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Random point fields revisited : Fock space associated with Poisson measures, fermion(determinantal) processes, and Gibbs measures

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Abstract: We prepare a general set-up of random point fields and introduce a densely defined operator T on the Fock space. Then, we will give a simple construction of the so-called Fock isomorphism associated with Poisson measures, an explicit formula between correlation functions and local density functions, and some applications to fermion(determinantal) and other processes. Furthermore, we will give the characterization of Gibbs measures by their Palm measures.

1 Introduction

The random point fields or point processes are the probability measures on the space of particle configurations. There are many famous classes of random point fields. The Poisson measure is the most typical, well investigated and widely applied. For instance, the equilibrium process consisting of infinitely many Markovian particles is a Poisson measure on the space of path configurations (cf., [SgT]) and its special case is the Sinai-Volkovisky ideal gass([SV]), the first infinite particle system for which the ergodicity was verified. The Gibbs measure is the basic notion indispensable to obtain rigorous results in classical statistical mechanics(cf., [R]). The fermion process or determinantal process(cf., [SrT1-3], [So]) is relatively new but includes interesting examples such as the Gaussian unitary ensemble(GUE) in random matrix theory(cf., [Me]), the zero distribution of certain Gaussian analytic functions(GAF)([HKPV]), etc. Also, the boson or permanental process and other processes are formulated and studies in [SrT1-3].

Although typical classes have been intensively studied, we have sometimes encountered with difficulties because of the lack of a general framework to study the random point fields and several problems are left open. The purpose of the present paper is to give such a framework and solve a few of the difficulties.

The first motivation originates in the use of the Fock space representation to compute the Dirichlet form of the symmetric equlibrium process([T3]) in order to discuss the scattering length ([K]) from the viewpoint of large deviation theory. It can be constructed through a lot of combinatorial arguments and so was hard to be applied to the computation, for instance, of Dirichlet forms although the Wiener-Ito decomposition of L^2-spaces on Wiener space is very useful in various situations. The reason is that the latter is not only a Fock space representation associated but gives the expansion of Wiener functionals by multiple Wiener-Ito integrals.

The Fock space associated with a Poisson measure, or Poissonian Fock space in short, seems to have a long history in physics and mathematics. It was known among specialists at least in 1960's soon after the Wiener-Ito decomposition was fully established by K.Ito and M.Nisio. The L^2-decomposition of functionals of Poisson processes was known to be constructed by applying Schmidt's orthogonalization method to polynomials of linear functionals via rather tedious combinatorial computations. Even more, T.Hida and N.Ikeda [HI] in 1967 developed a general theory for constructing the Fock spaces, including the one associated with Poissonian white noise, based on the characteristic functional by extending Aronszjan's work[A] on reproducing kernel Hilbert spaces and by using Bochner's theorem on nuclear spaces. But they mentioned nothing on naive combinatorial construction because such a Fock space representation was considered in lack of significant applications in those years.

It turns out that the introduction of a densely defined operator on the Fock space, denoted by T in Subsection 2.7, is the key point to the constuction of the so-called Fock isomorphism $\mathbf{I}$ ([T2]) stated in Section 4. Here the Fock space is realized as an L^2-space. The proof is simplified by using the fact that those functions on the Fock space called multiplicative in below are eigenfunctions of the operator T. The isomorphism $\mathbf{I}$ may not be so strong as the Wiener-Ito decomposition but is sufficient for certain computations. Some computation result will be published elsewhere ([T3]).

Unexpectedly, thus introduced operator T coincides with the wellknown operation to construct the correlation functions from local density functions in statistical mechanics. It has been given by a summation formula of multiple integrals with binomial coefficients([R], [HK]). We introduce the notion of the lift ξ_* of a configuration of particles ξ onto the configuration of finite configurations to simplify the arguments. Under certain natural conditions the operator T is invertible on its domain. Hence, we obtain an explicit formula between correlation functions and local density functions.

Moreover, the operator T also gives the derivatives of certain analytic functionals denoted by $\Phi_{\mathbf{f}}$ in Subsection 2.8. It then follows that T preserves certain classes of functions on the Fock space (up to constanst multipliers) which are called determinantal, permanental and α-determinantal in Section 5. For instance, a function on the Fock space is called determinantal if it is associated with the kernel of an integral operator K. Then its image under T is also determinantal but associated with integral operator $(I-K)^{-1}K$ with multiplier $\mathrm{Det}(I-K)$ if K is of trace class and $I-K$ is invertible. Thus, we may understand that such preservation by the operator T is the essence of the existence and uniqueness theorems in [SrT1-3].

Another motivation is a long standing (at least to the author) problem of the characterization of a Gibbs measure by its Palm measures. The Palm measure is one of the basic notions in random point fields which is a counterpart to the correlation function. The Palm measures of a Gibbs measure is absolutely continuous to the original Gibbs measure(stated as Theorem 47), as was pointed out in a lecture note [T0]. But the converse assertion is proved only for the restricted cases, including the lattices case. Thus, we could only studied the lattice case to prove that a fermion process is Gibbs in [SrT3] although the Palm measures of a fermion process are always fermion processes. The simple description by using operator T enables us to prove the converse assertion after some investigation on what should be the potential of the Gibbs measure. The converse assertion is true if the potential satisfies Conditions (C1) and (C2) in addition to the usual cocycle and other conditions (Theorem 50). This condition (C1) is natural in the sense that its sufficient conditions are the hard core condition and the sitewise bounded multiplicity conditions.

Contents

2 Preliminary on random point fields (or point processes)

2.1 Locally finite configuration space and finite configuration space

Let R be a Polish space and denote the space of continuous functions on R with compact support by $C_c(R)$. Its dual space is the space of Radon measures on R. We denote by $Q(R)$ the set of nonnegative integer-valued Radon measures on R. An element ξ of $Q(R)$ is called a locally finite configuration over R and is expressed as a finite or infinite sum of δ-measures

$$\xi = \sum_i \delta_{x_i} \qquad x_i \in R$$

where the number of i's such that $x_i \in \Lambda$ is finite for each compact subset Λ of R. For $f \in C_c(R)$ we write

$$\langle \xi, f \rangle = \sum_i f(x_i)$$

Notice that a configuration ξ may have multiple points, i.e., those points x with $\xi(x) \geq 2$. Thus, one may write

$$\langle \xi, f \rangle = \sum_{x \in R} \xi(x) f(x)$$

where $\xi(x)$ stands for the mass of ξ at x.

Also, we denote by $\hat{R}$ the set of finite configurations and for a subset Λ of R we write

$$\hat{\Lambda} = \{X \in \hat{R} : X(\Lambda) = X(R)\}.$$

For a compact subset Λ of R the set $\hat{\Lambda}$ is sometimes identified with $Q(\Lambda)$, the (necessarily finite) configuration space over Λ. An element $X \in \hat{R}$ is expressed as a finite sum of δ-measures and its total mass is denoted by $|X|$. The subset $R_n = \{X \in \hat{R} : |X| = n\}$ will be identified with the n-fold symmetric product of the space R and $\hat{R}$ with their union.

We write $X \le \xi$ if $X(x) \le \xi(x)$ for each $x \in R$ and then set

$$\binom{\xi}{X} = \prod_{x:X(x)>0} \binom{\xi(x)}{X(x)}$$

where the product over empty set is set to be 1, as usual. In other words, $\binom{\xi}{0} = 1$. Notice that $\binom{\xi}{X} = 1$ if ξ has no multiple point Also, if $X, Y \in \hat{R}$ and $Y \le X$, then

$$\binom{X}{Y} = \prod_{x \in R} \binom{X(x)}{Y(x)} = \binom{X}{X-Y}.$$

Definition 1. *For a $\xi \in Q(R)$ define the configuration ξ_* lifted onto $\hat{R}$ by*

$$\xi_* = \sum_{X \in \hat{R},\ X \le \xi} \binom{\xi}{X} \delta_X$$

and write

$$\langle \xi_*, \mathbf{f} \rangle = \sum_{X \in \hat{R},\ X \le \xi} \binom{\xi}{X} \mathbf{f}(X)$$

for a function $\mathbf{f}$ on $\hat{R}$ whenever the right hand side converges. Here δ_X stands for the delta measure on $\hat{R}$.

2.2 Convolution on $\mathbf{H}_0$ and invariant measure $\hat{\lambda}$

We need a space of test functions on $\hat{R}$.

Definition 2. *Let $\mathbf{H}_0$ be the set of functions $\mathbf{f}$ on $\hat{R}$ which satisfy the following conditions:*
(a)(compact support) $\mathbf{f}(X) = 0$ if $X(\Lambda^c) > 0$ for some compact subset Λ of R.
(b)(exponential growth) $\mid \mathbf{f}(X) \mid \le C^{|X|}$ for some positive constant C.

For a $\phi \in C_c(R)$ define $\hat{\phi} \in \mathbf{H}_0$ by

$$\hat{\phi}(X) = \prod_{x:X(x)>0} \phi(x)^{X(x)} = \prod_{i=1}^{n} \phi(x_i) \qquad \text{if} \qquad X = \sum_{i=1}^{n} \delta_{x_i}.$$

Such functions $\hat{\phi}$ are multiplicative with respect to the semigroup structure on $\hat{R}$:

$$\hat{\phi}(X+Y) = \hat{\phi}(X)\hat{\phi}(Y) \qquad X, Y \in \hat{R}.$$

Notice that $\hat{\phi} \in H_0$ if $\phi \in C_c(R)$.

Definition 3. *We define the convolution of functions* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *on* $\hat{R}$ *by*

$$\begin{aligned}\mathbf{f} * \mathbf{g}(X) &= \sum_{Y \leq X} \binom{X}{Y} \mathbf{f}(Y) \mathbf{g}(X - Y) \\ . &= \mathbf{g} * \mathbf{f}(X).\end{aligned}$$

Then, the binomial theorem shows that for ϕ, ψ on R

$$\hat{\phi} * \hat{\psi} = (\phi + \psi)^\wedge.$$

Let λ be a Radon measure on R. Then the n-fold product measure $\lambda^{\otimes n}$ on R^n defines a measure on R_n in a natural manner. Hence, we can define a Radon measure $\hat{\lambda}$ on $\hat{R}$ by

$$\hat{\lambda} = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{\otimes n}.$$

For simplicity, we write

$$\alpha(\phi) = \exp \int_R \lambda(dx) \phi(x).$$

Lemma 4. *(i) If* $\phi \in C_c(R)$*, then,* $\hat{\phi} \in \mathbb{H}_0$ *and*

$$\int_{\hat{R}} \hat{\lambda}(dX) \hat{\phi}(X) = \alpha(\phi)$$

(ii) If $\mathbf{f}, \mathbf{g} \in \mathbf{H}_0$*, then,*

$$\int_{\hat{R}} \hat{\lambda}(dX)(\mathbf{f} * \mathbf{g})(X) = \Big(\int_{\hat{R}} \hat{\lambda}(dX) \mathbf{f}(X) \Big) \Big(\int_{\hat{R}} \hat{\lambda}(dX) \mathbf{g}(X) \Big).$$

(iii) If $f, g, h \in \mathbf{H_0}$*, then,*

$$\int_{\hat{R}} \hat{\lambda}(dx) \int_{\hat{R}} \hat{\lambda}(dY) f(X) g(Y) h(X+Y) = \int_{\hat{R}} \hat{\lambda}(dZ)(f * g)(Z) h(Z)$$

Proof. It is immediate to see (i) from the definitions. It suffices to prove (ii) only when $\mathbf{f} = \hat{\phi}$ and $\mathbf{g} = \hat{\psi}$ with $\phi, \psi \in C_c' R)$. But then,

$$\begin{aligned}\int_{\hat{R}} \hat{\lambda}(dX)(\hat{\phi} * \hat{\psi})(X) = &\int_{\hat{R}} \hat{\lambda}(dX)(\phi + \psi)^\wedge(X) \\ =&\alpha(\phi + \psi)(x) = \alpha(\phi)\alpha(\psi).\end{aligned}$$

The assertion (iii) immediately follows if $f = \hat{\phi}, g = \hat{\psi}$ and $h = \hat{\theta}$ with $\phi, \psi, \theta \in C_c(R)$. Since such functions span $\mathbf{H_0}$, the rest of the proof is a routine work. □

2.3 The lift ξ_* and exponential function $\mathbf{e_f}$

For a given $f \in C_c(R)$ denote

$$\mathbf{e}_f = \hat{\phi} \qquad \text{with} \qquad \phi(x) = e^{-f(x)} - 1.$$

We call such functions exponential.

