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On a Schrödinger equation with a merging pair of a simple pole and a simple turning point — Alien calculus of WKB solutions through microlocal analysis

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On a Schrödinger equation with a merging pair of a simple pole and a simple turning point — Alien calculus of WKB solutions through microlocal analysis

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The purpose of this report is to present the core results of [KKKoT] and [KoT] with emphasis on their background. The object studied in these papers is, in somewhat rough description, a Schrödinger equation

$$(1)\qquad \left(\frac{d^2}{dx^2} - \eta^2 Q(x,\eta)\right)\psi(x,\eta) = 0 \quad (\eta:\ \text{a large parameter})$$

with one simple turning point and with a simple pole in the potential Q. Now that satisfactory results have been obtained by [AKT2] concerning the WKB theoretic structure of a Schrödinger equation with two simple turning points, it is high time for us to study the above equation in view of the fact that a simple pole in the potential gives the Borel transformed WKB solutions of (1) essentially the same effect as a simple turning point does ([Ko1], [Ko2]).

In studying this problem we have to analyse two (or more) singularities of the Borel transformed WKB solutions whose relative location is fixed (the so-called "fixed singularities" (cf. [DP]; see also [V]). This means that the usual technique (cf. [AKT1], [KT]) of relating Borel transformed WKB solutions through integral operators determined by some microdifferential operators (cf. [SKK], [K^3], [A]) requires the domain of definition of the relevant operators to be sufficiently large. To circumvent this problem, following the idea in [AKT2], we introduce an auxiliary parameter a to the potential Q so that the turning point and the pole in question merge as the parameter a tends to 0. Interestingly enough, we then naturally encounter the so-called ghost equation (cf. [Ko3], [KKKoT]) at $a = 0$, the top degree part $Q_0(x)$ of whose potential contains neither zeros nor poles. The transformation of a ghost equation to its canonical form is known ([Ko3]; see also [KKKoT; Section 1]), and by perturbing the transformation with respect to the parameter a we can find the WKB-theoretic canonical operator of an appropriately defined (Definition 1 below) class of

Schrödinger operators with a simple turning point and a simple pole (Theorem 1 below).

A mathematical formulation of the intuitive picture of such an "appropriate" class is given by the following

Definition 1. *The Schrödinger equation* (1) *is called an equation with a merging pair of a simple pole and a simple turning point, or, for short, an MPPT equation if its potential Q depends also on an auxiliary complex parameter a and has the following form:*

$$Q = \frac{Q_0(x,a)}{x} + \eta^{-1}\frac{Q_1(x,a)}{x} + \eta^{-2}\frac{Q_2(x,a)}{x^2}, \tag{2}$$

where $Q_j(x,a)$ $(j=0,1,2)$ are holomorphic near $(x,a)=(0,0)$ and $Q_0(x,a)$ satisfies the following conditions (3) *and* (4):

$$\left(\frac{\partial Q_0}{\partial a}\right)(0,0) \neq 0, \tag{3}$$

(4) $Q_0(x,0) = c_0^{(0)}x + O(x^2)$ *holds with $c_0^{(0)}$ being a constant different from* 0*.*

Remark 1. In [KKKoT] a slightly weaker condition

$$Q_0(0,a) \neq 0 \ \text{ if } \ a \neq 0 \tag{3$'$}$$

is imposed instead of (3).

It follows from the above definition that there exists a unique holomorphic function $x(a)$ near $a=0$ that satisfies

$$Q_0(x(a),a) = 0, \tag{5}$$

$$x(a) \neq 0 \ \text{ if } \ a \neq 0. \tag{6}$$

Then the assumption (4) guarantees that $x = x(a)$ $(a \neq 0, |a| \ll 1)$ is a simple turning point. Thus the above assumptions visualize our intuitive picture of the equation. The following Theorem 1 guarantees the appropriateness of the above definition. For the clarity of description we put ˜ to quantities relevanto to a general MPPT equation to distinguish them from those of the canonical equation (16).

