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On the universal sl_2 invariant of boundary bottom tangles

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Abstract

The universal sl_2 invariant of bottom tangles has a universality property for the colored Jones polynomial of links. Habiro conjectured that the universal sl_2 invariant of boundary bottom tangles takes values in certain subalgebras of the completed tensor powers of the quantized enveloping algebra $U_h(sl_2)$ of the Lie algebra sl_2. In the present paper, we prove an improved version of Habiro's conjecture. As an application, we prove a divisibility property of the colored Jones polynomial of boundary links.

1 Introduction

In the 80's, Jones [9] constructed a polynomial invariant of links. After that, Reshetikhin and Turaev [20] defined an invariant of framed links whose components are colored by finite dimensional representations of a ribbon Hopf algebra. The *colored Jones polynomial* is the Reshetikhin-Turaev invariant of links whose components are colored by finite dimensional representations of the quantized enveloping algebra $U_h(sl_2)$.

The *universal invariant* associated with a ribbon Hopf algebra is an invariant of framed links and tangles which are not colored by any representations, see Hennings [8], Lawrence [15, 14], Reshetikhin [20], Ohtsuki [19], Kauffman [10], and Kauffman and Radford [11]. The universal invariant has the universality property for the Reshetikhin-Turaev invariant. By the *universal sl_2 invariant*, we mean the universal invariant associated with $U_h(sl_2)$. In particular, one can obtain the colored Jones polynomial from the universal sl_2 invariant.

A *bottom tangle* is a tangle consisting of arc components in a cube such that each boundary point is on the bottom line, and the two boundary points of each component are adjacent to each other, see Figure 1 (a) for example. We can define the closure link of a bottom tangle, see Figure 1 (b). For each link L, there is a bottom tangle whose closure is L. In [4], Habiro studied the universal invariant of bottom tangles associated with a ribbon Hopf algebra, and in [6], he studied the universal sl_2 invariant in detail.

The universal sl_2 invariant of n-component bottom tangles takes values in the completed n-fold tensor power $U_h(sl_2)^{\hat{\otimes} n}$ of $U_h(sl_2)$. By using bottom tangles, we can restate the universality of the universal sl_2 invariant: the colored Jones polynomial of a link L is obtained from the universal sl_2 invariant of a bottom tangle whose closure is L, by taking the quantum traces associated with the representations attached to the components of links (cf. [4]).

We are interested in relationships between the algebraic properties of the colored Jones polynomial and the universal sl_2 invariant and the topological properties of links and bottom tangles.

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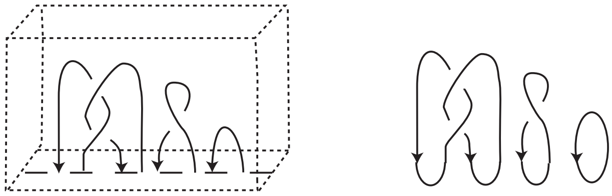


(a) (b)

Figure 1: (a) A bottom tangle T (b) The closure link of T

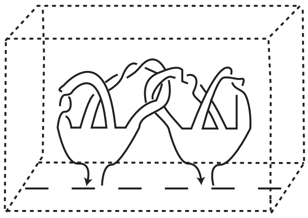


Figure 2: A boundary bottom tangle

Eisermann [2] proved that the Jones polynomial of an n-component ribbon link is divisible by the Jones polynomial of the n-component unlink. This result is generalized to links which are ribbon concordant to boundary links by Habiro [7]. Habiro [6] proved that the universal sl_2 invariant of n-component, algebraically-split, 0-framed bottom tangles takes values in certain small subalgebras of the completed tensor powers of $U_h(sl_2)$, and gave a divisibility property of the colored Jones polynomial of algebraically-split, 0-framed links.

In [21], the present author proved an improvement of Habiro's result for algebraically-split, 0-framed bottom tangles, in the special case of *ribbon bottom tangles* and ribbon links.

In the present paper, we study the universal sl_2 invariant of *boundary bottom tangles.* A bottom tangle is called *boundary* if its components admit mutually disjoint Seifert surfaces, see Figure 2 for example. We can obtain each boundary link from a boundary bottom tangle by closing. Habiro [6] conjectured that the universal sl_2 invariant of boundary bottom tangles takes values in certain subalgebras of the completed tensor powers of $U_h(sl_2)$. We prove an improved version of Habiro's conjecture (Theorem 1.2).

1.1 Main result

The quantized enveloping algebra $U_h = U_h(sl_2)$ is an h-adically completed $\mathbb{Q}[[h]]$-algebra (see Section 2.2 for the details). We set $q = \exp h$.

Habiro [6] proved that the universal sl_2 invariant J_T of an n-component, algebraically-split, 0-framed bottom tangle T is contained in the $\mathbb{Z}[q, q^{-1}]$-subalgebra $(\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} n}$ of $U_h^{\hat{\otimes} n}$. In [21], we defined another $\mathbb{Z}[q, q^{-1}]$-subalgebra $(\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n} \subset (\tilde{\mathcal{U}}_q^{ev})^{\tilde{\otimes} n}$, and prove the following theorem. (See Section 2.3 for the definition of $\bar{U}_q^{\mathrm{ev}}$, and see Sections 6.1–6.4 for the definition of the completion $(\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n}$ of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$.)

Theorem 1.1 ([21])**.** *Let T be an n-component ribbon bottom tangle with 0-framing. Then we have $J_T \in (\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n}$.*

The main result of the present paper is the following.

Theorem 1.2. *Let T be an n-component boundary bottom tangle with 0-framing. Then we have $J_T \in (\bar{U}_q^{\mathrm{ev}})\hat{}^{\hat{\otimes} n}$.*

Remark 1.3. Habiro [6, Conjecture 8.9] conjectured Theorem 1.2 with $(\bar{U}_q^{\mathrm{ev}})\hat{}^{\hat{\otimes} n}$ replaced with the $\mathbb{Z}[q, q^{-1}]$-subalgebra $(\bar{U}_q^{\mathrm{ev}})\tilde{}^{\tilde{\otimes} n}$, which includes $(\bar{U}_q^{\mathrm{ev}})\hat{}^{\hat{\otimes} n}$. We do not know whether the inclusion $(\bar{U}_q^{\mathrm{ev}})\hat{}^{\hat{\otimes} n} \subset (\bar{U}_q^{\mathrm{ev}})\tilde{}^{\tilde{\otimes} n}$ is proper or not. The definition of our algebra $(\bar{U}_q^{\mathrm{ev}})\hat{}^{\hat{\otimes} n}$ appears to be more natural.

Since every 1-component bottom tangle is boundary, Theorem 1.2 for $n = 1$ gives a possible improvement of the following theorem.

Theorem 1.4 (Habiro)**.** *Let T be an 1-component bottom tangle with 0-framing. Then we have $J_T \in (\bar{U}_q^{\mathrm{ev}})\tilde{}$.*

Theorem 1.4 follows from [6, Theorem 4.1] and the equalities

$$\mathrm{Inv}(\tilde{\mathcal{U}}_q^{ev}) = Z(\tilde{\mathcal{U}}_q^{ev}) = Z((\bar{U}_q^{\mathrm{ev}})\tilde{}),$$

which is implicit in [5, Section 9]. Here, for a subset $X \subset U_h$, we denote by $\mathrm{Inv}(X)$ the invariant part of X, and by $Z(X)$ the center of X.

If we use the one-to-one correspondence described in [4, Section 13] between the set of bottom tangles and the set of string links, then we can define the Milnor $\bar{\mu}$ invariants [17, 18] of a bottom tangle as that of the corresponding string link. See [3] for the Milnor $\bar{\mu}$ invariants of string links. In fact, all Milnor $\bar{\mu}$ invariants vanish both for ribbon bottom tangles and boundary bottom tangles. It is natural to expect the following conjecture.

Conjecture 1.5. *If T be an n-component bottom tangle with 0-framing with vanishing all Milnor $\bar{\mu}$ invariants, then we have $J_T \in (\bar{U}_q^{\mathrm{ev}})\hat{}^{\hat{\otimes} n}$.*

The converse of Conjecture 1.5 is also open.

1.2 Application to the colored Jones polynomial

We give an application (Theorem 1.6) of Theorem 1.2 to the colored Jones polynomial of boundary links. This result is parallel to the result for ribbon links [21].

We use the following q-integer notations:

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q\{i-1\}_q \cdots \{i-n+1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$
$$[i]_q = \{i\}_q/\{1\}_q, \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad \begin{bmatrix} i \\ n \end{bmatrix}_q = \{i\}_{q,n}/\{n\}_q!,$$

for $i \in \mathbb{Z}, n \geq 0$.

For $m \geq 1$, let V_m denote the m-dimensional irreducible representation of U_h. Let $\mathcal{R}$ denote the representation ring of U_h over $\mathbb{Q}(q^{\frac{1}{2}})$, i.e., $\mathcal{R}$ is the $\mathbb{Q}(q^{\frac{1}{2}})$-algebra

$$\mathcal{R} = \mathrm{Span}_{\mathbb{Q}(q^{\frac{1}{2}})}\{V_m \mid m \geq 1\}$$

with the multiplication induced by the tensor product. It is well known that $\mathcal{R} = \mathbb{Q}(q^{\frac{1}{2}})[V_2]$.

For $l \geq 0$, set

$$P_l = \prod_{i=0}^{l-1}(V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}) \in \mathcal{R},$$

$$\tilde{P}'_l = \frac{q^{\frac{1}{2}l}}{\{l\}_q!} P_l \in \mathcal{R},$$

which are used in [6] to construct the unified Witten-Reshetikhin-Turaev invariants for integral homology spheres. We denote by $J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}}$ the colored Jones polynomial of L with ith component L_i colored by $\tilde{P}'_{l_i}$. Habiro proved that Theorem 1.2 implied the following result. For $l \geq 0$, let I_l denote the ideal in $\mathbb{Z}[q, q^{-1}]$ generated by $\{l-k\}_q!\{k\}_q!$ for $k = 0, \dots, l$.

Theorem 1.6 ([6, Conjecture 8.10])**.** *Let L be an n-component boundary link with 0-framing. For $l_1, \dots, l_n \geq 0$, we have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_j+1\}_{q,l_j+1}}{\{1\}_q} I_{l_1} \cdots \hat{I}_{l_j} \cdots I_{l_n},$$

where j is an integer such that $l_j = \max\{l_i\}_{1 \leq i \leq n}$, and $\hat{I}_{l_j}$ denotes the omission of I_{l_j}.

Remark 1.7. For $m \geq 1$, let $\Phi_m(q) = \prod_{d|m}(q^d - 1)^{\mu(\frac{m}{d})} \in \mathbb{Z}[q]$ denote the mth cyclotomic polynomial, where $\prod_{d|m}$ denotes the product over all the positive divisors d of m, and μ is the Möbius function. It is not difficult to prove that for $l \geq 0$, I_l is contained in the principal ideal in $\mathbb{Z}[q]$ generated by $\prod_m \Phi_m(q)^{f(l,m)}$ with $f(l,m) = \max\{0, \lfloor \frac{l+1}{m} \rfloor - 1\}$, where, for $r \in \mathbb{Q}$, we denote by $\lfloor r \rfloor$ the largest integer smaller than or equal to r.

Theorem 1.6 is an improvement in the special case of boundary links of the following result.

Theorem 1.8 (Habiro [6, Theorem 8.2])**.** *Let L be an n-component, algebraically-split link with 0-framing. For $l_1, \dots, l_n \geq 0$, we have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_j+1\}_{q,l_j+1}}{\{1\}_q} \mathbb{Z}[q, q^{-1}].$$

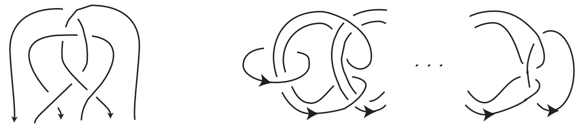
1.3 Examples

Let T_B be the Borromean bottom tangle depicted in Figure 3 (a), whose closure is the Borromean rings. Since we have $J_{T_B} \notin (\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} 3}$ (cf. [21]), it follows from Theorem 1.2 that the Borromean rings is neither boundary nor ribbon, as is well known.

More generally, for $n \geq 3$, let M_n be Milnor's n-component Brunnian link depicted in Figure 3 (b). Note that M_3 is the Borromean rings. Since there is a non-trivial Milnor $\bar{\mu}$ invariant of M_n of length n (cf. [17]), M_n is neither boundary nor ribbon. We can prove this fact also from Theorem 1.6 and

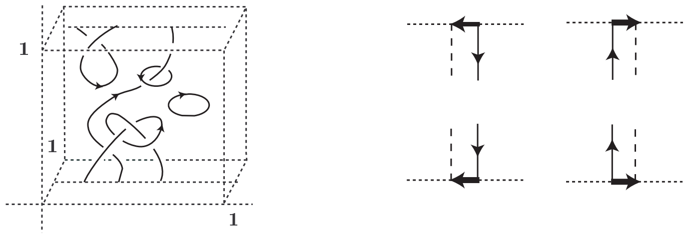
$$\begin{aligned} J_{M_n;\tilde{P}'_1,\dots,\tilde{P}'_1} &= (-1)^{n-2} q^{-2n+4} \Phi_1(q)^{n-2} \Phi_2(q)^{n-2} \Phi_3(q) \Phi_4(q)^{n-3} \\ &\notin \Phi_1(q)^n \Phi_2(q) \Phi_3(q) \mathbb{Z}[q, q^{-1}], \end{aligned}$$

which we will prove in a forthcoming paper [22].



(a) (b)

Figure 3: (a) Borromean rings (b) Milnor's link M_n



(a) (b)

Figure 4: (a) A tangle (b) The framing on the boundary

1.4 Organization of paper

The rest of the paper is organized as follows. Section 2 contains preliminary results about bottom tangles, the quantized enveloping algebra U_h, and the universal sl_2 invariant of bottom tangles. In Section 3, we recall from [6] Habiro's formula for the universal sl_2 invariant of boundary bottom tangles, and then give a modification of his formula. In Sections 4, 5, and 6, we prove Theorem 1.2.

2 Preliminaries

In this section, we recall basic things about bottom tangles, the universal enveloping algebra U_h, and the universal sl_2 invariant of bottom tangles.

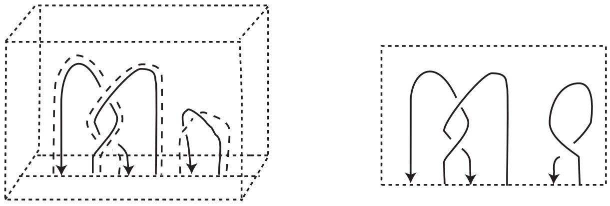
2.1 Bottom tangles and boundary bottom tangles

A *tangle* (cf. [12]) is the image of an embedding

$$\Big(\coprod^{m}[0,1]\Big)\sqcup\Big(\coprod^{n}S^1\Big)\hookrightarrow[0,1]^3,$$

with $m,n\geq 0$, whose boundary is on the two lines $[0,1]\times\{\frac{1}{2}\}\times\{0,1\}$ on the bottom and the top of the cube, see Figure 4 (a) for example. We equip the image with both an orientation and a framing. Here, at each boundary point, the framing is fixed on the lines $[0,1]\times\{\frac{1}{2}\}\times\{0,1\}$ as in Figure 4 (b), where the thin arrows represent the strands of the tangle, and the thick arrows represent the framing.

A *bottom tangle* (cf. [4, 6]) is a tangle consisting of arc components such that each boundary point is on the line $[0,1]\times\{\frac{1}{2}\}\times\{0\}$ on the bottom, and the two boundary points of each component are adjacent to each other. We give a preferred orientation of the tangle so that each component runs from its right boundary point to its left boundary point. For example, see Figure 5 (a), where the dotted lines represent the framing. We draw a diagram of a bottom tangle in a rectangle assuming the blackboard framing, see Figure 5 (b).



(a) (b)

Figure 5: (a) A 3-component bottom tangle T (b) A diagram of T

For each $n \geq 0$, let BT_n denote the set of the ambient isotopy classes, relative to boundary points, of n-component bottom tangles.

The *closure link* $\mathrm{cl}(T)$ of a bottom tangle T is defined as the link in $\mathbb{R}^3$ obtained from T by closing, see Figure 1 again. For each n-component link L, there is an n-component bottom tangle whose closure is L. For a bottom tangle, we can define its linking matrix as that of the closure link.

A *Seifert surface* of knot K is a compact, connected, orientable surface F in $\mathbb{R}^3$ bounded by K. An n-component link $L = L_1 \cup \cdots \cup L_n$ is called *boundary* if it has n mutually disjoint Seifert surfaces $F_1, \ldots, F_n$ in $\mathbb{R}^3$ such that L_i bounds F_i for $i = 1, \ldots, n$.

For a 1-component bottom tangle $T \in BT_1$, there is a knot $K_T = T \cup \gamma \in [0,1]^3$, where γ is the line segment in the bottom $[0,1] \times \{\frac{1}{2}\} \times \{0\}$ such that $\partial\gamma = \partial T$. A *Seifert surface* of a 1-component bottom tangle T is a Seifert surface of the knot K_T contained in $[0,1]^3$. A bottom tangle $T = T_1 \cup \cdots \cup T_n$ is called *boundary* if it has n mutually disjoint Seifert surfaces $F_1, \ldots, F_n$ in $[0,1]^3$ such that K_{T_i} bounds F_i for $i = 1, \ldots, n$. For example, see Figure 2 again. Obviously, for each boundary link L, there is a boundary bottom tangle whose closure is L.

2.2 Quantized enveloping algebra U_h

We recall the definition of the universal enveloping algebra $U_h(sl_2)$ of the Lie algebra sl_2, and its ribbon Hopf algebra structure. We follow the notations in [6].

We denote by $U_h = U_h(sl_2)$ the h-adically complete $\mathbb{Q}[[h]]$-algebra, topologically generated by H, E, and F, defined by the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}},$$

where we set

$$q = \exp h, \quad K = q^{H/2} = \exp \frac{hH}{2}.$$

We equip U_h with the topological $\mathbb{Z}$-graded algebra structure such that $\deg E = 1$, $\deg F = -1$, and $\deg H = 0$. For a homogeneous element x of U_h, the degree of x is denoted by $|x|$.

There is a complete ribbon Hopf algebra structure on U_h as follows. The comultiplication $\Delta\colon U_h \to U_h \hat{\otimes} U_h$, the counit $\varepsilon\colon U_h \to \mathbb{Q}[[h]]$, and the antipode $S\colon U_h \to U_h$ are given by

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \varepsilon(H) = 0, \quad S(H) = -H,$$
$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \varepsilon(E) = 0, \quad S(E) = -K^{-1}E,$$
$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \varepsilon(F) = 0, \quad S(F) = -FK.$$

Set

$$D = q^{\frac{1}{4}H\otimes H} = \exp\big(\frac{h}{4}H\otimes H\big) \in U_h^{\hat{\otimes}2}, \tag{1}$$

$$\tilde{F}^{(n)} = F^n K^n/[n]_q! \in U_h, \tag{2}$$

$$e = (q^{1/2} - q^{-1/2})E \in U_h, \tag{3}$$

for $n \geq 0$. The universal R-matrix and its inverse $R^{\pm 1} \in U_h \hat{\otimes} U_h$ are given by

$$R = D \sum_{n\geq 0} q^{\frac{1}{2}n(n-1)} \tilde{F}^{(n)} K^{-n} \otimes e^n,$$

$$R^{-1} = D^{-1} \sum_{n\geq 0} (-1)^n \tilde{F}^{(n)} \otimes K^{-n} e^n.$$

We have $R^{\pm 1} = \sum_{n\geq 0} \alpha_n^{\pm} \otimes \beta_n^{\pm}$, where for $n \geq 0$, we set formally

$$\alpha_n \otimes \beta_n (= \alpha_n^+ \otimes \beta_n^+) = D\Big(q^{\frac{1}{2}n(n-1)} \tilde{F}^{(n)} K^{-n} \otimes e^n\Big),$$

$$\alpha_n^- \otimes \beta_n^- = D^{-1}\Big((-1)^n \tilde{F}^{(n)} \otimes K^{-n} e^n\Big).$$

Note that the right hand sides are infinite sums of tensors of the form as $x \otimes y$ with $x, y \in U_h$. We denote them by $\alpha_n^{\pm} \otimes \beta_n^{\pm}$ for simplicity.

The ribbon element and its inverse $r^{\pm 1} \in U_h$ are given by

$$r = \sum_{n\geq 0} \alpha_n^- K^{-1} \beta_n^- = \sum_{n\geq 0} \beta_n^- K \alpha_n^-, \quad r^{-1} = \sum_{n\geq 0} \alpha_n K \beta_n = \sum_{n\geq 0} \beta_n K^{-1} \alpha_n.$$

We use a notation $D = \sum D' \otimes D''$. We use the following formulas.

$$\sum D'' \otimes D' = D, \tag{4}$$

$$(\Delta \otimes 1)D = D_{23}D_{13}, \quad (1 \otimes \Delta)D = D_{13}D_{12}, \tag{5}$$

$$(\varepsilon \otimes 1)(D) = 1 = (1 \otimes \varepsilon)(D), \tag{6}$$

$$(1 \otimes S)D = D^{-1} = (S \otimes 1)D, \tag{7}$$

$$D(1 \otimes x) = (K^{|x|} \otimes x)D, \quad D(x \otimes 1) = (x \otimes K^{|x|})D, \tag{8}$$

where $D_{13} = \sum D' \otimes 1 \otimes D''$, $D_{23} = 1 \otimes D$, $D_{12} = D \otimes 1$, and $x \in U_h$ homogeneous.

2.3 Subalgebras of U_h

In this section, we recall from [6] subalgebras $U_{\mathbb{Z},q}, \bar{U}_q$ and $\bar{U}_q^{\mathrm{ev}}$ of U_h. Recall from (2) and (3) the definitions of $\tilde{F}^{(n)} \in U_h$ and $e \in U_h$, respectively. Similarly, set

$$\tilde{E}^{(n)} = (q^{-1/2}E)^n/[n]_q! \in U_h,$$

$$f = (q-1)FK \in U_h,$$

for $n \geq 0$.

Let $U_{\mathbb{Z},q}$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra of U_h generated by $K, K^{-1}, \tilde{E}^{(n)}$, and $\tilde{F}^{(n)}$ for $n \geq 1$.

Let $\bar{U}_q$ denote the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $U_{\mathbb{Z},q}$ generated by K, K^{-1}, e and f. Let $\bar{U}_q^{\mathrm{ev}}$ be the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\bar{U}_q$ generated by K^2, K^{-2}, e and f.

Remark 2.1. Set $[i] = \frac{q^{i/2}-q^{-i/2}}{q^{1/2}-q^{-1/2}}$ for $i \in \mathbb{Z}$ and $[n]! = [n]\cdots[1]$ for $n \geq 0$. Let $U_{\mathbb{Z}}$ be the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-subalgebra of U_h generated by K, K^{-1}, $E^{(n)} = E^n/[n]!$, and $F^{(n)} = F^n/[n]!$ for $n \geq 1$ (Lusztig's integral form, cf. [16]). We have

$$U_{\mathbb{Z}} = U_{\mathbb{Z},q} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[q^{1/2}, q^{-1/2}].$$

Let $\bar{U}$ denote the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-subalgebra of U_h generated by K, K^{-1}, $(q^{1/2}-q^{-1/2})E$, and $(q^{1/2}-q^{-1/2})F$ (cf. [1]). We have

$$\bar{U} = \bar{U}_q \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[q^{1/2}, q^{-1/2}].$$

There is a Hopf $\mathbb{Z}[q, q^{-1}]$-algebra structure on $U_{\mathbb{Z},q}$ inherited from U_h (cf. [16, 21]). We have

$$\Delta(\tilde{E}^{(m)}) = \sum_{j=0}^{m} \tilde{E}^{(m-j)} K^j \otimes \tilde{E}^{(j)}, \tag{9}$$

$$\Delta(\tilde{F}^{(m)}) = \sum_{j=0}^{m} \tilde{F}^{(m-j)} K^j \otimes \tilde{F}^{(j)}, \tag{10}$$

$$S^{\pm 1}(\tilde{E}^{(m)}) = (-1)^m q^{\frac{1}{2}m(m\mp 1)} K^{-m} \tilde{E}^{(m)}, \tag{11}$$

$$S^{\pm 1}(\tilde{F}^{(m)}) = (-1)^m q^{-\frac{1}{2}m(m\mp 1)} K^{-m} \tilde{F}^{(m)}, \tag{12}$$

for $i \in \mathbb{Z}, m \geq 0$. Similarly, there is a Hopf $\mathbb{Z}[q, q^{-1}]$-algebra structure on $\bar{U}_q$ inherited from U_h (cf. [1, 6]).

Let U_h^0 denote the Cartan part of U_h, i.e., the subalgebra of U_h topologically generated by H. Let $\bar{U}_q^0$ denote the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\bar{U}_q$ generated by K and K^{-1}. Let $\bar{U}_q^{\mathrm{ev}\,0}$ be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $\bar{U}_q$ generated by K^2 and K^{-2}. We have

$$\bar{U}_q^0 = \bar{U}_q \cap U_h^0, \quad \bar{U}_q^{\mathrm{ev}\,0} = \bar{U}_q^{\mathrm{ev}} \cap U_h^0.$$

2.4 Adjoint action

In what follows, we use the following notations. For $m \geq 0$, let $\Delta^{[m]}\colon U_h \to U_h^{\hat{\otimes} m}$ denote the m-output comultiplication defined by $\Delta^{[0]} = \varepsilon, \Delta^{[1]} = \mathrm{id}_{U_h}$, and

$$\Delta^{[m]} = (\Delta \otimes \mathrm{id}_{U_h}^{\otimes m-2}) \circ \Delta^{[m-1]},$$

for $m \geq 2$. For $x \in U_h$, $m \geq 1$, we write

$$\Delta^{[m]}(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(m)}.$$

For $m_1, \ldots, m_l \geq 0$, set

$$\Delta^{[m_1,\ldots,m_l]} = \Delta^{[m_1]} \otimes \cdots \otimes \Delta^{[m_l]}\colon U_h^{\hat{\otimes} l} \to U_h^{\hat{\otimes} m_1+\cdots+m_l}. \tag{13}$$

We use the left adjoint action $\mathrm{ad}\colon U_h \hat{\otimes} U_h \to U_h$ defined by

$$\mathrm{ad}(x \otimes y) = x \triangleright y := \sum x_{(1)} y S(x_{(2)}), \tag{14}$$

for $x, y \in U_h$. We use the following proposition.

