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Dedicated to Professor Antonio Machi
on the occasion of his 70th birthday

By

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Abstract. In order to analyze the singularities of a power series $P(t)$ on the boundary of its convergent disc, we introduced the space $\Omega(P)$ of *opposite power series* in the opposite variable $s=1/t$, where $P(t)$ was, mainly, the growth function (Poincaré series) for a finitely generated group or a monoid [S1]. In the present paper, forgetting about that geometric or combinatorial background, we study the space $\Omega(P)$ abstractly for any suitably tame power series $P(t) \in \mathbb{C}\{t\}$. For the case when $\Omega(P)$ is a finite set and $P(t)$ is meromorphic in a neighbourhood of the closure of its convergent disc, we show *a duality between the set $\Omega(P)$ and the set of the highest order poles of $P(t)$ on the boundary of its convergent disc.*

Contents

1. Introduction

There seems a remarkable "resonance" between oscillation behavior[1] of a sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of complex numbers satisfying a tame condition (see §2(2.1.2)) and the singularities of its generating function $P(t)=\sum_{n=0}^{\infty}\gamma_n t^n$ on the boundary of the disc of convergence in $\mathbb{C}$. The idea was inspired and strongly used in the study of growth functions (Poincaré series) for finitely generated groups and monoids [S1, §11].

Let us explain this phenomena by a typical example due to Machì [M] (for details, see Examples in §3.3 and §5.4 of the present paper. Other simple examples are given in §3.4 (see [C, S2, S3]) and §3.5). By choosing generators of order 2 and 3 in $\mathrm{PSL}(2,\mathbb{Z})$, Machì has shown that the number γ_n of elements of $\mathrm{PSL}(2,\mathbb{Z})$ which are expressed in words of length less or equal than $n\in\mathbb{Z}_{\geq 0}$ w.r.t. the generators is given by $\gamma_{2k}=7\cdot 2^k-6$ and $\gamma_{2k+1}=10\cdot 2^k-6$ for $k\in\mathbb{Z}_{\geq 0}$. On one hand, this means that the sequence of ratios γ_{n-1}/γ_n $(n=1,2,\cdots)$ accumulates to *two distinct "oscillation" values* $\{\frac{5}{7},\frac{7}{10}\}$ according as n is even or odd. On the other hand, the generating function (or, so called, the growth function) can be expressed as a rational function $P(t)=\frac{(1+t)(1+2t)}{(1-2t^2)(1-t)}$, and it has *two poles* at $\{\pm\frac{1}{\sqrt{2}}\}$ on the boundary of its convergent disc of radius $\frac{1}{\sqrt{2}}$. *We see that there is a resonance between the set* $\{\frac{5}{7},\frac{7}{10}\}$ *of "oscillations" of the sequence* $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ *and the set* $\{\pm\frac{1}{\sqrt{2}}\}$ *of "poles" of the function* $P(t)$, in the way we shall explain in the present paper.

In order to analyze these phenomena, in [S1, §11], we introduced a space $\Omega(P)$ of *opposite power series* in the opposite variable $s=1/t$, as a compact subset of $\mathbb{C}[[s]]$, where each opposite series is defined by using "oscillations" of the sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ so that $\Omega(P)$ carries a comprehensive information of oscillations (see §2.2 Definition (2.2.2)). On the other hand, the space $\Omega(P)$ has duality with the singularities of the function $P(t)$ (§5 Theorem). Thus, $\Omega(P)$ becomes a bridge between the two subject: oscillations of $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ and singularities of $P(t)$. Since the method is independent of the group theoretic background and is extendable to a wider class of series, which we call *tame*, we separate the results and proofs in a self-contained way in the present paper. We study in details the case when $\Omega(P)$ is finite, where we have good understanding of the resonance phenomena by a use of *rational set* explained below, and Machì's example is explained in that frame.

One key concept introduced in the present paper is a *rational subset* U (§3), which is a subset of the positive integers $\mathbb{Z}_{\geq 0}$ such that the sum $\sum_{n\in U} t^n$ is a rational function in t. The concept is used twice in the

[1]Here, by an oscillation behavior, we mean that the sequence of the growth rate γ_{n-k}/γ_n $(n=1,2,3,\cdots)$ of period $k\in\mathbb{Z}_{>0}$ has several different accumulation values.

present paper. Firstly in §3, where we show that, if the space of opposite series $\Omega(P)$ is finite, then there is a finite partition $\mathbb{Z}_{\geq 0} = \amalg_i U_i$ of $\mathbb{Z}_{\geq 0}$ into rational sets so that there is no longer oscillation inside in each $\{\gamma_n : n \in U_i\}$. We call such phenomena "finite rational accumulation" (§3.2 Theorem) (such phenomena already appeared when we were studying the F-limit functions for monoids [S1, §11.5 Lemma]). Secondly in §5, where we introduce a rational operator T_U acting on a power series $P(t) \in \mathbb{C}[[t]]$ by letting $T_U P(t) := \sum_{n \in U} \gamma_n t^n$. The rational operators gives a machinery to "separate" singularities of the power series $P(t)$. In this way, the concept of a rational set combines the oscillation of a sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ and the singularities of the generating function $P(t) := \sum_{n=0}^{\infty} \gamma_n t^n$ for the case when $\Omega(P)$ is finite.

Contents of the present paper are as follows.

In §2, we introduce the space $\Omega(P)$ of opposite series as the accumulating subset in $\mathbb{C}[[s]]$ of the sequence $X_n(P) := \sum_{k=0}^{n} \frac{\gamma_{n-k}}{\gamma_n} s^k$ $(n = 0, 1, 2, \cdots)$ with respect to the coefficientwise convergence topology, where kth coefficient describes an oscillation of period k. Dividing by 1-period oscillation, we construct a shift action τ_Ω on the set $\Omega(P)$, which shifts k-period oscillations to $k-1$-oscillations.

In 3.1, we introduce the concept of a rational subset of $\mathbb{Z}_{\geq 0}$, and as an application, the key concept of *finite rational accumulation*. We show that if $\Omega(P)$ is a finite set, then $\Omega(P)$ is automatically a finite rational accumulation set and the τ_Ω-action becomes invertible and transitive.

After §4, we assume always finite rational accumulation for $\Omega(P)$. In §4, we analyze in details of the opposite series in $\Omega(P)$, showing that they become rational functions with the common denominator $\Delta^{op}(s)$ in 4.1, and that the rank of $\mathbb{C}\Omega(P)$ is equal to $\deg(\Delta^{op}(s))$ in 4.3.

In §5, we assume that the series $P(t)$ defines a meromorphic function in a neighbourhood of the closed convergent disc. Then we show that $\Delta^{op}(s)$ is opposite to the polynomial $\Delta^{top}(t)$ of the highest order part of poles of $P(t)$ (duality theorem in 5.3), and, in particular, the rank of the space $\mathbb{C}\Omega(P)$ is equal to the number of poles of the highest order of $P(t)$ on the boundary of the convergent disc. We get an identification of some transition matrices obtained in s-side and in t-side, which plays a crucial role in the trace formula in limit F-function [S1, 11.5.6].

Problems. The space $\Omega(P)$ for a study of the singularities of a series $P(t)$ is new. There seems some directions of its further study.

1. Generalize the space $\Omega(P)$ in order to capture lower order poles of $P(t)$ on the boundary of its convergent disc (c.f. [S1, §12, **2.**]).
2. Generalize the duality for the case when $\Omega(P)$ is infinite. Some probabilistic approach may be desirable (c.f. [S1, §12, **1.**]).

2. The space of opposite series.

In this section, we introduce the space $\Omega(P)$ of opposite series for a tame power series $P \in \mathbb{C}[[t]]$, and equip it with a τ_Ω-action.

2.1. **Tame power series.**

Let us call a complex coefficient power series in t

$$P(t) \ = \ \textstyle\sum_{n=0}^{\infty} \gamma_n t^n \tag{2.1.1}$$

to be *tame*, if there are positive real numbers $u, v \in \mathbb{R}_{>0}$ such that

$$u \ \leq \ |\gamma_{n-1}/\gamma_n| \ \leq \ v \tag{2.1.2}$$

for sufficiently large integers n. This implies that there exists a positive constant c so that

$$cv^{-n} \leq |\gamma_n| \leq cu^{-n} \tag{2.1.3}$$

for sufficiently large integer $n \in \mathbb{Z}_{\geq 0}$. Let us consider two limit values:

$$u \ \leq \ r_P := 1/\overline{\lim_{n\to\infty}}|\gamma_n|^{1/n} \ \leq \ R_P := 1/\underline{\lim_{n\to\infty}}|\gamma_n|^{1/n} \ \leq \ v. \tag{2.1.4}$$

Cauchy-Hadamard Theorem says that P is convergent of radius r_P.

Example. Let Γ be a group or a monoid with a finite generator system G. Then the length $l(g)$ of an element $g \in \Gamma$ is the shortest length of words expressing g in the letter G. Put $\Gamma_n := \{g \in \Gamma \mid l(g) \leq n\}$ and $\gamma_n := \#(\Gamma_n)$. Then the growth function (Poincaré series) for Γ with respect to G is defined by $P_{\Gamma,G}(t) := \sum_{n=0}^{\infty} \gamma_n t^n$. The sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is increasing and semi-multiplicative $\gamma_{m+n} \leq \gamma_m \gamma_n$. Therefore, by choosing $u = 1/\gamma_1$ and $v = 1$, the growth series is tame.

2.2. **The space $\Omega(P)$ of opposite series.**

Let P be a tame power series. An *opposite polynomial of degree n* for sufficiently large integer n is defined as

$$X_n(P) \ := \ \textstyle\sum_{k=0}^{n} \frac{\gamma_{n-k}}{\gamma_n} \ s^k. \tag{2.2.1}$$

We regard the sequence $\{X_n(P)\}_{n\gg 1}$ to be embedded in the space $\mathbb{C}[[s]]$ of formal power series, where $\mathbb{C}[[s]]$ is equipped with the classical topology, i.e. the product topology of coefficient-wise convergence in classical topology. Then, we define *the space of opposite series* by

$$\Omega(P) := \text{the set of accumulation points of the sequence (2.2.1)}. \tag{2.2.2}$$

The first statement on $\Omega(P)$ is the following.

