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A posteriori estimates of inverse operators for boundary value problems in linear elliptic partial differential equations

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Abstract

This paper presents constructive a posteriori estimates of inverse operators for boundary value problems in linear elliptic partial differential equations (PDEs) on a bounded domain. This type of estimates plays an important role in the numerical verification of the solutions for boundary value problems in nonlinear elliptic PDEs. In general, it is not easy to obtain the a priori estimates of the operator norm for inverse elliptic operators. Even if we can obtain these estimates, they are often over estimated. Our proposed a posteriori estimates are based on finite-dimensional spectral norm estimates for the Galerkin approximation and expected to converge to the exact operator norm of inverse elliptic operators. This provides more accurate estimates, and more efficient verification results for the solutions of nonlinear problems.

1 Introduction

The main aim of this paper is to provide the positive constant C_{L^2,H_0^1} satisfying the operator norm:

$$\left\|(-\Delta + b\cdot\nabla + c)^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)} \le C_{L^2,H_0^1}. \tag{1}$$

Here, $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ is a bounded polygonal or polyhedral domain, $b \in L^\infty(\Omega)^d$, $c \in L^\infty(\Omega)$. $H_0^1(\Omega) := \left\{u \in H^1(\Omega)\ ;\ u = 0 \text{ on } \partial\Omega\right\}$ is a Hilbert space with respect to the inner product is $(u,v)_{H_0^1(\Omega)} := (\nabla u, \nabla v)_{L^2(\Omega)^d}$ and the norm is $\|u\|_{H_0^1(\Omega)} := (u,u)^{\frac{1}{2}}_{H_0^1(\Omega)}$. The constant C_{L^2,H_0^1} plays an essential role in the verification of the solutions for the boundary value problems in nonlinear elliptic partial differential equations (PDEs) [8, 9] and must be numerically determined.

By defining $\mathscr{L} := -\Delta + b\cdot\nabla + c$, the problem of obtaining the estimates of (1) is equivalent to the norm estimation of the solution u for the following boundary value problems in linear elliptic PDEs such that

$$\begin{cases} \mathscr{L}u = f, & \text{in } \Omega, \quad (2a) \\ u = 0, & \text{on } \partial\Omega, \quad (2b) \end{cases}$$

for arbitrary $f \in L^2(\Omega)$. Here, the weak solution $u \in H_0^1(\Omega)$ of (2a) and (2b) is defined by the following variational equation:

$$L(u,v) = (f,v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega), \tag{3}$$

for a bilinear form $L : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$L(u,v) := (\nabla u, \nabla v)_{L^2(\Omega)^d} + ((b \cdot \nabla)u, v)_{L^2(\Omega)} + (cu, v)_{L^2(\Omega)}.$$

If we assume the coercivity of L, then by the Lax-Milgram theorem, there exists a unique solution for (3), indicating the existence of the inverse of $\mathscr{L}$. Nakao-Hashimoto-Watanabe [6] proposed the validated computational technique that demonstrates the existence of $\mathscr{L}^{-1}$ even if the coercivity of L is not assumed. They also derived a technique for obtaining the estimates of (1). In section 3, we introduce these results and discuss them in more detail.

However, the estimates of $\mathscr{L}^{-1}$ in [6] have an unavoidable lower bound. In this study, we propose a novel technique to obtain a posteriori estimates of (1) using $\mathscr{L}_h^{-1}$ that is defined by the Galerkin approximate integral operator for $\mathscr{L}^{-1}$. Our new approach has no restricted lower bound; therefore, it is expected that we can obtain C_{L^2,H_0^1} smaller than that of [6]. Moreover, we introduce a posteriori error estimates for $\mathscr{L}^{-1}$ and $\mathscr{L}_h^{-1}$.

The contents of this paper are as follows: In section 2, we introduce the necessary function spaces and calculate the a priori error estimates for their Galerkin approximations. In section 3, we present previously reported methods of error estimation. In section 4, we propose a posteriori estimates of (1). In section 5, we propose a posteriori error estimates for $\mathscr{L}^{-1}$ and $\mathscr{L}_h^{-1}$. Note that in this study, the term "a posteriori error estimates" is defined as the operator norm for integral operators. This suggests that these error estimates can be calculated whenever the Galerkin approximate spaces are given. Therefore, they do not depend on f. In section 6, we compare the constants given by [6] and propose a new value of C_{L^2,H_0^1} for the test problems.

2 Function spaces and Galerkin approximation

In this section, we introduce the function spaces and constructive error estimates of projections to finite dimensional subspaces. Let $X(\Omega) := \left\{u \in L^2(\Omega)\ ;\ \Delta u \in L^2(\Omega)\right\}$ be a Banach space with respect to the norm $\|u\|_{X(\Omega)} := \|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}$. We again define the linear elliptic partial differential operator $\mathscr{L} : H_0^1(\Omega) \cap X(\Omega) \to L^2(\Omega)$ by $\mathscr{L} := -\Delta + b \cdot \nabla + c$. The norms of Banach space $L^\infty(\Omega)^d$ and $L^\infty(\Omega)$ are defined by

$$\|b\|_{L^\infty(\Omega)^d} := \operatorname*{ess\,sup}_{x\in\Omega} \sqrt{b_1(x)^2 + \cdots + b_d(x)^2}, \quad \|c\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |c(x)|.$$

The following Theorem 2.1 is the Sobolev inequality.

Theorem 2.1 (Sobolev inequality) *Let the constant p satisfy $1 \le p \le 2^*$, where 2^* is the Sobolev conjugate index defined by $2^* := \frac{2d}{d-2}$. Then, there exists a positive constant $C_{s,p} > 0$ such that*

$$\|u\|_{L^p(\Omega)} \le C_{s,p} \|u\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{4}$$

Let $S_h(\Omega)$ be an approximate finite dimensional subspace of $H_0^1(\Omega)$ dependent on the parameter h. For example, $S_h(\Omega)$ is considered to be a finite element subspace with the mesh size h or a set of the finite polynomial expansion with polynomial degree. Let n be a degree of freedom for $S_h(\Omega)$ and ϕ_i be the basis function of $S_h(\Omega)$. This indicates that $S_h(\Omega) := \text{span}_{1\le i\le n}\{\phi_i\}$.