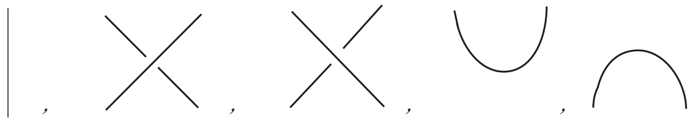


Figure 6: Fundamental tangles, where the orientations of the strands are arbitrary

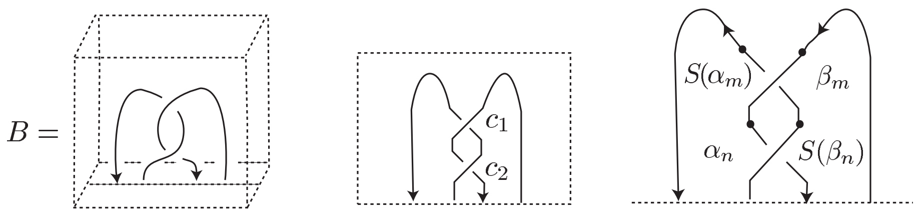


Figure 7: (a) A bottom tangle $B \in BT_2$ (b) A diagram of B (c) The labels which are put on the diagram of B

Proposition 2.2 ([21, Proposition 3.2])**.** *We have*

$$U_{\mathbb{Z},q} \triangleright \bar{U}_q^{\mathrm{ev}} \subset \bar{U}_q^{\mathrm{ev}}, \quad U_{\mathbb{Z},q} \triangleright K\bar{U}_q^{\mathrm{ev}} \subset K\bar{U}_q^{\mathrm{ev}}.$$

We also use a right action $\overline{\mathrm{ad}}\colon U_h \hat{\otimes} U_h \to U_h$, which is the continuous $\mathbb{Q}[[h]]$-linear map defined by

$$\begin{aligned}\overline{\mathrm{ad}}(y \otimes x) = y \triangleleft x :&= \sum S^{-1}(x_{(2)}) y x_{(1)}, \\ &= \sum S^{-1}(x) \triangleright y.\end{aligned}$$

for $x, y \in U_h$. Proposition 2.2 implies the following.

Corollary 2.3. *We have*

$$\bar{U}_q^{\mathrm{ev}} \triangleleft U_{\mathbb{Z},q} \subset \bar{U}_q^{\mathrm{ev}}, \quad K\bar{U}_q^{\mathrm{ev}} \triangleleft U_{\mathbb{Z},q} \subset K\bar{U}_q^{\mathrm{ev}}.$$

2.5 Universal sl_2 invariant of bottom tangles

For an n-component bottom tangle $T = T_1 \cup \cdots \cup T_n \in BT_n$, we define the universal sl_2 invariant $J_T \in U_h^{\hat{\otimes} n}$ as follows ([19, 4]).

We choose and fix a diagram of T obtained from the copies of the fundamental tangles depicted in Figure 6, by pasting horizontally and vertically. For example, for the bottom tangle B depicted in Figure 7 (a), we can take a diagram depicted in Figure 7 (b). We denote by $C(T)$ the set of the crossings of the diagram. We call a map

$$s\colon C(T) \quad \to \quad \{0, 1, 2, \ldots\}$$

a *state*. We denote by $\mathcal{S}(T)$ the set of states of the diagram.

Given a state $s \in \mathcal{S}(T)$, we attach labels on the copies of the fundamental tangles in the diagram following the rule described in Figure 8, where "S'" should be replaced with id if the string is oriented downward, and with S otherwise. For example, for a state $t \in \mathcal{S}(B)$, we put

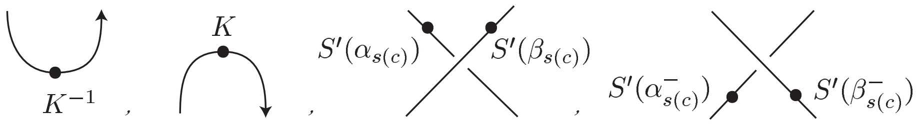


Figure 8: How to place labels on the fundamental tangles

labels on the diagram of B as in Figure 7 (c), where we set $m = t(c_1)$ and $n = t(c_2)$ for the upper and the lower crossings c_1 and c_2, respectively.

We define an element $J_{T,s} \in U_h^{\hat{\otimes} n}$ as follows. The ith tensorand of $J_{T,s}$ is defined to be the product of the labels put on the component corresponding to T_i, where the labels are read off along T_i reversing the orientation, and written from left to right. We identify the labels $S'(\alpha_i^{\pm})$ and $S'(\beta_i^{\pm})$ with the first and the second tensorands, respectively, of the element $S'(\alpha_i^{\pm}) \otimes S'(\beta_i^{\pm}) \in U_h^{\hat{\otimes} 2}$. Also we identify the label $K^{\pm 1}$ with the element $K^{\pm 1} \in U_h$. Then, $J_{T,s}$ is a well-defined element in $U_h^{\hat{\otimes} n}$. For example, we have

$$\begin{aligned} J_{B,t} &= S(\alpha_m)S(\beta_n) \otimes \alpha_n\beta_m \\ &= \sum q^{\frac{1}{2}m(m-1)} q^{\frac{1}{2}n(n-1)} S(D_1'\tilde{F}^{(m)}K^{-m})S(D_2''e^n) \otimes D_2'\tilde{F}^{(n)}K^{-n}D_1''e^m \\ &= (-1)^{m+n} q^{-n+2mn} D^{-2}(\tilde{F}^{(m)}K^{-2n}e^n \otimes \tilde{F}^{(n)}K^{-2m}e^m) \in U_h^{\hat{\otimes} 2}, \end{aligned}$$

where $D = \sum D_1' \otimes D_1'' = \sum D_2' \otimes D_2''$. Note that $J_{T,s}$ depends on the choice of the diagram.

Set

$$J_T = \sum_{s \in \mathcal{S}(T)} J_{T,s}.$$

For example, we have

$$J_B = \sum_{t \in \mathcal{S}(B)} J_{B,t} = \sum_{m,n \geq 0} (-1)^{m+n} q^{-n+2mn} D^{-2}(\tilde{F}^{(m)}K^{-2n}e^n \otimes \tilde{F}^{(n)}K^{-2m}e^m).$$

As is well known [19], J_T does not depend on the choice of the diagram, and defines an isotopy invariant of bottom tangles.

3 Universal invariant of boundary bottom tangles

In this section, we recall Habiro's formulas for boundary bottom tangles at the topological level (Proposition 3.1), and at the algebraic level on the universal sl_2 invariant (Proposition 3.3). Then, we modify these formulas into a form more convenient for our purpose. After that, we study the commutator maps of U_h. In the last section, we give an outline of the proof of Theorem 1.2.

In what follows, we use the following notations. Let $\eta\colon \mathbb{Q}[[h]] \to U_h$ be the unit morphism of U_h and $\mu\colon U_h^{\hat{\otimes} 2} \to U_h$ the multiplication of U_h. For $g \geq 0$, let $\mu^{[g]}\colon U_h^{\hat{\otimes} g} \to U_h$ denote the g-input multiplication defined by $\mu^{[0]} = \eta$, $\mu^{[1]} = \mathrm{id}_{U_h}$, and

$$\mu^{[g]} = \mu^{[g-1]} \circ (\mu \otimes \mathrm{id}_{U_h}^{\otimes g-2}),$$

for $g \geq 2$. For $g_1, \ldots, g_n \geq 0$, set

$$\mu^{[g_1,\ldots,g_n]} = \mu^{[g_1]} \otimes \cdots \otimes \mu^{[g_n]}\colon U_h^{\hat{\otimes} g_1+\cdots+g_n} \to U_h^{\hat{\otimes} n}. \tag{15}$$

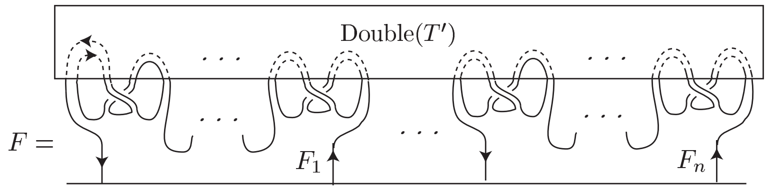


Figure 9: How to arrange Seifert surfaces

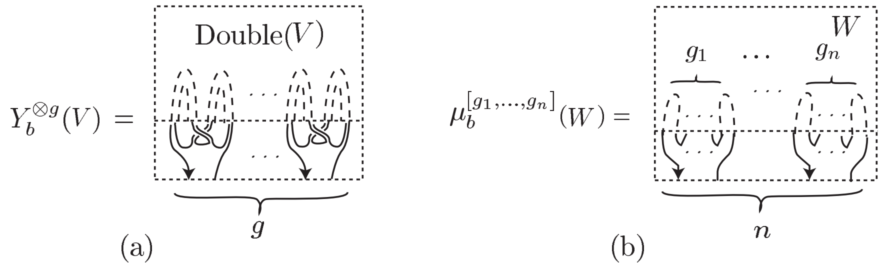


Figure 10: (a) $Y_b^{\otimes g}(V) \in BT_g$ for $V \in BT_{2g}$ (b) $\mu_b^{[g_1,\dots,g_n]}(W) \in BT_n$ for $W \in BT_g$

3.1 Habiro's formula (topological level)

Let $T = T_1 \cup \cdots \cup T_n \in BT_n$ be a boundary bottom tangle and $F_1, \dots, F_n$ mutually disjoint Seifert surfaces such that $\partial F_i = K_{T_i}$ for $i = 1, \dots, n$. We can arrange the surfaces $F_1, \dots, F_n$ as depicted in Figure 9, where Double(T') is the tangle obtained from a bottom tangle $T' \in BT_{2g}$, where $g = g_1 + \cdots + g_n$ with $g_i = \text{genus}(F_i)$, by first duplicating and then reversing the orientation of the inner component of each pair of duplicated components.

The above arrangement of the Seifert surfaces implies the following result, which appears in the proof of [4, Theorem 9.9].

Proposition 3.1. *For a bottom tangle $T \in BT_n$, the following conditions are equivalent.*

(1) *T is a boundary bottom tangle.*

(2) *There is a bottom tangle $T' \in BT_{2g}$, $g \geq 0$, and integers $g_1, \dots, g_n \geq 0$ satisfying $g_1 + \cdots + g_n = g$, such that*

$$T = \mu_b^{[g_1,\dots,g_n]} Y_b^{\otimes g}(T'), \tag{16}$$

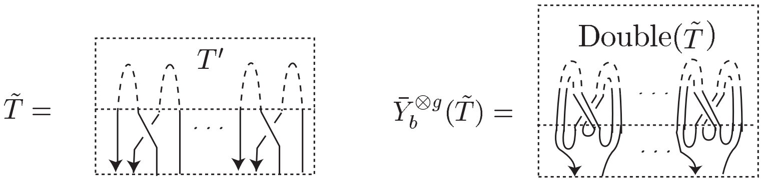
where $Y_b^{\otimes g} \colon BT_{2g} \to BT_g$ and $\mu_b^{[g_1,\dots,g_n]} \colon BT_g \to BT_n$ are defined as depicted in Figure 10 (a) and (b), respectively.

3.2 Habiro's formula (algebraic level)

Recall from [4, Proposition 9.7] the commutator morphism $Y_{\underline{H}} \colon H \otimes H \to H$ for a ribbon Hopf algebra H. In the present case $H = U_h$, the morphism $Y_{\underline{U_h}} \colon U_h \hat{\otimes} U_h \to U_h$ is the continuous $\mathbb{Q}[[h]]$-linear map defined by

$$Y_{\underline{U_h}}(x \otimes y) = \sum_{k \geq 0} x \triangleright \Big(\beta_k S\big((\alpha_k \triangleright y)_{(1)}\big)\Big)(\alpha_k \triangleright y)_{(2)}$$

for $x, y \in U_h$.



(a) (b)

Figure 11: (a) The tangle $\tilde{T} = \nu_b^{\otimes g}(T')$ (b) The bottom tangle $\bar{Y}_b^{\otimes g}(\tilde{T}) \in BT_g$

Lemma 3.2 (Habiro [4]). *For each bottom tangle $T \in BT_{2g}$, $g \geq 0$, we have*

$$J_{Y_b^{\otimes g}(T)} = Y_{\underline{U_h}}^{\otimes g}(J_T).$$

For each bottom tangle $T \in BT_{g_1+\cdots+g_n}$, $g_1, \ldots, g_n \geq 0$, we have

$$J_{\mu_b^{[g_1,\ldots,g_n]}(T)} = \mu^{[g_1,\ldots,g_n]}(J_T).$$

Proposition 3.1 and Lemma 3.2 imply the following.

Proposition 3.3 (Habiro [4]). *For a boundary bottom tangle $T \in BT_n$ and a bottom tangle $T' \in BT_{2g}$ satisfying (16), we have*

$$J_T = \mu^{[g_1,\ldots,g_n]}(Y_{\underline{U_h}})^{\otimes g}(J_{T'}).$$

3.3 Modification of Habiro's formula (topological level)

In this section, we modify Proposition 3.1.

Let $T \in BT_n$ be a boundary bottom tangle and T' a $2g$-component bottom tangle satisfying (16). We decompose the operation $Y_b^{\otimes g}$ into the two operations $\nu_b^{\otimes g}$ and $\bar{Y}_b^{\otimes g}$ as follows. Let $\tilde{T} = \nu_b^{\otimes g}(T')$ be the $2g$-component (non-bottom) tangle as depicted in Figure 11 (a). Set $\bar{Y}_b^{\otimes g}(\tilde{T}) = Y_b^{\otimes g}(T') \in BT_g$, i.e., $\bar{Y}_b^{\otimes g}(\tilde{T})$ is the bottom tangle as depicted in Figure 11 (b), where Double(T') is defined in the same way as that for bottom tangles.

We have

$$\begin{aligned} T &= \mu_b^{[g_1,\ldots,g_n]} Y_b^{\otimes g}(T') \\ &= \mu_b^{[g_1,\ldots,g_n]} (\bar{Y}_b^{\otimes g} \circ \nu_b^{\otimes g})(T') \\ &= \mu_b^{[g_1,\ldots,g_n]} \bar{Y}_b^{\otimes g}\big(\nu_b^{\otimes g}(T')\big) \\ &= \mu_b^{[g_1,\ldots,g_n]} \bar{Y}_b^{\otimes g}(\tilde{T}). \end{aligned}$$

Thus, we can modify Proposition 3.1 by replacing (2) with (2') as follows.

(2') There is a $2g$-component tangle $\tilde{T} = \nu_b^{\otimes g}(T')$ with $T' \in BT_{2g}$, $g \geq 0$, and integers $g_1, \ldots, g_n \geq 0$ satisfying $g_1 + \cdots + g_n = g$, such that

$$T = \mu_b^{[g_1,\ldots,g_n]} \bar{Y}_b^{\otimes g}(\tilde{T}).$$

For a boundary bottom tangle $T \in BT_n$, we call $(\tilde{T}; g, g_1, \ldots, g_n)$ as in (2') a *boundary data* for T.

3.4 Modification of Habiro's formula (algebraic level)

Let $\bar{Y}\colon U_h\hat{\otimes}U_h \to U_h$ be the continuous $\mathbb{Q}[[h]]$-linear map defined by

$$\bar{Y}(x\otimes y) = \sum x_{(1)}KS(y_{(2)})KS(x_{(2)})y_{(1)}, \tag{17}$$

for $x, y \in U_h$.

We modify Proposition 3.3 as follows.

Proposition 3.4. *Let $T \in BT_n$ be a boundary bottom tangle and $(\tilde{T}; g, g_1, \ldots, g_n)$ a boundary data for T. We have*

$$J_T = \mu^{[g_1,\ldots,g_n]}\bar{Y}^{\otimes g}(J_{\tilde{T}}).$$

Here, we can define the universal sl_2 invariant $J_{\tilde{T}} \in U_h^{\hat{\otimes}2g}$ of the tangle $\tilde{T}$ in a similar way to that of bottom tangles (cf. [4]).

Let $\nu\colon U_h\hat{\otimes}U_h \to U_h\hat{\otimes}U_h$ be the continuous $\mathbb{Q}[[h]]$-linear map defined by

$$\nu(x\otimes y) = \sum_{k\geq 0} x\beta_k \otimes \alpha_k y,$$

for $x, y \in U_h$. We reduce Proposition 3.4 to the following lemma.

Lemma 3.5. *We have*

$$Y_{\underline{U_h}} = \bar{Y}\circ\nu. \tag{18}$$

Proof of Proposition 3.4 by assuming Lemma 3.5. Let $T' \in BT_g$ the bottom tangle such that $\tilde{T} = \nu_b^{\otimes g}(T')$. We depict in Figure 12 the labels put on the new crossings $c_1, \ldots, c_g$ at the bottom of $\tilde{T}$ associated a state $s \in \mathcal{S}(\tilde{T})$. Since $(1\otimes S)(R^{-1}) = R$, we have

$$J_{\tilde{T}} = \nu^{\otimes g}(J_{T'}). \tag{19}$$

By Proposition 3.3, Lemma 3.5, and (19), we have

$$\begin{aligned}
J_T &= \mu^{[g_1,\ldots,g_n]}Y_{\underline{U_h}}^{\otimes g}(J_{T'}) \\
&= \mu^{[g_1,\ldots,g_n]}(\bar{Y}\circ\nu)^{\otimes g}(J_{T'}) \\
&= \mu^{[g_1,\ldots,g_n]}\bar{Y}^{\otimes g}\big(\nu^{\otimes g}(J_{T'})\big) \\
&= \mu^{[g_1,\ldots,g_n]}\bar{Y}^{\otimes g}(J_{\tilde{T}}).
\end{aligned}$$

□

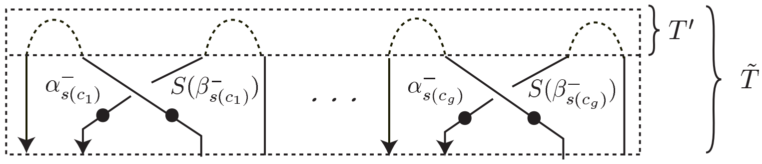


Figure 12: The labels which are put on the crossings $c_1, \ldots, c_g$ at the bottom of $\tilde{T}$

Proof of Lemma 3.5. For $x, y \in U_h$, we have

$$
\begin{aligned}
&(\bar{Y} \circ \nu)(x \otimes y) \\
&\quad = \sum_{k \geq 0} (x\beta_k)_{(1)} K S((\alpha_k y)_{(2)}) K S((x\beta_k)_{(2)})(\alpha_k y)_{(1)} \\
&\quad = \sum_{k \geq 0} x_{(1)} \beta_{k(1)} K S(y_{(2)}) S(\alpha_{k(2)}) K S(\beta_{k(2)}) S(x_{(2)}) \alpha_{k(1)} y_{(1)} \\
&\quad \overset{\text{(i)}}{=} \sum_{k_1,k_2,k_3,k_4 \geq 0} x_{(1)} \beta_{k_1} \beta_{k_2} K S(y_{(2)}) S(\alpha_{k_4}\alpha_{k_2}) K S(\beta_{k_3}\beta_{k_4}) S(x_{(2)}) \alpha_{k_3}\alpha_{k_1} y_{(1)} \\
&\quad = \sum_{k_1,k_2,k_3,k_4 \geq 0} x_{(1)} \beta_{k_1} \beta_{k_2} K S(y_{(2)}) S(\alpha_{k_2}) S(\alpha_{k_4}) K S(\beta_{k_4}) S(\beta_{k_3}) S(x_{(2)}) \alpha_{k_3}\alpha_{k_1} y_{(1)} \\
&\quad \overset{\text{(ii)}}{=} \sum_{k_1,k_2,k_3,k_4 \geq 0} x_{(1)} \beta_{k_1} \beta_{k_2} K S(y_{(2)}) S(\alpha_{k_2}) \alpha_{k_4} K \beta_{k_4} S(\beta_{k_3}) S(x_{(2)}) \alpha_{k_3}\alpha_{k_1} y_{(1)} \\
&\quad = \sum_{k_1,k_2,k_3 \geq 0} x_{(1)} \beta_{k_1} \beta_{k_2} K S(y_{(2)}) S(\alpha_{k_2}) r^{-1} S(\beta_{k_3}) S(x_{(2)}) \alpha_{k_3}\alpha_{k_1} y_{(1)} \\
&\quad \overset{\text{(iii)}}{=} \sum_{k_1,k_2,k_3,k_5,k_6 \geq 0} x_{(1)} \beta_{k_1} \beta_{k_2} K S(\beta^-_{k_5} y_{(1)} \beta_{k_6}) S(\alpha_{k_2}) r^{-1} S(\beta_{k_3}) S(x_{(2)}) \alpha_{k_3}\alpha_{k_1} \alpha^-_{k_5} y_{(2)} \alpha_{k_6} \\
&\quad = \sum_{k_1,k_2,k_3,k_5,k_6 \geq 0} x_{(1)} \beta_{k_1} \beta_{k_2} K S(\beta_{k_6}) S(y_{(1)}) S(\beta^-_{k_5}) S(\alpha_{k_2}) r^{-1} S(\beta_{k_3}) S(x_{(2)}) \alpha_{k_3}\alpha_{k_1} \alpha^-_{k_5} y_{(2)} \alpha_{k_6} \\
&\quad \overset{\text{(iv)}}{=} \sum_{k_1,k_2,k_3,k_5,k_6 \geq 0} x_{(1)} \beta_{k_1} \beta_{k_2} S^{-1}(\beta_{k_6}) K S(y_{(1)}) S(\beta^-_{k_5}) S(\alpha_{k_2}) r^{-1} S(\beta_{k_3}) S(x_{(2)}) \alpha_{k_3}\alpha_{k_1} \alpha^-_{k_5} y_{(2)} \alpha_{k_6} \\
&\quad = \sum_{k_1,k_2,k_3,k_5,k_6 \geq 0} x_{(1)} \beta_{k_1} \beta_{k_2} S^{-1}(\beta_{k_6}) r^{-1} K S(y_{(1)}) S(\beta^-_{k_5}) S(\alpha_{k_2}) S(\beta_{k_3}) S(x_{(2)}) \alpha_{k_3}\alpha_{k_1} \alpha^-_{k_5} y_{(2)} \alpha_{k_6} \\
&\quad \overset{\text{(v)}}{=} \sum_{k_1,k_2,k_3,k_4,k_5,k_6 \geq 0} x_{(1)} \beta_{k_1} \beta_{k_2} \beta_{k_6} \beta_{k_4} K^{-1} \alpha_{k_4} K S(y_{(1)}) S(\beta^-_{k_5}) S(\beta_{k_3}\alpha_{k_2}) S(x_{(2)}) \alpha_{k_3}\alpha_{k_1} \alpha^-_{k_5} y_{(2)} S(\alpha_{k_6}) \\
&\quad \overset{\text{(vi)}}{=} \sum_{k_1,k_2,k_3,k_4,k_5,k_6 \geq 0} x_{(1)} \beta_{k_1} \beta_{k_2} \beta_{k_6} \beta_{k_4} S^2(\alpha_{k_4}) S(y_{(1)}) S(\beta^-_{k_5}) S(\beta_{k_3}\alpha_{k_2}) S(x_{(2)}) \alpha_{k_3}\alpha_{k_1} \alpha^-_{k_5} y_{(2)} S(\alpha_{k_6}) \\
&\quad \overset{\text{(vii)}}{=} \sum_{k,k_3,k_5 \geq 0} x_{(1)} \beta_k S^2(\alpha_{k(4)}) S(y_{(1)}) S(\beta^-_{k_5}) S(\beta_{k_3}\alpha_{k(2)}) S(x_{(2)}) \alpha_{k_3} \alpha_{k(1)} \alpha^-_{k_5} y_{(2)} S(\alpha_{k(3)}) \\
&\quad \overset{\text{(viii)}}{=} \sum_{k,k_3,k_5 \geq 0} x_{(1)} \beta_k S^2(\alpha_{k(4)}) S(y_{(1)}) S(\beta^-_{k_5}) S(\alpha_{k(1)}\beta_{k_3}) S(x_{(2)}) \alpha_{k(2)} \alpha_{k_3} \alpha^-_{k_5} y_{(2)} S(\alpha_{k(3)}) \\
&\quad = \sum_{k,k_3,k_5 \geq 0} x_{(1)} \beta_k S^2(\alpha_{k(4)}) S(y_{(1)}) S(\beta_{k_3}\beta^-_{k_5}) S(\alpha_{k(1)}) S(x_{(2)}) \alpha_{k(2)} \alpha_{k_3} \alpha^-_{k_5} y_{(2)} S(\alpha_{k(3)}) \\
&\quad = \sum_{k \geq 0} x_{(1)} \beta_k S^2(\alpha_{k(4)}) S(y_{(1)}) S(\alpha_{k(1)}) S(x_{(2)}) \alpha_{k(2)} y_{(2)} S(\alpha_{k(3)}) \\
&\quad = \sum_{k \geq 0} x_{(1)} \beta_k S(\alpha_{k(1)} y_{(1)} S(\alpha_{k(4)})) S(x_{(2)}) \alpha_{k(2)} y_{(2)} S(\alpha_{k(3)}) \\
&\quad \overset{\text{(ix)}}{=} \sum_{k \geq 0} x_{(1)} \beta_k S\Big(\big(\alpha_{k(1)} y S(\alpha_{k(2)})\big)_{(1)} \Big) S(x_{(2)}) \big(\alpha_{k(1)} y S(\alpha_{k(2)})\big)_{(2)} = Y_{\underline{U_h}}(x \otimes y).
\end{aligned}
$$

Here, recall that $r^{-1} = \sum_{k_4 \geq 0} \alpha_{k_4} K \beta_{k_4} = \sum_{k_4 \geq 0} \beta_{k_4} K^{-1} \alpha_{k_4}$ is the inverse of the ribbon element, which is central.