Assertion 1. *Let P be a tame series. Then the space $\Omega(P)$ of its opposite series is a non-empty compact closed subset of $\mathbb{C}[[s]]$.*

Proof. For each $k \in \mathbb{Z}_{\geq 0}$, the kth coefficient $\frac{\gamma_{n-k}}{\gamma_n}$ of the polynomial $X_n(P)$ for sufficiently (with respect to P and k) large $n \in \mathbb{Z}_{\geq 0}$ has the approximation $u^k \leq |\frac{\gamma_{n-k}}{\gamma_n}| = |\frac{\gamma_{n-1}}{\gamma_n}||\frac{\gamma_{n-2}}{\gamma_{n-1}}| \cdots |\frac{\gamma_{n-k}}{\gamma_{n-k+1}}| \leq v^k$, i.e. it lies in the compact annuli

$$\bar{D}(0,u^k,v^k) := \{a \in \mathbb{C} \mid u^k \leq |a| \leq v^k\}.$$

Thus, for each fixed $m \in \mathbb{Z}_{\geq 0}$, the image of the sequence (2.2.1) under the projection map $\pi_m : \mathbb{C}[[s]] \to \mathbb{C}^{m+1}$, $\sum_{k=0}^{\infty} a_k s^k \mapsto (a_0, \cdots, a_m)$ accumulates to an non-empty compact set, say Ω_m. Then, we have:

$$\Omega(P) = \cap_{m=0}^{\infty}\big((\pi_m)^{-1}\Omega_m \cap \textstyle\prod_{k=0}^{\infty} \bar{D}(0,u^k,v^k)\big),$$

where RHS is an intersection of decreasing sequence of compact sets, so that their intersection is a non-empty compact set. □

Any element $a(s) = \Sigma_{k=0}^{\infty} a_k s^k \in \Omega(P)$ is called an *opposite series*, whose coefficients $\{a_k\}_{k=0}^{\infty}$ satisfy $a_k \in \bar{D}(0,u^k,v^k)$. By the definition, the constant term a_0 is equal to 1. The coefficient a_1 of the linear term of a is called the *initial* of the opposite series a, and denoted by $\iota(a)$.

For later use, let us introduce the space of the initials:

(2.2.3) $\quad \Omega_1(P) :=$ the accumulation set of the sequence $\left\{\dfrac{\gamma_{n-1}}{\gamma_n}\right\}_{n \gg 0}$,

which is a compact subset in $\bar{D}(0,u,v)$. The projection map $\Omega(P) \to \Omega_1(P)$, $a \mapsto \iota(a)$ is surjective but may not be injective (see §3.5 Ex.).

2.3. **The τ_Ω-action on $\Omega(P)$.**

We introduce a continuous map τ_Ω of $\Omega(P)$ to itself.

Assertion 2. a. *Let $\{n_m\}_{m \in \mathbb{Z}_{\geq 0}}$ be a subsequence of $\mathbb{Z}_{\geq 0}$ tending to ∞. If the sequence $\{X_{n_m}(P)\}_{m \in \mathbb{Z}_{\geq 0}}$ converges to an opposite series a, then the sequence $\{X_{n_m-1}(P)\}_{m \in \mathbb{Z}_{\geq 0}}$ also converges to an opposite series, whose limit depends only on a and is denoted by $\tau_\Omega(a)$. Then, we have*

(2.3.1) $$\tau_\Omega(a) = (a-1)/\iota(a)s.$$

b. *Consider a map*

(2.3.2) $$\tau : \Omega(P) \longrightarrow \mathbb{C}\Omega(P), \quad a \mapsto \iota(a)\tau_\Omega(a) = (a-1)/s$$

where $\mathbb{C}\Omega(P)$ is a closed $\mathbb{C}$-linear subspace of $\mathbb{C}[[s]]$ generated by $\Omega(P)$. Then, the map τ naturally extends to an endomorphism of $\mathbb{C}\Omega(P)$.

(2.3.3) $$\tau \in End_{\mathbb{C}}(\mathbb{C}\Omega(P))$$

Proof. a. By definition, for any $k\in\mathbb{Z}_{\geq 0}$, the sequence $\frac{\gamma_{n_m-k}}{\gamma_{n_m}}$ converges to a constant $a_k\in\bar{D}(u^k,v^k)$. Then, $\frac{\gamma_{(n_m-1)-(k-1)}}{\gamma_{n_m-1}}=\frac{\gamma_{n_m-k}}{\gamma_{n_m}}/\frac{\gamma_{n_m-1}}{\gamma_{n_m}}$ converges to a_k/a_1. That is, the sequence $\{X_{n_m-1}(P)\}_{m\in\mathbb{Z}_{\geq 0}}$ converges to an opposite series, whose $(k-1)$th coefficient is equal to a_k/a_1.

b. Let $\sum_{i\in I}c_ia^{(i)}(s)=0$ be a linear relation among opposite sequences $a^{(i)}(s)$ $(i\in I)$ with $\#I<\infty$. Then we also have a linear relation $\sum_{i\in I}c_i\iota(a^{(i)})\tau_\Omega(a^{(i)}(s))=0$, since, using expression (2.3.1), this follows from the original relation $\sum_{i=1}^\infty c_ia_i(s)=0$ and another one $\sum_{i=1}^\infty c_i=0$, which is obtained by substituting $s=0$ in the first relation. This implies that τ_Ω is extended to a linear map: $\mathbb{C}\Omega(P)\to\mathbb{C}\Omega(P)$. □

2.4. **Stability of $\Omega(P)$.**

In the present subsection, we are (mainly) concerned with following type of questions: for a given tame series P, under which assumptions on another power series Q, is $P+Q$ again tame and $\Omega(P)=\Omega(P+Q)$? Or, if $\Omega(P+Q)$ changes from $\Omega(P)$, how does it change? These sort of questions, we shall call *stability questions of* $\Omega(P)$.

We discuss some miscellaneous results related to stability questions, but we do not pursue full generalities. The results, except for the Assertion 3, are not used in the present article. Therefore, hurrying readers are suggested to skip this subsection after reading Assertion 3.

Assertion 3. *Let $Q=\sum_{n=0}^\infty q_nt^n$ converge in the disc of radius r_Q such that $r_Q>R_P$. Then $P+Q$ is tame and $\Omega(P)=\Omega(P+Q)$.*

Proof. Let c be a real number satisfying $r_Q>c>R_P$. Then, one has $\lim_{n\to\infty}q_nc^n=0$ and $c^n\geq 1/|\gamma_n|$ for sufficiently large n. This implies $\lim_{n\to\infty}\frac{\gamma_n+q_n}{\gamma_n}=1+\lim_{n\to\infty}\frac{q_n}{\gamma_n}=1$. The required properties follows. □

Assertion 4. *Let r be a positive real number with $r<R_P$. If $\Omega_1(P)\cap\{z\in\mathbb{C}:|z|=r\}=\emptyset$. Then there exists a power series $Q(t)$ of radius r_Q of convergence equal to r such that $P{+}Q$ is tame and $\Omega(P{+}Q)\not\subset\Omega(P)$.*

Proof. We define the coefficients of $Q(t)=\sum_{n=0}^\infty q_nt^n$ by the following conditions: $|q_n|=r^{-n}$ and $\arg(q_n)=\arg(\gamma_n)$. Then, for tameness of $P+Q$, we have to show some positive bounds $0<U\leq A_n\leq V$ for $A_n=|\frac{\gamma_{n-1}+q_{n-1}}{\gamma_n+q_n}|$. Since $|\gamma_n+q_n|=|\gamma_n|+r^{-n}$, we have $A_n=\frac{|\gamma_{n-1}/\gamma_n|+r/(|\gamma_n|r^n)}{1+1/(|\gamma_n|r^n)}$. Then, evaluating term-wisely in the numerator, one gets $A_n\leq v+r=:V$. On the other hand, according as $1\geq 1/(|\gamma_n|r^n)$ or not, we have $A_n\geq u/2$ or $A_n\geq r/2$. So, we may put $U:=\min\{u/2,r/2\}$.

Let us find a particular element $d\in\Omega(P+Q)$ such that $d\notin\Omega(P)$. For a small positive real number ε satisfying the inequality $(1-\varepsilon)/r>1/R_P$, there exists an increasing infinite sequence of integers n_m $(m\in$

$\mathbb{Z}_{\geq 0}$) such that $((1-\varepsilon)/r)^{n_m} > |\gamma_{n_m}|$ for $m \in \mathbb{Z}_{\geq 0}$. Choosing suitably a sub-sequence (denoted by the same n_m), we may assume that $X_{n_m}(P+Q)$ converges to an element, say d, in $\Omega(P+Q)$. Its kth coefficient d_k is equal to the limit of the sequence $(\gamma_{n_m-k}+q_{n_m-k})/(\gamma_{n_m}+q_{n_m})$ for $n_m \to \infty$. For each fixed n_m, dividing the numerator and the denominator by q_{n_m}, we get an expression $(X+r^kY)/(Z+1)$ where $|X| = |\gamma_{n_m-k}/\gamma_{n_m}| \cdot |\gamma_{n_m}r^{n_m}| \leq v^k \cdot (1-\varepsilon)^{n_m}$ (for $n >> k$), $Y \in S^1$, and $|Z| = |\gamma_{n_m}r^{n_m}| < (1-\varepsilon)^{n_m}$. Thus, taking the limit $n_m \to \infty$, we have $X \to 0$, $Y \to e^{i\theta_k}$ for some $\theta_k \in \mathbb{R}$ and $Z \to 0$ so that $d_k = r^k e^{i\theta_k}$. On the other hand, we see that $d \notin \Omega(P)$, since $\iota(d) = re^{i\theta_1} \notin \Omega_1(P)$ by assumption. □

We do not use following Assertion in the present paper, since we know more precise information for the cases $\#\Omega(P) < \infty$. However, it may have a significance when we study the general case with $\#\Omega(P) = \infty$.

Assertion 5. *An opposite series converges with radius* $1/\sup\{|a| : a \in \Omega_1(P)\} \leq 1/R_P$.

Proof. Let $a(s) = \lim_{m\to\infty} X_{n_m}(P)$ for an increasing sequence $\{n_m\}_{m\in\mathbb{Z}_{\geq 0}}$ be an opposite series. By the Cauchy-Hadmard theorem, the radius of convergence of a is given by

$$r_a = 1/\overline{\lim_{k\to\infty}} |a_k|^{1/k} = 1/\overline{\lim_{k\to\infty}}\, |\lim_{m\to\infty} \gamma_{n_m-k}/\gamma_{n_m}|^{1/k},$$

where RHS is bounded from below by $1/\sup\{|a| : a \in \Omega_1(P)\}$ from below. □

Question. When can we replace $\sup\{|a| : a \in \Omega_1(P)\}$ by R_P?

Finally, we state a result, which is not related to the stability.

Assertion 6. *For any positive integer* m*, we have the equality*

$$\Omega(P) \;=\; \Omega\left(\tfrac{d^m P}{dt^m}\right) \tag{2.4.1}$$

which is equivariant with the action of τ_Ω

Proof. It is sufficient to show the case $m = 1$. We show slightly a stronger statement: *the subsequence* $\{X_{n_m}(P)\}_{m\in\mathbb{Z}_{\geq 0}}$ *converges to a series* $a(s)$ *if and only if* $\{X_{n_m}\left(\frac{dP}{dt}\right)\}_{m\in\mathbb{Z}_{\geq 0}}$ *also converges to* $a(s)$.

For an increasing sequence $\{n_m\}_{m\in\mathbb{Z}_{\geq 0}}$ and for any fixed $k \in \mathbb{Z}_{\geq 0}$, the convergence of the sequence $\frac{\gamma_{n_m-k}}{\gamma_{n_m}}$ to c is equivalent to the convergence of the sequence $\frac{(n_m-k)\gamma_{n_m-k}}{n_m\gamma_{n_m}} = (1-k/n_m)\frac{\gamma_{n_m-k}}{\gamma_{n_m}}$ to the same c. □

3. Finite rational accumulation

We show that, if $\Omega(P)$ is a finite set, then it has a strong structure, which we call the *finite rational accumulation* (§3.2 Lemma and its Corollary). The whole sequel of the present paper focuses on its study.

