B}^{-1}\big(\mathcal{A}\big(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha_0}\big)\psi_{+,B}(x, y; \alpha_0, c)\big)\Big)_B(x, y)\Big|_{c=c(\alpha_0)} \\
=&\bigg(\mathcal{A}\big(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha_0}\big)\big(\big(\Delta_{y=-y_+(x)+2m\pi i\alpha_0}\psi_+\big)_B\big(x, y, \alpha_0, c\big)\big)(x, y)\bigg)\bigg|_{c=c(\alpha_0)}.
\end{aligned}$$

Then it follows from Theorem 3.1 that the rightmost term of (4.20) coincides with

$$\begin{aligned}
\Big(\mathcal{A}\big(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha_0}\big)\Big[&\frac{\exp(2m\pi i\gamma) + \exp(-2m\pi i\gamma)}{2m} \\
&\times \psi_{+,B}(x, y - 2m\pi i\alpha_0; \alpha_0, c)\Big]\Big)\Big|_{c=c(\alpha_0)}.
\end{aligned} \tag{4.21}$$

To relate this function with $\tilde{\psi}_{+,B}(x, y - 2m\pi i\alpha_0)$ we use the technique of [AKT4]; we introduce the following coordinate transformation from (y, α_0) to $(\tilde{y}, \tilde{\alpha}_0)$:

$$\begin{cases} \tilde{y} = y - 2m\pi i\alpha_0 \\ \tilde{\alpha}_0 = \alpha_0. \end{cases} \tag{4.22}$$

Correspondingly $\tilde{\eta} = \sigma(\partial/\partial\tilde{y})$ and $\tilde{\theta} = \sigma(\partial/\partial\tilde{\alpha}_0)$ are related with η and θ in the following manner:

$$\begin{cases} \eta = \tilde{\eta} \\ \theta = -2m\pi i\tilde{\eta} + \tilde{\theta}. \end{cases} \tag{4.23}$$

Using $(\tilde{y}, \tilde{\alpha}_0)$-variable, we then find

(4.24)

$$\begin{aligned}
&\Big(\mathcal{A}\big(\alpha_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha_0}\big)\psi_{+,B}(x, y - 2m\pi i\alpha_0; \alpha_0, c)\Big)\Big|_{c=c(\alpha_0)} \\
&= \Big(: \sum_{n\geq 0} \frac{(\alpha_1\tilde{\eta}^{-1} + \alpha_2\tilde{\eta}^{-2} + \cdots)^n(\tilde{\theta} - 2m\pi i\tilde{\eta})^n}{n!} : \\
&\qquad\qquad\qquad \times \psi_{+,B}\big(x, \tilde{y}; \tilde{\alpha}_0, c\big)\Big)\Big|_{c=c(\tilde{\alpha}_0)} \\
&= \Big(: \sum_{n\geq 0} \frac{1}{n!}(\alpha_1\tilde{\eta}^{-1} + \alpha_2\tilde{\eta}^{-2} + \cdots)^n \sum_{\substack{k+l=n \\ k,l\geq 0}} \frac{n!}{k!l!}\tilde{\theta}^k(-2m\pi i\tilde{\eta})^l : \\
&\qquad\qquad\qquad \times \psi_{+,B}\big(x, \tilde{y}; \tilde{\alpha}_0, c\big)\Big)\Big|_{c=c(\tilde{\alpha}_0)} \\
&= \Big(: \sum_{l\geq 0} \frac{1}{l!}\big(-2m\pi i(\alpha_1 + \alpha_2\tilde{\eta}^{-1} + \cdots)\big)^l : \\
&\times : \sum_{k\geq 0} \frac{1}{k!}(\alpha_1\tilde{\eta}^{-1} + \alpha_2\tilde{\eta}^{-2} + \cdots)^k\tilde{\theta}^k : \ \psi_{+,B}\big(x, \tilde{y}; \tilde{\alpha}_0, c\big)\Big)\Big|_{c=c(\tilde{\alpha}_0)} \\
&= \Big(: \exp(-2m\pi i(\alpha_1 + \alpha_2\tilde{\eta}^{-1} + \cdots)) : \\
&\qquad\qquad \times \mathcal{A}\big(\tilde{\alpha}_0, \frac{\partial}{\partial \tilde{y}}, \frac{\partial}{\partial \tilde{\alpha}_0}\big)\psi_{+,B}\big(x, \tilde{y}; \tilde{\alpha}_0, c\big)\Big)\Big|_{c=c(\tilde{\alpha}_0)} \\
&=: \exp(-2m\pi i(\alpha_1 + \alpha_2\eta^{-1} + \cdots)) : \ \tilde{\psi}_{+,B}\big(x, y - 2m\pi i\alpha_0\big).
\end{aligned}$$

Combining (4.20), (4.21) and (4.24), we obtain (4.17). The proof of (4.18) can be given in exactly the same manner.

Q.E.D.

5 Analytic properties of Borel transformed WKB solutions of an MPPT equation for $a \neq 0$

In the preceding section we have seen that the Borel transform ψ_B of a WKB solution of the ∞-Whittaker equation

(5.1)
$$\Big(\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{4} + \frac{\alpha(a,\eta)}{x} + \eta^{-2}\frac{c(a)}{x^2}\Big)\Big)\psi\big(x,\eta;\alpha(a,\eta),c(a)\big) = 0$$

can be represented in the form

$$\big(\mathcal{A}\big(\alpha_0, \partial/\partial y, \partial/\partial\alpha_0\big)\psi_{0,B}(x,y;\alpha_0,c)\big)\Big|_{c=c(\alpha_0)}, \tag{5.2}$$

where $\mathcal{A}$ is a microdifferential operator and $\psi_{0,B}$ is a Borel transformed WKB solution ψ_0 of the Whittaker equation

$$\Big(\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{4} + \frac{\alpha_0}{x} + \eta^{-2}\frac{c}{x^2}\Big)\Big)\psi_0(x,\eta;\alpha_0,c) = 0, \tag{5.3}$$

where α_0 and c are complex numbers. We note that we have changed the notation $(\tilde{\psi},\psi)$ used in Section 4 to (ψ,ψ_0) for the convenience of the presentation in this section. On the other hand, Theorem 2.2 shows that the study of a WKB solution $\tilde{\psi}_+(\tilde{x},a,\eta)$ of an MPPT equation for $a \neq 0$ can be reduced to that of a WKB solution ψ_+ of the ∞-Whittaker equation in that they are related as in (5.4) below with the infinite series $x(\tilde{x},a,\eta)$ and $\alpha(a,\eta)$ constructed in Theorem 2.1:

(5.4)
$$\tilde{\psi}_+(\tilde{x},a,\eta) = \Big(\frac{\partial x(\tilde{x},a,\eta)}{\partial\tilde{x}}\Big)^{-1/2}\psi_+\big(x(\tilde{x},a,\eta),\eta;\alpha(a,\eta),\tilde{Q}_2(0,a)\big).$$

Furthermore, as is noted in Remark 2.2, the growth order condition (B.4) that $\{x_k(\tilde{x}, a)\}_{k\geq 0}$ satisfies has enabled us to rewrite (5.4) as the following microdifferential relation between $\tilde{\psi}_{+,B}$ and $\psi_{+,B}$:

$$\tilde{\psi}_{+,B}(x, a, y) = \mathcal{X}\psi_{+,B}(x, y), \tag{5.5}$$

where

$$\mathcal{X} =: \left(\frac{\partial g}{\partial x}(x, a)\right)^{1/2} \left(1 + \frac{\partial r}{\partial x}\right)^{-1/2} \exp\big(r(x, a, \eta)\xi\big) : \tag{5.6}$$

with the notations in Section 2. (See (2.58).) In view of the concrete expression (2.60) of $\mathcal{X}$ as an integro-differential operator, we find by Theorem 4.1 that the singularities of $\tilde{\psi}_{+,B}(x, a, y)$ are confined to

$$y = -y_+(x, a) + 2m\pi i\alpha_0(a) \quad (m = 0, \pm 1, \pm 2, \cdots) \tag{5.7}$$

in a sufficiently small neighborhood of the origin $(x, a, y) = (0, 0, 0)$, where

$$y_+(x, a) = \int_{-4\alpha_0(a)}^{x} \sqrt{\frac{x + 4\alpha_0(a)}{4x}}\, dx. \tag{5.8}$$

Then it follows from the comparison of degree 0 part of (2.8) that the corresponding point is expressed in $(\tilde{x}, a, y)$-coordinate as follows:

$$y = -y_+(\tilde{x}, a) + 2m\pi i\alpha_0(a) \tag{5.9}$$

where

$$y_+(\tilde{x}, a) = \int_{\tilde{x}_0(a)}^{\tilde{x}} \sqrt{\frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}}}\, d\tilde{x} \tag{5.10}$$

with $\tilde{x}_0(a)$ in (2.30) (i.e., the simple turning point of the MPPT equation in question.) Since the alien derivative of $\psi_{+,B}$ at the point is given by (4.17), the application of the operator $\mathcal{X}$ entails the following

Theorem 5.1. *Let $\tilde{\psi}_+(\tilde{x}, a, \eta)$ be a WKB solution of a generic ($i.e., a \neq 0$) MPPT equation that is normalized as in* (2.32)*. Then for each positive integer m the following relation* (5.11) *holds for sufficiently small $a(\neq 0)$:*

(5.11)

$$\Big(\Delta_{y=-y_+(\tilde{x},a)+2m\pi i\alpha_0(a)}\tilde{\psi}_+\Big)_B(\tilde{x}, a, y)$$

$$= \frac{\exp(2m\pi i\gamma(a)) + \exp(-2m\pi i\gamma(a))}{2m}$$

$$\times : \exp\big(-2m\pi i(\alpha_1(a) + \alpha_2(a)\eta^{-1} + \cdots)\big) : \tilde{\psi}_{+,B}\big(\tilde{x}, a, y - 2m\pi i\alpha_0(a)\big)$$

where

$$y_+(\tilde{x}, a) = \int_{\tilde{x}_0(a)}^{\tilde{x}} \sqrt{\frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}}}\, d\tilde{x}, \tag{5.12}$$

$$\gamma(a)^2 + \gamma(a) = \tilde{Q}_2(0, a) \tag{5.13}$$

and

$$\alpha_j(a) = \frac{1}{2\pi i}\oint_{\tilde{\gamma}(a)} \tilde{S}_{j-1}(\tilde{x}, a) d\tilde{x} \tag{5.14}$$

with $\tilde{\gamma}(a)$ being the closed curve in Figure 2.2 and with $\tilde{S}_k$ designating the degree k part of $\tilde{S}_{\rm odd}$, the odd part of $\tilde{S}$ that satisfies

$$\tilde{S}^2 + \frac{d\tilde{S}}{d\tilde{x}} = \eta^2\tilde{Q}(\tilde{x}, a). \tag{5.15}$$

A Convergence of the top order part of the transformation which brings an MPPT equation to its canonical form

In Appendix A and Appendix B, we give the estimates of the transformation

$$x(\tilde{x}, a, \eta) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} x_k^{(j)}(\tilde{x}) a^j \eta^{-k} \tag{A.1}$$

that appears in Section 2, which brings an MPPT equation

(A.2)
$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 \left(\frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}} + \eta^{-1} \frac{\tilde{Q}_1(\tilde{x}, a)}{\tilde{x}} + \eta^{-2} \frac{\tilde{Q}_2(\tilde{x}, a)}{\tilde{x}^2} \right) \right) \tilde{\psi}(\tilde{x}, \eta) = 0$$

to its canonical form

$$\left(\frac{d^2}{dx^2} - \eta^2 \left(\frac{1}{4} + \frac{\alpha(a, \eta)}{x} + \eta^{-2} \frac{\gamma(a)}{x^2} \right) \right) \psi(x, \eta) = 0 \tag{A.3}$$

with

$$\alpha(a, \eta) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \alpha_k^{(j)} a^j \eta^{-k}. \tag{A.4}$$

Here we assume that $\tilde{Q}_j (j = 0, 1, 2)$ are holomorphic in a neighborhood of $(\tilde{x}, a) = (0, 0)$ and satisfy

$$\tilde{Q}_0(0, 0) = 0, \tag{A.5}$$

$$\frac{\partial \tilde{Q}_0}{\partial \tilde{x}}(0, 0) \neq 0, \tag{A.6}$$

$$\gamma(a) = \tilde{Q}_2(0, a). \tag{A.7}$$

We also obtain the estimates of $\alpha(a, \eta)$ in the course of the estimation of $x(\tilde{x}, a, \eta)$.