The identity (i) follows from

$$\sum_{k\geq 0}\alpha_{k(1)}\otimes\alpha_{k(2)}\otimes\beta_{k(1)}\otimes\beta_{k(2)}=\sum_{k_1,k_2,k_3,k_4\geq 0}\alpha_{k_3}\alpha_{k_1}\otimes\alpha_{k_4}\alpha_{k_2}\otimes\beta_{k_1}\beta_{k_2}\otimes\beta_{k_3}\beta_{k_4}.$$

(ii) and (v) follow from $(S\otimes S)R=R$.
(iii) and (viii) follow from $R\big(\sum x_{(1)}\otimes x_{(2)}\big)=\big(\sum x_{(2)}\otimes x_{(1)}\big)R$ for $x\in U_h$.
(iv) and (vi) follow from $KxK^{-1}=S^{-2}(x)$ for $x\in U_h$.
(vii) follows from

$$\sum_{k_1,k_2,k_4,k_6\geq 0}\beta_{k_1}\beta_{k_2}\beta_{k_6}\beta_{k_4}\otimes\alpha_{k_1}\otimes\alpha_{k_2}\otimes\alpha_{k_6}\otimes\alpha_{k_4}=\sum_{k\geq 0}\beta_k\otimes\alpha_{k(1)}\otimes\alpha_{k(2)}\otimes\alpha_{k(3)}\otimes\alpha_{k(4)}.$$

(ix) follows from $\sum S(x_{(1)})\otimes S(x_{(2)})=\sum S(x)_{(2)}\otimes S(x)_{(1)}$ for $x\in U_h$. □

3.5 Commutator maps

In this section, we study the commutator map $\bar{Y}$ of U_h.

Let $\dot{Y}\colon U_h\hat{\otimes}U_h\to U_h$ be the continuous $\mathbb{Q}[[h]]$-linear map defined by

$$\dot{Y}(x\otimes y)=\sum x_{(1)}S^{-1}(y_{(2)})S(x_{(2)})y_{(1)},$$

for $x,y\in U_h$. Note that

$$\dot{Y}(x\otimes y)=\sum\Big(x\triangleright S^{-1}(y_{(2)})\Big)y_{(1)} \tag{20}$$

$$=\sum x_{(1)}\Big(S(x_{(2)})\triangleleft y\Big) \tag{21}$$

$$=\sum x_{(1)}\Big(S^{-1}(y)\triangleright S(x_{(2)})\Big). \tag{22}$$

By the following lemma, we can study $\bar{Y}$ by using $\dot{Y}$, $\triangleright$ and $\triangleleft$.

Lemma 3.6. *For $x,y\in U_h$, we have*

$$\bar{Y}(x\otimes y)=\sum\dot{Y}(x_{(1)}\otimes y_{(2)})\big((x_{(2)}\triangleright K^2)\triangleleft y_{(1)}\big).$$

Proof. We have

$$\begin{aligned}\bar{Y}(x\otimes y)&=\sum x_{(1)}KS(y_{(2)})KS(x_{(2)})y_{(1)}\\&=\sum x_{(1)}S^{-1}(y_{(2)})K^2S(x_{(2)})y_{(1)}\\&=\sum x_{(1)}S^{-1}(y_{(2)})S(x_{(2)})x_{(3)}K^2S(x_{(4)})y_{(1)}\\&=\sum x_{(1)}S^{-1}(y_{(4)})S(x_{(2)})y_{(3)}S^{-1}(y_{(2)})x_{(3)}K^2S(x_{(4)})y_{(1)}\\&=\sum\dot{Y}(x_{(1)}\otimes y_{(2)})\big((x_{(2)}\triangleright K^2)\triangleleft y_{(1)}\big),\end{aligned}$$

where the second identity follows from $Kz=S^{-2}(z)K$ for $z\in U_h$. □

The rest of this section is devoted to studying the map $\dot{Y}$.

Lemma 3.7. *For $x, y, z \in U_h$, we have*

$$\dot{Y}(xy \otimes z) = \sum \big(x_{(1)} \triangleright \dot{Y}(y \otimes z_{(2)})\big)\dot{Y}(x_{(2)} \otimes z_{(1)}), \tag{23}$$

$$\dot{Y}(x \otimes yz) = \sum \dot{Y}(x_{(1)} \otimes z_{(2)})\big(\dot{Y}(x_{(2)} \otimes y) \triangleleft z_{(1)}\big). \tag{24}$$

Proof. We have

$$\begin{aligned}
\dot{Y}(xy \otimes z) &= \sum (xy)_{(1)} S^{-1}(z_{(2)}) S\big((xy)_{(2)}\big) z_{(1)} \\
&= \sum x_{(1)} y_{(1)} S^{-1}(z_{(2)}) S(y_{(2)}) S(x_{(2)}) z_{(1)} \\
&= \sum x_{(1)} y_{(1)} S^{-1}(z_{(4)}) S(y_{(2)}) z_{(3)} S^{-1}(z_{(2)}) S(x_{(2)}) z_{(1)} \\
&= \sum x_{(1)} y_{(1)} S^{-1}(z_{(4)}) S(y_{(2)}) z_{(3)} S(x_{(2)}) x_{(3)} S^{-1}(z_{(2)}) S(x_{(4)}) z_{(1)} \\
&= \sum \big(x_{(1)} \triangleright \dot{Y}(y \otimes z_{(2)})\big)\dot{Y}(x_{(2)} \otimes z_{(1)}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\dot{Y}(x \otimes yz) &= \sum x_{(1)} S^{-1}\big((yz)_{(2)}\big) S(x_{(2)}) (yz)_{(1)} \\
&= \sum x_{(1)} S^{-1}(z_{(4)}) S(x_{(2)}) z_{(3)} S^{-1}(z_{(2)}) x_{(3)} S^{-1}(y_{(2)}) S(x_{(4)}) y_{(1)} z_{(1)} \\
&= \sum \dot{Y}(x_{(1)} \otimes z_{(2)})\big(\dot{Y}(x_{(2)} \otimes y) \triangleleft z_{(1)}\big).
\end{aligned}$$

□

Lemma 3.8. *For $x, y \in U_h^0$, we have*

$$\dot{Y}(x \otimes y) = \varepsilon(x)\varepsilon(y).$$

Proof. It is enough to prove

$$\dot{Y}(H^m \otimes H^n) = \delta_{m,0}\delta_{n,0},$$

for $m, n \geq 0$. By using the formula

$$\Delta(H^m) = \sum_{i=0}^{m} \binom{m}{i} H^i \otimes H^{m-i},$$

for $m \geq 0$, we have

$$\begin{aligned}
\dot{Y}(H^m \otimes H^n) &= \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} H^i(-H)^j(-H)^{m-i}H^{n-j} \\
&= \left(\sum_{i=0}^{m}(-1)^i \binom{m}{i}\right)\left(\sum_{j=0}^{n}(-1)^j \binom{n}{j}\right)(-1)^m H^{n+m} \\
&= \delta_{n,0}\delta_{m,0}.
\end{aligned}$$

□

Lemma 3.9. *We have*

$$\dot{Y}(U_{\mathbb{Z},q} \otimes \bar{U}_q) \subset \bar{U}_q^{\mathrm{ev}}, \tag{25}$$

$$\dot{Y}(\bar{U}_q \otimes U_{\mathbb{Z},q}) \subset \bar{U}_q^{\mathrm{ev}}. \tag{26}$$

Proof. We prove (25). Then (26) is similar. Note that

$$(1 \otimes S^{\pm 1})\Delta(\bar{U}_q) \subset \bigoplus_{i=0,1} \Big(K^i \bar{U}_q^{\mathrm{ev}} \otimes K^i \bar{U}_q^{\mathrm{ev}}\Big), \tag{27}$$

since we have

$$(1 \otimes S^{\pm 1})\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\mp 1}, \tag{28}$$

$$(1 \otimes S^{\pm 1})\Delta(\tilde{F}^{(n)}) = \sum_{j=0}^{n} (-1)^n q^{-\frac{1}{2}n(n\mp 1)} \tilde{F}^{(n-j)} K^j \otimes K^{-j} \tilde{F}^{(j)}, \tag{29}$$

$$(1 \otimes S^{\pm 1})\Delta(e) = e \otimes 1 - K \otimes K^{-1} e. \tag{30}$$

Then, (20) and (27) imply

$$\dot{Y}(U_{\mathbb{Z},q} \otimes \bar{U}_q) \subset \sum_{i=0,1} \Big(U_{\mathbb{Z},q} \triangleright K^i \bar{U}_q^{\mathrm{ev}}\Big) K^i \bar{U}_q^{\mathrm{ev}}.$$

By Proposition 2.2, we have

$$\sum_{i=0,1} \Big(U_{\mathbb{Z},q} \triangleright K^i \bar{U}_q^{\mathrm{ev}}\Big) K^i \bar{U}_q^{\mathrm{ev}} \subset \sum_{i=0,1} \big(K^i \bar{U}_q^{\mathrm{ev}}\big) \cdot \big(K^i \bar{U}_q^{\mathrm{ev}}\big) \subset \bar{U}_q^{\mathrm{ev}}.$$

This completes the proof. □

In what follows, we use the notations $D^{\pm 1} = \sum D'_{\pm} \otimes D''_{\pm}$.

Lemma 3.10. *We have*

$$\sum \dot{Y}(U_{\mathbb{Z},q} \otimes \bar{U}_q^0 D'_{\pm}) \otimes \dot{Y}(\bar{U}_q \otimes \bar{U}_q^0 D''_{\pm}) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes 2}, \tag{31}$$

$$\sum \dot{Y}(U_{\mathbb{Z},q} \otimes \bar{U}_q^0 D'_{\pm}) \otimes \dot{Y}(\bar{U}_q^0 D''_{\pm} \otimes \bar{U}_q) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes 2}, \tag{32}$$

$$\sum \dot{Y}(\bar{U}_q^0 D'_{\pm} \otimes U_{\mathbb{Z},q}) \otimes \dot{Y}(\bar{U}_q \otimes \bar{U}_q^0 D''_{\pm}) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes 2}, \tag{33}$$

$$\sum \dot{Y}(\bar{U}_q^0 D'_{\pm} \otimes U_{\mathbb{Z},q}) \otimes \dot{Y}(\bar{U}_q^0 D''_{\pm} \otimes \bar{U}_q) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes 2}. \tag{34}$$

Proof. First, we prove (32) with D. Let us assume a weaker inclusion

$$\sum \dot{Y}(U_{\mathbb{Z},q} \otimes D') \otimes \dot{Y}(D'' \otimes \bar{U}_q) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes 2}, \tag{35}$$

which we prove later. We have

$$\begin{aligned}
&\sum \dot{Y}(U_{\mathbb{Z},q} \otimes \bar{U}_q^0 D') \otimes \dot{Y}(\bar{U}_q^0 D'' \otimes \bar{U}_q) \\
=&\sum \dot{Y}(U_{\mathbb{Z},q} \otimes D' \bar{U}_q^0) \otimes \dot{Y}(\bar{U}_q^0 D'' \otimes \bar{U}_q) \\
\subset&\sum \dot{Y}(U_{\mathbb{Z},q} \otimes \bar{U}_q^0)\big(\dot{Y}(U_{\mathbb{Z},q} \otimes D') \triangleleft \bar{U}_q^0\big) \otimes \big(\bar{U}_q^0 \triangleright \dot{Y}(D'' \otimes \bar{U}_q)\big)\dot{Y}(\bar{U}_q^0 \otimes \bar{U}_q) \\
\subset&\sum \dot{Y}(U_{\mathbb{Z},q} \otimes \bar{U}_q^0)\big(\bar{U}_q^{\mathrm{ev}} \triangleleft \bar{U}_q^0\big) \otimes \big(\bar{U}_q^0 \triangleright \bar{U}_q^{\mathrm{ev}}\big)\dot{Y}(\bar{U}_q^0 \otimes \bar{U}_q) \\
\subset&\sum \dot{Y}(U_{\mathbb{Z},q} \otimes \bar{U}_q^0) \cdot \bar{U}_q^{\mathrm{ev}} \otimes \bar{U}_q^{\mathrm{ev}} \cdot \dot{Y}(\bar{U}_q^0 \otimes \bar{U}_q) \\
\subset&(\bar{U}_q^{\mathrm{ev}})^{\otimes 2},
\end{aligned} \tag{36}$$

where the identity follows from (8), the first inclusion follows from Lemma 3.7, $\Delta(X)\subset X^{\otimes 2}$, for $X=\bar{U}_q,\bar{U}_q^0,U_{\mathbb{Z},q}$, and the last inclusion follows from Lemma 3.9.

Now, we prove (35). By (5) and (7), we have

$$(1\otimes S^{\pm 1}\otimes 1\otimes S^{\pm 1})(\Delta\otimes\Delta)(D)=\sum D'_{1,-}D'_1\otimes D'_2D'_{2,-}\otimes D''_{2,-}D''_1\otimes D''_2D''_{1,-},$$

where $D=\sum D'_1\otimes D''_1=\sum D'_2\otimes D''_2$ and $D^{-1}=\sum D'_{1,-}\otimes D''_{1,-}=\sum D'_{2,-}\otimes D''_{2,-}$.

For $a\in U_{\mathbb{Z},q}$ and $b\in \bar{U}_q$ homogeneous, we have

$$\begin{aligned}
&\sum \dot{Y}(a\otimes D')\otimes \dot{Y}(D''\otimes b)\\
&=\sum a_{(1)}D'_2D'_{2,-}S(a_{(2)})D'_{1,-}D'_1\otimes D''_{2,-}D''_1S^{-1}(b_{(2)})D''_2D''_{1,-}b_{(1)}\\
&=\sum a_{(1)}D'_2D'_{2,-}K^{-|b_{(2)}|}S(a_{(2)})K^{|b_{(2)}|}D'_{1,-}D'_1\otimes S^{-1}(b_{(2)})D''_{2,-}D''_1D''_2D''_{1,-}b_{(1)}\\
&=\sum a_{(1)}K^{-|b_{(2)}|}S(a_{(2)})K^{|b_{(2)}|}\otimes S^{-1}(b_{(2)})b_{(1)}\\
&=\sum (a\triangleright K^{-|b_{(2)}|})K^{|b_{(2)}|}\otimes S^{-1}(b_{(2)})b_{(1)},
\end{aligned}\tag{37}$$

where by (28)–(30), we can assume that $S^{-1}(b_{(2)})b_{(1)}\in\bar{U}_q^{\mathrm{ev}}$, with $b_{(1)},b_{(2)}\in\bar{U}_q$ homogeneous. By Corollary 2.3, we have $a\triangleright K^{-|b_{(2)}|}\in K^{|b_{(2)}|}\bar{U}_q^{\mathrm{ev}}$. Hence we have

$$\sum (a\triangleright K^{-|b_{(2)}|})K^{|b_{(2)}|}\otimes S^{-1}(b_{(2)})b_{(1)}\subset (K^{|b_{(2)}|}\bar{U}_q^{\mathrm{ev}})K^{|b_{(2)}|}\otimes\bar{U}_q^{\mathrm{ev}}\subset(\bar{U}_q^{\mathrm{ev}})^{\otimes 2},$$

which completes the proof of (35).

We can prove (31), (32) with D^{-1}, (33), and (34) almost in the same way by using

$$\begin{aligned}
\sum \dot{Y}(a\otimes D'_{\pm})\otimes\dot{Y}(b\otimes D''_{\pm})&=\sum (a\triangleright K^{\pm|b_{(2)}|})K^{\mp|b_{(2)}|}\otimes b_{(1)}S(b_{(2)}),\\
\sum \dot{Y}(a\otimes D'_{-})\otimes\dot{Y}(D''_{-}\otimes b)&=\sum (a\triangleright K^{|b_{(2)}|})K^{-|b_{(2)}|}\otimes S^{-1}(b_{(2)})b_{(1)},\\
\sum \dot{Y}(D'_{\pm}\otimes a)\otimes\dot{Y}(b\otimes D''_{\pm})&=\sum K^{\mp|b_{(2)}|}(K^{\pm|b_{(2)}|}\triangleleft a)\otimes b_{(1)}S(b_{(2)}),\\
\sum \dot{Y}(D'_{\pm}\otimes a)\otimes\dot{Y}(D''_{\pm}\otimes b)&=\sum K^{\pm|b_{(2)}|}(K^{\mp|b_{(2)}|}\triangleleft a)\otimes S^{-1}(b_{(2)})b_{(1)},
\end{aligned}$$

for $a\in U_{\mathbb{Z},q}$ and $b\in\bar{U}_q$ homogeneous. □

3.6 Outline of the proof of Theorem 1.2

We give an outline of the proof of Theorem 1.2. There are two steps. The first step is in Section 5. We prove the following proposition.

Proposition 3.11. *Let $T\in BT_n$ be a boundary bottom tangle and $(\tilde{T};g,g_1,\ldots,g_n)$ a boundary data for T. For each state $s\in\mathcal{S}(\tilde{T})$ we have*

$$\mu^{[g_1,\ldots,g_n]}\bar{Y}^{\otimes g}(J_{\tilde{T},s})\in(\bar{U}_q^{\mathrm{ev}})^{\otimes n}.$$

The second step is in Section 6. We define a completion $(\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}$ of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$ and prove Theorem 1.2, i.e.,

$$J_T=\sum_{s\in\mathcal{S}(\tilde{T})}\mu^{[g_1,\ldots,g_n]}\bar{Y}^{\otimes g}(J_{\tilde{T},s})\in(\bar{U}_q^{\mathrm{ev}})^{\hat{\otimes} n}.$$

In the above two steps, we use "graphical calculus" because the proof is too complicated to be written down by using expressions. In order to do so, in Section 4, we define two symmetric monoidal categories $\mathcal{A}$, $\mathcal{M}$, and a functor $\mathcal{F}\colon\ \mathcal{A}\to\mathcal{M}$.

4 The categories $\mathcal{M}, \mathcal{A}$ and the functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{M}$

In what follows, we use strict symmetric monoidal categories and strict symmetric monoidal functors. Since we use only strict ones, we omit the word "strict". For the definition of symmetric monoidal categories, see [10], [13].

4.1 The category $\mathcal{M}$

We define the symmetric monoidal category $\mathcal{M}$. The objects in $\mathcal{M}$ are non-negative integers. For $k, l \geq 0$, the morphisms from k to l in $\mathcal{M}$ are $\mathbb{Z}[q,q^{-1}]$-submodules of the $\mathbb{Q}[[h]]$-module $\mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Q}[[h]]}(U_h^{\hat{\otimes} k}, U_h^{\hat{\otimes} l})$ of continuous $\mathbb{Q}[[h]]$-linear maps from $U_h^{\hat{\otimes} k}$ to $U_h^{\hat{\otimes} l}$.

We equip $\mathcal{M}$ with a symmetric monoidal category structure as follows.

- The identity of an object k in $\mathcal{M}$ is defined by $\mathrm{id}_k = \mathbb{Z}[q,q^{-1}]\,\mathrm{id}_{U_h^{\hat{\otimes} k}}$.

  The composition of morphisms $k \xrightarrow{X} l \xrightarrow{Y} m$ in $\mathcal{M}$ is defined by
  $$Y \circ X = \mathrm{Span}_{\mathbb{Z}[q,q^{-1}]}\{y \circ x \mid x \in X,\ y \in Y\}.$$
- The unit object is 0, and the tensor product of objects k and l in $\mathcal{M}$ is define by $k+l$. The tensor product of morphisms $F\colon\ k \to l$ and $F'\colon\ k' \to l'$ in $\mathcal{M}$ is defined by
  $$Z \otimes Z' = \mathrm{Span}_{\mathbb{Z}[q,q^{-1}]}\{\varphi(z \otimes z') \mid z \in Z,\ z' \in Z'\},$$
  where φ is the natural $\mathbb{Q}[[h]]$-linear map
  $$\varphi\colon\ \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Q}[[h]]}(U_h^{\hat{\otimes} k}, U_h^{\hat{\otimes} l}) \otimes \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Q}[[h]]}(U_h^{\hat{\otimes} k'}, U_h^{\hat{\otimes} l'}) \to \mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Q}[[h]]}(U_h^{\hat{\otimes} k+k'}, U_h^{\hat{\otimes} l+l'}).$$
- The symmetry $c_{k,l}\colon\ k \otimes l \to l \otimes k$ of objects k and l in $\mathcal{M}$ is defined by
  $$c_{k,l} = \mathbb{Z}[q,q^{-1}]\tau_{U_h^{\hat{\otimes} k}, U_h^{\hat{\otimes} l}},$$
  where $\tau_{U_h^{\hat{\otimes} k}, U_h^{\hat{\otimes} l}}\colon\ U_h^{\hat{\otimes} k+l} \to U_h^{\hat{\otimes} l+k}$ is the continuous $\mathbb{Q}[[h]]$-linear map defined by
  $$\tau_{U_h^{\hat{\otimes} k}, U_h^{\hat{\otimes} l}}(x \otimes y) = y \otimes x,$$
  for $x \in U_h^{\hat{\otimes} k}$ and $y \in U_h^{\hat{\otimes} l}$.

It is straightforward to check the axioms of a symmetric monoidal category.

4.2 The category $\mathcal{A}$ and the functor $\mathcal{F}\colon\ \mathcal{A} \to \mathcal{M}$

Let $\mathcal{A}$ be the symmetric monoidal category with the unit object I, freely generated by an object A and morphisms

$$\begin{aligned}
&\langle \{i\}_q! \rangle \in \mathrm{Hom}_{\mathcal{A}}(I, I),\\
&\langle \eta \rangle, \langle \tilde{E}^{(i)} \rangle, \langle \tilde{F}^{(i)} \rangle, \langle \bar{U}_q^0 \rangle, \langle \bar{U}_q^{\mathrm{ev}\,0} \rangle \in \mathrm{Hom}_{\mathcal{A}}(I, A),\\
&\langle D^{\pm 1} \rangle \in \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes 2}),\\
&\langle \varepsilon \rangle \in \mathrm{Hom}_{\mathcal{A}}(A, I),\\
&\langle \Delta \rangle \in \mathrm{Hom}_{\mathcal{A}}(A, A^{\otimes 2}),\\
&\langle \mu \rangle, \langle \dot{Y} \rangle, \langle \mathrm{ad} \rangle, \langle \overline{\mathrm{ad}} \rangle \in \mathrm{Hom}_{\mathcal{A}}(A^{\otimes 2}, A),
\end{aligned}$$

for $i \geq 0$. (Here $\langle D^{\pm 1}\rangle$ is one morphism, not two morphisms $\langle D^{+1}\rangle$ and $\langle D^{-1}\rangle$.) We denote by $c_{X,Y}\colon X\otimes Y \to Y\otimes X$ the symmetry of objects X, Y in $\mathcal{A}$.

We define the symmetric monoidal subcategories $\mathcal{A}_{\mathfrak{S}}, \mathcal{A}_{\mu}, \mathcal{A}_{\Delta}$, and $\mathcal{A}_{\mu,\Delta}$ of $\mathcal{A}$ as follows. On objects, we define $\mathrm{Ob}(\mathcal{A}_{\mathfrak{S}}) = \mathrm{Ob}(\mathcal{A}_{\mu}) = \mathrm{Ob}(\mathcal{A}_{\Delta}) = \mathrm{Ob}(\mathcal{A}_{\mu,\Delta}) = \mathrm{Ob}(\mathcal{A})$. On morphisms, $\mathcal{A}_{\mathfrak{S}}$ is generated by no morphism as a symmetric monoidal category, i.e., for $k, l \geq 0$, $k \neq l$, we have $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{S}}}(A^{\otimes k}, A^{\otimes l}) = \emptyset$, and for $l \geq 0$, the monoid $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{S}}}(A^{\otimes l}, A^{\otimes l})$ is isomorphic to the symmetric group $\mathfrak{S}(l)$ in a natural way. On morphisms, $\mathcal{A}_{\mu}$ is generated by $\langle\mu\rangle$, $\mathcal{A}_{\Delta}$ is generated by $\langle\Delta\rangle$, and $\mathcal{A}_{\mu,\Delta}$ is generated by $\langle\mu\rangle$ and $\langle\Delta\rangle$, as symmetric monoidal categories.

Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{M}$ be the symmetric monoidal functor defined by $\mathcal{F}(A) = 1$ on the objects and

$$\begin{aligned}
\mathcal{F}(\langle \{i\}_q!\rangle) &= \mathbb{Z}[q,q^{-1}]\{i\}_q!,\\
\mathcal{F}(\langle \eta\rangle) &= \mathbb{Z}[q,q^{-1}]\eta,\\
\mathcal{F}(\langle \varepsilon\rangle) &= \mathbb{Z}[q,q^{-1}]\varepsilon,\\
\mathcal{F}(\langle \mu\rangle) &= \mathbb{Z}[q,q^{-1}]\mu,\\
\mathcal{F}(\langle \Delta\rangle) &= \mathbb{Z}[q,q^{-1}]\Delta,\\
\mathcal{F}(\langle \dot{Y}\rangle) &= \mathbb{Z}[q,q^{-1}]\dot{Y},\\
\mathcal{F}(\langle \mathrm{ad}\rangle) &= \mathbb{Z}[q,q^{-1}]\,\mathrm{ad},\\
\mathcal{F}(\langle \overline{\mathrm{ad}}\rangle) &= \mathbb{Z}[q,q^{-1}]\overline{\mathrm{ad}},\\
\mathcal{F}(\langle \tilde{E}^{(i)}\rangle) &= \bar{U}_q^0\tilde{E}^{(i)},\\
\mathcal{F}(\langle \tilde{F}^{(i)}\rangle) &= \bar{U}_q^0\tilde{F}^{(i)},\\
\mathcal{F}(\langle \bar{U}_q^0\rangle) &= \bar{U}_q^0,\\
\mathcal{F}(\langle \bar{U}_q^{\mathrm{ev}\,0}\rangle) &= \bar{U}_q^{\mathrm{ev}\,0},\\
\mathcal{F}(\langle D^{\pm 1}\rangle) &= (\bar{U}_q^0)^{\otimes 2}D + (\bar{U}_q^0)^{\otimes 2}D^{-1},
\end{aligned}$$

for $i \geq 0$, on the morphisms. Here, for a $\mathbb{Z}[q,q^{-1}]$- submodule $X \subset U_h^{\hat{\otimes} n}$, we identify X with a $\mathbb{Z}[q,q^{-1}]$-submodule of $\mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Q}[[h]]}(\mathbb{Q}[[h]], U_h^{\hat{\otimes} n})$ by identifying each $x \in X$ with the map $f_x\colon \mathbb{Q}[[h]] \to U_h^{\hat{\otimes} n}$ such that $f_x(a) = ax$ for $a \in \mathbb{Q}[[h]]$.