Theorem 1. *Let*

$$\tilde{L}\tilde{\psi} = \left(\frac{d^2}{d\tilde{x}^2} - \eta^2\tilde{Q}(\tilde{x},a,\eta)\right)\tilde{\psi}(\tilde{x},a,\eta) = 0 \tag{7}$$

be an MPPT equation in the sense of Definition 1, that is, the potential $\tilde{Q}(\tilde{x},a,\eta)$ *is of the form* (2) *and the conditions* (3) *and* (4) *are satisfied. Then there exist an open neighborhood* U *of* $\tilde{x} = 0$*, holomorphic functions* $x_k^{(j)}(\tilde{x})$ *defined on* U *and constants* $\alpha_k^{(j)}$ $(j, k \geq 0)$ *for which the following conditions* (8) $\sim$ (12) *are satisfied:*

$$\frac{dx_0^{(0)}}{d\tilde{x}}(0) \neq 0, \tag{8}$$

$$x_k^{(j)}(0) = 0 \quad \textit{for every } j \textit{ and } k, \tag{9}$$

$$\alpha_0^{(0)} = 0, \tag{10}$$

$$\sup_{\tilde{x}\in U} |x_k^{(j)}(\tilde{x})|,\ |\alpha_k^{(j)}| \leq AC_1^j C_2^k k! \tag{11}$$

with some positive constants A, C_1 *and* C_2,

(12)

$$\tilde{Q}(\tilde{x},a,\eta) = \left(\frac{\partial x(\tilde{x},a,\eta)}{\partial\tilde{x}}\right)^2 \left(\frac{1}{4} + \frac{\alpha(a,\eta)}{x(\tilde{x},a,\eta)} + \eta^{-2}\frac{\tilde{Q}_2(0,a)}{x(\tilde{x},a,\eta)^2}\right) - \frac{1}{2}\eta^{-2}\{x;\tilde{x}\},$$

where

$$x(\tilde{x},a,\eta) = \sum_{k\geq 0}\sum_{j\geq 0} x_k^{(j)}(\tilde{x})a^j\eta^{-k}, \tag{13}$$

(14) $$\alpha(a,\eta) = \sum_{k\geq 0}\sum_{j\geq 0} \alpha_k^{(j)} a^j \eta^{-k}$$

and $\{x;\tilde{x}\}$ *denotes the Schwarzian derivative*

(15) $$\frac{d^3x/d\tilde{x}^3}{dx/d\tilde{x}} - \frac{3}{2}\left(\frac{d^2x/d\tilde{x}^2}{dx/d\tilde{x}}\right)^2.$$

This theorem combined with the general WKB theory (cf. [KT]) asserts that the WKB theoretically canonical equation of an MPPT equation $\tilde{L}\tilde{\psi} = 0$ is given by the following

(16) $$M\psi = \left(\frac{d^2}{dx^2} - \eta^2\Big(\frac{1}{4} + \frac{\alpha(a,\eta)}{x} + \eta^{-2}\frac{\tilde{Q}_2(0,a)}{x^2}\Big)\right)\psi = 0.$$

In parallel with the usage of the name "∞-Weber equation" in [AKT2], we call the equation $M\psi = 0$ an ∞-Whittaker equation.

An important point is that in the double series $x(\tilde{x},a,\eta)$ and $\alpha(a,\eta)$ in Theorem 1 the growth order property of $|x_k^{(j)}|$ and $|\alpha_k^{(j)}|$ as j tends to ∞ and that as k tends to ∞ are substantially different despite the fact that their construction is done in a symmetric way with respect to indexes j and k (cf. [KKKoT; Remark 2.1]). In particular,

(17) $$x_k(\tilde{x},a) = \sum_{j\geq 0} x_k^{(j)}(\tilde{x})a^j$$

and

(18) $$\alpha_k(a) = \sum_{j\geq 0} \alpha_k^{(j)} a^j$$

are holomorphic respectively on $U \times V$ and on V for some open neighborhood V of $a = 0$, while $x(\tilde{x},a,\eta)$ and $\alpha(a,\eta)$ are only Borel transformable series in the sense of [KT]. Although the problem is of singular perturbative character, it seems that it is of regular perturbative character in the variable a. Actually our reasoning indicates that the singular perturbative character originates from the part

$\eta^{-2}(d^3x_k^{(j)}/d\tilde{x}^3)/(dx_k^{(j)}/d\tilde{x})$ in the defining equation of $x_k^{(j)}$, which does not affect much the behavior of $x_k^{(j)}$ as j tends to infinity. (See [KKKoT; (B.64)].)