We denote the symmetric positive definite matrices D_ϕ and L_ϕ in $\mathbb{R}^{n,n}$ by

$$D_{\phi,i,j} := \left(\nabla\phi_j, \nabla\phi_i\right)_{L^2(\Omega)^d}, \quad 1 \le i, j \le n, \tag{5}$$

$$L_{\phi,i,j} := \left(\phi_j, \phi_i\right)_{L^2(\Omega)}, \quad 1 \le i, j \le n. \tag{6}$$

Let $D_\phi^{1/2}$ and $L_\phi^{1/2}$ be the Cholesky factors of D_ϕ and L_ϕ, respectively, i.e.,

$$D_\phi = D_\phi^{1/2} D_\phi^{T/2}, \quad \text{and} \quad L_\phi = L_\phi^{1/2} L_\phi^{T/2}.$$

We define the H_0^1 projection $P_h^1 : H_0^1(\Omega) \to S_h(\Omega)$ by

$$\left(u - P_h^1 u, v_h\right)_{H_0^1(\Omega)} = 0, \quad \forall v_h \in S_h(\Omega). \tag{7}$$

Therefore, the problems of the solvability of the variational equation (7) and the nonsingularity of D_ϕ become equivalent. Because the matrix D_ϕ is positive definite, the projection P_h^1 is well defined. Similarly, we define the L^2 projection $P_h^0 : L^2(\Omega) \to S_h(\Omega)$ by

$$\left(u - P_h^0 u, v_h\right)_{L^2(\Omega)} = 0, \quad \forall v_h \in S_h(\Omega). \tag{8}$$

Now, we assume that the following estimates of P_h^1 hold.

Assumption 2.2 *There exist a positive constant $C(h) > 0$ satisfying*

$$\left\|u - P_h^1 u\right\|_{H_0^1(\Omega)} \le C(h)\left\|\Delta u\right\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega) \cap X(\Omega), \tag{9}$$

$$\left\|u - P_h^1 u\right\|_{L^2(\Omega)} \le C(h)\left\|u - P_h^1 u\right\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{10}$$

Assumption 2.2 is the most basic error estimates in the Galerkin method. For example, in the case of a finite element space used piecewise bilinear polynomial approximation of $H_0^1(\Omega)$, the value $C(h)$ is known by $C(h) = \frac{h}{\pi}$. Alternatively, in the case of piecewise biquadratic polynomial approximation, Assumption 2.2 is satisfied by $C(h) = \frac{h}{2\pi}$. Moreover, these approximations give the optimal constants (e.g., [5]). In the case of N degree polynomial approximation is used, Assumption 2.2 is satisfied by $C(h) = O(\frac{h}{N})$. However, in these cases, the optimal constants are unknown (e.g., [3]).

For arbitrary $f \in L^2(\Omega)$, we define the Galerkin approximate solution $u_h \in S_h(\Omega)$ of (3) such that

$$(\nabla u_h, \nabla v_h)_{L^2(\Omega)^d} + ((b\cdot\nabla)u_h, v_h)_{L^2(\Omega)} + (cu_h, v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)}, \quad \forall v_h \in S_h(\Omega). \tag{11}$$

Let G_ϕ be a matrix in $\mathbb{R}^{n,n}$, where each element is defined by

$$G_{\phi,i,j} := L(\phi_j,\phi_i) = \left(\nabla\phi_j,\nabla\phi_i\right)_{L^2} + \left((b\cdot\nabla)\phi_j,\phi_i\right)_{L^2} + \left(c\phi_j,\phi_i\right)_{L^2}, \quad 1\le i,j\le n. \tag{12}$$

Then, the nonsingularity of G_ϕ and the unique existence of the solution u_h in (11) become equivalent. Therefore, we assume the nonsingularity of G_ϕ. However, when applying the proposed a posteriori estimates, it is necessary to confirm the nonsingularity of G_ϕ by validated computations.

Next, we define the L projection $P_h^L : H_0^1(\Omega) \to S_h(\Omega)$ by

$$L(u - P_h^L u, v_h) = 0, \quad \forall v_h \in S_h(\Omega). \tag{13}$$

From the nonsingularity of G_ϕ, P_h^L is well defined. If for an arbitrary $f \in L^2(\Omega)$ there exists u that is a unique solution for (3), then we denote the operator $\mathscr{L}^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ by $u = \mathscr{L}^{-1} f$. By defining the operator $\mathscr{L}_h^{-1} : L^2(\Omega) \to S_h(\Omega)$, we obtain u_h, the solution of (11). Thus, we obtain $\mathscr{L}_h^{-1} = P_h^L \mathscr{L}^{-1}$ from the definition of P_h^L.

3 Known results

In this section, we introduce the result for the invertibility condition of the operator $\mathscr{L}$ and its previously determined estimates. We define the following constants:

$$\begin{aligned}
C_1 &:= \|b\|_{L^\infty(\Omega)^d} + C_{s,2}\|c\|_{L^\infty(\Omega)}, \quad K_1(h) := C(h)\left(C_{s,2}\|\mathrm{div}\, b\|_{L^\infty(\Omega)} + C_1\right),\\
C_2 &:= \|b\|_{L^\infty(\Omega)^d} + C(h)\|c\|_{L^\infty(\Omega)}, \quad K_2(h) := \sqrt{d}C_{s,2}\|b\|_{L^\infty(\Omega)^d} + C(h)C_{s,2}\|c\|_{L^\infty(\Omega)},\\
M_\phi^{11}(h) &:= \left\| D_\phi^{T/2} G_\phi^{-1} D_\phi^{1/2} \right\|_2,
\end{aligned}$$

where $\|\cdot\|_2$ is the matrix two-norm i.e., the maximum singular value.