In what follows, we use diagrams of morphisms in $\mathcal{A}$ as follows. The generating morphisms in $\mathcal{A}$ are depicted as in Figure 13. The composition $y \circ x$ of morphisms x and y in $\mathcal{A}$ is represented as the diagram obtained by placing the diagram of x on the top of the diagram of y, see Figure 14 (a). The tensor product $z \otimes z'$ of morphisms z and z' in $\mathcal{A}$ is represented as the diagram obtained by placing the diagram of z' to the right of the diagram of z, see Figure 14 (b).

For a diagram of a morphism $b\colon A^{\otimes k} \to A^{\otimes l}$ in $\mathcal{A}$, we call the k edges at the top of the diagram the *input edges* of b, and the l edges at the bottom of the diagram the *output edges* of b.

For simplicity, a copy of a generating morphism f of $\mathcal{A}$ appearing in a diagram will be called 'an f' in the diagram.

4.3 Some morphisms in $\mathcal{A}$

In this section, we define morphisms $\langle\mu\rangle^{[g_1,\dots,g_n]}, \langle\Delta\rangle^{[m_1,\dots,m_l]}, \langle\alpha_i^{\pm}\otimes\beta_i^{\pm}\rangle$, and $\langle\bar{Y}\rangle$ in $\mathcal{A}$.

For $g_1, \dots, g_n \geq 0$, we define

$$\langle\mu\rangle^{[g_1,\dots,g_n]} \in \mathrm{Hom}_{\mathcal{A}}(A^{\otimes g_1+\dots+g_n}, A^{\otimes n})$$

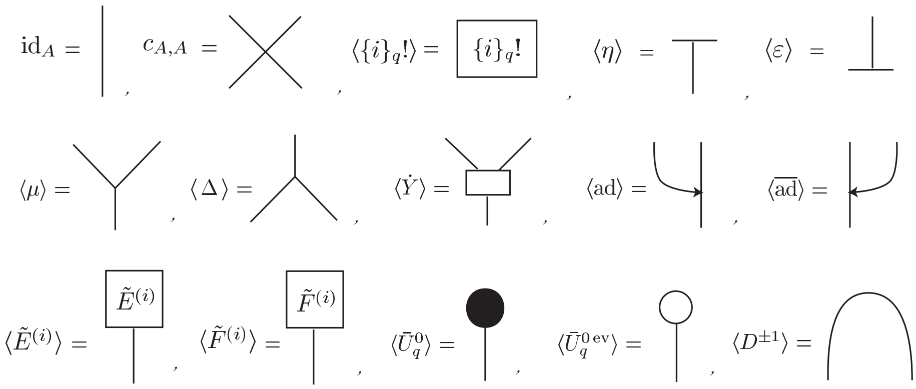
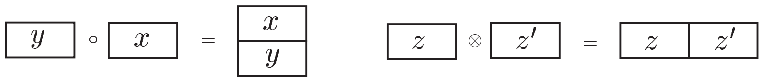


Figure 13: The diagrams of the generator morphisms in $\mathcal{A}$



(a) (b)

Figure 14: (a) Composition (b) Tensor product

.

in a similar way to (15), and for $m_1, \ldots, m_l \geq 0$, we define

$$\langle \Delta \rangle^{[m_1,\ldots,m_l]} \in \mathrm{Hom}_{\mathcal{A}}(A^{\otimes l}, A^{\otimes m_1+\cdots+m_l})$$

in a similar way to (13), see Figure 15. Clearly we have

$$\mathcal{F}\big(\langle \mu \rangle^{[g_1,\ldots,g_n]}\big) = \mathbb{Z}[q,q^{-1}]\mu^{[g_1,\ldots,g_n]}, \tag{38}$$

$$\mathcal{F}\big(\langle \Delta \rangle^{[m_1,\ldots,m_l]}\big) = \mathbb{Z}[q,q^{-1}]\Delta^{[m_1,\ldots,m_l]}. \tag{39}$$

For $i \geq 0$, set

$$\langle \Theta_i \rangle = \langle \{i\}_q! \rangle \otimes \langle \tilde{F}^{(i)} \rangle \otimes \langle \tilde{E}^{(i)} \rangle \in \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes 2}).$$

We represent $\langle \Theta_i \rangle$ as in Figure 16 (a). We define

$$\langle \alpha_i^{\pm} \otimes \beta_i^{\pm} \rangle \in \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes 2})$$

as in Figure 16 (b), i.e.,

$$\langle \alpha_i^{\pm} \otimes \beta_i^{\pm} \rangle = \big(\langle \mu \rangle \otimes \langle \mu \rangle\big) \circ \big(\mathrm{id}_A \otimes c_{A,A} \otimes \mathrm{id}_A\big) \circ \big(\langle D^{\pm 1} \rangle \otimes \langle \Theta_i \rangle\big).$$

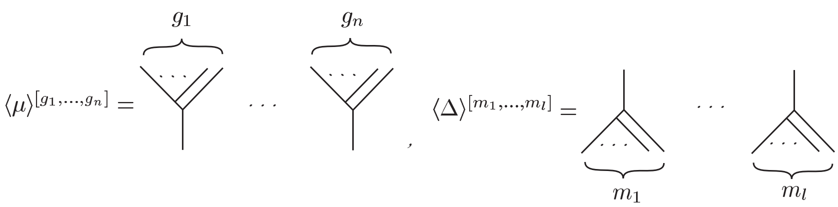


Figure 15: $\langle \mu \rangle^{[g_1,\ldots,g_n]}$ and $\langle \Delta \rangle^{[m_1,\ldots,m_l]}$

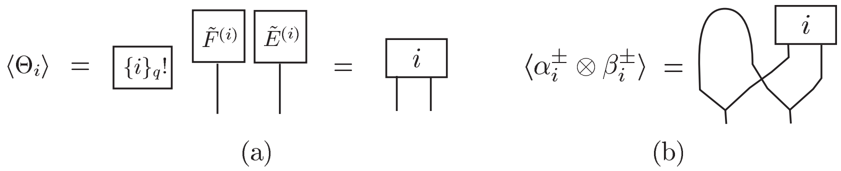


Figure 16: (a) A diagram of $\langle \Theta_i \rangle$ (b) A diagram of $\langle \alpha_i^{\pm} \otimes \beta_i^{\pm} \rangle$

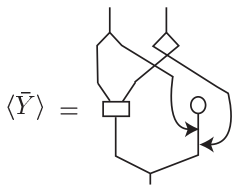


Figure 17: A diagram of $\langle \bar{Y} \rangle$

In $U_h^{\hat{\otimes} 2}$, we have

$$\mathcal{F}\big(\langle \alpha_i^{\pm} \otimes \beta_i^{\pm} \rangle\big)(1) = \big(\bar{U}_q^0 \otimes \bar{U}_q^0\big) D \big(\tilde{F}^{(i)} \otimes e^i\big) + \big(\bar{U}_q^0 \otimes \bar{U}_q^0\big) D^{-1} \big(\tilde{F}^{(i)} \otimes e^i\big),$$

which implies

$$\alpha_i \otimes \beta_i, \ \alpha_i^{-} \otimes \beta_i^{-} \in \mathcal{F}\big(\langle \alpha_i^{\pm} \otimes \beta_i^{\pm} \rangle\big)(1).$$

Since we have

$$(S^j \otimes S^k)\Big(\mathcal{F}\big(\langle \alpha_i^{\pm} \otimes \beta_i^{\pm} \rangle\big)(1)\Big) = \mathcal{F}\big(\langle \alpha_i^{\pm} \otimes \beta_i^{\pm} \rangle\big)(1),$$

for $j, k \in \mathbb{Z}$, it follows that

$$S^j(\alpha_i) \otimes S^k(\beta_i), \ S^j(\alpha_i^{-}) \otimes S^k(\beta_i^{-}) \in \mathcal{F}\big(\langle \alpha_i^{\pm} \otimes \beta_i^{\pm} \rangle\big)(1). \tag{40}$$

We define

$$\langle \bar{Y} \rangle \in \mathrm{Hom}_{\mathcal{A}}(A^{\otimes 2}, A^{\otimes 1})$$

as in Figure 17, i.e.,

$$\begin{aligned} \langle \bar{Y} \rangle = & \langle \mu \rangle \circ \big(\mathrm{id}_A \otimes \langle \overline{\mathrm{ad}} \rangle\big) \circ \big(\mathrm{id}_A \otimes \langle \mathrm{ad} \rangle \otimes \mathrm{id}_A\big) \\ & \circ \big(\langle \dot{Y} \rangle \otimes \mathrm{id}_A \otimes \langle \bar{U}_q^{\mathrm{ev}\, 0} \rangle \otimes \mathrm{id}_A\big) \circ \big(\mathrm{id} \otimes c_{A^{\otimes 2}, A}\big) \circ \big(\langle \Delta \rangle \otimes \langle \Delta \rangle\big). \end{aligned}$$

By Lemma 3.6, we have

$$\bar{Y} = \mu \circ (\mathrm{id}_{U_h} \otimes \overline{\mathrm{ad}}) \circ (\mathrm{id}_{U_h} \otimes \mathrm{ad} \otimes \mathrm{id}_{U_h}) \circ (\dot{Y} \otimes \mathrm{id}_{U_h} \otimes K^2 \otimes \mathrm{id}_{U_h}) \circ (\mathrm{id}_{U_h} \otimes \tau_{U_h^{\hat{\otimes} 2}, U_h}) \circ (\Delta \otimes \Delta).$$

Hence we have

$$\bar{Y} \in \mathcal{F}\big(\langle \bar{Y} \rangle\big). \tag{41}$$

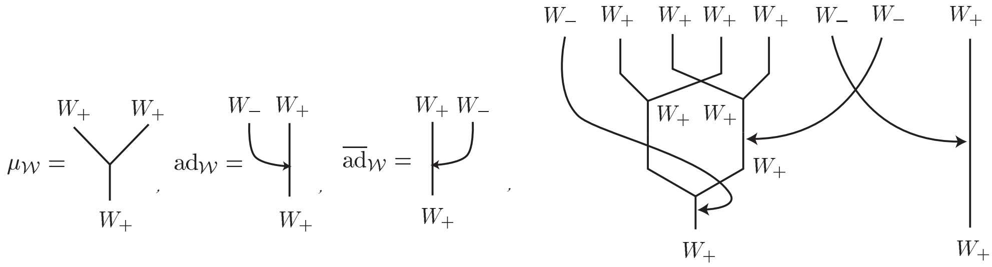


Figure 18: A morphism in $\mathcal{W}$

5 Proof of Proposition 3.11

The goal of this section is to prove Proposition 3.11. In Section 5.1, we define a subset $\Gamma_g \subset \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes g})$, and prove $\bar{Y}^{\otimes g}(J_{\tilde{T},s}) \in \mathcal{F}(\Gamma_g)(1)$. Here, for a subset $X \subset \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes g})$, we set

$$\mathcal{F}(X)(1) = \bigcup_{B \in X} \mathcal{F}(B)(1) \subset U_h^{\hat{\otimes} g}.$$

In Section 5.2–5.8, we prove $\mathcal{F}(\Gamma_g)(1) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes g}$. Thus we have

$$\bar{Y}^{\otimes g}(J_{\tilde{T},s}) \in (\bar{U}_q^{\mathrm{ev}})^{\otimes g},$$

which implies Proposition 3.11.

5.1 The set $\Gamma_g \subset \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes g})$

Let $\mathcal{W}$ be the symmetric monoidal category freely generated by two objects W_+ and W_- and three morphisms

$$\mu_{\mathcal{W}} \colon (W_+)^{\otimes 2} \to W_+, \quad \mathrm{ad}_{\mathcal{W}} \colon W_- \otimes W_+ \to W_+, \quad \overline{\mathrm{ad}}_{\mathcal{W}} \colon W_+ \otimes W_- \to W_+,$$

see Figure 18 for example. We define the symmetric monoidal functor $\mathcal{F}_{\mathcal{W}} \colon \mathcal{W} \to \mathcal{A}$ by $\mathcal{F}_{\mathcal{W}}(W_+) = \mathcal{F}_{\mathcal{W}}(W_-) = A$ on objects, and

$$\mathcal{F}_{\mathcal{W}}(\mu_{\mathcal{W}}) = \langle \mu \rangle, \quad \mathcal{F}_{\mathcal{W}}(\mathrm{ad}_{\mathcal{W}}) = \langle \mathrm{ad} \rangle, \quad \mathcal{F}_{\mathcal{W}}(\overline{\mathrm{ad}}_{\mathcal{W}}) = \langle \overline{\mathrm{ad}} \rangle,$$

on morphisms.

For $g \geq 0$, let $\tilde{\Gamma}_g$ be the set of quadruples $b = (b_1, b_2, b_3, b_4) \in \mathrm{Hom}(\mathcal{A})^{\times 4}$ of composable morphisms

$$I \xrightarrow{b_1} A^{\otimes 2l_1 + 2l_2 + l_3} \xrightarrow{b_2} A^{\otimes l_4 + 2l_5} \xrightarrow{b_3} A^{\otimes l_4 + l_5 + l_6} \xrightarrow{b_4} A^{\otimes g}$$

in $\mathcal{A}$ such that

$$\begin{aligned}
b_1 &= \langle D^{\pm 1} \rangle^{\otimes l_1} \otimes \langle \Theta_{s_1} \rangle \otimes \cdots \otimes \langle \Theta_{s_{l_2}} \rangle \otimes \langle \bar{U}_q^0 \rangle^{\otimes l_3}, \\
b_2 &\in \mathrm{Hom}_{\mathcal{A}_{\mu,\Delta}}(A^{\otimes 2l_1 + 2l_2 + l_3}, A^{\otimes l_4 + 2l_5}), \\
b_3 &= \mathrm{id}_A^{\otimes l_4} \otimes \langle \dot{Y} \rangle^{\otimes l_5} \otimes \langle \bar{U}_q^{\mathrm{ev}\, 0} \rangle^{\otimes l_6}, \\
b_4 &\in \mathcal{F}_{\mathcal{W}}\big(\mathrm{Hom}_{\mathcal{W}}(W_-^{\otimes l_4} \otimes W_+^{\otimes l_5 + l_6}, W_+^{\otimes g}) \big),
\end{aligned}$$

for $l_1, \ldots, l_6, s_1, \ldots, s_{l_2} \geq 0$, satisfying Condition A below.

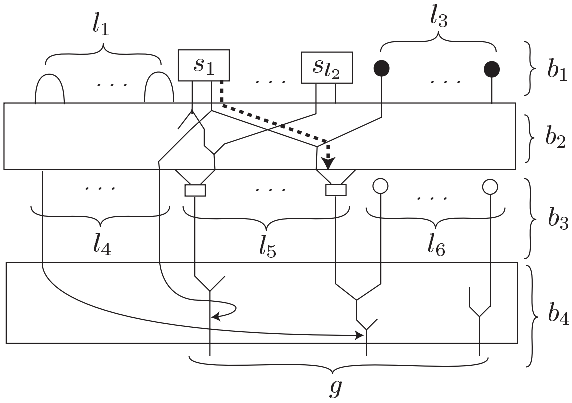


Figure 19: An example of $b_4 \circ b_3 \circ b_2 \circ b_1$ with $(b_1, b_2, b_3, b_4) \in \tilde{\Gamma}_g$

Condition A: On a diagram of $b_4 \circ b_3 \circ b_2 \circ b_1$, from each output edge of $\langle \Theta_{s_p} \rangle$ for $p = 1, \ldots, l_2$, we can find a descending path to an input edge of a $\langle \dot{Y} \rangle$.

For example, see Figure 19, where the dotted arrow denotes a path as in Condition A from the right output edge of $\langle \Theta_{s_1} \rangle$.

Let $\lambda\colon \tilde{\Gamma}_g \to \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes g})$ be the composition map defined by

$$\lambda(b_1, b_2, b_3, b_4) = b_4 \circ b_3 \circ b_2 \circ b_1.$$

Set

$$\Gamma_g = \lambda(\tilde{\Gamma}_g) \subset \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes g}).$$

We consider the following sequence of maps

$$\tilde{\Gamma}_g \xrightarrow{\lambda} \Gamma_g \subset \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes g}) \xrightarrow{\mathcal{F}} \mathrm{Hom}_{\mathcal{M}}(0, g).$$

Lemma 5.1. *Let $T \in BT_n$ be a boundary bottom tangle and $(\tilde{T}; g, g_1, \ldots, g_n)$ a boundary data for T. For each state $s \in \mathcal{S}(\tilde{T})$, we have $\bar{Y}^{\otimes g}(J_{\tilde{T},s}) \in \mathcal{F}(\Gamma_g)(1)$.*

Proof. It is enough to construct an element $B \in \Gamma_g$ such that $\bar{Y}^{\otimes g}(J_{\tilde{T},s}) \in \mathcal{F}(B)(1)$.

Recall from Section 2.5 the definition of $J_{\tilde{T},s}$ associated with a fixed diagram of $\tilde{T}$ with the crossings $c_1, \ldots, c_l$. We put the labels $S'(\alpha^{\pm}_{s(c_i)})$ and $S'(\beta^{\pm}_{s(c_i)})$ on the crossing c_i for $i = 1, \ldots, l$, and put the labels K and K^{-1} on the maximal and minimal points, respectively, on strands running from left to right. After that, we multiply the labels on each strand, and take the tensor product. Thus, there is $k \geq 0$ and a permutation $\sigma \in \mathfrak{S}(2l + k)$ such that

$$J_{\tilde{T},s} \in \left(\mu^{[N_1,\ldots,N_{2g}]} \circ \sigma\right)\Big(S'(\alpha^{\pm}_{s(c_1)}) \otimes S'(\beta^{\pm}_{s(c_1)}) \otimes \cdots \otimes S'(\alpha^{\pm}_{s(c_l)}) \otimes S'(\beta^{\pm}_{s(c_l)}) \otimes (\bar{U}^0_q)^{\otimes k}\Big),$$

where, for each $i = 1, \ldots, 2g$, $N_i \geq 0$ is the number of labels put on ith strand of $\tilde{T}$.

By (38) and (40), we have $J_{\tilde{T},s} \in \mathcal{F}(u)(1)$, where

$$u = \langle \mu \rangle^{[N_1,\ldots,N_{2g}]} \circ \sigma \circ \Big(\langle \alpha^{\pm}_{s(c_1)} \otimes \beta^{\pm}_{s(c_1)} \rangle \otimes \cdots \otimes \langle \alpha^{\pm}_{s(c_l)} \otimes \beta^{\pm}_{s(c_l)} \rangle \otimes \langle \bar{U}^0_q \rangle^{\otimes k}\Big).$$

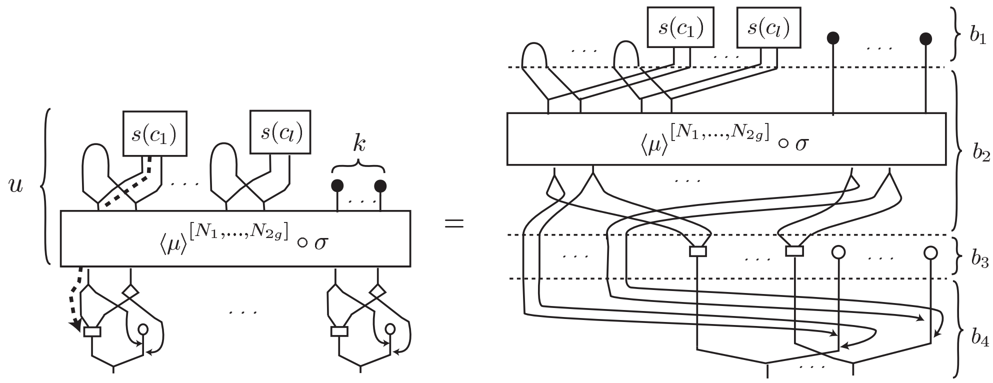


Figure 20: $\langle \bar{Y} \rangle^{\otimes g} \circ u = b_4 \circ b_3 \circ b_2 \circ b_1$ with $(b_1, b_2, b_3, b_4) \in \tilde{\Gamma}_g$

Here we identify $\sigma \in \mathfrak{S}(2l+k)$ with the corresponding morphism in $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{S}}}(A^{\otimes 2l+k}, A^{\otimes 2l+k})$.

By (41), we have

$$\bar{Y}^{\otimes g}(J_{\tilde{T},s}) \in \mathcal{F}\big(\langle \bar{Y} \rangle^{\otimes g} \circ u\big)(1).$$

Set $B = \langle \bar{Y} \rangle^{\otimes g} \circ u \in \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes g})$. It is not difficult to check that $B \in \Gamma_g$ as in Figure 20. In particular, B satisfies Condition A, since for each $i = 1, \ldots, l$, the output edges of $\langle \Theta_{s(c_l)} \rangle$ go down to the output edges of u, and there is a descending path from each output edge of u to an input edge of a $\langle \dot{Y} \rangle$, see the dotted lines in Figure 20 for example. □

Proposition 3.11 follows from Lemma 5.1 and the following lemma.

Lemma 5.2. *For $g \geq 0$, we have $\mathcal{F}(\Gamma_g)(1) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes g}$.*

The outline of the proof of Lemma 5.2 is as follows. We define two subsets $\Gamma'_g, \Gamma''_g \subset \Gamma_g$ such that $\Gamma''_g \subset \Gamma'_g \subset \Gamma_g$, and prove the following inclusions

$$\mathcal{F}(\Gamma_g)(1) \subset \mathcal{F}(\Gamma'_g)(1) \subset \mathcal{F}'\big(\Gamma'_g\big)(1) \subset \mathcal{F}'\big(\Gamma''_g\big)(1) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes g}, \tag{42}$$

where $\mathcal{F}'$ is a modification of the functor $\mathcal{F}$, which is defined in Section 5.4.

5.2 The subset $\Gamma'_g \subset \Gamma_g$

In this section, we define the subset $\Gamma'_g \subset \Gamma_g$.

For $g \geq 0$, let $\dot{\Gamma}'_g$ be the set of 7-tuples $(b_1, d, w, k, \sigma, b_3, b_4)$ of morphisms in $\mathcal{A}$, such that $(b_1, \sigma \circ (d \otimes w \otimes k), b_3, b_4) \in \tilde{\Gamma}_g$ is well-defined, $\sigma \in \mathrm{Hom}(\mathcal{A}_{\mathfrak{S}})$ and

$$b_1 = \langle D^{\pm 1} \rangle^{\otimes l_1} \otimes \langle \Theta_{s_1} \rangle \otimes \cdots \otimes \langle \Theta_{s_{l_2}} \rangle \otimes \langle \bar{U}_q^0 \rangle^{\otimes l_3},$$

$$d \in \mathrm{Hom}_{\mathcal{A}_\mu}(A^{\otimes 2l_1}, A^{\otimes l_4}), \quad w = \bigotimes_{p=1}^{l_2} \big(\langle \Delta \rangle^{[m_p, n_p]}\big), \quad k = \langle \Delta \rangle^{[r_1, \ldots, r_{l_3}]},$$

for $l_1, \ldots, l_4, s_1, \ldots, s_{l_2} \geq 0$, $m_1, \ldots, m_{l_2}, n_1, \ldots, n_{l_2}, r_1, \ldots, r_{l_3} \geq 1$. See Figure 21 for an example of $\sigma \circ (d \otimes w \otimes k) \circ b_1$.

Let $\kappa \colon \dot{\Gamma}_g \to \tilde{\Gamma}_g$ be the map defined by

$$\kappa(b_1, d, w, k, \sigma, b_3, b_4) = (b_1, \sigma \circ (d \otimes w \otimes k), b_3, b_4).$$

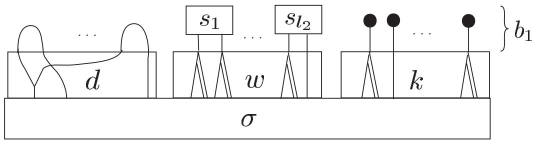


Figure 21: An example of $\sigma \circ (d \otimes w \otimes k) \circ b_1$ for $(b_1, d, w, k, \sigma, b_3, b_4) \in \dot{\Gamma}'_g$

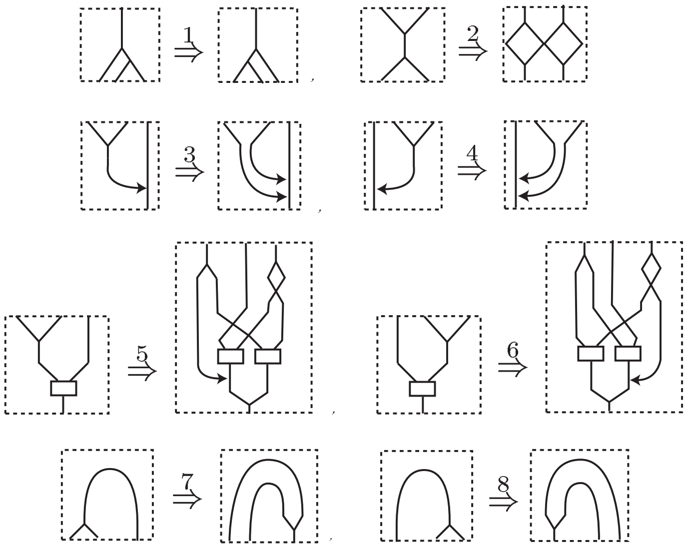


Figure 22: Local moves $\stackrel{i}{\Rightarrow}$ for $i = 1, \ldots, 8$

Set

$$\tilde{\Gamma}'_g = \kappa(\dot{\Gamma}'_g) \subset \tilde{\Gamma}_g,$$
$$\Gamma'_g = \lambda(\tilde{\Gamma}'_g) \subset \Gamma_g.$$

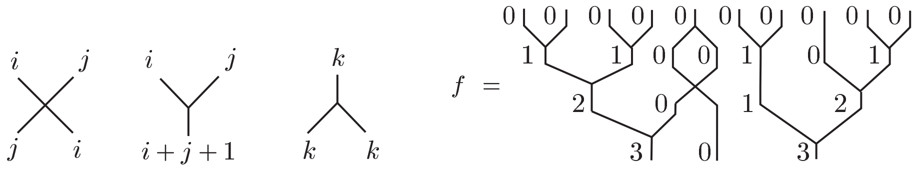
5.3 Proof of $\mathcal{F}(\Gamma_g)(1) \subset \mathcal{F}(\Gamma'_g)(1)$

In this section, we define a preorder $\preceq$ on Γ_g, and prove the following two lemmas, which imply $\mathcal{F}(\Gamma_g)(1) \subset \mathcal{F}(\Gamma'_g)(1)$.