3.1. **Finite rational accumulation.**

We start with a preliminary concept of rational subsets of $\mathbb{Z}_{\geq 0}$, and then introduce the concept of *finite rational accumulation.*

Definition. **1.** A subset U of $\mathbb{Z}_{\geq 0}$ is called a *rational subset* if the sum $U(t) := \sum_{n\in U} t^n$ is the Taylor expansion at 0 of a rational function in t.

2. A *finite rational partition* of $\mathbb{Z}_{\geq 0}$ is a finite collection $\{U_a\}_{a\in\Omega}$ of rational subsets $U_a \subset \mathbb{Z}_{\geq 0}$ indexed by a finite set Ω such that there is a finite subset D of $\mathbb{Z}_{\geq 0}$ so that one has the disjoint decomposition

$$\mathbb{Z}_{\geq 0} \setminus D = \amalg_{a\in\Omega}(U_a \setminus D).$$

Assertion 7. *For any rational subset U of $\mathbb{Z}_{\geq 0}$, there exist a positive integer h, a subset $u \subset \mathbb{Z}/h\mathbb{Z}$ and a finite subset $D \subset \mathbb{Z}_{\geq 0}$ such that $U \setminus D = \cup_{[e]\in u} U^{[e]} \setminus D$, where $[e] \in \mathbb{Z}/h\mathbb{Z}$ is the class of $e \in \mathbb{Z}$ and*

$$U^{[e]} := \{n \in \mathbb{Z}_{\geq 0} \mid n \equiv e \bmod h\}. \tag{3.1.1}$$

We call $\cup_{[e]\in u} U^{[e]}$ the standard expression of U.

Proof. The fact that $U(t)$ is rational implies that the characteristic function χ_U of U is recursive, i.e. there exist $N \in \mathbb{Z}_{\geq 1}$ and numbers $\alpha_1, \cdots, \alpha_N$ such that one has the recursive relation $\chi_U(n) + \chi_U(n-1)\alpha_1 + \cdots + \chi_U(n-N)\alpha_N = 0$ for sufficiently large $n \gg 0$. Since the range of χ_U is finite (i.e. $\{0,1\}$), there are only finite possible patterns of values of χ on an interval $[n-N, n]$ for $n \gg 0$. Therefore, there exists two large numbers $n > m \gg 0$ such that $\chi_U(n-i) = \chi_U(m-i)$ for $i = 0, \cdots, N$. Due to the recursive relation, this means that χ_U is $h := (n-m)$-periodic after m. □

Corollary. *Any finite rational partition of $\mathbb{Z}_{\geq 0}$ has a subdivision of the form $\mathcal{U}_h := \{U^{[e]}\}_{[e]\in\mathbb{Z}/h\mathbb{Z}}$ for some $h \in \mathbb{Z}_{>0}$, called a period of the partition. The smallest period h is called the period of the partition, and $\mathcal{U}_h$ is called the standard subdivision of the partition.*

In the present paper, the concept of a finite rational partition of $\mathbb{Z}_{\geq 0}$ is used twice: once, in the following definition of a finite rational accumulation, and once in the definition of a rational operator in §5.

Definition. A sequence $\{X_n\}_{n\in\mathbb{Z}_{\geq 0}}$ in a Hausdorff space is *finite rationally accumulating* if the sequence accumulates to a finite set, say Ω, such that for a system of open neighborhoods $\mathcal{V}_a$ for $a \in \Omega$ with $\mathcal{V}_a \cap \mathcal{V}_b = \emptyset$ if $a \neq b$, the system $\{U_a\}_{a\in\Omega}$ for $U_a := \{n \in \mathbb{Z}_{\geq 0} \mid X_n \in \mathcal{V}_a\}$ is a finite rational partition of $\mathbb{Z}_{\geq 0}$. The (resp. a) period of the partition is called the (resp. a) *period of the finite rationally accumulation set* Ω.

3.2. **τ_Ω-periodic point in $\Omega(P)$.**

Generally speaking, finiteness of the accumulation set Ω of a sequence does not imply that it is finite rationally accumulating (see §3.5 Example a). Therefore, the following theorem says a distinguished property of the accumulation set $\Omega(P)$. This justifies the introduction of the concept of "finite rational accumulation".

Theorem. *Let $P(t)$ be a tame power series in t. If the τ_Ω-action on $\Omega(P)$ has an isolated periodic point, then $\Omega(P)$ is a finite rational accumulation set, whose period h_P is equal to $\#\Omega(P)$. We have a natural bijection:*

$$\begin{array}{ccc} \mathbb{Z}/h_P\mathbb{Z} & \simeq & \Omega(P) \\ e \bmod h_P & \mapsto & a^{[e]} := \lim_{n\to\infty} X_{e+h_P\cdot n}(P), \end{array} \tag{3.2.1}$$

where the standard subdivision $\mathcal{U}_{h_P}$ of the partition of $\mathbb{Z}_{\geq 0}$ is the exact partition for the space $\Omega(P)$ of the opposite series of P. The shift action $[e] \mapsto [e-1]$ in LHS is equivariant with the τ_Ω action in RHS.

Proof. Assumption means that i) there exists an element $a \in \Omega(P)$ and a positive integer $h \in \mathbb{Z}_{>0}$ such that $(\tau_\Omega)^h a = a \neq (\tau_\Omega)^{h'} a$ for $0 < h' < h$ and ii) there exists an open neighbourhood $\mathcal{V}_a$ of a such that $\Omega(P) \cap \mathcal{V}_a = \{a\}$. Since $\Omega(P)$ is a compact Hausdorff space, it is a regular space. So, we may assume further that $\Omega(P) \cap \overline{\mathcal{V}_a} = \{a\}$. Then, by putting $U_a := \{n \in \mathbb{Z}_{\geq 0} \mid X_n(P) \in \mathcal{V}_a\}$, the sequence $\{X_n(P)\}_{n \in U_a}$ converges to the unique limit element a. By the definition of τ_Ω in §2, the relation $(\tau_\Omega)^h a = a$ implies that the sequence $\{X_{n-h}(P)\}_{n \in U_a}$ converges to a. That is, there exists a positive number N such that for any $n \in U_a$ with $n > N$, $X_{n-h}(P) \in \mathcal{V}_a$, and hence $n-h$ belongs to U_a.

Consider the set $A := \{[e] \in \mathbb{Z}/h\mathbb{Z} \mid$ there are infinitely many elements of U_a which are congruent to $[e]$ modulo h $\}$. Actually, if $[e] \in A$, then U_a contains $U^{[e]} \cap \mathbb{Z}_{\geq N}$ (*Proof.* For any $m \in \mathbb{Z}_{\geq N}$ with $m \bmod h \equiv [e]$, there exists an integer $m' \in U_a$ such that $m' > m$ and $m' \bmod h = [e]$ by the definition of the set A. Then, by the definition of N, $m' - h \in U_a$. Obviously, either $m' - h = m$ or $m' - h > m$ occurs. If $m' - h > m$ then we repeat the argument so that $m' - 2h \in U_a$. Repeating, similar steps, after finite k-steps, we show that $m' - kh = m \in U_a$).

Thus, U_a is, up to a finite number of elements, equal to the rational set $\cup_{[e] \in A} U^{[e]}$. This implies $A \neq \emptyset$. Consider the rational set $U_{(\tau_\Omega)^i a} := \{n - i \mid n \in U_a\}$ for $i = 0, 1, \cdots, h-1$. Due to §2.3 Assertion 2, $\{X_n(P)\}_{n \in U_{(\tau_\Omega)^i a}}$ converges to $(\tau_\Omega)^i a$. By the definition, $U_{(\tau_\Omega)^i a}$ is, up to a finite number of elements, equal to the rational set $\cup_{[e] \in A} U^{[e-i]}$. By assumption $a \neq \tau_\Omega^i a$ for $0 \leq i < h$, there should not be an infinite intersection between two rational sets $U_{(\tau_\Omega)^i a}$ $(0\leq i < h)$ so that we have $\#A=1$, say $A=\{[e_0]\}$ and $U_{(\tau_\Omega)^i a}=U^{[e_0-i]}$ up to a finite number of elements. On the other hand, since the union $\cup_{i=0}^{h-1}U_{(\tau_\Omega)^i a}$ already covers $\mathbb{Z}_{\geq 0}$ up to finite elements and since each $\{X_n(P)\}_{n\in U_{(\tau_\Omega)^i a}}$ converges only to $(\tau_\Omega)^i a$, the opposite sequence (2.2.1) can have no other accumulating point than the set $\{a,\tau_\Omega a,\cdots,(\tau_\Omega)^{h-1}a\}$. That is, $\Omega(P)$ is a finite rational accumulation set with the h_P-periodic action of τ_Ω. ☐

Corollary. *If $\Omega(P)$ is a finite set, then it is automatically a finite rational accumulation set with the presentation* (3.2.1)*.*

Proof. If $\Omega(P)$ is finite, then any point is isolated and the action τ_Ω should have a periodic point. ☐

3.3. **Example by Machì** [M]**.**

Let $\Gamma:=\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/3\mathbb{Z}\simeq \mathrm{PSL}(2,\mathbb{Z})$ with the generator system $G:=\{a,b^{\pm1}\}$ where a,b are the generators of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, respectively. Then, the number $\#\Gamma_n$ of elements of Γ expressed by the words in the letters G of length less or equal than n for $n\in\mathbb{Z}_{\geq0}$ is given by

$$\#\Gamma_{2k}=7\cdot 2^k-6 \quad\text{and}\quad \#\Gamma_{2k+1}=10\cdot 2^k-6 \quad \text{for } k\in\mathbb{Z}_{\geq0}.$$

Therefore, we get the following expression of the growth function:

$$P_{\Gamma,G}(t) \quad := \quad \textstyle\sum_{k=0}^{\infty}\#\Gamma_k t^k \quad = \quad \frac{(1+t)(1+2t)}{(1-2t^2)(1-t)}.$$

Then, we see that $\Omega_1(P_{\Gamma,G})$ and, hence, $\Omega(P_{\Gamma,G})$ are finite rationally accumulating of period 2. Explicitly, they are given as follows.

$$\Omega_1(P_{\Gamma,G})=\Big\{a_1^{[0]}:=\lim_{n\to\infty}\tfrac{\#\Gamma_{2n-1}}{\#\Gamma_{2n}}=\tfrac{5}{7},\ a_1^{[1]}:=\lim_{n\to\infty}\tfrac{\#\Gamma_{2n}}{\#\Gamma_{2n+1}}=\tfrac{7}{10}\Big\}$$

$$\Omega(P_{\Gamma,G})=\Big\{\ a^{[0]}(s)\ ,\ a^{[1]}(s)\ \Big\}$$

where

$$a^{[0]}(s):=\sum_{k=0}^{\infty}2^{-k}s^{2k}+\frac{5}{7}s\sum_{k=0}^{\infty}2^{-k}s^{2k}$$
$$=\frac{(1+\frac{5}{7}s)}{(1-\frac{s^2}{2})}=\frac{1}{2}\cdot\frac{1+\frac{5}{7}\sqrt{2}}{1-\frac{s}{\sqrt{2}}}+\frac{1}{2}\cdot\frac{1-\frac{5}{7}\sqrt{2}}{1+\frac{s}{\sqrt{2}}},$$

$$a^{[1]}(s):=\sum_{k=0}^{\infty}2^{-k}s^{2k}+\frac{7}{10}s\sum_{k=0}^{\infty}2^{-k}s^{2k}$$
$$=\frac{(1+\frac{7}{10}s)}{(1-\frac{s^2}{2})}=\frac{1}{2}\cdot\frac{1+\frac{7}{5}\frac{1}{\sqrt{2}}}{1-\frac{s}{\sqrt{2}}}+\frac{1}{2}\cdot\frac{1-\frac{7}{5}\frac{1}{\sqrt{2}}}{1+\frac{s}{\sqrt{2}}}.$$

In §5.4, these coefficients of fractional expansion are recovered due to §5.3 Theorem ii). We calculate also $r_P^2=R_P^2=a_1^{[0]}a_1^{[1]}=\frac{5}{7}\frac{7}{10}=\frac{1}{2}$.