The series $x(\tilde{x}, a, \eta)$ and $\alpha(a, \eta)$ are constructed so that they satisfy (2.8), that is,

$$\begin{aligned}(\mathrm{A}.8)\qquad & \frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}} + \eta^{-1}\frac{\tilde{Q}_1(\tilde{x}, a)}{\tilde{x}} + \eta^{-2}\frac{\tilde{Q}_2(\tilde{x}, a)}{\tilde{x}^2} \\ &= \left(\frac{\partial x}{\partial \tilde{x}}\right)^2 \left(\frac{1}{4} + \frac{\alpha(a, \eta)}{x} + \eta^{-2}\frac{\gamma(a)}{x^2}\right) - \frac{1}{2}\eta^{-2}\{x; \tilde{x}\}.\end{aligned}$$

For simplicity, we use the following notations: For multi-indices $\tilde{\kappa} = (\kappa_1, \cdots, \kappa_\mu)$ and $\tilde{\lambda} = (\lambda_1, \cdots, \lambda_\mu)$ in $\mathbb{N}_0^\mu$ with $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$, we define

$$(\mathrm{A}.9)\qquad |\tilde{\lambda}|_\mu := \sum_{j=1}^{\mu} \lambda_j,$$

$$(\mathrm{A}.10)\qquad \tilde{\lambda}! := \prod_{j=1}^{\mu} \lambda_j!,$$

$$(\mathrm{A}.11)\qquad C(\tilde{\lambda}) := \prod_{j=1}^{\mu} C(\lambda_j),\ C(\lambda_j) := \frac{3}{2\pi^2(\lambda_j + 1)^2}.$$

For (λ_j, κ_j)-dependent $(j = 1, 2, \cdots, \mu)$ quantities $\rho_{\kappa_j}^{\lambda_j}$ and σ_{κ_j} we also use the following notations:

$$(\mathrm{A}.12)\qquad \rho_{\tilde{\kappa}}^{\tilde{\lambda}} := \prod_{j=1}^{\mu} \rho_{\kappa_j}^{\lambda_j},$$

$$(\mathrm{A}.13)\qquad \sum_{|\tilde{\kappa}|_\mu = l}^{*} \sigma_{\tilde{\kappa}} := \begin{cases} 1 & \text{for } \mu = 0, \\ \displaystyle\sum_{\substack{|\tilde{\kappa}|_\mu = l, \\ \kappa_j \geq 1}} \prod_{j=1}^{\mu} \sigma_{\kappa_j} & \text{for } \mu \geq 1. \end{cases}$$

In what follows $x_k^{(j)}$ or functions related to it such as $dx_k^{(j)}/d\tilde{x}$ etc. typically stands for ρ_k^j. We also use the notation $\displaystyle\sum_{|\tilde{\lambda}|_\mu = l}^{*} \rho_{\tilde{\kappa}}^{\tilde{\lambda}}$ to mean imposing the constraint on λ_j exactly in the same way as in (A.13). We denote the supremum of a function $f(\tilde{x})$ on $\{\tilde{x} \in \mathbb{C}; |\tilde{x}| \leq r\}$ by

$$\|f\|_{[r]} := \sup_{|\tilde{x}|\leq r} |f(\tilde{x})|. \tag{A.14}$$

As in Section 2, we introduce $z(\tilde{x}, a, \eta)$ given by

$$z(\tilde{x}, a, \eta) := \tilde{x}^{-1} x(\tilde{x}, a, \eta). \tag{A.15}$$

The purpose of Appendix A is to confirm (2.5) and (2.6) for $k = 0$, that is, to prove Proposition A.1 below. As we will see in Appendix B the convergence of the series $x_0(\tilde{x}, a)$ and $\alpha_0(a)$ plays a central role in our subsequent discussions.

Proposition A.1. *Let*

$$x_0(\tilde{x}, a) = \sum_{j=0}^{\infty} x_0^{(j)}(\tilde{x}) a^j \quad \textit{and} \quad \alpha_0(a) = \sum_{j=0}^{\infty} \alpha_0^{(j)} a^j \tag{A.16}$$

be the top order part (with respect to η^{-1}) of the transformation and the coefficient of the canonical form constructed in Section 2 respectively. Then, $x_0(\tilde{x}, a)$ and $\alpha_0(a)$ converge in a neighborhood of $(\tilde{x}, a) = (0, 0)$.

Proof. To begin with, we briefly recall how to construct $x_0^{(j)}$ and $\alpha_0^{(j)}$.

Comparing the coefficients of η^0 in (A.8), we have

$$\frac{\tilde{Q}_0(\tilde{x}, a)}{\tilde{x}} = \left(\frac{\partial x_0}{\partial \tilde{x}}\right)^2 \left(\frac{1}{4} + \frac{\alpha_0(a)}{x_0}\right). \tag{A.17}$$

Further, by comparing the coefficients of a^0 in (A.17), we find

$$\tilde{Q}_0^{(0)}(\tilde{x}) = \left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^2 \left(\frac{\tilde{x}}{4} + \frac{\alpha_0^{(0)}}{z_0^{(0)}}\right), \tag{A.18}$$

where $\tilde{Q}_k{}^{(j)}$ denotes the Taylor coefficient (with respect to a) of $\tilde{Q}_k$ at $a = 0$ (cf. (2.17)). Our choice of $x_0^{(0)}$ and $\alpha_0^{(0)}$ are as follows:

$$\alpha_0^{(0)} = 0,\ x_0^{(0)}(\tilde{x}) = \int_0^{\tilde{x}} 2\sqrt{\frac{\tilde{Q}_0{}^{(0)}(y)}{y}}dy. \tag{A.19}$$

It follows from (A.6) that $x_0^{(0)}$ thus chosen is holomorphic in a neighborhood of 0 and satisfies

$$x_0^{(0)}(0) = 0, \tag{A.20}$$

$$\frac{dx_0^{(0)}}{d\tilde{x}}(0) \neq 0. \tag{A.21}$$

By a similar procedure, we determine $x_0^{(j)}$ and $\alpha_0^{(j)}$ succesively in the following way: first comparing the coefficients of a^j in (A.17), we have

(A.22)

$$\tilde{Q}_0{}^{(j)}(\tilde{x}) = \sum_{j_1+j_2+j_3=m} \frac{dx_0^{(j_1)}}{d\tilde{x}}\frac{dx_0^{(j_2)}}{d\tilde{x}} \times \left(\delta_{0,j_3}\frac{\tilde{x}}{4} + \sum_{j_1'+j_2'=j_3} \frac{\alpha_0^{(j_1')}}{z_0^{(0)}} \sum_{\mu=\min\{1,j_2'\}}^{j_2'} \sum_{|\tilde{\lambda}|_\mu = j_2'}^{*} \frac{(-1)^\mu z_0^{(\tilde{\lambda})}}{\left(z_0^{(0)}\right)^\mu} \right).$$

Here, and in what follows, $\delta_{p,q}$ designates Kronecker's delta (i.e., $= 1$ for $p = q$ and $= 0$ if $p \neq q$). By multiplying (A.22) by $-2z_0^{(0)}\left(dx_0^{(0)}/d\tilde{x}\right)^{-2}$ and taking $w = x_0^{(0)}(\tilde{x})$ as a new independent variable, we can rewrite (A.22) as follows:

$$w\frac{d}{dw}x_0^{(j)} + 2\alpha_0^{(j)} = 2\Phi^{(j)}(w). \tag{A.23}$$

Here the explicit form of $\Phi^{(j)}(w)$ is given by the following:

(A.24)

$$\begin{aligned}\Phi^{(j)}(w) := &-\sum_{\substack{j_1+j_2+j_3=j\\ 1\le j_3\le j-1}} \frac{dx_0^{(j_1)}}{dw}\frac{dx_0^{(j_2)}}{dw} \sum_{j_1'+j_2'=j_3} \alpha_0^{(j_1')} \\ &\times \sum_{\mu=\min\{1,j_2'\}}^{j_2'} \sideset{}{^*}\sum_{|\tilde{\lambda}|_\mu=j_2'} \frac{(-1)^\mu z_0^{(\tilde{\lambda})}}{\left(z_0^{(0)}\right)^\mu} \\ &- \sideset{}{^*}\sum_{j_1'+j_2'=m} \alpha_0^{(j_1')} \sum_{\mu=\min\{1,j_2'\}}^{j_2'} \sideset{}{^*}\sum_{|\tilde{\lambda}|_\mu=j_2'} \frac{(-1)^\mu z_0^{(\tilde{\lambda})}}{\left(z_0^{(0)}\right)^\mu} \\ &- \frac{w}{4} \sideset{}{^*}\sum_{j_1+j_2=m} \frac{dx_0^{(j_1)}}{dw}\frac{dx_0^{(j_2)}}{dw} + z_0^{(0)}\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^{-2} \tilde{Q}_0^{(j)}(w).\end{aligned}$$

We then define $\alpha_0^{(j)}$ by

$$\alpha_0^{(j)} := \Phi^{(j)}(0). \tag{A.25}$$

With this choice of $\alpha_0^{(j)}$ we solve (A.23) to obtain

$$x_0^{(j)}(w) = 2\int_0^w \frac{\Phi^{(j)}(\tilde{w}) - \alpha_0^{(j)}}{\tilde{w}} d\tilde{w}. \tag{A.26}$$

In view of the definition of $\alpha_0^{(j)}$, we find that $x_0^{(j)}(w)$ is holomorphic in some neighborhood of $\{w \in \mathbb{C}; |w| \le r\}$ for some $r > 0$.

To verify the convergence of the series $x_0(\tilde{x}, a)$ and $\alpha_0(a)$, we use the majorant series method; that is, we construct a majorant series $A(a) = \sum_{j\ge 0} A^{(j)} a^j$ of $x_0(\tilde{x}, a)$ and $\alpha_0(a)$. Hence our task is to find a sequence $\{A^{(j)}\}_{j\ge 0}$ of complex numbers so that they satisfy the following relation (A.27.j) for every $j \geq 0$:

$$
\text{(A.27.}j\text{)} \qquad \begin{cases} \left|\alpha_0^{(j)}\right| & \leq \dfrac{A^{(j)}}{4}, \\ \left\|x_0^{(j)}\right\|_{[r]} & \leq A^{(j)}, \\ \left\|\dfrac{dx_0^{(j)}}{dw}\right\|_{[r]}, \left\|z_0^{(j)}\right\|_{[r]} & \leq \dfrac{A^{(j)}}{r}. \end{cases}
$$

To begin with, we choose $A^{(0)}$ and $A^{(1)}$ so that they respectively satisfy (A.27.0) and (A.27.1). To define $A^{(j)}$ ($j \geq 2$) we introduce an auxiliary constant C so that the following relations may be satisfied:

$$
\text{(A.28)} \qquad \left\|\tilde{Q}_0{}^{(j)}\right\|_{[r]} \leq C^{j+1},
$$

$$
\text{(A.29)} \qquad \left\|\left(\frac{dx_0^{(0)}}{dw}\right)^{-1}\right\|_{[r]}, \left\|\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^{-1}\right\|_{[r]}, \left\|\left(z_0^{(0)}\right)^{-1}\right\|_{[r]} \leq C.
$$

Since $\tilde{Q}_0(w,a)$ is holomorphic at $(w,a) = (0,0)$ and $(dx_0^{(0)}/d\tilde{x})(0)$, $z_0^{(0)}(0) \neq 0$, we can find such a constant C by taking $r(>0)$ sufficiently small. Using this constant C we recursively define $A^{(j)}$ by the following:

(A.30)

$$
\begin{aligned}
A^{(j)} := \sum_{\substack{j_1+j_2+j_3=j \\ 1\leq j_3\leq j-1}} \frac{A^{(j_1)}A^{(j_2)}}{r^2} \sum_{\substack{j_1'+j_2'=j_3 \\ j_1'\geq 1}} A^{(j_1')} \sum_{\mu=\min\{1,j_2'\}}^{j_2'} \sum_{|\tilde{\lambda}|_\mu=j_2'}^{*} \left(\frac{C}{r}\right)^{\mu} A^{(\tilde{\lambda})} \\
+ \sum_{j_1'+j_2'=j}^{*} A^{(j_1')} \sum_{\mu=\min\{1,j_2'\}}^{j_2'} \sum_{|\tilde{\lambda}|_\mu=j_2'}^{*} \left(\frac{C}{r}\right)^{\mu} A^{(\tilde{\lambda})} \\
+ \sum_{j_1+j_2=j}^{*} \frac{A^{(j_1)}A^{(j_2)}}{r} + 4C^{j+3}\frac{A^{(0)}}{r}.
\end{aligned}
$$

By using the induction on j we prove that $A^{(j)}$ satisfies (A.27.j).

Let us now suppose $A^{(j)}$ satisfies (A.27.j) for $0 \le j \le m-1$. Then by using (A.24), (A.28), (A.29) and (A.30) we find

(A.31)

$$\begin{aligned}
\left\|\Phi^{(m)}\right\|_{[r]} \le & \sum_{\substack{j_1+j_2+j_3=m\\ 1\le j_3\le m-1}} \left\|\frac{dx_0^{(j_1)}}{dw}\right\|_{[r]} \left\|\frac{dx_0^{(j_2)}}{dw}\right\|_{[r]} \sum_{j_1'+j_2'=j_3} |\alpha_0^{(j_1')}| \\
& \times \sum_{\mu=\min\{1,j_2'\}}^{j_2'} \sum\nolimits^*_{|\tilde{\lambda}|_\mu=j_2'} \left\|z_0^{(\tilde{\lambda})}\right\|_{[r]} \left\|\left(z_0^{(0)}\right)^{-1}\right\|_{[r]}^{\mu} \\
& + \sum\nolimits^*_{j_1'+j_2'=m} |\alpha_0^{(j_1')}| \sum_{\mu=\min\{1,j_2'\}}^{j_2'} \sum\nolimits^*_{|\tilde{\lambda}|_\mu=j_2'} \left\|z_0^{(\tilde{\lambda})}\right\|_{[r]} \left\|\left(z_0^{(0)}\right)^{-1}\right\|_{[r]}^{\mu} \\
& + \frac{|w|}{4} \sum\nolimits^*_{j_1+j_2=m} \left\|\frac{dx_0^{(j_1)}}{dw}\right\|_{[r]} \left\|\frac{dx_0^{(j_2)}}{dw}\right\|_{[r]} \\
& + \left\|z_0^{(0)}\right\|_{[r]} \left\|\left(\frac{dx_0^{(0)}}{d\tilde{x}}\right)^{-1}\right\|_{[r]}^{2} \left\|\tilde{Q}_0{}^{(m)}(w)\right\|_{[r]} \\
\le & \sum_{\substack{j_1+j_2+j_3=m\\ 1\le j_3\le m-1}} \frac{A^{(j_1)}A^{(j_2)}}{r^2} \sum_{\substack{j_1'+j_2'=j_3\\ j_1'\ge 1}} \frac{A^{(j_1')}}{4} \\
& \times \sum_{\mu=\min\{1,j_2'\}}^{j_2'} \sum\nolimits^*_{|\tilde{\lambda}|_\mu=j_2'} \left(\frac{C}{r}\right)^{\mu} A^{(\tilde{\lambda})} \\
& + \sum\nolimits^*_{j_1'+j_2'=m} \frac{A^{(j_1')}}{4} \sum_{\mu=\min\{1,j_2'\}}^{j_2'} \sum\nolimits^*_{|\tilde{\lambda}|_\mu=j_2'} \left(\frac{C}{r}\right)^{\mu} A^{(\tilde{\lambda})}
\end{aligned}$$

$$+\frac{r}{4}\sum_{j_1+j_2=m}^{*}\frac{A^{(j_1)}A^{(j_2)}}{r^2}+C^{m+3}\frac{A^{(0)}}{r}$$
$$=\frac{A^{(m)}}{4}.$$