Lemma 5.3. *For $B \preceq B'$ in Γ_g, we have $\mathcal{F}(B)(1) \subset \mathcal{F}(B')(1)$.*

Lemma 5.4. *For each $B \in \Gamma_g$, there exists $B' \in \Gamma'_g$ such that $B \preceq B'$.*

The preorder $\preceq$ on Γ_g is generated by the binary relations $\stackrel{i}{\Rightarrow}$ for $i = 1, \ldots, 8$ on $\mathrm{Hom}(\mathcal{A})$ defined by the local moves on diagrams as depicted in Figure 22, where in each relation, the outsides of the two rectangles are the same. Note that Γ_g is *closed* under $\stackrel{i}{\Rightarrow}$ for $i = 1, \ldots, 8$, i.e., for $B \stackrel{i}{\Rightarrow} B'$ in $\mathrm{Hom}(\mathcal{A})$, if $B \in \Gamma_g$, then $B' \in \Gamma_g$. In particular, we can check that each $\stackrel{i}{\Rightarrow}$ preserves Condition A.



(a) (b)

Figure 23: (a) How to color the edges (b) An example of the coloring

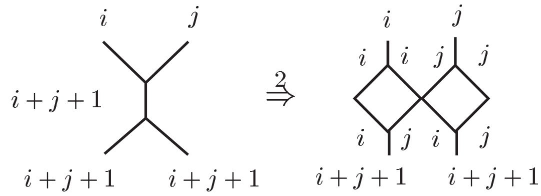


Figure 24: The coloring before and after we apply $\overset{2}{\Rightarrow}$

Proof of Lemma 5.3. It is enough to prove that, for $B \overset{i}{\Rightarrow} B'$ in Γ_g with $i \in \{1, \ldots, 8\}$, we have $\mathcal{F}(B)(1) \subset \mathcal{F}(B')(1)$.

The cases $i = 1, 2, 3, 4$ are clear. The cases $i = 5, 6$ follow from Lemma 3.7. The cases $i = 7, 8$ follow from (5), $\Delta(\bar{U}_q^0) \subset (\bar{U}_q^0)^{\otimes 2}$ and $\bar{U}_q^0 \subset \mu((\bar{U}_q^0)^{\otimes 2})$. □

The rest of this section is devoted to the proof of Lemma 5.4. We divide Lemma 5.4 to the following two claims.

Claim 1. For $b = (b_1, b_2, b_3, b_4) \in \tilde{\Gamma}_g$, there exists $b' = (b'_1, b'_2, b'_3, b'_4) \in \tilde{\Gamma}_g$ with $b'_2 \in \mathrm{Hom}(\mathcal{A}_\Delta)$ such that $\lambda(b) \preceq \lambda(b')$.

Claim 2. For $b' = (b'_1, b'_2, b'_3, b'_4) \in \tilde{\Gamma}_g$ with $b'_2 \in \mathrm{Hom}(\mathcal{A}_\Delta)$, there exists $b'' = (b''_1, b''_2, b''_3, b''_4) \in \tilde{\Gamma}'_g$ such that $\lambda(b') \preceq \lambda(b'')$.

Roughly speaking, we prove Claim 1 by reducing the number of the $\langle\mu\rangle$'s of b_2 by using $\overset{i}{\Rightarrow}$ for $i = 3, \ldots, 6$. For that purpose, we define "$\langle\mu\rangle$-complexity" functions

$$| \ |, m\colon \ \mathrm{Hom}(\mathcal{A}_{\mu,\Delta}) \to \mathbb{Z}_{\geq 0}$$

as follows. Given an element $b \in \mathrm{Hom}(\mathcal{A}_{\mu,\Delta})$, we color each edge of a diagram of b with an non-negative integer. First, we color each edge on the top with 0. Then, we color the edges below inductively as in Figure 23 (a). We define $|b|$ as the maximal integer on the edges on the bottom. We define $m(b)$ as the number of the edges on the bottom colored with $|b|$. For example, for the colored morphism $f \in \mathrm{Hom}(\mathcal{A}_{\mu,\Delta})$ in Figure 23 (b), we have $|f| = 3$, and $m(f) = 2$.

We use the following lemma.

Lemma 5.5. *For every $B \in \mathrm{Hom}(\mathcal{A}_{\mu,\Delta})$, there exists $B_\mu \in \mathrm{Hom}(\mathcal{X}_\mu)$ and $B_\Delta \in \mathrm{Hom}(\mathcal{X}_\Delta)$ such that $B \preceq B_\mu \circ B_\Delta$ and $|B| = |B_\mu \circ B_\Delta|$.*

Proof. We can realize a path from B to $B_\mu \circ B_\Delta$ with some $B_\mu \in \mathrm{Hom}(\mathcal{X}_\mu)$ and $B_\Delta \in \mathrm{Hom}(\mathcal{X}_\Delta)$ by using $\overset{2}{\Rightarrow}$, which preserves $| \ |$ as in Figure 24. □

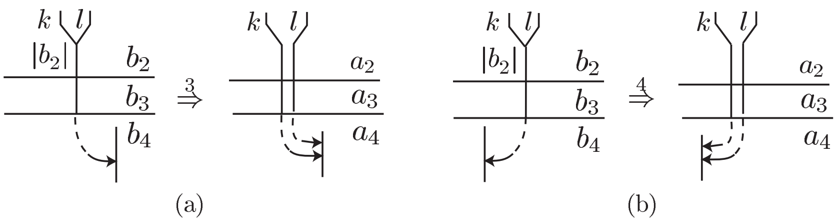


Figure 25: How to obtain $a = (a_1, a_2, a_3, a_4) \in \tilde{\Gamma}_g$ from b, where $k, l \geq 0$, $k + l + 1 = |b_2|$

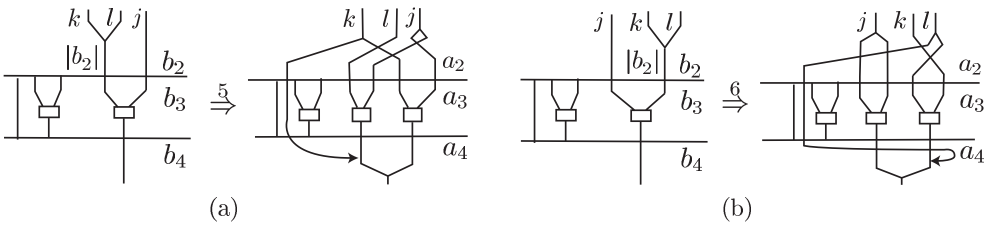


Figure 26: How to obtain $a = (a_1, a_2, a_3, a_4) \in \tilde{\Gamma}_g$ from b, where $j, k, l \geq 0$, $k + l + 1 = |b_2|$

Proof of Claim 1. We use double induction on $|b_2|$ and $m(b_2)$. If $|b_2| = 0$, then we have $b_2 \in \mathrm{Hom}(\mathcal{A}_\Delta)$. We assume $|b_2| > 0$. It is enough to prove that there exists $a = (a_1, a_2, a_3, a_4) \in \tilde{\Gamma}_g$ such that $\lambda(b) \preceq \lambda(a)$ satisfying either $|b_2| > |a_2|$, or $|b_2| = |a_2|$ and $m(b_2) > m(a_2)$.

By Lemma 5.5, we can assume $b_2 = b_{2,\mu} \circ b_{2,\Delta}$ with $b_{2,\mu} \in \mathrm{Hom}(\mathcal{A}_\mu)$ and $b_{2,\Delta} \in \mathrm{Hom}(\mathcal{A}_\Delta)$. Since $|b_{2,\mu}| = |b_2| > 0$, there is a $\langle\mu\rangle$ at the bottom of b_2 whose output edge is colored by $|b_2|$. We define $a = (a_1, a_2, a_3, a_4) \in \tilde{\Gamma}_g$ as follows.

(1) If the output edge of the $\langle\mu\rangle$ is connected to the left input edge of an $\langle \mathrm{ad} \rangle$ (resp. the right input edge of an $\langle \overline{\mathrm{ad}} \rangle$), then let a be the element obtained from b by applying $\overset{3}{\Rightarrow}$ to the $\langle \mathrm{ad} \rangle$ (resp. $\overset{4}{\Rightarrow}$ to the $\langle \overline{\mathrm{ad}} \rangle$) in $\lambda(b)$ as in Figure 25 (a) (resp. (b)).

(2) If the output edge of the $\langle\mu\rangle$ is connected to the left (resp. right) input edge of a $\langle \dot{Y} \rangle$, then let a be the element obtained from b by applying $\overset{5}{\Rightarrow}$ (resp. $\overset{6}{\Rightarrow}$) on the $\langle \dot{Y} \rangle$ in $\lambda(b)$ as in Figure 26 (a) (resp. (b)).

If $m(b_2) = 1$, then we have $|b_2| > |a_2|$. If $m(b_2) > 1$, then we have $|b_2| = |a_2|$ and $m(b_2) > m(a_2)$. This completes the proof. □

Proof of Claim 2. We transform $b'_2 \circ b'_1$ into $b''_2 \circ b''_1$ such that $b'' = (b''_1, b''_2, b'_3, b'_4) \in \tilde{\Gamma}'_g$ by the two steps as in Figure 27. That is,

(i) we can transform b'_2 into $\sigma \circ \langle\Delta\rangle^{[m_1,\ldots,m_l]}$ for some $\sigma \in \mathrm{Hom}(\mathcal{A}_{\mathfrak{S}})$, $l \geq 0$, $m_1, \ldots, m_l \geq 1$ by using $\overset{1}{\Rightarrow}$, and

(ii) we can transform $(\langle\Delta\rangle^{[m]} \otimes \langle\Delta\rangle^{[n]}) \circ \langle D^{\pm 1} \rangle$, $m, n \geq 1$, into $a \circ \langle D^{\pm 1} \rangle^{\otimes mn}$, for some $a \in \mathrm{Hom}_{\mathcal{A}_\mu}(2mn, m+n)$, by using $\overset{2}{\Rightarrow}$, $\overset{7}{\Rightarrow}$, and $\overset{8}{\Rightarrow}$ as depicted in Figure 28.

Hence we have the assertion. □

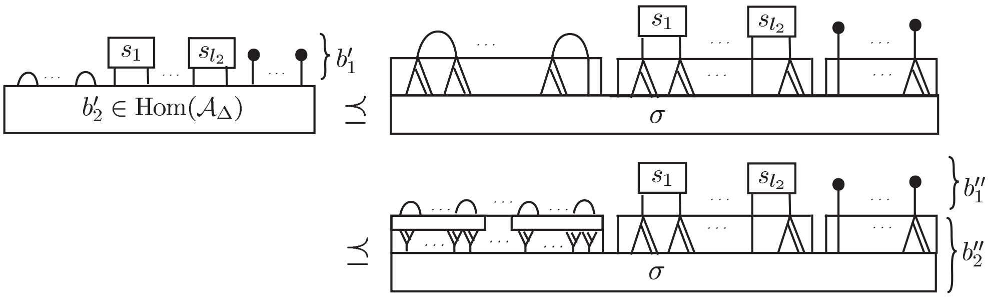


Figure 27: How to transform $b_2' \circ b_1'$ to $b_2'' \circ b_1''$

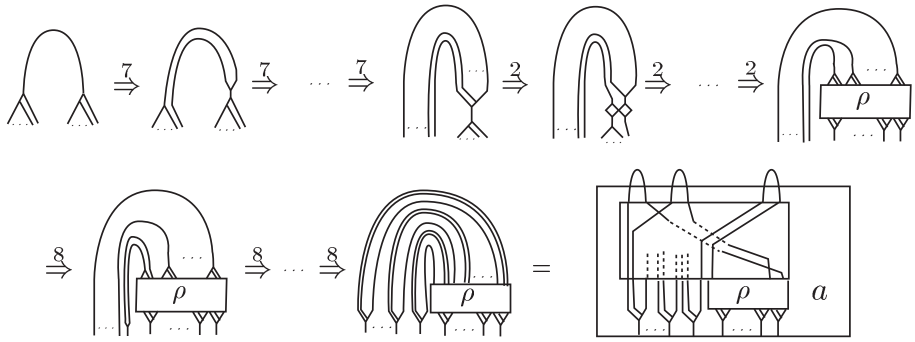


Figure 28: How to transform $(\langle\Delta\rangle^{[m]} \otimes \langle\Delta\rangle^{[n]}) \circ \langle D^{\pm 1}\rangle$ into $a \circ \langle D^{\pm 1}\rangle^{\otimes mn}$, where $\rho \in \mathrm{Hom}(\mathcal{A}_{\mathfrak{S}})$

5.4 The functor $\mathcal{F}'$ and proof of $\mathcal{F}\big(\Gamma_g'\big)(1) \subset \mathcal{F}'\big(\Gamma_g'\big)(1)$

In this section, we define the symmetric monoidal functor $\mathcal{F}' \colon \mathcal{A} \to \mathcal{M}$ and prove $\mathcal{F}\big(\Gamma_g'\big)(1) \subset \mathcal{F}'\big(\Gamma_g'\big)(1)$.

For $n \geq 0$, we equip $U_h^{\hat{\otimes} n}$ with the topological $\mathbb{Z}^n$-graded algebra structure such that

$$\deg(x_1 \otimes \cdots \otimes x_n) = \big(|x_1|, \ldots, |x_n|\big),$$

for homogeneous elements $x_1, \ldots, x_n \in U_h$ with respect to the topological $\mathbb{Z}$-grading of U_h defined in Section 2.2.

For $k, l \geq 0$, we call a map $f \colon U_h^{\hat{\otimes} k} \to U_h^{\hat{\otimes} l}$ *homogeneous* if it sends each homogeneous element to a homogeneous element. We call an object in $\mathcal{M}$ *homogeneous* if it is generated by homogeneous maps as a $\mathbb{Z}[q, q^{-1}]$-submodule of $\mathrm{Hom}^{\mathrm{cts}}_{\mathbb{Q}[[h]]}(U_h^{\hat{\otimes} k}, U_h^{\hat{\otimes} l})$. Note that the image by the functor $\mathcal{F}$ of each generator morphism in $\mathcal{A}$ in Section 4.2, except for $\langle\Delta\rangle$, is homogeneous.

We define $\mathcal{F}'$ in the same way as $\mathcal{F}$ except that we set $\mathcal{F}'(\langle\Delta\rangle) = \sum_{j \in \mathbb{Z}} \mathbb{Z}[q, q^{-1}]\Delta_j$ instead of $\mathcal{F}(\langle\Delta\rangle) = \mathbb{Z}[q, q^{-1}]\Delta$. Here, for $j \in \mathbb{Z}$, $\Delta_j \colon U_h \to U_h \hat{\otimes} U_h$ is the continuous $\mathbb{Q}[[h]]$-linear map defined by

$$\Delta_j(x) = \sum x_{(1)} \otimes p_j(x_{(2)}),$$

for $x \in U_h$, where $p_j \colon U_h \to U_h$ is the projection map defined by

$$p_j(y) = \begin{cases} y & \text{if } |y| = j, \\ 0 & \text{otherwise,} \end{cases}$$

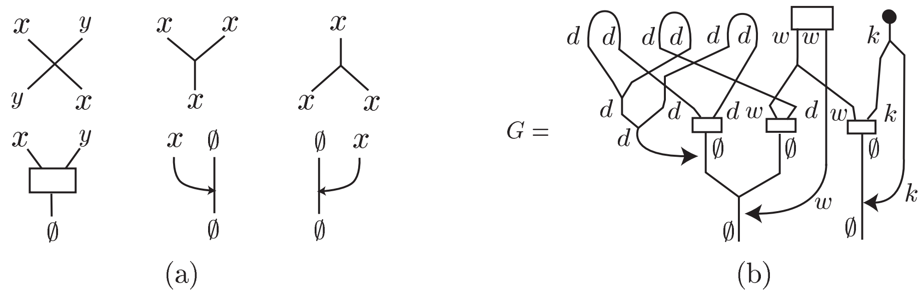


Figure 29: (a) How to color the edges (b) An example of the coloring

for $y \in U_h$ homogeneous. Since $\mathcal{F}'(\langle\Delta\rangle)$ is homogeneous, $\mathcal{F}'$ sends each generator morphism in $\mathcal{A}$ to a homogeneous module. Moreover, since the compositions and the tensor products of homogeneous objects in $\mathcal{M}$ are also homogeneous, the image by $\mathcal{F}'$ of each morphism in $\mathcal{A}$ is homogeneous.

We prove $\mathcal{F}\big(\Gamma'_g\big)(1) \subset \mathcal{F}'\big(\Gamma'_g\big)(1)$. For $x \in U_h$ a finite linear combination of homogeneous elements, it is easy to check that

$$\Delta(x) = \sum_{j\in\mathbb{Z}} \Delta_j(x), \tag{43}$$

$$\mathcal{F}(\langle\Delta\rangle)(x) = \big(\mathbb{Z}[q,q^{-1}]\Delta\big)(x) \subset \Big(\sum_{j\in\mathbb{Z}} \mathbb{Z}[q,q^{-1}]\Delta_j\Big)(x) = \mathcal{F}'(\langle\Delta\rangle)(x). \tag{44}$$

(In fact, (43) is true for all $x \in U_h$. However, (44) is not, since $\sum_{j\in\mathbb{Z}} \mathbb{Z}[q,q^{-1}]\Delta_j$ consists of finite linear combinations of Δ_j for $j \in \mathbb{Z}$.)

Note that each $\langle\Delta\rangle$ in a diagram of $B \in \Gamma'_g$ is contained in a $\langle\Delta\rangle^{[n]}\langle\tilde{E}^{(m)}\rangle$, in a $\langle\Delta\rangle^{[n]}\langle\tilde{F}^{(m)}\rangle$, or in a $\langle\Delta\rangle^{[n]}\langle\bar{U}_q^0\rangle$ for $m,n \geq 0$. By (44), we can prove that

$$\mathcal{F}(\langle\Delta\rangle^{[n]}\langle\tilde{E}^{(m)}\rangle)(1) \subset \mathcal{F}'(\langle\Delta\rangle^{[n]}\langle\tilde{E}^{(m)}\rangle)(1),$$
$$\mathcal{F}(\langle\Delta\rangle^{[n]}\langle\tilde{F}^{(m)}\rangle)(1) \subset \mathcal{F}'(\langle\Delta\rangle^{[n]}\langle\tilde{F}^{(m)}\rangle)(1),$$

for $m,n \geq 0$, by using induction on n. For $y \in U_h^0$, we have

$$\mathcal{F}(\langle\Delta\rangle)(y) = \big(\mathbb{Z}[q,q^{-1}]\Delta_0\big)(y) = \mathcal{F}'(\langle\Delta\rangle)(y).$$

Thus, we have $\mathcal{F}\big(B\big)(1) \subset \mathcal{F}'\big(B\big)(1)$, which completes the proof.

5.5 The Subset $\Gamma''_g \subset \Gamma'_g$

In this section, we define the subset $\Gamma''_g \subset \Gamma'_g$.

In what follows, we color each edge of a diagram of $B \in \Gamma'_g$ with d, w, k or $\emptyset$ as follows. First, we color the output edges of $\langle D^{\pm1}\rangle$'s, $\langle\Theta_i\rangle$'s, and $\langle\bar{U}_q^0\rangle$'s with d, w, and k, respectively. Then, we color the edges below as in Figure 29 (a). See Figure 29 (b) for an example of $G \in \Gamma'_2$ with the coloring.

For $g \geq 0$, let $\dot{\Gamma}''_g \subset \dot{\Gamma}'_g$ be the subset consisting of $b = (b_1, d, w, k, \sigma, b_3, b_4)$ such that

(C_{dk}) d and k are the identity morphisms in $\mathcal{A}$,

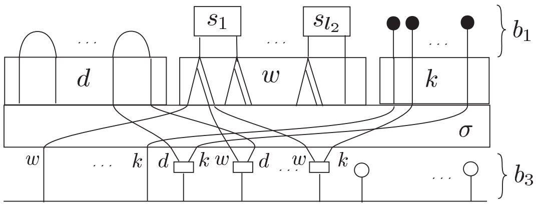


Figure 30: An example of $b_3 \circ \sigma \circ (d \otimes w \otimes k) \circ b_1$ for $(b_1, d, w, k, \sigma, b_3, b_4) \in \dot{\Gamma}''_g$

(C_{ad}) in $(\lambda \circ \kappa)(b)$, there is no $\langle \mathrm{ad} \rangle$ (resp. $\langle \overline{\mathrm{ad}} \rangle$) with the d-colored left (resp. right) input edge, i.e., the first l input edges of $b_3 = \mathrm{id}_A^{\otimes l} \otimes \langle \dot{Y} \rangle^{\otimes m} \otimes \langle \bar{U}_q^{\mathrm{ev}\, 0} \rangle^{\otimes n}$ are not colored by d,

(C_Y) there is no $\langle \dot{Y} \rangle$ with the left and right input edges both colored by d.

See Figure 30 for an example of $b_3 \circ \sigma \circ (d \otimes w \otimes k) \circ b_1$.

Set

$$\tilde{\Gamma}''_g = \kappa(\dot{\Gamma}''_g) \subset \tilde{\Gamma}'_g,$$
$$\Gamma''_g = \lambda(\tilde{\Gamma}''_g) \subset \Gamma'_g.$$

5.6 Proof of $\mathcal{F}'\big(\Gamma'_g\big)(1) \subset \mathcal{F}'\big(\Gamma''_g\big)(1)$

Similarly to Section 5.3, we define a preorder $\preceq'$ on Γ'_g, and prove the following two lemmas, which imply $\mathcal{F}'\big(\Gamma'_g\big)(1) \subset \mathcal{F}'\big(\Gamma''_g\big)(1)$.

Lemma 5.6. *For $B \preceq' B'$ in Γ'_g, we have $\mathcal{F}'(B)(1) \subset \mathcal{F}'(B')(1)$.*

Lemma 5.7. *For each element $B \in \Gamma'_g$, there exists $B' \in \Gamma''_g$, such that $B \preceq' B'$.*

The preorder $\preceq'$ on Γ'_g is generated by binary relations $\overset{i}{\Rightarrow}$ for $i = 9, \ldots, 13$ on Γ'_g. In the present case, we divide the definitions of the binary relations into three. Correspondingly, the proof of Lemma 5.6 is divided into that of Lemmas 5.9, 5.11, and 5.13.

For $B \in \Gamma'_g$, let $N_{dk}(B) \geq 0$ be the number of the $\langle \mu \rangle$'s colored by d and the $\langle \Delta \rangle$'s colored by k, $N_{\mathrm{ad}}(B) \geq 0$ the number of the $\langle \mathrm{ad} \rangle$'s with d-colored left input edges and the $\langle \overline{\mathrm{ad}} \rangle$'s with d-colored right input edges, and $N_Y(B) \geq 0$ the number of the $\langle \dot{Y} \rangle$'s with the left and right input edges both d-colored. For example, for $G \in \Gamma'_2$ as in Figure 29 (b), we have $N_{dk}(G) = 3$, $N_{\mathrm{ad}}(G) = 1$, and $N_Y(G) = 1$.

Note that for $B \in \Gamma'_g$, we have $B \in \Gamma''_g$ if and only if $N_{dk}(B) = N_{\mathrm{ad}}(B) = N_Y(B) = 0$. By using inductions on $N_{\mathrm{ad}}(B), N_Y(B)$ and $N_{dk}(B)$, Lemma 5.7 follows from Lemmas 5.8, 5.10, and 5.12.

5.6.1 Binary relation $\overset{9}{\Rightarrow}$

Let $\overset{i}{\rightsquigarrow}$ for $i = 1, \ldots, 8$ be the local moves on diagrams of morphisms in $\mathcal{A}$ as depicted in Figure 31, where in each relation, the outsides of the two rectangles are the same. For $B, B'' \in \Gamma'_g$, we write $B \overset{9}{\Rightarrow} B'$ if there exists $B'' \in \mathrm{Hom}(\mathcal{A})$ such that either $B \overset{1}{\rightsquigarrow} B''$ or $B \overset{2}{\rightsquigarrow} B''$, and there exists a sequence from B'' to B' in $\mathrm{Hom}(\mathcal{A})$ of moves $\overset{i}{\rightsquigarrow}$ for $i = 3, \ldots, 8$.

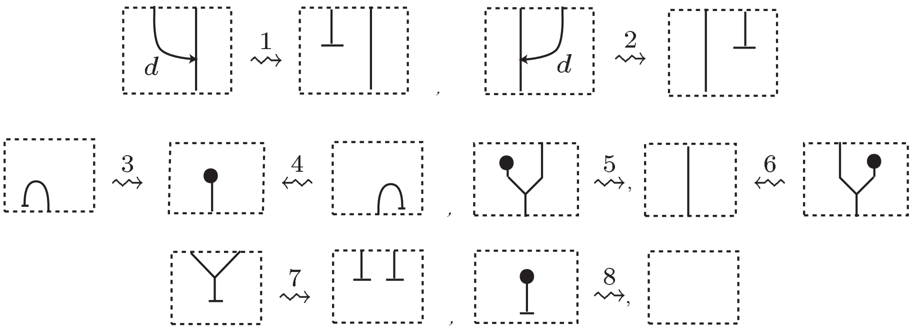


Figure 31: Local moves $\overset{i}{\rightsquigarrow}$ for $i = 1, \ldots, 8$

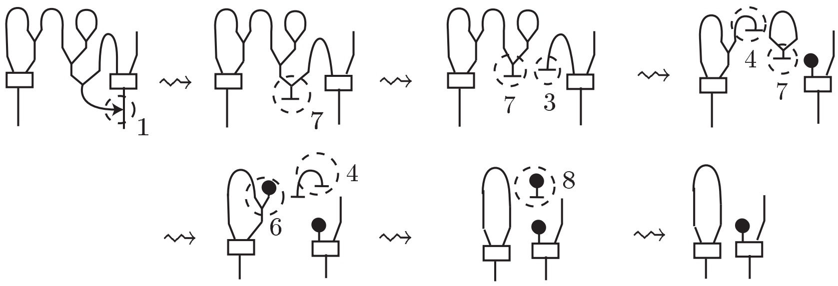


Figure 32: Binary relation $\overset{10}{\Rightarrow}$

Lemma 5.8. *For $B \in \Gamma'_g$ with $N_{\mathrm{ad}}(B) > 0$, there exists $B' \in \Gamma'_g$ such that $N_{\mathrm{ad}}(B) > N_{\mathrm{ad}}(B')$ and $B \overset{9}{\Rightarrow} B'$.*

Proof. We can transform B into $B' \in \Gamma'_g$ as in Lemma 5.8 as follows. Since $N_{\mathrm{ad}}(B) > 0$, there exists B'' obtained from B by applying $\overset{1}{\rightsquigarrow}$ or $\overset{2}{\rightsquigarrow}$. There is an $\langle\varepsilon\rangle$ in B'', and we continue the transformation as follows.