3.4. **Simply accumulating Examples.**

A tame power series $P(t)$ is called *simply accumulating* if $\#\Omega(P)=1$ (e.g. growth functions $P_{\Gamma,G}(t)$ for surface groups [C]). Growth functions for Artin monoids are simply accumulating, which enables to determine the F-function of the Cayley graph (Γ, G) [S2, S3, S4].

3.5. **Miscellaneous Examples.**

Before going further, using a simple model of oscillating sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$, we give some examples of the power series $P(t)$ such that

a) $\Omega_1(P)$ is finite but is not finite rationally accumulating,

b) $\Omega_1(P)$ is finite rationally accumulating but $\#\Omega_1(P)<\#\Omega(P)$,

c) $\Omega(P)\neq\Omega(P+Q)$ for a power series $Q(t)$ for any $R_P>r_Q>r_P$.

We do not use these results in sequel. Hurrying readers may skip present paragraph.

With a triple $\mathfrak{U}:=(U,a,b)$, where $U\subset\mathbb{Z}_{\geq 1}$ is any subset such that $\#U=\infty$ and $\#(U^c:=\mathbb{Z}_{\geq 1}\setminus U)=\infty$ and $a,b\in\mathbb{C}\setminus\{0\}$, we associate a sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ defined by an induction on n: $\gamma_0:=1$ and $\gamma_n:=\gamma_{n-1}\cdot a$ if $n\in U$ and $\gamma_{n-1}\cdot b$ if $n\notin U$. Put $P_{\mathfrak{U}}(t):=\sum_{n=0}^{\infty}\gamma_n t^n$. Then:

Fact i) *The $P_{\mathfrak{U}}(t)$ is tame and $\Omega_1(P_{\mathfrak{U}})=\{a^{-1},b^{-1}\}$.*

ii) *The $P_{\mathfrak{U}}(t)$ is finite rational accumulating if and only if U is rational.*

Proof. i) The inequalities: $\min\{|a|,|b|\}\leq|\gamma_n/\gamma_{n-1}|\leq\max\{|a|,|b|\}$ imply the tameness of $P_{\mathfrak{U}}$. The latter half is trivial since the proportion γ_n/γ_{n-1} takes only the values a or b.

ii) This follows from: $P_{\mathfrak{U}}$ is rational $\Leftrightarrow$ The sets $\{n\in\mathbb{Z}_{\geq 1}\mid\gamma_n/\gamma_{n-1}=a\}=U$ and $\{n\in\mathbb{Z}_{\geq 1}\mid\gamma_n/\gamma_{n-1}=b\}=U^c$ are rational$\Leftrightarrow$ U is rational. $\square$

a) By choosing a non-rational set U, we obtain an example a).

b) Even U (and, hence, U^c also) is a rational set, if $\{U,U^c\}$ is not the standard partition of $\mathbb{Z}_{\geq 0}$ of period 2, then the period of the partition $\{U,U^c\}=\#\Omega(P_{\mathfrak{U}})>2=\#\Omega_1(P_{\mathfrak{U}})$. This gives an example b).

c) To get an example c), we need a bit more consideration. Define $p_U:=\overline{\lim_{n\to\infty}}\frac{\#(U\cap\mathbb{Z}_{1\leq\cdot\leq n})}{n}$ and $q_U:=\underline{\lim_{n\to\infty}}\frac{\#(U\cap\mathbb{Z}_{1\leq\cdot\leq n})}{n}$. If U is a rational subset, then $p_U=q_U$ is a rational number. In general, the pair (p_U,q_U) can be any of $\{(p,q)\in[0,1]^2\mid p\geq q\}$. Suppose $|a|\geq|b|$.

$$1/r_P:=\overline{\lim_{n\to\infty}}\,|a|^{\frac{\#(U\cap\mathbb{Z}_{1\leq\cdot\leq n})}{n}}\cdot|b|^{1-\frac{\#(U\cap\mathbb{Z}_{1\leq\cdot\leq n})}{n}}=|a|^{p_U}|b|^{1-p_U},$$

$$1/R_P:=\underline{\lim_{n\to\infty}}\,|a|^{\frac{\#(U\cap\mathbb{Z}_{1\leq\cdot\leq n})}{n}}\cdot|b|^{1-\frac{\#(U\cap\mathbb{Z}_{1\leq\cdot\leq n})}{n}}=|a|^{q_U}|b|^{1-q_U}.$$

Thus, r_P and R_P can take any values, satisfying: $|a|^{-1}\leq r_P\leq R_P\leq|b|^{-1}$. If there is a gap $r_P<R_P$, then for any $r\in\mathbb{R}_{>0}$ such that $r_P<r<R_P$, $Q(t):=\sum_{n=0}^{\infty}e^{i\theta_n}(t/r)^n$ for $\theta_n=\#(U\cap\mathbb{Z}_{1\leq\cdot\leq n})\arg(a)+(n-\#(U\cap\mathbb{Z}_{1\leq\cdot\leq n}))\arg(b)$ gives example c) (since $\Omega_1(P_{\mathfrak{U}})\cap\{z\in\mathbb{C}:|z|=r\}=\emptyset$ and §2.4 Assertion4).

4. Rational expression of opposite series

From this section, we restrict our attention only to a tame power series having the finite rational accumulation set $\Omega(P)$.

4.1. **Rational expression.**

We show that the opposite series become a rational function of a special form, whose analysis is the theme of the present section.

We start with a characterization of a finite rational accumulation.

Assertion 8. *Let $P(t)$ be a tame power series in t. The set $\Omega(P)$ is a finite rationally accumulation set of period $h_P \in \mathbb{Z}_{\geq 1}$ if and only if $\Omega_1(P)$ is so. We say P is finite rationally accumulating of period h_P.*

Proof. If $\Omega(P)$ is finite rationally accumulating, then, in particular, the sequence $\frac{\gamma_{n-1}}{\gamma_n}$ is finite rationally accumulating. To show the converse and to show the coincidence of the periods, assume that $\{\gamma_{n-1}/\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ accumulate finite rationally of period h_1. Consider the standard subdivision $\mathcal{U}_{h_1} := \{U^{[e]}\}_{[e]\in\mathbb{Z}/h_1\mathbb{Z}}$ (recall §3.1 Corollary), and let the subsequence $\{\gamma_{n-1}/\gamma_n\}_{n\in U^{[e]}}$ converge to $a_1^{[e]} \in \mathbb{C}$ for $[e] \in \mathbb{Z}/h_1\mathbb{Z}$.

For any $k\in\mathbb{Z}_{\geq 0}$ and sufficiently large (depending on k) n, one has

$$\frac{\gamma_{n-k}}{\gamma_n} \ = \ \frac{\gamma_{n-1}}{\gamma_n}\frac{\gamma_{n-2}}{\gamma_{n-1}}\cdots\frac{\gamma_{n-k}}{\gamma_{n-k+1}}.$$

For $n\in U^{[e]}$ with $[e]\in\mathbb{Z}/h_1\mathbb{Z}$, we see that RHS converges to $a_1^{[e]}a_1^{[e-1]}\dots a_1^{[e-k+1]}$. Then, for $[e]\in\mathbb{Z}/h_1\mathbb{Z}$ and $k\in\mathbb{Z}_{\geq 0}$, by putting

$$a_k^{[e]} \ := \ a_1^{[e]}a_1^{[e-1]}\dots a_1^{[e-k+1]}, \tag{4.1.1}$$

the sequence $\{X_n(P)\}_{n\in U_{[e]}}$ converges to $a^{[e]} := \sum_{k=0}^{\infty} a_k^{[e]}s^k$ with $a_1^{[e]} = \iota(a^{[e]})$ so that $\Omega(P)$ is finite rational accumulating. Its period h_P is a divisor of h_1, but it cannot be strictly smaller than h_1, since otherwise the sequence $\{\gamma_{n-1}/\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ gets a period shorter than h_1. □

Remark. That the period of the rational accumulation of $\Omega_1(P)$ is equal to h does not imply $\#\Omega_1(P)=h$. That is, the map $a\in\Omega(P)\mapsto\iota(a)\in\Omega_1(P)$ is surjective but may not be injective (see §4.2 Example b).

Assertion 9. *Let P be finite rationally accumulating of period $h_P \in \mathbb{Z}_{\geq 1}$. Then the opposite series $a^{[e]} = \sum_{k=0}^{\infty} a_k^{[e]}s^k$ in $\Omega(P)$ associated with the rational subset $U^{[e]}$ converges to a rational function*

$$a^{[e]}(s) \ = \ \frac{A^{[e]}(s)}{1-A_P s^{h_P}}, \tag{4.1.2}$$

where the numerator $A^{[e]}(s)$ is a polynomial in s of degree h_P-1:

(4.1.3) $$A^{[e]}(s) \ := \ \sum_{j=0}^{h_P-1} \left(\prod_{i=1}^{j} a_1^{[e-i+1]}\right) s^j$$

and

(4.1.4) $$A_P := \prod_{i=0}^{h_P-1} a_1^{[i]} = a_{h_P}^{[0]} = \cdots = a_{h_P}^{[h_P-1]}.$$

We have a relation

(4.1.5) $$(r_P)^{h_P} \ = \ (R_P)^{h_P} \ = \ |A_P|,$$

where r_P is the radius of convergence of $P(t)$ and R_P is given by (2.1.4).

Proof. Due to the h_P-periodicity of the sequence $a_1^{[e]}$ $(e \in \mathbb{Z})$, formula (4.1.1) implies the "*semi-periodicity*" with respect to the factor (4.1.4):

$$a_{mh_P+k}^{[e]} = (A_P)^m a_k^{[e]} \quad \text{for } m \in \mathbb{Z}_{\geq 0}, \ k = 0, \cdots, h_P-1.$$

This implies a factorization $a^{[e]} = A^{[e]} \cdot \sum_{m=0}^{\infty} (A_P s^{h_P})^m$ and hence (4.1.2).

To show (4.1.5), it is sufficient to show the existence of positive real constants c_1 and c_2 such that for any $k \in \mathbb{Z}_{\geq 0}$ there exists $n(k) \in \mathbb{Z}_{\geq 0}$ and for any integer $n \geq n(k)$, one has $c_1 r^k \leq \left|\frac{\gamma_{n-k}}{\gamma_n}\right| \leq c_2 r^k$.