It is readily imagined, and we can really confirm, that the canonical equation $M\psi = 0$ is further reduced to the following Whittaker equation with a large parameter:

$$M_0\chi = \left(\frac{d^2}{dx^2} - \eta^2\left(\frac{1}{4} + \frac{\alpha_0}{x} + \eta^{-2}\frac{\gamma(\gamma+1)}{x^2}\right)\right)\chi = 0, \tag{19}$$

where α_0 and γ are complex numbers. Concerning the Whittaker equation with a large parameter for $\alpha_0 \neq 0$ we know ([KoT]) the following Theorem 2: Let $\chi_\pm(x,\eta)$ be WKB solutions of the Whittaker equation normalized as

$$\chi_\pm(x,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_{-4\alpha_0}^{x} S_{\mathrm{odd}}dx\right), \tag{20}$$

where S_{odd} is the odd part of the formal power series solution $S = \eta S_{-1}(x) + S_0(x) + \eta^{-1}S_1(x) + \cdots$ of the associated Riccati equation (cf. [KKKoT]). Then the following holds.

Theorem 2. *Suppose $\alpha_0 \neq 0$. Then the Borel transform $\chi_{+,B}(x,y)$ of χ_+ has fixed singularities at $y = -y_+(x) + 2m\pi i\alpha_0$ $(m = \pm 1, \pm 2, \cdots)$, where*

$$y_+(x) = \int_{-4\alpha_0}^{x} S_{-1}dx = \int_{-4\alpha_0}^{x}\sqrt{\frac{x+4\alpha_0}{4x}}dx \tag{21}$$

and its alien derivative is explicitly given by

$$\begin{aligned}&\left(\Delta_{y=-y_+(x)+2m\pi i\alpha_0}\chi_+\right)_B(x,y)\\ &= \frac{\exp(2m\pi i\gamma)+\exp(-2m\pi i\gamma)}{2m}\,\chi_{+,B}(x, y-2m\pi i\alpha_0).\end{aligned} \tag{22}$$

Note that the relative location between two singular points $-y_+(x) + 2m\pi i\alpha_0$ and $-y_+(x) + 2m'\pi i\alpha_0$ does not vary, that is, their difference $2(m-m')\pi i\alpha_0$ is a constant independent of x. The proof of Theorem 2 can be done by using the following expression of the Borel transform of the Voros coefficient ϕ:

$$
\begin{aligned}
(23) \quad & \phi_B(\alpha_0, \gamma; y) \\
& = \frac{1}{2y}\left(\frac{\exp(y/\alpha_0)+1}{\exp(y/\alpha_0)-1}\right)\cosh\left(\frac{\gamma y}{\alpha_0}\right) - \frac{\alpha_0}{y^2} + \frac{1}{2y}\sinh\left(\frac{\gamma y}{\alpha_0}\right),
\end{aligned}
$$

where the Voros coefficient of the Whittaker equation (19) is defined by

$$
(24) \qquad \phi(\alpha_0, \gamma; \eta) = \int_{-4\alpha_0}^{\infty} (S_{\mathrm{odd}} - \eta S_{-1})dx.
$$

See [KoT] for the details. Since the concrete computation in alien calculus is normally performed on the Borel plane (cf. [P], [DP]), we have to study the Borel transformed version of Theorem 1. To employ Theorem 2, we assume $a \neq 0$ in what follows. Thanks to the estimate (11), we have the following Theorem 3 and Theorem 4. To state them we make the following notational preparations: Let $g(x, a)$ be the inverse function of $x_0(\tilde{x}, a)$, i.e., a holomorphic function that satisfies

$$
(25) \qquad x = x_0\big(g(x,a), a\big), \quad \tilde{x} = g\big(x_0(\tilde{x}, a), a\big)
$$

on a neighborhood of $(x, a) = (0, 0)$. Then we consider the Borel transform of $\tilde{L}$ in (x, y, a)-variable:

$$
\begin{aligned}
(26) \quad \mathcal{L} & \stackrel{\mathrm{def}}{=} \left(\frac{\partial g}{\partial x}\right)^2 \times \big(\text{Borel transform of } \tilde{L}\big)\big|_{\tilde{x}=g(x,a)} \\
& = \frac{\partial^2}{\partial x^2} - \left(\frac{\partial^2 g/\partial x^2}{\partial g/\partial x}\right)\frac{\partial}{\partial x} - \left(\frac{\partial g}{\partial x}\right)^2 \tilde{Q}(g(x,a), a, \frac{\partial}{\partial y}).
\end{aligned}
$$

