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Monoids in the fundamental groups of the complement of logarithmic free divisors in $\mathbb{C}^3$

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Contents

Abstract

We study monoids generated by Zariski-van Kampen generators in the 17 fundamental groups of the complement of logarithmic free divisors in $\mathbb{C}^3$ listed by Sekiguchi (Theorem 1). Five of them are Artin monoids and eight of them are free abelian monoids. *The remaining four monoids are not Gaußian and, hence, are neither Garside nor Artin* (Theorem 2). However, *we introduce, similarly to Artin monoids, fundamental elements and show their existence* (Theorem 3). One of the four non-Gaussian monoids satisfies the cancellation condition (Theorem 4).

1 Introduction

A hypersurface D in $\mathbb{C}^l$ ($l \in \mathbb{Z}_{\geq 0}$) is called a *logarithmic free divisor* ([S1]), if the associated module $Der_{\mathbb{C}^l}(-\log(D))$ of logarithmic vector fields is a free $\mathcal{O}_{\mathbb{C}^l}$-module. Classical example of logarithmic free divisors is the discriminant loci of a finite reflection group ([S1,2,3,4]). The fundamental group of the complement of the discriminant loci is presented (Brieskorn [B]) by certain positive homogeneous relations, called Artin braid relations. The group (resp. monoid) defined by that presentation is called an *Artin group* (resp. *Artin monoid*) of finite type [B-S], for which the word problem and other problems are solved using a particular element Δ, the *fundamental elements*, in the monoids ([B-S],[D],[G]).

*The present paper is a complete version with proofs of [S-I1].

In [Se1], Sekiguchi listed up 17 weighted homogeneous polynomials, defining logarithmic free divisors in $\mathbb{C}^3$, whose weights coincide with those of the discriminant of types A_3, B_3 or H_3. Then, the fundamental groups of the complements of the divisors are presented by Zariski-van Kampen method by [I] (we recall the result in §3). It turns out that the defining relations can be reformulated by a system of positive homogeneous relations in the sense explained in §4 of the present paper, so that we can introduce monoids defined by them. We show that, among 17 monoids, 5 are Artin monoids, and 8 are free abelian monoids. However, four remaining monoids are not Gaussian, and hence are neither Garside nor Artin (§5). Nevertheless, we show that they carry certain particular elements similar to the fundamental elements in Artin monoids (§6).

Let us explain more details of the contents. The 17 Sekiguchi-polynomials $\Delta_X(x,y,z)$ are labeled by the type $X \in \{\mathrm{A_i}, \mathrm{A_{ii}}, \mathrm{B_i}, \mathrm{B_{ii}}, \mathrm{B_{iii}}, \mathrm{B_{iv}}, \mathrm{B_v}, \mathrm{B_{vi}}, \mathrm{B_{vii}}, \mathrm{H_i}, \mathrm{H_{ii}}, \mathrm{H_{iii}}, \mathrm{H_{iv}}, \mathrm{H_v}, \mathrm{H_{vi}}, \mathrm{H_{vii}}, \mathrm{H_{viii}}\}$ (§2). They are monic polynomials of degree 3 in the variable z. We calculate the fundamental group of the complement of the divisor $D_X := \{\Delta_X(x,y,z) = 0\}$ in $\mathbb{C}^3$ by choosing Zariski-pencils l in z-coordinate direction, which intersect with the divisor D_X by 3 points. Zariski-van Kampen method gives a presentation of the fundamental group $\pi_1(\mathbb{C}^3 \setminus D_X, *)$ with respect to three generators a, b and c presented by a choice of paths in the pencil turning once around each of three intersection points.

We rewrite the Zariski-van Kampen relations into a system of positive homogeneous relations (not unique, §4 Theorem 1), and study the group G_X and the monoid M_X defined by the relations as well as the localization homomorphism $M_X \to G_X$, where G_X is naturally isomorphic to $\pi_1(\mathbb{C}^3 \setminus D_X, *)$. We denote by G_X^+ the image of M_X in G_X, that is, the monoid generated by the Zariski-van Kampen generators $\{a,b,c\}$ in $\pi_1(\mathbb{C}^3 \setminus D_X, *)$. The G_X^+ depends on the choice of generators but not on homogeneous relations, whereas the monoid M_X does. It turns out that M_X are Artin monoids for the types $\mathrm{A_i}, \mathrm{B_i}, \mathrm{H_i}, \mathrm{A_{ii}}, \mathrm{B_{iv}}$, and are free abelian monoid for the types $\mathrm{B_v}, \mathrm{B_{vii}}, \mathrm{H_{iv}}, \mathrm{H_v}, \mathrm{H_{vi}}, \mathrm{H_{vii}}, \mathrm{H_{viii}}, \mathrm{B_{iii}}$ so that one has natural isomorphisms: $M_X \simeq G_X^+$. However, for any of the remaining four types $\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}, \mathrm{H_{iii}}$, the monoids G_X^+ does not admit the divisibility theory (see [B-S, §5], or §5 Theorem 2 of present paper). That is, they are not Gaussian groups [D-P, §2], and, hence, they are neither Artin nor Garside groups (actually, we have an isomorphism $M_{\mathrm{B_{vi}}} \simeq M_{\mathrm{H_{iii}}}$ and hence $G^+_{\mathrm{B_{vi}}} \simeq G^+_{\mathrm{H_{iii}}}$).

On the other hand, as one main result of the present paper, we show that the monoid M_X carries some distinguished elements, which we call *fundamental* (§6 Theorem 3). Namely, we call an element $\Delta \in M_X$ fundamental if there exists a permutation σ_Δ of the set $\{a,b,c\}/\sim$ (see §6) such that for any $d \in \{a,b,c\}/\sim$, there exists $\Delta_d \in M_X$ such that the following relation holds:

$$\Delta \;=\; d \cdot \Delta_d \;=\; \Delta_d \cdot \sigma_\Delta(d).$$

The set $\mathcal{F}(M_X)$ of fundamental elements in M_X form a submonoid of M_X such that $\mathcal{QZ}(M_X)\mathcal{F}(M_X) = \mathcal{F}(M_X)\mathcal{QZ}(M_X) = \mathcal{F}(M_X)$ (see §6 **Fact 3.**) where $\mathcal{QZ}(M_X)$ is the quasi-center of M_X.[1] For an Artin monoid of finite type, $\mathcal{F}(M_X)$ is generated by a single element Δ and $\mathcal{F}(M_X) = \Delta^{\mathbb{Z}_{\geq 1}}$ ([B-S]). Since the

[1]An element $\Delta \in M_X$ is called quasi-central ([B-S, 7.1]) if $d \cdot \Delta = \Delta \cdot \sigma_\Delta(d)$ for $d \in \{a,b,c\}$.

localization morphism induces a map $\mathcal{F}(M_X)\to\mathcal{F}(G_X^+)$, the fact $\mathcal{F}(M_X)\neq\emptyset$ for all 17 monoids (§6 Theorem3) implies $\mathcal{F}(G_X^+)\neq\emptyset$. We ask, more generally, *whether the monoid generated by Zariski-van Kampen generators in the local fundamental group of the complement of a free divisor has always a fundamental element* (see §6 Remark 6.4). In the 4 types $\mathrm{B_{ii}},\mathrm{B_{vi}},\mathrm{H_{ii}},\mathrm{H_{iii}}$, we observe that $\mathcal{F}(G_X^+)$ is not singly generated. Therefore, we ask, also, *whether the set of fundamental elements $\mathcal{F}(G_X^+)$ is finitely generated over $\mathcal{QZ}(G_X^+)$ or not.*

In §7, we discuss about the cancellation condition on the monoid M_X. In fact, this condition together with the existence of fundamental elements (shown in §6), imply that the localization morphism $M_X\to G_X^+$ is an isomorphism. An Artin monoid or a free abelian monoid satisfies already this condition ([B-S]). We show that *the monoid $M_{\mathrm{B_{ii}}}$ satisfies the cancellation condition* (Theorem4). For the remaining three types $\mathrm{B_{vi}},\mathrm{H_{ii}},\mathrm{H_{iii}}$, we do not know whether the localization map is $M_X\to G_X^+$ is injective or not. That is, we don't know whether we have sufficiently many defining relations to assert the cancellation condition or not.

Finally in §8, we construct non-abelian representations of the groups $G_{\mathrm{B_{ii}}},G_{\mathrm{B_{vi}}}$, $G_{\mathrm{H_{ii}}}$ and $G_{\mathrm{H_{iii}}}$ into $\mathrm{GL}_2(\mathbb{C})$ (Theorem 5). Actually, this result is independent of §5, 6 and 7, and is used in the proof of Theorem 2 in §5.

2 Sekiguchi's Polynomial

J. Sekiguchi [Se1,2] listed the following 17 weighted homogeneous polynomials Δ in three variables (x,y,z) satisfying freeness criterion by K.Saito [S1].

$$\begin{aligned}
\Delta_{\mathrm{A_i}}(x,y,z) &:= -4x^3y^2-27y^4+16x^4z+144xy^2z-128x^2z^2+256z^3\\
\Delta_{\mathrm{A_{ii}}}(x,y,z) &:= 2x^6-3x^4z+18x^3y^2-18xy^2z+27y^4+z^3\\
\Delta_{\mathrm{B_i}}(x,y,z) &:= z(x^2y^2-4y^3-4x^3z+18xyz-27z^2)\\
\Delta_{\mathrm{B_{ii}}}(x,y,z) &:= z(-2y^3+4x^3z+18xyz+27z^2)\\
\Delta_{\mathrm{B_{iii}}}(x,y,z) &:= z(-2y^3+9xyz+45z^2)\\
\Delta_{\mathrm{B_{iv}}}(x,y,z) &:= z(9x^2y^2-4y^3+18xyz+9z^2)\\
\Delta_{\mathrm{B_v}}(x,y,z) &:= xy^4+y^3z+z^3\\
\Delta_{\mathrm{B_{vi}}}(x,y,z) &:= 9xy^4+6x^2y^2z-4y^3z+x^3z^2-12xyz^2+4z^3\\
\Delta_{\mathrm{B_{vii}}}(x,y,z) &:= (1/2)xy^4-2x^2y^2z-y^3z+2x^3z^2+2xyz^2+z^3\\
\Delta_{\mathrm{H_i}}(x,y,z) &:= -50z^3+(4x^5-50x^2y)z^2+(4x^7+60x^4y^2+225xy^3)z\\
&\qquad -(135/2)y^5-115x^3y^4-10x^6y^3-4x^9y^2\\
\Delta_{\mathrm{H_{ii}}}(x,y,z) &:= 100x^3y^4+y^5+40x^4y^2z-10xy^3z+4x^5z^2-15x^2yz^2+z^3\\
\Delta_{\mathrm{H_{iii}}}(x,y,z) &:= 8x^3y^4+108y^5-36xy^3z-x^2yz^2+4z^3\\
\Delta_{\mathrm{H_{iv}}}(x,y,z) &:= y^5-2xy^3z+x^2yz^2+z^3\\
\Delta_{\mathrm{H_v}}(x,y,z) &:= x^3y^4-y^5+3xy^3z+z^3\\
\Delta_{\mathrm{H_{vi}}}(x,y,z) &:= x^3y^4+y^5-2x^4y^2z-4xy^3z+x^5z^2+3x^2yz^2+z^3\\
\Delta_{\mathrm{H_{vii}}}(x,y,z) &:= xy^3z+y^5+z^3\\
\Delta_{\mathrm{H_{viii}}}(x,y,z) &:= x^3y^4+y^5-8x^4y^2z-7xy^3z+16x^5z^2+12x^2yz^2+z^3.
\end{aligned}$$

Here, the polynomials are classified into three types A, B and H according as the numerical data $(\deg(x), \deg(y), \deg(z); \deg(\Delta))$ is equal to $(2, 3, 4; 12)$, $(2, 4, 6; 18)$ or $(2, 6, 10; 30)$, respectively. In each type, the polynomials are numbered by small Roman numerals i, ii,... etc. We remark that, in all cases, the polynomial is a monic polynomial of degree 3 in the variable z.

3 Zariski-van Kampen method

Let X be one of the 17 types $\mathrm{A_i}$,$\mathrm{A_{ii}}$,$\mathrm{B_i}$,..., $\mathrm{B_{vii}}$, $\mathrm{H_i}$,...,$\mathrm{H_{viii}}$. In the present section, we recall from [I] the calculation of the fundamental group $\pi_1(S_X \setminus D_X, *_X)$ of the complement of the free divisor D_X in the space S_X by Zarisik-van Kampen method, where we put $S_X := \mathbb{C}^3$ and

$$D_X := \{(x, y, z) \in \mathbb{C}^3 \mid \Delta_X(x, y, z) = 0\}. \tag{3.1}$$

The first step is the following reduction from the space S_X to a plane H_X.

Lemma 3.1 (Lefschetz Theorem [H-L]). *Let $H_X \subset S_X$ be a hyperplane defined by $x = \varepsilon$ for a general $\varepsilon \in \mathbb{C}^\times$. Then, the natural inclusion induces an isomorphism:*

$$\pi_1(H_X \setminus (H_X \cap D_X), *_X) \xrightarrow{\sim} \pi_1(S_X \setminus D_X, *_X) \tag{3.2}$$

for any choice of a base point $*_X \in H_X \setminus (H_X \cap D_X)$.

The second step is to apply Zariski-van Kampen method, using pencils.