Lemma 5. *(i) There holds the identity*

$$\langle \xi_*, \mathbf{e}_f \rangle = e^{-\langle \xi, f \rangle}$$

for any $f \in C_c(R)$.
(ii) For $\phi, \psi \in C_c(R)$

$$\langle \xi_*, \hat{\phi} \rangle \langle \xi_*, \hat{\psi} \rangle = \langle \xi_*, (\phi\psi + \phi + \psi)^\wedge \rangle.$$

Proof. The assertion (i) is nothing but the expansion formula for the product $\prod_{i=1}^n a_i = \prod_{i=1}^n (1 + (a_i - 1))$. Applying (i) to $\phi = e^{-f} - 1$ and $\psi = e^{-g} - 1$ we obtain

$$\langle \xi_*, \hat{\phi} \rangle \langle \xi_*, \hat{\psi} \rangle = e^{-\langle \xi, f+g \rangle} = \langle \xi_*, \mathbf{e}_{f+g} \rangle = \langle \xi_*, (\phi\psi + \phi + \psi)^\wedge \rangle$$

since $e^{-f-g} - 1 = \phi\psi + \phi + \psi$. The rest is a routine work. □

2.4 Poisson measure

The following theorem is wellkown.

Theorem 6. *Let λ be a nonnegative Radon measure on R. Then there exists a unique probability Borel measure π_λ on the locally finite configuration space $Q(R)$ over R whose Laplace transform is given by*

$$\int_{Q(R)} \pi_\lambda(d\xi) e^{-\langle \xi, f \rangle} = \exp \int_R (e^{-f(x)} - 1) \lambda(dx)$$

for any $f \in C_c(R)$.

Definition 7. *The probability measure π_λ is called the Poisson measure with intensity λ.*

Here we omit the proof of the above theorem but note that, if *suppf* $\subset \Lambda$, then

$$\begin{aligned} &\exp \int_R (e^{-f(x)} - 1) \lambda(dx) \\ =&\exp(\int_\Lambda e^{-f(x)} \lambda(dx) - \lambda(\Lambda)) \\ =&e^{-\lambda(\Lambda)} \int_{\hat{\Lambda}} (e^{-f})^\wedge(X) \hat{\lambda}(dX). \end{aligned}$$

The last hand side is an integral with respect to the probability measure $e^{-\lambda(\Lambda)}\hat{\lambda}(dX)$ on $\hat{\Lambda}$, which is nothing but the Poisson measure restricted to $Q(\Lambda)$.

Recall that the simplest examples of Poisson measures are the cases where
(a) $R = \mathbb{R}^1$ and $\lambda(dx) = cdx$ with $c > 0$.
(b) $R = \mathbb{Z}^1$ and λ is the counting measure.
So we consider the measure λ which may have both atomic and continuous parts although λ is assumed nonatomic in previous papers [SgT, T0, T2]. We had thought that the nonatomic case is simpler but it is not from the viewpoint of algebraic structure on the configuration spaces. Anyway, the nonatomic case can be recovered under a slight modification.

Also we should note here that the intensity λ can be a σ-finite measure. Indeed, then there exists an countable increasing sequence of Borel subsets Λ_n with $\lambda(\Lambda_n) < \infty$ and $\bigcup \Lambda_n = R$. Then the Poisson measure $e^{-\lambda(\Lambda_n)}\hat{\lambda}(dX)$ on $Q(\Lambda_n)$ converges as $n \to \infty$ to the desired Poisson measures. Since this generalization can easily be done if necessary, we will restrict ourselves to the case where λ is a nonnegative Radon measure in the present paper.

2.5 L^2-realization of the Fock space

Our arguments will start from the L^2-realization of the Fock space stated as Lemma 8 below which may considered as the definition of our Fock space. But we should begin with the usual definition of the abstract symmetric (or boson) Fock space.

Given a Hilbert space H with norm $|\cdot|_H$, denote the symmetric n-fold tensor product of H by

$$H_n = H \otimes_s H \otimes_s \cdots \otimes_s H \qquad (n\text{-}fold)$$

for $n \geq 1$ and $H_0 = \mathbb{C}$ where $\otimes_s$ stands for the symmetric tensor product. Then the Fock space $\mathbf{H}$ is defined as the direct Hilbert sum of the Hilbert spaces $H_n,\ n \geq 0$ with weighted norms $\frac{1}{\sqrt{n!}}|\cdot|_{H_n}$. Thus, an element $\mathbf{f}$ of $\mathbf{H}$ is given by a sequence $(f_n)_{n\geq 0}$ where $f_n \in H_n,\ n \geq 0$ and

$$\|\mathbf{f}\|_{\mathbf{H}} = \Big(\sum_{n=0}^{\infty} \frac{1}{n!} \mid f_n \mid^2_{H_n} \Big)^{\frac{1}{2}} < \infty.$$

Now let λ be a nonnegative Radon measure on R and take $H = L^2(R, \lambda)$. Then, the Hilbert space H_n is isomorphic to the subspace $L^2_{sym}(R^n, \otimes^n \lambda)$ consisting of symmetric square integrable functions on R^n with respect to the n-fold direct product measure $\otimes^n\lambda$. The weight $\frac{1}{\sqrt{n!}}$ put on the Hilbert norm $|\cdot|_{H_n}$ fits for the factor $\frac{1}{n!}$ in the definition of the measure $\hat{\lambda}$ and we obtain the following L^2-realization.

Lemma 8. *If $H = L^2(R, \lambda)$, the Fock space $\mathbf{H}$ is isomorphic to the space $L^2(\hat{R}, \hat{\lambda})$.*

Henceforth, we identify $\mathbf{H}$ with $L^2(\hat{R}, \hat{\lambda})$. Thus, $\mathbf{f} \in \mathbf{H}$ is a function on $\hat{R}$ which is square integrable with respect to $\hat{\lambda}$ and its norm is given by

$$\|\mathbf{f}\|_{\mathbf{H}} = \Big(\int_{\hat{R}} | \, \mathbf{f}(X)|^2 \hat{\lambda}(dX) \Big)^{\frac{1}{2}}.$$

We will sometimes denote the restriction of $\mathbf{f}$ to R_n by f_n and write $\mathbf{f} = (f_n)_{n \geq 0}$.

2.6 Operators S and S^{-1}

For a function $\mathbf{f}$ on $\hat{R}$ define

$$S\mathbf{f}(X) = \sum_{Y \leq X} \binom{X}{Y} \mathbf{f}(Y).$$

For instance, it is immediate to see that

$$S\hat{\phi}(X) = (\phi + 1)^{\wedge}(X).$$

Lemma 9. *The operator S is invertible and*

$$S^{-1}\mathbf{f}(X) = \sum_{Y \leq X} (-1)^{|X|-|Y|} \binom{X}{Y} \mathbf{f}(Y).$$

Proof. The invertibility of S follows from the inclusion-exclusion formula for finite sets. □

Notice that for an $X \in \hat{R}$ and an $\mathbf{f} \in \mathbf{H}_0$,

$$\langle X_*, \mathbf{f} \rangle = S\mathbf{f}(X).$$

Furthermore, if we introduce an involution operator J (i.e., $J^2 = I$) by

$$J\mathbf{f}(X) = (-1)^{|X|}\mathbf{f}(X) \qquad X \in \hat{R}$$

then, $S^{-1} = JSJ$.

Finally, we notice that S and S^{-1} map the space of functions with exponential growth on $\hat{R}$ to itself but they are not operators on $\mathbf{H}_0$.

2.7 Operators T and T^{-1}

From now on we fix a nonnegative Radon measure λ on R and keep in mind the three L^2-spaces

$$H = L^2(R, \lambda), \qquad \mathbf{H} = L^2(\hat{R}, \hat{\lambda}) \qquad \text{and} \qquad \mathcal{H} = L^2(Q(R), \pi_\lambda).$$

Recall that the space $\mathbf{H}_0$ is dense in $\mathbf{H}$ and that the space $\mathbf{H}$ is an L^2-realization of the symmetric(or boson) Fock space over H.

Lemma 10. *For a function* $\mathbf{f} \in \mathbf{H}_0$ *define*

$$T\mathbf{f}(X) = \int_{\hat{R}} \mathbf{f}(X+Y)\hat{\lambda}(dY).$$

Then, T *defines an invertible operator from* $\mathbf{H}_0$ *to itself and*

$$T^{-1}\mathbf{f}(X) = \int_{\hat{R}} (-1)^{|Y|}\mathbf{f}(X+Y)\hat{\lambda}(dY).$$

Moreover, T *and* T^{-1} *are formally adjoint operators of* S *and* S^{-1} *in* $\mathbf{H}$*, respectively.*

Proof. The point of the proof is the following observation: for $\phi \in C_c(R)$ the mulitiplicativity of $\hat{\phi}$ implies

$$T\hat{\phi}(X) = \alpha(\phi)\hat{\phi}(X) \qquad \text{and} \qquad \alpha(\phi) = T\hat{\phi}(0).$$

It then follows that for $\phi, \psi \in C_c(R)$

$$\begin{aligned} \langle T\hat{\phi}, \hat{\psi}\rangle_{\mathbf{H}} &= \alpha(\phi)\alpha(\phi\psi) = \alpha(\phi(\psi+1)) \\ &= \langle \hat{\phi}, S\hat{\psi}\rangle_{\mathbf{H}}. \end{aligned}$$

Thus, formally, $T = S^*$ in $\mathbf{H}$. It then follows that T is invertible and $T^{-1} = JTJ$ takes the desired form. □

2.8 A characterization of the operator T

From the viewpoint of functional analysis the operator T can be characterized as the derivative of a certain functional. This viewpoint will bring us several interesting applications stated in Section 5. We denote the space of bounded continuous functions on $\hat{R}$ by $C_b(R)$.

Theorem 11. *For a given* $\mathbf{f} \in \mathbf{H}_0$ *consider the functional*

$$\Phi_{\mathbf{f}}(\phi) = \int_{\hat{R}} \hat{\lambda}(dX)\mathbf{f}(X)\hat{\phi}(X), \qquad \phi \in C_b(R).$$

Then, it is an analytic functional and its Taylor expansion at $\phi = 1$ *is given by*

$$\Phi_{\mathbf{f}}(1+\psi) = \Phi_{T\mathbf{f}}(\psi) = \int_{\hat{R}} \hat{\lambda}(dX)T\mathbf{f}(X)\hat{\psi}(X), \qquad \psi \in C_b(R).$$

Proof.

$$\begin{aligned} \Phi_{\mathbf{f}}(1+\psi) &= \int_{\hat{R}} \hat{\lambda}(dX)\mathbf{f}(X)(1+\psi)^{\wedge}(X) \\ &= \int_{\hat{R}} \hat{\lambda}(dX)\mathbf{f}(X)S\hat{\psi}(X) \\ &= \int_{\hat{R}} \hat{\lambda}(dX)T\mathbf{f}(X)\hat{\psi}(X) = \Phi_{T\mathbf{f}}(\psi). \end{aligned}$$

□

3 Correlation functions and Palm measures

3.1 Correlation functions

The correlation function is one of the key notions in the theory of random point fields or point processes. Here we restrict ourselves to treat correlation functions of exponential growth. Fix a nonnegative Radon measure λ on R as a reference measure.