Similarly let $\mathcal{M}$ (resp. $\mathcal{M}_0$) be the Borel transform of M (resp. M_0):

$$\mathcal{M} = \frac{\partial^2}{\partial x^2} - \left(\frac{1}{4} + \frac{\alpha(a, \partial/\partial y)}{x}\right)\frac{\partial^2}{\partial y^2} - \frac{\tilde{Q}_2(0,a)}{x^2}, \tag{27}$$

$$\mathcal{M}_0 = \frac{\partial^2}{\partial x^2} - \left(\frac{1}{4} + \frac{\alpha_0}{x}\right)\frac{\partial^2}{\partial y^2} - \frac{\gamma(\gamma+1)}{x^2}. \tag{28}$$

Theorem 3. *Suppose $a \neq 0$. Let ω_0 be a sufficiently small open neighborhood of $x = 0$, and set*

$$\Omega_0 = \{(x, y; \xi, \eta) \in T^*\mathbb{C}^2_{(x,y)}; x \in \omega_0, \eta \neq 0\}. \tag{29}$$

Then there exist microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ defined on Ω_0 that satisfy

$$\mathcal{L}\mathcal{X} = \mathcal{Y}\mathcal{M} \tag{30}$$

for $x \neq 0$. The concrete form of operators $\mathcal{X}$ and $\mathcal{Y}$ is as follows:

$$\mathcal{X} = :\left(\frac{\partial g}{\partial x}\right)^{1/2}\left(1 + \frac{\partial r}{\partial x}\right)^{-1/2}\exp\big(r(x,a,\eta)\xi\big):, \tag{31}$$

$$\mathcal{Y} = :\left(\frac{\partial g}{\partial x}\right)^{1/2}\left(1 + \frac{\partial r}{\partial x}\right)^{3/2}\exp\big(r(x,a,\eta)\xi\big):, \tag{32}$$

where

$$r(x,a,\eta) = \sum_{k\geq 1} x_k\big(g(x,a),a\big)\eta^{-k} \tag{33}$$

and $:\ :$ designates the normal ordered product (cf. [A]).

Theorem 3 implies that the operators $\mathcal{L}$ and $\mathcal{M}$ are microlocally equivalent. This fact indicates that the singularities of $\tilde{\psi}_B(g(x,a),y)$ that satisfies $\mathcal{L}\tilde{\psi}_B = 0$ and those of $\psi_B(x,y)$ that satisfies $\mathcal{M}\psi_B = 0$ are the same. This is really visualized by the following Theorem 4:

Theorem 4. *The action of the microdifferential operator $\mathcal{X}$ upon the Borel transformed WKB solution $\psi_{+,B}$ of the ∞-Whittaker* *equation is expressed as an integro-differential operator of the following form*:

$$\mathcal{X}\psi_{+,B} = \int_{y_0}^{y} K(x,a,y-y',\partial/\partial x)\psi_{+,B}(x,a,y')dy', \tag{34}$$

where $K(x,a,y,\partial/\partial x)$ is a differential operator of infinite order that is defined on $\{(x,a,y) \in \mathbb{C}^3;\ (x,a) \in \omega$ for an open neighborhood ω of the origin and $|y| < C$ for some positive constant $C\}$, and y_0 is a constant that fixes the action of $(\partial/\partial y)^{-1}$ as an integral operator.

Since a differential operator of infinite order acts on the sheaf of holomorphic functions as a sheaf homomorphism, we can immediately locate the singularities of $\mathcal{X}\psi_{+,B}$ through the integral representation (34). Another important point in the integral representation (34) is that its domain of definition enjoys the uniformity with respect to the parameter a, that is, the open neighborhood ω is taken to be of the form

$$\{x \in \mathbb{C}; |x| < \delta_1\} \times \{a \in \mathbb{C}; |a| < \delta_2\} \tag{35}$$

for some positive constants δ_1 and δ_2. Note that since $\alpha_0(a)$ tends to 0 as a tends to 0 by (10), (δ_1, δ_2) can be chosen so that $\{|x| < \delta_1\}$ contains $x = -4\alpha_0(a)$ for every a in $\{|a| < \delta_2\}$. This is the precise meaning of saying "To circumvent the problem (of the existence of a large domain of definition of relevant integral operators)" at the beginning of this report.