Proof. Choose $c_1, c_2 \in \mathbb{C}_{>0}$ satisfying $c_1 < \min\{\left|\frac{a_i^{[e]}}{r^i}\right| \mid [e] \in \mathbb{Z}/h\mathbb{Z}, i \in \mathbb{Z} \cap [0, h-1]\}$ and $c_2 > \max\{\left|\frac{a_i^{[e]}}{r^i}\right| \mid [e] \in \mathbb{Z}/h\mathbb{Z}, i \in \mathbb{Z} \cap [0, h-1]\}$. □

This completes a proof of Assertion 9. □

Corollary. *Let $\Omega(P)$ be finite. For any power series $Q(t)$ of radius r_Q of convergence larger than r_P, $P+Q$ is tame and $\Omega(P) = \Omega(P+Q)$.*

4.2. **Linear dependence relations among opposite series.**

Though the opposite series $a^{[e]}(s)$ for $[e] \in \mathbb{Z}/h_P\mathbb{Z}$ are mutually distinct, they may be linearly dependent. This phenomenon occurs when the matrix

(4.2.1) $$M_h := \left(\prod_{i=1}^{f} a_1^{[e-i+1]}\right)_{e,f \in \{0,1,\cdots,h-1\}}$$

of the coefficients of (4.1.3) degenerates, i.e. $\det(M_h) = 0$. Regarding $a_1^{[0]}, \cdots, a_1^{[h-1]}$ as variables, $D_h(a_1^{[0]}, \cdots, a_1^{[h-1]}) := \det(M_h) \in \mathbb{Z}[a_1^{[0]}, \cdots, a_1^{[h-1]}]$ is an irreducible homogeneous polynomial of degree $h(h-1)/2$ with sign changes $D_h \circ \sigma = (-1)^{h-1} D_h$ under the cyclic permutation $\sigma = (1, \cdots, h-1)$ of the variables.

In the present paragraph, we show a formula (4.2.4) on the rank of the matrix M_h, where we may take an arbitrary coefficient field K. In particular, for the case of $K = \mathbb{R}$, we give a stratification of the positive real parameter space $(\mathbb{R}_{>0})^h$ of the parameter $(a_1^{[0]}, \cdots, a_1^{[h-1]})$, where each stratum is labeled by the cyclotomic polynomial i.e. an integral factor of $1-s^h$ which contains also the factor $1-s$ (see Assertion 10.iv).

Assertion 10. *Fix* $h \in \mathbb{Z}_{>0}$. *Using expressions* (4.1.3) *and* (4.1.4), *define polynomials* $A^{[e]}(s)$ *indexed by* $[e]\in\mathbb{Z}/h\mathbb{Z}$ *and a constant* $A\in K^\times$ *associated with any* h*-tuple* $\bar{a}=(a_1^{[0]},\cdots,a_1^{[h-1]})\in(K^\times)^h$.

i) *In* $K[s]$, *we have the equality of the greatest common divisors:*

$$\begin{aligned} & \gcd(A^{[0]}(s),1-As^h) &= \cdots &= \gcd(A^{[h-1]}(s),1-As^h) \\ = & \gcd(A^{[0]}(s),A^{[1]}(s)) &= \cdots &= \gcd(A^{[h-1]}(s),A^{[h]}(s)) \end{aligned}$$

after normalizing their constant terms to be equal to 1.

Let us denote by $\delta_{\bar{a}}(s)$ *the common divisor, and we put*

$$\Delta^{op}_{\bar{a}}(s) := (1-As^h)/\delta_{\bar{a}}(s). \tag{4.2.2}$$

ii) *For* $[e]\in\mathbb{Z}/h\mathbb{Z}$, *put*

$$b^{[e]}(s) := A^{[e]}(s)/\delta_{\bar{a}}(s). \tag{4.2.3}$$

The polynomials $b^{[e]}(s)$ *for* $[e]\in\mathbb{Z}/h\mathbb{Z}$ *span the space* $K[s]_{<\deg(\Delta^{op}_{\bar{a}})}$ *of polynomials of degree less than* $\deg(\Delta^{op}_{\bar{a}})$. *Hence, one has the equality:*

$$\operatorname{rank}(M_h) \ = \ \deg(\Delta^{op}_{\bar{a}}). \tag{4.2.4}$$

iii) *For* $\varphi(s)\in K[s]$, $\varphi(s)\mid\Delta^{op}_{\bar{a}}$ *if and only if* $\varphi(s)\mid 1-As^h$ *and* $\gcd(\varphi(s),A^{[e]}(s))=1$. *In particular, if* $\bar{a}\in(\mathbb{R}_{>0})^h$, *then* $\Delta^{op}_{\bar{a}}$ *is always divisible by* $1-\sqrt[h]{A}s$.

iv) *Let* $h\in\mathbb{Z}_{>0}$. *There exists a stratification* $\mathbb{R}^h_{>0}=\amalg_{\Delta^{op}}C_{\Delta^{op}}$, *where the index set is equal to*

$$\{\Delta^{op}\in\mathbb{R}[s] : 1-s \mid \Delta^{op}(s) \mid 1-s^h \ \& \ \Delta^{op}(0)=1\}, \tag{4.2.5}$$

and $C_{\Delta^{op}}$ *is a smooth semi-algebraic set of* $\mathbb{R}$*-dimension* $\deg(\Delta^{op})-1$, *such that* $\Delta^{op}_{\bar{a}}(s)=\Delta^{op}(\sqrt[h]{A}s)$ *for* $\forall\bar{a}\in C_{\Delta^{op}}$ *and* $\overline{C_{\Delta^{op}_1}}\supset C_{\Delta^{op}_2}\Leftrightarrow\Delta^{op}_1|\Delta^{op}_2$

Proof. i) By Definitions (4.1.3), (4.1.4) and (4.1.1), we have relations:

$$a_1^{[e+1]}sA^{[e]}(s)+(1-As^h) \ = \ A^{[e+1]}(s) \tag{4.2.6}$$

for $[e]\in\mathbb{Z}/h\mathbb{Z}$. This implies $\gcd(A^{[e]}(s),1-As^h)\mid\gcd(A^{[e+1]}(s),1-As^h)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ so that one concludes that all the elements $\gcd(A^{[e]}(s),1-As^h)=\gcd(A^{[e]}(s),A^{[e+1]}(s))$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ are the same up to a constant factor. It is obvious that a factor of $1-As^h$ contains a nontrivial constant term.

ii) Let V be the subspace of $K[s]/(\Delta^{op}_{\bar{a}})$ spanned by the images of $b^{[e]}(s):=A^{[e]}(s)/\delta_{\bar{a}}(s)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$. Relation (4.2.6) implies that V is closed under the multiplication of s. On the other hand, $b^{[e]}(s)$ and $\Delta^{op}_{\bar{a}}$ are relatively prime so that they generate 1 as a $K[s]$-module. That is, V contains the class [1] of 1. Hence, $V=K[s]\cdot[1]=K[s]/(\Delta^{op}_{\bar{a}})$. Since $\deg(b^{[e]}(s))=h-1-\deg(\delta_{\bar{a}}(s))=\deg(\Delta^{op}_{\bar{a}})-1$, $V\cap K[s]\Delta^{op}_{\bar{a}}=0$. This means that the polynomials $b^{[e]}(s)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ span the space of polynomials of degree less than $\deg(\Delta_{\bar{a}}^{op})$. In particular, one has $\mathrm{rank}(M_h)=\mathrm{rank}_K V=\deg(\Delta_{\bar{a}}^{op})$.

iii) The first half is the reformulation of the definition of $\delta_{\bar{a}}$ and (4.2.2). Then we see that if $1-rs \not| \Delta_{\bar{a}}^{op})$ then $1-rs \mid A^{[e]}(s)$ (4.2.3) so that $A^{[e]}(1/r)=0$. This is impossible, since all coefficients of $A^{[e]}$ and $1/r$ are positive reals.

iv) Let Δ^{op} be a polynomial as given in (4.2.5) and put $d=\deg(\Delta^{op})$. Consider the set $\overline{C}_{\Delta^{op}} := \{c(s) = 1+c_1s+\cdots+c_{d-1}s^d \in \mathbb{R}[s] \mid \exists r \in \mathbb{R}_{>0}$ s.t. all coefficients of $A_c^{[0]} := c(s)(1-r^hs^h)/\Delta^{op}(rs)$ are positive$\}$. Then $\overline{C}_{\Delta^{op}}$ is an open semi-algebraic set in $\mathbb{R}^d$, which is a non-empty since $\Delta^{op}(rs)/(1-rs)$ belongs to $\overline{C}_{\Delta^{op}}$. In particular, it is pure dimensional of dimension $\dim_{\mathbb{R}} \overline{C}_{\Delta^{op}} = d-1$. To any $c \in \overline{C}_{\Delta^{op}}$, one can associate a unique $\bar{a} \in (\mathbb{R}_{>0})^h$ such that the associated polynomial $A^{[0]}$ (4.1.3) is equal to $A_c^{[0]}$. We identify $\overline{C}_{\Delta^{op}}$ with the semi-algebraic subset $\{a \in (\mathbb{R}_{>0})^h \mid a \leftrightarrow c \in \overline{C}_{\Delta^{op}}\}$ of pure dimension $d-1$ embedded in $(\mathbb{R}_{>0})^h$. Similarly, for any factor Δ' of Δ^{op} (over $\mathbb{R}$) divisible by $1-s$, we consider the semi-algebraic subsets $\overline{C}_{\Delta'}$ in $\mathbb{R}_{>0}^h$ of pure dimension $\deg(\Delta')$. Then, the multiplication of Δ^{op}/Δ' induces the inclusion $\overline{C}_{\Delta'} \subset \overline{C}_{\Delta^{op}}$. Then we define the semi-algebraic set $C_{\Delta^{op}}$ inductively by $\overline{C}_{\Delta^{op}} \setminus \cup_{\Delta'} C_{\Delta'}$, where the index Δ' runs over all factors of Δ^{op} which are not equal to Δ^{op} and are divisible by $1-rs$. By the induction hypothesis, $d-1 > \dim_{\mathbb{R}}(C_{\Delta'})$ so that the difference $C_{\Delta^{op}}$ is non-empty open semi-algebraic set with pure $\dim_{\mathbb{R}} C_{\Delta^{op}} = d-1$.

This completes the proof of Assertion 10. □

Suppose $char(K) \not| h$, and let $\tilde{K}$ be the splitting field of $\Delta_{\bar{a}}^{op}$ with the decomposition $\Delta_{\bar{a}}^{op} = \prod_{i=1}^d (1-x_is)$ in $\tilde{K}$ for $d := \deg(\Delta_{\bar{a}}^{op})$. Then, one has the partial fraction decomposition:

$$\text{(4.2.7)} \qquad \frac{A^{[e]}(s)}{1-As^h} \ = \ \sum_{i=1}^d \frac{\mu_{x_i}^{[e]}}{1-x_is}$$

for $[e] \in \mathbb{Z}/h\mathbb{Z}$, where $\mu_{x_i}^{[e]}$ is a constant in $\tilde{K}$ given by the residue:

$$\text{(4.2.8)} \qquad \mu_{x_i}^{[e]} \ = \ \frac{A^{[e]}(s)(1-x_is)}{1-As^h}\Big|_{s=(x_i)^{-1}} \ = \ \frac{1}{h}A^{[e]}(x_i^{-1}).$$

Corollary. *The matrix* $\left((\mu_{x_i}^{[e]})_{[e]\in\mathbb{Z}/h\mathbb{Z}, x_i^{-1}\in V(\Delta_{\bar{a}}^{op})}\right)$ *is of maximal rank* d.