Theorem 3.1 ([6, Theorem 2.1 & Corollary 1]) *Let $K(h) > 0$ be defined by*

$$K(h) := \begin{cases} K_1(h), & \text{if } b \in W^{1,\infty}(\Omega)^d,\\ K_2(h), & \text{if } b \in L^\infty(\Omega)^d. \end{cases}$$

Let $\kappa_\phi > 0$ satisfy

$$\kappa_\phi := C(h)\left(C_1 M_\phi^{11}(h)K(h) + C_2\right) < 1. \tag{15}$$

Then, under Assumption 2.2, the operator $\mathscr{L}$ is invertible.

We denote the symmetric positive definite matrix R in $\mathbb{R}^{2,2}$ by

$$R := \frac{1}{(1-\kappa_\phi)^2}\begin{pmatrix} M_\phi^{11}(h)^2\left(C_1^2 C(h)^2 + \left(1 - C_2C(h)\right)^2\right) & \text{symmetry}\\ M_\phi^{11}(h)\left(C_1C(h) + \left(1-C_2C(h)\right)M_\phi^{11}(h)K(h)\right) & 1 + M_\phi^{11}(h)^2K(h)^2 \end{pmatrix}.$$

We can obtain the estimates of $\mathscr{L}^{-1}$ used R.

Theorem 3.2 ([6, Theorem 2.3]) *By using the same assumptions as those in Theorem 3.1, we obtain the following estimates,*

$$\left\|\mathscr{L}^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)} \leq C_{s,2}\,\|R\|_2^{\frac{1}{2}}. \tag{16}$$

Even if b has sufficient regularity, the estimates (16) is expected to converge to $C_{s,2}\max\{M_\phi^{11},1\}$ as $h \to 0$. As a result, this a posteriori method over estimates the operator norm and fails to converge to its exact operator norm. Further discussion of the error in the previously reported a posteriori estimates for $\mathscr{L}^{-1}$ and $\mathscr{L}_h^{-1}$ are discussed in [7]. Next, we will improve this estimation method (16), and propose the new a posteriori estimates of $\mathscr{L}^{-1}$ that converges to the exact operator norm.

Theorem 3.3 ([7, Theorem 6]) *By using the same assumptions as those in Theorem 3.1, we obtain the following error estimates:*

$$\left\|\mathscr{L}^{-1}-\mathscr{L}_h^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)} \leq C(h)\frac{1+C_{s,2}M_\phi^{11}(h)C_1}{1-\kappa_\phi}\sqrt{1+\left(M_\phi^{11}(h)K(h)\right)^2}, \tag{17}$$

$$\left\|\mathscr{L}^{-1}-\mathscr{L}_h^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),L^2(\Omega)\right)} \leq C(h)\frac{1+C_{s,2}M_\phi^{11}(h)C_1}{1-\kappa_\phi}\left(C(h)+C_{s,2}M_\phi^{11}(h)K(h)\right). \tag{18}$$

The proof of Theorem 3.3 can be obtained by using the proof of Theorem 3.2. Therefore, if the estimates of (16) can be improved, then the error estimates of Theorem 3.3 can also be improved. In Section 6, we use numerical examples to describe the results of improving these error estimates.

Remark 3.4 (Aubin-Nitsche trick) *In the case of $b \in W^{1,\infty}(\Omega)^d$, the convergence order of* (18) *is $O(h^2)$. Because we can apply the L^2 error estimates by applying the Aubin-Nitsche trick, the convergence order of $K(h)\left(=K_1(h)\right)$ is $O(h)$. On the other hand, in the case of $b \in L^\infty(\Omega)^d$ and $b \notin W^{1,\infty}(\Omega)^d$, the convergence order of* (18) *is $O(h)$. Because the solution for the dual problem of* (2a) *and* (2b) *does not have sufficient regularity, we cannot apply the Aubin-Nitsche trick. Therefore, $K(h)\left(=K_2(h)\right)$ does not have the order of h. Thus, when the dual problem becomes singular, it is difficult to obtain the L^2 error estimates whose convergence order is $O(h^2)$. To address this difficulty, we have previously proposed a technique for obtaining L^2 error estimates by using validated computations in [4]. When this technique is used, it is expected that $K_2(h)$ will have the order h.*

4 A posteriori estimates for inverse linear elliptic operators

In this section, we improve the previously reported estimates of (16) by proposing the new a posteriori estimates of $\mathscr{L}^{-1}$, which converges to the exact operator norm. To this end, let $M_\phi^{00}(h)$, $M_\phi^{10}(h)$, and $M_\phi^{01}(h)$ be the positive constants defined by

$$M_\phi^{00}(h) := \left\|L_\phi^{T/2}G_\phi^{-1}L_\phi^{1/2}\right\|_2, \quad M_\phi^{10}(h) := \left\|D_\phi^{T/2}G_\phi^{-1}L_\phi^{1/2}\right\|_2, \quad M_\phi^{01}(h) := \left\|L_\phi^{T/2}G_\phi^{-1}D_\phi^{1/2}\right\|_2,$$

respectively. The following lemma consists of the constants M_ϕ^{00} and M_ϕ^{10}.

Lemma 4.1 *The operator norm of $\mathscr{L}_h^{-1}$ satisfies the following equalities*

$$\|\mathscr{L}_h^{-1}\|_{\mathscr{L}\left(L^2(\Omega),L^2(\Omega)\right)} = M_\phi^{00}(h), \tag{19}$$

$$\|\mathscr{L}_h^{-1}\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)} = M_\phi^{10}(h). \tag{20}$$

Proof. —— Note that we only discuss the proof of (20). The proof of (19) is omitted because it is almost the same. For arbitrary $f \in L^2(\Omega)$, let $u_h := \mathscr{L}_h^{-1} f \in S_h(\Omega)$. The values from u_h to $P_h^0 f$ are the elements of $S_h(\Omega)$, and can be expressed by the linear combination of the basis of $S_h(\Omega)$. This indicates that $\alpha := (\alpha_1, \cdots, \alpha_n)^T$ and $\beta := (\beta_1, \cdots, \beta_n)^T \in \mathbb{R}^n$ exists such that

$$u_h(x) = \sum_{i=1}^n \alpha_i \phi_i(x), \qquad P_h^0 f(x) = \sum_{i=1}^n \beta_i \phi_i(x).$$