In parallel with Theorem 3, we can show that $\mathcal{M}$ and $\mathcal{M}_0$ are also microlocally equivalent. For simplicity we employ $\alpha_0(a)$ as an independent variable in substitution for a (this substitution of variable is guaranteed by (3)). Thanks to the estimate (11) we obtain the following

Theorem 5. *Let $\mathcal{A}$ be a microdifferential operator on*

$$\{(\alpha_0, y; \theta, \eta) \in T^*\mathbb{C}^2; |\alpha_0| < \delta_0, \eta \neq 0\} \tag{36}$$

for some positive constant δ_0 defined by

$$\mathcal{A} = :\exp\big(\big(\alpha_1(\alpha_0)\eta^{-1} + \alpha_2(\alpha_0)\eta^{-2} + \cdots\big)\theta\big): \ . \tag{37}$$

Here θ and η are respectively identified with the symbol $\sigma(\partial/\partial\alpha_0)$ and the symbol $\sigma(\partial/\partial y)$. Then the following holds:

$$\mathcal{M}\mathcal{A} = \big(\mathcal{A}\mathcal{M}_0\big)\big|_{\gamma(\gamma+1)=\tilde{Q}_2(0,a)} \tag{38}$$

for $x \neq 0$.

Although the target variable is α_0, not x, as is the case for the microdifferential operator $\mathcal{X}$, the operator $\mathcal{A}$ also has a concrete expression as an integro-differential operator stated in Theorem 4. On the other hand, as is indicated in Theorem 2, a fixed singular point of $\psi_{+,B}(x,y)$ ("fixed" with respect to $y = -y_+(x)$) is located at $y = -y_+(x) + 2m\pi i\alpha$. Thus, by the same reasoning for the case of $\mathcal{X}$, each individual fixed singular point of $\tilde{\psi}_{+,B}(x,y)$ is contained, for sufficiently small a, in the domain of definition of the integro-differential operator $\mathcal{A}$.

Summing up all these results, we finally obtain

Theorem 6. *Suppose $a \neq 0$ and let $\tilde{\psi}_+(\tilde{x}, a, \eta)$ be a WKB solution of an MPPT equation normalized at its turning point $\tilde{x}_0(a)$ as follows:*

$$\tilde{\psi}_+(x, a, \eta) = \frac{1}{\sqrt{\tilde{S}_{\mathrm{odd}}}} \exp\Big(\int_{\tilde{x}_0(a)}^{x} \tilde{S}_{\mathrm{odd}}\, dx\Big) \tag{39}$$

where $\tilde{S}_{\mathrm{odd}}$ is the odd part of the formal power series solution $\tilde{S}$ of the associated Riccati equation. Then for each positive integer m

the following relation (40) *holds for sufficiently small* a:

(40)

$$\left(\Delta_{y=-y_+(\tilde{x},a)+2m\pi i\alpha_0(a)}\tilde{\psi}_+\right)_B(\tilde{x},a,y)$$
$$= \frac{\exp(2m\pi i\gamma(a))+\exp(-2m\pi i\gamma(a))}{2m}\times$$
$$:\exp\left(-2m\pi i(\alpha_1(a)+\alpha_2(a)\eta^{-1}+\cdots)\right):\tilde{\psi}_{+,B}\big(\tilde{x},a,y-2m\pi i\alpha_0(a)\big),$$

where

$$y_+(\tilde{x},a)=\int_{\tilde{x}_0(a)}^{\tilde{x}}\sqrt{\frac{\tilde{Q}_0(\tilde{x},a)}{\tilde{x}}}\,d\tilde{x}, \tag{41}$$

$$\gamma(a)^2+\gamma(a)=\tilde{Q}_2(0,a) \tag{42}$$

and

$$\alpha_j(a)=\frac{1}{2\pi i}\oint_{\tilde{\Gamma}(a)}\tilde{S}_{\mathrm{odd},j-1}(\tilde{x},a)d\tilde{x} \tag{43}$$

with $\tilde{\Gamma}(a)$ *being a closed curve encircling* $\tilde{x}_0(a)$ *and the origin as in Figure 1 and with* $\tilde{S}_{\mathrm{odd},k}$ *designating the degree* k *part of* $\tilde{S}_{\mathrm{odd}}$.

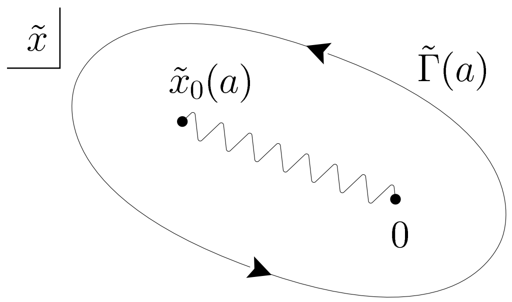

Figure 1.

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