To define pencils, we consider the projection map π from S_X to the space $T_X := \mathbb{C}^2$ of coordinates x, y by forgetting the coordinate z. The fibers of the projection π shall be called the Zariski-pencils. The $\pi|D_X$ is a triple covering map, whose branching loci (or, bifurcation set) B_X is defined by

$$B_X := \{(x, y) \in T_X = \mathbb{C}^2 \mid \omega_X(x, y) = 0\}, \tag{3.3}$$

where $\omega_X(x, y) := \delta\Big(\Delta_X, \frac{\partial \Delta_X}{\partial z}\Big)$ is the resultant of Δ_X and $\frac{\partial \Delta_X}{\partial z}$ with respect to the variable z. In fact, ω_X is a weighted homogeneous polynomial which is monic in the variable y. As we can see explicitly from Table of the equations below, the restriction of ω_X to the line $L_X := \{(x, y) \in T_X \mid x = \varepsilon\}$, where $\varepsilon = -1$ for the type A and $\varepsilon = 1$ for the types B and H, is totally real, i.e. all roots of the equation $\omega_X(\varepsilon, y) = 0$ in y are real numbers, except for $\mathrm{B_{vii}}$ and $\mathrm{H_{vi}}$.

$$\begin{array}{rclrcl}
\omega_{\mathrm{A_i}}(-1, y) &=& -cy^2(27y^2 - 8)^3, & \omega_{\mathrm{A_{ii}}}(-1, y) &=& cy^6(27y^2 - 4), \\
\omega_{\mathrm{B_i}}(1, y) &=& cy^4(1 - 4y)^2(1 - 3y)^3, & \omega_{\mathrm{B_{ii}}}(1, y) &=& cy^6(2 + 3y)^2(1 + 3y), \\
\omega_{\mathrm{B_{iii}}}(1, y) &=& cy^8(9 + 40y), & \omega_{\mathrm{B_{iv}}}(1, y) &=& cy^7(9 - 4y)^2, \\
\omega_{\mathrm{B_v}}(1, y) &=& cy^8(27 + 4y), & \omega_{\mathrm{B_{vi}}}(1, y) &=& cy^5(3 - 64y)^2(2 - y)^3, \\
\omega_{\mathrm{B_{vii}}}(1, y) &=& cy^7(16y^2 + 13y + 8), & \omega_{\mathrm{H_i}}(1, y) &=& cy^2(2 - 5y)^5(2 + 27y)^3, \\
\omega_{\mathrm{H_{ii}}}(1, y) &=& cy^5(4 - 27y)^5(12 - y)^4, & \omega_{\mathrm{H_{iii}}}(1, y) &=& cy^7(1 - 54y)^3, \\
\omega_{\mathrm{H_{iv}}}(1, y) &=& -cy^9(4 + 27y), & \omega_{\mathrm{H_v}}(1, y) &=& -cy^8(1 + y)^2, \\
\omega_{\mathrm{H_{vi}}}(1, y) &=& -cy^8(27y^2 + 14y + 3), & \omega_{\mathrm{H_{vii}}}(1, y) &=& cy^9(4 + 27y), \\
\omega_{\mathrm{H_{viii}}}(1, y) &=& -cy^7(3 + y)^2(32 + 27y). & & &
\end{array}$$

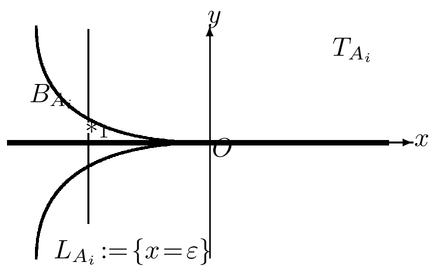

Figure 1: bifurcation set B_{A_i} in T_{A_i}

Remember that $H_X = \pi^{-1}(L_X)$. We apply, now, Zariski-van Kampen method (see [Ch],[T-S] for instance) to calculate the fundamental group $\pi_1(H_X \setminus C_X, *_X)$ of the complement of the plane curve $C_X := H_X \cap D_X$ in the yz-plane. Let us explain the step wisely the process more in details.

1. Choose a base point $*_X$ in $L_X \setminus (L_X \cap B_X)$ and call the associated pencil $l_{*_1} := \pi^{-1}(*_1)$ the *basic pencil*.

2. Choose and fix (i) the base point $*_X \in l_{*_1} \setminus (l_{*_1} \cap D_X)$ and (ii) three mutually disjoint (except at $*_X$) path connecting $*_X$ with the three points $l_{*_1} \cap D_X$ in the basic pencil. Accordingly, fix three the generators, say a, b and c, of the free group $F_3 := \pi_1(l_{*_1} \setminus (l_{*_1} \cap D_X), *_X)$ (they are presented by the movements from $*_X$ to close to the points on D_X along paths, then turn once around the end points of the paths counterclockwise, and then return to $*_X$ along the paths).

3. Move the pencils $l_t := \pi^{-1}(t)$ by moving t along a closed path γ in $L_X \setminus (L_X \cap B_X)$ turning around a bifurcation point in $L_X \cap B_X$. This induces a (braid) action $\gamma_* : F_3 \to F_3$, and we define the relations: $\gamma_*(a) = a, \gamma_*(b) = b, \gamma_*(c) = c$. Running γ over all generators of $\pi_1(L_x \setminus (L_X \setminus (L_X \cap B_X), *_1)$, we obtain all list of defining relations of the group $\pi_1(H_X \setminus C_X, *_X)$.

Actually, in most of the cases except for the cases $X \in \{\mathrm{B_{vii}}, \mathrm{H_{ii}}, \mathrm{H_{vi}}, \mathrm{H_{viii}}\}$, we can find totally real region in L_X in the sense that, if $*_1 \in$ {totally real region}, three roots $l_{*_1} \cap D_X$ of the equation $\Delta_X(\varepsilon, *_1, z) = 0$ with respect to the coordinate z of the pencil are real numbers. In such case, we choose the base point $*_X$ and then the paths a, b, c in the basic pencil l_{*_1} as in Figure 2. along three intervals connecting $*_X$ with the points in $l_{*_1} \cap D_X$.

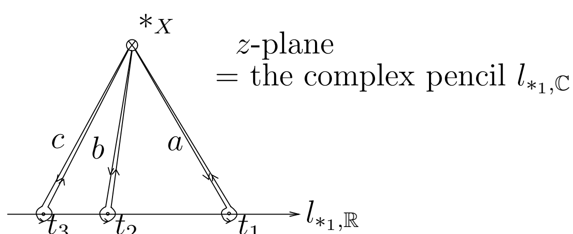

Figure 2. The generators a, b and c (see also Figures 3.1-3.11).

The following Figure 3. briefly describes the real plane curve $C_{X,\mathbb{R}} = H_{X,\mathbb{R}} \cap D_X$ and the real basic pencil $l_{*_1,\mathbb{R}}$ inside the real plane $H_{X,\mathbb{R}}$ for all X except for the cases $X \in \{\mathrm{B_{vii}}, \mathrm{H_{ii}}, \mathrm{H_{vi}}, \mathrm{H_{viii}}\}$.

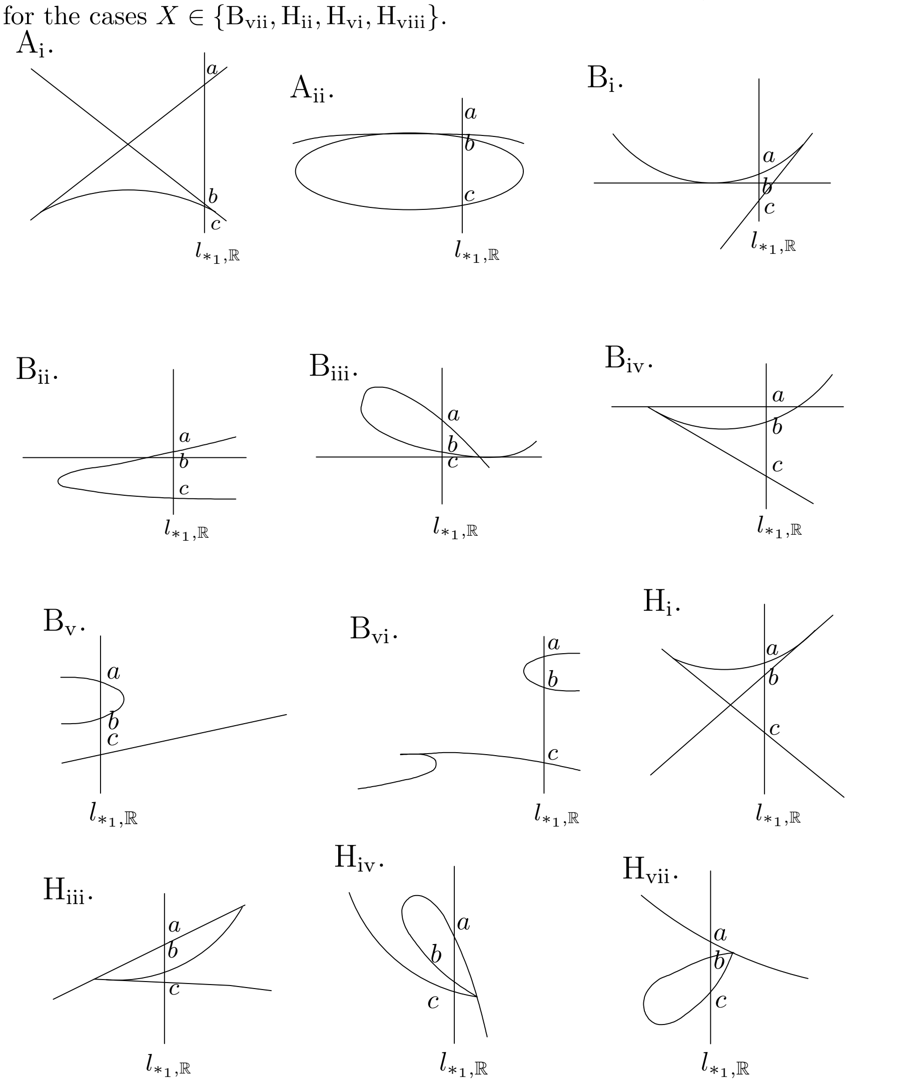

Figure 3. Real plane curve $C_{X,\mathbb{R}}$ and the pencil $l_{*_1,\mathbb{R}}$ in the real plane $H_{X,\mathbb{R}}$.

For the remaining cases $X \in \{\mathrm{B_{vii}}, \mathrm{H_{ii}}, \mathrm{H_{vi}}, \mathrm{H_{viii}}\}$, some more careful considerations are necessary. We briefly indicate the choices of $*_1$ in the (complex) line L_X, the base point $*_X$ and then the paths a, b, c in the basic (complex) pencil l_{*_1} along the intervals connecting $*_X$ and the three points $l_{*_1} \cap D_X$ as in Figure 4.1-4.5. We indicate also the bifurcation points $L_X \cap B_X$ and the paths γ_i which shall be used in the step 3. of Zariski-van Kampen method.

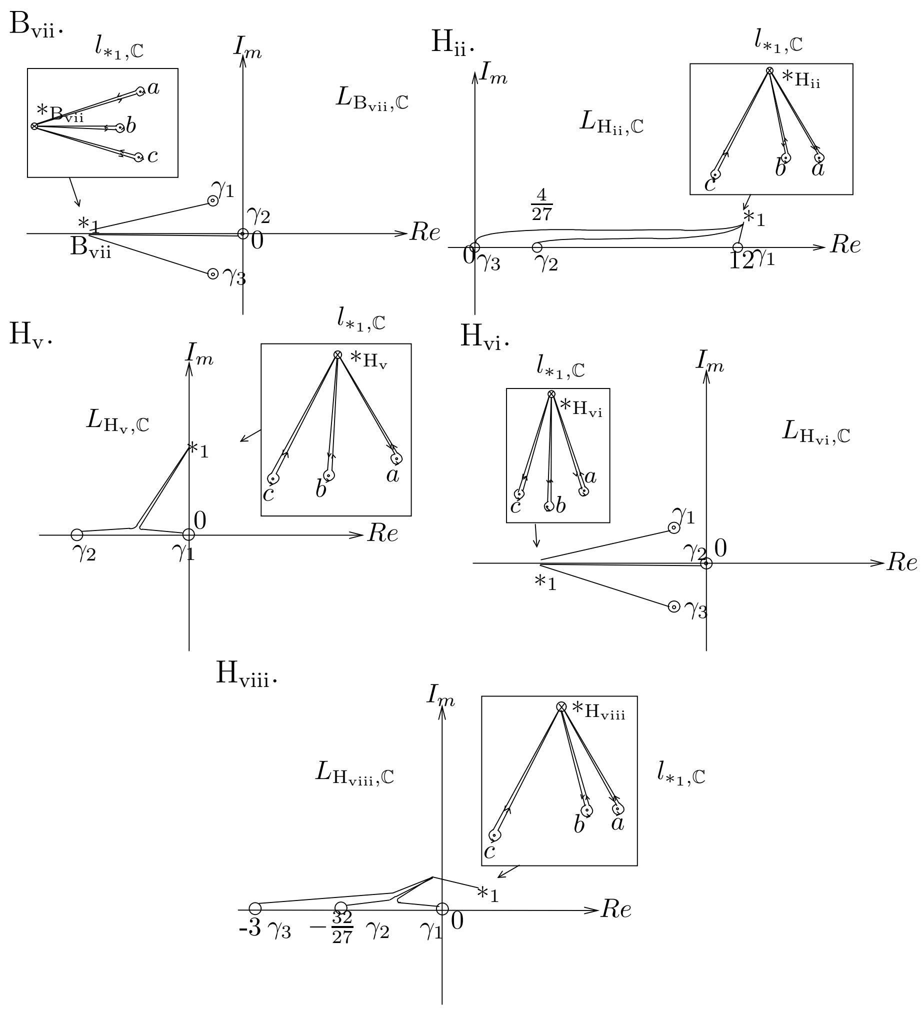

Figure 4. Complex line $B_{X,\mathbb{C}}$ and complex pencil $l_{*_1,\mathbb{C}}$.