(1) If the $\langle\varepsilon\rangle$ is connected to the left (resp. right) output edge of a $\langle D^{\pm 1}\rangle$, then we apply $\overset{3}{\rightsquigarrow}$ (resp. $\overset{4}{\rightsquigarrow}$). If the new $\langle \bar{U}_q^0\rangle$ is connected to the left (resp. right) input edge of a $\langle\mu\rangle$, then we apply $\overset{5}{\rightsquigarrow}$ (resp. $\overset{6}{\rightsquigarrow}$), otherwise we put its edge into the k-part.

(2) If the $\langle\varepsilon\rangle$ is connected to an output edge of a $\langle\mu\rangle$, then we apply $\overset{7}{\rightsquigarrow}$. Then, for each new $\langle\varepsilon\rangle$, we continue the transformation starting from (1). If there appears $(\langle\varepsilon\rangle\otimes\langle\varepsilon\rangle)\circ\langle D^{\pm 1}\rangle$, then we apply $\overset{3}{\rightsquigarrow}$ or $\overset{4}{\rightsquigarrow}$, and then we apply $\overset{8}{\rightsquigarrow}$.

For example, see Figure 32, where a dotted circle with a number i attached is a place to where we apply $\overset{i}{\rightsquigarrow}$. It is easy to check that the procedure terminates, and the result B' is contained in Γ'_g. One can check that $N_{\mathrm{ad}}(B') = N_{\mathrm{ad}}(B) - 1$. □

Lemma 5.9. *For $B \overset{9}{\Rightarrow} B'$ in Γ'_g , we have $\mathcal{F}'\big(B\big)(1) \subset \mathcal{F}'\big(B'\big)(1)$.*

Proof. It is enough to prove that, for $C \overset{j}{\rightsquigarrow} C'$ with $j \in \{1, \ldots, 8\}$ in the sequence of the local moves from B to B', we have $\mathcal{F}'(C)(1) \subset \mathcal{F}'(C')(1)$.

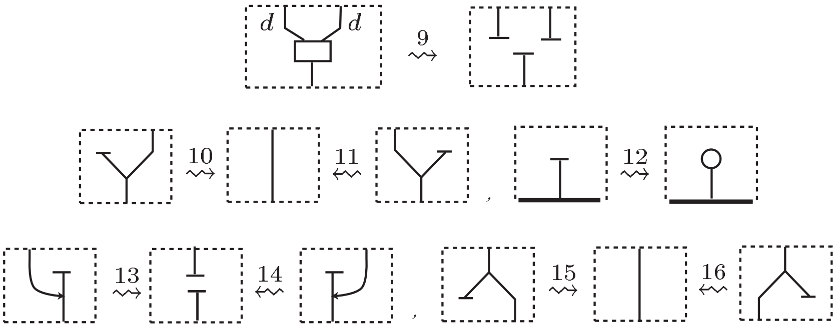


Figure 33: Local moves $\overset{i}{\rightsquigarrow}$ for $i = 9, \dots, 16$

Consider the case $j = 1$. The case $j = 2$ is similar. Recall from Section 5.4 that the image by $\mathcal{F}'$ of each morphism in $\mathcal{A}$ is homogeneous. This implies that, for each $b \in \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes l})$, $l \geq 0$, the $\mathbb{Z}[q, q^{-1}]$-submodule $\mathcal{F}'(b)(1)$ of $U_h^{\hat{\otimes} l}$ is generated by homogeneous elements of $U_h^{\hat{\otimes} l}$. Thus, the case $j = 1$ follows from

$$\begin{aligned}
&\sum \mathrm{ad}(\bar{U}_q^0 D'_{1,\pm} \cdots D'_{n,\pm}(D'_\pm D''_\pm)^m \otimes x) \otimes \bar{U}_q^0 D''_{1,\pm} \otimes \cdots \otimes \bar{U}_q^0 D''_{n,\pm} \\
\subset & x \otimes (\bar{U}_q^0)^{\otimes n} \\
\subset & \sum (\varepsilon \otimes \mathrm{id}_{U_h})(\bar{U}_q^0 D'_{1,\pm} \cdots D'_{n,\pm}(D'_\pm D''_\pm)^m \otimes x) \otimes \bar{U}_q^0 D''_{1,\pm} \otimes \cdots \otimes \bar{U}_q^0 D''_{n,\pm},
\end{aligned}$$

for $m, n \geq 0$ and $x \in U_h$ homogeneous, where we set $D^{\pm 1} = \sum D'_{i,\pm} \otimes D''_{i,\pm}$ for $1 \leq i \leq n$. Here, we use from [21, Lemma 5.2] the identities

$$\begin{aligned}
&\sum \mathrm{ad}(D'_\pm \otimes x) \otimes D''_\pm = x \otimes K^{\pm|x|}, \\
&\sum \mathrm{ad}(D'_\pm D''_\pm \otimes x) = q^{|x|^2} x,
\end{aligned}$$

for $x \in U_h$ homogeneous.

The cases $j = 3, 4$ follow from

$$(\varepsilon \otimes \mathrm{id}_{U_h}) \circ \Big((\bar{U}_q^0)^{\otimes 2} D^{\pm 1} \Big) = \bar{U}_q^0 = (\mathrm{id}_{U_h} \otimes \varepsilon) \circ \Big((\bar{U}_q^0)^{\otimes 2} D^{\pm 1} \Big).$$

The other cases $j = 5, 6, 7, 8$ are clear. Hence we have the assertion. □

5.6.2 Binary relation $\overset{10}{\Rightarrow}$

Let $\overset{i}{\rightsquigarrow}$ for $i = 9, \dots, 16$ be the local moves as depicted in Figure 33, where in each relation, the outsides of the two rectangles are the same. Here, the bottom lines in $\overset{12}{\rightsquigarrow}$ is the bottom lines of the morphisms. For $B, B' \in \Gamma'_g$, we write $B \overset{10}{\Rightarrow} B'$ if there exist $B'' \in \mathrm{Hom}(\mathcal{A})$ such that $B \overset{9}{\rightsquigarrow} B''$ and there is a sequence from B'' to B' in $\mathrm{Hom}(\mathcal{A})$ of moves $\overset{i}{\rightsquigarrow}$ for $i = 3, \dots, 8, 10, \dots, 16$.

Lemma 5.10. *For $B \in \Gamma'_g$ with $N_Y(B) > 0$ and $N_{\mathrm{ad}}(B) = 0$, there exists $B' \in \Gamma'_g$ such that $N_Y(B) > N_Y(B')$, $N_{\mathrm{ad}}(B') = 0$, and $B \overset{10}{\Rightarrow} B'$.*

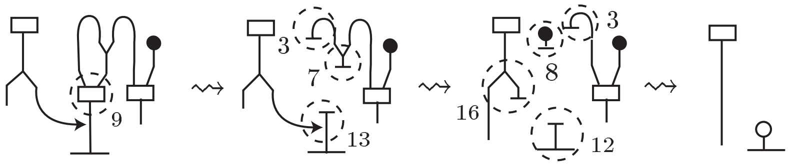


Figure 34: Binary relation $\overset{10}{\Rightarrow}$

Proof. We can transform B into $B' \in \Gamma'_g$ as in Lemma 5.10 as follows. Since $N_Y(B) > 0$, there exists B'' obtained from B by applying $\overset{9}{\rightsquigarrow}$. For the two $\langle \varepsilon \rangle$ in B'', we apply the local moves $\overset{i}{\rightsquigarrow}$ for $i = 3, \dots, 8$ as in the proof of Lemma 5.8. For the $\langle \eta \rangle$ in B'', we continue the transformation as follows.

(1) If the $\langle \eta \rangle$ is connected to the left (resp. right) input edge of a $\langle \mu \rangle$, then we apply $\overset{10}{\rightsquigarrow}$ (resp. $\overset{11}{\rightsquigarrow}$).

(2) If the $\langle \eta \rangle$ is connected to the bottom of the diagram, then we replace the $\langle \eta \rangle$ with $\langle \bar{U}_q^{\mathrm{ev}\,0} \rangle$ by using $\overset{12}{\rightsquigarrow}$.

(3) If the $\langle \eta \rangle$ is connected to the right (resp. left) input edge of an $\langle \mathrm{ad} \rangle$ (resp. $\langle \overline{\mathrm{ad}} \rangle$), then we apply $\overset{13}{\rightsquigarrow}$ (resp. $\overset{14}{\rightsquigarrow}$). Then, there appears an $\langle \eta \rangle$ and an $\langle \varepsilon \rangle$. For the $\langle \eta \rangle$, we continue the transformation starting from (1). If the $\langle \varepsilon \rangle$ is colored by d, then we apply $\overset{i}{\rightsquigarrow}$ for $i = 3, \dots, 8$ as in the proof of Lemma 5.8. Let us assume that the $\langle \varepsilon \rangle$ is colored by w or k. By Condition A in the definition of $\tilde{\Gamma}_g$, the $\langle \varepsilon \rangle$ cannot be connected directly to any output edge of the $\langle \Theta_i \rangle$'s. Hence the $\langle \varepsilon \rangle$ is connected to either an output edge of a $\langle \Delta \rangle$ or the output edge of a $\langle \bar{U}_q^0 \rangle$. If the $\langle \varepsilon \rangle$ is connected to the left (resp. right) output edge a $\langle \Delta \rangle$, then we apply $\overset{15}{\rightsquigarrow}$ (resp. $\overset{16}{\rightsquigarrow}$). If the $\langle \varepsilon \rangle$ is connected to a $\langle \bar{U}_q^0 \rangle$, we apply $\overset{8}{\rightsquigarrow}$.

For example, see Figure 34. It is easy to check that the procedure terminates, and the result B' is contained in Γ'_g. One can check that $N_Y(B') = N_Y(B) - 1$ and $N_{\mathrm{ad}}(B') = N_{\mathrm{ad}}(B) = 0$. □

Lemma 5.11. *For $B \overset{10}{\Rightarrow} B'$ in Γ'_g, we have $\mathcal{F}'\big(B\big)(1) \subset \mathcal{F}'\big(B'\big)(1)$.*

Proof. It is enough to prove that, for $C \overset{j}{\rightsquigarrow} C'$ with $j \in \{9, \dots, 16\}$ in the sequence of the local moves from B to B', we have $\mathcal{F}'(C)(1) \subset \mathcal{F}'(C')(1)$.

The case $j = 9$ follows from Lemma 3.8.

The case $j = 15, 16$ follows from

$$\Big((\varepsilon \otimes \mathrm{id}_{U_h}) \circ \Delta_k\Big)(\tilde{F}^{(l)} \bar{U}_q^0 \tilde{E}^{(m)}) = \begin{cases} \tilde{F}^{(l)} \bar{U}_q^0 \tilde{E}^{(m)} & \text{if } k = m - l, \\ 0 & \text{otherwise}, \end{cases}$$

$$\Big((\mathrm{id}_{U_h} \otimes \varepsilon) \circ \Delta_k\Big)(\tilde{F}^{(l)} \bar{U}_q^0 \tilde{E}^{(m)}) = \begin{cases} \tilde{F}^{(l)} \bar{U}_q^0 \tilde{E}^{(m)} & \text{if } k = 0, \\ 0 & \text{otherwise}, \end{cases}$$

respectively, for $k, l, m \geq 0$.

The other cases $j = 10, \dots, 14$ are clear. Hence we have the assertion. □

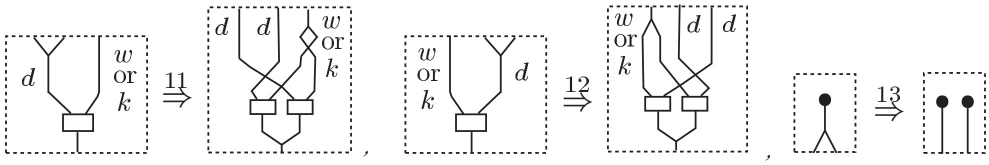


Figure 35: Local moves $\overset{i}{\Rightarrow}$ for $i = 11, 12, 13$

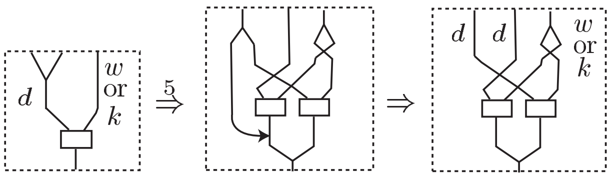


Figure 36: Graphical proof of the case $i = 11$

5.6.3 Binary relation $\overset{i}{\Rightarrow}$ for $i = 11, 12, 13$

Let $\overset{i}{\Rightarrow}$ for $i = 11, 12, 13$ be the binary relations on Γ'_g defined by the local moves on diagrams as in Figure 35, where in each relation, the outsides of the two rectangles are the same. It is easy to check that Γ'_g is closed under $\overset{i}{\Rightarrow}$ for $i = 11, 12, 13$.

Lemma 5.12. *For $B \in \Gamma'_g$ with $N_{dk}(B) > 0, N_{\mathrm{ad}}(B) = N_Y(B) = 0$, there exists $B' \in \Gamma'_g$ such that $N_{dk}(B) > N_{dk}(B')$, $N_{\mathrm{ad}}(B') = N_Y(B') = 0$, and $B \overset{i}{\Rightarrow} B'$ with $i \in \{11, 12, 13\}$.*

Proof. Since $N_{dk}(B) > 0$ and $N_Y(B) = 0$, there is a part in B as in the left hand side of $\overset{i}{\Rightarrow}$ with $i \in \{11, 12, 13\}$. We can obtain B' as in Lemma 5.12 from B as follows. If there is a $\langle\Delta\rangle \circ \langle\bar{U}_q^0\rangle$, then we obtain B' from B by applying $\overset{13}{\Rightarrow}$. If there is no $\langle\Delta\rangle \circ \langle\bar{U}_q^0\rangle$, then we obtain B' from B by applying $\overset{11}{\Rightarrow}$ or $\overset{12}{\Rightarrow}$, and then applying $\overset{13}{\Rightarrow}$ if necessary. □

Lemma 5.13. *For $B \overset{i}{\Rightarrow} B'$ in Γ'_g with $i \in \{11, 12, 13\}$, we have $\mathcal{F}'\big(B\big)(1) \subset \mathcal{F}'\big(B'\big)(1)$.*

Proof. Consider the case $i = 11$. The case $i = 12$ is similar. We can prove the assertion by two steps as in Figure 36, i.e., we have $F'(C)(1) \subset F'(C')(1)$ for $C \overset{5}{\Rightarrow} C'$ in Γ'_g by (23) and (44), and we have

$$\begin{aligned}
&\sum \mathrm{ad}\Big((\bar{U}_q^0 D'_{1,\pm} \cdots D'_{n,\pm}(D'_\pm D''_\pm)^m)_{(1)} \otimes x\Big) \\
&\qquad\qquad \otimes (\bar{U}_q^0 D'_{1,\pm} \cdots D'_{n,\pm}(D'_\pm D''_\pm)^m)_{(2)} \otimes \bar{U}_q^0 D''_{1,\pm} \otimes \cdots \otimes \bar{U}_q^0 D''_{n,\pm} \\
&\subset x \otimes \bar{U}_q^0 D'_{1,\pm} \cdots D'_{n,\pm}(D'_\pm D''_\pm)^m \otimes \bar{U}_q^0 D''_{1,\pm} \otimes \cdots \otimes \bar{U}_q^0 D''_{n,\pm},
\end{aligned}$$

for $m, n \geq 0$.

The case $\overset{13}{\Rightarrow}$ follows from $\Delta(\bar{U}_q^0) \subset (\bar{U}_q^0)^{\otimes 2}$. Hence we have the assertion. □

5.7 Modification of the elements in Γ''_g

In the next section, we prove $\mathcal{F}'\big(\Gamma''_g\big)(1) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes g}$, which completes the proof of the sequence (42). Before that, we modify the elements in Γ''_g.

First of all, we define notations. For $m \geq 0, n \geq 1$, set

$$\mathcal{I}(m,n) = \{(i_1, \dots, i_n) \mid i_1, \dots, i_n \geq 0, i_1 + \dots + i_n = m\}.$$

For $\mathbf{i} = (i_1, \dots, i_n) \in \mathcal{I}(m,n)$, set

$$\begin{aligned}
\tilde{E}^{\mathbf{i}} &= (\bar{U}_q^0)^{\otimes n}(\tilde{E}^{(i_1)} \otimes \dots \otimes \tilde{E}^{(i_n)}) \subset (U_{\mathbb{Z},q})^{\otimes n},\\
\tilde{F}^{\mathbf{i}} &= (\bar{U}_q^0)^{\otimes n}(\tilde{F}^{(i_1)} \otimes \dots \otimes \tilde{F}^{(i_n)}) \subset (U_{\mathbb{Z},q})^{\otimes n},\\
\langle \tilde{E}^{\mathbf{i}} \rangle &= \langle \tilde{E}^{(i_1)} \rangle \otimes \dots \otimes \langle \tilde{E}^{(i_n)} \rangle \in \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes n}),\\
\langle \tilde{F}^{\mathbf{i}} \rangle &= \langle \tilde{F}^{(i_1)} \rangle \otimes \dots \otimes \langle \tilde{F}^{(i_n)} \rangle \in \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes n}).
\end{aligned}$$

Clearly, we have

$$\tilde{E}^{\mathbf{i}} = \mathcal{F}'(\langle \tilde{E}^{\mathbf{i}} \rangle)(1), \quad \tilde{F}^{\mathbf{i}} = \mathcal{F}'(\langle \tilde{F}^{\mathbf{i}} \rangle)(1).$$

We use the following lemma.

Lemma 5.14. *For $m \geq 0$, $n \geq 1$, we have*

$$\begin{aligned}
\mathcal{F}'\big(\langle \Delta \rangle^{[n]} \circ \langle \tilde{E}^{(m)} \rangle\big)(1) &\subset \sum_{\mathbf{i} \in \mathcal{I}(m,n)} \mathcal{F}'\big(\langle \tilde{E}^{\mathbf{i}} \rangle\big)(1),\\
\mathcal{F}'\big(\langle \Delta \rangle^{[n]} \circ \langle \tilde{F}^{(m)} \rangle\big)(1) &\subset \sum_{\mathbf{i} \in \mathcal{I}(m,n)} \mathcal{F}'\big(\langle \tilde{F}^{\mathbf{i}} \rangle\big)(1).
\end{aligned}$$

Proof. The assertion follows from (9) and (10), by using induction on $n \geq 1$. □

Let $B = b_4 \circ b_3 \circ b_2 \circ b_1$ with $b = (b_1, b_2, b_3, b_4) \in \tilde{\Gamma}''_g$. By the condition (C_{dk}) in the definition of Γ''_g, we can write $b_2 \circ b_1 = \sigma \circ \tilde{b}_1$ with $\sigma \in \mathrm{Hom}(\mathcal{A}_{\mathfrak{S}})$ and

$$\tilde{b}_1 = \langle D^{\pm 1} \rangle^{\otimes l_1} \otimes \Big(\bigotimes_{p=1}^{l_2} \big(\langle \Delta \rangle^{[m_p, n_p]}\big) \circ \langle \Theta_{s_p} \rangle \Big) \otimes \langle \bar{U}_q^0 \rangle^{\otimes l_3}, \tag{45}$$

for $l_1, l_2, l_3 \geq 0$, $s_1, \dots, s_{l_2} \geq 0$, $m_1, \dots, m_{l_2}, n_1, \dots, n_{l_2} \geq 1$. Note that

$$\Big(\bigotimes_{p=1}^{l_2} \big(\langle \Delta \rangle^{[m_p, n_p]}\big) \circ \langle \Theta_{s_p} \rangle \Big) = \bigotimes_{p=1}^{l_2} \Big(\langle \{s_p\}_q! \rangle \otimes \big(\langle \Delta \rangle^{[m_p]} \circ \tilde{F}^{(s_p)}\big) \otimes \big(\langle \Delta \rangle^{[n_p]} \circ \tilde{E}^{(s_p)}\big) \Big).$$

Let us start the modification of B. For $\mathbf{i}_p \in \mathcal{I}(s_p, m_p)$ and $\bar{\mathbf{i}}_p \in \mathcal{I}(s_p, n_p)$, $p = 1, \dots, l_2$, set

$$\tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \dots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2}) = \langle D^{\pm 1} \rangle^{\otimes l_1} \otimes \left(\bigotimes_{p=1}^{l_2} \Big(\langle \{s_p\}_q! \rangle \otimes \langle \tilde{F}^{\mathbf{i}_p} \rangle \otimes \langle \tilde{E}^{\bar{\mathbf{i}}_p} \rangle \Big) \right) \otimes \langle \bar{U}_q^0 \rangle^{\otimes l_3}. \tag{46}$$

In other words, $\tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \dots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})$ is obtained from $\tilde{b}_1$ by replacing the $\langle \Delta \rangle^{[m_p]} \circ \tilde{F}^{(s_p)}$ with a $\langle \tilde{F}^{\mathbf{i}_p} \rangle$, and the $\langle \Delta \rangle^{[n_p]} \circ \tilde{E}^{(s_p)}$ with a $\langle \tilde{E}^{\bar{\mathbf{i}}_p} \rangle$, for $p = 1, \dots, l_2$, see Figure 37.

Thus, we obtain the modification $b_4 \circ b_3 \circ b_2 \circ \tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \dots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})$ of B with respect to $b \in \tilde{\Gamma}''_g$ and $\mathbf{i}_p \in \mathcal{I}(s_p, m_p)$, $\bar{\mathbf{i}}_p \in \mathcal{I}(s_p, n_p)$, for $p = 1, \dots, l_2$.

By Lemma 5.14, we have

$$\mathcal{F}'(B)(1) \subset \sum_{\substack{\mathbf{i}_p \in \mathcal{I}(s_p, m_p) \\ \bar{\mathbf{i}}_p \in \mathcal{I}(s_p, n_p) \\ p = 1, \dots, l_2}} \mathcal{F}'\big(b_4 \circ b_3 \circ \sigma \circ \tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \dots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})\big)(1). \tag{47}$$

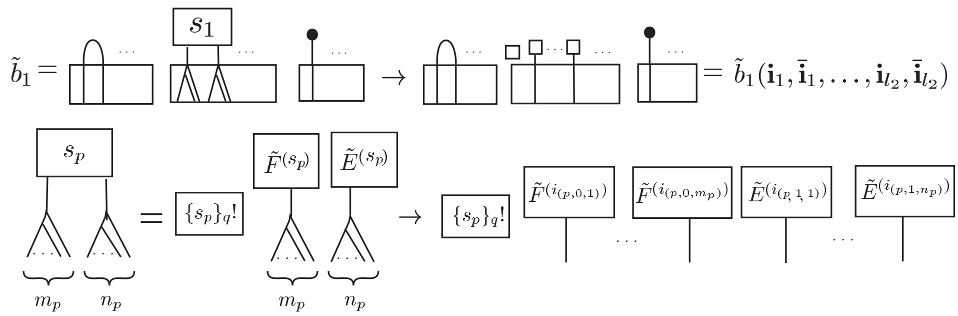
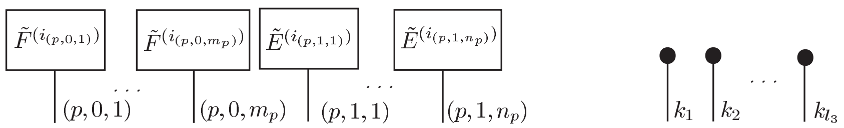


Figure 37: How to modify $\tilde{b}_1$ with respect to $\mathbf{i}_p = \big(i_{(p,0,1)}, \ldots, i_{(p,0,m_p)}\big) \in \mathcal{I}(s_p, m_p), \bar{\mathbf{i}}_p = \big(i_{(p,1,1)}, \ldots, i_{(p,1,n_p)}\big) \in \mathcal{I}(s_p, n_p)$ for $p = 1, \ldots, l_2$



(a) (b)

Figure 38: How to color the output edges of $\tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})$

5.8 Proof of $\mathcal{F}'\big(\Gamma''_g\big)(1) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes g}$

We prove the inclusion $\mathcal{F}'\big(\Gamma''_g\big)(1) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes g}$. Let $B = b_4 \circ b_3 \circ b_2 \circ b_1$ with $(b_1, b_2, b_3, b_4) \in \Gamma''_g$ such that $b_2 \circ b_1 = \sigma \circ \tilde{b}_1$ with $\sigma \in \mathrm{Hom}(\mathcal{A}_{\mathfrak{S}})$ and $\tilde{b}_1$ as in (45). By (47), it is enough to prove

$$\mathcal{F}'\big(b_4 \circ b_3 \circ \sigma \circ \tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})\big)(1) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes g}, \tag{48}$$

for $\mathbf{i}_p \in \mathcal{I}(s_p, m_p)$, $\bar{\mathbf{i}}_p \in \mathcal{I}(s_p, n_p)$, $p = 1, \ldots, l_2$.