To deduce (A.27.m) from (A.31) we use the following

Lemma A.2. *Let $v(w)$ be a holomorphic function on $D_r=\{w;|w|\leq r\}$. We consider the following differential equation for $u(w)$:*

$$w\frac{du}{dw}(w)+2\alpha=2v(w), \tag{A.32}$$

where α is a constant. Then there exist a constant α and a holomorphic function $u(w)$ on D_r that vanishes at $w=0$ so that (A.32) *and the following inequalities are satisfied:*

$$|\alpha|\leq\|v\|_{[r]}, \tag{A.33}$$

$$\|u\|_{[r]}\leq 4\|v\|_{[r]}, \tag{A.34}$$

$$\left\|\frac{du}{dw}\right\|_{[r]},\left\|\frac{u}{w}\right\|_{[r]}\leq\frac{4}{r}\|v\|_{[r]}. \tag{A.35}$$

Proof. By setting w to be 0 in (A.32) we find

$$\alpha=v(0), \tag{A.36}$$

and then we define

$$u(w)=2\int_0^w\frac{v(\tilde{w})-\alpha}{\tilde{w}}d\tilde{w}. \tag{A.37}$$

Then we easily see that $u(w)$ is a holomorphic solution of (A.32) on D_r that vanishes at $w=0$. For this choice of α and $u(w)$, (A.33) is clearly satisfied, and the first inequality of (A.35) is an immediate consequence of the Schwarz lemma, because $u(w)$ satisfies (A.38) below as a solution of (A.32).

$$\left\| w\frac{du}{dw} \right\|_{[r]} \leq 2\left\| v \right\|_{[r]} + 2|\alpha| \leq 4\left\| v \right\|_{[r]}. \tag{A.38}$$

Since $u(0) = 0$, we also find the following:

$$\begin{aligned} \left\| u \right\|_{[r]} &\leq \left\| \int_0^w \frac{du}{dw}(\tilde{w})d\tilde{w} \right\|_{[r]} \\ &\leq r\left\| \frac{du}{dw} \right\|_{[r]} \\ &\leq 4\left\| v \right\|_{[r]}. \end{aligned} \tag{A.39}$$

We thus obtain the second inequality of (A.35) by using the Schwarz lemma again.

Q.E.D.

By applying Lemma A.2 to $\alpha_0^{(m)}$ and $x_0^{(m)}$, we obtain (A.27.m). Thus the induction proceeds. This means that we have confirmed that

$$A(a) := \sum_{j\geq 0} A^{(j)} a^j \tag{A.40}$$

is a majorant series of $\alpha_0(a)$ and $x_0(\tilde{x}, a)$. Hence what we should show is the convergence of the series (A.40). The required convergence follows from the implicit function theorem by the following reasoning: first, by comparing the coefficients of a^j, we observe that $A(a)$ satisfies the following equation:

(A.41)

$$\begin{aligned} A =& A^{(0)} + A^{(1)}a + \frac{1}{r^2}(A^2 - (A^{(0)})^2)(A - A^{(0)})\left(\frac{1}{1-(A-A^{(0)})C/r}\right) \\ &+ (A - A^{(0)})\left(\frac{1}{1-(A-A^{(0)})C/r} - 1\right) \end{aligned}$$

$$+\frac{1}{r}(A-A^{(0)})^2+4C^3\frac{A^{(0)}}{r}\frac{(Ca)^2}{1-Ca}.$$

Therefore if we define $\Xi(a,A)$ by

(A.42)

$$\begin{aligned}\Xi(a,A) :=&(A-A^{(0)}-A^{(1)}a)\\ &-\frac{1}{r^2}(A^2-(A^{(0)})^2)(A-A^{(0)})\left(\frac{1}{1-(A-A^{(0)})C/r}\right)\\ &-(A-A^{(0)})\left(\frac{1}{1-(A-A^{(0)})C/r}-1\right)\\ &-\frac{1}{r}(A-A^{(0)})^2-4C^3\frac{A^{(0)}}{r}\frac{(Ca)^2}{1-Ca},\end{aligned}$$

then we find that $A(a)$ is a solution of $\Xi(a,A)=0$. Since Ξ is holomorphic in a neighborhood of $(a,A)=(0,A^{(0)})$ and satisfies

$$\Xi(0,A^{(0)})=0,\ \left(\frac{\partial\Xi}{\partial A}\right)(0,A^{(0)})=1\neq 0, \tag{A.43}$$

it follows from the implicit function theorem that $\Xi(a,A)=0$ has a unique holomorphic solution satisfying $A(0)=A^{(0)}$ near $(a,A)=(0,A^{(0)})$. Hence $A(a)$ is convergent. This implies the convergence of the series $\alpha_0(a)$ and $x_0(\tilde{x},a)$.

Q.E.D.

B Estimation of the transformation which brings an MPPT equation to its canonical form

The purpose of this subsection is to prove (2.5), (2.6) and (2.7), that is, to prove the following

Theorem B.1. *Let*

$$x(\tilde{x}, a, \eta) = \sum_{k=0}^{\infty} x_k(\tilde{x}, a)\eta^{-k} \tag{B.1}$$

be the transformation that brings an MPPT equation (2.33) *to the canonical form* (2.35) *with*

$$\alpha(a, \eta) = \sum_{k=0}^{\infty} \alpha_k(a)\eta^{-k}. \tag{B.2}$$

Then, x and α satisfy the following conditions for some positive constants r_0 and A_0:

(i) *x_k and $\alpha_k (k = 0, 1, 2, \cdots)$ are holomorphic respectively on $\{(\tilde{x}, a); |\tilde{x}| \le r_0, |a| \le r_0\}$ and $\{a; |a| \le r_0\}$.*

(ii) *the following inequalities hold for $k = 1, 2, \cdots$:*

$$\sup_{|a| \le r_0} |\alpha_k(a)| \le k! A_0^k, \tag{B.3}$$

$$\sup_{|\tilde{x}|, |a| \le r_0} |x_k(\tilde{x}, a)| \le k! A_0^k, \tag{B.4}$$

$$\sup_{|\tilde{x}|, |a| \le r_0} \left| \frac{\partial x_k}{\partial \tilde{x}}(\tilde{x}, a) \right| \le k! A_0^k. \tag{B.5}$$

In order to prove Theorem B.1, we use the following lemmas frequently:

Lemma B.2. *For $l, \mu \in \mathbb{N} = \{1, 2, 3, \cdots\}$ with $\mu \le l$, the following inequality holds:*

$$\sum_{|\tilde{\lambda}|_\mu = l}^{*} \tilde{\lambda}! \le 4^{\mu-1}(l - \mu + 1)!. \tag{B.6}$$

Proof. We shall verify (B.6) by induction on $\mu \geq 1$. For the case $\mu = 1$, (B.6) is trivial. For $\mu = 2$, we have

(B.7)

$$\begin{aligned}
\sum_{|\tilde{\lambda}|_2 = l}^{*} \lambda_1!\lambda_2! &= (l-1)! \left(2 + \sum_{\substack{\lambda_1+\lambda_2=l \\ \lambda_1,\lambda_2 \geq 2}} \frac{\lambda_1!\lambda_2!}{(l-1)!} \right) \\
&= (l-1)! \left(2 + \frac{2}{l-1} \sum_{\lambda=2}^{l-2} \frac{\lambda(\lambda-1)\cdots 3}{(l-2)(l-3)\cdots(l-\lambda+1)} \right) \\
&\leq 2(l-1)! \left(1 + \frac{l-2}{l-1} \right) \\
&\leq 4(l-1)!.
\end{aligned}$$

If we assume that (B.6) holds for $\mu - 1 \geq 1$, then we obtain

$$\begin{aligned}
\text{(B.8)} \qquad \sum_{|\tilde{\lambda}|_\mu = l}^{*} \tilde{\lambda}! &= \sum_{\substack{l'+\lambda_\mu = l \\ l' \geq \mu-1, \lambda_\mu \geq 1}} \lambda_\mu! \sum_{\lambda_1+\cdots+\lambda_{\mu-1}=l'}^{*} \lambda_1! \cdots \lambda_{\mu-1}! \\
&\leq \sum_{\substack{l'+\lambda_\mu = l \\ l' \geq \mu-1, \lambda_\mu \geq 1}} 4^{\mu-2}(l'-\mu+2)!\lambda_\mu! \\
&= 4^{\mu-2} \sum_{\substack{l'+\lambda_\mu = l-\mu+2 \\ l' \geq 1, \lambda_\mu \geq 1}} l'!\lambda_\mu! \\
&\leq 4^{\mu-1}(l-\mu+1)!.
\end{aligned}$$

Q.E.D.

Lemma B.3. *For* $\tilde{\lambda} = (\lambda_1, \cdots, \lambda_\mu) \in \mathbb{N}_0^\mu$ *with* $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$, *the following inequality holds for* $C(\tilde{\lambda})$ *given by* (A.11)*:*

$$\text{(B.9)} \qquad \sum_{|\tilde{\lambda}|_\mu = l} C(\tilde{\lambda}) \leq C(l).$$

Proof. We first prove (B.9) for the case $\mu = 2$:

$$\left(\frac{3}{2\pi^2}\right)^2 \sum_{\lambda_1+\lambda_2=l} \frac{1}{(\lambda_1+1)^2}\frac{1}{(\lambda_2+1)^2} \leq \frac{3}{2\pi^2}\frac{1}{(l+1)^2}. \tag{B.10}$$

Since

$$\sum_{\lambda=0}^{\infty} \frac{1}{(\lambda+1)^2} = \frac{\pi^2}{6}, \tag{B.11}$$

we have

(B.12)

$$\begin{aligned}
&\sum_{\lambda_1+\lambda_2=l} \frac{(l+2)^2}{(\lambda_1+1)^2(\lambda_2+1)^2} \\
&= \sum_{\lambda_1+\lambda_2=l} \left(\frac{1}{\lambda_1+1}+\frac{1}{\lambda_2+1}\right)^2 \\
&= \sum_{\lambda_1=0}^{l} \frac{1}{(\lambda_1+1)^2} + \sum_{\lambda_1+\lambda_2=l} \frac{2}{(\lambda_1+1)(\lambda_2+1)} + \sum_{\lambda_2=0}^{l} \frac{1}{(\lambda_2+1)^2} \\
&\leq 2\sum_{\lambda=0}^{\infty} \frac{1}{(\lambda+1)^2} + 2\left(\sum_{\lambda_1=0}^{l} \frac{1}{(\lambda_1+1)^2}\right)^{1/2} \left(\sum_{\lambda_2=0}^{l} \frac{1}{(\lambda_2+1)^2}\right)^{1/2} \\
&\leq 4\sum_{\lambda=0}^{\infty} \frac{1}{(\lambda+1)^2} = \frac{2\pi^2}{3}.
\end{aligned}$$

Then (B.10) immediately follows from this. Since (B.9) is trivial for the case $\mu = 1$, we obtain (B.9) for $\mu \geq 2$ by the succesive use of (B.10).

Q.E.D.