For each type X of 17 polynomials, applying Zariski-van Kampen method to the generators $a,\ b$ and c explained above, we obtain the following presentations of the fundamental group $\pi_1(S_X \setminus D_X, *_X) \cong \pi_1(H_X \setminus C_X, *_X)$.

Table 1.

$$\pi_1(S_{\mathrm{A_i}} \setminus D_{\mathrm{A_i}}, *_{\mathrm{A_i}}) \cong \pi_1(H_{\mathrm{A_i}} \setminus C_{\mathrm{A_i}}, *_{\mathrm{A_i}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} ab = ba, \\ bcb = cbc, \\ aca = cac \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{A_{ii}}} \setminus D_{\mathrm{A_{ii}}}, *_{\mathrm{A_{ii}}}) \cong \pi_1(H_{\mathrm{A_{ii}}} \setminus C_{\mathrm{A_{ii}}}, *_{\mathrm{A_{ii}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} ababab = bababa, \\ aba = bab, \\ b = c \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_i}} \setminus D_{\mathrm{B_i}}, *_{\mathrm{B_i}}) \cong \pi_1(H_{\mathrm{B_i}} \setminus C_{\mathrm{B_i}}, *_{\mathrm{B_i}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} abab = baba, \\ bc = cb, \\ aca = cac, \\ cbac = baca \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_{ii}}} \setminus D_{\mathrm{B_{ii}}}, *_{\mathrm{B_{ii}}}) \cong \pi_1(H_{\mathrm{B_{ii}}} \setminus C_{\mathrm{B_{ii}}}, *_{\mathrm{B_{ii}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} ababab = bababa, \\ bc = ab, \\ ac = ca \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_{iii}}} \setminus D_{\mathrm{B_{iii}}}, *_{\mathrm{B_{iii}}}) \cong \pi_1(H_{\mathrm{B_{iii}}} \setminus C_{\mathrm{B_{iii}}}, *_{\mathrm{B_{iii}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} a = b, \\ a = cbab^{-1}c^{-1}, \\ b = cbacbc^{-1}a^{-1}b^{-1}c^{-1}, \\ c = cbacbcb^{-1}c^{-1}a^{-1}b^{-1}c^{-1} \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_{iv}}} \setminus D_{\mathrm{B_{iv}}}, *_{\mathrm{B_{iv}}}) \cong \pi_1(H_{\mathrm{B_{iv}}} \setminus C_{\mathrm{B_{iv}}}, *_{\mathrm{B_{iv}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} acb = cba, \\ bcba = cbac, \\ cbac = bacb, \\ ab = ba \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_v}} \setminus D_{\mathrm{B_v}}, *_{\mathrm{B_v}}) \cong \pi_1(H_{\mathrm{B_v}} \setminus C_{\mathrm{B_v}}, *_{\mathrm{B_v}}) \cong \left\langle a,b,c \;\middle|\; a = b = c \right\rangle.$$

$$\pi_1(S_{\mathrm{B_{vi}}} \setminus D_{\mathrm{B_{vi}}}, *_{\mathrm{B_{vi}}}) \cong \pi_1(H_{\mathrm{B_{vi}}} \setminus C_{\mathrm{B_{vi}}}, *_{\mathrm{B_{vi}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} aba = bab, \\ aca = bac, \\ acaca = cacac \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{B_{vii}}} \setminus D_{\mathrm{B_{vii}}}, *_{\mathrm{B_{vii}}}) \cong \pi_1(H_{\mathrm{B_{vii}}} \setminus C_{\mathrm{B_{vii}}}, *_{\mathrm{B_{vii}}})$$

$$\cong \left\langle a,b,c \;\middle|\; \begin{array}{c} a = b^{-1}cbab^{-1}cbab^{-1}cbab^{-1}cba^{-1}b^{-1}c^{-1}ba^{-1}b^{-1}c^{-1}ba^{-1}b^{-1}c^{-1}b, \\ c = bab^{-1}cbab^{-1}cbab^{-1}cbab^{-1}c^{-1}ba^{-1}b^{-1}c^{-1}ba^{-1}b^{-1}c^{-1}ba^{-1}b^{-1}, \\ a = ba^{-1}b^{-1}c^{-1}bab^{-1}cbab^{-1}, \\ cba = bab, cba = bcb, cba = bab^{-1}c^{-1}b^{-1}cbcb \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_i}} \setminus D_{\mathrm{H_i}}, *_{\mathrm{H_i}}) \cong \pi_1(H_{\mathrm{H_i}} \setminus C_{\mathrm{H_i}}, *_{\mathrm{H_i}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} ababa = babab, \\ bc = cb, \\ aca = cac \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{ii}}} \setminus D_{\mathrm{H_{ii}}}, *_{\mathrm{H_{ii}}}) \cong \pi_1(H_{\mathrm{H_{ii}}} \setminus C_{\mathrm{H_{ii}}}, *_{\mathrm{H_{ii}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} abab = baba, \\ aca = bac, \\ acaca = cacac \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{iii}}} \setminus D_{\mathrm{H_{iii}}}, *_{\mathrm{H_{iii}}}) \cong \pi_1(H_{\mathrm{H_{iii}}} \setminus C_{\mathrm{H_{iii}}}, *_{\mathrm{H_{iii}}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} aba = bab, \\ bcba = cbac, \\ cba = acb \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{iv}}} \setminus D_{\mathrm{H_{iv}}}, *_{\mathrm{H_{iv}}}) \cong \pi_1(H_{\mathrm{H_{iv}}} \setminus C_{\mathrm{H_{iv}}}, *_{\mathrm{H_{iv}}}) \cong \left\langle a,b,c \;\middle|\; a = b = c \right\rangle.$$

$$\pi_1(S_{\mathrm{H_v}} \setminus D_{\mathrm{H_v}}, *_{\mathrm{H_v}}) \cong \pi_1(H_{\mathrm{H_v}} \setminus C_{\mathrm{H_v}}, *_{\mathrm{H_v}}) \cong \left\langle a,b,c \;\middle|\; \begin{array}{c} acba = cbac, \\ bcbac = cbacb, \\ bacb = cbac, \\ bc = cb \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{vi}}} \setminus D_{\mathrm{H_{vi}}}, *_{\mathrm{H_{vi}}}) \cong \pi_1(H_{\mathrm{H_{vi}}} \setminus C_{\mathrm{H_{vi}}}, *_{\mathrm{H_{vi}}}) \cong \left\langle a,b,c \,\middle|\, \begin{array}{c} abababab = babababa, \\ ba = cb, \\ ac = ba \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{vii}}} \setminus D_{\mathrm{H_{vii}}}, *_{\mathrm{H_{vii}}}) \cong \pi_1(H_{\mathrm{H_{vii}}} \setminus C_{\mathrm{H_{vii}}}, *_{\mathrm{H_{vii}}}) \cong \left\langle a,b,c \,\middle|\, \begin{array}{c} a = cbaca^{-1}b^{-1}c^{-1}, \\ b = cbacbc^{-1}a^{-1}b^{-1}c^{-1}, \\ c = cbacbab^{-1}c^{-1}a^{-1}b^{-1}c^{-1}, \\ b = c \end{array} \right\rangle.$$

$$\pi_1(S_{\mathrm{H_{viii}}} \setminus D_{\mathrm{H_{viii}}}, *_{\mathrm{H_{viii}}}) \cong \pi_1(H_{\mathrm{H_{viii}}} \setminus C_{\mathrm{H_{viii}}}, *_{\mathrm{H_{viii}}}) \cong \left\langle a,b,c \,\middle|\, \begin{array}{c} abababa = bababab, \\ ab = bc, \\ ac = ca \end{array} \right\rangle.$$

4 Positive Homogeneous Presentation

In the present section, we rewrite the presentations of the fundamental groups in section 3 to a positive homogeneous form. We, first, prepare some terminology.

Definition. 1. Let $G = \langle L \mid R \rangle$ be a presentation of a group G, where L is the set of generators (called alphabets) and R is the set of relations. We call that the presentation is *positive homogeneous*, if R consists of relations of the form $R_i = S_i$ where R_i and S_i are positive words in the letters L (i.e. words consisting of only non-negative powers of the letters in L) of the same length.

2. If a positive homogeneous presentation $\langle L \mid R \rangle$ of a group G is given, then we associate a monoid M defined as the quotient of free monoid L^* generated by L by the equivalence relation $\simeq$ defined as follows:

1) two words U and V in L^* are called *elementarily equivalent* if either $U = V$ or V is obtained from U by substituting a substring R_i of U by S_i where $R_i = S_i$ is a relation of R ($S_i = R_i$ is also a relation if $R_i = S_i$ is a relation),

2) two words U and V in L^* are called *equivalent*, denoted by $U \simeq V$, if there exists a sequence $U = W_0, W_1, \cdots, W_n = V$ of words in L^* for $n \in \mathbb{Z}_{\geq 0}$ such that W_i is elementarily equivalent to W_{i-1} for $i = 1, \cdots, n$.

3. The natural homomorphism $M \to G$ will be called the *localization morphism.* The image of the localization homomorphism is denoted by G^+.

Note. 1. The monoid G^+ depends on the choice of the generators for the group G. Even if we choose the same generators for the same group G, the monoid M depends on the choice of the relations R.

2. Due to the homogeneity of the relations, one defines a homomorphism:

$$l \; : \; G \longrightarrow \mathbb{Z}$$

by associating 1 to each letter in L. The restriction of the homomorphism on G^+ and its pull-back to M by the localization homomorphism are called *length functions.* Length functions have the additivity: $l(UV) = l(U) + l(V)$ and the conicity: $l(U) = 1$ implies $U = 1$. The existence of such length functions implies that the monoids M and G^+ are *atomic* ([D-P, §2]).

Theorem 1. *The fundamental group in* **Table 1.** *of type X is naturally isomorphic to the following positive homogeneously presented group G_X by identifying the generators $\{a, b, c\}$ in the both groups.*

$$\mathrm{A_i} \quad : G_{\mathrm{A_i}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} ab = ba, \\ bcb = cbc, \\ aca = cac \end{array} \right\rangle .$$

$$\mathrm{A_{ii}} \quad : G_{\mathrm{A_{ii}}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} aba = bab, \\ b = c \end{array} \right\rangle .$$

$$\mathrm{B_i} \quad : G_{\mathrm{B_i}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} abab = baba, \\ bc = cb, \\ aca = cac \end{array} \right\rangle .$$

$$\mathrm{B_{ii}} \quad : G_{\mathrm{B_{ii}}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} cbb = bba, \\ bc = ab, \\ ac = ca \end{array} \right\rangle .$$

$$\mathrm{B_{iii}} \quad : G_{\mathrm{B_{iii}}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} a = b, \\ ac = ca \end{array} \right\rangle .$$

$$\mathrm{B_{iv}} \quad : G_{\mathrm{B_{iv}}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} ab = ba, \\ bcb = cbc, \\ ac = ca \end{array} \right\rangle .$$

$$\mathrm{B_v} \quad : G_{\mathrm{B_v}} := \left\langle a,b,c \,\middle|\, a = b = c \right\rangle .$$

$$\mathrm{B_{vi}} \quad : G_{\mathrm{B_{vi}}} := \left\langle a,b,c \,\middle|\, \begin{array}{l} aba=bab,\ bcb=cbc,\ aca=bac,\ cab=bca,\ acb=cac, \\ abb = bbc,\ bcca = ccac,\ bbac = caab,\ cbbb = bbba, \\ acbcb=bccca, accbb=bccba, accaa=ccaac, caacc=aacca, \\ acccc = bcccb,\ bbaac = cbaab,\ caaab = abaac, \\ a^5 = b^5 = c^5,\ ccbaac = accbaa \end{array} \right\rangle .$$

$$\mathrm{B_{vii}} \quad : G_{\mathrm{B_{vii}}} := \left\langle a,b,c \,\middle|\, a = b = c \right\rangle .$$

$$\mathrm{H_i} \quad : G_{\mathrm{H_i}} := \left\langle a,b,c \,\middle|\, \begin{array}{c} ababa = babab, \\ bc = cb, \\ aca = cac \end{array} \right\rangle .$$

$$\mathrm{H_{ii}} \quad : G_{\mathrm{H_{ii}}} := \left\langle a,b,c \,\middle|\, R_{\mathrm{H_{ii}}} \right\rangle \quad (R_{\mathrm{H_{ii}}} \text{ is given at the end of present Table}).$$

$$\mathrm{H_{iii}} \quad : G_{\mathrm{H_{iii}}} := \left\langle a,b,c \,\middle|\, \begin{array}{l} aba=bab,\ aca=cac,\ bcb=abc,\ cba=acb,\ bca=cbc, \\ baa = aac,\ accb = ccbc,\ aabc = cbba,\ caaa = aaab, \\ bcaca=acccb, bccaa=accab, bccbb=ccbbc, cbbcc=bbccb, \\ bcccc = accca,\ aabbc = cabba,\ cbbba = babbc, \\ a^5 = b^5 = c^5,\ ccabbc = bccabb \end{array} \right\rangle .$$

$$\mathrm{H_{iv}} \quad : G_{\mathrm{H_{iv}}} := \left\langle a,b,c \,\middle|\, a = b = c \right\rangle .$$

$$\mathrm{H_v} \quad : G_{\mathrm{H_v}} := \left\langle a,b,c \,\middle|\, a = b = c \right\rangle .$$