We prove (48). First, we study $\mathcal{F}'\big(b_3 \circ \sigma \circ \tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})\big)(1)$. Fix

$$\mathbf{i}_p = \big(i_{(p,0,1)}, \ldots, i_{(p,0,m_p)}\big) \in \mathcal{I}(s_p, m_p), \quad \bar{\mathbf{i}}_p = \big(i_{(p,1,1)}, \ldots, i_{(p,1,n_p)}\big) \in \mathcal{I}(s_p, n_p),$$

for $p = 1, \ldots, l_2$. On a diagram of $\tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})$, we color the output edge of the $\langle \tilde{F}^{(i_{(p,0,t)})} \rangle$ (resp. $\langle \tilde{E}^{(i_{(p,1,u)})} \rangle$) with a label $(p, 0, t)$ (resp. $(p, 1, u)$) for $t = 1, \ldots, m_p$ (resp. $u = 1, \ldots, n_p$), $p = 1, \ldots, l_2$, see Figure 38 (a). We also color the output edges of the $\langle \bar{U}_q^0 \rangle$'s in $\tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_l, \bar{\mathbf{i}}_l)$ with symbols $k_1, \ldots, k_{l_3}$ from the left to right, see Figure 38 (b). Let $\mathcal{P}$ be the set of the all labels, i.e., set

$$\begin{aligned}\mathcal{P} =& \{(p, 0, t) \mid 1 \le t \le m_p, 1 \le p \le l_2\} \sqcup \{(p, 1, u) \mid 1 \le u \le n_p, 1 \le p \le l_2\} \\ & \sqcup \{k_1, \ldots, k_{l_3}\}.\end{aligned}$$

In what follows, since $\mathcal{F}'(\langle \bar{U}_q^0 \rangle)(1) = \mathcal{F}'(\langle \tilde{E}^{(0)} \rangle)(1) = \bar{U}_q^0$, we identify $\langle \bar{U}_q^0 \rangle$ with $\langle \tilde{E}^{(0)} \rangle$, and set $i_{k_j} = 0$ for $j = 1, \ldots, l_3$, see figure 39. We call the diagram of $\langle \tilde{X}^{(i)} \rangle$ for $i \ge 0$ with $X \in \{E, F\}$ an *X-box*.

After we color the edges, we arrange the diagram of $b_3 \circ \sigma \circ \tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})$ keeping b_3 so that each X-box is connected to b_3 directly without any crossings as in Figure 40, where we set $b_3 = \mathrm{id}_A^{\otimes l_4} \otimes \langle \dot{Y} \rangle^{\otimes l_5} \otimes \langle \bar{U}_q^{\mathrm{ev}\,0} \rangle^{\otimes l_6}$, and the floating boxes is the diagrams of $\langle \{s_p\}_q! \rangle$ for

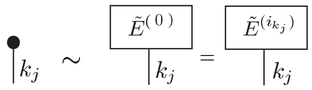


Figure 39: How to treat the jth $\langle \bar{U}_q^0 \rangle$ for $j = 1, \dots, l_3$

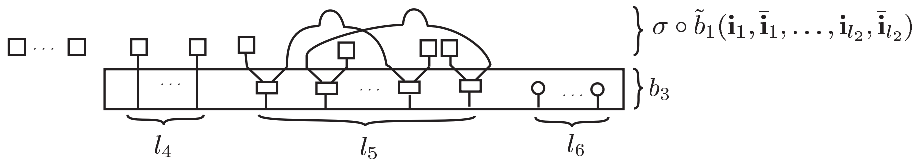


Figure 40: How to arrange the diagram of $b_3 \circ \sigma \circ \tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \dots, \mathbf{i}_l, \bar{\mathbf{i}}_l)$

$p = 1, \dots, l_2$. Here, by the condition (C_{ad}) in the definition of Γ''_g, the first l_4 input edges of b_3 are connected to X-boxes, and by the condition (C_Y), at least one of the input edges of each $\langle \dot{Y} \rangle$ in b_3 is connected to an X-box. Note that there are five cases as depicted in Figure 41 (c1)–(c5), how a $\langle \dot{Y} \rangle$ is connected to the X-boxes and the $\langle D^{\pm 1} \rangle$'s.

Thus, we have

$$b_3 \circ \sigma \circ \tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \dots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2}) = \bigotimes_{p=1}^{l_2} \langle \{s_p\}_q! \rangle \otimes \langle \tilde{X}_1^{(i_{a(1)})} \rangle \otimes \cdots \otimes \langle \tilde{X}_{l_4}^{(i_{a(l_4)})} \rangle \otimes Z \otimes \langle \bar{U}_q^{\mathrm{ev}\,0} \rangle^{\otimes l_6}, \tag{49}$$

for $a(1), \dots, a(l_4) \in \mathcal{P}$, $X_1, \dots, X_{l_4} \in \{E, F\}$, and $Z \in \langle \dot{Y} \rangle^{\otimes l_5} \circ \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes 2 l_5})$.

For $j = 1, \dots, l_4$, we call the label $a(j)$ *isolated*. We say the labels a and b as in Figure 41 (c1)–(c5) are *adjacent* to each other.

Note that the identity (49) implies

$$\mathcal{F}'\big(b_3 \circ \sigma \circ \tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \dots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})\big)(1) \subset \Big(\prod_{p=1}^{l_2} \{s_p\}_q! \Big) \cdot \Big(U_{\mathbb{Z},q}^{\otimes l_4} \otimes \mathcal{F}'(Z)(1) \otimes (\bar{U}_q^{\mathrm{ev}\,0})^{\otimes l_6} \Big). \tag{50}$$

Let us consider $\sum z_1 \otimes \cdots \otimes z_{l_5} \in \mathcal{F}'(Z)(1)$. If the mth $\langle \dot{Y} \rangle$ (from the left in b_3) is as in

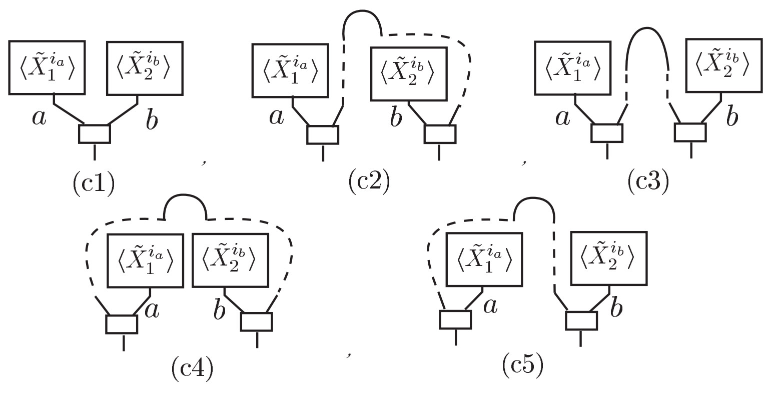


Figure 41: The $\langle \dot{Y} \rangle$'s in b_3, where $X_1, X_2 \in \{E, F\}$

(c1), then we can assume $z_m \in \dot{Y}(\bar{U}_q^0 \tilde{X}_1^{(i_a)} \otimes \bar{U}_q^0 \tilde{X}_2^{(i_b)})$. By Proposition 3.9, we have

$$\dot{Y}(\bar{U}_q^0 \tilde{X}_1^{(i_a)} \otimes \bar{U}_q^0 \tilde{X}_2^{(i_b)}) \subset (\{\min(i_a, i_b)\}_q!)^{-1} \cdot \bar{U}_q^{\mathrm{ev}}, \tag{51}$$

where $(\{\min(i,j)\}_q!)^{-1} \cdot \bar{U}_q^{\mathrm{ev}} \subset \bar{U}_q^{\mathrm{ev}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$. For example, we have

$$\begin{aligned}\dot{Y}(\bar{U}_q^0 \tilde{E}^{(2)} \otimes \bar{U}_q^0 \tilde{F}^{(3)}) &= (\{2\}_q!)^{-1} \dot{Y}(\bar{U}_q^0 e^2 \otimes \bar{U}_q^0 \tilde{F}^{(3)}) \\ &\subset (\{2\}_q!)^{-1} \bar{U}_q^{\mathrm{ev}}.\end{aligned}$$

If the mth and nth $\langle \dot{Y} \rangle$'s are as in (c2), then we can assume

$$\sum z_m \otimes z_n \in \sum \dot{Y}(\bar{U}_q^0 \tilde{X}_1^{(i_a)} \otimes \bar{U}_q^0 D'_\pm) \otimes \dot{Y}(\bar{U}_q^0 \tilde{X}_2^{(i_b)} \otimes \bar{U}_q^0 D''_\pm).$$

By Lemma 3.10, we have

$$\sum \dot{Y}(\bar{U}_q^0 \tilde{X}_1^{(i_a)} \otimes \bar{U}_q^0 D'_\pm) \otimes \dot{Y}(\bar{U}_q^0 \tilde{X}_2^{(i_b)} \otimes \bar{U}_q^0 D''_\pm) \subset (\{\min(i_a, i_b)\}_q!)^{-1} \cdot (\bar{U}_q^{\mathrm{ev}})^{\otimes 2}. \tag{52}$$

Similarly, for (c3), (c4), (c5), we have

$$\sum \dot{Y}(\bar{U}_q^0 \tilde{X}_1^{(i_a)} \otimes \bar{U}_q^0 D'_\pm) \otimes \dot{Y}(\bar{U}_q^0 D''_\pm \otimes \bar{U}_q^0 \tilde{X}_2^{(i_b)}) \subset (\{\min(i_a, i_b)\}_q!)^{-1} \cdot (\bar{U}_q^{\mathrm{ev}})^{\otimes 2}, \tag{53}$$

$$\sum \dot{Y}(\bar{U}_q^0 D'_\pm \otimes \bar{U}_q^0 \tilde{X}_1^{(i_a)}) \otimes \dot{Y}(\bar{U}_q^0 \tilde{X}_2^{(i_b)} \otimes \bar{U}_q^0 D''_\pm) \subset (\{\min(i_a, i_b)\}_q!)^{-1} \cdot (\bar{U}_q^{\mathrm{ev}})^{\otimes 2}, \tag{54}$$

$$\sum \dot{Y}(\bar{U}_q^0 D'_\pm \otimes \bar{U}_q^0 \tilde{X}_1^{(i_a)}) \otimes \dot{Y}(\bar{U}_q^0 D''_\pm \otimes \bar{U}_q^0 \tilde{X}_2^{(i_b)}) \subset (\{\min(i_a, i_b)\}_q!)^{-1} \cdot (\bar{U}_q^{\mathrm{ev}})^{\otimes 2}, \tag{55}$$

respectively.

Let $\mathcal{P}_{\mathrm{A}}^2$ denote the set of unordered pairs $\{a, b\}$ of mutually adjacent elements $a, b \in \mathcal{P}$. By the above inclusions (51)–(54), we have

$$\sum z_1 \otimes \cdots \otimes z_{l_5} \in \Big(\prod_{\{a,b\} \in \mathcal{P}_{\mathrm{A}}^2} \big(\{ \min(i_a, i_b) \}_q ! \big)^{-1} \Big) \cdot (\bar{U}_q^{\mathrm{ev}})^{\otimes l_5}.$$

Thus, by (50), we have

$$\begin{aligned}\mathcal{F}'\big(b_3 \circ \sigma \circ \tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \dots, \mathbf{i}_l, \bar{\mathbf{i}}_l)\big)(1) &\subset I \cdot \Big(U_{\mathbb{Z},q}^{\otimes l_4} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes l_5} \otimes (\bar{U}_q^{\mathrm{ev}\,0})^{\otimes l_6} \Big) \\ &\subset I \cdot \Big(U_{\mathbb{Z},q}^{\otimes l_4} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes l_5 + l_6} \Big),\end{aligned} \tag{56}$$

where we set

$$I = \Big(\prod_{p=1}^{l_2} \{s_p\}_q! \Big) \cdot \Big(\prod_{\{a,b\} \in \mathcal{P}_{\mathrm{A}}^2} \big(\{ \min(i_a, i_b) \}_q ! \big)^{-1} \Big) \in \mathbb{Q}(q).$$

Let us study $\mathcal{F}'(b_4)\Big(U_{\mathbb{Z},q}^{\otimes l_4} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes l_5+l_6}\Big)$. Since the first l_4 input edges of b_4 are connected to the left (resp. right) input edges of the $\langle \mathrm{ad} \rangle$'s (resp. $\langle \overline{\mathrm{ad}} \rangle$'s), and the next $l_5 + l_6$ input edges of b_4 go down to the edges of the $\langle \mu \rangle$'s and to the right (resp. left) input edges of the $\langle \mathrm{ad} \rangle$'s (resp. $\langle \overline{\mathrm{ad}} \rangle$'s), by Proposition 2.2, (resp. Corollary 2.3) we have

$$\mathcal{F}'(b_4)\Big(U_{\mathbb{Z},q}^{\otimes l_4} \otimes (\bar{U}_q^{\mathrm{ev}})^{\otimes l_5+l_6}\Big) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes g}. \tag{57}$$

By (56) and (57), we have

$$\mathcal{F}'\big(b_4 \circ b_3 \circ \sigma \circ \tilde{b}_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})\big)(1) \subset I \cdot (\bar{U}_q^{\mathrm{ev}})^{\otimes g}.$$

Thus, for the proof of (48), it is enough to prove

$$I \in \mathbb{Z}[q, q^{-1}]. \tag{58}$$

For $k \geq 1$, let $\Phi_k(q)$ be the kth cyclotomic polynomial in q. For $f(q) \in \mathbb{Z}[q, q^{-1}]$, $f(q) \neq 0$, let $d_k\big(f(q)\big)$ be the largest integer i such that $f(q) \in \Phi_k^i(q)\mathbb{Z}[q, q^{-1}]$. Since both $\prod_{p=1}^{l_2}\{s_p\}_q!$ and $\prod_{\{a,b\}\in\mathcal{P}_{\mathrm{A}}^2}\big\{\min(i_a, i_b)\big\}_q!$ are products of the cyclotomic polynomials, in order to prove (58), it is enough to prove

$$d_k\Big(\prod_{p=1}^{l_2}\{s_p\}_q!\Big) \geq d_k\Big(\prod_{\{a,b\}\in\mathcal{P}_{\mathrm{A}}^2}\big\{\min(i_a, i_b)\big\}_q!\Big), \tag{59}$$

for $k \geq 1$.

We prove (59). Fix $k \geq 1$. Note that for $i \in \mathbb{Z}$, we have

$$d_k\big(\{i\}_q\big) = d_k(q^i - 1) = \begin{cases} 1 & \text{if } k|i, \\ 0 & \text{otherwise.} \end{cases}$$

This identity and $s_p = \sum_{t=1}^{m_p} i_{(p,0,t)} = \sum_{u=1}^{n_p} i_{(p,1,u)}$ imply

$$\begin{aligned} d_k\big(\{s_p\}_q!\big) &\geq \sum_{t=1}^{m_p} d_k\big(\{i_{(p,0,t)}\}_q!\big), \\ d_k\big(\{s_p\}_q!\big) &\geq \sum_{u=1}^{n_p} d_k\big(\{i_{(p,1,u)}\}_q!\big). \end{aligned}$$

Thus, we have

$$\begin{aligned} d_k\Big(\prod_{p=1}^{l_2}\{s_p\}_q!\Big) &\geq \sum_{p=1}^{l_2}\Big(\sum_{t=1}^{m_p} d_k\big(\{i_{(p,0,t)}\}_q!\big) + \sum_{u=1}^{n_p} d_k\big(\{i_{(p,1,u)}\}_q!\big)\Big)/2 \\ &= \sum_{a\in\mathcal{P}} d_k\big(\{i_a\}_q!\big)/2 \\ &= \Big(\sum_{\{a,b\}\in\mathcal{P}_{\mathrm{A}}^2} d_k\big(\{i_a\}_q!\{i_b\}_q!\big) + \sum_{c\in\mathcal{P}_{\mathrm{iso}}} d_k\big(\{i_c\}_q!\big)\Big)/2 \\ &\geq \sum_{\{a,b\}\in\mathcal{P}_{\mathrm{A}}^2} d_k\big(\{i_a\}_q!\{i_b\}_q!\big)/2 \\ &= d_k\Big(\prod_{\{a,b\}\in\mathcal{P}_{\mathrm{A}}^2}\{i_a\}_q!\{i_b\}_q!\Big)/2 \\ &\geq d_k\Big(\prod_{\{a,b\}\in\mathcal{P}_{\mathrm{A}}^2}\big\{\min(i_a, i_b)\big\}_q!\Big). \end{aligned}$$

Here $\mathcal{P}_{\mathrm{iso}} \subset \mathcal{P}$ denotes the subset consisting of isolated labels. Hence we have (59), which completes the proof of $\mathcal{F}'\big(\Gamma''_g\big)(1) \subset (\bar{U}_q^{\mathrm{ev}})^{\otimes g}$.

6 Completions

In this section, we define the completion $(\bar{U}_q^{\mathrm{ev}})^{\hat{}\hat{\otimes} n}$ of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$, and prove Theorem 1.2.

6.1 Filtrations of $\bar{U}_q^{\mathrm{ev}}$ with respect to ad and $\overline{\mathrm{ad}}$

For a subset $X \subset \bar{U}_q^{\mathrm{ev}}$, let $\langle X\rangle_{\mathrm{ideal}}$ denote the two-sided ideal of $\bar{U}_q^{\mathrm{ev}}$ generated by X. Set

$$\begin{aligned}
&A_p = \langle U_{\mathbb{Z},q} \triangleright e^p\rangle_{\mathrm{ideal}},\\
&C_p = \langle \sum_{p'\geq p} (U_{\mathbb{Z},q}\tilde{E}^{(p')} \triangleright \bar{U}_q^{\mathrm{ev}})\rangle_{\mathrm{ideal}}, \quad C'_p = \langle \sum_{p'\geq p} (U_{\mathbb{Z},q}\tilde{F}^{(p')} \triangleright \bar{U}_q^{\mathrm{ev}})\rangle_{\mathrm{ideal}},\\
&\tilde{C}_p = \langle \sum_{p'\geq p} K(U_{\mathbb{Z},q}\tilde{E}^{(p')} \triangleright K\bar{U}_q^{\mathrm{ev}})\rangle_{\mathrm{ideal}}, \quad \tilde{C}'_p = \langle \sum_{p'\geq p} K(U_{\mathbb{Z},q}\tilde{F}^{(p')} \triangleright K\bar{U}_q^{\mathrm{ev}})\rangle_{\mathrm{ideal}},
\end{aligned}$$

for $p \geq 0$. For $X = A, C, C', \tilde{C}, \tilde{C}'$, the $X_p, p \geq 0$ form a decreasing filtration of $\bar{U}_q^{\mathrm{ev}}$, i.e., we have $X_p \supset X_{p+1}$ for $p \geq 0$.

Lemma 6.1 ([21, Proposition 5.5])**.** (i) *For $p \geq 0$, we have $C_p = C'_p$.*

(ii) *For $p \geq 0$, we have $C_{2p} \subset A_p$.*

(iii) *If $p \geq 0$ is even, then we have $C_{2p} = A_p$.*

Lemma 6.2. (i) *For $p \geq 0$, we have $\tilde{C}_p = \tilde{C}'_p$.*

(ii) *For $p \geq 0$, we have $\tilde{C}_{2p} \subset A_p$.*

(iii) *If $p \geq 0$ is odd, then we have $\tilde{C}_{2p} = A_p$.*

Proof. The proof is almost the same as that of Lemma 6.1. ☐

For $p \geq 0$, set

$$G_p = C_p + \tilde{C}_p = C'_p + \tilde{C}'_p.$$

Corollary 6.3. *For $p \geq 0$, we have $G_{2p} = A_p$.*

Proof. For $p \geq 0$, by Lemma 6.1 (ii) and 6.2 (ii), we have $G_{2p} = C_{2p} + \tilde{C}_{2p} \subset A_p$.
If $p \geq 0$ is even, then by Lemma 6.1 (iii), we have $G_{2p} \supset C_{2p} = A_p$.
If $p \geq 0$ is odd, then by Lemma 6.2 (iii), we have $G_{2p} \supset \tilde{C}_{2p} = A_p$.
Thus, we have the assertion. ☐

Corollary 6.4. *The filtrations $\{A_p\}_{p\geq 0}, \{C_p\}_{p\geq 0}, \{\tilde{C}_p\}_{p\geq 0}, \{G_p\}_{p\geq 0}$ are all cofinal with each other.*

6.2 Filtrations of $\bar{U}_q^{\mathrm{ev}}$ and $(\bar{U}_q^{\mathrm{ev}})^{\otimes 2}$ with respect to $\dot{Y}$

For $p \geq 0$, let $\mathcal{Y}_p$ be the two-sided ideal in $\bar{U}_q^{\mathrm{ev}}$ generated by the elements in

$$\sum_{p'\geq p} \dot{Y}(\bar{U}_q^0\tilde{E}^{(p')} \otimes \bar{U}_q), \ \sum_{p'\geq p} \dot{Y}(\bar{U}_q \otimes \bar{U}_q^0\tilde{E}^{(p')}), \ \sum_{p'\geq p} \dot{Y}(\bar{U}_q^0\tilde{F}^{(p')} \otimes \bar{U}_q), \ \sum_{p'\geq p} \dot{Y}(\bar{U}_q \otimes \bar{U}_q^0\tilde{F}^{(p')}).$$

Lemma 6.5. *For $p \geq 0$, we have $\mathcal{Y}_p \subset G_p$.*

Proof. It is enough to prove that all the generators of $\mathcal{Y}_p$ are contained in G_p.

By (20), (22), and (27), we have

$$\begin{aligned}\dot{Y}(\bar{U}_q^0\tilde{E}^{(p')}\otimes\bar{U}_q) &\subset \sum_{i=0,1}\Big(\bar{U}_q^0\tilde{E}^{(p')}\triangleright K^i\bar{U}_q^{\mathrm{ev}}\Big)K^i\bar{U}_q^{\mathrm{ev}}\subset C_p+\tilde{C}_p\subset G_p,\\ \dot{Y}(\bar{U}_q\otimes\bar{U}_q^0\tilde{E}^{(p')}) &= \sum_{i=0,1}K^i\bar{U}_q^{\mathrm{ev}}\Big(S^{-1}(\bar{U}_q^0\tilde{E}^{(p')})\triangleright K^i\bar{U}_q^{\mathrm{ev}}\Big)\\ &\subset \sum_{i=0,1}K^i\bar{U}_q^{\mathrm{ev}}\Big(\bar{U}_q^0\tilde{E}^{(p')}\triangleright K^i\bar{U}_q^{\mathrm{ev}}\Big)\subset C_p+\tilde{C}_p\subset G_p,\end{aligned}$$

for $p'\geq p$. Similarly, we have

$$\begin{aligned}\dot{Y}(\bar{U}_q^0\tilde{F}^{(p')}\otimes\bar{U}_q)&\subset C'_p+\tilde{C}'_p\subset G_p,\\ \dot{Y}(\bar{U}_q\otimes\bar{U}_q^0\tilde{F}^{(p')})&\subset C'_p+\tilde{C}'_p\subset G_p,\end{aligned}$$

for $p'\geq p$. Hence we have the assertion. □

Let $(\mathcal{Y}^D)_p$ be the two-sided ideal in $(\bar{U}_q^{\mathrm{ev}})^{\otimes 2}$ generated by the elements in

$$\begin{aligned}&\sum_{p'\geq p}\dot{Y}(\bar{U}_q^0\tilde{E}^{(p')}\otimes\bar{U}_q^0D'_\pm)\otimes\dot{Y}(\bar{U}_q\otimes\bar{U}_q^0D''_\pm), &&\sum_{p'\geq p}\dot{Y}(\bar{U}_q^0\tilde{F}^{(p')}\otimes\bar{U}_q^0D'_\pm)\otimes\dot{Y}(\bar{U}_q\otimes\bar{U}_q^0D''_\pm),\\ &\sum_{p'\geq p}\dot{Y}(\bar{U}_q^0\tilde{E}^{(p')}\otimes\bar{U}_q^0D'_\pm)\otimes\dot{Y}(\bar{U}_q^0D''_\pm\otimes\bar{U}_q), &&\sum_{p'\geq p}\dot{Y}(\bar{U}_q^0\tilde{F}^{(p')}\otimes\bar{U}_q^0D'_\pm)\otimes\dot{Y}(\bar{U}_q^0D''_\pm\otimes\bar{U}_q),\\ &\sum_{p'\geq p}\dot{Y}(\bar{U}_q^0D'_\pm\otimes\bar{U}_q^0\tilde{E}^{(p')})\otimes\dot{Y}(\bar{U}_q\otimes\bar{U}_q^0D''_\pm), &&\sum_{p'\geq p}\dot{Y}(\bar{U}_q^0D'_\pm\otimes\bar{U}_q^0\tilde{F}^{(p')})\otimes\dot{Y}(\bar{U}_q\otimes\bar{U}_q^0D''_\pm),\\ &\sum_{p'\geq p}\dot{Y}(\bar{U}_q^0D'_\pm\otimes\bar{U}_q^0\tilde{E}^{(p')})\otimes\dot{Y}(\bar{U}_q^0D''_\pm\otimes\bar{U}_q), &&\sum_{p'\geq p}\dot{Y}(\bar{U}_q^0D'_\pm\otimes\bar{U}_q^0\tilde{F}^{(p')})\otimes\dot{Y}(\bar{U}_q^0D''_\pm\otimes\bar{U}_q).\end{aligned}$$

Note that these sets are all contained in $(\bar{U}_q^{\mathrm{ev}})^{\otimes 2}$ by Lemma 3.10.

6.3 Filtrations of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$

For $n\geq 1$ and a filtration $\{X_p\}_{p\geq 0}$ of $\bar{U}_q^{\mathrm{ev}}$, we define a filtration $\{X_p^{(n)}\}_{p\geq 0}$ of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$ by

$$X_p^{(n)}=\sum_{j=1}^{n}(\bar{U}_q^{\mathrm{ev}})^{\otimes j-1}\otimes X_p\otimes(\bar{U}_q^{\mathrm{ev}})^{\otimes n-j}.$$

For $n\geq 1$, we define the filtration $\{(\mathcal{Y}^D)_p^{(n)}\}_{p\geq 0}$ of $(\bar{U}_q^{\mathrm{ev}})^{\otimes n}$ by

$$\begin{aligned}(\mathcal{Y}^D)_p^{(n)}=&\{\sum_{1\leq i<j\leq n}(\bar{U}_q^{\mathrm{ev}})^{\otimes i-1}\otimes y'\otimes(\bar{U}_q^{\mathrm{ev}})^{\otimes j-i-1}\otimes y''\otimes(\bar{U}_q^{\mathrm{ev}})^{\otimes n-j}\mid\sum y'\otimes y''\in(\mathcal{Y}^D)_p\}\\ &+\{\sum_{1\leq i<j\leq n}(\bar{U}_q^{\mathrm{ev}})^{\otimes i-1}\otimes y''\otimes(\bar{U}_q^{\mathrm{ev}})^{\otimes j-i-1}\otimes y'\otimes(\bar{U}_q^{\mathrm{ev}})^{\otimes n-j}\mid\sum y'\otimes y''\in(\mathcal{Y}^D)_p\}\\ &+\{\sum_{k=1}^{n}(\bar{U}_q^{\mathrm{ev}})^{\otimes k-1}\otimes y'y''\otimes(\bar{U}_q^{\mathrm{ev}})^{\otimes n-k}\mid\sum y'\otimes y''\in(\mathcal{Y}^D)_p\}\\ &+\{\sum_{k=1}^{n}(\bar{U}_q^{\mathrm{ev}})^{\otimes k-1}\otimes y''y'\otimes(\bar{U}_q^{\mathrm{ev}})^{\otimes n-k}\mid\sum y'\otimes y''\in(\mathcal{Y}^D)_p\}.\end{aligned}$$

Lemma 6.6. *For $n \geq 1, p \geq 0$, we have $(\mathcal{Y}^D)_p^{(n)} \subset G^{(n)}_{\lfloor \frac{p}{2} \rfloor}$.*

Proof. It is enough to prove that all the generators of $(\mathcal{Y}^D)_p$ are contained in $G^{(2)}_{\lfloor \frac{p}{2} \rfloor}$.