The equation (11) is rewritten using α and β to give

$$G_\phi \alpha = L_\phi \beta, \tag{21}$$

where the matrices G_ϕ and L_ϕ are defined by (12) and (6), respectively. Because L_ϕ and D_ϕ are symmetric positive definite matrices, they can be factorized by the Cholesky decomposition. From (21), we have

$$\begin{aligned}
\|u_h\|_{H_0^1(\Omega)}^2 &= \alpha^T D_\phi \alpha = \left(D_\phi^{T/2}\alpha\right)^T \left(D_\phi^{T/2}\alpha\right) \\
\|u_h\|_{H_0^1(\Omega)} &= \left\|D_\phi^{T/2}\alpha\right\|_2 \\
&= \left\|\left(D_\phi^{T/2} G_\phi^{-1} L_\phi^{1/2}\right)\left(L_\phi^{T/2}\beta\right)\right\|_2 \\
&\le \left\|D_\phi^{T/2} G_\phi^{-1} L_\phi^{1/2}\right\|_2 \left\|L_\phi^{T/2}\beta\right\|_2 \qquad (22)\\
&= \left\|D_\phi^{T/2} G_\phi^{-1} L_\phi^{1/2}\right\|_2 \|P_h^0 f\|_{L^2(\Omega)} \\
&\le \left\|D_\phi^{T/2} G_\phi^{-1} L_\phi^{1/2}\right\|_2 \|f\|_{L^2(\Omega)}\,. \qquad (23)
\end{aligned}$$

Therefore, we obtain

$$\|\mathscr{L}_h^{-1}\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)} = \sup_{L^2(\Omega)\ni f\neq 0} \frac{\|\mathscr{L}_h^{-1} f\|_{H_0^1(\Omega)}}{\|f\|_{L^2(\Omega)}} \le \left\|D_\phi^{T/2} G_\phi^{-1} L_\phi^{1/2}\right\|_2. \tag{24}$$

Next, we consider the existence of $f_0 \in L^2(\Omega)$ that satisfies the equalities of (22) and (23). Let $B_\phi := D_\phi^{T/2} G_\phi^{-1} L_\phi^{1/2}$, $\lambda > 0$ be a maximum eigenvalue of $B_\phi^T B_\phi$, and$\gamma \neq 0$ be an eigenvector associated to λ. Note that λ satisfies $\sqrt{\lambda} = \left\|D_\phi^{T/2} G_\phi^{-1} L_\phi^{1/2}\right\|_2$. Because $L_\phi^{T/2}$ is nonsingular, we denote $\beta_0 := \left(L_\phi^{T/2}\right)^{-1}\gamma$. Let $f_0 \in S_h(\Omega)$ be defined by $f_0 := \sum_{i=1}^n \beta_{0,i}\phi_i$. Then, f_0 satisfies the equalities (22) and (23). Practically, we obtain the equality of (22) by

$$\begin{aligned}\left\|B_\phi L_\phi^{T/2}\beta_0\right\|_2^2 &= \gamma^T B_\phi^T B_\phi \gamma\\ &= \lambda \|\gamma\|_2^2\\ \left\|B_\phi L_\phi^{T/2}\beta_0\right\|_2 &= \left\|D_\phi^{T/2} G_\phi^{-1} L_\phi^{1/2}\right\|_2 \|\gamma\|_2.\end{aligned}$$

Furthermore, $P_h^0 f_0 = f_0$ is clear from $f_0 \in S_h(\Omega)$. Therefore, we have the equality of (23). As a result, (24) satisfies the equality. □

From Lemma 4.1, we can expect that accurate estimates of $\mathscr{L}^{-1}$ can be obtained using $M_\phi^{10}(h)$. Practically, we have the following theorem.

Theorem 4.2 *Let $\hat{\kappa}_\phi > 0$ satisfy*

$$\hat{\kappa}_\phi := C(h)C_2\left(1 + M_\phi^{10}(h)C_1\right) < 1. \tag{25}$$

Then under the same assumptions in as those in Theorem 3.1, we have the following estimates

$$\|\mathscr{L}^{-1}\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)} \leq \frac{\sqrt{M_\phi^{10}(h)^2 + C(h)^2\left(1 + M_\phi^{10}(h)C_1\right)^2}}{1-\hat{\kappa}_\phi}. \tag{26}$$

Proof. —— By assuming (15), we find that the bounded linear operator $\mathscr{L}^{-1} : L^2(\Omega) \to H_0^1(\Omega) \cap X(\Omega)$ exists. For arbitrary $f \in L^2(\Omega)$, let $u := \mathscr{L}^{-1} f \in H_0^1(\Omega) \cap X(\Omega)$. By using the definition of u, u satisfies the following integral equation

$$u = (-\Delta)^{-1}\left(-(b\cdot\nabla)u - cu + f\right),$$

where $(-\Delta)^{-1} : L^2(\Omega) \to H_0^1(\Omega) \cap X(\Omega)$ denotes the solution operator of the Poisson equation with homogeneous Dirichlet boundary conditions. We can decompose the finite and infinite dimensional parts using the projection P_h^1 such that

$$\begin{cases} P_h^1 u = P_h(-\Delta)^{-1}\left(-(b\cdot\nabla)u - cu + f\right), & \text{(27a)}\\ (I-P_h^1)u = (I-P_h^1)(-\Delta)^{-1}\left(-(b\cdot\nabla)u - cu + f\right). & \text{(27b)}\end{cases}$$

In short, we denote $u_\perp := u - P_h^1 u$. From (27a), for arbitrary $v_h \in S_h(\Omega)$, we obtain

$$\begin{aligned}\left(\nabla P_h^1 u, \nabla v_h\right)_{L^2(\Omega)^d} &= \left(\nabla P_h^1(-\Delta)^{-1}\left(-(b\cdot\nabla)u - cu + f\right), \nabla v_h\right)_{L^2(\Omega)^d}\\ &= (-(b\cdot\nabla)u - cu + f, v_h)_{L^2(\Omega)}\\ L(P_h^1 u, v_h) &= (-(b\cdot\nabla)u_\perp - cu_\perp + f, v_h)_{L^2(\Omega)}\\ &= \left(P_h^0\left(-(b\cdot\nabla)u_\perp - cu_\perp + f\right), v_h\right)_{L^2(\Omega)}.\end{aligned} \tag{28}$$