$$\mathrm{H_{vi}} \quad : G_{\mathrm{H_{vi}}} := \left\langle a,b,c \,\middle|\, a = b = c \right\rangle .$$

$$\mathrm{H_{vii}} \quad : G_{\mathrm{H_{vii}}} := \left\langle a,b,c \,\middle|\, a = b = c \right\rangle .$$

$$\mathrm{H_{viii}} \quad : G_{\mathrm{H_{viii}}} := \left\langle a,b,c \,\middle|\, a = b = c \right\rangle .$$

$$
R_{\mathrm{H_{ii}}} := \left\{ \begin{array}{l}
abab = baba, aca = bac, bcbc = cbcb, acb = cac, bbcaba = abccac, \\
abbbca = baaaac, bbbabb = abbaaa, baaaaba = abbbbab, \\
baabbb = aaabaa, abccc = cccab, bbcbab = cccaac, \\
cccbcaa = bbbccab, bccbbb = cccbcc, bbccab = caaccc, \\
ccaac = bccaa, ccaab = accaa, ccabaac = accbcaa, \\
caaccab = bcaacca, aabaaa = bbbaab, bbbaaa = aaabbb, \\
abaaaab = babbbba, aaabba = bbabbb, baabbaa = aabbaab, \\
baabaabaa = abbabbabb, aabbaac = babbcaa, aaabc = bcaaa, \\
abbaabaac = babbabbca, cccaaa = aaaccc, cccbbb = bbbccc, \\
caacaac = aabccba, bbbcbb = cbbccc, abacbc = cbcaba, \\
cbbbbcb = bccccbc, cabbbc = accccb, bcccccaa = cbbbcaac, \\
ccbccc = bbbccb, cbcaaab = bcccaba, caabcb = baccca, \\
bcbaab = aaccba, baaccbbc = caccabcb, bccabb = accaaa, \\
babcbab = cabcaca, caabbbbcb = bacccccca, cbaacc = bccbab, \\
abcbaa = ccbabb, bcbbaa = ccbbab, caacac = babcca, \\
cbbaaaacc = acacbbcba, caaaacc = aabccca, bcabbcc = aabbcbb, \\
bbcaabc = cccaabb, cbbcaab = bccabba, bbaabba = abbaabb, \\
abaabcc = bbabbcb, bacbcab = cabcaba, cbcabca = bcaacab, \\
caaccbba = bcabcaab, babbcbb = aabccbc, bbcbbb = cccbbc, \\
bcbbbbc = cbccccb, bccbbabbc = abcabccba, bbabcbbab = cbbabbccc, \\
cabaaccc = abbcbbab, bacabc = cbcabb, bcabaab = abcabaa, \\
aaccbcab = bcabaacc, cbaabcc = baccbca, cccbaabc = baaccaba, \\
bccbaabc = cabacbca, abaabcaba = bbaabcabb, ccbbaaa = aaaccbb, \\
ccbbaabca = abcabbaac, baabcabba = abacabaab, bcaabb = aaacca, \\
accbbcc = ccabbcb, bbcabbccc = abbabbccb, bcaaccbc = abcabcca, \\
cabaabcc = babccbca, babccba = cbbabcb
\end{array} \right\}
$$

Proof. Except for the types $\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}, \mathrm{H_{iii}}, \mathrm{H_{viii}}$, the relations are obtained by elementary reductions of the Zariski-van Kampen relations, and we omit details.

Some new relations for the cases of types $\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}, \mathrm{H_{iii}}$ are obtained by cancelling common factors from the left or from the right of equivalent expressions of the same fundamental elements (introduced in §6 6.1. See §7 Definition 7.1), where these equivalent expressions of a fundamental element are obtained by the help of Hayashi's computer program (see http://www.kurims.kyoto-u.ac.jp/ saito/SI/). In the following, we sketch how some of them are obtained by hand calculations. In the proof, "the first relation, the second relation, ...", mean "the relation which is at the first place, the second place, ... in Table 1. of Zariski-van Kampen relations in §3".

The case for the type $\mathrm{H_{viii}}$ needs to be treated separately because its calculations are non-trivial. Detailed verifications are left to the reader.

$\mathrm{B_{ii}}$: Using $ab = bc$, rewrite the LHS $ababab$ (resp. RHS $bababa$) of the first relation to $bcabbc$ (resp. $babbca$). Then, using the commutativity of a and c, we cancel ba from left and c from right so that we obtain a new relation $cbb = bba$.

$\mathrm{B_{vi}}$: Using $aca = bac$, rewrite the LHS $acaca$ of the third relation to $acbac$ so that the relation turns to $acbac = cacac$. We cancel ac from right and obtain a new relation $acb = cac$. Using this, one has $bcbac = bcaca = bacba = acaba = cacab = cbacb = cbcac$. We cancel ac from right and obtain $bcb = cbc$. Using this, one has $acabc = bacbc = babcb = abacb = abcac$. Cancelling a and c for left and right, we obtain a new relation $cab = bca$. Using this, one has $cabba = bcaba = bcbab = cbcab = cbbca$. Cancelling c and a for left and right, we obtain a new relation $abb = bbc$. The last relation of length 4 is obtained by cancelling a from left of the equality: $abbac = abbac = bbcac = bbacb = bacab = acaab$.

H_{ii}: Using $aca = bac$, rewrite the LHS $acaca$ of the third relation to $acbac$ so that the relation turns to $acbac = cacac$. We cancel ac from right so that we obtain a new relation $acb = cac$.

$\mathrm{H}_{\mathrm{iii}}$: Multiply b to the second relation from the right, and rewire the LHS to $bcaba$ (by a use of $bab = aba$ and rewrite the RHS to $cbcba$ (by a use of $acb = cba$). Cancelling by ba from right, we obtain a new relation $bca = cbc$.

Using the length 3 relations, on has $acabc = acbcb = cbacb = cbcba = cabca = cacbc$. Cancelling by bc from right, we obtain a new relation $aca = cac$.

Using the length 3 relations, on has $bcaac = cbcac = cbaca = acbca = abcaa = bcbaa$. Cancelling by bc from left, we obtain a new relation $aac = baa$.

In the above sequence, the middle term $acbca$ is also equivalent to $accbc$. Thus, cancelling c from right, we obtain a new relation $accb = cbca (= bcaa)$.

$\mathrm{H}_{\mathrm{viii}}$: From the defining relations, we have $abababa = bcbcbca$, $bababab = bbcbcbc$, and, hence, $bcbcbca = bbcbcbc$. Dividing by b from the left, we get $cbcbca = bcbcbc$. The left hand side of this equality is equivalent to $cabbca = acbbca$, and the right hand side of the equality is equivalent to $abbcbc$ so that $acbbca \simeq abbcbc$. dividing by a from the left, we get $bbcbc \simeq cbbca \simeq cbbac$. Dividing by c from the light, we get $cbba \simeq bbcb (\simeq babb)$ (1). Multiplying $cbcb$ from the right, we get $cbbacbcb \simeq bbcbcbcb$. The right hand side is equivalent to $bbcbcbcb \simeq bcbcbcbc \simeq cbcbcbcc \simeq cbcbcabc \simeq cbcbacbc$. The left hand side is equivalent to $cbbacbcb \simeq cbbcabcb \simeq cbababcb$, and hence $cbababcb \simeq cbcbacbc$. dividing by cb from the left, we get $cbacbc \simeq ababcb$. The left hand side is equivalent to $cbacab \simeq cbaacb$. Dividing by cb from the right, we get $abab \simeq cbaa$ (2). Mutiplying b from the right, the left hand side is equivalent to $acbba \simeq cabba \simeq cbcba$ so that $cbcba \simeq cbaab$. Dividing by cb from the left, we get $cba \simeq aab$ (3). Applying (3) to the equality (2), we get $abab \simeq cbaa \simeq aaba$. Dividing by a from the left, we get $bab \simeq aba \simeq bca$. Dividing by b from the left, we get $ab \simeq ca = ac$, and hence $b = c$. □

Notation. *For each type* $X \in \{\mathrm{A_i}, \mathrm{A_{ii}}, \mathrm{B_i}, \mathrm{B_{ii}}, \mathrm{B_{iii}}, \mathrm{B_{iv}}, \mathrm{B_v}, \mathrm{B_{vi}}, \mathrm{B_{vii}}, \mathrm{H_i}, \mathrm{H_{ii}}, \mathrm{H_{iii}}, \mathrm{H_{iv}}, \mathrm{H_v}, \mathrm{H_{vi}}, \mathrm{H_{vii}}, \mathrm{H_{viii}}\}$*, we denote by* G_X*,* M_X *and* G_X^+ *the* **group***, the* **monoid** *and the* **image of localization***:* $M_X \to G_X$*, respectively, associated with the positive homogeneous relations of type* X *given in* **Theorem. 1**.

From the presentations, we immediately observe the followings.

Corollary. i) *For the type* $X \in \{\mathrm{A_i}, \mathrm{A_{ii}}, \mathrm{B_i}, \mathrm{B_{iv}}, \mathrm{H_i}\}$*, the monoid* M_X *and the group* G_X *is an Artin monoid and an Artin group of type* A_3, A_2, B_3, A_3, $\mathrm{A}_1 \times \mathrm{A}_2$ *and* H_3*, respectively. We have the natural isomorphisms:* $M_X \simeq G_X^+$.

ii) *For the type* $X \in \{\mathrm{B_v}, \mathrm{B_{vii}}, \mathrm{H_{iv}}, \mathrm{H_v}, \mathrm{H_{vi}}, \mathrm{H_{vii}}, \mathrm{H_{viii}}\}$*, the monoid* M_X *and the group* G_X *is the infinite cyclic monoid* $\mathbb{Z}_{\geq 0}$ *and group* $\mathbb{Z}$*, respectively. The monoid* $M_{\mathrm{B_{iii}}}$ *and the group* $G_{\mathrm{B_{iii}}}$ *is a free abelian monoid* $(\mathbb{Z}_{\geq 0})^2$ *and group* $\mathbb{Z}^2$ *of rank 2. We have the natural isomorphisms:* $M_X \simeq G_X^+$.

iii) *The correspondence:* $\{a \mapsto b, b \mapsto a, c \mapsto c\}$ *induces an isomorphism:*

$$M_{\mathrm{B_{vi}}} \quad \simeq \quad M_{\mathrm{H_{iii}}}$$

and, hence, also the isomorphisms: $G_{\mathrm{B_{vi}}} \simeq G_{\mathrm{H_{iii}}}$ *and* $G^+_{\mathrm{B_{vi}}} \simeq G^+_{\mathrm{H_{iii}}}$.
(*Proof.* We can show that the Zariski-van Kampen relations of one of the two types can be deduced, up to the transposition of a and b, from that of the other type. □) Note that the isomophism does not identify the Coxeter elements.

As the consequence of **Corollary**, in the rest of the present paper, we shall focus our attention to the remaining 4 types $\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}$ and $\mathrm{H_{iii}}$ together with the "constraint $\mathrm{B_{vi}} \simeq \mathrm{H_{iii}}$".

Remark 4.1. The group G_X is naturally isomorphic to the fundamental group, which does not depend on the choice of Zariski-van Kampen generators $\{a, b, c\}$, but the monoid G_X^+ depends on that choice (see next Remark 4.2).

Further more, the monoid M_X, a priori, depends on the choice of relations in Theorem 1. The isomorphism $M_X \simeq G_X^+$ in the above corollary follows from cancellation conditions on M_X (see [B-S]). We shall show that, also for $M_{\mathrm{B_{ii}}}$ in §7, the cancellation condition holds, implying $M_{\mathrm{B_{ii}}} \simeq G^+_{\mathrm{B_{ii}}}$. Thus, for these cases as a consequence of the cancellation condition, M_X does not depend on the choice of relations in Theorem 1. However, for the remaining types $\mathrm{B_{vi}}, \mathrm{H_{ii}}$ and $\mathrm{H_{iii}}$, it may be still possible that we need more relations in order to obtain the isomorphism $M_X \simeq G_X^+$.

Remark 4.2. Recall that, in the present paper, the generators a, b, c are presented by the paths, which start from the base point $*_X$ and move along the intervals connecting $*_X$ and the three points $D_X \cap l_{*1}$ in the pencil $l_{*1,\mathbb{C}}$ and turn once counterclockwise the points $D_X \cap l_{*1}$ and then return to $*_X$ along the interval (see Fig. 2). Then, the set of the tuples generator system a, b, c explained in §3.3 admits the action of the braid group $B(3)$ of three strings, which changes associated relations. Here is a remarkable observation.

Assertion. *Recall the projection* $\pi : S_{\mathrm{B_{ii}}} \simeq \mathbb{C}^3 \to T_{\mathrm{B_{ii}}} \simeq \mathbb{C}^2$*. Then, for any choice of Zariski-van Kampen generator system* $\{a, b, c\}$ *(up to a permutation) in a pencil with respect to* π *(i.e. a fiber of* π*) admit only one of the following two presentations I. and II.*

$$\mathrm{I}: \qquad \left\langle a, b, c \,\middle|\, \begin{array}{c} cbb = bba, \\ bc = ab, \\ ac = ca \end{array} \right\rangle .$$

$$\mathrm{II}: \qquad \left\langle a, b, c \,\middle|\, \begin{array}{c} ababab = bababa, \\ b = c, \\ aabab = baaba \end{array} \right\rangle .$$

Corollary. *The groups $G_{\mathrm{B_{vi}}}$ and $G_{\mathrm{H_{iii}}}$ do not admit Artin group presentation with respect to any Zariski-van Kampen type generator system.*

Proof. Due to Theorem 1., both groups have the relations: $a^5 = b^5 = c^5$, which are invariant by the change of generator system by the braid group $B(3)$. □

5 Non-division property of the monoid G_X^+

In the present section, we show that none of the monoids G_X^+ of the four types $\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}$ and $\mathrm{H_{iii}}$ does admit the divisibility theory ([B-S, §4]), and therefore the monoid is neither Gaussian, Garside nor Artin.