Proof. LHS of (4.2.7) for $[e] \in \mathbb{Z}/h\mathbb{Z}$ span a vector space of rank $d := \deg(\Delta_{\bar{a}}^{op})$. So, the coefficient matrix in RHS has rank equal to d. □

Remark. 1. One has the equivariance $\sigma(\mu_{x_i}^{[e]}) = \mu_{\sigma(x_i)}^{[e]}$ with respect to the action $\sigma \in \mathrm{Gal}(\tilde{K}, K)$ of the Galois group of the splitting field.

2. The index x_i in (4.2.8) may run over all roots x of the equation $x^h - A = 0$. However, if $x^{-1} \notin V(\Delta_{\bar{a}}^{op})$ (i.e. $\Delta_{\bar{a}}^{op}(x^{-1}) \neq 0$), then $\mu_x^{[e]} = 0$.

4.3. **Module $\mathbb{C}\Omega(P)$.**

We return to a tame power series $P(t)$ (2.1.1). Suppose $P(t)$ is finite rationally accumulating of a period h_P. Let $a_1^{[e]}$ be the initial of the opposite series $a^{[e]} \in \Omega(P)$ for $[e] \in \mathbb{Z}/h_P\mathbb{Z}$. Since $\Delta_{\bar{a}}^{op}(s)$ (4.2.2) for $\bar{a} := (a_1^{[0]}, \cdots, a_1^{[h-1]})$ depends only on P but not on the choice of a period h_P, we shall denote it by $\Delta_P^{op}(s)$. Then, §4.2 Assertion 10.ii) says that we have the $\mathbb{C}$-isomorphism:

$$(4.3.1) \qquad \begin{array}{ccc} \mathbb{C}\Omega(P) & \simeq & \mathbb{C}[s]/(\Delta_P^{op}(s)), \\ a^{[e]} & \mapsto & b^{[e]} := \Delta_P^{op} \cdot a^{[e]} \bmod \Delta_P^{op}. \end{array}$$

Let us rewrite equality (4.2.3) and introduce the key number:

$$(4.3.2) \qquad d_P := \mathrm{rank}_{\mathbb{C}}\big(\mathbb{C}\Omega(P)\big) = \deg(\Delta_P^{op}).$$

Define an endomorphism σ on $\mathbb{C}\Omega(P)$ by letting

$$(4.3.3) \qquad \sigma(a^{[e]}) := \tau_\Omega^{-1}(a^{[e]}) = \frac{1}{a_1^{[e+1]}} a^{[e+1]}.$$

Assertion 11. *The actions of σ on LHS and the multiplication of s on RHS of* (4.3.1) *are equivariant. Hence, the linear dependence relations among the generators $a^{[e]}$ ($[e] \in \mathbb{Z}/h\mathbb{Z}$) are obtained by the linear dependence relations $\Delta_P^{op}(\sigma)a^{[e]}$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$.*

Proof. The first part of Assertion 11 is a matter of calculation.
$\sum_{[e]\in\mathbb{Z}/h\mathbb{Z}} c_{[e]} b^{[e]} \equiv 0 \bmod \Delta_P^{op}(\sigma) b^{[e]} = 0$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$. □

Note that the σ-action on $\mathbb{C}\Omega(P)$ is not $s|_{\mathbb{C}\Omega(P)}$ in the ring $\mathbb{C}[[s]]$.

5. Duality theorem

In this section, we restrict the class of function $P(t)$ to that of analytically continuable to a meromorphic function in a neighbourhood of the closed disc of convergence.[2] Under this setting, we show a duality between $\Omega(P)$ and poles of $P(t)$ on the boundary of the disc.

[2]This assumption is necessary, since *the finite rational accumulation of $P(t)$ does not imply that $P(t)$ is meromorphic on the boundary of its convergent disc.*

Example. Consider the function $P(t) := \sqrt{\frac{1+t}{1-t}} = \sum_{n=0}^{\infty} \frac{(n-1)!}{2^n [n/2]! [(n-1)/2]!} t^n$ which is tame. We see that the sequence of the proportion γ_{n-1}/γ_n of its coefficients accumulates to the unique values 1, i.e. $\Omega_1(P) = \{1\}$ and $\Omega(P) = \{1/(1-s)\}$. On the other hand, we watch that the function $P(t)$ has two singular points on the boundary of the unit disc $D(0,1)$ which are not meromorphic but algebraic. Such algebraic branching cases shall be treated in a forthcoming paper.

5.1. **Functions of class $\mathbb{C}\{t\}_r$.**

For $r \in \mathbb{R}_{>0}$, we introduce a class

$$(5.1.1)\qquad \mathbb{C}\{t\}_r := \left\{ P(t) \in \mathbb{C}[[t]] \;\middle|\; \begin{array}{l} \text{i) } P(t) \text{ converges on the open disc } D(0,r). \\ \text{ii) } P(t) \text{ is analytically continuable to a meromorphic} \\ \quad \text{function on an open neighbourhood of } \overline{D(0,r)}. \end{array} \right\}$$

For an element $P(t)$ of $\mathbb{C}\{t\}_r$, let us introduce a monic polynomial $\Delta_P(t)$, called the *polar part polynomial* of $P(t)$, characterized by

i) $\Delta_P(t)P(t)$ is holomorphic in a neighbourhood of the circle $|t| = r$,

ii) $\Delta_P(t)$ has lowest degree among all polynomials satisfying i).

Next, we decompose

$$(5.1.2)\qquad \Delta_P(t) = \textstyle\prod_{i=1}^{N} (t - x_i)^{d_i}$$

where x_i $(i=1,\cdots,N,\ N \in \mathbb{Z}_{\geq 0})$ are mutually distinct complex numbers with $|x_i| = r$ and $d_i \in \mathbb{Z}_{>0}$ $(i=1,\cdots,N)$.

Definition. The *top polar part polynomial* $\Delta_P^{top}(t)$ of $P(t)$ is defined by

$$(5.1.3)\qquad \Delta_P^{top}(t) := \textstyle\prod_{i,d_i=d_m} (t - x_i) \quad \text{where} \quad d_m := \max\{d_i\}_{i=1}^{N}.$$

Note that $\Delta_P(t)$ may be equal to 1, and then $\Delta_P^{top}(t) = 1$. The converse: *if* $\Delta_P(t) \neq 1$, *then* $\Delta_P^{top}(t) \neq 1$, is also true.

5.2. **The rational operator T_U.**

We introduce an linear operator T_U on $\mathbb{C}\{t\}_r$ associated with a rational subset U of $\mathbb{Z}_{\geq 0}$, which we call a *rational operator* or a *rational action* of U.

Definition. The action T_U on $\mathbb{C}[[t]]$ of a rational subset U of $\mathbb{Z}_{\geq 0}$ is

$$(5.2.1)\qquad T_U \ : \ P = \textstyle\sum_{n \in \mathbb{Z}_{\geq 0}} \gamma_n t^n \quad \mapsto \quad T_U P := \sum_{n \in U} \gamma_n t^n.$$

One may regard $T_U P$ as a product of P with the rational function $U(t)$ (§3.1 Definition) in the sense of Hadamard [?]. Clearly, the radius of convergence of $T_U P$ is not less than that of P.

Assertion 12. *The action of T_U preserves the space $\mathbb{C}\{t\}_r$ for any $r \in \mathbb{R}_{>0}$. The highest order of the poles on $|t| = r$ of $T_U f$ does not exceed that of $f \in \mathbb{C}\{t\}_r$.*

Proof. For $P \in \mathbb{C}\{t\}_r$, let us consider its partial fractional expansion:

$$(5.2.2)\qquad P(t) = \textstyle\sum_{i=1}^{N} \sum_{j=1}^{d_i} \frac{c_{i,j}}{(t-x_i)^j} \ + \ Q(t),$$

where x_i is a place of a pole of order $d_i > 0$ with $|x_i| = r$ for $i = 1,\ldots,N$, $c_{i,j}$ are constants $\in \mathbb{C}$ with $c_{i,d_i} \neq 0$ for $i = 1,\ldots,N$, and $Q(t)$ is a holomorphic function on a disc of radius $> r$. Then, $T_U P = \sum_{i,j} T_U \frac{c_{i,j}}{(t-x_i)^j} + T_U Q$ where $T_U Q$ is a holomorphic function on a disc of radius $> r$. It is sufficient to show the result for each term $T_U \frac{1}{(t-x_i)^j}$ when U is a standard rational set $U^{[e]} := \{n \in \mathbb{Z}_{\geq 0} \mid n \equiv [e] \bmod h\}$ of period $h \in \mathbb{Z}_{>0}$ and $[e] \in \mathbb{Z}/h\mathbb{Z}$ (recall (3.1.1)). Let us show

Assertion 13. *For $h \in \mathbb{Z}_{\geq 0}$ and $[e] \in \mathbb{Z}/h\mathbb{Z}$, let us define the rational operator $T^{[e]} := T_{U^{[e]}}$. Then, we have*

$$T^{[e]} \cdot \frac{d}{dt} = \frac{d}{dt} \cdot T^{[e+1]} \tag{5.2.3}$$

$$T^{[e]} \frac{1}{(t-x_i)^j} = \frac{B_{i,j}(t)}{(t^h - x_i^h)^j}, \tag{5.2.4}$$

where $B_{i,j}(t)$ is a homogeneous polynomial in t and x_i of degree $(h-1)j$.

Proof. As for (5.2.3): for any monomial t^m ($m \in \mathbb{Z}_{\geq 0}$), both hand sides returns the same $mt^{m-1}\delta_{[e],[m-1]} = mt^{m-1}\delta_{[e+1],[m]}$.

As for (5.2.4): using (5.2.3), we calculate

$$\begin{aligned} T_{U^{[e]}} \tfrac{1}{(t-x_i)^j} &= T_{U^{[e]}} \tfrac{(-1)^{j-1}}{(j-1)!} (\tfrac{d}{dt})^{j-1} \tfrac{1}{t-x_i} \\ &= \tfrac{(-1)^{j-1}}{(j-1)!} (\tfrac{d}{dt})^{j-1} T_{U^{[e+j-1]}} \tfrac{1}{t-x_i} = \tfrac{(-1)^{j-1}}{(j-1)!} (\tfrac{d}{dt})^{j-1} \tfrac{t^f x^g}{t^h - x_i^h} \end{aligned}$$

where $f := e+j-1-h[(e+j-1)/h]$ and $g := h-f-1 = h-e-j+h[(e+j-1)/h]$. □

Expression (5.2.3) implies Assertion 12. In particular, the latter half of the statement follows from the fact that the equation $t^h - x_i^h = 0$ does not have a multiple root (in characteristic 0).

This completes the proof of Assertion 12. □

5.3. Duality theorem.

The following is the goal of the present paper.