Because $P_h^1 u$ and $P_h^0\big(-(b\cdot\nabla)u_\perp - cu_\perp + f\big)$ are the elements of $S_h(\Omega)$, they are expressible by the linear combination of the basis of $S_h(\Omega)$. This indicates that $\alpha := (\alpha_1,\cdots,\alpha_n)^T$ and $\beta := (\beta_1,\cdots,\beta_n)^T \in \mathbb{R}^n$ exists such that

$$P_h^1 u = \sum_{i=1}^{n} \alpha_i \phi_i, \quad P_h^0\big(-(b\cdot\nabla)u_\perp - cu_\perp + f\big) = \sum_{i=1}^{n} \beta_i \phi_i.$$

(28) is rewritten using α and β to give

$$G_\phi \alpha = L_\phi \beta. \tag{29}$$

From (29), we have

$$\begin{aligned}\left\|P_h^1 u\right\|_{H_0^1(\Omega)}^2 &= \alpha^T D_\phi \alpha \\ &= \left(D_\phi^{T/2}\alpha\right)^T \left(D_\phi^{T/2} G_\phi^{-1} L_\phi^{1/2}\right)\left(L_\phi^{T/2}\beta\right) \\ &\le \left\|P_h^1 u\right\|_{H_0^1(\Omega)} \left\|D_\phi^{T/2} G_\phi^{-1} L_\phi^{1/2}\right\|_2 \left\|P_h^0\big(-(b\cdot\nabla)u_\perp - cu_\perp + f\big)\right\|_{L^2(\Omega)}.\end{aligned}$$

By using Assumption 2.2 and the fact that P_h^0 is L^2 projection, we have

$$\begin{aligned}\left\|P_h^1 u\right\|_{H_0^1(\Omega)} &\le M_\phi^{10}(h)\left\|-(b\cdot\nabla)u_\perp - cu_\perp + f\right\|_{L^2(\Omega)} \\ &\le M_\phi^{10}(h)\left(\|b\|_{L^\infty(\Omega)^d}\|\nabla u_\perp\|_{L^2(\Omega)^d} + \|c\|_{L^\infty(\Omega)}\|u_\perp\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right) \\ &\le M_\phi^{10}(h) C_2 \|\nabla u_\perp\|_{L^2(\Omega)^d} + M_\phi^{10}(h)\|f\|_{L^2(\Omega)}.\end{aligned} \tag{30}$$

Next, by calculating the H_0^1 norm of (27b) from Assumption 2.2, we obtain

$$\begin{aligned}\|u_\perp\|_{H_0^1(\Omega)} &\le C(h)\left\|-(b\cdot\nabla)u - cu + f\right\|_{L^2(\Omega)} \\ &\le C(h)\left(\|b\|_{L^\infty(\Omega)^d}\|\nabla u\|_{L^2(\Omega)^d} + \|c\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\right) \\ &\le C(h)\left(\|b\|_{L^\infty}\left(\left\|\nabla P_h^1 u\right\|_{L^2} + \|\nabla u_\perp\|_{L^2}\right) + \|c\|_{L^\infty}\left(\left\|P_h^1 u\right\|_{L^2} + \|u_\perp\|_{L^2}\right) + \|f\|_{L^2}\right) \\ &\le C(h)C_1\left\|\nabla P_h^1 u\right\|_{L^2(\Omega)^d} + C(h)C_2\|\nabla u_\perp\|_{L^2(\Omega)^d} + C(h)\|f\|_{L^2(\Omega)}.\end{aligned} \tag{31}$$

From (31) and (30), we obtain

$$\|u_\perp\|_{H_0^1} \le C(h)C_1\left(M_\phi^{10}(h)C_2\|u_\perp\|_{H_0^1} + M_\phi^{10}(h)\|f\|_{L^2}\right) + C(h)C_2\|u_\perp\|_{H_0^1} + C(h)\|f\|_{L^2}.$$

By using Assumption (25), we obtain

$$\|u_\perp\|_{H_0^1(\Omega)} \le C(h)\frac{1 + M_\phi^{10}(h)C_1}{1 - \hat{\kappa}_\phi}\|f\|_{L^2(\Omega)}. \tag{32}$$

From (30) and (32), we have

$$\left\|P_h^1 u\right\|_{H_0^1(\Omega)} \le \frac{M_\phi^{10}(h)}{1 - \hat{\kappa}_\phi}\|f\|_{L^2(\Omega)}. \tag{33}$$

Finally, from (33), (32), and the fact that P_h^1 is H_0^1 projection, we have

$$\begin{aligned}
\|u\|_{H_0^1(\Omega)}^2 &= \left\|P_h^1 u\right\|_{H_0^1(\Omega)}^2 + \|u_\perp\|_{H_0^1(\Omega)}^2 \\
&\le \frac{M_\phi^{10}(h)^2}{(1-\hat{\kappa}_\phi)^2}\|f\|_{L^2(\Omega)}^2 + C(h)^2\frac{\left(1+M_\phi^{10}(h)C_1\right)^2}{(1-\hat{\kappa}_\phi)^2}\|f\|_{L^2(\Omega)}^2 \\
&= \frac{M_\phi^{10}(h)^2 + C(h)^2\left(1+M_\phi^{10}(h)C_1\right)^2}{(1-\hat{\kappa}_\phi)^2}\|f\|_{L^2(\Omega)}^2 \\
\left\|\mathscr{L}^{-1}f\right\|_{H_0^1(\Omega)} &\le \frac{\sqrt{M_\phi^{10}(h)^2 + C(h)^2\left(1+M_\phi^{10}(h)C_1\right)^2}}{1-\hat{\kappa}_\phi}\|f\|_{L^2(\Omega)},
\end{aligned}$$

Therefore, this proof is completed. □

The L^2 estimates are obtained by providing a proof similar to that of Theorem 4.2.