We first recall some terminology and concepts on the monoid G^+.
An element $U \in G^+$ is said to *divide* $V \in G^+$ from the left (resp. right), denoted by $U|_l V$ (resp. $U|_r V$), if there exists $W \in G^+$ such that $V = UW$ (resp. $V = WU$). We also say V is *left-divisible* by U, or V is a *left-multiple* of U.

We say that G^+ *admits the left* (resp. *right*) *divisibility theory*, if for any two elements U, V of G_X^+, there always exists their left (resp. right) least common multiple, i.e. a left (resp. right) common multiple which divides any other left (resp. right) common multiple, denoted by $\mathrm{lcm_l}(U,V)$ (resp. $\mathrm{lcm_r}(U,V)$).

Theorem 2. *The monoids $G^+_{\mathrm{B_{ii}}}, G^+_{\mathrm{B_{vi}}}, G^+_{\mathrm{H_{ii}}}, G^+_{\mathrm{H_{iii}}}$ admits neither the left-divisibility theory nor the right divisibility theory.*

Proof. We claim a fact, which shall be proven in §8 Theorem 5 ii) independent of the results of §5, 6 and 7.

Fact. None of the groups $G_{\mathrm{B_{ii}}}, G_{\mathrm{B_{vi}}}, G_{\mathrm{H_{ii}}}$ and $G_{\mathrm{H_{iii}}}$ is abelian.

Assuming that the monoid G_X^+ admits the left division theory, we show that G_X becomes an abelian group: a contradiction! to **Fact**. The case for the right-division theory can be shown similarly.

1) $G^+_{\mathrm{B_{ii}}}$: It is immediate to see $l(\mathrm{lcm}_l(b,c)) > 2$ from the defining relations in Theorem 1. Then, $bba = cbb$ is a common multiple of b and c of the shortest length 3, and, hence, should be equal to $\mathrm{lcm}_l(b,c)$. On the other hand, we have the following sequence of elementary equivalent words: $bcba$, $bbba$, $acbb$, $cabb$. That is, $bcba = cabb$ in $G^+_{\mathrm{B_{ii}}}$ is another common left-multiple of b and c. If $bba = cbb$ divides $bcba = cabb$ from the left, there exists $d \in \{a,b,c\}$ such that $bcba = bbad$. So, in $G^+_{\mathrm{B_{ii}}}$, we have $cba = bad$ which is again a common left-multiple of b and c. Thus, we have the equality: $cba = cbb$ in $G^+_{\mathrm{B_{ii}}}$. That is, $a = b$ in $G^+_{\mathrm{B_{ii}}}$. By adding this relation $a = b$ to the set of the defining relations of the group $G_{\mathrm{B_{ii}}}$, we get $G_{\mathrm{B_{ii}}} \simeq \mathbb{Z}$. A contradiction!

2) $G^+_{\mathrm{B_{vi}}}$: Due to the first defining relation in Theorem 1., we have $l(\mathrm{lcm}_l(a,b)) \leq 3$. Let us consider 3 cases:

i) $l(\mathrm{lcm}_l(a,b)) = 1$. This means $l(\mathrm{lcm}_l(a,b)) = a = b$. By adding this relation to the defining relation of the group $G_{\mathrm{B_{vi}}}$, we get $G_{\mathrm{B_{vi}}} \simeq \mathbb{Z}$. A contradiction!

ii) $l(\mathrm{lcm}_l(a,b)) = 2$. This means that there exists $u, v \in \{a,b,c\}$ such that $l(\mathrm{lcm}_l(a,b)) = au = bv$. Depending on each choice of u and v, one can show that this assumption leads to a contradictory conclusion $G_{\mathrm{B_{vi}}} \simeq \mathbb{Z}$. Details are left to the reader.

iii) $l(\mathrm{lcm}_l(a,b)) = 3$. In view of the first two defining relations in Theorem 1., one has $aba = bab = aca = bac$. By adding this relation to the set of the defining relations of the group $G_{\mathrm{B_{vi}}}$, we get $G_{\mathrm{B_{vi}}} \simeq \mathbb{Z}$. A contradiction!.

3) $G^+_{\mathrm{H_{ii}}}$: Due to the second defining relation in Theorem 1., we have $l(\mathrm{lcm}_l(a,b)) \leq 3$. Let us consider 3 cases:

i) $l(\mathrm{lcm}_l(a,b)) = 1$. This means $l(\mathrm{lcm}_l(a,b)) = a = b$. By adding this relation to the defining relation of the group $G_{\mathrm{H_{ii}}}$, we get a contradiction $G_{\mathrm{H_{ii}}} \simeq \mathbb{Z}$.

ii) $l(\mathrm{lcm}_l(a,b)) = 2$. This means that there exists $u, v \in \{a,b,c\}$ such that $l(\mathrm{lcm}_l(a,b)) = au = bv$. Depending on each choice of u and v, one can show that this assumption leads to a contradictory conclusion $G_{\mathrm{H_{ii}}} \simeq \mathbb{Z}$. Details are left to the reader.

iii) $l(\mathrm{lcm}_l(a,b)) = 3$. In view of the first two defining relations, one has $\mathrm{lcm}_l(a,b) = aca = bac$, and it divides $abab = baba$ (from left). This means that there exist $d \in \{a,b,c\}$ such that $cd = ba$ in $G_{\mathrm{H_{ii}}}$. For each case $d = a, b$ or c separately, one can show that $G_{\mathrm{H_{ii}}} \simeq \mathbb{Z}$. A contradiction!.

4) $G^+_{\mathrm{H_{iii}}}$: Due to the first defining relation in Theorem 1., we have $l(\mathrm{lcm}_r(a,b)) \leq 3$. Let us consider 3 cases:

i) $l(\mathrm{lcm}_r(a,b)) = 1$. This means $l(\mathrm{lcm}_r(a,b)) = a = b$. By adding this relation to the defining relation of the group $G_{\mathrm{H_{iii}}}$, we get a contradiction $G_{\mathrm{H_{iii}}} \simeq \mathbb{Z}$.

ii)$l(\mathrm{lcm}_r(a,b)) = 2$. This means that there exists $u, v \in \{a,b,c\}$ such that $l(\mathrm{lcm}_r(a,b)) = ua = vb$. Depending on each choice of u and v, one can show that this assumption leads to a contradictory conclusion $G_{\mathrm{H_{iii}}} \simeq \mathbb{Z}$. Details are left to the reader.

iii) $l(\mathrm{lcm}_r(a,b)) = 3$. In view of the first two defining relations, one has $\mathrm{lcm}_r(a,b) = aba = bab = cba = acb$. This leads to a conclusion $G_{\mathrm{H_{iii}}} \simeq \mathbb{Z}$, which is a contradiction!. These complete the proof of Theorem 2. □

Corollary 5.1. *The monoids $G^+_{\mathrm{B_{ii}}}, G^+_{\mathrm{B_{vi}}}, G^+_{\mathrm{H_{ii}}}, G^+_{\mathrm{H_{iii}}}$ are not Gaussian, where a monoid is Gaussian if it is atomic, cancellative and admits divisibility theory ([D-P, §2]). Hence, they are neither Artin groups nor Garside groups.*

6 Fundamental elements of the monoid M_X

Artin monoid of finite type has a particular element, denoted by Δ and called the *fundamental element* ([B-S] §6). We want to generalize the concept for our new setting. However, in view of Theorem 2, we cannot employ the original definition: the left and right least common multiple of the generators. Analyzing equivalent defining properties of the fundamental element for Artin monoid case, we consider two classes of elements in the monoid M: quasi-central elements and fundamental elements, forming submonoids $\mathcal{QZ}(M)$ and $\mathcal{F}(M)$ in

M, respectively, with $\mathcal{F}(M) \subset \mathcal{QZ}(M)$. The goal of the present section is to show $\mathcal{F}(M_X) \neq \emptyset$ for all types X, implying also $\mathcal{F}(G_X^+) \neq \emptyset$ for all types X.

Let M be a monoid given in §4, i.e. defined by a positive homogeneous relations on a generator set L. Let us denote by $L/\sim$ the quotient set of L divided by the equivalence relation generated by the equalities between two alphabets (in the relation set R). An element $\Delta \in M$ is called *quasi-central* ([B-S] 7.1), if there exists a permutation σ_Δ of $L/\sim$ such that

$$a \cdot \Delta = \Delta \cdot \sigma_\Delta(a)$$

holds for all generators $a \in L/\sim$. The set of all quasi-central elements is denoted by $\mathcal{QZ}(M)$. The following is an immediate consequence of the definition.

Fact 2. *The $\mathcal{QZ}(M)$ is closed under the product. For two elements $\Delta_1, \Delta_2 \in \mathcal{QZ}(M)$, we have $\sigma_{\Delta_1 \cdot \Delta_2} = \sigma_{\Delta_2} \cdot \sigma_{\Delta_1}$.*

According to **Fact 2.**, we introduce an anti-homomorphism:

$$\sigma \;:\; \mathcal{QZ}(M) \longrightarrow \mathfrak{S}(L/\sim), \quad \Delta \mapsto \sigma_\Delta.$$

The kernel of σ is the center $\mathcal{Z}(M)$ of the monoid M.

Next, we introduce the concept of a fundamental element.

Definition 6.1. An element $\Delta \in M$ is called *fundamental* if there exists a permutation σ_Δ of $L/\sim$ such that, for any $a \in L/\sim$, there exists $\Delta_a \in G_X^+$ satisfying the following relation:

$$\Delta \;=\; a \cdot \Delta_a \;=\; \Delta_a \cdot \sigma_\Delta(a).$$

We denote by $\mathcal{F}(M)$ the set of all fundamental elements of M. Note that $1 \in \mathcal{QZ}(M)$ but $1 \notin \mathcal{F}(M)$

Fact 3. *The $\mathcal{F}(M)$ is an idealistic submonoid of $\mathcal{QZ}(M)$. That is, the following two properties hold.*

i) *A fundamental element is a quasi-central element: $\mathcal{F}(M) \subset \mathcal{QZ}(M)$. The associated permutation of $L/\sim$ as a fundamental element coincides with that as a quasi-central element.*

ii *Products $\Delta \cdot \Delta'$ and $\Delta' \cdot \Delta$ of a fundamental element Δ and a quasi-central element Δ' are again fundamental elements whose permutation of $L/\sim$ is given in* **Fact 2**. *We have $(\Delta\Delta')_a = \Delta_a\Delta'$, and $(\Delta'\Delta)_a = \Delta'\Delta_{\sigma_{\Delta'}(a)}$.*

$$\mathcal{F}(M)\mathcal{QZ}(M) = \mathcal{QZ}(M)\mathcal{F}(M) = \mathcal{F}(M).$$

Proof. i) We have $a \cdot \Delta = a \cdot \Delta_a \cdot \sigma_\Delta(a) = \Delta \cdot \sigma_\Delta(a)$ for all $a \in L/\sim$.

ii) We prove only the case $\Delta \cdot \Delta'$.

On one side, one has:

$$\Delta \cdot \Delta' \simeq (a \cdot \Delta_a) \cdot \Delta' \simeq a \cdot (\Delta_a \cdot \Delta').$$

On the other side, one has:

$$\begin{aligned}\Delta \cdot \Delta' &\simeq (\Delta_a \cdot \sigma_\Delta(a)) \cdot \Delta' \simeq \Delta_a \cdot (\sigma_\Delta(a) \cdot \Delta') \simeq \Delta_a \cdot (\Delta' \cdot \sigma_{\Delta'}(\sigma_\Delta(a))) \\ &\simeq (\Delta_a \cdot \Delta') \cdot \sigma_{\Delta'}(\sigma_\Delta(a)) \simeq (\Delta_a \cdot \Delta') \cdot \sigma_{\Delta\Delta'}(a)).\end{aligned}$$

□

One basic property of a fundamental element is that it can be a universal denominator for the localization morphism (c.f. §7 Lemma7.2 2.).

Fact 4. *Let Δ be a fundamental element of M. Then, for any $U \in M$, U divides $\Delta^{l(U)}$ from the left and from the right.*

Proof. We prove only for the left division. Right division can be shown similarly. We show the statement by induction on $l(U)$, where the case $l(U)=1$ follows from the definition of a fundamental element. Let $l(U)>1$ and $U \simeq U' \cdot a$. By induction hypothesis, we have $\Delta^{l(U)-1} \simeq U' \cdot V$ for some V. Then, multiplying Δ from the right, we have $\Delta^{l(U)} \simeq U' \cdot V \cdot \Delta \simeq U' \cdot \Delta \cdot \sigma(V) \simeq U' \cdot a \cdot \Delta_a \cdot \sigma(V)$. Here, if V is a word $v_1 \cdots v_n$ then $\sigma(V)$ is a word $\sigma(v_1) \cdots \sigma(v_n)$ □

Remark 6.2. If M is an indecomposable Artin monoid (of finite type), then any non-trivial quasi-central element is fundamental ([B-S] 5.2 and 7.1). That is, one has the "opposite" inclusion: $(\mathcal{QZ}(M) \setminus \{1\}) \subset \mathcal{F}(M)$.

Remark 6.3. By the definition, any fundamental element is divisible from both left and right by all generators in L. However, (non-trivial) quasi-central element in general may not have this property.