Proof of Theorem B.1. We rewrite (A.8) as (2.10), that is,

(B.13)

$$\begin{aligned}
&\tilde{Q}_0(\tilde{x},a)+\eta^{-1}\tilde{Q}_1(\tilde{x},a)\\
&=\left(\frac{dx}{d\tilde{x}}\right)^2\left(\frac{\tilde{x}}{4}+\frac{\alpha(a,\eta)}{z}\right)\\
&\quad+\eta^{-2}\left(-\tilde{R}_2(\tilde{x},a)+2\frac{dz}{d\tilde{x}}\frac{\gamma(a)}{z}+\tilde{x}\left(\frac{dz}{d\tilde{x}}\right)^2\frac{\gamma(a)}{z^2}\right)-\frac{1}{2}\eta^{-2}\{x;\tilde{x}\}\tilde{x}.
\end{aligned}$$

Here $\tilde{R}_2(\tilde{x},a)$ is the function given by (2.11), that is,

$$\tilde{R}_2(\tilde{x},a)=\frac{\tilde{Q}_2(\tilde{x},a)-\gamma(a)}{\tilde{x}}. \tag{B.14}$$

The choice (A.7) of $\gamma(a)$ guarantees that $\tilde{R}_2(\tilde{x},a)$ is holomorphic in a neighborhood of $(\tilde{x},a)=(0,0)$. By comparing the coefficients of $\eta^{-k}(k\geq 1)$, we obtain

(B.15)

$$\begin{aligned}
&\delta_{k,1}\tilde{Q}_1(\tilde{x},a)\\
&=\sum_{k_1+k_2+k_3=k}\frac{dx_{k_1}}{d\tilde{x}}\frac{dx_{k_2}}{d\tilde{x}}\sum_{k_1'+k_2'=k_3}\frac{\alpha_{k_1'}}{z_0}\sum_{\nu=\min\{1,k_2'\}}^{k_2'}\sum_{|\tilde{\kappa}|_\nu=k_2'}^{*}\frac{(-1)^\nu z_{\tilde{\kappa}}}{z_0^\nu}\\
&\quad+\frac{\tilde{x}}{2}\frac{dx_0}{d\tilde{x}}\frac{dx_k}{d\tilde{x}}+\frac{\tilde{x}}{4}\sum_{k_1+k_2=k}^{*}\frac{dx_{k_1}}{d\tilde{x}}\frac{dx_{k_2}}{d\tilde{x}}-\delta_{k,2}\tilde{R}_2(\tilde{x},a)\\
&\quad+2\gamma(a)\sum_{k_1+k_2=k-2}\frac{dz_{k_1}}{d\tilde{x}}\frac{1}{z_0}\sum_{\nu=\min\{1,k_2\}}^{k_2}\sum_{|\tilde{\kappa}|_\nu=k_2}^{*}\frac{(-1)^\nu z_{\tilde{\kappa}}}{z_0^\nu}\\
&\quad+\tilde{x}\gamma(a)\sum_{k_1+k_2+k_3=k-2}\frac{dz_{k_1}}{d\tilde{x}}\frac{dz_{k_2}}{d\tilde{x}}\frac{1}{z_0^2}\sum_{\nu=\min\{1,k_3\}}^{k_3}\sum_{|\tilde{\kappa}|_\nu=k_3}^{*}(-1)^\nu(\nu+1)\frac{z_{\tilde{\kappa}}}{z_0^\nu}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\tilde{x}}{2} \sum_{k_1+k_2=k-2} \frac{d^3 x_{k_1}}{d\tilde{x}^3} \left(\frac{dx_0}{d\tilde{x}} \right)^{-1} \sum_{\nu=\min\{1,k_2\}}^{k_2} \sum_{|\tilde{\kappa}|_\nu = k_2}^{*} (-1)^\nu \frac{dx_{\tilde{\kappa}}}{d\tilde{x}} \left(\frac{dx_0}{d\tilde{x}} \right)^{-\nu} \\
&+ \frac{3}{4}\tilde{x} \sum_{k_1+k_2+k_3=k-2} \frac{d^2 x_{k_1}}{d\tilde{x}^2} \frac{d^2 x_{k_2}}{d\tilde{x}^2} \left(\frac{dx_0}{d\tilde{x}} \right)^{-2} \\
&\times \sum_{\nu=\min\{1,k_3\}}^{k_3} \sum_{|\tilde{\kappa}|_\nu = k_3}^{*} (-1)^\nu (\nu+1) \frac{dx_{\tilde{\kappa}}}{d\tilde{x}} \left(\frac{dx_0}{d\tilde{x}} \right)^{-\nu}.
\end{aligned}
$$

Further, by comparing the coefficients of a^j in (B.15) and taking $w = x_0^{(0)}(\tilde{x})$ as a new independent variable, we have

$$
w \frac{dx_k^{(j)}}{dw} + 2\alpha_k^{(j)} = 2 \left(\frac{dx_0^{(0)}}{d\tilde{x}} \right)^{-2} z_0^{(0)} \Phi_k^{(j)}, \tag{B.16}
$$

where $\Phi_k^{(j)}$ is

$$
\Phi_k^{(j)} = \Phi_{k,1}^{(j)} + \Phi_{k,2}^{(j)} + \Phi_{k,3}^{(j)} \tag{B.17}
$$

and $\Phi_{k,i}^{(j)}$ $(i = 1, 2, 3)$ are defined as follows:

(B.18)

$$
\begin{aligned}
\Phi_{k,1}^{(j)} = &- 2 \sum_{k_1+k_2=k-2} \sum_{j_1+j_2+j_3+j_4=j} \gamma^{(j_1)} \frac{dz_{k_1}^{(j_2)}}{d\tilde{x}} \\
&\times \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^\nu (z_0^{-\nu-1})^{(j_3)} \sum_{|\tilde{\kappa}|_\nu = k_2}^{*} \sum_{|\tilde{\lambda}|_\nu = j_4} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \\
&- \tilde{x} \sum_{k_1+k_2+k_3=k-2} \sum_{j_1+j_2+j_3+j_4+j_5=j} \gamma^{(j_1)} \frac{dz_{k_1}^{(j_2)}}{d\tilde{x}} \frac{dz_{k_2}^{(j_3)}}{d\tilde{x}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{\nu=\min\{1,k_3\}}^{k_3} (-1)^\nu(\nu+1)(z_0^{-\nu-2})^{(j_4)} \sum_{|\tilde{\kappa}|_\nu=k_3}^{*} \sum_{|\tilde{\lambda}|_\nu=j_5} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \\
&+ \frac{\tilde{x}}{2} \sum_{k_1+k_2=k-2} \sum_{j_1+j_2+j_3=j} \frac{d^3 x_{k_1}^{(j_1)}}{d\tilde{x}^3} \\
&\times \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^\nu \left(\left(\frac{dx_0}{d\tilde{x}} \right)^{-\nu-1} \right)^{(j_2)} \sum_{|\tilde{\kappa}|_\nu=k_2}^{*} \sum_{|\tilde{\lambda}|_\nu=j_3} \frac{dx_{\tilde{\kappa}}^{(\tilde{\lambda})}}{d\tilde{x}} \\
&- \frac{3}{4}\tilde{x} \sum_{k_1+k_2+k_3=k-2} \sum_{j_1+j_2+j_3+j_4=j} \frac{d^2 x_{k_1}^{(j_1)}}{d\tilde{x}^2} \frac{d^2 x_{k_2}^{(j_2)}}{d\tilde{x}^2} \\
&\times \sum_{\nu=\min\{1,k_3\}}^{k_3} (-1)^\nu(\nu+1) \left(\left(\frac{dx_0}{d\tilde{x}} \right)^{-\nu-2} \right)^{(j_3)} \\
&\times \sum_{|\tilde{\kappa}|_\nu=k_3}^{*} \sum_{|\tilde{\lambda}|_\nu=j_4} \frac{dx_{\tilde{\kappa}}^{(\tilde{\lambda})}}{d\tilde{x}} \\
&+ \delta_{k,2} \tilde{R}_2^{(j)}(w),
\end{aligned}
$$

(B.19)

$$
\begin{aligned}
\Phi_{k,2}^{(j)} =& \delta_{k,1} \tilde{Q}_1^{(j)}(w) \\
&- \frac{\tilde{x}}{4} \sum_{k_1+k_2=k}^{*} \sum_{j_1+j_2=j} \frac{dx_{k_1}^{(j_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(j_2)}}{d\tilde{x}} \\
&- \sum_{\substack{k_1+k_2+k_3=k \\ 1\le k_3\le k-1}} \sum_{l_1+l_2+l_3=j} \frac{dx_{k_1}^{(l_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(l_2)}}{d\tilde{x}} \sum_{k'_1+k'_2=k_3} \sum_{l'_1+l'_2+l'_3=l_3} \alpha_{k'_1}^{(l'_1)}
\end{aligned}
$$

$$\times \sum_{\nu=\min\{1,k_2'\}}^{k_2'} (-1)^\nu (z_0^{-\nu-1})^{(l_2')} \sum_{|\tilde\kappa|_\nu=k_2'}^{*} \sum_{|\tilde\lambda|_\nu=l_3'} z_{\tilde\kappa}^{(\tilde\lambda)}$$

$$- \sum_{k_1+k_2=k}^{*} \sum_{j_1+j_2+j_3+j_4=j} \frac{dx_0^{(j_1)}}{d\tilde x} \frac{dx_0^{(j_2)}}{d\tilde x} \alpha_{k_1}^{(j_3)}$$

$$\times \sum_{\nu=1}^{k_2} \sum_{j_1'+j_2'=j_4} (-1)^\nu (z_0^{-\nu-1})^{(j_1')} \sum_{|\tilde\kappa|_\nu=k_2}^{*} \sum_{|\tilde\lambda|_\nu=j_2'} z_{\tilde\kappa}^{(\tilde\lambda)},$$

(B.20)

$$\Phi_{k,3}^{(j)} = - \sum_{\substack{j_1+j_2+j_3+j_4=j \\ j_3\le j-1}} \frac{dx_0^{(j_1)}}{d\tilde x} \frac{dx_0^{(j_2)}}{d\tilde x} \alpha_k^{(j_3)} (z_0^{-1})^{(j_4)}$$

$$- \frac{\tilde x}{2} \sum_{\substack{j_1+j_2=j \\ j_2\le j-1}} \frac{dx_0^{(j_1)}}{d\tilde x} \frac{dx_k^{(j_2)}}{d\tilde x}$$

$$- \sum_{k_1+k_2=k} \sum_{\substack{j_1+j_2+j_3+j_4=j \\ 1\le j_3}} \frac{dx_{k_1}^{(j_1)}}{d\tilde x} \frac{dx_{k_2}^{(j_2)}}{d\tilde x} \alpha_0^{(j_3)} (z_0^{-1})^{(j_4)}$$

$$- \sum_{\substack{j_1+j_2+j_3+j_4=j \\ 1\le j_3}} \frac{dx_0^{(j_1)}}{d\tilde x} \frac{dx_0^{(j_2)}}{d\tilde x} \alpha_0^{(j_3)}$$

$$\times \sum_{\nu=1}^{k} \sum_{j_1'+j_2'=j_4} (-1)^\nu (z_0^{-\nu-1})^{(j_1')} \sum_{|\tilde\kappa|_\nu=k}^{*} \sum_{|\tilde\lambda|_\nu=j_2'} z_{\tilde\kappa}^{(\tilde\lambda)}.$$

Here we denote the coefficients of a^j of $z_0^{-\nu}$ and $(dx_0/d\tilde x)^{-\nu}$ respectively by $(z_0^{-\nu})^{(j)}$ and $\big((dx_0/d\tilde x)^{-\nu}\big)^{(j)}$.

The above decomposition of $\Phi_k^{(j)}$ into three parts $\Phi_{k,i}^{(j)}$ $(i = 1, 2, 3)$ is made so that we may dominate each term in $\Phi^{(j)}_{k,i}$ by constants of the uniform form

$$c_i M^{(j)}_k, \tag{B.21}$$

where c_i and $M^{(j)}_k$ are described with the notations to be given later in the following manner:

$$c_1 = \delta_0/A, \tag{B.22}$$
$$c_2 = \delta_0, \tag{B.23}$$
$$c_3 = B/C, \tag{B.24}$$
$$M^{(j)}_k = k!(A\varepsilon^{-1})^k C(j) C^j \delta_0 M. \tag{B.25}$$

We also note that $\Phi^{(j)}_{1,1}$ is regarded to be 0 as a convention. As we discussed in the proof of Theorem 2.1, $\alpha^{(j)}_k$ and $x^{(j)}_k$ are determined by

$$\alpha^{(j)}_k = \left(z^{(0)}_0(0)\right)^{-1} \Phi^{(j)}_k(0) \tag{B.26}$$

$$x^{(j)}_k = \int_0^w \frac{2}{\tilde{w}} \left(\left(\frac{dx^{(0)}_0}{d\tilde{x}} \right)^{-2} z^{(0)}_0 \Phi^{(j)}_k(\tilde{w}) - \alpha^{(j)}_k \right) d\tilde{w}. \tag{B.27}$$

We now estimate the growth order of $x^{(j)}_k$ and $\alpha^{(j)}_k$ as j and k tend to infinity, by using the induction on the double index (j, k) appropriately ordered. Since we proved in Appendix A that $\sum_{j\geq 0} x^{(j)}_0(\tilde{x})a^j$ and $\sum_{j\geq 0} \alpha^{(j)}_0 a^j$ are convergent near the origin, we can find constants C_0, B and ρ so that the following relations (B.28) $\sim$ (B.31) hold:

(B.28)

$$\left\| x^{(0)}_0 \right\|_{[r]}, \left\| z^{(0)}_0 \right\|_{[r]}, \left\| \frac{dx^{(0)}_0}{d\tilde{x}} \right\|_{[r]}, \left\| \left(\frac{dx^{(0)}_0}{d\tilde{x}} \right)^{-1} \right\|_{[r]}, \left\| \left(z^{(0)}_0 \right)^{-1} \right\|_{[r]}$$
$$\leq C_0 C(0),$$

(B.29)

$$\|\tilde{x}(w)\|_{[r]}, \left\| \left(\frac{d\tilde{x}}{dw} \right)^{-1} \right\|_{[r]}, \sup_{|w|\le r, |a|\le\rho} \left| \left(\frac{dx_0}{d\tilde{x}} \right)^{-1} \right|,$$
$$\sup_{|w|\le r, |a|\le\rho} \left| (z_0)^{-1} \right|, \sup_{|a|\le\rho} |\alpha_0| \le C_0,$$

(B.30)

$$\left\| z_0^{(j)} \right\|_{[r]}, \left\| \frac{dx_0^{(j)}}{d\tilde{x}} \right\|_{[r]}, |\alpha_0^{(j)}|, \left\| \tilde{Q}_1^{(j)} \right\|_{[r]}, \left\| \tilde{R}_2^{(j)} \right\|_{[r]}, |\gamma^{(j)}| \le C_0 C(j) B^j,$$

(B.31)

$$\left\| \left(\left(\frac{dx_0}{d\tilde{x}} \right)^{-\nu} \right)^{(j)} \right\|_{[r]}, \left\| \left(z_0^{-\nu} \right)^{(j)} \right\|_{[r]} \le C_0^{\nu} C(j) B^j.$$

We now try to show that the following dominance relation (B.32.$k.j$) ($k \ge 1, j \ge 0$) holds for some constants A, C and δ_0 which satisfy (B.33) and (B.34) below:

(B.32.$k.j$)

$$\left\| x_k^{(j)} \right\|_{[r-\varepsilon]}, \left\| z_k^{(j)} \right\|_{[r-\varepsilon]}, \left\| \frac{dx_k^{(j)}}{dw} \right\|_{[r-\varepsilon]}, |\alpha_k^{(j)}| \le k!(A\varepsilon^{-1})^k C(j) C^j \delta_0$$

for any ε that satisfies (B.35) below:

(B.33) $$1 < \sqrt{A}\delta_0, \ 0 < \delta_0 \ll 1,$$

(B.34) $$0 < B \ll C,$$

(B.35) $$0 < \varepsilon < r/3.$$

We note that (B.32.1.0) is validated by (B.33) if we choose A sufficiently large.