Theorem 4.3 *By using the same assumptions as those in Theorem 4.2, we obtain the following estimates*

$$\left\|\mathscr{L}^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),L^2(\Omega)\right)} \le \frac{M_\phi^{00}(h) + C(h)^2\left(1+M_\phi^{10}(h)C_1\right)}{1-\hat{\kappa}_\phi}. \tag{34}$$

Proof. —— For arbitrary $f \in L^2(\Omega)$, let $u := \mathscr{L}^{-1}f \in H_0^1(\Omega)\cap X(\Omega)$. From (29), we obtain

$$\begin{aligned}
\left\|P_h^1 u\right\|_{L^2(\Omega)}^2 &= \left(L_\phi^{T/2}\alpha\right)^T\left(L_\phi^{T/2}G_\phi^{-1}L_\phi^{1/2}\right)\left(L_\phi^{T/2}\beta\right) \\
&\le \left\|P_h^1 u\right\|_{L^2(\Omega)}\left\|L_\phi^{T/2}G_\phi^{-1}L_\phi^{1/2}\right\|_2\left\|P_h^0\left(-(b\cdot\nabla)u_\perp - cu_\perp + f\right)\right\|_{L^2(\Omega)}.
\end{aligned}$$

By using Assumption 2.2 and (32), we obtain

$$\begin{aligned}
\left\|P_h^1 u\right\|_{L^2(\Omega)} &\le M_\phi^{00}(h)C_2\|\nabla u_\perp\|_{L^2(\Omega)^d} + M_\phi^{00}(h)\|f\|_{L^2(\Omega)} \\
&\le M_\phi^{00}(h)C_2C(h)\frac{1+M_\phi^{10}(h)C_1}{1-\hat{\kappa}_\phi}\|f\|_{L^2(\Omega)} + M_\phi^{00}(h)\|f\|_{L^2(\Omega)} \\
&= \frac{M_\phi^{00}(h)}{1-\hat{\kappa}_\phi}\|f\|_{L^2(\Omega)}.
\end{aligned} \tag{35}$$

Similarly, for the estimates of $\|u_\perp\|_{L^2(\Omega)}$, by using Assumption 2.2 and (32), we obtain

$$\|u_\perp\|_{L^2(\Omega)} \le C(h)\|u_\perp\|_{H_0^1(\Omega)} \le C(h)^2\frac{1+M_\phi^{10}(h)C_1}{1-\hat{\kappa}_\phi}\|f\|_{L^2(\Omega)}. \tag{36}$$

From (35) and (36), we obtain

$$\begin{aligned}
\|u\|_{L^2(\Omega)} &\le \left\|P_h^1 u\right\|_{L^2(\Omega)} + \|u_\perp\|_{L^2(\Omega)} \\
&\le \frac{M_\phi^{00}(h)}{1-\hat{\kappa}_\phi}\|f\|_{L^2(\Omega)} + C(h)^2\frac{1+M_\phi^{10}(h)C_1}{1-\hat{\kappa}_\phi}\|f\|_{L^2(\Omega)},
\end{aligned}$$

Therefore, this proof is completed. □

To obtain the L^p estimates, the following theorem is necessary.

Theorem 4.4 (Gagliardo-Nirenberg) *Let the constants p and q satisfy* $1 \le p \le q^* \le \infty$. *Then, for arbitrary* $0 \le \theta \le 1$, *there exists the positive constant* $C_{g,r,p,q} > 0$ *such that*

$$\|u\|_{L^r(\Omega)} \le C_{g,r,p,q} \|u\|_{L^p(\Omega)}^{\theta} \|u\|_{W^{1,q}(\Omega)}^{1-\theta}, \quad \forall u \in W^{1,q}(\Omega), \tag{37}$$

where $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q^*}$.

It is known that the optimal constants of $C_{g,r,p,q}$ in Theorem 4.4 become the minimum eigenvalue of the certain nonlinear elliptic boundary value problems (e.g., [1]). Moreover, we can obtain the upper bounds of $C_{g,r,p,q}$ by Sobolev constants. For example, if we can calculate the Sobolev constants for $C_{s,2^*} > 0$ in (4), then for arbitrary $2 \le p \le 2^*$, we obtain

$$\begin{aligned}\|u\|_{L^p(\Omega)} &\le \|u\|_{L^2(\Omega)}^{1-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|u\|_{L^{2^*}(\Omega)}^{d\left(\frac{1}{2}-\frac{1}{p}\right)} \\ &\le C_{s,2^*}^{d\left(\frac{1}{2}-\frac{1}{p}\right)} \|u\|_{L^2(\Omega)}^{1-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|u\|_{H_0^1(\Omega)}^{d\left(\frac{1}{2}-\frac{1}{p}\right)}.\end{aligned}$$

Therefore, we obtain $C_{g,p,2,2} \le C_{s,2^*}^{d\left(\frac{1}{2}-\frac{1}{p}\right)}$.

Finally, in this section, we present the L^p estimates.

Corollary 4.5 *Assume that the following two inequalities are provided:*

$$\begin{aligned}\left\|\mathscr{L}^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),L^2(\Omega)\right)} &\le C_{L^2,L^2} \\ \left\|\mathscr{L}^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)} &\le C_{L^2,H_0^1}\end{aligned}$$

then, for arbitrary $2 \le p \le 2^*$, *we obtain*

$$\left\|\mathscr{L}^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),L^p(\Omega)\right)} \le C_{g,p,2,2} C_{L^2,L^2}^{1-d\left(\frac{1}{2}-\frac{1}{p}\right)} C_{L^2,H_0^1}^{d\left(\frac{1}{2}-\frac{1}{p}\right)}. \tag{38}$$

Proof. —— For arbitrary $f \in L^2(\Omega)$, let $u := \mathscr{L}^{-1} f \in H_0^1(\Omega) \cap X(\Omega)$. From Gagliardo-Nirenberg inequality and assumptions, we have

$$\begin{aligned}\|u\|_{L^p(\Omega)} &\le C_{g,p,2,2} \|u\|_{L^2(\Omega)}^{1-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|u\|_{H_0^1(\Omega)}^{d\left(\frac{1}{2}-\frac{1}{p}\right)} \\ &\le C_{g,p,2,2} C_{L^2,L^2}^{1-d\left(\frac{1}{2}-\frac{1}{p}\right)} \|f\|_{L^2(\Omega)}^{1-d\left(\frac{1}{2}-\frac{1}{p}\right)} C_{L^2,H_0^1}^{d\left(\frac{1}{2}-\frac{1}{p}\right)} \|f\|_{L^2(\Omega)}^{d\left(\frac{1}{2}-\frac{1}{p}\right)}.\end{aligned}$$

Therefore, this proof is completed. □

5 A posteriori error estimates for inverse linear elliptic operators

In this section, we consider the error estimates for $\mathscr{L}^{-1}$ and $\mathscr{L}_h^{-1}$. We obtain the following estimates corresponding to P_h^1 and P_h^L.