(i) $b^3 \in \mathcal{QZ}(M_{\mathrm{B_{ii}}})$ is central. However, it is not divisible by a and c from the left and right.

(ii) $ababa \in M_{\mathrm{B_{ii}}}$ is divisible by all generators from both sides, but it does not belong to $\mathcal{QZ}(M_{\mathrm{B_{ii}}})$.

We state the second main result of the present paper.

Theorem 3. *The following elements Δ_X belong to $\mathcal{F}(M_X)$ for any type X.*

$\mathrm{A_i}$:	$\Delta_{\mathrm{A_i}}$	:=	$(cba)^2$	$\sigma : \binom{a,\ b,\ c}{c,\ b,\ a}$
$\mathrm{A_{ii}}$:	$\Delta_{\mathrm{A_{ii}}}$	:=	aba	$\sigma : \binom{a,\ b{=}c}{b{=}c,\ a}$
$\mathrm{B_i}$:	$\Delta_{\mathrm{B_i}}$	:=	$(cba)^3$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
$\mathrm{B_{ii}}$:	$\Delta_{\mathrm{B_{ii}1}}$	:=	$(ab)^3$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{B_{ii}2}}$	:=	$(bcc)^3 \simeq (cba)^3$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
$\mathrm{B_{iii}}$:	$\Delta_{\mathrm{B_{iii}}}$	:=	ac	$\sigma : \binom{a{=}b,\ c}{a{=}b,\ c}$
$\mathrm{B_{iv}}$:	$\Delta_{\mathrm{B_{iv}}}$	:=	$abcb$	$\sigma : \binom{a,\ b,\ c}{a,\ c,\ b}$
$\mathrm{B_v}$:	$\Delta_{\mathrm{B_v}}$	:=	a	$\sigma : \binom{a{=}b{=}c}{a{=}b{=}c}$
$\mathrm{B_{vi}}$:	$\Delta_{\mathrm{B_{vi}}1}$	:=	$a^5 \simeq b^5 \simeq c^5$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{B_{vi}}2}$	:=	$(aba)^2$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{B_{vi}}3}$	:=	$bccabcb$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{B_{vi}}4}$	:=	$(bbac)^2$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{B_{vi}}5}$	:=	$(acaca)^2$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{B_{vi}}6}$	:=	$(cba)^3$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{B_{vi}}7}$	:=	$(cab)^5$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$

$\mathrm{B_{vii}}$:	$\Delta_{\mathrm{B_{vii}}}$	$:= a$	$\sigma : \binom{a=b=c}{a=b=c}$
$\mathrm{H_{i}}$:	$\Delta_{\mathrm{H_{i}}}$	$:= (cba)^5$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
$\mathrm{H_{ii}}$:	$\Delta_{\mathrm{H_{ii}1}}$	$:= (acaca)^2 \simeq (ac)^5$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{H_{ii}2}}$	$:= (babac)^3 \simeq (cba)^5$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
$\mathrm{H_{iii}}$:	$\Delta_{\mathrm{H_{iii}}1}$	$:= a^5 \simeq b^5 \simeq c^5$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{H_{iii}}2}$	$:= (aba)^2$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{H_{iii}}3}$	$:= accbaca$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{H_{iii}}4}$	$:= (bcba)^2$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{H_{iii}}5}$	$:= (bcbcb)^2 \simeq (bc)^5$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{H_{iii}}6}$	$:= (abc)^3$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
	$\Delta_{\mathrm{H_{iii}}7}$	$:= (cba)^5$	$\sigma : \binom{a,\ b,\ c}{a,\ b,\ c}$
$\mathrm{H_{iv}}$:	$\Delta_{\mathrm{H_{iv}}}$	$:= a$	$\sigma : \binom{a=b=c}{a=b=c}$
$\mathrm{H_{v}}$:	$\Delta_{\mathrm{H_{v}}}$	$:= a$	$\sigma : \binom{a=b=c}{a=b=c}$
$\mathrm{H_{vi}}$:	$\Delta_{\mathrm{H_{vi}}}$	$:= a$	$\sigma : \binom{a=b=c}{a=b=c}$
$\mathrm{H_{vii}}$:	$\Delta_{\mathrm{H_{vii}}}$	$:= a$	$\sigma : \binom{a=b=c}{a=b=c}$
$\mathrm{H_{viii}}$:	$\Delta_{\mathrm{H_{viii}}}$	$:= a$	$\sigma : \binom{a=b=c}{a=b=c}$

Proof. Since the cases for an Artin monoid or a free abelian monoid are classical, we show only the 4 exceptional cases.

$\mathrm{B_{ii}}$:

$\Delta_{\mathrm{B_{ii}}1} := ababab.$
$\Delta_{\mathrm{B_{ii}}1} = a(babab),$
$\Delta_{\mathrm{B_{ii}}1} \simeq bcabab \simeq bacbab \simeq bacbbc \simeq babbac \simeq babbca \simeq (babab)a.$

$\Delta_{\mathrm{B_{ii}}1} \simeq b(ababa),$
$\Delta_{\mathrm{B_{ii}}1} = (ababa)b.$

$\Delta_{\mathrm{B_{ii}}1} \simeq bababa \simeq bbcaba \simeq bbacba \simeq c(bbcba),$
$\Delta_{\mathrm{B_{ii}}1} \simeq bcbcbc \simeq bcabbc \simeq bacbbc \simeq babbac \simeq (bbcba)c.$

$\Delta_{\mathrm{B_{ii}}2} := (bcc)^3.$

$$\begin{aligned}\Delta_{\mathrm{B_{ii}}2} &= b(ccbccbcc) \simeq abcbccbcc \simeq aabbccbcc \simeq aabbccabc\\ &\simeq aabbaccbc \simeq aacbbccbc \simeq caabbccbc \simeq caabbccab\\ &\simeq caabbaccb \simeq caacbbccb \simeq ccaabbccb \simeq ccabcbccb\\ &\simeq ccbccbccb = (ccbccbcc)b.\end{aligned}$$

$$\begin{aligned}\Delta_{\mathrm{B_{ii}}2} &\simeq a(bcbccbcc) \simeq bccbccbcc \simeq bccabcbcc \simeq bccaabbcc\\ &\simeq bcaacbbcc \simeq bcaabbacc \simeq bcabcbacc \simeq bcbccbacc\\ &\simeq bcbccbcca = (bcbccbcc)a.\end{aligned}$$

$$\begin{aligned}\Delta_{\mathrm{B_{ii}}2} &\simeq c(aacbbccb) \simeq aaccbbccb \simeq aacbbaccb \simeq aacbbcacb\\ &\simeq aacbbccab \simeq aacbbccbc = (aacbbccb)c.\end{aligned}$$

B_{vi} : Due to **Remark.** after **Theorem 1.** in §4, we may reduce the proof to the case H_{iii}.

H_{ii} : First, let us show a relation: $acaca = cacac$ ($acaca \simeq acbac \simeq cacac$), which shall be used in the sequel.

$\Delta_{H_{ii}1} := acacaacaca$.
$\Delta_{H_{ii}1} = a(cacaacaca) \simeq cacacacaca \simeq (cacaacaca)a$.
$\Delta_{H_{ii}1} \simeq c(acacacaca) \simeq acacaacaca \simeq (acacacaca)c$.

$\Delta_{H_{ii}1} \simeq acacaacaca \simeq b(accaacaca) \simeq acacaacaca \simeq acacacacac$
$\simeq accacaccac \simeq accaacbcac \simeq accaacbacb \simeq (accaacaca)b$,
$\Delta_{H_{ii}2} := babacbabacbabac \simeq ababcbabacbabac$.

$\Delta_{H_{ii}2} = a(babcbabacbabac) \simeq bababcbabacbabac \simeq babcacabacbabac$
$\simeq babcbacbacbabac \simeq babcbacacababac \simeq babcbaacbababac$
$\simeq babcbaacababbac \simeq babcbabacbabbac \simeq (babcbabacbabac)a$.
$\Delta_{H_{ii}2} = b(abacbabacbabac) \simeq ababcbabacbabac \simeq ababcababcbabac$
$\simeq ababcababcbaaca \simeq ababcbabacbaaca \simeq ababacbaacabaaca$
$\simeq ababcbaacababac \simeq ababcbaacbabaac \simeq ababcbacacabaac$
$\simeq ababcacaacabaac \simeq ababacbaacabaac \simeq abaacabaacabaac$
$\simeq abaacababacbaac \simeq abaacbabaacbaac \simeq abacacabaacbaac$
$\simeq abacbacbaacbaac \simeq abacbacbacacaac \simeq abacbacacaacaac$
$\simeq abacbaacbaacac \simeq abacbaacbabacac \simeq abacbaacababcac$
$\simeq abacbabacbabcac \simeq (abacbabacbabac)b$.
$\Delta_{H_{ii}2} = abacbabacbabacb \simeq aacababacbabacb \simeq aacbabaacbabacb$
$\simeq acacabaacbabacb \simeq acbacbaacbabacb \simeq c(acacbaacbabacb)$
$\simeq acbacbaacbabacb \simeq acacabaacababcb \simeq acacababacbabcb$
$\simeq acacbabaacbabcb \simeq acacbabacacabcb \simeq acacbaacaacabcb$
$\simeq acacbaacabacbcb \simeq acacbaacababcbc \simeq (acacbaacbabacb)c$.

H_{iii} :
$\Delta_{H_{iii}2} := (aba)^2$.
$\Delta_{H_{iii}2} = a(baaba) \simeq bababa \simeq (baaba)a$.
$\Delta_{H_{iii}2} = b(ababa) \simeq abaaba \simeq (ababa)b$.
$\Delta_{H_{iii}2} = abaaba \simeq aaacba \simeq aacbaa \simeq aacaac$

$\Delta_{H_{iii}3} := accbaca$.
$\Delta_{H_{iii}3} = a(ccbaca) \simeq cbcaaca \simeq ccbcaca \simeq (ccbaca)a$.
$\Delta_{H_{iii}3} = accbaca \simeq cbcaaca \simeq b(caaaca)$.
$\Delta_{H_{iii}3} = accbaca \simeq cbcaaca \simeq ccbcaca \simeq caccbca \simeq cacbcaa$
$\simeq caaccba \simeq caacacb \simeq (caaaca)b$.
$\Delta_{H_{iii}3} = accbaca \simeq c(bcaaca)$.
$\Delta_{H_{iii}3} = accbaca \simeq cbcaaca \simeq bcaaaca \simeq= (bcaaca)c$.

$\Delta_{H_{iii}4} := bcbabcba$.
$\Delta_{H_{iii}4} \simeq a(bcabcba) \simeq bcabacba \simeq (bcabcba)a$.

$\Delta_{H_{iii}4} = b(cbabcba) \simeq bcabacba \simeq cbcbacba \simeq cbacbcba$
$\simeq cbabcaba \simeq (cbabcba)b$.

$\Delta_{H_{iii}4} \simeq bcabacba \simeq c(bcbacba) \simeq bcbaabca \simeq bcbaacbc \simeq bcbacbac$.

$\Delta_{\mathrm{H_{iii}}5} := bcbcbbcbcb.$
$\Delta_{\mathrm{H_{iii}}5} \simeq abccbbcbcb \simeq abccbcbcbc \simeq a(bcbcbcbbc),$
$\Delta_{\mathrm{H_{iii}}5} \simeq bcbcbcbcbc \simeq (bcbcbcbbc)a.$

$\Delta_{\mathrm{H_{iii}}5} = b(cbcbbcbcb) \simeq cbcbcbcbcb \simeq (cbcbbcbcb)b.$

$\Delta_{\mathrm{H_{iii}}5} \simeq c(bcbcbcbcb) \simeq (bcbcbcbcb)c.$

$\Delta_{\mathrm{H_{iii}}6} := (abc)^3$
$\Delta_{\mathrm{H_{iii}}6} = a(bcabcabc) \simeq bcbabcab \simeq bcabacabc \simeq bcabcacbc \simeq (bcabcabc)a$
$\Delta_{\mathrm{H_{iii}}6} \simeq (abcabcab)c \simeq abcabcbcb \simeq abcabbcab \simeq acbcbbcab$
$\simeq acabcbcab \simeq cacbcbcab \simeq c(abcabcab).$

$\Delta_{\mathrm{H_{iii}}7} := (cba)^5$
$\Delta_{\mathrm{H_{iii}}7} = (cba)^5 \simeq (acb)^5 \simeq (bac)^5.$ □

As a consequence of Theorem 3, we have the folloing fact.

Fact 5. *There exists a positive integer $k \in \mathbb{Z}_{>0}$ such that the k-th power of the Coxeter element $C := cba$ (= a homotopy class which turns once around all the three points $C_X \cap l_{*_1,\mathbb{C}}$ counterclockwise) is a fundamental element.*

Finally, we ask a few questions related to the fundamental elements.

Let M be a monoid defined by positive homogeneous relations. Recall (§4 Definition) that G^+ is the image of M in the group G by the localization homomorphism. We define quasi-central elements and fundamental elements of G^+ exactly by the same defining relations for M^+. Let us denote by $\mathcal{QZ}(G^+)$ and $\mathcal{F}(G^+)$ the set of quasi-central elements and fundamental elements in G^+, respectively. Then, the localization morphism induces homomorphisms: $\mathcal{QZ}(M) \to \mathcal{QZ}(G^+)$ and $\mathcal{F}(M) \to \mathcal{F}(G^+)$, which may be neither injective nor surjective. However, Theorem 3 implies the following fact.

Fact 6. *For any type X, the set of fundamental elements $\mathcal{F}(G_X^+)$ is non-empty.*

We observe that $\mathcal{F}(G_X^+)$ may not be singly generated. On the other hand, the list in Theorem 3 may not be sufficient to generate whole $\mathcal{F}(M_X)$ or $\mathcal{F}(G_X^+)$.