Definition 12. *A probability Borel measure μ on the locally finite configuration space $Q(R)$ is said to admit the correlation function $\boldsymbol{\rho}$ with respect to λ (precisely, $\hat{\lambda}$) if the following two conditions are satisfied:*
(a)(exponential growth) $\boldsymbol{\rho}$ is a continuous function on $\hat{R}$ and there exists a positive constant C such that

$$\boldsymbol{\rho}(X) \leq C^{|X|} \text{ for any } X \in \hat{R}.$$

(b) (correlation equation)

$$\int_{Q(R)} \mu(d\xi)\langle \xi_*, \mathbf{f}\rangle = \int_{\hat{R}} \hat{\lambda}(dX)\rho(X)\mathbf{f}(X)$$

for any $\mathbf{f} \in \mathbf{H}_0$. The condition (b) may be written in a symbolical manner as

$$\Big(\int_{Q(R)} \mu(d\xi)\xi_*\Big)(dX) = \hat{\lambda}(dX)\boldsymbol{\rho}(X), \qquad dX \subset \hat{R}.$$

The restriction $\boldsymbol{\rho}_n$ of $\boldsymbol{\rho}$ to R_n is the so-called n-point correlation function.

Example 13. *For $\mu = \pi_\lambda$, the correlation function is given by $\boldsymbol{\rho}(X) = 1$ for any $X \in \hat{R}$.*

Indeed, for any $f \in C_c(R)$,

$$\begin{aligned}
&\int_{Q(R)} \pi_\lambda(d\xi)\langle \xi_*, \mathbf{e}_f\rangle \\
=&\int_{Q(R)} \pi_\lambda(d\xi)e^{-\langle \xi, f\rangle} \\
=&\exp\int_R (e^{-f(x)} - 1)\lambda(dx) \\
=&\int_{\hat{R}} \mathbf{e}_f(X)\hat{\lambda}(dX).
\end{aligned}$$

3.2 Palm measures

The Palm measure is another key notion which is a counterpart of the correlation function. Recall that the Palm measures(or Palm-Khintchin measures or

Kendall measures) $\mu^x,\ x \in R$ of a given probability Borel measure μ on $Q(R)$ is usually defined as the measures which satisfy the Palm formula

$$\int_{Q(R)} \mu(d\xi) \int_R \xi(dx) u(x,\xi) = \int_R \lambda(dx)\rho_1(x) \int_{Q(R)} \mu^x(d\xi) u(x, \xi + \delta_x)$$

for any continuous functions $u(x,\xi)$ which has compact support in x if μ admits the one-point correlation function ρ_1. Indeed, the existence of ρ_1 implies that the left hand side defines a measure on $Q(R) \times R$ whose marginal on R is absolutely continuous with respect to λ. Then the Radon-Nykodim derivative gives the Palm measures by a standard argument on regular conditional measures or by a disintegration theorem.

Moreover, if, in addition, μ admits the two-point correlation function ρ_2, then, the Palm measures μ^x admit their Palm measures and there holds the identity $(\mu^x)^y = (\mu^y)^x$ for almost all x and y. Thus the second Palm measure can safely be written as $\mu^{x,y} = (\mu^x)^y = (\mu^y)^x$. Similarly, one can define higher order Palm measure $\mu^{x_1,\cdots,x_n}$ for $n \geq 3$ if the n-point correlation function ρ_n exists.

Thus we employ the following.

Definition 14. *Let μ be a probability Borel measure on the locally finite configuration space $Q(R)$ and assume that it admits the correlation function $\boldsymbol{\rho}$ on $\hat{R}$ with respect to the meaure $\hat{\lambda}$. In other words, assume that it admits all the n-point correlation functions $\rho_n,\ n \geq 1$. We call the probability Borel measures $\mu^X,\ X \in \hat{R}$ Palm measures if they satisfy the formula*

$$\int_{Q(R)} \mu(d\xi) \int_{\hat{R}} \xi_*(dX) F(X,\xi) = \int_{\hat{R}} \hat{\lambda}(dX) \boldsymbol{\rho}(dX) \int_{Q(R)} \mu^X(d\xi) F(X, \xi + X)$$

for any continuous functions $F(X,\xi)$ on $\hat{R} \times Q(R)$ whenever it is compactly supported in X in the sense that there exists a compact subset Λ such that $F(X,\xi) = 0$ if $X(\Lambda^c) > 0$. We call the above formula generalized Palm formula.

Here we must remark that the Palm measure is defined in some literatures so that the variable $\xi + X$ is replaced by ξ in the above formula. But our definition often simplifies the description.

Example 15. *For $\mu = \pi_\lambda$, the Palm measures coincide with the original measure:*

$$(\pi_\lambda)^X = \pi_\lambda.$$

Indeed, for any $f, g \in C_c(R)$,

$$\begin{aligned}
&\int_{Q(R)} \pi_\lambda(d\xi)\langle \xi_*, \mathbf{e}_f\rangle e^{-\langle \xi, g\rangle} \\
=&\int_{Q(R)} \pi_\lambda(d\xi) e^{-\langle \xi, f+g\rangle} \\
=&\exp \int_R (e^{-f(x)-g(x)} - 1)\lambda(dx) \\
=&\exp\Big(\int_R (e^{-f(x)} - 1)e^{-g(x)}\lambda(dx) + \int_R (e^{-g(x)} - 1)\lambda(dx)\Big) \\
=&\int_{\hat{R}} \mathbf{e}_f(X)(e^{-g})^\wedge(X)\hat{\lambda}(dX) \int_{Q(R)} e^{-\langle \xi, f\rangle}\pi_\lambda(d\xi) \\
=&\int_{\hat{R}} \mathbf{e}_f(X)\hat{\lambda}(dX) \int_{Q(R)} e^{-\langle \xi+X, g\rangle}\pi_\lambda(dx).
\end{aligned}$$

It is also wellknown that the converse assertion holds: if μ has intensity λ and satisfies $\mu^x = \mu$ for any x, then $\mu = \pi_\lambda$.

Indeed, then, for $f \in C_c(R)$, $f \geq 0$ and $t \geq 0$

$$\begin{aligned}
&\frac{d}{dt}\int_{Q(R)} \mu(d\xi) e^{-\langle \xi, tf\rangle} \\
=&-\int_{Q(R)} \mu(d\xi)\langle \xi, f\rangle e^{-\langle \xi, tf\rangle} \\
=&-(\int_R \lambda(dx) f(x) e^{-tf(x)}) \int_{Q(R)} \mu(d\xi) e^{-\langle \xi, tf\rangle}
\end{aligned}$$

because $\mu^x = \mu$. Solving this differential equation, we find that

$$\int_{Q(R)} \mu(d\xi)\rangle e^{-\langle \xi, f\rangle} = \exp \int_R (e^{-f(x)} - 1)\lambda(dx)$$

at $t = 1$.

3.3 Correlation functions and Palm measures of a Palm measure

The correlation functions of a Palm measure satisfies the following "chain rule".

Theorem 16. *Let μ be a probability Borel measure on the locally finite configuration space $Q(R)$ and assume that it admits the correlation function $\boldsymbol{\rho}$ on $\hat{R}$ with respect to the measure $\hat{\lambda}$. Denote its Palm measures by μ^X.*
(i) Then for $\hat{\lambda} -$ a.e. X the Palm measure μ^X admit correlation function $\boldsymbol{\rho}^X$ and there hold the identity

$$\boldsymbol{\rho}(X)\boldsymbol{\rho}^X(Y) = \boldsymbol{\rho}(X+Y) \qquad \hat{\lambda} - a.e.\ X, Y \in \hat{R}$$

(ii) The Palm measures $(\mu^X)^Y$ of μ^X exists and satisfy the relation

$$(\mu^X)^Y = (\mu^Y)^X = \mu^{X+Y} \quad \hat{\lambda} - a.e.\ X,\ Y \in \hat{R}$$

whenever $\boldsymbol{\rho}(X+Y) > 0$ (and then, automatically, $\boldsymbol{\rho}(X) > 0$ and $\boldsymbol{\rho}(Y) > 0$).

Proof. Recall Lemma (iii), and observe this

$$e^{-f-g} - 1 = (e^{-f} - 1) + e^{-f}(e^{-g} - 1)$$

and so

$$\boldsymbol{e}_{f+g} = \boldsymbol{e}_f * (\boldsymbol{e}_g(e^{-f})^\wedge).$$

Hence

$$\int \mu(d\xi) e^{-\langle \xi, f+g\rangle} = \int \mu(d\xi)\langle \xi_*, \boldsymbol{e}_{f+g}\rangle = \int \hat{\lambda}(dX)\boldsymbol{\rho}(X)\boldsymbol{e}_{f+g}(X)$$

$$= \int \hat{\lambda}(dX)\boldsymbol{\rho}(X)(\boldsymbol{e}_f * (\boldsymbol{e}_g(e^{-f})^\wedge)(X)$$

$$= \int \hat{\lambda}(dX) \int \hat{\lambda}(dY)\boldsymbol{\rho}(X+Y)\boldsymbol{e}_f(Y)\boldsymbol{e}_g(X)(e^{-f})^\wedge(X).$$

On the other hand,

$$\int \mu(d\xi) e^{-\langle \xi, f+g\rangle} = \int \mu(d\xi)\langle \xi_*, \boldsymbol{e}_g\rangle e^{-\langle \xi, -f\rangle}$$

$$= \int \hat{\lambda}(dX)\boldsymbol{\rho}(X)\boldsymbol{e}_g(X) \int \mu^x(d\xi) e^{-\langle \xi + X, f\rangle}$$

$$= \int \hat{\lambda}(dX)\boldsymbol{\rho}(X)\boldsymbol{e}_g(X)(e^{-f})^\wedge(X) \int \mu^\lambda(d\xi)\langle \xi_*, \boldsymbol{e}_f\rangle$$

Consequently,

$$\int \hat{\lambda}(dY)\boldsymbol{\rho}(X+Y)\boldsymbol{e}_f(Y) = \boldsymbol{\rho}(X) \int \mu^X(d\xi)\langle \xi_*, \boldsymbol{e}_f\rangle$$

which shows that the correlation functions $\boldsymbol{\rho}^X$ of μ^X exist and satisfy

$$\boldsymbol{\rho}(X+Y) = \boldsymbol{\rho}(X)\boldsymbol{\rho}^X(Y).$$

Hence, we proved (i).

The assertion (ii) can be proved by a similar, but more refined arguments as above. But, if one observe that the measures are completely determined by their correlation functions of exponential growth, it suffices to notice that the correlation functions

$$(\boldsymbol{\rho}^X)^Y(Z) = \frac{\boldsymbol{\rho}^X(Y+Z)}{\boldsymbol{\rho}^X(Y)} = \frac{\boldsymbol{\rho}(X+Y+Z)}{\boldsymbol{\rho}(X+Y)} = \boldsymbol{\rho}^{X+Y}(Z)$$

whenever $\boldsymbol{\rho}(X+Y) > 0$ since then one obtains $\boldsymbol{\rho}(X) > 0$ and $\boldsymbol{\rho}(Y) > 0$ by (i). □

4 Poisson measure and Fock space

4.1 Isomorphism I on $\mathbf{H_0}$

In this section we state the essence of the results in [T2] for the completeness of the paper.

The operator $\boldsymbol{T}$ is useful to define the isomorphism from the Fock space $\boldsymbol{H}$ to the L^2-space $\boldsymbol{H}$. Firstly, we define $\boldsymbol{I}$ on $\boldsymbol{H}_0$.