Lemma 5.1 *We obtain the following error estimates:*

$$\left\|P_h^1 u - P_h^L u\right\|_{L^2(\Omega)} \le M_\phi^{01}(h) \left\|P_h^1 \Delta^{-1}(b\cdot\nabla + c)(u - P_h^1 u)\right\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega), \tag{39}$$

$$\left\|P_h^1 u - P_h^L u\right\|_{H_0^1(\Omega)} \le M_\phi^{11}(h) \left\|P_h^1 \Delta^{-1}(b\cdot\nabla + c)(u - P_h^1 u)\right\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{40}$$

Proof. —— For arbitrary $u \in H_0^1(\Omega)$, let $u_\perp := u - P_h^1 u$ and $g := \Delta^{-1}(b\cdot\nabla + c)u_\perp$. For arbitrary $v_h \in S_h(\Omega)$, we have

$$\begin{aligned} L\left(P_h^1 u - P_h^L u, v_h\right) &= -\left((b\cdot\nabla + c)u_\perp, v_h\right)_{L^2(\Omega)} \\ &= \left(\nabla\Delta^{-1}(b\cdot\nabla + c)u_\perp, \nabla v_h\right)_{L^2(\Omega)^d} \\ &= \left(\nabla P_h^1 g, \nabla v_h\right)_{L^2(\Omega)^d}. \end{aligned} \tag{41}$$

Because $P_h^1 u - P_h^L u$ and $P_h^1 g$ are the elements of $S_h(\Omega)$, they can be expressed by the linear combination of the basis of $S_h(\Omega)$. This indicates that $\alpha := (\alpha_1, \cdots, \alpha_n)^T$ and $\beta := (\beta_1, \cdots, \beta_n)^T \in \mathbb{R}^n$ exists such that

$$P_h^1 u - P_h^L u = \sum_{i=1}^n \alpha_i \phi_i, \quad P_h^1 g = \sum_{i=1}^n \beta_i \phi_i.$$

(41) is written using α and β to give

$$G_\phi \alpha = D_\phi \beta.$$

Therefore, we have the following L^2 error estimates

$$\begin{aligned} \left\|P_h^1 u - P_h^L u\right\|_{L^2(\Omega)}^2 &= \alpha^T L_\phi \alpha \\ &= \left(L_\phi^{T/2}\alpha\right)^T \left(L_\phi^{T/2} G_\phi^{-1} D_\phi^{1/2}\right)\left(D_\phi^{T/2}\beta\right) \\ &\le \left\|P_h^1 u - P_h^L u\right\|_{L^2(\Omega)} \left\|L_\phi^{T/2} G_\phi^{-1} D_\phi^{1/2}\right\|_2 \left\|P_h^1 g\right\|_{H_0^1(\Omega)}. \end{aligned}$$

Similarly, we have the following H_0^1 error estimates

$$\begin{aligned} \left\|P_h^1 u - P_h^L u\right\|_{H_0^1(\Omega)}^2 &= \alpha^T D_\phi \alpha \\ &= \left(D_\phi^{T/2}\alpha\right)^T \left(D_\phi^{T/2} G_\phi^{-1} D_\phi^{1/2}\right)\left(D_\phi^{T/2}\beta\right) \\ &\le \left\|P_h^1 u - P_h^L u\right\|_{H_0^1(\Omega)} \left\|D_\phi^{T/2} G_\phi^{-1} D_\phi^{1/2}\right\|_2 \left\|P_h^1 g\right\|_{H_0^1(\Omega)}. \end{aligned}$$

Therefore, this proof is completed. □

Theorem 5.2 *By using the same assumptions as those in Theorem 4.2, we obtain the following error estimates:*

$$\left\|\mathscr{L}^{-1}-\mathscr{L}_h^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)} \le C(h)\frac{1+M_\phi^{10}(h)C_1}{1-\hat{\kappa}_\phi}\sqrt{1+\left(M_\phi^{11}(h)K(h)\right)^2}, \tag{42}$$

$$\left\|\mathscr{L}^{-1}-\mathscr{L}_h^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),L^2(\Omega)\right)} \le C(h)\frac{1+M_\phi^{10}(h)C_1}{1-\hat{\kappa}_\phi}\left(C(h)+M_\phi^{01}(h)K(h)\right). \tag{43}$$

Proof. —— For arbitrary $f \in L^2(\Omega)$, let $u := \mathscr{L}^{-1}f \in H_0^1(\Omega)\cap X(\Omega)$ and $u_h := \mathscr{L}_h^{-1}f \in S_h(\Omega)$. By the definition of u and u_h, we have $u_h = P_h^L u$. Let $u_\perp := u - P_h^1 u$.