Now we will confirm (B.32.$k.j$) for every (k, j) ($k \ge 1, j \ge 0$) by using the following induction procedure:

[I] We first confirm (B.32.$n.m$) by assuming that (B.32.$n'.m'$)$(0 \le m', 1 \le n' \le n-1)$ and (B.32.$n.m'$) $(0 \le m' \le m-1)$ are all validated,
and then
[II] we confirm (B.32.n.0) by assuming that (B.32.n'.0) $(0 \le n' \le n-1)$ are validated.
As we know (B.32.1.0) is valid for a sufficiently large A, these confirmations suffice for our purpose. To attain this goal we first note that application of Lemma A.2 to (B.16) entails the following relations:

$$\left\| x_k^{(j)} \right\|_{[r-\varepsilon]} \le 4(C_0C(0))^3 \left\| \Phi_k^{(j)} \right\|_{[r-\varepsilon]}, \tag{B.36}$$

$$\left\| \frac{dx_k^{(j)}}{dw} \right\|_{[r-\varepsilon]}, \left\| z_k^{(j)} \right\|_{[r-\varepsilon]} \le \frac{4}{r-\varepsilon}(C_0C(0))^3 \left\| \Phi_k^{(j)} \right\|_{[r-\varepsilon]}. \tag{B.37}$$

From (B.26) we also find

$$|\alpha_k^{(j)}| \le C_0C(0) \left\| \Phi_k^{(j)} \right\|_{[r-\varepsilon]}. \tag{B.38}$$

Thus it suffices for us to estimate $\Phi_k^{(j)}$ under the appropriate induction hypothesis.

Let us first consider the case [I]; we assume that (B.32.$n'.m'$) $(0 \le m', 1 \le n' \le n-1)$ and (B.32.$n.m'$) $(0 \le m' \le m-1)$ have been validated, and we try to prove the following estimates:

$$\left\| \Phi_{n,i}^{(m)} \right\|_{[r-\varepsilon]} \le c_i M_n^{(m)} \tag{B.39.i}$$

for $i = 1, 2, 3$. Here c_i and $M_n^{(m)}$ are given by (B.22) $\sim$ (B.25) with M in (B.25) being a constant independent of n, m, δ_0, C, A.

Before embarking on the estimation we note the following

Lemma B.4. *Suppose that* (B.32.$k.j$) *holds. Then we find*

$$\left\| \frac{d^2 x_k^{(j)}}{dw^2} \right\|_{[r-\varepsilon]}, \left\| \frac{dz_k^{(j)}}{dw} \right\|_{[r-\varepsilon]} \le e(k+1)!A^k \varepsilon^{-k-1} C(j) C^j \delta_0, \tag{B.40}$$

$$\left\| \frac{d^3 x_k^{(j)}}{dw^3} \right\|_{[r-\varepsilon]} \le e^2(k+2)!A^k \varepsilon^{-k-2} C(j) C^j \delta_0. \tag{B.41}$$

Proof. Let $\tilde{\varepsilon}$ denote $k\varepsilon/(k+1)$. Then (B.32.$k.j$) entails

$$\begin{aligned} \sup_{|w| \le r - \tilde{\varepsilon}} |z_k^{(j)}(w)| &\le k! A^k \tilde{\varepsilon}^{-k} C(j) C^j \delta_0 \\ &= k! A^k \left(1 + \frac{1}{k}\right)^k \varepsilon^{-k} C(j) C^j \delta_0 \\ &\le e k! A^k \varepsilon^{-k} C(j) C^j \delta_0, \end{aligned} \tag{B.42}$$

where $e = \exp(1)$. On the other hand, Cauchy's formula tells us

$$\frac{dz_k^{(j)}(w)}{dw} = \frac{1}{2\pi\sqrt{-1}} \int_{|\tilde{w} - w| = (k+1)^{-1}\varepsilon} \frac{z_k^{(j)}(\tilde{w})}{(\tilde{w} - w)^2} d\tilde{w}. \tag{B.43}$$

In view of the definition of $\tilde{\varepsilon}$ we find $\tilde{w}$ that appears in the above contour integral satisfies the following (B.44) for w with $|w| \le r - \varepsilon$:

$$\begin{aligned} |\tilde{w}| &\le |\tilde{w} - w| + |w| \\ &\le (k+1)^{-1}\varepsilon + r - \varepsilon \\ &= r - \tilde{\varepsilon}. \end{aligned} \tag{B.44}$$

Hence (B.42) shows (B.40) for $dz_k^{(j)}/dw$. The estimation of $d^2 x_k^{(j)}/dw^2$ and $d^3 x_k^{(j)}/dw^3$ can be done in exactly the same manner.

Q.E.D.

Remark B.1. For a holomorphic function $f(\tilde{x})$ of $\tilde{x}$ and a change of variables $\tilde{x} = \tilde{x}(w)$, the following relations hold for the differentiation of $f(\tilde{x})$ with respect to the two variables $\tilde{x}$ and w:

$$\text{(B.45)}\quad \frac{df}{d\tilde{x}}(\tilde{x}(w)) = \left(\frac{d\tilde{x}(w)}{dw}\right)^{-1} \frac{d}{dw} f(\tilde{x}(w)),$$

$$\text{(B.46)}\quad \frac{d^2 f}{d\tilde{x}^2}(\tilde{x}(w)) = \left(\frac{d\tilde{x}(w)}{dw}\right)^{-2} \frac{d^2}{dw^2} f(\tilde{x}(w)) + \frac{1}{2}\frac{d}{dw}\left(\frac{d\tilde{x}(w)}{dw}\right)^{-2} \frac{d}{dw} f(\tilde{x}(w)),$$

$$\text{(B.47)}\quad \frac{d^3 f}{d\tilde{x}^3}(\tilde{x}(w)) = \left(\frac{d\tilde{x}(w)}{dw}\right)^{-3} \frac{d^3}{dw^3} f(\tilde{x}(w)) + \frac{d}{dw}\left(\frac{d\tilde{x}(w)}{dw}\right)^{-3} \frac{d^2}{dw^2} f(\tilde{x}(w)) + \frac{1}{2}\left(\frac{d\tilde{x}(w)}{dw}\right)^{-1} \frac{d^2}{dw^2}\left(\frac{d\tilde{x}(w)}{dw}\right)^{-2} \frac{d}{dw} f(\tilde{x}(w)).$$

Since $(d\tilde{x}/dw)^{-1}$ satisfy (B.29) we obtain the following estimate from Cauchy's inequality:

$$\text{(B.48)}\qquad \left\| \frac{d^k}{dw^k}\left(\frac{d\tilde{x}(w)}{dw}\right)^{-l} \right\|_{[r-\varepsilon]} \le k!\varepsilon^{-k} \left\| \left(\frac{d\tilde{x}(w)}{dw}\right)^{-l} \right\|_{[r]} \le k!\varepsilon^{-k} C_0^l.$$

Using the relations (B.45) $\sim$ (B.47) and the estimate (B.48) we obtain the following inequalities:

$$\text{(B.49)}\qquad \left\| \frac{df}{d\tilde{x}}(\tilde{x}(w)) \right\|_{[r-\varepsilon]} \le C_0 \left\| \frac{d}{dw} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]},$$

$$\text{(B.50)}\qquad \left\| \frac{d^2 f}{d\tilde{x}^2}(\tilde{x}(w)) \right\|_{[r-\varepsilon]} \le C_0^2 \left\| \frac{d^2}{dw^2} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]} + \frac{\varepsilon^{-1}}{2} C_0^2 \left\| \frac{d}{dw} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]},$$

$$\left\| \frac{d^3 f}{d\tilde{x}^3}(\tilde{x}(w)) \right\|_{[r-\varepsilon]} \leq C_0^3 \left\| \frac{d^3}{dw^3} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]} + \varepsilon^{-1} C_0^3 \left\| \frac{d^2}{dw^2} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]} + \varepsilon^{-2} C_0^3 \left\| \frac{d}{dw} f(\tilde{x}(w)) \right\|_{[r-\varepsilon]}. \tag{B.51}$$

Then the following estimates immediately follow from the above inequalities (B.49) $\sim$ (B.51) and Lemma B.4 for $k \geq 1$:

$$\left\| \frac{dz_k^{(j)}}{d\tilde{x}} \right\|_{[r-\varepsilon]} \leq C_0 e (k+1)! A^k \varepsilon^{-k-1} C(j) C^j \delta_0, \tag{B.52}$$

$$\left\| \frac{d^l x_k^{(j)}}{d\tilde{x}^l} \right\|_{[r-\varepsilon]} \leq l C_0^l e^{l-1} (k+l-1)! A^k \varepsilon^{-k-l+1} C(j) C^j \delta_0 \ (l = 1, 2, 3). \tag{B.53}$$

For $k = 0$, we have the following estimates from (B.30) by the same discussion of Lemma B.4:

$$\left\| \frac{dz_0^{(j)}}{d\tilde{x}} \right\|_{[r-\varepsilon]} \leq e \varepsilon^{-1} C(j) B^j C_0^2, \tag{B.54}$$

$$\left\| \frac{d^l x_0^{(j)}}{d\tilde{x}^l} \right\|_{[r-\varepsilon]} \leq l e^{l-1} (l-1)! \varepsilon^{-l+1} C(j) B^j C_0^{l+1} \ (l = 1, 2, 3). \tag{B.55}$$

Remark B.2. Lemma B.4 explains the background reason of the asymmetry of the estimate of $|x_k^{(j)}|$ with respect to j and k; we dominate $|x_k^{(j)}|$ by C^{j+1} as j tends to infinity, whereas we include a much worse factor $k!$ to control their behavior as k tends to infinity. As the estimate

(B.64) below shows, the seemingly innocent term

$$-\frac{\tilde{x}}{2}\frac{d^3 x_{k-2}}{d\tilde{x}^3}\left(\frac{dx_0}{d\tilde{x}}\right)^{-1} \tag{B.56}$$

in (B.15) forces us to introduce the $k!$-factor for making the induction reasoning run smoothly. This observation indicates that the singular perturbative character of the problem in question originates mainly from the Schwarzian derivative multiplied by η^{-2} in (B.13).

Now we begin the estimation of $\Phi_{n,i}^{(m)}$ $(i = 1, 2, 3)$.

1) The estimation of $\Phi_{n,1}^{(m)}$.

First we estimate $\Phi_{n,1}^{(m)}$. The background of the expected form (B.39.1) is as follows: we observe that the sum of suffixes in each term that are relevant to η^{-1}, that is, the sum of k_p's, is $n-2$. Hence by using (B.32.$k.j$) we will encounter the factor A^{n-2} in the resulting estimate. Then (B.33) may be used to rewrite it as follows:

$$A^{n-2} = A^n A^{-2} < A^n A^{-1}\delta_0^2. \tag{B.57}$$

Thus we expect the extra factor A^{-1} in our estimation. Let us concretely check whether this argument really goes well. We shall estimate the first term of (B.18) for $n \geq 2$: By using (B.30), (B.31), induction hypothesis (B.32), (B.52) and (B.54) we have the following estimate:

(B.58)

$$\begin{aligned}
&\Bigg\| 2\sum_{k_1+k_2=n-2}\ \sum_{l_1+l_2+l_3+l_4=m} \gamma^{(l_1)}\frac{dz_{k_1}^{(l_2)}}{d\tilde{x}} \\
&\quad\times \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^{\nu}(z_0^{-\nu-1})^{(l_3)} \sum_{|\tilde{\kappa}|_\nu=k_2}^{*}\ \sum_{|\tilde{\lambda}|_\nu=l_4} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \Bigg\|_{[r-\varepsilon]} \\
&\leq 2\sum_{k_1+k_2=n-2}\ \sum_{l_1+l_2+l_3+l_4=m} C_0^3 C(l_1)B^{l_1} e(k_1+1)! A^{k_1}\varepsilon^{-k_1-1}C(l_2)C^{l_2}
\end{aligned}$$

$$\times \sum_{\nu=\min\{1,k_2\}}^{k_2} C_0^{\nu+1} C(l_3) B^{l_3} \sum_{|\tilde{\kappa}|_\nu = k_2}^{*} \sum_{|\tilde{\lambda}|_\nu = l_4} \tilde{\kappa}! (A\varepsilon^{-1})^{k_2} C(\tilde{\lambda}) C^{l_4} \delta_0^\nu .$$

Here we applied (B.52) to $dz_{k_1}^{(l_2)}/d\tilde{x}$ for $k_1 \geq 1$ with replacing δ_0 of (B.52) by C_0 in order to estimate $dz_{k_1}^{(l_2)}/d\tilde{x}$ $(k_1 \geq 1)$ and $dz_0^{(l_2)}/d\tilde{x}$ in the same form. Further, by applying Lemma B.3 to the summation on $l_1, \cdots, l_4$ and $\tilde{\lambda}$ and also by using (B.34), we find