Question 1. Is $\mathcal{F}(M)$ (resp. $\mathcal{F}(G^+)$) finitely generated over $\mathcal{QZ}(M)$ (resp. $\mathcal{QZ}(G^+)$)? That is, are there finitely many elements $\Delta_1, \cdots, \Delta_k \in \mathcal{F}(M)$ (resp. $\mathcal{F}(G^+)$) such that following holds?

$$\mathcal{F}(M) = \mathcal{QZ}(M)\Delta_1 \cup \cdots \cup \mathcal{QZ}(M)\Delta_k.$$
$$\mathcal{F}(G^+) = \mathcal{QZ}(G^+)\Delta_1 \cup \cdots \cup \mathcal{QZ}(G^+)\Delta_k.$$

Question 2. The following five cases 1, 2, 3, 4, and 5. give or may give example of *an indecomposable logarithmic free divisor such that the local fundamental group of its compliment admits positive homogeneous presentation by a suitable choice of Zariski-van Kampen generators and a power of the Coxeter element gives a fundamental element of the monoid generated by them.* We ask whether this property holds for any indecomposable logarithmic free divisor or not (for a more precise formulation of the question, see [S-I2]).

1. The discriminant of a finite irreducible reflection group ([B-S, S2, S3]).

2. The discriminant of a finite irreducible complex refrection group (except for type G_{31}) ([B-M-R, Be]).

3. The Sekiguchi polynomials (Theorems 1. and 3. of the present paper).

4. A plane curve is locally logarithmic free (see [S1]). The local fundamental group of the complement of a plane curve seems to be presented by positive homogeneous relations in [K]. It seems likely that a power of the Coxeter element is a fundamental element of the associated monoid (to be confirmed yet).

5. The discriminant of elliptic Weyl group is a free divisor ([S4]II). A Zariski-van Kampen presentation of the fundamental group of the complement of the divisor is not yet given. However, the hyperbolic Coxeter element in the elliptic Weyl group ([S4]I,III) may (conjecturally) be lifted to the fundamental group, whose power of order m_Γ, gives a fundamental element.

7 Cancellation conditions on M_X

In the present section, we study the *cancellation condition* on a monoid M. In the first half, we show some general consequences on the monoid M under the cancellation condition, or under its weaker version: a *weak cancellation condition.* In the latter half, we prove that the monoid $M_{\mathrm{B_{ii}}}$ satisfies the cancellation condition, however, we do not know whether the monoids $M_{\mathrm{B_{vi}}}$, $M_{\mathrm{H_{ii}}}$ and $M_{\mathrm{H_{iii}}}$ satisfy it or not.

Definition 7.1. A monoid M is said to *satisfy the cancellation condition*, if an equality $AXB=AYB$ for $A,B,X,Y\in M$ implies $X=Y$.

It is well-known that an Artin monoid satisfies the cancellation condition [B-S, Prop.2.3]. Let us state some important consequences of the cancellation condition on a monoid defined by positive homogeneous relations.

Lemma 7.2. *Let M be a monoid defined by positive homogeneous relations. Suppose it satisfies the cancellation condition. Then, we have the followings.*

1. *For any $\Delta\in\mathcal{QZ}(M)$, the associated permutation σ_Δ of $L/\sim$ extends to an isomorphism, denoted by the same σ_Δ, of M. The correspondence: $\Delta\mapsto\sigma_\Delta$ induces an anti-homomorphism:*

$$\mathcal{QZ}(M)\longrightarrow Aut(M).$$

2. *If $\mathcal{F}(M)\neq\emptyset$, then the localization homomorphism is injective and, hence, one has an isomorphism:*

$$M\ \simeq\ G^+.$$

3. *For any element $A\in G$ and any $\Delta\in\mathcal{F}(M)$, there exists $B\in G^+$ and $n\in\mathbb{Z}_{\geq 0}$ such that, in G, one has equalities:*

$$A=B\cdot(\Delta)^{-n}=(\Delta^{-n})\cdot\sigma_\Delta^n(B).$$

Proof. 1. First, we note that the permutation σ_Δ induces an isomorphism of the free monoid $(L/\sim)^*$, denote by the same σ_Δ. Let U and V be words in $(L/\sim)^*$ which are equivalent by the relations R (i.e. give the same element in M). Then, by definition, $U\Delta \simeq \Delta\sigma_\Delta(U)$ and $V\Delta \simeq \Delta\sigma_\Delta(V)$ are equivalent. That is, $\Delta\sigma_\Delta(U)$ and $\Delta\sigma_\Delta(V)$ give the same element in M. Then, cancelling Δ from the left, we see that $\sigma_\Delta(U)$ and $\sigma_\Delta(V)$ give the same element in M. Thus σ_Δ induces a homomorphism from M to M. The homomorphism is invertible, since a finite power of it is an identity. By the definition, for any $U \in M$ and $\Delta_1, \Delta_2 \in \mathcal{QZ}(M)$, one has:

$$U \cdot \Delta_1\Delta_2 \ \simeq \ \Delta_1 \cdot \sigma_{\Delta_1}(U) \cdot \Delta_2 \simeq \Delta_1\Delta_2 \cdot \sigma_{\Delta_2}(\sigma_{\Delta_1}(U)).$$

2. For a localization morphism to be injective, it is sufficient to show that the monoid satisfies the cancellation condition and that any two elements of the monoid have (at least) one (left and right) common multiple (Öre's condition, see [C-P]). In view of Fact 4. in §6, for any two elements $U, V \in M$ and $\Delta \in \mathcal{F}(M)$, $\Delta^{\max\{l(U),l(V)\}}$ is a common multiple of U and V from both sides.

3. Owing to the previous **2.**, it is sufficient to show that, for any element $A \in G$ and any $\Delta \in \mathcal{F}(M)$, there exists $k \in \mathbb{Z}_{\geq 0}$ such that $\Delta^k \cdot A \in G^+$. This can be easily shown by an induction on $k(A) \in \mathbb{Z}_{\geq 0}$ where $k(A)$ is the (minimal) number of letters of negative power in a word expression of A in $(L \cup L^{-1})^*$. Details are left to the reader. □

Next, we formulate a weak cancellation condition and its consequences.

Definition 7.3. An element $\Delta \in M$ is called left (resp. right) *weakly cancellative*, if an equality $\Delta = U \cdot V = U \cdot W$ (resp. $\Delta = V \cdot U = W \cdot U$) holds in M for some $U, V, W \in M$, then $V = W$ holds in M.

Notation. For an element $\Delta \in M$, we put
$Div_l(\Delta) := \{U \in M : \ U \mid_l \Delta\}$ and $Div_r(\Delta) := \{U \in M : \ U \mid_r \Delta\}$.

Fact 7. *Let a fundamental element $\Delta \in \mathcal{F}(M)$ be left weakly cancellative. Then the following* i), ii), iii) *and* iv) *hold.*

i) *For any element $U \in Div_l(\Delta)$, let $\tilde{U} \in (L/\sim)^*$ be a lifting to a word. Then, the class of $\sigma_\Delta(\tilde{U})$ in M depends only on the class U but not on the lifting $\tilde{U}$. Let us denote the class in M by $\sigma_\Delta(U)$.*

ii) *The divisor set $Div_l(\Delta)$ is invariant under the action of σ_Δ. In particular, the unique longest element Δ is fixed by σ_Δ.*

iii) *The fundamental element Δ is right weakly cancellative.*

iv) *We have the equality:* $\quad Div_l(\Delta) \ = \ Div_r(\Delta)$.

Proof. i) Suppose one has a decomposition $\Delta \simeq U \cdot V$ for $U, V \in M$, and let $\tilde{U}$ be a lifting of U into a word in $L/\sim$. Then, $\sigma_\Delta(\tilde{U})$ is well-defined as a word and hence induce an element in M, which we denote by the same $\sigma_\Delta(\tilde{U})$. We claim that Δ is equivalent to $V \cdot \sigma_\Delta(\tilde{U})$. This is shown by induction on $l(U)$. If $l(U) = 1$, this is the definition of quasi-centrality. Let $l(U) > 1$, $\tilde{U} = \tilde{U}' \cdot a$ and $\Delta \simeq \tilde{U}' \cdot a \cdot V$. By induction hypothesis, we have $\Delta \simeq a \cdot V \cdot \sigma_\Delta(\tilde{U}')$. Due to the weak cancellativity, $V \cdot \sigma_\Delta(\tilde{U}')$ is equivalent to Δ_a. Then, by definition of quasi-centrality, Δ is equivalent to $V \cdot \sigma_\Delta(\tilde{U}') \cdot \sigma_\Delta(a) \simeq V \cdot \sigma_\Delta(\tilde{U})$.

Let $\tilde{U}_1$ and $\tilde{U}_2$ be liftings of U. Then, applying the above result, we see that Δ is equal to $V \cdot \sigma_\Delta(\tilde{U}_1)$ and $V \cdot \sigma_\Delta(\tilde{U}_2)$. Then, applying the weak cancellativity of Δ, we see that $\sigma_\Delta(\tilde{U}_1)$ and $\sigma_\Delta(\tilde{U}_2)$ define the same element in M, which we shall denote by $\sigma_\Delta(U)$.

ii) In the proof of i), taking $U = \Delta$ and $V = 1$, we obtain $\Delta = \sigma_\Delta(\Delta)$. Then, $\sigma_\Delta(Div_l(\Delta)) = Div_l(\sigma_\Delta(\Delta)) = Div_l(\Delta)$.

iii) Suppose $\Delta = V \cdot U = W \cdot U$. then according to i), we have $\Delta = U \cdot \sigma_\Delta(V) = U \cdot \sigma_\Delta(W)$. Then the left cancellation condition implies $\sigma_\Delta(V) = \sigma_\Delta(W)$. On the other hand, according to ii), $\sigma_\Delta(V) = \sigma_\Delta(W)$ are again elements of $Div_l(\Delta)$ so that we can apply σ_Δ to the equality. Since σ_Δ is of finite order, after repeating this several times, we obtain the equality $V = W$.

iv) If Δ is left divisible by U, Δ is right divisible by $\sigma_\Delta(U)$. That is, the set $Div_r(\Delta)$ of the right divisors of Δ is equal to $\sigma_\Delta(Div_l(\Delta)) = Div_l(\Delta)$. □

Conjecture. Let C^k of the element in §6 **Fact 5.** If $C^{k\cdot \mathrm{ord}(\sigma_{C^k})}$ is weakly cancellative, then M satisfies the cancellation condition.

The following Theorem shows that we have already enough relations for type $\mathrm{B_{ii}}$.

Theorem 4. *The monoid $M_{\mathrm{B_{ii}}}$ satisfies the cancellation condition.*

Proof. We, first, remark the following.

Fact 8. *The left cancellation condition on $\mathrm{M_{B_{ii}}}$ implies the right cancell. condition.*

Proof. Consider a map $\varphi : \mathrm{M_{B_{ii}}} \to \mathrm{M_{B_{ii}}}$, $W \mapsto \varphi(W) := \sigma(rev(W))$, where σ is a permutation $\binom{a\ b\ c}{c\ b\ a}$ and $rev(W)$ is the reverse of the word $W = x_1 x_2 \cdots x_t$ (x_i is a letter or an inverse of a letter) given by the word $x_t \cdots x_2 x_1$. In view of the defining relation of $\mathrm{M_{B_{ii}}}$ in Theorem 1., φ is well defined and is an anti-isomorphism. If $\beta\alpha \simeq \gamma\alpha$, then $\varphi(\beta\alpha) \simeq \varphi(\gamma\alpha)$, i.e., $\varphi(\alpha)\varphi(\beta) \simeq \varphi(\alpha)\varphi(\gamma)$. Using left cancellation condition, we obtain $\varphi(\beta) = \varphi(\gamma)$ and, hence, $\beta \simeq \gamma$. □

The following is sufficient to show the left cancellation condition on $\mathrm{M_{B_{ii}}}$.

Proposition. *Let X and Y be positive words in $\mathrm{M_{B_{ii}}}$ of word-length $r \in \mathbb{Z}_{\geq 0}$.*
(i) *If $uX \simeq uY$ for some $u \in \{a, b, c\}$, then $X \simeq Y$.*
(ii) *If $aX \simeq bY$, then $X \simeq bZ$, $Y \simeq cZ$ for some positive word Z.*
(iii) *If $aX \simeq cY$, then $X \simeq cZ$, $Y \simeq aZ$ for some positive word Z.*
(iv) *If $bX \simeq cY$, then there exists an integer k ($0 \leq k < r-1$) and a word Z such that $X \simeq c^k baZ$ and $Y \simeq a^k bbZ$.*

Proof. Let us denote by $H(r,t)$ the statement in Theorem for all pairs of words X and Y such that their word-lengths are r and for all $u, v \in \{a, b, c\}$ such that $uX \simeq vY$ and the number of elementary transformasions to bring uX to vY is less or equal than t. It is easy to see that $H(r,t)$ is true if $r \leq 1$ or $t \leq 1$.

For $r,t\in\mathbb{Z}_{>1}$, we prove $H(r,t)$ under the induction hypotheis that $H(s,u)$ holds for (s,u) such that either $s<r$ and arbitrary u or $s=r$ and $u<t$.

Let X,Y be of word-length r, and let $u_1X\simeq u_2W_2\simeq\cdots\simeq u_tW_t\simeq u_{t+1}Y$ be a sequence of elementary transformations of t steps, where $u_1,\cdots,u_{t+1}\in\{a,b,c\}$ and $W_2,\cdots,W_t$ are positive words of length r. By assumption $t>1$, there exists an index $i\in\{2,...,t\}$ so that we decompose the sequence into two steps $u_1X\simeq u_iW_i\simeq u_{t+1}Y$, where each step satsifies the induction hypothesis.