Definition 17. *For an* $\mathbf{f} \in \mathbf{H}_0$ *define a function* $\mathbf{I}(\mathbf{f})(\xi)$ *on* $Q(R)$ *by*

$$\mathbf{I}(\mathbf{f})(\xi) = \langle \xi_*, T^{-1}\mathbf{f} \rangle.$$

Example 18. *Let* $\boldsymbol{f} = \boldsymbol{e}_f$ *with* $f \in C_c(R)$*. Then*

$$I(\boldsymbol{e}_f)(\xi) = e^{-\int_R (e^{-f(x)}-1)\lambda(dx)} e^{-\langle \xi, f\rangle}$$

Indeed,

$$T^{-1}\boldsymbol{e}_f(X) = (1/\alpha(e^{-f}-1))\boldsymbol{e}_f(X).$$

$$\langle \xi_*, \boldsymbol{e}_f \rangle = e^{-\langle \xi, f\rangle}$$

and $1/\alpha(e^{-f}-1) = e^{-(\int_R (e^{-f(x)}-1)\lambda(dx)}$.

As a consequence we obtain

$$\int_Q I(\boldsymbol{e}_f)(\xi)\pi_\lambda(d\xi) = 1$$

and

$$\|I(\boldsymbol{e}_f)\|^2_{\mathcal{H}} = \|\boldsymbol{e}_f\|^2_{\boldsymbol{H}} = e^{|e^{-f}-1|^2_H}$$

because

$$-2\int_R (e^{-f}-1)d\lambda + \int_R (e^{-2f}-1)d\lambda = \int_R (e^{-f}-1)^2 d\lambda.$$

More generally, we can show the following.

Lemma 19. *Let* $\phi, \psi \in C_c(R)$*. Then,*

$$\langle \mathbf{I}(\hat{\phi}), \mathbf{I}(\hat{\psi})\rangle_{\mathcal{H}} = \langle \hat{\phi}, \hat{\psi}\rangle_{\mathbf{H}}.$$

Proof. It follows from the proceeding lemmas that the inner products are computed as follows.

$$\begin{aligned}\langle \mathbf{I}(\hat{\phi}), \mathbf{I}(\hat{\psi})\rangle_{\mathcal{H}} =& (1/\alpha(\phi))(1/\alpha(\psi)) \int_{Q(R)} \pi_\lambda(d\xi)\langle \xi_*, \hat{\phi}\rangle\langle \xi_*, \hat{\psi}\rangle \\ =& (1/\alpha(\phi))(1/\alpha(\psi))\alpha(\phi\psi + \phi + \psi) \\ =& \alpha(\phi\psi) \\ =& \exp\langle \phi, \psi\rangle_H = \langle \hat{\phi}, \hat{\psi}\rangle_{\mathbf{H}}\end{aligned}$$

because $1/\alpha(\phi) = \alpha(-\phi)$. ∎

4.2 Isomorphism I on H

Now we obtain the following from Lemma 19 since the functions $\hat{\phi},\ \phi \in C_c(R)$ are dense in $\mathbf{H}_0$ with respect to the norm $\|\cdot\|_{\mathbf{H}}$.

Lemma 20. *(i) If* $\mathbf{f} \in \mathbf{H}_0$*, then* $\mathbf{I}(\mathbf{f}) \in \mathcal{H}$.
(ii) If $\mathbf{f}, \mathbf{g} \in \mathbf{H}_0$*, then,*

$$\langle \mathbf{I}(\mathbf{f}), \mathbf{I}(\mathbf{g}) \rangle_{\mathcal{H}} = \langle \mathbf{f}, \mathbf{g} \rangle_{\mathbf{H}}.$$

In particular,

$$\|\mathbf{I}(\mathbf{f})\|_{\mathcal{H}} = \|\mathbf{f}\|_{\mathbf{H}}.$$

Now we can prove the following.

Theorem 21. *The operator* $\mathbf{I}$ *is extended to the unitary operator from* $\mathbf{H}$ *onto* $\mathcal{H}$*.*

Proof. From the proceeding lemma we can extend $\mathbf{I}$ to the norm preserving operator from $\mathbf{H} = L^2(\hat{R}, \hat{\lambda})$ to $\mathcal{H} = L^2(Q(R), \pi_\lambda)$. It only remains to prove that $\mathbf{I}$ is onto. But it is obvious because the functions

$$\alpha(e^{-f} - 1)\mathbf{I}(\mathbf{e}_f) = e^{-\langle \xi, f \rangle}, \qquad f \in C_c(R)$$

span the Hilbert space $\mathcal{H} = L^2(Q(R), \pi_\lambda)$. □

4.3 Product $\diamond$ in $\mathbf{H}_0$

Since our Fock space $\mathbf{H}$ is isomorphic to the space $\mathcal{H} = L^2(Q(R), \pi_\lambda)$, the pointwise multiplication of functions on $Q(R)$ induces a product structure on the Hilbert space $\mathbf{H}$ which is, of course, densely defined. Hence, the Fock space admits a densely defined, algebra structure. If we start from another measure, for instance, a Gaussian measure in place of a Poisson measure, we obtain another densely defined algebra structure on the Fock space. The author would like to recall here that G.Maruyama studied the product of Wiener-Ito expansions and gave some applications in his last three papers [M1-3]. It is the motivation of this small subsection although he had had a further plan to construct an applicable analysis based on the Wiener-Ito expansions.

Definition 22. *For* $f, g \in \mathbf{H}_0$ *set*

$$\begin{aligned}
&(\mathbf{f} \diamond \mathbf{g})(X) \\
&= \sum_{X_0 + X_1 + X_2 = X} \binom{X}{X_0, X_1, X_2} \int_{\hat{R}} \hat{\lambda}(dY) \mathbf{f}(X_0 + X_1 + Y) \mathbf{g}(X_0 + X_2 + Y)
\end{aligned}$$

where we set

$$\binom{X}{X_0, X_1, X_2} = \binom{X}{X_0} \binom{X - X_0}{X_1} \qquad \text{if} \qquad X = X_0 + X_1 + X_2.$$

It is immediate to see

$$\mathbf{f} \diamond \mathbf{g} \in \mathbf{H}_0.$$

The following theorem implies that the product $\diamond$ cannot be extended to the whole space $\mathbf{H}$.

Theorem 23. *If* $\mathbf{f}, \mathbf{g} \in \mathbf{H}_0$*, then*

$$\mathbf{I}(\mathbf{f} \diamond \mathbf{g})(\xi) = \mathbf{I}(\mathbf{f})(\xi)\mathbf{I}(\mathbf{g})(\xi).$$

Proof. It suffices to prove

$$\mathbf{I}(\hat{\phi} \diamond \hat{\psi})(\xi) = \mathbf{I}(\hat{\phi})(\xi)\mathbf{I}(\hat{\psi})(\xi)$$

for $\phi, \psi \in C_c(R)$.

Recall that

$$\begin{aligned}\mathbf{I}(\hat{\phi})(\xi) =&\langle \xi_*, T^{-1}\hat{\phi}\rangle \\ =&(1/\alpha(\phi))\langle \xi_*, \hat{\phi}\rangle,\end{aligned}$$

$$\alpha(\phi + \psi) = \alpha(\phi)\alpha(\psi).$$

and

$$\langle \xi_*, \hat{\phi}\rangle\langle \xi_*, \hat{\psi}\rangle = \langle \xi_*, (\phi\psi + \phi + \psi)^\wedge\rangle.$$

Now the above definition implies that

$$\begin{aligned}&(\hat{\phi} \diamond \hat{\psi})(X) \\ =&\sum_{X_0+X_1+X_2=X} \\ &\binom{X}{X_0, X_1, X_2} \int_{\hat{R}} \hat{\lambda}(dY)\hat{\phi}(X_0 + X_1 + Y)\hat{\psi}(X_0 + X_2 + Y) \\ =&\sum_{X_0+X_1+X_2=X} \\ &\binom{X}{X_0, X_1, X_2}(\phi\psi)^\wedge(X_0)\hat{\phi}(X_1)\hat{\psi}(X_2) \int_{\hat{R}} \hat{\lambda}(dY)(\phi\psi)^\wedge(Y) \\ =&(\phi\psi + \phi + \psi)^\wedge(X).\end{aligned}$$

Hence follows the desired identity. □

5 Fermion (determinantal) and other processes

5.1 Explicit formula between correlation functions and density functions

The following is essentially known for Gibbs measures(cf., [R], [HK], [T0]) but is presented in a general and neat form using the operator T.

Theorem 24. *Assume that a probability measure μ on $Q(R)$ admits the correlation function $\boldsymbol{\rho}$ with respect to λ. Then for any compact subset Λ of R its restriction μ_Λ to $Q(\Lambda) \cong \hat{\Lambda}$ is absolutely continuous with respect to the restriction of the Poisson measure $\pi_{\lambda,\Lambda} = \hat{\lambda}_\Lambda$. Moreover, its density function $\boldsymbol{\sigma}_\Lambda(\xi)$ on $Q(\Lambda)$ is given by the formula*

$$\boldsymbol{\sigma}_\Lambda(X) = T^{-1}\boldsymbol{\rho}_\Lambda(X) \quad \hat{\lambda} - a.e. X \in \hat{\Lambda}$$

where $\boldsymbol{\rho}_\Lambda$ is the restriction(or cutoff) of $\boldsymbol{\rho}$ to $\hat{\Lambda}$:

$$\boldsymbol{\rho}_\Lambda(X) = \begin{cases} \boldsymbol{\rho}(X) & \text{if } X \in \hat{\Lambda}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $f \in C_c(R)$ and take a compact Λ such that *supp*$f \subset \Lambda$. Recall that

$$S\mathbf{e}_f(X) = (e^{-f})^\wedge(X) = \exp\langle X, f\rangle.$$

Thus, $T = S^*$ shows that

$$\begin{aligned} \int_{Q(R)} \mu(d\xi)e^{-\langle \xi, f\rangle} &= \int_{\hat{R}} \hat{\lambda}(dX)\boldsymbol{\rho}(X)\mathbf{e}_f(X) \\ &= \int_{\hat{R}} \hat{\lambda}(dX)\boldsymbol{\rho}_\Lambda(X)\mathbf{e}_f(X) \\ &= \int_{\hat{R}} \hat{\lambda}(dX)(T^{-1}\boldsymbol{\rho}_\Lambda)(X)(S\mathbf{e}_f)(X) \\ &= \int_{\hat{R}} \hat{\lambda}(dX)(T^{-1}\boldsymbol{\rho}_\Lambda)(X)e^{-\langle X, f\rangle}. \end{aligned}$$

Consequently, μ_Λ is absolutely continuous with respect to π_Λ and the density is $\boldsymbol{\sigma}_\Lambda$ stated in Theorem. □

5.2 Determinantal functions and fermion processes

Besides the exponential functions $\mathbf{e}_f$, there exist some interesting classes of functions on $\hat{R}$. The first one is the following.

Definition 25. *(determinatal functions). Let $K(x, y)$ be a function on $R \times R$. For $X \in \hat{R}$ denote*

$$K_X = (K(x_i, x_j))_{i,j=1}^n \qquad \text{if} \qquad X = \sum_{i=1}^n \delta_{x_i}.$$

and define a function $\mathbf{d}_K$ on $\hat{R}$ by

$$\mathbf{d}_K(X) = \det K_X$$

and $K_0 = 1$.

By definition, $\mathbf{d}_K(X) = 0$ if X has a multiple point.

Theorem 26. *Let $K(x, y)$ be a continuous kernel of a trace class integral operator K on $L^2(R, \lambda)$. Assume that $I - K$ is invertible for any compact subset Λ of R. Then,*

$$T\mathbf{d}_K(X) = \mathbf{d}_{(I-K)^{-1}K}(X), \qquad X \in \hat{R}.$$

The proof will be given in below after the proof of Corollary 27.

If, in addition, the nonnegativity is guaranteed, the existence theorem of fermion point processes(or determinantal processes) follows from the above theorem.