(B.59)

$$\begin{aligned}
&2 \sum_{k_1+k_2=n-2} \sum_{l_1+l_2+l_3+l_4=m} C_0^3 C(l_1) B^{l_1} e (k_1+1)! A^{k_1} \varepsilon^{-k_1-1} C(l_2) C^{l_2} \\
&\quad \times \sum_{\nu=\min\{1,k_2\}}^{k_2} C_0^{\nu+1} C(l_3) B^{l_3} \sum_{|\tilde{\kappa}|_\nu = k_2}^{*} \sum_{|\tilde{\lambda}|_\nu = l_4} \tilde{\kappa}! (A\varepsilon^{-1})^{k_2} C(\tilde{\lambda}) C^{l_4} \delta_0^\nu \\
&\leq 2e C_0^4 C(m) C^m \varepsilon^{-n+1} A^{n-2} \\
&\quad \times \sum_{k_1+k_2=n-2} (k_1+1)! \sum_{\nu=\min\{1,k_2\}}^{k_2} (C_0\delta_0)^\nu \sum_{|\tilde{\kappa}|_\nu = k_2}^{*} \tilde{\kappa}! .
\end{aligned}$$

Then we obtain the following estimation from Lemma B.2:

(B.60)

$$\begin{aligned}
&2e C_0^4 C(m) C^m \varepsilon^{-n+1} A^{n-2} \\
&\quad \times \sum_{k_1+k_2=n-2} (k_1+1)! \sum_{\nu=\min\{1,k_2\}}^{k_2} (C_0\delta_0)^\nu \sum_{|\tilde{\kappa}|_\nu = k_2}^{*} \tilde{\kappa}! \\
&\leq 2e C_0^4 C(m) C^m \varepsilon^{-n+1} A^{n-2}
\end{aligned}$$

$$
\begin{aligned}
&\times \left((n-1)! + \sum_{\substack{k_1+k_2=n-2 \\ 1\le k_2}} (k_1+1)!k_2! \sum_{\nu=1}^{k_2} (C_0\delta_0)^\nu 4^{\nu-1} \frac{(k_2-\nu+1)!}{k_2!} \right) \\
&\le 2eC_0^4 C(m) C^m (A\varepsilon^{-1})^n \varepsilon A^{-2} \\
&\quad \times \left((n-1)! + \sideset{}{^*}\sum_{k_1'+k_2=n-1} k_1'!k_2! C_0\delta_0 \sum_{\nu=1}^{\infty} (4C_0\delta_0)^{\nu-1} \frac{1}{\nu!} \right) \\
&\le 2eC_0^4 C(m) C^m (A\varepsilon^{-1})^n \varepsilon A^{-2} \\
&\quad \times \left((n-1)! + C_0\delta_0 e^{4C_0\delta_0} \sideset{}{^*}\sum_{k_1'+k_2=n-1} k_1'!k_2! \right) \\
&\le 2eC_0^4 C(m) C^m (A\varepsilon^{-1})^n \varepsilon A^{-2} \left((n-1)! + 4C_0\delta_0 e^{4C_0\delta_0} (n-2)! \right).
\end{aligned}
$$

Consequently, since we can assume that δ_0 is sufficiently small as

$$
C_0\delta_0 e^{4C_0\delta_0} < 1, \tag{B.61}
$$

we obtain the following inequality from (B.57):

$$
\begin{aligned}
&\Bigg\| 2 \sum_{k_1+k_2=n-2} \sum_{l_1+l_2+l_3+l_4=m} \gamma^{(l_1)} \frac{dz_{k_1}^{(l_2)}}{d\tilde{x}} \\
&\quad \times \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^\nu (z_0^{-\nu-1})^{(l_3)} \sideset{}{^*}\sum_{|\tilde{\kappa}|_\nu=k_2} \sum_{|\tilde{\lambda}|_\nu=l_4} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \Bigg\|_{[r-\varepsilon]} \\
&\le n!(A\varepsilon^{-1})^n C(m) C^m \delta_0^2 A^{-1} 2eC_0^4 \varepsilon \left(\frac{1}{n} + \frac{4}{n(n-1)} \right).
\end{aligned} \tag{B.62}
$$

We find that similar estimates hold for other terms:

(B.63)

$$\Bigg\| \tilde{x} \sum_{k_1+k_2+k_3=n-2} \sum_{l_1+l_2+l_3+l_4+l_5=m} \gamma^{(l_1)} \frac{dz_{k_1}^{(l_2)}}{d\tilde{x}} \frac{dz_{k_2}^{(l_3)}}{d\tilde{x}}$$

$$\times \sum_{\nu=\min\{1,k_3\}}^{k_3} (-1)^\nu (\nu+1)(z_0^{-\nu-2})^{(l_4)} \sum^*_{|\tilde{\kappa}|_\nu=k_3} \sum_{|\tilde{\lambda}|_\nu=l_5} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \Bigg\|_{[r-\varepsilon]}$$

$$\leq \sum_{k_1+k_2+k_3=n-2} \sum_{l_1+l_2+l_3+l_4+l_5=m} C_0^6 C(l_1)C(l_2)C(l_3)A^{k_1+k_2}B^{l_1}C^{l_2+l_3}$$

$$\times e^2(k_1+1)!(k_2+1)!\varepsilon^{-k_1-k_2-2}$$

$$\times \sum_{\nu=\min\{1,k_3\}}^{k_3} (\nu+1)C_0^{\nu+2}C(l_4)B^{l_4} \sum^*_{|\tilde{\kappa}|_\nu=k_3} \sum_{|\tilde{\lambda}|_\nu=l_5} \tilde{\kappa}!(A\varepsilon^{-1})^{k_3}C(\tilde{\lambda})C^{l_5}\delta_0^\nu$$

$$\leq e^2 C_0^8 A^{-2}(A\varepsilon^{-1})^n C(m)C^m \left(\sum^*_{k_1'+k_2'=n} k_1'!k_2'! \right.$$

$$\left. + \sum^*_{k_1'+k_2'+k_3=n} k_1'!k_2'!k_3! \sum_{\nu=1}^{k_3} (\nu+1)(C_0\delta_0)^\nu 4^{\nu-1} \frac{(k_3-\nu+1)!}{k_3!} \right)$$

$$\leq e^2 C_0^8 A^{-2}(A\varepsilon^{-1})^n C(m)C^m$$

$$\times \left(4(n-1)! + 16(n-2)!C_0\delta_0 \sum_{\nu=1}^{\infty} (4C_0\delta_0)^{\nu-1} \frac{2}{(\nu-1)!} \right)$$

$$\leq n!(A\varepsilon^{-1})^n C(m)C^m \delta_0^2 A^{-1} \left(\frac{4}{n} + \frac{32}{n(n-1)} \right) C_0^8 e^2,$$

(B.64)
$$
\begin{aligned}
&\left\| \frac{\tilde{x}}{2} \sum_{k_1+k_2=n-2} \sum_{l_1+l_2+l_3=m} \frac{d^3 x_{k_1}^{(l_1)}}{d\tilde{x}^3} \right.\\
&\quad \left. \times \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^\nu \left(\left(\frac{dx_0}{d\tilde{x}} \right)^{-\nu-1} \right)^{(l_2)} \sum^{*}_{|\tilde{\kappa}|_\nu=k_2} \sum_{|\tilde{\lambda}|_\nu=l_3} \frac{dx_{\tilde{\kappa}}^{(\tilde{\lambda})}}{d\tilde{x}} \right\|_{[r-\varepsilon]}\\
&\leq \sum_{k_1+k_2=n-2} \sum_{l_1+l_2+l_3=m} \frac{3}{2} C_0^4 C(l_1) A^{k_1} C^{l_1} e^2 (k_1+2)! \varepsilon^{-k_1-2}\\
&\quad \times \sum_{\nu=\min\{1,k_2\}}^{k_2} C_0^{\nu+1} C(l_2) B^{l_2} \sum^{*}_{|\tilde{\kappa}|_\nu=k_2} \sum_{|\tilde{\lambda}|_\nu=l_3} C_0^\nu \tilde{\kappa}! (A\varepsilon^{-1})^{k_2} C(\tilde{\lambda}) C^{l_3} \delta_0^\nu\\
&\leq \frac{3}{2} e^2 C_0^5 C(m) C^m (A\varepsilon^{-1})^n A^{-2}\\
&\quad \times \left(n! + \sum^{*}_{k_1'+k_2=n} k_1'! k_2! C_0^2 \delta_0 \sum_{\nu=1}^{\infty} (4C_0^2\delta_0)^{\nu-1} \frac{1}{\nu!} \right)\\
&\leq \frac{3}{2} e^2 C_0^5 C(m) C^m (A\varepsilon^{-1})^n A^{-2} \left(n! + 4C_0^2 \delta_0 e^{4C_0^2\delta_0} (n-1)! \right)\\
&\leq n! (A\varepsilon^{-1})^n C(m) C^m \delta_0^2 A^{-1} \left(1 + \frac{4}{n} \right) \frac{3C_0^5 e^2}{2},
\end{aligned}
$$

(B.65)
$$
\begin{aligned}
&\left\| \frac{3}{4} \tilde{x} \sum_{k_1+k_2+k_3=n-2} \sum_{l_1+l_2+l_3+l_4=m} \frac{d^2 x_{k_1}^{(l_1)}}{d\tilde{x}^2} \frac{d^2 x_{k_2}^{(l_2)}}{d\tilde{x}^2} \right.\\
&\quad \left. \times \sum_{\nu=\min\{1,k_3\}}^{k_3} (-1)^\nu (\nu+1) \left(\left(\frac{dx_0}{d\tilde{x}} \right)^{-\nu-2} \right)^{(l_3)} \sum^{*}_{|\tilde{\kappa}|_\nu=k_3} \sum_{|\tilde{\lambda}|_\nu=l_4} \frac{dx_{\tilde{\kappa}}^{(\tilde{\lambda})}}{d\tilde{x}} \right\|_{[r-\varepsilon]}
\end{aligned}
$$

$$\leq \sum_{k_1+k_2+k_3=n-2} \sum_{l_1+l_2+l_3+l_4=m} 3C_0^5 C(l_1)C(l_2)A^{k_1+k_2}C^{l_1+l_2}$$

$$\times e^2(k_1+1)!(k_2+1)!\varepsilon^{-k_1-k_2-2} \sum_{\nu=\min\{1,k_3\}}^{k_3} (\nu+1)C_0^{\nu+2}C(l_3)B^{l_3}$$

$$\times \sum_{|\tilde{\kappa}|_\nu=k_3}^{*} \sum_{|\tilde{\lambda}|_\nu=l_4} C_0^\nu \tilde{\kappa}!(A\varepsilon^{-1})^{k_3}C(\tilde{\lambda})C^{l_4}\delta_0^\nu$$

$$\leq 3e^2C_0^7A^{-2}(A\varepsilon^{-1})^nC(m)C^m$$

$$\times \left(\sum_{k_1'+k_2'=n}^{*} k_1'!k_2'! + 2C_0^2\delta_0 e^{4C_0^2\delta_0} \sum_{k_1'+k_2'+k_3=n}^{*} k_1'!k_2'!k_3! \right)$$

$$\leq n!(A\varepsilon^{-1})^nC(m)C^m\delta_0^2A^{-1}3\left(\frac{4}{n}+\frac{32}{n(n-1)}\right)C_0^7e^2,$$

(B.66)

$$\left\| \delta_{n,2}\tilde{R}_2^{(m)}(z) \right\|_{[r-\varepsilon]} \leq \delta_{n,2}(A\varepsilon^{-1})^2C(m)C^m\delta_0^2A^{-1}C_0.$$

In the estimation of (B.64) and (B.65), we assumed that δ_0 is sufficiently small as

$$C_0^2\delta_0e^{4C_0^2\delta_0} < 1. \tag{B.67}$$

Since $n \geq 2$ and $A^{-1}, \varepsilon < 1$, we obtain (B.39.1).

The worst estimate in the above appears in (B.64) since no factor that weakens $n!$ is contained. This is the reason why (B.3) $\sim$ (B.5) must contain the factor $k!$.

2) The estimation of $\Phi_{n,2}^{(m)}$.

The appearance of the extra factor δ_0 in the estimate

$$\left\| \delta_{n,1}\tilde{Q}_1^{(m)}(z) \right\|_{[r-\varepsilon]} \leq A\varepsilon^{-1}C(m)C^m\delta_0^2C_0\varepsilon \tag{B.68}$$

is an immediate consequence of the assumption (B.33). To obtain this extra factor in the estimation of other terms of $\Phi_{n,2}^{(m)}$, we note each term in the summation contains two factors each of whose suffix k is greater than or equal to 1. It then follows from the induction hypothesis that we find the extra δ_0 factor. Let us confirm the estimation of the most complicated term in $\Phi_{n,2}^{(m)}$. Since $x_k^{(j)}, \alpha_k^{(j)}$ $(k \geq 1)$ and $x_0^{(j)}, \alpha_0^{(j)}$ respectively satisfy different type of estimation (B.32.$k.j$) and (B.30), we have to separate its summand depending on its suffix. However the procedure of its estimation is essentially the same with that of (B.58).