If there exists i such that u_i is equal either to u_1 or u_{t+1}, then by induction hypothesis, W_i is equivalent either to X or to Y. Then, again, applying the induction hypothesis to the remaining step, we obtain the statement for the $u_1X\simeq u_{t+1}Y$. Thus, we assume from now on $u_i\neq u_1,u_{t+1}$ for $1<i\leq t$.

Suppose $u_1=u_{t+1}$. If there exists i such that $\{u_1=u_{t+1},u_i\}\neq\{b,c\}$, then each of the equivalence says the existence of $\alpha,\beta\in\{a,b,c\}$ and words Z_1,Z_2 such that $X\simeq\alpha Z_1$, $W_i\simeq\beta Z_1\simeq\beta Z_2$ and $Y\simeq\alpha Z_2$. Applying the induction hypothesis for r to $\beta Z_1\simeq\beta Z_2$, we get $Z_1\simeq Z_2$ and, hence, we obtained the statement $X\simeq\alpha Z_1\simeq\alpha Z_2\simeq Y$. Thus, we exclude these cases from our considerations. Next, we consider the case $\{u_1=u_{t+1},u_i\}=\{b,c\}$. However, due to the above consideration, we have only the case $u_2=u_3=\cdots=u_t$. Then, by induction hypothesis, we have $W_2\simeq\cdots\simeq W_t$. On the other hand, since the equivalences $u_1X\simeq u_2W_2$ and $u_{t+1}Y\simeq u_tW_t$ are the elementary trasformations at the tops of the words, there exists again $\alpha,\beta\in\{a,b,c\}$ and words Z_1,Z_2 with the similar descriptions as above hold, implying again $X\simeq Y$.

To complete the proof, we have to examine three more cases $(u_1,u_2,u_3)=(a,b,c),(a,c,b)$ and (b,a,c) for $t=2$, where we shall put $W:=W_2$.

(I) Case (a,b,c). We have $aX\simeq bW\simeq cY$.
Since the equivalences are single elementary transformations, there exists words Z_1 and Z_2 such that $X\simeq bZ_1$, $W\simeq cZ_1\simeq baZ_2$ and $Y\simeq bbZ_2$. Applying the induction hypothesis for r to the two equivalent expressions of W, we see that there exists k and a word Z_3 such that $0\leq k<r-2$ such that $Z_1\simeq a^kbbZ_3$ and $aZ_2\simeq c^kbaZ_3$. We can apply k-times the induction hypothesis to the last two equivalent expressions and we see that there exists a word Z_4 such that $Z_2\simeq c^kZ_4$ and $baZ_3\simeq aZ_4$. Applying again the induction hypothesis to the last equivalence relation, there exists a word Z_5 such that $Z_4\simeq bZ_5$ and $aZ_3\simeq cZ_5$. Once again applying the induction hypothesis to the last equivalence relation, we finally obtain $Z_3\simeq cZ_6$ and $Z_5\simeq aZ_6$ for a word Z_6. Reversing the procedure, obtain the descriptions:

$$\begin{aligned} X &\simeq bZ_1\simeq ba^kbbZ_3\simeq ba^kbbcZ_6,\\ Y &\simeq bbZ_2\simeq bbc^kZ_4\simeq bbc^kbZ_5\simeq bbc^kbaZ_6. \end{aligned}$$

By using the relations of $\mathrm{M}_{\mathrm{B}_{\mathrm{ii}}}$, we can show $ba^kbbc\simeq cbbc^kb$ and $bbc^kba\simeq abbc^kb$. So, we conclude that $X\simeq cZ, Y\simeq aZ$ for $Z\simeq bbc^kbZ_6$.

(II) Case (a,c,b). We have $aX\simeq cW\simeq bY$.
Since the equivalences are single elementary transformations, there exists words Z_1 and Z_2 such that $X\simeq cZ_1$, $W\simeq aZ_1\simeq bbZ_2$ and $Y\simeq baZ_2$. Applying the induction hypothesis for r to the two equivalent expressions of W, we see that

there exists a word Z_3 such that $Z_1 \simeq bZ_3$ and $bZ_2 \simeq cZ_3$. Again applying the induction hypothesis to the last two equivalent expressions, we see that there exists an integer k with $0 \leq k < r-3$ and a word Z_4 such that $Z_2 \simeq c^k baZ_4$ and $Z_3 \simeq a^k bbZ_4$. Reversing the procedure, obtain the descriptions:

$$X \simeq cZ_1 \simeq cbZ_3 \simeq cba^k bbZ_4 \quad \text{and} \quad Y \simeq baZ_2 \simeq bac^k baZ_4.$$

It is not hard to show the equivalences $cba^k bb \simeq bbac^k b$ and $bac^k ba \simeq cbac^k b$. Thus, we obtain $X \simeq bZ, Y \simeq cZ$ for $Z := bac^k bZ_4$.

(III) Case (b, a, c). We have $bX \simeq aW \simeq cY$.
By induction hypothesis, there exists words Z_1 and Z_2 such that $X \simeq cZ_1$, $W \simeq bZ_1 \simeq cZ_2$ and $Y \simeq aZ_2$. Applying the induction hypothesis for r to the two equivalent expressions of W, we see that here exists k and a word Z_3 such that $0 \leq k < r-2$ such that $Z_1 \simeq c^k baZ_3$ and $Z_2 \simeq a^k bbZ_3$. Thus, we obtain the descriptions:

$$X \simeq cZ_1 \simeq cc^k baZ_3 \quad \text{and} \quad Y \simeq aZ_2 \simeq aa^k bbZ_3.$$

This is the conclusion in Proposition (iv) with $0 \leq k+1 < r-1$, which we looked for.

This completes the proof of Proposition. □

This completes the proof of Theorem 4. □

8 2×2-matrix representation of the group G_X

We construct non-abelian representations of the groups $G_{\mathrm{B_{ii}}}, G_{\mathrm{B_{vi}}}, G_{\mathrm{H_{ii}}}, G_{\mathrm{H_{iii}}}$.

Theorem 5. *For each type $X \in \{\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}, \mathrm{H_{iii}}\}$, consider matrices $A, B, C \in \mathrm{GL}(2, \mathbb{C})$ listed below. Then we have the following i) and ii).*

i) The correspondence $a \mapsto A, b \mapsto B, c \mapsto C$ induces representations

$$\rho \ : \ G_X \longrightarrow \mathrm{GL}(2, \mathbb{C}).$$

ii) The image $\rho(G_X)$ is not an abelian group if $l^2 \neq 1$.

1. Type $\mathrm{B_{ii}}$*:* $A = u\begin{pmatrix} 1 & l^2 \\ 0 & 1 \end{pmatrix}, \quad B = v\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}, \quad C = u\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$
where $l^6 = 1$ and $u, v \in \mathbb{C}^\times$.

2. Type $\mathrm{B_{vi}}$*:* $A = u\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}, \quad B = u\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad C = u\begin{pmatrix} p & q \\ r & s \end{pmatrix},$
where $l^{10} = 1$ ($l^2 \neq 1$) and $u \in \mathbb{C}^\times$

$$a = -\frac{1}{l(l^2-1)}, \quad bc = \frac{-l^4 + l^2 - 1}{(1-l^2)^2}, \quad d = \frac{l^3}{l^2-1}$$
$$p = -l^4 a, \quad q = -\frac{b}{l^4}, \quad r = -l^4 c, \quad s = -\frac{d}{l^4}$$

3. Type $\mathrm{H_{ii}}$*:* $A = u\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}, \quad B = u\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad C = u\begin{pmatrix} p & q \\ r & s \end{pmatrix}$

where $u \in \mathbb{C}^\times$ *and one of the following two cases holds.*

i) $l^2 + l + 1 = 0$ *and* $3p^2 + 3p + 2 = 0$

$$a = \frac{l-1}{3},\ d = \frac{-l-2}{3},\ bc = -\frac{2}{3},\ q = \frac{-b(l+2)}{3p},\ r = \frac{p(1-l)}{3b},\ s = \frac{2}{3p}.$$

ii) $l^2 - l + 1 = 0$ *and* $3p^2 - 3p + 2 = 0$.

$$a = \frac{l+1}{3},\ d = \frac{-l+2}{3}, bc = -\frac{2}{3},\ q = \frac{b(-l+2)}{3p},\ r = \frac{-p(l+1)}{3b},\ s = \frac{2}{3p}.$$

4. Type $\mathrm{H_{iii}}$*:* $A = u\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}$, $B = u\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $C = u\begin{pmatrix} p & q \\ r & s \end{pmatrix}$,
where $l^{10} = 1$ $(l^2 \neq 1)$ *and* $u \in \mathbb{C}^\times$

$$a = -\frac{1}{l(l^2-1)},\quad bc = \frac{-l^4+l^2-1}{(l^2-1)^2},\quad d = \frac{l^3}{l^2-1}$$

$$p = a,\quad q = \frac{b}{l^4},\quad r = l^4 c,\quad s = d$$

Proof. It is sufficient to prove only for the case $u = v = 1$.

We present the matrices A, B and C by the indeterminates a, b, c, d, p, q, r, s and l as in Theorem, and then solve the polynomial equation on them defined by the relations listed in Theorem 1. It is unnecessary to check all relations, since some relations are included in the list because of the cancellation condition, whereas $\mathrm{GL}(2,\mathbb{C})$ is a group where the cancellation condition is automatically satisfied. However, as we shall see, it is sometimes convenient to take these "superfluous" relations in account. Detailed calculations are left to the reader.

1.Type $\mathrm{B_{ii}}$: We need to show $CBB = BBA, BC = AB, AC = CA$, whose verifications are left to the reader. We have $\det(A) = \det(C) = u^2 \neq 0, \det(B) = v^2 \neq 0$. Since $ABA^{-1}B^{-1} = \begin{pmatrix} 1 & l^2(1-l^2) \\ 0 & 1 \end{pmatrix}$ and $BCB^{-1}C^{-1} = \begin{pmatrix} 1 & l^2-1 \\ 0 & 1 \end{pmatrix}$, $\rho(G_{\mathrm{B_{ii}}})$ is abelian if and only if $l^2 = 1$.

2. Type $\mathrm{B_{vi}}$: We need to show $ABA = BAB, ACA = BAC, ACB = CAC$. Actually, solving the (1,1) entry of the equation $ABA = BAB$, $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\det(B) = 1$, w obtain the expressions for a, b, c, d. Then, using the relation $C = ABA^{-2}B$, we obtain the expressions for p, q, r, s. Further more, comparing the $(1,1)$-entry of $A^5 = B^5$, we get $l^8 + l^6 + l^4 + l^2 + 1 = 0$.

3. Type $\mathrm{H_{ii}}$: We need to show $ABAB = BABA, ACA = BAC, ACB = CAC$.

$$ABAB = \begin{pmatrix} bc + a^2l^2 & bd + abl^2 \\ ac + cd/l^2 & bc + d^2/l^2 \end{pmatrix}, BABA = \begin{pmatrix} bc + a^2l^2 & ab + bd/l^2 \\ cd + acl^2 & bc + d^2/l^2 \end{pmatrix}$$

By these calculations, we have $d + al^2 = 0$. By $TrA = TrB = TrC$ and $detA = detB = detC$, we have $a + d = l + l^{-1} = p + s$, $ad - bc = ps - qr = 1$.

$$a = \frac{l^2+1}{l(1-l^2)}, d = \frac{l(l^2+1)}{l^2-1}, bc = \frac{-2(l^4+1)}{(l^2-1)^2}$$

$$ACA = \begin{pmatrix} l^2p & q \\ r & s/l^2 \end{pmatrix}, BAC = \begin{pmatrix} alp + br/l & alq + bs/l \\ clp + dr/l & clq + ds/l \end{pmatrix}$$

$$q = \frac{b}{lp(1-l^2)}, r = \frac{l(l^4+1)p}{b(l^2-1)}, s = \frac{-2l^2}{(1-l^2)^2p}$$

$ACB = CAC \Leftrightarrow (1-l+l^2) = 0$ and $3p^2-3p+2 = 0$, or , $(1+l+l^2) = 0$ and $3p^2+3p+2 = 0$

We calculate each cases separately and obtain the result.

4. Type $\mathrm{H_{iii}}$: We need to show $ABA = BAB, CBA = ACB, BCA = CBC$. As in case of $\mathrm{B_{vi}}$, already the first relation $ABA = BAB$ (in particular $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\det(B) = 1$) implies the expressions for a, b, c, d. Using further the relation $ACA = CAC$, we obtain $a = p$, $d = s$ and $bc = qr$. Then applying the relation $A^2C = BA^2$, we get $q = l^4b$ and $r = l^{-4}c$. Further, using the relation $CA^3 = A^3B$, we obtain $l^{10} = 1$. □

Corollary. For $X \in \{\mathrm{B_{ii}}, \mathrm{B_{vi}}, \mathrm{H_{ii}}, \mathrm{H_{iii}}\}$, $\sigma(\mathcal{QZ}(G_X^+))$ *consists only of the identity.*

Sketch of Proof. For $\sigma \in \mathfrak{S}(L)$, we consider a matrix $X \in \mathrm{Mat}(2, \mathbb{C})$ satisfying the equations: $AX = X\sigma(A)$, $BX = X\sigma(B)$, $CX = X\sigma(C)$. If $\sigma = 1$, then the solutions are *constant* $\times$ id. If $\sigma \neq 1$, then $X = 0$.

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