Corollary 27. ([SrT],[So]) *Let $K(x, y)$ be a continuous kernel of a nonnegative definite locally trace class integral operator K on $L^2(R, \lambda)$. Assume that the restriction K_Λ of K satisfies the spectral condition*

$$Spec(K_\Lambda) \subset [0, 1) \qquad \text{for any compact } \Lambda.$$

Then there exists a unique probability Borel measure μ_K on $Q(R)$ which admits $\mathbf{d}_K$ as its correlation function. The measure μ_K is called the fermion point process associated with K.

Proof. The spectral condition implies that $I - K_\Lambda$ is invertible. Then the nonnegative definiteness of the operator K implies that the nonnegativity of determinantal function $\mathbf{d}_K$ and $\mathbf{J}[\boldsymbol{\Lambda}]$ with

$$J[\Lambda] = (I - K_\Lambda)^{-1}K_\Lambda.$$

Consequently, there exists a probability Borel measure, say $\mu_{K,\Lambda}$ for each compact subset Λ of R such that it admits $\mathbf{K}|_{\hat{\Lambda}}$ as its correlation function. Moreover, the measures $\mu_{K,\Lambda}$ are consitent. Hence, a Kolmogorov extension theorem shows the existence and uniqueness of the fermion measure μ_K. □

Here we should remark that the measure μ_{K_Λ} exists even in the degenerated case when 1 is an eigenvalue of $I - K_\Lambda$ for a compact Λ as is shown in [SrT2]. But then the formula for $T\mathbf{d}_{K_\Lambda}$ fails to hold.

Now we proceed to

Proof of Theorem 26. The essence of the proof is as follows. Since K is a local trace class operator, the operator $K\phi$ is a trace class operator whenever $\phi \in C_c(R)$. Here $K\phi$ stands for the composition of K and the multiplication operator by the function ϕ. Consider the functional

$$\Phi(\phi) = \mathrm{Det}(I + K\phi), \qquad \phi \in C_c(R).$$

Recall that the Fredholm determinant is defined by

$$\mathrm{Det}(I + K\phi) = 1 + \sum_{n=1}^{\infty} \mathrm{Tr}(\bigwedge^n (K\phi)).$$

Since

$$\begin{aligned}\operatorname{Tr}(\bigwedge^n(K\phi)) =&\frac{1}{n!}\int_{R^n} \det((K(x_i,x_j)\phi(x_j))_{i,j=1}^n)\lambda(dx_1)\cdots\lambda(dx_n)\\ =&\frac{1}{n!}\int_{R^n} \det((K(x_i,x_j))_{i,j=1}^n)(\prod_{j=1}^n \phi(x_j))\lambda(dx_1)\cdots\lambda(dx_n),\end{aligned}$$

we obtain the expansion formula

$$\operatorname{Det}(I+K\phi) = \int_{\hat{R}} \hat{\lambda}(dX)\mathbf{d}_K(X)\hat{\phi}(X).$$

It is nothing but the Taylor expansion of the analytic functional $\Phi(\phi)$ in ϕ.

Let us compute $\Phi(\phi+\psi)$ in two ways.

On one hand, a similar argument to the proof of Theorem 11 in 2.7 (or by setting $\psi\lambda$ in place of λ) shows

$$\Phi(\phi+\psi) = \int_{\hat{R}} \hat{\lambda}(dX)\hat{\psi}(X)(T\mathbf{d}_{K\phi})(X).$$

On the other hand, so far as $I+K\phi$ is invertible,

$$\begin{aligned}\Phi(\phi+\psi) &= \operatorname{Det}(I+K\phi+K\psi) = \operatorname{Det}(I+K\phi)\operatorname{Det}(I+(I+K\phi)^{-1}K\psi)\\ &= \operatorname{Det}(I+K\phi)\int_{\hat{R}} \hat{\lambda}(dX)\mathbf{d}_{(I+K\phi)^{-1}K}(X)\hat{\psi}(X).\end{aligned}$$

Consequently, we obtain

$$(T\mathbf{d}_{K\phi})(X) = \operatorname{Det}(I+K\phi)\mathbf{d}_{(I+K\phi)^{-1}K}(X).$$

Similarly, we obtain

$$(T^{-1}\mathbf{d}_{K\phi})(X) = \operatorname{Det}(I-K\phi)\mathbf{d}_{(I-K\phi)^{-1}K}(X).$$

Now Theorem follows by taking $K = K_\Lambda$ and $\phi = 1$ since $I - K_\Lambda$ is assumed invertible. □

5.3 Permanental functions and boson processes

As in the previous subsection we denote

$$K_X = (K(x_i,x_j))_{i,j=1}^n \qquad \text{if} \qquad X = \sum_{i=1}^n \delta_{x_i}$$

for a given kernel $K(x,y)$ on $R\times R$. Now we consider the permanent in stead of the determinant.

Definition 28. *(permanental functions). Define a function* $\mathbf{p}_K$ *on* $\hat{R}$ *by*

$$\mathbf{p}_K(X) = \operatorname{perm} K_X$$

and $K_0 = 1$.

Now we consider the analytic functional

$$\Phi(\phi) = \operatorname{Det}(I - K\phi)^{-1}.$$

Then it is known (cf. [SrT], also, [VJ]) that

$$\begin{aligned}&\operatorname{Det}(I - K\phi)^{-1}\\ =&1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{R^n} \operatorname{perm}((K(x_i, x_j)\phi(x_j))_{i,j=1}^n)\lambda(dx_1)\cdots\lambda(dx_n)\end{aligned}$$

and so

$$\operatorname{Det}(I - K\phi)^{-1} = \int_{\hat{R}} \hat{\lambda}(dX)\mathbf{p}_K(X)\hat{\phi}(X).$$

In the permanent case, we obtain

$$\operatorname{Det}(I - K(\phi + \psi))^{-1} = \operatorname{Det}(I - K\phi)^{-1} \operatorname{Det}(I - (I - K\phi)^{-1}K\psi))^{-1}$$

whenever $I - K\phi$ is invertible. Hence follows

Theorem 29.

$$T\mathbf{p}_{K_\Lambda} = \operatorname{Det}(I + K_\Lambda)^{-1}\mathbf{J}_{-\mathbf{1}}[\mathbf{\Lambda}]$$

where

$$J_{-1}[\Lambda] = (I + K_\Lambda)^{-1}K_\Lambda$$

whenever $I + K_\Lambda$ *is invertible.*

The above arguments show

Corollary 30. ([SrT]) Let $K(x, y)$ be a continuous kernel of a nonnegative definite locally trace class integral Then there exists a unique probability Borel measure μ_K on $Q(R)$ which admits $\mathbf{p}_K$ as its correlation function. The measure μ_K is called the boson process associated with K.

5.4 α-determinantal functions

In [SrT] we introduced the α-determinant of a square matrix $A = (a_{ij})_{i,j=1}^n$ by

$$\det_\alpha(A) = \sum_{\boldsymbol{\sigma} \in S_n} \alpha^{n-\nu(\boldsymbol{\sigma})} \prod_{i=1}^n a_{i\boldsymbol{\sigma}(i)}$$

where $\alpha \in R$, S_n is the symmetric group of order n and $\nu(\boldsymbol{\sigma})$ stands for the number of cycles in the permutation $\boldsymbol{\sigma}$. It is immediate to see that $\det_{-1} = \det$

and $\det_1 = \text{perm}$. If $-1 < \alpha < 1$, $\det_\alpha$ does not coincide with the so-called q-analogue of determinants and permanents with $-1 < q < 1$ which is defined using the invertion number $\iota(\boldsymbol{\sigma})$ of a permutation $\boldsymbol{\sigma}$ in place of $n - \nu(\boldsymbol{\sigma})$.

The α-determinant is defined so that the following identity holds:

$$\text{Det}(I - \alpha K)^{1/\alpha} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{R^n} \det_\alpha((K(x_i, x_j))_{i,j=1}^n)\lambda(dx_1) \cdots \lambda(dx_n)$$

for a trace class integral operator on $L^2(R, \lambda)$.

For simplicity, let us introduce the following notation.

Definition 31. *For a kernel of locally trace class integral operator K on $L^2(R, \lambda)$ we call a function given by*

$$\mathbf{d}_{\alpha,K}(X) = \det_\alpha(K_X)$$

α-determinantal where $K_X = (K(x_i, x_j))_{i,j=1}^n$ if $X = \sum_{i=1}^n \delta_{x_i}$, as before.

Then,

$$\text{Det}(I - \alpha K)^{1/\alpha} = \int_{\hat{R}} \mathbf{d}_{\alpha,K}(X)\hat{\lambda}(dX).$$

Definition 32. ([SrT]). *Let α be a real number and K be an integral operator on $L^2(R, \lambda)$. A probability Borel measure μ on the configuration space $Q(R)$ is called an α-determinantal process if its Laplace transform is given by*

$$\int_{Q(R)} \mu(d\xi)e^{-\langle \xi, f \rangle} = \text{Det}(I + \alpha K(1 - e^{-f}))^{-1/\alpha}$$

for any $f \in C_c(R)$, or equivalently, if μ admits the correlation function $\boldsymbol{\rho}$ which is given by

$$\boldsymbol{\rho}(X) = \mathbf{d}_{\alpha,K}(X), \qquad X \in \hat{R}.$$

Similarly to previous two subsections we obtain

Theorem 33. *Let K be of tracce class and $-1 \leq \alpha \leq 1$*

$$T\mathbf{d}_{\alpha,K} = \det(I - \alpha K)^{1/\alpha}\mathbf{J}_{\boldsymbol{\alpha}}[\boldsymbol{\Lambda}]$$

where

$$J_\alpha[\Lambda] = (I - \alpha K)^{-1/\alpha}K$$

whenever $I - \alpha K$ is invertible.

This is one of the main ingredient in the proof of the existence theorem on α-determinantal processes. But we need some assumptions to ensure the nonnegativity of correlation functions and/or density functions.

A sufficient condition is given and called Condition A in [SrT]. Firstly, K is assumed a locally trace class nonnegative definite integral operators. Secondly, the following condition on the parameter α is imposed:

$$\alpha \in \{-\frac{1}{n} \mid n = 1, 2, \dots\} \cup \{\frac{2}{n} \mid n = 1, 2, \dots\}$$

If the kernel $K(x, y)$ is nonnegative-valued, the second assumption is relaxed to

$$\alpha \in \{-\frac{1}{n} \mid n = 1, 2, \dots\} \cup [0, 2].$$

Concerning the rest cases we raised a conjecture on the nonnengativity in [SrT2] which is still open up to the present.

Conjecture 34. *Let $0 \leq \alpha \leq 2$. Then $\det_\alpha A$ is nonnegative whenever A is a nonnegative definite matrix.*

Notice that there is no problem on the nonnegativity of the q-analogue of determinants by virtue of the representation theoretic nature of the invertion number of a permutation while the α-determinant comes from analysis and probability theory. But our α-determinants also have certain meanings in representation theory [WKM].