Theorem. (Duality) *Let $P(t)$ be a tame power series belonging to $\mathbb{C}\{t\}_r$ for $r = r_P$ (= the radius of convergence of P). Suppose that $P(t)$ is finite rationally accumulating of period h_P. Then*

i) *The denominator $\Delta_P^{op}(s)$ (4.2.2) of opposite sequences and the top part $\Delta_P^{top}(t)$ (5.1.3) of $P(t)$ are opposite to each other. That is,*

$$\deg_t(\Delta_P^{top}(t)) = d_P = \deg_s(\Delta_P^{op}(s)), \tag{5.3.1}$$

and

$$t^{d_P} \Delta_P^{op}(t^{-1}) = \Delta_P^{top}(t), \text{ equivalently } s^{d_P} \Delta_P^{top}(s^{-1}) = \Delta_P^{op}(s). \tag{5.3.2}$$

ii) *We have an equality of transition matrices:*

(5.3.3)

$$\Big(\frac{P(t)}{T^{[e]}P(t)}\Big|_{t=x_i} \Big)_{[e] \in \mathbb{Z}/h_P\mathbb{Z},\ x_i \in V(\Delta_P^{top}(t))} = \Big(A^{[e]}\Big|_{s=x_i^{-1}} \Big)_{[e] \in \mathbb{Z}/h_P\mathbb{Z},\ x_i^{-1} \in V(\Delta_P^{op}(s))}.$$

In particular, $\Big(\frac{P(t)}{T^{[e]}P(t)}\Big|_{t=x_i} \Big)_{[e] \in \mathbb{Z}/h_P\mathbb{Z},\ x_i \in V(\Delta_P^{top}(t))}$ *is of maximal rank d_P.*

Proof. We start with the following obvious remark.

Assertion 14. *Let $c \in \mathbb{C}^\times$ be any non-zero complex constant. Change the variable t to $\tilde{t} := t/c$ and the opposite variable s to $\tilde{s} := cs$, and, for any tame series P, define a new tame series $\tilde{P} := P|_{t=c\tilde{t}}$.*

Then we have,

$$\begin{aligned} \Omega(\tilde{P}) &= \Omega(P)|_{s=\tilde{s}/c} := \{a(\tilde{s}/c) \mid a(t) \in \Omega(P)\} , \\ \Omega_1(\tilde{P}) &= \Omega_1(P)/c := \{a_1/c \mid a_1 \in \Omega_1(P)\}. \end{aligned}$$

Proof. The equalities follows immediately by direct calculations. □

According to this Assertion, we prove the theorem by changing the variable t to $\tilde{t} = t/c$ for $c = \sqrt[h_P]{A_P}$ (recall (4.1.4)) so that new tame series has the constant $A_{\tilde{P}}$ equal to 1. Therefore, from now on, in the present proof, we shall assume that P is a finite rationally accumulating tame series with $A_P = 1$. In particular, this implies that the radius r_P of convergence of P is equal to 1 (recall (4.1.5)). Hence, we have $|x_i| = 1$ for all the places of poles in expression (5.2.2).

We first prove the theorem for a special but the key case when $\#\Omega(P) = 1$.

Assertion 15. *If $P(t)$ is simply accumulating then $\Delta_P^{top} = t - 1$.*

Proof. We apply stability: Corollary to §4.1 Assertion to the partial fractional expansion (5.2.2), so that we obtain $\Omega(P) = \Omega(P - Q)$. That is, the principal part $P_0 := P–Q$ gives arise a simply accumulating power series. That is, $X_n(P_0) = \sum_{k=0}^{n} \frac{\sum_{i=1}^{N} \sum_{1\le j\le d_m} c_{i,j} x_i^{k-n-1}(n-k;j)/(j-1)!}{\sum_{i=1}^{N} \sum_{1\le j\le d_m} c_{i,j} x_i^{-n-1}(n;j)/(j-1)!} s^k$ $(n = 0, 1, 2, \cdots)$ converges to $\frac{1}{1-s} = \sum_{k=0}^{\infty} s^k$. Then, under this assumption, we'll show that if $c_{i,d_m} \neq 0$ then $x_i = 1$.

For each fixed $k \in \mathbb{Z}_{\ge 0}$, the numerator and the denominator of the coefficient of s^k in $X_n(P_0)$ are polynomials in n of degree $\le d_m$. Let $v_n := \sum_{i=1}^{N} c_{i,d_m} x_i^{-n-1}$ be the coefficients of the top term $n^{d_m}/(d_m - 1)!$ in the denominator. Since the range of v_n is bounded (i.e. $|v_n| \le \sum_i |c_{i,d_m}|$ due to the assumption $|x_i| = 1$), the sequence for $n = 0, 1, 2, \cdots$ accumulates to a non-empty compact set in $\mathbb{C}$.

First, consider the case when the sequence $\{v_n\}_{n \in \mathbb{Z}_{ge0}}$ has a unique accumulating value v_0. Let us show that v_0 is non-zero and the result of Assertion is true. (*Proof.* The mean sequence: $\{(\sum_{n=0}^{M-1} v_n)/M\}_{M \in \mathbb{Z}_{>0}}$ also converges to $v_0 = \lim_{n\to\infty} v_n$. This means that $\sum_{i=1}^{N} c_{i,d_m} \frac{\sum_{n=0}^{M-1} x_i^{-n-1}}{M}$ converges to v_0. If $x_i \neq 1$, the mean sum $\frac{\sum_{n=0}^{M-1} x_i^{-n-1}}{M} = \frac{1 - x_i^{-M}}{(x_i - 1)M}$ tends to 0 as $M \to \infty$. That is, $v_0 = c_{1,d_m}$, where we assume $x_1 = 1$ (even if, possibly $c_{1,d_m} = 0$). That is, the sequence $v'_n := v_n - c_{1,d_m} =$

$\sum_{i=2}^{N} c_{i,d_m} x_i^{-n-1}$ converges to 0. For a fixed $n_0 \in \mathbb{Z}_{>0}$, consider the relations: $v'_{n_0+k} = \sum_{i=2}^{N}(c_{i,d_m}x_i^{-n_0})x_i^{-k+1}$ for $k = 1, \cdots, N-1$. Regarding $c_{i,d_m}x_i^{-n_0}$ $(i = 2,\cdots, N)$ as the unknown, we can solve the linear equation for them, since Vandermonde determinant for the matrix $(x_i^{-k+1})_{i=2,\cdots,N,k=1,\cdots,N-1}$ does not vanish. So, we obtain a linear approximation: $|c_{i,d_m}| = |c_{i,d_m}x_i^{-n_0}| \le c \cdot \max\{|v'_{n_0+k}|\}_{k=1}^{N-1}$ $(i = 2,\cdots, N)$ for a constant $c > 0$ which depends only on $x_i's$ and N but not on n_0. The RHS tend to zero as $n_0 \to \infty$, whereas LHS are unchanged. This implies $|c_{i,d_m}| = 0$, i.e. $d_i < d_m$ for $i = 2,\cdots, N$. As we have already remarked $\Delta_P(t) \ne 1$ implies $\Delta_P^{top}(t) := \prod_{d_i=d_m}(t - x_i) \ne 1$, and hence c_{1,d_m} cannot be 0. So $\Delta_P^{top}(t) = t - 1$.

Next, consider the case when the sequence v_n has more than two accumulating values. Then, one of them is non-zero. Suppose the subsequence $\{v_{n_m}\}_{m\in\mathbb{Z}_{>0}}$ converges to a non-zero value, say c. Recall the assumption that the sequence γ_{n-1}/γ_n converges to 1. So, the subsequence $\frac{\gamma_{n_m-1}}{\gamma_{n_m}} = \frac{v_{n_m-1}+\text{lower terms}}{v_{n_m}+\text{lower terms}}$ should also converges to 1 as $m \to \infty$. In the denominator, the first term tends to $c \ne 0$ and the second term (= (a polynomial in n of degree d_m–1)$/n^{d_m}$) tends to zero. Similarly, in the numerator, the second term tends to zero. This implies that the first term in the numerator also converges to $c \ne 0$. Repeating the same argument, we see that for any $k \in \mathbb{Z}_{\ge 0}$, the subsequence $\{v_{n_m-k}\}_{m\in\mathbb{Z}_{\gg 0}}$ converges to the same c. Then, for each fixed $M \in \mathbb{Z}_{>0}$, the average sequence $\{(\sum_{k=0}^{M-1} v_{n_m-k})/M\}_{m\in\mathbb{Z}_{\gg 0}}$ converges to c, whereas the values is given by $\sum_{i=2}^{N} c_{i,d_m} x_i^{-n_m} \frac{1-x_i^{-M}}{(1-x_i^{-1})M} + c_{1,d_m}$ which is close to c_{1,d_m} for sufficiently large M and $n_m \gg M$. This implies $c = c_{1,d_m}$. Thus, the sequences $\{v'_{n_m-k} = \sum_{i=2}^{N} c_{i,d_m} x_i^{n_m-k}\}_{m\in\mathbb{Z}_{\gg 0}}$ for any $k \ge 0$ converge to 0. Then, an argument similar to that of the previous case implies $|c_{i,d_m}| = 0$, i.e. $d_i < d_m$ $(i = 2,\cdots, N)$. Hence we have $\Delta_P^{top}(t) = t - 1$.

The proof of Assertion 15 is complete. □

We return to the proof of the general case, when P is finite rational accumulating of period h, but may no longer be simply accumulating.

The rational operators $T^{[e]} := T_{U^{[e]}}$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ give a partition of unity:

$$\sum_{[e]\in\mathbb{Z}/h\mathbb{Z}} T^{[e]} = 1.$$

Since h is a period of P, the series $T^{[f]}P = t^f \sum_{m=0}^{\infty} \gamma_{f+mh}\tau^m$ for any $0 \le f < h$, considered as a series in $\tau = t^h$ where t^f is regarded as a constant factor, is simple accumulating such that $\Omega_1(T^{[f]}P) = \{1\}$ (since $\lim_{m\to\infty} \gamma_{f+(m-1)h}/\gamma_{f+mh} = 1$). Then Assertion 15 implies that the highest order poles of $T^{[f]}P$ (in the variable τ) are only at solutions x of the equation $\tau-1=0$, i.e. $t^h-1=0$, where the equation is common for all $[f]\in\mathbb{Z}/h\mathbb{Z}$. In view of the fact that the highest order of poles of $T^{[f]}P$ cannot exceed that of P (recall Assertion 12) and the fact $P=\sum_{[e]\in\mathbb{Z}/h\mathbb{Z}}T^{[e]}P$ where poles at t^h-1 do not cancel each other out since each term has distinct factor t^f such that $0\le f<h=\deg(t^h-1)$, the highest order poles of P are also only at solutions x of the equation $t^h-1=0$. That is; $\Delta_P^{top}(t)$ is a factor of t^h-1.

For $0\le e,f<h$, the value of the proportion $\frac{T^{[f]}P}{T^{[e]}P}(t)$ at a root x of the equation t^h-1 is defined by cancelling the poles. The value is the limit of the proportion of the values of functions at the sequence of points in the variable t (resp. τ) converging to x (resp. $x^h=1$) from inside the convergence disc $|t|<1$ (resp. $|\tau|<1$). Thus, we obtain:

$$*)\qquad \frac{T^{[f]}P}{T^{[e]}P}(t)\Big|_{t=x} = x^{f-e}\lim_{\tau\to1}\frac{\sum_{m=0}^{\infty}\gamma_{f+mh}\tau^m}{\sum_{m=0}^{\infty}\gamma_{e+mh}\tau^m}$$

where the second factor of RHS may be considered as the evaluation of the power series in τ at $\tau=1$. In order to calculate this value, we prepare an elementary Fact.