We prove

$$\sum_{p' \geq p} \dot{Y}(\bar{U}_q^0 \tilde{E}^{(p')} \otimes \bar{U}_q^0 D') \otimes \dot{Y}(\bar{U}_q^0 D'' \otimes \bar{U}_q) \subset G^{(2)}_{\lfloor \frac{p}{2} \rfloor}.$$

Similarly for the other generators of $(\mathcal{Y}^D)_p$. For $p \geq 0$, let us assume

$$\sum \dot{Y}(\bar{U}_q^0 \tilde{E}^{(p)} \otimes D') \otimes \dot{Y}(D'' \otimes \bar{U}_q) \subset G^{(2)}_p. \tag{60}$$

Then, similarly to (36), for $p' \geq p$, we have

$$\begin{aligned}
&\sum \dot{Y}(\bar{U}_q^0 \tilde{E}^{(p')} \otimes \bar{U}_q^0 D') \otimes \dot{Y}(\bar{U}_q^0 D'' \otimes \bar{U}_q) \\
=&\sum \dot{Y}(\bar{U}_q^0 \tilde{E}^{(p')} \otimes D' \bar{U}_q^0) \otimes \dot{Y}(\bar{U}_q^0 D'' \otimes \bar{U}_q) \\
\subset&\sum \dot{Y}\big(\bar{U}_q^0 (\tilde{E}^{(p')})_{(1)} \otimes \bar{U}_q^0\big)\big(\dot{Y}\big(\bar{U}_q^0 (\tilde{E}^{(p')})_{(2)} \otimes D'\big) \triangleleft \bar{U}_q^0\big) \otimes \big(\bar{U}_q^0 \triangleright \dot{Y}(D'' \otimes \bar{U}_q)\big)\dot{Y}(\bar{U}_q^0 \otimes \bar{U}_q) \\
\subset&\sum \dot{Y}\big(\bar{U}_q^0 (\tilde{E}^{(p')})_{(1)} \otimes \bar{U}_q^0\big)\dot{Y}\big(\bar{U}_q^0 (\tilde{E}^{(p')})_{(2)} \otimes D'\big) \otimes \dot{Y}(D'' \otimes \bar{U}_q)\dot{Y}(\bar{U}_q^0 \otimes \bar{U}_q) \\
=&\sum_{p'_1 + p'_2 = p'} \dot{Y}(\bar{U}_q^0 \tilde{E}^{(p'_1)} \otimes \bar{U}_q^0)\dot{Y}(\bar{U}_q^0 \tilde{E}^{(p'_2)} \otimes D') \otimes \dot{Y}(D'' \otimes \bar{U}_q)\dot{Y}(\bar{U}_q^0 \otimes \bar{U}_q) \\
\subset&\sum_{p'_1 + p'_2 = p'} \Big(\mathcal{Y}_{p'_1} \cdot \dot{Y}(\bar{U}_q^0 \tilde{E}^{(p'_2)} \otimes D')\Big) \otimes \Big(\dot{Y}(D'' \otimes \bar{U}_q) \cdot \bar{U}_q^{\mathrm{ev}}\Big) \\
\subset&\sum_{p'_1 + p'_2 = p'} \mathcal{Y}^{(2)}_{p'_1} \cdot \Big(\dot{Y}(\bar{U}_q^0 \tilde{E}^{(p'_2)} \otimes D') \otimes \dot{Y}(D'' \otimes \bar{U}_q) \cdot \bar{U}_q^{\mathrm{ev}}\Big) \\
\subset&\sum_{p'_1 + p'_2 = p'} \mathcal{Y}^{(2)}_{p'_1} \cdot G^{(2)}_{p'_2} \subset \sum_{p'_1 + p'_2 = p'} G^{(2)}_{p'_1} \cdot G^{(2)}_{p'_2} \subset \sum_{p'_1 + p'_2 = p'} G^{(2)}_{\max(p'_1, p'_2)} \subset G^{(2)}_{\lfloor \frac{p'}{2} \rfloor} \subset G^{(2)}_{\lfloor \frac{p}{2} \rfloor}.
\end{aligned}$$

Now, we prove (60). Similar to (37), for $b \in \bar{U}_q$ homogeneous, we have

$$\sum \dot{Y}(\bar{U}_q^0 \tilde{E}^{(p)} \otimes D') \otimes \dot{Y}(D'' \otimes b) = \sum (\bar{U}_q^0 \tilde{E}^{(p)} \triangleright K^{-|b_{(2)}|})K^{|b_{(2)}|} \otimes S^{-1}(b_{(2)})b_{(1)},$$

with $b_{(1)}, b_{(2)} \in \bar{U}_q$ homogeneous such that $S^{-1}(b_{(1)})b_{(2)} \in \bar{U}_q^{\mathrm{ev}}$.

If $|b_{(2)}| \in 2\mathbb{Z}$, then we have

$$(\bar{U}_q^0 \tilde{E}^{(p)} \triangleright K^{-|b_{(2)}|})K^{|b_{(2)}|} \subset C_p \subset G_p.$$

If $|b_{(2)}| \in \mathbb{Z} \setminus 2\mathbb{Z}$, then we have

$$(\bar{U}_q^0 \tilde{E}^{(p)} \triangleright K^{-|b_{(2)}|})K^{|b_{(2)}|} \subset \tilde{C}_p \subset G_p.$$

Thus, we have

$$\sum (\bar{U}_q^0 \tilde{E}^{(p)} \triangleright K^{-|b_{(2)}|})K^{|b_{(2)}|} \otimes S^{-1}(b_{(2)})b_{(1)} \subset G_p \otimes \bar{U}_q^{\mathrm{ev}} \subset G^{(2)}_p.$$

Hence we have the assertion. □

6.4 Completions

Let $(\bar{U}_q^{\rm ev})\hat{}$ denote the completion of $\bar{U}_q^{\rm ev}$ in U_h with respect to the filtration $\{G_p\}_{p\geq 0}$, i.e., $(\bar{U}_q^{\rm ev})\hat{}$ is the image of the map

$$\varprojlim_p \big(\bar{U}_q^{\rm ev}/G_p\big) \to U_h$$

induced by the inclusion $\bar{U}_q^{\rm ev} \subset U_h$. Since $G_{2p} = A_p \subset h^p U_h$, this map is well-defined.

Let $(\bar{U}_q^{\rm ev})^{\hat{}\hat{\otimes} n}$ denote the completion of $(\bar{U}_q^{\rm ev})^{\otimes n}$ in $U_h^{\hat{\otimes} n}$ with respect to the filtration $\{G_p^{(n)}\}_{p\geq 0}$. For $n=0$, it is natural to set

$$G_p^{(0)} = \begin{cases} \mathbb{Z}[q,q^{-1}] & \text{if } \ p=0, \\ 0 & \text{if } \ p>0. \end{cases}$$

Thus, we have

$$(\bar{U}_q^{\rm ev})^{\hat{}\hat{\otimes} 0} = \mathbb{Z}[q,q^{-1}].$$

6.5 Proof of Theorem 1.2

Let $T \in BT_n$ be a boundary bottom tangle and $(\tilde{T}; g, g_1, \dots, g_n)$ a boundary data for T. Let $C(\tilde{T}) = \{c_1, \dots, c_l\}$ be the set of crossings of the diagram of $\tilde{T}$ which we fix in the definition of $J_{\tilde{T}}$. We fix these notations in this section.

By Proposition 3.11, we have

$$\mu^{[g_1,\dots,g_n]} \bar{Y}^{\otimes g}(J_{\tilde{T},s}) \in (\bar{U}_q^{\rm ev})^{\otimes n},$$

for $s \in \mathcal{S}(\tilde{T})$. In this section, we prove Theorem 1.2, i.e., we prove

$$J_T = \sum_{s\in\mathcal{S}(\tilde{T})} \mu^{[g_1,\dots,g_n]} \bar{Y}^{\otimes g}(J_{\tilde{T},s}) \in (\bar{U}_q^{\rm ev})^{\hat{}\hat{\otimes} n}.$$

Since $\mu^{[g_1,\dots,g_n]}\big(G_p^{(g)}\big) \subset G_p^{(n)}$ for $p \geq 0$, it is enough to prove the following lemma.

Lemma 6.7. *For each $p \geq 0$, there are only finitely many states $s \in \mathcal{S}(\tilde{T})$ such that $\bar{Y}^{\otimes g}(J_{\tilde{T},s}) \notin G_p^{(n)}$.*

We use the setting in in Section 5, where we can consider a state $s \in \mathcal{S}(\tilde{T})$ as a parameter.

Lemma 6.8. *There is a map $B\colon \mathcal{S}(\tilde{T}) \to \Gamma_g''$, $s \mapsto B^s$, satisfying the following conditions: For all $s \in \mathcal{S}(\tilde{T})$, we have $\bar{Y}^{\otimes g}(J_{\tilde{T},s}) \in \mathcal{F}'\big(B^s\big)(1)$, and we have $B^s = b_4 \circ b_3 \circ b_2 \circ b_1^s$ with $(b_1^s, b_2, b_3, b_4) \in \tilde{\Gamma}_g''$ such that $b_2 \circ b_1^s = \sigma \circ \tilde{b}_1^s$ with $\sigma \in \mathrm{Hom}(\mathcal{A}_{\mathfrak{S}})$ and $\tilde{b}_1^s$ as in (45) replacing s_p with $s(c_p)$ for $p = 1, \dots, l_2$, where any part of B^s except $\langle \Theta_{s(c_1)}\rangle \otimes \dots \otimes \langle \Theta_{s(c_{l_2})}\rangle$ in $\tilde{b}_1^s$ does not depend on $s \in \mathcal{S}(\tilde{T})$.*

Proof. We can define B as in Lemma 6.8 by constructing B^s as follows. First, we choose a state $x \in \mathcal{S}(\tilde{T})$, and construct $B_0^x \in \Gamma_g$ so that $\bar{Y}^{\otimes g}(J_{\tilde{T},x}) \in \mathcal{F}\big(B_0^x\big)(1)$ as in the proof of Lemma 5.1. By the definition, we have $\bar{Y}^{\otimes g}(J_{\tilde{T},s}) \in \mathcal{F}\big(B_0^s\big)(1)$ for all state $s \in \mathcal{S}(\tilde{T})$, where $B_0^s \in \Gamma_g$ is obtained from B_0^s by replacing $\langle \Theta_{x(c_p)}\rangle$ with $\langle \Theta_{s(c_p)}\rangle$ for $p = 1, \dots, l_2$.

Second, we transform B_0^x into some $B^x \in \Gamma_g''$ by using the preorder $\preceq$ and $\preceq'$. We have $\bar{Y}^{\otimes g}(J_{\tilde{T},x}) \in \mathcal{F}'\big(B^x\big)(1)$ by Lemma 5.3 and 5.6. Here, since $\preceq$ and $\preceq'$ each does not depend on any $\langle \Theta_{x(c_p)} \rangle$, we can obtain the desired $B^s \in \Gamma_g''$ from B^x by replacing $\langle \Theta_{x(c_p)} \rangle$ with $\langle \Theta_{s(c_p)} \rangle$ for $p = 1, \ldots, l_2$. $\square$

We take $B\colon \mathcal{S}(\tilde{T}) \to \Gamma_g''$, $s \mapsto B^s$, as in Lemma 6.8. For $s \in \mathcal{S}(\tilde{T})$, recall from (46) the modification $\tilde{b}_1^s(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})$ of $\tilde{b}_1^s$ with respect to $\mathbf{i}_p \in \mathcal{I}(s(c_p), m_p)$ and $\bar{\mathbf{i}}_p \in \mathcal{I}(s(c_p), n_p)$ for $p = 1, \ldots, l_2$.

For $s \in \mathcal{S}(\tilde{T})$, and

$$\mathbf{i}_p = \big(i_{(p,0,1)}, \ldots, i_{(p,0,m_p)}\big) \in \mathcal{I}(s(c_p), m_p), \quad \bar{\mathbf{i}}_p = \big(i_{(p,1,1)}, \ldots, i_{(p,1,n_p)}\big) \in \mathcal{I}(s(c_p), n_p),$$

for $p = 1, \ldots, l_2$, set

$$\begin{aligned} N^s(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_l, \bar{\mathbf{i}}_l) &= \max\{i_{(p,0,i)}, i_{(p,1,j)} \mid 1 \le i \le m_p,\ 1 \le j \le n_p,\ 1 \le p \le l_2\}, \\ N^s &= \min\{N^s(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_l, \bar{\mathbf{i}}_l) \mid \mathbf{i}_p \in \mathcal{I}(s(c_p), m_p),\ \bar{\mathbf{i}}_p \in \mathcal{I}(s(c_p), n_p),\ 1 \le p \le l_2\}. \end{aligned}$$

We use the following lemma.

Lemma 6.9. *For $r \ge 0$, there are only finitely many states $s \in \mathcal{S}(\tilde{T})$ such that $N^s \le r$.*

Proof. Note that for $\mathbf{i} = (i_1, \ldots, i_l) \in \mathcal{I}(k, l)$, $k \ge 0, l \ge 1$ we have

$$\frac{k}{l} \le \max(i_1, \ldots, i_l).$$

Thus we have

$$w^s := \max\left\{\frac{s(c_p)}{m_p}, \frac{s(c_p)}{n_p} \;\middle|\; 1 \le p \le l_2\right\} \le N^s(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_l, \bar{\mathbf{i}}_l),$$

for all $\mathbf{i}_p \in \mathcal{I}(s(c_p), m_p)$, $\bar{\mathbf{i}}_p \in \mathcal{I}(s(c_p), n_p)$, $p = 1, \ldots, l_2$. Hence we have

$$w^s \le N^s. \tag{61}$$

It is not difficult to prove that, for $r \ge 0$, there are only finitely many states $s \in \mathcal{S}(\tilde{T})$ such that $w^s \le r$. This and (61) imply the assertion. $\square$

Lemma 6.7 follows from Lemma 6.9 and the following lemma.

Lemma 6.10. *For $s \in \mathcal{S}(\tilde{T})$ and $r \ge 0$ such that $N^s \ge 2r$, we have*

$$\mathcal{F}'\big(b_4 \circ b_3 \circ \sigma \circ \tilde{b}_1^s(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})\big)(1) \subset G_r^{(g)},$$

for $\mathbf{i}_p \in \mathcal{I}(s(c_p), m_p)$, $\bar{\mathbf{i}}_p \in \mathcal{I}(s(c_p), n_p)$, $p = 1, \ldots, l_2$.

Proof. The proof is similar to that of Lemma 5.2. By replacing s_p with $s(c_p)$ for $p = 1, \ldots, l_2$, we use the notations and results in the proof of Lemma 5.2.

Fix $\mathbf{i}_p \in \mathcal{I}(s(c_p), m_p)$ and $\bar{\mathbf{i}}_p \in \mathcal{I}(s(c_p), n_p)$, for $p = 1, \ldots, l_2$. Recall that we color the output edges of $\tilde{b}_1^s(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})$ with the labels in $\mathcal{P}$ as in Figure 38. Note that

$$M := N^s(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_l, \bar{\mathbf{i}}_l) = \max\{i_a \mid a \in \mathcal{P}\} \ge N^s \ge 2r.$$

Since the filtration $\{G_p\}_{p\geq 0}$ is decreasing, it is enough to prove

$$\mathcal{F}'\big(b_4\circ b_3\circ\sigma\circ\tilde{b}_1^s(\mathbf{i}_1,\bar{\mathbf{i}}_1,\ldots,\mathbf{i}_{l_2},\bar{\mathbf{i}}_{l_2})\big)(1)\subset G^{(g)}_{\lfloor M/2\rfloor}. \tag{62}$$

We prove (62). Recall that $\mathcal{P}_{\rm iso}\subset\mathcal{P}$ denotes the set of isolated labels, and $\mathcal{P}^2_{\rm A}$ denotes the set of unordered pairs $\{a,b\}$ of mutually adjacent labels $a,b\in\mathcal{P}$. Set

$$\mathcal{P}_Y=\mathcal{P}\setminus\mathcal{P}_{\rm iso}=\bigcup\mathcal{P}^2_{\rm A}.$$

Set $M_{\rm iso}=\max\{i_a|\ a\in\mathcal{P}_{\rm iso}\}$ and $M_Y=\max\{i_a|\ a\in\mathcal{P}_Y\}$. It is enough to prove

(i) $\mathcal{F}'\big(b_4\circ b_3\circ\sigma\circ\tilde{b}_1^s(\mathbf{i}_1,\bar{\mathbf{i}}_1,\ldots,\mathbf{i}_{l_2},\bar{\mathbf{i}}_{l_2})\big)(1)\subset G^{(g)}_{M_{\rm iso}}$ $(\subset G^{(g)}_{\lfloor M_{\rm iso}/2\rfloor})$, and

(ii) $\mathcal{F}'\big(b_4\circ b_3\circ\sigma\circ\tilde{b}_1^s(\mathbf{i}_1,\bar{\mathbf{i}}_1,\ldots,\mathbf{i}_{l_2},\bar{\mathbf{i}}_{l_2})\big)(1)\subset G^{(g)}_{\lfloor M_Y/2\rfloor}$.

Let us prove (i). Recall from (56) that

$$\mathcal{F}'\big(b_3\circ\sigma\circ\tilde{b}_1^s(\mathbf{i}_1,\bar{\mathbf{i}}_1,\ldots,\mathbf{i}_{l_2},\bar{\mathbf{i}}_{l_2})\big)(1)\subset U_{\mathbb{Z},q}^{\otimes l_4}\otimes(\bar{U}_q^{\rm ev})^{\otimes l_5+\otimes l_6}.$$

Thus, it is enough to prove

$$\mathcal{F}'(b_4)\Big(U_{\mathbb{Z},q}^{\otimes l_4}\otimes(\bar{U}_q^{\rm ev})^{\otimes l_5+\otimes l_6}\Big)\subset G^{(g)}_{M_{\rm iso}}. \tag{63}$$

Recall that the first l_4 input edges of b_4 are connected to the left (resp. right) input edges of the $\langle\mathrm{ad}\rangle$'s (resp. $\langle\overline{\mathrm{ad}}\rangle$'s), and the next l_5+l_6 input edges of b_4 go down to the edges of the $\langle\mu\rangle$'s and to the right (resp. left) input edges of the $\langle\mathrm{ad}\rangle$'s (resp. $\langle\overline{\mathrm{ad}}\rangle$'s). By the definition of C_p and C'_p, we have

$$\mathrm{ad}(U_{\mathbb{Z},q}\tilde{E}^{(p)}\otimes\bar{U}_q^{\rm ev})\subset C_p\subset G_p, \tag{64}$$
$$\mathrm{ad}(U_{\mathbb{Z},q}\tilde{F}^{(p)}\otimes\bar{U}_q^{\rm ev})\subset C'_p\subset G_p, \tag{65}$$

for $p\geq 0$. We also have

$$\begin{aligned}\overline{\mathrm{ad}}(\bar{U}_q^{\rm ev}\otimes U_{\mathbb{Z},q}\tilde{E}^{(p)})&\subset\overline{\mathrm{ad}}(\bar{U}_q^{\rm ev}\otimes\tilde{E}^{(p)})\\&\subset\mathrm{ad}\left(S^{-1}(\tilde{E}^{(p)})\otimes\bar{U}_q^{\rm ev}\right)\\&\subset\mathrm{ad}(\bar{U}_q^0\tilde{E}^{(p)}\otimes\bar{U}_q^{\rm ev})\subset C_p\subset G_p,\end{aligned} \tag{66}$$

for $p\geq 0$. Similarly, we have

$$\overline{\mathrm{ad}}(\bar{U}_q^{\rm ev}\otimes U_{\mathbb{Z},q}\tilde{F}^{(p)})\subset C'_p\subset G_p, \tag{67}$$

for $p\geq 0$. Thus, (63) follows from (64)–(67) and the inclusions

$$\mu(U_{\mathbb{Z},q}\otimes G_p)=\mu(G_p\otimes U_{\mathbb{Z},q})\subset G_p, \tag{68}$$
$$\mathrm{ad}(U_{\mathbb{Z},q}\otimes G_p)\subset G_p,\quad\overline{\mathrm{ad}}(G_p\otimes U_{\mathbb{Z},q})\subset G_p. \tag{69}$$

for $p\geq 0$. We have finished the proof of (i).

Let us prove (ii). Recall from (50) that

$$\mathcal{F}'\big(b_3\circ\sigma\circ\tilde{b}_1^s(\mathbf{i}_1,\bar{\mathbf{i}}_1,\ldots,\mathbf{i}_{l_2},\bar{\mathbf{i}}_{l_2})\big)(1)\in\Big(\prod_{p=1}^{l_2}\{s_p\}_q!\Big)\cdot\Big(U_{\mathbb{Z},q}^{\otimes l_4}\otimes\mathcal{F}'(Z)(1)\otimes(\bar{U}_q^{\mathrm{ev}\,0})^{\otimes l_6}\Big), \tag{70}$$

where $Z \in \langle \dot{Y} \rangle^{\otimes l_5} \circ \mathrm{Hom}_{\mathcal{A}}(I, A^{\otimes 2l_5})$. We study $\mathcal{F}'(Z)(1)$ by using the following inclusions instead of (51)–(54).

For $X_1, X_2 \in \{E, F\}$ and $i_a, i_b \geq 0$ for $\{a,b\} \in \mathcal{P}^2_{\mathrm{A}}$, we have

$$\dot{Y}(\bar{U}^0_q \tilde{X}_1^{(i_a)} \otimes \bar{U}^0_q \tilde{X}_2^{(i_b)}) \subset (\{\min(i,j)\}_q!)^{-1} \cdot \mathcal{Y}_{\max(i_a,i_b)}. \tag{71}$$

For example, we have

$$\begin{aligned}\dot{Y}(\bar{U}^0_q \tilde{E}^{(2)} \otimes \bar{U}^0_q \tilde{F}^{(3)}) &= (\{2\}_q!)^{-1} \dot{Y}(\bar{U}^0_q e^2 \otimes \bar{U}^0_q \tilde{F}^{(3)}) \\ &\subset (\{2\}_q!)^{-1} \mathcal{Y}_3.\end{aligned}$$

We also have

$$\sum \dot{Y}(\bar{U}^0_q \tilde{X}_1^{(i_a)} \otimes \bar{U}^0_q D'_{\pm}) \otimes \dot{Y}(\bar{U}^0_q \tilde{X}_2^{(i_b)} \otimes \bar{U}^0_q D''_{\pm}) \subset (\{\min(i_a,i_b)\}_q!)^{-1} \cdot (\mathcal{Y}^D)_{\max(i_a,i_b)}, \tag{72}$$

$$\sum \dot{Y}(\bar{U}^0_q \tilde{X}_1^{(i_a)} \otimes \bar{U}^0_q D'_{\pm}) \otimes \dot{Y}(\bar{U}^0_q D''_{\pm} \otimes \bar{U}^0_q \tilde{X}_2^{(i_b)}) \subset (\{\min(i_a,i_b)\}_q!)^{-1} \cdot (\mathcal{Y}^D)_{\max(i_a,i_b)}. \tag{73}$$

$$\sum \dot{Y}(\bar{U}^0_q D'_{\pm} \otimes \bar{U}^0_q \tilde{X}_1^{(i_a)}) \otimes \dot{Y}(\bar{U}^0_q \tilde{X}_2^{(i_b)} \otimes \bar{U}^0_q D''_{\pm}) \subset (\{\min(i_a,i_b)\}_q!)^{-1} \cdot (\mathcal{Y}^D)_{\max(i_a,i_b)}, \tag{74}$$

$$\sum \dot{Y}(\bar{U}^0_q D'_{\pm} \otimes \bar{U}^0_q \tilde{X}_1^{(i_a)}) \otimes \dot{Y}(\bar{U}^0_q D''_{\pm} \otimes \bar{U}^0_q \tilde{X}_2^{(i_b)}) \subset (\{\min(i_a,i_b)\}_q!)^{-1} \cdot (\mathcal{Y}^D)_{\max(i_a,i_b)}. \tag{75}$$

By the above inclusions (71)–(75), and by Lemmas 6.5 and 6.6, we have

$$\mathcal{F}'(Z)(1) \in \prod_{\{a,b\} \in \mathcal{P}^2_{\mathrm{A}}} \left(\left\{ \min(i_a,i_b)\right\}_q!\right)^{-1} \cdot G^{(l_5)}_{\lfloor M_Y/2 \rfloor}. \tag{76}$$

Thus, by (70), (76) and (58), we have

$$\begin{aligned}\mathcal{F}'\big(b_3 \circ \sigma \circ \tilde{b}^s_1(\mathbf{i}_1, \bar{\mathbf{i}}_1, \ldots, \mathbf{i}_{l_2}, \bar{\mathbf{i}}_{l_2})\big)(1) &\subset U^{\otimes l_4}_{\mathbb{Z},q} \otimes G^{(l_5)}_{\lfloor M_Y/2 \rfloor} \otimes (\bar{U}^{\mathrm{ev}\,0}_q)^{\otimes l_6} \\ &\subset U^{\otimes l_4}_{\mathbb{Z},q} \otimes G^{(l_5+l_6)}_{\lfloor M_Y/2 \rfloor}.\end{aligned}$$

For the proof of the claim, it is enough to prove the inclusion

$$\mathcal{F}'(b_4)\Big(U^{\otimes l_4}_{\mathbb{Z},q} \otimes G^{(l_5+l_6)}_{\lfloor M_Y/2 \rfloor}\Big) \subset G^{(g)}_{\lfloor M_Y/2 \rfloor}, \tag{77}$$

which follows from (68) and (69). This completes the proof. □

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