(B.69)

$$
\begin{aligned}
&\Bigg\| \sum_{\substack{k_1+k_2+k_3=n \\ 1\leq k_3 \leq n-1}} \sum_{l_1+l_2+l_3=m} \frac{dx_{k_1}^{(l_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(l_2)}}{d\tilde{x}} \sum_{k_1'+k_2'=k_3} \sum_{l_1'+l_2'+l_3'=l_3} \alpha_{k_1'}^{(l_1')} \\
&\quad \times \sum_{\nu=\min\{1,k_2'\}}^{k_2'} (-1)^\nu (z_0^{-\nu-1})^{(l_2')} {\sum}^{*}_{|\tilde{\kappa}|_\nu = k_2'} \sum_{|\tilde{\lambda}|_\nu = l_3'} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \Bigg\|_{[r-\varepsilon]} \\
&= \Bigg\| \sum_{l_1+l_2+l_3=m} \left({\sum_{k_1+k_2+k_3=n}}^{*} \frac{dx_{k_1}^{(l_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(l_2)}}{d\tilde{x}} + 2 {\sum_{k+k_3=n}}^{*} \frac{dx_0^{(l_1)}}{d\tilde{x}} \frac{dx_k^{(l_2)}}{d\tilde{x}} \right) \\
&\quad \times \sum_{l_1'+l_2'+l_3'=l_3} \left(\alpha_0^{(l_1')} \sum_{\nu=1}^{k_3} {\sum}^{*}_{|\tilde{\kappa}|_\nu = k_3} + \sum_{\substack{k_1'+k_2'=k_3 \\ 1\leq k_1'}} \alpha_{k_1'}^{(l_1')} \sum_{\nu=\min\{1,k_2'\}}^{k_2'} {\sum}^{*}_{|\tilde{\kappa}|_\nu = k_2'} \right) \\
&\quad \times (-1)^\nu (z_0^{-\nu-1})^{(l_2')} \sum_{|\tilde{\lambda}|_\nu = l_3'} z_{\tilde{\kappa}}^{(\tilde{\lambda})} \Bigg\|_{[r-\varepsilon]} \\
&\leq \sum_{l_1+l_2+l_3=m} C_0^2 C(l_1) C(l_2) \Bigg({\sum_{k_1+k_2+k_3=n}}^{*} C^{l_1+l_2} k_1! k_2! (A\varepsilon^{-1})^{k_1+k_2} \delta_0^2
\end{aligned}
$$

$$
\left. +2 \sum_{k+k_3=n}^{*} B^{l_1} C^{l_2} k! (A\varepsilon^{-1})^k \delta_0 \right)
$$

$$
\times \sum_{l'_1+l'_2+l'_3=l_3} \left(C_0 C(l'_1) B^{l'_1} \sum_{\nu=1}^{k_3} \sum_{|\tilde{\kappa}|_\nu = k_3}^{*} \right.
$$

$$
\left. + \sum_{\substack{k'_1+k'_2=k_3 \\ 1\le k'_1}} k'_1! (A\varepsilon^{-1})^{k'_1} C(l'_1) C^{l'_1} \delta_0 \sum_{\nu=\min\{1,k'_2\}}^{k'_2} \sum_{|\tilde{\kappa}|_\nu = k'_2}^{*} \right)
$$

$$
\times C_0^{\nu+1} C(l'_2) B^{l'_2} \sum_{|\tilde{\lambda}|_\nu = l'_3} \tilde{\kappa}! (A\varepsilon^{-1})^{|\tilde{\kappa}|_\nu} C(\tilde{\lambda}) C^{l'_3} \delta_0^\nu
$$

$$
\le (A\varepsilon^{-1})^n C(m) C^m C_0^2 \left(\sum_{k_1+k_2+k_3=n}^{*} k_1! k_2! \delta_0^2 + 2 \sum_{k+k_3=n}^{*} k! \delta_0 \right)
$$

$$
\times \left(C_0 \sum_{\nu=1}^{k_3} \sum_{|\tilde{\kappa}|_\nu = k_3}^{*} C_0^{\nu+1} \tilde{\kappa}! \delta_0^\nu \right.
$$

$$
\left. + \sum_{\substack{k'_1+k'_2=k_3 \\ 1\le k'_1}} k'_1! \delta_0 \sum_{\nu=\min\{1,k'_2\}}^{k'_2} \sum_{|\tilde{\kappa}|_\nu = k'_2}^{*} C_0^{\nu+1} \tilde{\kappa}! \delta_0^\nu \right)
$$

$$
\le (A\varepsilon^{-1})^n C(m) C^m C_0^2 \left(\sum_{k_1+k_2+k_3=n}^{*} k_1! k_2! \delta_0^2 + 2 \sum_{k+k_3=n}^{*} k! \delta_0 \right)
$$

$$
\times \left(C_0^3 k_3! \delta_0 \sum_{\nu=1}^{\infty} \frac{(4C_0\delta_0)^{\nu-1}}{\nu!} + k_3! C_0 \delta_0 \right.
$$

$$+\sum^*_{k_1'+k_2'=k_3} k_1'!k_2'!(C_0\delta_0)^2\sum_{\nu=1}^{\infty}\frac{(4C_0\delta_0)^{\nu-1}}{\nu!}\Bigg)$$

$$\le n!(A\varepsilon^{-1})^n C(m)C^m\delta_0^2$$

$$\times C_0^3\left(\frac{16\delta_0}{n(n-1)}+\frac{8}{n}\right)\left((C_0^2+4C_0\delta_0)e^{4C_0\delta_0}+1\right).$$

Similarly we can estimate the other terms as follows:

(B.70)

$$\left\|\frac{\tilde{x}}{4}\sum^*_{k_1+k_2=n}\sum_{l_1+l_2=m}\frac{dx_{k_1}^{(l_1)}}{d\tilde{x}}\frac{dx_{k_2}^{(l_2)}}{d\tilde{x}}\right\|_{[r-\varepsilon]}$$

$$\le\frac{C_0}{4}\sum^*_{k_1+k_2=n}\sum_{l_1+l_2=m}k_1!k_2!(A\varepsilon^{-1})^{k_1+k_2}C(l_1)C(l_2)C^{l_1+l_2}\delta_0^2C_0^2$$

$$\le n!(A\varepsilon^{-1})^nC(m)C^m\frac{\delta_0^2C_0^3}{n},$$

(B.71)

$$\Bigg\|\sum^*_{k_1+k_2=n}\sum_{l_1+l_2+l_3+l_4=m}\frac{dx_0^{(l_1)}}{d\tilde{x}}\frac{dx_0^{(l_2)}}{d\tilde{x}}\alpha_{k_1}^{(l_3)}$$

$$\times\sum_{\nu=1}^{k_2}\sum_{l_1'+l_2'=l_4}(-1)^\nu(z_0^{-\nu-1})^{(l_1')}\sum^*_{|\tilde{\kappa}|_\nu=k_2}\sum_{|\tilde{\lambda}|_\nu=l_2'}z_{\tilde{\kappa}}^{(\tilde{\lambda})}\Bigg\|_{[r-\varepsilon]}$$

$$\le\sum^*_{k_1+k_2=n}\sum_{l_1+l_2+l_3+l_4=m}C_0^2C(l_1)C(l_2)C(l_3)B^{l_1+l_2}k_1!(A\varepsilon^{-1})^{k_1}C^{l_3}\delta_0$$

$$\times\sum_{\nu=1}^{k_2}\sum_{l_1'+l_2'=l_4}C_0^{\nu+1}C(l_1')B^{l_1'}\sum^*_{|\tilde{\kappa}|_\nu=k_2}\sum_{|\tilde{\lambda}|_\nu=l_2'}\tilde{\kappa}!C(\tilde{\lambda})(A\varepsilon^{-1})^{k_2}C^{l_2'}\delta_0^\nu$$

$$\leq n!(A\varepsilon^{-1})^n C(m) C^m \delta_0^2 \frac{4C_0^4 e^{4C_0\delta_0}}{n}.$$

Therefore we obtain (B.39.2).

3) The estimation of $\Phi_{n,3}^{(m)}$.

To find the extra factor BC^{-1} in the estimate of each term in $\Phi_{n,3}^{(m)}$, we first note that the constant B is dominated by the inverse of the radius of convergence of z_0, α_0, etc. (cf. (B.30)) and that the constant C is relevant to the radius of convergence of z_m, α_m, etc. Hence we obtain this factor thanks to the fact that each term in the summation in $\Phi_{n,3}^{(m)}$ contains a factor that originates from the coefficient of $\eta^0 a^j$ ($j \geq 1$); for example, we find

(B.72)

$$\begin{aligned} &\left\| \sum_{\substack{l_1+l_2+l_3+l_4=m \\ l_3 \leq m-1}} \frac{dx_0^{(l_1)}}{d\tilde{x}} \frac{dx_0^{(l_2)}}{d\tilde{x}} \alpha_n^{(l_3)} (z_0^{-1})^{(l_4)} \right\|_{[r-\varepsilon]} \\ &\leq \sum_{\substack{l_1+l_2+l_3+l_4=m \\ l_3 \leq m-1}} C_0^3 C(l_1) C(l_2) C(l_3) C(l_4) B^{l_1+l_2+l_4} C^{l_3} n! (A\varepsilon^{-1})^n \delta_0 \\ &\leq n!(A\varepsilon^{-1})^n C(m) C^m \delta_0 \frac{B}{C} C_0^3, \end{aligned}$$

because $l_1+l_2+l_4 = m-l_3 \geq 1$ holds by the constraint of the range of indexes which is due to the fact that $\alpha_n^{(m)}$ is excluded in the summation. Similarly we find

$$\text{(B.73)} \qquad \left\| \sum_{k_1+k_2=n} \sum_{\substack{l_1+l_2+l_3+l_4=m \\ 1 \leq l_3}} \frac{dx_{k_1}^{(l_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(l_2)}}{d\tilde{x}} \alpha_0^{(l_3)} (z_0^{-1})^{(l_4)} \right\|_{[r-\varepsilon]}$$

$$= \left\| \sum_{\substack{l_1+l_2+l_3+l_4=m \\ 1\le l_3}} \left({\sum_{k_1+k_2=n}}^{*} \frac{dx_{k_1}^{(l_1)}}{d\tilde{x}} \frac{dx_{k_2}^{(l_2)}}{d\tilde{x}} \right. \right.$$

$$\left. \left. +2\frac{dx_0^{(l_1)}}{d\tilde{x}} \frac{dx_n^{(l_2)}}{d\tilde{x}} \right) \alpha_0^{(l_3)} (z_0^{-1})^{(l_4)} \right\|_{[r-\varepsilon]}$$

$$\le \sum_{\substack{l_1+l_2+l_3+l_4=m \\ 1\le l_3}} C_0^4 C(l_1)C(l_2)C(l_3)C(l_4)(A\varepsilon^{-1})^n$$

$$\times \left({\sum_{k_1+k_2=n}}^{*} C^{l_1+l_2} k_1! k_2! \delta_0^2 + 2B^{l_1}C^{l_2} n! \delta_0 \right) B^{l_3+l_4}$$

$$\le n!(A\varepsilon^{-1})^n C(m) C^m \delta_0 \frac{B}{C} C_0^4 \left(\frac{4\delta_0}{n} + 2 \right).$$

This time the condition that $l_3 \ge 1$ is due to the fact that $\alpha_0^{(0)}$ vanishes.

By the same reasoning we also find

(B.74)

$$\left\| \frac{\tilde{x}}{2} \sum_{\substack{l_1+l_2=m \\ l_2\le m-1}} \frac{dx_0^{(l_1)}}{d\tilde{x}} \frac{dx_n^{(l_2)}}{d\tilde{x}} \right\|_{[r-\varepsilon]}$$

$$\le \frac{C_0}{2} \sum_{\substack{l_1+l_2=m \\ l_2\le m-1}} C_0 C(l_1) B^{l_1} C_0 n! (A\varepsilon^{-1})^n C(l_2) C^{l_2} \delta_0$$

$$\le n!(A\varepsilon^{-1})^n C(m) C^m \delta_0 \frac{B}{C} \frac{C_0^3}{2},$$

(B.75)

$$\begin{aligned}&\left\| \sum_{\substack{l_1+l_2+l_3+l_4=m\\ 1\le l_3}} \frac{dx_0^{(l_1)}}{d\tilde{x}}\frac{dx_0^{(l_2)}}{d\tilde{x}}\alpha_0^{(l_3)}\right.\\ &\qquad\left.\times\sum_{\nu=1}^{n}\sum_{l_1'+l_2'=l_4}(-1)^{\nu}(z_0^{-\nu-1})^{(l_1')}\sum_{|\tilde{\kappa}|_\nu=n}^{*}\sum_{|\tilde{\lambda}|_\nu=l_2'} z_{\tilde{\kappa}}^{(\tilde{\lambda})}\right\|_{[r-\varepsilon]}\\ &\le \sum_{\substack{l_1+l_2+l_3+l_4=m\\ 1\le l_3}} C_0^3C(l_1)C(l_2)C(l_3)B^{l_1+l_2+l_3}\\ &\qquad\times\sum_{\nu=1}^{n}\sum_{l_1'+l_2'=l_4}C_0^{\nu+1}C(l_1')B^{l_1'}\sum_{|\tilde{\kappa}|_\nu=n}^{*}\sum_{|\tilde{\lambda}|_\nu=l_2'}\tilde{\kappa}!C(\tilde{\lambda})(A\varepsilon^{-1})^nC^{l_2'}\delta_0^{\nu}\\ &\le n!(A\varepsilon^{-1})^nC(m)C^m\delta_0\frac{B}{C}C_0^5e^{4C_0\delta_0}.\end{aligned}$$

Hence we obtain (B.39.3).