6 Gibbs measures and their Palm measures

6.1 Gibbs measures in the sense of Dobrushin-Lanford-Ruelle

The concept of Gibbs measures is originated in statistical mechanics. They consider the Hamiltonian of a finite configuration $\{x_1, x_2, \dots, x_n\}$ defined by the sum

$$H(x_1, x_2, \dots, x_n) = \sum_{k=1}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \Phi_k(x_{i_1}, \dots, x_{i_k})$$

where Φ_k is a symmetric function on R^k with $R = \mathbb{R}^3$ or $R = \mathbb{R}^3 \otimes \mathbb{R}^3$ called an n-point interaction. Roughly to say, the canonical Gibbs measure $\mu_{\Lambda,n}$ on n-point configurations over a compact subset Λ is the probability measure defined by

$$\frac{1}{Z(\Lambda, n)} \exp(-H(x_1, x_2, \dots, x_n)) dx_1 \dots dx_n$$

where $dx_1 \dots dx_n$ is the Lebesgue measure on R^n and $Z(\Lambda, n)$ is the normalizing constant. And the grandcanonical Gibbs measure μ_Λ on finite configurarion over Λ is defined by

$$\frac{1}{Z(\Lambda)} \frac{1}{n!} \exp(-H(x_1, x_2, \dots, x_n)) dx_1 \dots dx_n$$

where $Z(\Lambda)$ is the normalizing constant $Z(\Lambda) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} Z(\Lambda, n)$. The main concern in statistical mechanics lies in the infinite volume limits and a limit point of μ_Λ as Λ goes to R is called a limit grandcanonical Gibbs measures, or simply, a Gibbs measure in the terminology of recent decades. (One may consider limit canonical Gibbs measures but they are known to be a little bit more complicated.)

R.L.Dobrushin[] and O.Lanford and D.Ruelles[] gave a beautiful definition of the Gibbs measure (which is sometimes called the DLR measures) as follows.

In the rest of this subsection, we only consider configurations without multiple points following the tradition in statistical mechanics. Thus, a configuration ξ on R is identified with a subset of R.

For a finite configuration X and a locally finite configuration ξ, let

$$H(X|\ \xi) = \sum_{Y\,\text{finite}, \subset X \cup \xi, X \cap \xi \neq \emptyset} \Phi(Y)$$

where

$$\Phi(Y) = \Phi_n(y_1, \ldots, y_n) \qquad \text{if} \qquad Y = \{y_1, \ldots, y_n\}.$$

Under certain summability conditions (called lower regularity and superstability in Ruelle[]) $H(X|\ \xi)$ is a continuous function in $(X, \xi) \in \hat{R} \times Q(R)$ and

$$Z(\Lambda, \xi) = \int_{\hat{\Lambda}} \hat{\lambda}(dX) \exp(-H(X|\ \xi \cap \Lambda^c))$$

is finite and defines a continuous function in $\xi \in Q(R)$ where $\xi \cap \Lambda^c$ is the restriction of ξ outside of Λ.

Now define a probability measure $q_{\Lambda,\xi}$ on $\hat{\Lambda}$ by

$$q_{\Lambda,\xi}(dX) = \frac{1}{Z(\Lambda, \xi)} \hat{\lambda}(dX) \exp\{-H(X|\ \xi \cap \Lambda^c)\}.$$

Definition 35. *A probability measure μ on $Q(R)$ is called a DLR measure if it satisfies the following equation*

$$\int_{Q(R)} \mu(d\xi) F(\xi) = \int_{Q(R)} \mu(d\xi) \int_{\hat{\Lambda}} \hat{\lambda}(dX) q_{\Lambda,\xi}(dX) F(X \cup (\xi \cap \Lambda^c))$$

for any compact subset Λ and any bounded continuous function F on $Q(R)$.

6.2 Cocycles and Contitions (C1) and (C2)

Now we formulate Gibbs measures on $Q(R)$ in our framework. The definition of the Gibbs measure given in 6.3 is a generalized version of the one given by Dobrushin-Lanford-Ruelle (Definition 35). For this sake we must think of what the potential should be.

We start from the following definition of the cocycle.

Definition 36. *Let $U(X|\ \xi)$ be a function on $\hat{R}\times Q(R)$ with values in$(-\infty,\infty]$. We call U a cocycle if the following conditions* (a) *and* (b) *are satisfied:*
(a) Set

$$Dom(U)=\{(X,\xi)\in\hat{R}\times Q(R)\mid U(X|\ \xi)<\infty\}.$$

Then U is a continuous function on Dom(U).
(b) There holds the equation

$$U(X+Y\mid\xi)=U(X|\ \xi+Y)+U(Y\mid\xi)$$

for any $X,Y\in\hat{R}$ and any $\xi\in Q(R)$. Here and here after we use the convention:

$$a+\infty=\infty+a=\infty\quad\textit{for any}\quad a\in(-\infty,\infty].$$

We remark that the condition (a) guarantees the existence of $U(X|\xi+Y)$ in (b) and that (a) implies that

$$(Y,\xi)\in Dom(U)\quad\text{if}\quad(X,\xi)\in Dom(U)\ \ \text{and}\ \ Y\leqslant\xi.$$

Furthermore, we will consider the following conditions on U.
(C1). *There exists a subset Q_U of $Q(R)$ such that*

$$Dom(U)=\{(X,\xi)\in\hat{R}\times Q(R)\mid\xi+X\in Q_U\}.$$

(C2). *There exists a constant B such that*

$$U(X|\ \xi)\geqslant B|X|\quad\text{if}\quad X\in R\quad\text{and}\quad\xi\in W(R)$$

and the integral

$$Z(\Lambda,\xi)=\int_{\hat{\Lambda}}\hat{\lambda}(dX)e^{-U(X|\ \xi)},\quad\xi\in Q_U$$

defines a positive continuous function for each compact subset Λ of R.

We remark here that (a) implies that the set Q_U necessarily satisfies $\eta\in Q_U$ if $\eta+X\in Q_U$ for some $X\in\hat{R}$.

Example 37. (a). *Let $u:Q(R)\to\mathbb{R}$ be a bounded continuous functions and*

$$U(X|\ \xi)=u(\xi+X)-u(\xi).$$

Then U is a cocycle with $Dom(U)=\hat{R}\times Q(R)$ and satisfies (C1) and (C2) with $Q_U=Q(R)$ and $B=-||u||_\infty$. (Such U may be called a boundary.)
(b).*Let $m\geqslant 1$ and*

$$Q_{\leqq m}=\{\xi\in Q(R)\mid\xi(x)\leqslant m\quad\textit{for any}\quad x\in R\}.$$

Set

$$U(X|\ \xi)=\begin{cases}0 & \textit{if}\ \ X+\xi\in Q_{\leq m},\\ \infty & \textit{otherwise.}\end{cases}$$

Then U is a cocycle and satisfies (c1) and (C2) with $Q_U = Q_{\leq m}$ and $B = 0$.

(c). *Let R be a metric space with distance d. For $r > 0$, set*

$$Q(r) = \{\xi \in Q_{\leqq 1} \mid \xi\{x, y\} = 0 \quad whenever \quad d(x, y) > r\}$$

and

$$U(X|\ \xi) = \begin{cases} 0 & if \quad X + \xi \in Q(r) \\ \infty & otherwise. \end{cases}$$

Then U is a cocycle and satisfies (C1)and (C2) with $Q_U = Q(r)$ and $B = 0$.

(d). *A typical example in classical statistical mechanics is the following. Let $R = \mathbb{R}^3$, $\Phi_1(x)$ be a continuous function on R (self potential) and $\Phi_2(x, y)$ be a symmetric continuous function defined for $(x, y) \in R \times R$ with $|x - y| > 2r$(pair potential). Here $r \geq 0$(radius of a particle) and set $\Phi_2(x, y) = \infty$ if $|x - y| \leq 2r$. Assume that $\Phi_2(x, \cdot)$ has a compact support for each fixed x. Set*

$$U(X|\ \xi) = \sum_{i=1}^{n} \Phi_1(x_i) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \Phi(x_i, x_j) + \sum_{i=1}^{n} \langle \xi, \Phi_2(x_i, \cdot) \rangle$$

for $X = \sum_{i=1}^{n} \delta_{x_i} \in \hat{R}$ and $\xi \in Q(R)$. Then $U(X|\ \xi)$ is a cocycle with

$$Q_U = \{\xi \in Q_{\leqslant 1} \quad | \quad \langle \xi_2, f_2 \rangle = 0\}$$

where $f_2(x, y) = 1(|x - y| > 2r)$.

Lemma 38. *(i) Let U_1, U_2 be cocycles satisfying (C1) and (C2). For $c_1, c_2 > 0$ define*

$$U(X|\ \xi) = \begin{cases} c_1 U_1(X|\ \xi) + c_2 U_2(X|\ \xi) & for \quad (X, \xi) \in Dom(U) \\ +\infty & otherwise. \end{cases}$$

where $Dom(U) = Dom(U_1) \cap Dom(U_2)$. Then U is a cocycle satisfying (C1) and (C2),

(ii) For $X \in \hat{R}$ define U^X by

$$U^X(\cdot \mid \xi) = U(\cdot \mid \xi + X).$$

Then, U^X is a cocycle with

$$Q_{U^X} = \{\xi \mid \xi + X \in Q_U\}.$$

and satisfies (C1) and (C2).

Proof. Both (i) and (ii) are immediate. □

The above definition is rather complicated and we might have employed the notion of multiplicative cocycle by considering $M(X|\ \xi) = e^{-U(X|\ \xi)}$.

Definition 39. *Let $M(X|\ \xi)$ be a nonnegative function defined for $(X,\xi) \in \hat{R} \times Q(R)$. We call it a multiplicative cocycle if the following conditions are satisfied:*
(a) $M(X|\ \xi)$ is continuous on the set where $M(X|\ \xi) > 0$ and there exists a constant B such that

$$M(X|\ \xi) \leq e^{nB|X|},\ X \in \hat{R},\ \xi \in Q(R).$$

(b) $M(X+Y|\xi) = M(Y|\ \xi + X)M(X|\ \xi), \qquad X, Y \in \hat{R},\ \xi \in Q(R).$
(c) $\int_{\hat{\Lambda}} \hat{\lambda}(dX)M(X|\ \xi)$ converges and defines a positive continuous function in X.

But we prefer the traditional, additive cocycles and will give a few remarks. The cocycle $U(X|\xi)$ is completely determined by its values for $|X| = 1$.

Lemma 40. *(i) Let $U(X|\ \xi),\ X \in \hat{R},\ \xi \in Q(R)$ be a cocycle. Then it is completely determined by $u(x \mid \xi) = U(\delta_x \mid \xi),\ x \in R,\ \xi \in Q(R),\ \xi + X \in Q_0$ and $u(x \mid \xi)$ must satisfy the condition*

$$u(x \mid \xi + \delta_y) - u(y \mid \xi + \delta_x) = u(x \mid \xi) - u(y \mid \xi)$$

(whenever both hand sides are well-defined.) (ii) Conversely, if $u(x|\xi)$ satisfies the above condition, then it can be extended to a cocycle except for the condition (c). .

Proof. Obvious. □

To define Gibbs measures (Definition 43) it apparently suffices to assume the following for cocycles.

Definition 41. *Let $U(X|\xi)$ be a function with values in $(-\infty, \infty]$ defined on the set for $(X,\xi) \in \hat{R} \times Q(R)$ such that supp(X) and supp(ξ) are disjoint. We say that $U(X|\xi)$ satisfy the restricted cocycle condition if it satisfies the condition provided that supp(X), supp(Y) and supp(ξ) are mutually disjoint.*

But such restriction is not essential.

Lemma 42. *Let $H(X|\xi)$ satisfy the restricted cocycle condition. Then it can be extended to a cocycle in a unique manner.*

Proof. For given $X \in \hat{R}$ and $\xi \in Q(R)$, let

$$X = \sum_{i=1}^{k} n_i \delta_{x_i} \qquad x_1, \ldots, x_k, \quad \text{mutually distinct}$$

and decompose ξ as

$$\xi = \sum_{i=1}^{k} m_i \delta_{x_i} + \eta, \qquad \eta(x_i) = 0, \quad i = 1, \ldots, k.$$

Define

$$U(X|\ \xi) = H(\sum_{i=1}^{k}(n_i + m_i)\delta_{x_i}|\ \eta) - H(\sum_{i=1}^{k} m_i\delta_{x_i}|\ \eta)$$

if $(X, \xi) \in Dom(H)$ and set $U(X|\ \xi) = \infty$ otherwise. Then it is immediate to see that $U(X|\ \xi)$ is a cocycle. □

6.3 Gibbs measure and its correlation functions

Definition 43. *For a given measurable function $U(X|\ \xi)$ of $(X, \xi) \in \hat{R} \times Q(R)$ with values in $(-\infty, \infty]$ we denote as before*

$$q_{\Lambda,\xi}(dX) = \frac{1}{Z(\Lambda, \xi)}\hat{\lambda}(dX)\exp(-U(X|\ \xi)).$$

whenever the integral $Z(\Lambda, \xi)$ is definite and is positive where Λ is a compact subset of R and $\xi \in Q(\Lambda^c)$.