Fact. *Let* $A(\tau)=\sum_{m=0}^{\infty}a_m\tau^m, B(\tau)=\sum_{m=0}^{\infty}b_m\tau^m\in\mathbb{C}\{\tau\}_1$ *such that their highest order poles of the same order d exist only at $\tau=1$. Then,*

$$**)\qquad \frac{A(\tau)}{B(\tau)}\Big|_{\tau=1} = \lim_{m\to\infty}\frac{a_m}{b_m}.$$

Proof. Replacing t and c_{ij} in (5.2.2) with τ and a_{ij} or b_{ij}, respectively, RHS of $**$) is written as $\lim_{m\to\infty}\frac{\sum_{i=1}^N\sum_{j\le d}a_{i,j}x_i^{k-m-1}(m-k;j)/(j-1)!}{\sum_{i=1}^N\sum_{j\le d}b_{i,j}x_i^{-m-1}(m;j)/(j-1)!}$. The numerator and denominator are polynomials in m of degree d so that the limit is the proportion $a_{1,d}/b_{1,d}$ of the coefficients of $(\tau-1)^{-d}$ in the fractional expansions of A and B, which is equal to LHS of $**$) □

Applying this Fact, RHS of $*$) is equal to $x^{f-e}\lim_{m\to\infty}\frac{\gamma_{f+mh}}{\gamma_{e+mh}}$. Then, applying to this expression a similar argument for (4.1.1), we obtain:

$$(5.3.4)\qquad \frac{T^{[f]}P}{T^{[e]}P}(t)\Big|_{t=x} = \begin{cases} x^{f-e}/a_1^{[f]}a_1^{[f-1]}\cdots a_1^{[e+1]} & \text{if } e<f\\ 1 & \text{if } e=f\\ x^{f-e}a_1^{[e]}a_1^{[e-1]}\cdots a_1^{[f+1]} & \text{if } e>f.\end{cases}$$

Since RHS are non-zero, this implies that the order of the poles of $T^{[e]}P(t)$ at a solution x of the equation t^h-1 is independent of $[e]\in\mathbb{Z}/h\mathbb{Z}$. Summing up BHS of (5.3.4) for $0\le f<h$, we obtain

$$(5.3.5)\qquad \frac{P}{T^{[e]}P}(t)\Big|_{t=x} = A^{[e]}(x^{-1}).$$

(recall the $A^{[e]}(s)$ (4.1.3)). Let x be a solution of $t^h - r^h = 0$ but $\Delta_P^{op}(x^{-1}) \neq 0$. Then $\delta_a(x^{-1}) = 0$ (see (4.2.2)) and $A^{[e]}(x^{-1}) = 0$ for all $[e] \in \mathbb{Z}/h\mathbb{Z}$ (see Assertion 10. i)). That is, $\frac{T^{[e]}P}{P}(t)$ has a pole at $t = x$. This implies that $P(t)$ cannot have a pole of order d_m at $t = x$ (otherwise, due to Assertion 12, the pole at $t = x$ of $T^{[e]}P$ is at most of order d_m, which is cancelled in $\frac{T^{[e]}P}{P}(t)$ by dividing by P. A contradiction!). That is, we get one division relation.

Assertion 16. $\Delta_P^{top}(t) \mid t^{d_P}\Delta_P^{op}(t^{-1})$ *and* $\deg(\Delta_P^{top}) \leq d_P$.

Finally, let us show the opposite division relation.

Assertion 17. *Let $P(t)$ be a tame power series belonging to $\mathbb{C}\{t\}_r$, which is finite rational accumulating of period h. Then*

i) There exists a constant $c \in \mathbb{R}_{>0}$ such that $|\gamma_n| \geq c r^{-n} n^{d_m}$ for $n \gg 0$.

ii) $$t^d \Delta_P^{op}(t^{-1}) \mid \Delta_P^{top}(t) \ .$$

Proof. i) Consider the Taylor expansion of the partial fractional (5.2.2)). Using notation v_n in Assertion 15, we have $\gamma_n = -v_n \frac{r^{-n-1}(n;d_m)}{(d_m-1)!}$ +(terms coming from poles of order $< d_m$)+(terms coming from $Q(t)$), where $v_n = \sum_i c_{i,d_m}(x_i/r)^{-n-1}$ depends only on n mod h since x_i is the root of the equation $t^h - r^h = 0$. They cannot all be zero (otherwise, by solving the equations $v_n = 0$ $(0 \leq n < h)$, we get $c_{i,d_m} = 0$ for all i, which contradicts to the vanishing of d_m). Let us show that none of the v_n is zero. Suppose the contrary and $v_e = 0 \neq v_f$ for some integers $0 \leq e, f < h$. Then, one observes easily that $\lim_{m\to\infty} \frac{\gamma_{e+mh}}{\gamma_{f+mh}} = 0$. This contradicts to formula (5.3.4) and the non-vanishing of $a_1^{[e]}$ $([e] \in \mathbb{Z}/h\mathbb{Z})$.

ii) By definition, the fractional expansion of $\Delta_P^{top}(t)P(t)$ has poles of order at most $d_m - 1$. Put $\Delta_P^{top}(t) = t^l + \alpha_1 t^{l-1} + \cdots + \alpha_l$. Then, this means that the sequence $\{\gamma_N\}$ (Taylor coefficients of P) satisfies

$$***) \qquad \gamma_N \cdot \alpha_l + \gamma_{N-1} \cdot \alpha_{l-1} + \cdots + \gamma_{N-l} \cdot 1 \quad \sim \quad o(N^{d_m} r^{-N})$$

as $N \to \infty$. Let $\sum_k a_k s^k \in \Omega(P)$ be an opposite series given by a sequence $\{X_{n_m}(P)\}_{m \in \mathbb{Z}_{\geq 0}}$ (2.2.1). For each fixed $k \in \mathbb{Z}_{\geq l}$, substitute N by $n_m - k + l$ in $***$) and divide it by γ_{n_m}. Then, taking the limit $m \to \infty$ using the part i), RHS converges to 0, so that we get

$$a_{k-l}\alpha_l + a_{k-l+1}\alpha_{l-1} + \cdots + a_k \ = \ 0.$$

Thus $s^l \Delta_P^{top}(s^{-1}) a(s)$ is a polynomial of degree $< l$ and the denominator $\Delta_P^{op}(s)$ of $a(s)$ divides $s^l \Delta_P^{top}(s^{-1})$. So, $d_P \leq l$ and ii) is proved.

This completes a proof of Assertion 17. □

The proof of the theorem: (5.3.1) and (5.3.2) are already shown by Assertions 16 and 17, and (5.3.3) is shown by (4.2.8) and (5.3.5). □

5.4. **Example by Machì (continued).**

Recall §3.3 Machì's example, where we learned that the growth function $P_{\Gamma,G}(t)=\sum_{n=0}^{\infty}\#\Gamma_n t^n$ for the modular group $\Gamma=\mathrm{PSL}(2,\mathbb{Z})$ with respect to a generator system G is equal to $\frac{(1+t)(1+2t)}{(1-2t^2)(1-t)}$ and that it is finite rational accumulating of period $h=2$.

Using this data, we calculate further the rational actions on it.

$$T^{[0]}P_{\Gamma,G}(t)=\quad \sum_{k=0}^{\infty}\#\Gamma_{2k}t^{2k}\quad =\frac{1+5t^2}{(1-2t^2)(1-t^2)},$$

$$T^{[1]}P_{\Gamma,G}(t)=\sum_{k=0}^{\infty}\#\Gamma_{2k+1}t^{2k+1}=\frac{2t(2+t^2)}{(1-2t^2)(1-t^2)},$$

The denominator polynomial for the opposite series $a^{[e]}$ ($[e]\in\mathbb{Z}/2\mathbb{Z}$) and the top polar part polynomial of $P_{\Gamma,G}(t)$ are given as follows.

$$\Delta^{op}_{P_{\Gamma,G}}(s)=1-\frac{1}{2}s^2 \qquad \& \qquad \Delta^{top}_{P_{\Gamma,G}}(t)=t^2-\frac{1}{2}.$$

Then the transformation matrix is given by

$$\begin{bmatrix} \frac{P_{\Gamma,G}(t)}{T^{[0]}P(t)}=\frac{(1+t)^2(1+2t)}{1+5t^2}\big|_{t=\frac{1}{\sqrt{2}}} & \frac{P_{\Gamma,G}(t)}{T^{[1]}P(t)}=\frac{(1+t)^2(1+2t)}{2t(2+t^2)}\big|_{t=\frac{1}{\sqrt{2}}} \\ \frac{P_{\Gamma,G}(t)}{T^{[0]}P(t)}=\frac{(1+t)^2(1+2t)}{1+5t^2}\big|_{t=\frac{-1}{\sqrt{2}}} & \frac{P_{\Gamma,G}(t)}{T^{[1]}P(t)}=\frac{(1+t)^2(1+2t)}{2t(2+t^2)}\big|_{t=\frac{-1}{\sqrt{2}}} \end{bmatrix}=\begin{bmatrix} 1+\frac{5}{7}\sqrt{2} & 1+\frac{7}{5}\frac{1}{\sqrt{2}} \\ 1-\frac{5}{7}\sqrt{2} & 1-\frac{7}{5}\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Actually, *this matrix coincides with the matrix* $2\cdot\big(\mu^{[e]}_{x_i}\big)_{[e]\in\mathbb{Z}/2\mathbb{Z},x_i\in\{\pm\sqrt{2}^{-1}\}}$ (4.2.8), which was already calculated in §3.3 Example as the coefficient of fractional expansion of opposite series $a^{[0]}$ and $a^{[1]}$. In particular, its determinant, equal to $\frac{\sqrt{2}}{35}$, is non-zero. The matrix is an essential ingredient of the trace formula for limit F-functions [S1, (11.5.6)]

References

[C] Cannon, J.W.: The growth of the closed surface groups and the compact hyperbolic Coxeter groups (unpublished, 1980's)

[H] Hadamard, J.: Théoreme sur les séries entieres, Acta math. **22** (1899), 55-63.

[M] Machì, Antonio: The growth function for PSL(2, $\mathbb{Z}$), Chap.VI 7. of the book "Topics in geometric group theory" by de la Harpe, the university of Chicago Press, ISBN: 0-226-31719-6 (paper), ISBN 0-226-31721-8 (paper)

[S1] Saito, Kyoji: Limit Elements in the Configuration Algebra for a Discrete monoid, Publ. RIMS Kyoto Univ. **46** (2010), 31-113. DOI 10.2977/PRIMS/2

[S2] Saito, Kyoji: Growth functions associated with Artin monoids of finite type, Proc. Japan Acad. Ser. A **84** (2008), 179-183. Zbl 1159.20330 MR 2483563

[S3] Saito, Kyoji: Growth functions for Artin monoids, Proc. Japan Acad. Ser. A **85** (2009), 84-88. Zbl pre05651160 MR 2548018

[S4] Saito, Kyoji: Growth F-function for canncellative monoids, submitted.

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