In conclusion, $\Phi_n^{(m)}$ satisfies the following inequality:

$$\left\|\Phi_n^{(m)}\right\|_{[r-\varepsilon]} \le n!(A\varepsilon^{-1})^nC(m)C^m\delta_0\left(\frac{\delta_0}{A}+\delta_0+\frac{B}{C}\right)M. \tag{B.76}$$

By taking δ_0 sufficiently small at first and then, A and C sufficiently large, we can assume that the following holds:

$$6r^{-1}(C_0C(0))^3M\left(\frac{\delta_0}{A}+\delta_0+\frac{B}{C}\right)<1. \tag{B.77}$$

Since $0<\varepsilon<r/3$, from (B.36) $\sim$ (B.38), (B.76) and (B.77), we obtain (B.32.$k.j$). Thus the induction proceeds in the case [I], and it remains to consider the case [II]; we are to confirm (B.32.n.0) under the assumption (B.32.k.0) $(1\le k\le n-1)$. But, we can readily confirm this fact by the same estimation as in the case [I]. Actually

$\Phi^{(0)}_{n,3}$ vanishes in this case, and the estimation is easier than before. Therefore we obtain (B.32.$k.j$) for every $k \geq 1$ and $j \geq 0$. Then by fixing $\varepsilon > 0$ and taking r_0 and A_0 in Theorem B.1 as $\min\{r-\varepsilon, C^{-1}\}$ and $A\varepsilon^{-1}$, respectively, we obtain Theorem B.1.

Q.E.D.

C Representation of the action of $\mathcal{X}$ as an integro-differential operator

Using the results obtained in Appendix B we now study how the microdifferential operator $\mathcal{X}$ constructed in Theorem 1.6 and Theorem 2.6 acts upon multi-valued analytic functions. Although the situation where this operator appears is different from the situation where its counterpart (also denoted by $\mathcal{X}$) appeared in [AKT4], their structures are essentially the same; the reasoning in [AKT4, Appendix C] applies to our case almost word for word. But, in order to make this paper self-contained, we describe the core part of the argument in this appendix. As the following reasoning indicates, the operator $\mathcal{X}$ constructed in Theorem 1.6 and that in Theorem 2.6 can be dealt with in exactly the same manner. In what follows we discuss the operator $\mathcal{X}$ constructed in Theorem 2.6 for the sake of definiteness. It then follows from (2.58) that it has the following form:

(C.1) $$\mathcal{X} =: \left(\frac{\partial g}{\partial x}\right)^{1/2} \left(1 + \frac{\partial r}{\partial x}\right)^{-1/2} \exp(r(x,a,\eta)\xi) :,$$

where

(C.2) $$r = r(x,a,\eta) = \sum_{k\geq 1} r_k(x,a)\eta^{-k}$$

(C.3) $$r_k = x_k(g(x,a),a)$$

and $g(x,a)$ is the inverse function of $x = x_0(\tilde{x}, a)$ given in (2.52), that is,

$$x = x_0(g(x,a), a). \tag{C.4}$$

Here $x_k (k \geq 0)$ is the function given in (2.5) and ξ stands for the symbol $\sigma(\partial/\partial x)$ of the differential operator $\partial/\partial x$. For the sake of convenience, we introduce $r_k^\dagger(x)$ by

$$\left(\frac{\partial g}{\partial x}\right)^{-1}\left(1 + \frac{\partial r}{\partial x}\right) = \sum_{k=0}^{\infty} r_k^\dagger(x,a)\eta^{-k}. \tag{C.5}$$

Then the coefficients $\{h_k\}_{k\geq 0}$ and $\{f_{l,k}\}_{1\leq l\leq k}$ in the expansion (C.6) and (C.7) below can be explicitly expressed in terms of $\{r_k\}$ and $\{r_k^\dagger\}$ as undermentioned in (C.8) and (C.9):

$$\left(\frac{\partial g}{\partial x}\right)^{1/2}\left(1 + \frac{\partial r}{\partial x}\right)^{-1/2} = \sum_{k=0}^{\infty} h_k(x,a)\eta^{-k}, \tag{C.6}$$

$$\exp(r(x,a,\eta)\xi) = 1 + \sum_{1\leq l\leq k} \eta^{-k}\xi^l f_{l,k}(x,a), \tag{C.7}$$

$$\begin{cases} h_0 = (r_0^\dagger)^{1/2}, \\ h_k = (r_0^\dagger)^{1/2} \displaystyle\sum_{l=1}^{k} \frac{(-1)^l\Gamma(l+\frac{1}{2})}{l!\Gamma(\frac{1}{2})} \sum_{|\tilde{\lambda}|_l = k}^{*} \frac{r_{\tilde{\lambda}}^\dagger}{(r_0^\dagger)^l} \ (k \geq 1), \end{cases} \tag{C.8}$$

and

$$f_{l,k} = \frac{1}{l!}\sum_{|\tilde{\lambda}|_l = k}^{*} r_{\tilde{\lambda}}. \tag{C.9}$$

Hence it follows from the definition (C.1) of $\mathcal{X}$ that its total symbol $\sigma(\mathcal{X})$ is written down as follows:

$$\sum_{k=0}^{\infty} \eta^{-k}\left(h_k + \sum_{k'=1}^{k}\sum_{l=1}^{k'} \xi^l h_{k-k'} f_{l,k'}\right). \tag{C.10}$$

As the parameter a does not play an important role in the following discussion, we omit to write a for the sake of simplicity.

Since r_k and $r_k^\dagger$ are respectively given by (C.3) and (C.5), Theorem B.1 and its proof tell us that there exist a neighborhood ω_1 of $(x,a) = (0,0)$ and a constant $C_0 > 0$ such that

$$\sup_{\omega_1} |r_k| \leq k! C_0^k \ (k = 1, 2, \cdots), \tag{C.11}$$

$$\sup_{\omega_1} |r_k^\dagger| \leq k! C_0^k \ (k = 1, 2, \cdots), \tag{C.12}$$

and

$$\max \left\{ \sup_{\omega_1} |r_0^\dagger|, \sup_{\omega_1} |(r_0^\dagger)^{-1}| \right\} \leq C_0. \tag{C.13}$$

Then it follows from Lemma B.2 that the following holds:

$$\begin{aligned} \sup_{\omega_1} |h_k| &\leq C_0^{1/2} \sum_{l=1}^{k} \frac{\Gamma(l+\frac{1}{2})}{l!\Gamma(\frac{1}{2})} \sum_{|\tilde{\lambda}|_l = k}^{*} \tilde{\lambda}! C_0^{k+l} \\ &\leq C_0^{k+1/2} \sum_{l=1}^{k} 4^{l-1} (k-l+1)! C_0^l \\ &\leq C_0^{3/2} k! C_0^k \sum_{l=1}^{k} \frac{4^{l-1} C_0^{l-1}}{(l-1)!} \\ &\leq C_0^{3/2} e^{4C_0} k! C_0^k \end{aligned} \tag{C.14}$$

for $k \geq 1$ and

$$\sup_{\omega_1} |f_{l,k}| \leq \frac{(k-l+1)!}{l!} 4^{l-1} C_0^k \ (1 \leq l \leq k). \tag{C.15}$$

Using these estimates together with Proposition C.1 below we obtain Theorem 2.7. Although the following Proposition C.1 is the same as Proposition C.1 in [AKT4], we include it here for the convenience of the reader.

Proposition C.1. *For a domain U in $\mathbb{C}_x$, let Ω denote*

$$\Omega = \{(x, y; \xi, \eta) \in T^*(U \times \mathbb{C}_y); \eta \neq 0\}, \tag{C.16}$$

and let $P = P(x, \partial/\partial x, \partial/\partial y)$ be a microdifferential operator of order 0 on Ω with the total symbol

$$\sigma(P) = \sum_{k=0}^{\infty} P_k(x, \eta^{-1}\xi)\eta^{-k}. \tag{C.17}$$

Here, we assume that each $P_k(x, \zeta)$ is an entire function of ζ and that the following growth order condition should hold: There exists a constant $C_0 > 0$ so that, for any compact subset K of $U \times \mathbb{C}_\zeta$, we can find another constant M_K satisfying

$$\sup_{(x,\zeta)\in K} |P_k(x, \zeta)| \leq M_K k! C_0^k \tag{C.18}$$

for $k = 0, 1, 2, \cdots$. Then, the action of P upon a (multi-valued) analytic function $\phi(x, y)$ is represented in the following form:

$$P\phi(x, y) = \int_{y_0}^{y} K(x, y - y', d/dx)\phi(x, y')dy', \tag{C.19}$$

where $K(x, y, d/dx)$ is a differential operator of infinite order that is defined on $\{(x, y); x \in U$ and $|y| < 1/C_0\}$ and y_0 is an arbitrarily chosen point that fixes the action of $(\partial/\partial y)^{-1}$ as an integral operator.

Although we omit the proof of Proposition C.1 and refer the reader to [AKT4] for it, we describe below how the differential operator K is expressed in terms of P_k: Let $a_{l,k}(x)$ denote the coefficient of ζ^l in the Taylor expansion of P_k, i.e.,

$$P_k(x, \zeta) = \sum_{l=0}^{\infty} a_{l,k}(x)\zeta^l. \tag{C.20}$$

Then we find
(C.21)

$$
\begin{aligned}
P\phi(x,y) &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} :\eta^{-k-l}a_{l,k}(x): \left(\frac{\partial}{\partial x}\right)^{l}\phi(x,y) \\
&= \int_{y_0}^{y}\left(\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} a_{l,k}(x)\frac{(y-y')^{k+l-1}}{(k+l-1)!}\left(\frac{\partial}{\partial x}\right)^{l}\right)\phi(x,y')dy'
\end{aligned}
$$

for some reference point y_0 that fixes the action of $:\eta^{-k-l}:$ upon $\phi(x,y)$. Hence the operator K should have the form

(C.22)
$$\sum_{l=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{l,k}(x)\frac{y^{k+l-1}}{(k+l-1)!}\right)\left(\frac{\partial}{\partial x}\right)^{l}$$

and our task is to show that

(C.23)
$$c_l(x,y) = \sum_{k=0}^{\infty} a_{l,k}(x)\frac{y^{k+l-1}}{(k+l-1)!}$$

enjoys the following property:

(C.24) For any compact subset K' of U, any constant r that is smaller than C_0^{-1} and any positive constant ε, there exists a constant M for which

$$\sup_{x\in K', |y|\le r} |c_l(x,y)| \le M\frac{\varepsilon^l}{(l-1)!}$$

holds for $l = 1, 2, \cdots$.

This fact can be confirmed by the assumption (C.18) (cf. [AKT4]).

In order to apply Proposition C.1 to the microdifferential operator $\mathcal{X}$ in question, we rewrite the total symbol (C.10) of $\mathcal{X}$ in the following manner:

(C.25)
$$\left(\sum_{j=0}^{\infty} h_j\eta^{-j}\right)\left(1+\sum_{1\le l\le k} f_{l,k}\eta^{-k}\xi^{l}\right)$$

$$= \left(\sum_{j=0}^{\infty} h_j\eta^{-j}\right)\left(1 + \sum_{k=0}^{\infty}\eta^{-k}\sum_{l=1}^{\infty} f_{l,l+k}(\eta\xi)^{-l}\right)$$
$$= \sum_{j=0}^{\infty} h_j\eta^{-j} + \sum_{j,k=0}^{\infty}\eta^{-(j+k)}h_j\sum_{l=1}^{\infty} f_{l,l+k}(\eta^{-1}\xi)^{l}$$
$$= \sum_{m=0}^{\infty}\eta^{-m}\left[h_m + \sum_{l=1}^{\infty}\left(\sum_{j+k=m} h_j f_{l,l+k}\right)(\eta^{-1}\xi)^l\right].$$

Thus, if we define $P_m(x,\zeta)$ by

$$P_m(x,\zeta) = h_m + \sum_{l=1}^{\infty}\left(\sum_{j+k=m} h_j f_{l,l+k}\right)\zeta^l, \tag{C.26}$$

we find that the total symbol of $\mathcal{X}$ has the form (C.17). Then (C.14) and (C.15) entail the following:

$$\begin{aligned} |P_m| &\le |h_m| + \sum_{l=1}^{\infty}\left(\sum_{\substack{j+k=m,\\ j,k\ge 0}} |h_j f_{l,l+k}|\right)|\zeta|^l \\ &\le C_0^{3/2}e^{4C_0}m!C_0^m \\ &\quad + \sum_{l=1}^{\infty}\left(\sum_{j+k=m} C_0^{3/2}e^{4C_0}\frac{j!(k+1)!}{l!}4^{l-1}C_0^{j+k+l}\right)|\zeta|^l. \end{aligned} \tag{C.27}$$

Then the application of Lemma B.2 shows that this is further dominated in the following way:

$$
\text{(C.28)} \quad C_0^{3/2} e^{4C_0} C_0^m \left[m! + \sum_{l=1}^{\infty} \frac{4^{l-1} C_0^l |\zeta|^l}{l!} \left(\sum_{\substack{j+\tilde{k}=m+1, \\ j \geq 0, \tilde{k} \geq 1}} j! \tilde{k}! \right) \right]
$$

$$
\begin{aligned}
&\leq C_0^{3/2} e^{4C_0} C_0^m \left[m! + \frac{1}{4} \sum_{l=1}^{\infty} \frac{(4C_0|\zeta|)^l}{l!} ((m+1)! + 4m!) \right] \\
&\leq C_0^{3/2} e^{4C_0} C_0^m (m+1)! \left(1 + \frac{5}{4} \sum_{l=1}^{\infty} \frac{(4C_0|\zeta|)^l}{l!} \right) \\
&= C_0^{3/2} e^{4C_0} \left(1 + \frac{5}{4} (e^{4C_0|\zeta|} - 1) \right) (m+1)! C_0^m.
\end{aligned}
$$

Therefore $P_m(x, \zeta)$ given by (C.26) is an entire function of ζ and it satisfies the growth order condition (C.18). Hence Proposition C.1 entails that the operator $\mathcal{X}$ is represented as in (C.19) with a differential operator K of infinite order. This completes the proof of Theorem 2.7.

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