Lemma 44. *Let $U(X|\ \xi)$ be as above.*
(i) If $U(X|\ \xi)$ is a cocycle and $Z(\Lambda, \xi) < \infty$, then the probability measures $q_{\Lambda,\xi}$ form a consistent family in the sense that if Λ_0 and Λ are compact and $\Lambda_0 \subset \Lambda$ then the conditional probability of $q_{\Lambda,\xi}$ given the configuration X_1 on $\Lambda_1 = \Lambda \setminus \Lambda_0$ coincides with $q_{\Lambda_0,\xi+X_1}$. In other words,

$$\int_{\hat{\Lambda}} q_{\Lambda,\xi}(dX)F(X_1)G(X_0) = \int_{\hat{\Lambda}} q_{\Lambda,\xi}(dX)F(X_1)\int_{\hat{\Lambda}_0} q_{\Lambda_0,\xi+X_1}(dY)G(Y)$$

for any continuous function F depending only on the configuration restricted to Λ_1 and any continuous function G depending only on the configuration restricted to Λ_0. Here X_i stands for the restriction $X_i = X|_{\Lambda_i},\ i = 0, 1$.
(ii) Conversely, if $q_{\Lambda,\xi}$ form a consistent family, then $U(X|\ \xi)$ is a cocycle.

Proof. The consistent condition can be written in a little bit symbolical manner as

$$q_{\Lambda,\xi}(dX_1dX_0) = q_{\Lambda,\xi}(dX_1 \times (\Lambda_0)^\wedge)q_{\Lambda,\xi+X_1}(dX_0), \qquad X_i = X\ |_{\Lambda_i},\ i = 0, 1$$

since

$$\hat{\lambda}(dX_1dX_0) = \hat{\lambda}(dX_1)\hat{\lambda}(dX_0).$$

If $U(X|\ \xi)$ is a cocycle, then,

$$\begin{aligned}
&Z(\Lambda, \xi)q_{\Lambda,\xi}(dX_1 \times (\Lambda_0)^\wedge)\\
=&\hat{\lambda}(dX_1)\int_{(\Lambda_0)^\wedge}\hat{\lambda}(dY)\exp(-U(X_1 + Y\ |\ \xi))\\
=&\hat{\lambda}(dX_1)\int_{(\Lambda_0)^\wedge}\hat{\lambda}(dY)\exp(-U(Y\ |\ \xi + X_1))\exp(-U(X_1\ |\ \xi))\\
=&\hat{\lambda}(dX_1)Z(\Lambda_0, \xi + X_1)\exp(-U(X_1\ |\ \xi)).
\end{aligned}$$

Hence,

$$
\begin{aligned}
&q_{\Lambda,\xi}(dX_1 \times (\Lambda_0)^\wedge) q_{\Lambda,\xi+X_1}(dX_0) \\
=&\frac{1}{Z(\Lambda,\xi)}\hat{\lambda}(dX_1)\hat{\lambda}(dX_0)\exp(-U(X_1 \mid \xi))\exp(-U(X_0 \mid \xi + X_1)) \\
=&\hat{\lambda}(dX_1)\hat{\lambda}(dX_0)\exp(-U(X_1 + X_0 \mid \xi)).
\end{aligned}
$$

Consequently, (i) is proved. Conversely, if the consitency condition holds, then there must hold the equation

$$\exp(-U(X_1 \mid \xi))\exp(-U(X_0 \mid \xi + X_1)) = \exp(-U(X_1 + X_0 \mid \xi))$$

for $\hat{\lambda} - a.e. X_0$ and $\hat{\lambda} - a.e. X_1$, and so for all X_1 and X_0 by the continuity. This shows the restricted cocycle condition. Consequently, $U(X|\ \xi)$ is a cocycle. □

Now we give our definition of the Gibbs measure. We fix a nonnegative Radon measure λ.

Definition 45. *A probability Borel measure μ is called a Gibbs measure if there exists a cocycle $U(X|\ \xi)$ satisfying Conditions (C1) and (C2) such that the conditional probability $\mu_{\Lambda,\xi}$ given the configuration $\xi\ |_{\Lambda^c}$ outside Λ coincides with $q_{\Lambda,\xi\ |_{\Lambda^c}}$ for μ-almost every ξ and each compact set Λ. In other words, a Gibbs measure given a cocycle $U(X|\ \xi)$ is a probability measure μ such that the conditional measure $\mu_{\Lambda,\xi}(dX)$ on $Q(\Lambda) = \hat{\Lambda}$ is absolutely continuous with respect to $\hat{\lambda}(dX)$ and that its density $\boldsymbol{\sigma}_{\Lambda,\xi}(X)$ is given by*

$$\boldsymbol{\sigma}_{\Lambda,\xi}(X) = \frac{1}{Z(\Lambda,\xi)}\hat{\lambda}(dX)\exp(-U(X|\ \xi)).$$

We do not discuss the existence and the uniqueness problems for Gibbs measure here.

Theorem 46. *Let μ be a Gibbs measure associated with a cocycle $U(X|\xi)$. Then it admits the correlation function $\boldsymbol{\rho}(X)$ and is given by the formula*

$$\boldsymbol{\rho}(X) = \int_{Q(R)} \mu(d\xi)\exp(-U(X|\xi)).$$

Proof. Firstly, we compute the correlation function $\boldsymbol{\rho}_{\Lambda,\xi}(X)$ of $\mu_{\Lambda,\xi}$ for a configuration ξ outside Λ by using the operator T.

$$\begin{aligned}
\boldsymbol{\rho}_{\Lambda,\xi}(X) =& T\boldsymbol{\sigma}_{\Lambda,\xi}(X) \\
=& \frac{1}{Z(\Lambda,\xi)} \int_{\hat{\Lambda}} \hat{\lambda}(dY) \exp(-U(X+Y|\ \xi)) \\
=& \frac{1}{Z(\Lambda,\xi)} \int_{\hat{\Lambda}} \hat{\lambda}(dY) \exp(-U(X+Y|\ \xi+Y)) \exp(-U(Y|\ \xi)) \\
=& \frac{1}{Z(\Lambda,\xi)} \int_{\hat{\Lambda}} \hat{\lambda}(dY) \exp(-U(X|\ \xi+Y)) \exp(-U(Y|\ \xi)) \\
=& \int_{\hat{\Lambda}} q_{\Lambda,\xi}(dY) \exp(-U(X+Y|\ \xi+Y)) \\
=& \int_{\hat{\Lambda}} \mu_{\Lambda,\xi}(dY) \exp(-U(X+Y|\ \xi+Y)).
\end{aligned}$$

Integrating the both sides of this equation with respect to μ, we obtain the desired equation. □

6.4 Palm measures of a Gibbs measure

Now the Palm measure in our sense can easily be computed.

Theorem 47. *Let μ is a Gibbs measure associated with a cocycle $U(X|\xi)$. Then its Palm measure μ^X exists whenever its correlation function $\boldsymbol{\rho}(X)$ is positive and satisfies the equation*

$$\boldsymbol{\rho}(X) \int_{Q(R)} \mu^X(d\xi) F(\xi) = \int_{Q(R)} \mu(d\xi) e^{-U(Y|\ \xi)} F(\xi + X).$$

Moreover, one can take a version of the Palm measure μ^X which satisfies the above equation for all $X \in \hat{R}$ though it is originally defined $\hat{\lambda} - a.e.X$ on the set $\{X \in \hat{R} | \boldsymbol{\rho}(X) > 0\}$.

Proof. If $\mathbf{f} \in \mathbf{H}_0$ is supported by $\hat{\Lambda}$ and a bounded continuous function $F(\xi)$ depends only on $X = \xi|_\Lambda$, then,

$$\begin{aligned}
& \int_{\hat{\Lambda}} q_{\Lambda,\xi}(dX) F(X) \langle X_*, \mathbf{f} \rangle \\
=& \frac{1}{Z(\Lambda,\xi)} \int_{\hat{\Lambda}} \hat{\lambda}(dX) F(X) e^{-U(X|\ \xi)} \langle X_*, \mathbf{f} \rangle \\
=& \frac{1}{Z(\Lambda,\xi)} \int_{\hat{\Lambda}} \hat{\lambda}(dY) \mathbf{f}(Y) \int_{\hat{\Lambda}} \hat{\lambda}(dX) F(X+Y) e^{-U(X+Y|\ \xi)} \\
=& \frac{1}{Z(\Lambda,\xi)} \int_{\hat{\Lambda}} \hat{\lambda}(dY) \mathbf{f}(Y) \int_{\hat{\Lambda}} \hat{\lambda}(dX) F(X+Y) e^{-U(Y|\ \xi+X)} e^{-U(X|\ \xi)} \\
=& \int_{\hat{\Lambda}} \hat{\lambda}(dY) \mathbf{f}(Y) \int_{\hat{\Lambda}} q_{\Lambda,\xi}(dX) F(X+Y) e^{-U(Y|\ \xi+X)}.
\end{aligned}$$

Hence,

$$\int_{Q(R)} \mu(d\xi)F(\xi)\langle\xi_*, \mathbf{f}\rangle$$
$$= \int_{\hat{\Lambda}} \hat{\lambda}(dY)\mathbf{f}(Y) \int_{Q(R)} \mu(d\xi)F(\xi+Y)e^{-U(Y|\ \xi)}.$$

This formula is nothing but a generalization of so-called Nguyen-Zessin formula([NZ]).
Consequently, if $\boldsymbol{\rho}(X) > 0$, then the Palm measure μ^X exists and is given by the formula

$$\int_{Q(R)} \mu^Y(d\xi)F(\xi) = \int_{Q(R)} \frac{1}{\boldsymbol{\rho}(Y)}\mu(d\xi)e^{-U(Y|\ \xi)}F(\xi+X).$$

If $\boldsymbol{\rho}(X) = 0$, then the Palm measure is, originally. not defined but one may define it arbitrarily to satisfy the required equation.

□

Lemma 48. *Let μ be a probability Borel measure on $Q(R)$. Assume that μ admits correlation function $\boldsymbol{\rho}(X)$, $X \in \hat{R}$ which grows exponentially. Then, there exists a family of probability Borel measures μ^X, $X \in \hat{R}$ on $Q(R)$ such that*

$$\int_{Q(R)} \mu(d\xi) \int_{\hat{R}} \xi_*(dX)F(X,\xi) = \int_{\hat{R}} \hat{\lambda}(dX)\boldsymbol{\rho}(X) \int_{Q(R)} \mu^X(d\xi)F(X,\xi+X)$$

for any bounded continuous function $F(X,\xi)$ which has compact support in X.

Definition 49. *We call μ^X the generalized Palm measure of μ and the above formula generalized Palm formula.*

Proof. Let us prove the above lemma. Firstly, fixing a bounded continuous function F on $Q(R)$, consider the linear functional

$$\mathbf{f} \mapsto \int_{Q(R)} \mu(d\xi) \int_{\hat{R}} \xi_*(dX)\mathbf{f}(X)F(\xi).$$

it is well-defined at least for $\mathbf{f} \in \mathbf{H}_0$ and defines a measure on $\hat{R}$ which is absolutely continuous with respect to $\boldsymbol{\rho}(X)\hat{\lambda}(dX)$. Denote its density by $\tilde{\mu}^X(F)$.

Next consider the functional $F \mapsto \tilde{\mu}^X(F)$. Then for $\boldsymbol{\rho}(X)\hat{\lambda}(dX) - a.e.\ X$, it is positivity preserving and $\tilde{\mu}^X(1) = 1$. Hence, it defines a probability measure on $Q(R)$. Denote it by $\tilde{\mu}^X(d\xi)$. By definition, $X \leq \xi$ for $\tilde{\mu}^X(d\xi) - a.e.\ \xi$. Consequently, we can define a probability measure μ^X so that

$$\tilde{\mu}^X(F) = \int_{Q(R)} \tilde{\mu}^X(d\xi)F(\xi) = \int_{Q(R)} \mu^X(d\xi)F(\xi+X).$$

Finally, we can suitably define μ^X for X with $\boldsymbol{\rho}(X) = 0$ so that the desired relation holds. □

6.5 Characterization of a Gibbs measure by its Palm measures

Now we will prove that Gibbs measures are characterized by generalized Palm formula.

The following theorem shows that μ is a Gibbs measure if its Palm measures μ^X are absolutely continuous with respect to μ and their Radon-Mykodim densities satisfy some regularity conditions, namely, continuity and Conditions (C1) and (C2).

Theorem 50. *Let $U(X \mid \xi)$ be a cycle the conditions (C1) and (C2) and μ be a probability which admits the correlation function $\boldsymbol{\rho}$. Assume that it satisfies the condition*

$$\boldsymbol{\rho}(X) \int_{Q(R)} \mu^X(d\xi) F(\xi) = \int_{Q(R)} \mu(d\xi) F(\xi) e^{-U(X|\xi)}$$

for all $F \in C_b(Q(R)$. Then μ is a Gibbs measure associated with the cycle $U(X \mid \xi)$.

Proof. We compute the both hand sides of the equation

$$\int_{Q(R)} \mu(d\xi) \int_{\hat{R}} \xi_*(dX) F(X, \xi) = \int_{\hat{R}} \hat{\lambda}(dX) \int_{Q(R)} \mu(d\xi) F(X, \xi + X) e^{-U(X|\xi)}$$

when

$$F(X, \xi) = \mathbf{e}_f(X) e^{-\langle \xi, g \rangle} H(\xi_{\Lambda^c})$$

with $f, g \in C_c(\Lambda)$ and $H \in C_b(Q(\Lambda^c))$ for some compact Λ.

Step 1. Observing

$$\int_{\hat{R}} \xi_*(dX) F(X, \xi) = e^{-\langle \xi, f \rangle} e^{-\langle \xi, g \rangle} H(\xi_{\Lambda^c}),$$

we obtain

$$\begin{aligned}(LHS) &= \int_{Q(R)} \mu(d\xi) e^{-\langle \xi, f+g \rangle} H(\xi_{\Lambda^c}) \\ &= \int_{Q(\Lambda^c)} \mu_{\Lambda^c}(d\eta) H(\eta) \int_{\hat{\Lambda}} \mu_{\Lambda,\eta}(dX) (e^{-f-g})^\wedge(X).\end{aligned}$$

On the other hand,

$$
\begin{aligned}
(RHS) =& \int_{\hat{R}} \hat{\lambda}(dX) \int_{Q(R)} \mu(d\xi) \mathbf{e}_f(X)(e^{-g})^\wedge(X) e^{-\langle \xi, g\rangle} H(\xi_{\Lambda^c}) e^{-U(X|\xi)} \\
=& \int_{\hat{R}} \hat{\lambda}(dX) \int_{Q(\Lambda^c)} \mu_{\Lambda^c}(\eta) H(\eta) \\
& \times \int_{\hat{\Lambda}} \mu_{\Lambda,\eta}(dY) \mathbf{e}_f(X)(e^{-g})^\wedge(X+Y) e^{-\langle \eta, g\rangle} e^{-U(X|\eta+Y)} \\
=& \int_{Q(\Lambda^c)} \mu_{\Lambda^c}(\eta) H(\eta) \int_{\hat{R}} \hat{\lambda}(dX) \mathbf{e}_f(X)(e^{-g})^\wedge(X) \\
& \times \int_{\hat{\Lambda}} \mu_{\Lambda,\eta}(dY)(e^{-g})^\wedge(Y) e^{-\langle \eta, g\rangle}) e^{-U(X|\eta+Y)}
\end{aligned}
$$

Hence, for $\mu_{\Lambda^c} - a.e.\eta$,we obtain the equation

$$
\begin{aligned}
& \int_{\hat{\Lambda}} \mu_{\Lambda,\eta}(dX)(e^{-f-g})^\wedge(X) \\
=& \int_{\hat{\Lambda}} \mu_{\Lambda,\eta}(dY)(e^{-g})^\wedge(Y) \int_{\hat{R}} \hat{\lambda}(dX) \mathbf{e}_f(X)(e^{-g})^\wedge(X) e^{-U(X|\eta+Y)}
\end{aligned}
$$

Step 2. Now recall that $\mathbf{e}_f = S^{-1}((e^{-f})^\wedge)$ and $(S^{-1})^* = T^{-1}$. Thus,

$$
\begin{aligned}
& \int_{\hat{R}} \hat{\lambda}(dX) \mathbf{e}_f(X)(e^{-g})^\wedge(X) e^{-U(X|\eta+Y)} \\
=& \int_{\hat{R}} \hat{\lambda}(dX)(e^{-f})^\wedge(X) T^{-1}(\chi_\Lambda (e^{-g})^\wedge e^{-U(\cdot|\eta+Y)})(X)
\end{aligned}
$$

where χ_Λ stands for the indicator of the set $\{X|X(\Lambda^c) = 0\}$. Here,

$$
\begin{aligned}
T^{-1}(\chi_\Lambda (e^{-g})^\wedge e^{-U(\cdot|\eta+Y)})(X) =& \int_{\hat{\Lambda}} \hat{\lambda}(dZ)(e^{-g})^\wedge(X+Z) e^{-U(X+Z|\eta+Y)} \\
=& (e^{-g})^\wedge(X) \int_{\hat{\Lambda}} \hat{\lambda}(dZ)(-1)^{|Z|}(e^{-g})^\wedge(Z) e^{-U(X+Z|\eta+Y)}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
& \int_{\hat{\Lambda}} \mu_{\Lambda,\eta}(dX)(e^{-f-g})^\wedge(X) \\
=& \int_{\hat{R}} \hat{\lambda}(dX)(e^{-f})^\wedge(X)(e^{-g})^\wedge(X) \mu_{\Lambda,\eta}(dY) \\
& \times \int_{\hat{\Lambda}} \hat{\lambda}(dZ)(-1)^{|Z|}(e^{-g})^\wedge(Z) e^{-U(X+Z|\eta+Y)}.
\end{aligned}
$$

Step 3. Now we consider the above equation as an equality between two functionals in $(e^{-f})^\wedge \in C_b(\hat{\Lambda})$. Then we finds that $\mu_{\Lambda,\eta}(dX)$ is absolutely continuous with respect to $\hat\lambda(dX)$ and its density $\boldsymbol{\sigma}_{\Lambda,\eta}(X)$ satisfies the equation

$$\begin{aligned}\boldsymbol{\sigma}_{\Lambda,\eta}(X) &= \int_{\hat\Lambda} \hat\lambda(dY)\boldsymbol{\sigma}_{\Lambda,\eta}(Y) \int_{\hat\Lambda} \hat\lambda(dZ)(-1)^{|Z|}(e^{-g})^\wedge(Z)e^{-U(X+Z|\eta+Y)} \\ &= \int_{\hat\Lambda} \hat\lambda(dY) \int_{\hat\Lambda} \hat\lambda(dZ)(-1)^{|Z|}(e^{-g})^\wedge(Y+Z)\boldsymbol{\sigma}_{\Lambda,\eta}(Y)e^{-U(X+Z|\eta+Y)}.\end{aligned}$$

Now we set

$$\boldsymbol{\tau}_\Lambda, \eta(Y) = \begin{cases}\boldsymbol{\sigma}_{\Lambda,\eta}(Y)e^{U(Y|\xi)} & \text{if } U(Y|\eta) < \infty, \\ 0 & \text{otherwise.}\end{cases}$$

Notice that

$$U(X+Z|\eta+Y) = \begin{cases} U(X+Y+Z \mid \eta) - U(Y \mid \eta) & \text{if } U(Y|\ \eta) < \infty, \\ \infty & \text{otherwise}\end{cases}$$

for any $X, Y, Z \in \hat R$ and any η by the definition of a cocycle and the condition (C1).

On one hand,

$$\boldsymbol{\tau}_{\Lambda,\eta}(X) = 0 \quad \text{if} \;\; U(X|\ \eta) = \infty.$$

On the other hand, we obtain

$$\begin{aligned}\boldsymbol{\tau}_{\Lambda,\eta}(X) &= \int_{\hat\Lambda} \hat\lambda(dY) \int_{\hat\Lambda} \hat\lambda(dZ)(-1)^{|Z|}(e^{-g})^\Lambda(Y+Z)e^{-U(X+Y+Z|\eta)}\boldsymbol{\tau}_{\Lambda,\eta}(Y) \\ &= \int_{\hat\Lambda} \hat\lambda(dW)(e^{-g})^\Lambda(W)e^{-U(X+W|\eta)}S^{-1}\boldsymbol{\tau}_{\Lambda,\eta}(W)\end{aligned}$$

by Lemma 4 (iii) and the identity

$$S^{-1}\boldsymbol{f} = (-1)^{|\cdot|} * \boldsymbol{f}.$$

Since $g \in C_c(\Lambda)$ is arbitrary chosen, we conclude that

$$e^{-U(X+W|\eta)}S^{-1}\boldsymbol{\tau}_{\Lambda,\eta}(W) = 0 \quad \text{if} \;\; W \neq 0$$

for any X, W and any η. Hence,

$$e^{-U(X|\eta)}\xi^{-1}\boldsymbol{\tau}_{\Lambda,\eta}(X) = 0 \quad \text{if} \;\; X \neq 0.$$

Step 4. Assume $U(X|\eta) < \infty$. Then it follows from Step 3 that

$$S^{-1}\boldsymbol{\tau}_{\Lambda,\eta}(Y) = 0 \quad \text{if} \;\; Y \leqslant X \;\; \text{and} \;\; Y \neq 0.$$

Hence,

$$\boldsymbol{\tau}_{\Lambda,\eta}(Y) = const. \;\; \text{if} \;\; Y \leqslant X.$$

The *const*. here must satisfy

$$const. = \boldsymbol{\tau}_{\Lambda,\eta}(0) = \boldsymbol{\sigma}_{\Lambda,\eta}(0).$$

In particular, it does not depends on the choice of X. Thus, for any $X \in \hat{\Lambda}$

$$\boldsymbol{\tau}_{\Lambda,\eta}(X) = \boldsymbol{\sigma}_{\Lambda,\eta}(0).$$

equivalently,

$$\boldsymbol{\sigma}_{\Lambda,\eta}(X) = \boldsymbol{\sigma}_{\Lambda,\eta}(0)e^{-U(X|\eta)} \quad \text{if} \quad U(X|\eta) < \infty$$

whenever $U(X \mid \eta) < \infty$.

Recall that we already proved that

$$\boldsymbol{\sigma}_{\Lambda,\eta}(X) = 0 \quad \text{if} \quad U(X|\eta) = \infty.$$

Consequently,

$$\boldsymbol{\sigma}_{\Lambda,\eta}(X) = \frac{1}{Z(\Lambda,\eta)}e^{-U(X|\eta)}$$

where

$$Z(\Lambda,\eta) = \int_{\hat{\Lambda}} \hat{\lambda}(dX)e^{-U(X|\eta)}.$$

In other words, μ is a Gibbs measure.

□

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