First, we derive (42). By using the definition of H_0^1 projection and (40), we have

$$\begin{aligned}\|u-u_h\|_{H_0^1(\Omega)}^2 &= \left\|u-P_h^1 u\right\|_{H_0^1(\Omega)}^2 + \left\|P_h^1 u - P_h^L u\right\|_{H_0^1(\Omega)}^2 \\ &\le \left\|u-P_h^1 u\right\|_{H_0^1(\Omega)}^2 + M_\phi^{11}(h)^2\left\|P_h^1\Delta^{-1}(b\cdot\nabla+c)u_\perp\right\|_{H_0^1(\Omega)}^2.\end{aligned}$$

Then, from [4, Theorem 3.3.], we obtain the following estimates:

$$\left\|P_h^1\Delta^{-1}(b\cdot\nabla+c)(u-P_h^1 u)\right\|_{H_0^1(\Omega)} \le K(h)\left\|u-P_h^1 u\right\|_{H_0^1(\Omega)}. \tag{44}$$

Furthermore, from (32), we obtain

$$\begin{aligned}\|u-u_h\|_{H_0^1(\Omega)}^2 &\le \left(1+\left(M_\phi^{11}(h)K(h)\right)^2\right)\|\nabla u_\perp\|_{L^2(\Omega)^d}^2 \\ \|u-u_h\|_{H_0^1(\Omega)} &\le \sqrt{1+\left(M_\phi^{11}(h)K(h)\right)^2}C(h)\frac{1+M_\phi^{10}(h)C_1}{1-\hat{\kappa}_\phi}\|f\|_{L^2(\Omega)}.\end{aligned}$$

Therefore, we obtain (42).

Next, we derive (43). By using Assumption 2.2 and (39), we obtain

$$\begin{aligned}\|u-u_h\|_{L^2(\Omega)} &\le \left\|u-P_h^1 u\right\|_{L^2(\Omega)} + \left\|P_h^1 u - P_h^L u\right\|_{L^2(\Omega)} \\ &\le C(h)\left\|u-P_h^1 u\right\|_{H_0^1(\Omega)} + M_\phi^{01}(h)\left\|P_h^1\Delta^{-1}(b\cdot\nabla+c)(u-P_h u)\right\|_{H_0^1(\Omega)}.\end{aligned}$$

From (44) and (32), we have

$$\begin{aligned}\|u-u_h\|_{L^2(\Omega)} &\le \left(C(h)+M_\phi^{01}(h)K(h)\right)\left\|u-P_h^1 u\right\|_{H_0^1(\Omega)} \\ &\le \left(C(h)+M_\phi^{01}(h)K(h)\right)C(h)\frac{1+M_\phi^{10}(h)C_1}{1-\hat{\kappa}_\phi}\|f\|_{L^2(\Omega)},\end{aligned}$$

Therefore, we obtain (43). □

Finally in this section, we present the L^p error estimates.

Corollary 5.3 *Assume that the following two inequalities are provided:*

$$\left\|\mathscr{L}^{-1}-\mathscr{L}_h^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),L^2(\Omega)\right)} \le E_{L^2,L^2}$$

$$\left\|\mathscr{L}^{-1}-\mathscr{L}_h^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)} \le E_{L^2,H_0^1}$$

For arbitrary $2 \le p \le 2^*$, *we have*

$$\left\|\mathscr{L}^{-1}-\mathscr{L}_h^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),L^p(\Omega)\right)} \le C_{g,p,2,2} E_{L^2,L^2}^{1-d\left(\frac{1}{2}-\frac{1}{p}\right)} E_{L^2,H_0^1}^{d\left(\frac{1}{2}-\frac{1}{p}\right)}. \tag{45}$$

The proof is similar to Corollary 4.5.

Remark 5.4 *From the results of Lemma 4.1 and Theorem 5.2,* $M_\phi^{00}(h)$ *and* $M_\phi^{10}(h)$ *converge to* $\left\|\mathscr{L}^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),L^2(\Omega)\right)}$ *and* $\left\|\mathscr{L}^{-1}\right\|_{\mathscr{L}\left(L^2(\Omega),H_0^1(\Omega)\right)}$ *as* $h \to 0$, *respectively.*

6 Numerical example

In this section, we apply the described method to numerical experiments on test problems. First, we compared (26) with (16). For simplicity, in this section, the domain Ω is fixed as the unit square $(0,1)\times(0,1) \subset \mathbb{R}^2$. We assume that the finite element partition of Ω is a uniform triangular mesh and the basis of $S_h(\Omega)$ is a set of piecewise linear polynomials (P1 element). Therefore, Assumption 2.2 is realized by $C(h) = 0.49293h$ (e.g., [2]).

6.1 Steady-state convection diffusion equation

We show a computational result in the case of the steady-state convection diffusion equation $\mathscr{L} = -\Delta + b\cdot\nabla + c$. In particular, we consider $b(x_1,x_2) = 5\left(-x_2+\frac{1}{2}, x_1-\frac{1}{2}\right)^T \in W^{1,\infty}(\Omega)^2$.

Table 1: Convection diffusion equation for $c = 0$.

$1/h$	$M_\phi^{00}(h)$	$M_\phi^{01}(h)$	$M_\phi^{10}(h)$	$M_\phi^{11}(h)$	(16)	(26)
10	0.04943	0.22232	0.22232	1.00002	2.12100	0.34720
20	0.05034	0.22435	0.22435	1.00001	1.73512	0.27102
30	0.05051	0.22473	0.22473	1.00001	1.63742	0.25304
40	0.05057	0.22487	0.22487	1.00001	1.59292	0.24512
50	0.05060	0.22493	0.22493	1.00001	1.56748	0.24068
order	-0.01035	-0.00516	-0.00516	0.00000	0.15014	0.17978

First, we consider the case of $c = 0$. Table 1 shows its verification results. The column for $1/h$ lists the reciprocal number of the mesh size. These values denote the number of partitions for the domain Ω. From $b \neq 0$, the matrix G_ϕ is nonsymmetric and has complex eigenvalues. However, $M_\phi^{01}(h)$ and $M_\phi^{10}(h)$ are always equal. Our proposed new estimates in (26) are smaller than previous estimates (16) for any mesh size.

Table 2: Convection diffusion equation for $c = -10$.

$1/h$	$M_\phi^{00}(h)$	$M_\phi^{01}(h)$	$M_\phi^{10}(h)$	$M_\phi^{11}(h)$	(16)	(26)
10	0.09772	0.43953	0.43953	1.97692	fail	fail
20	0.10133	0.45167	0.45167	2.01302	6.81287	0.95000
30	0.10203	0.45400	0.45400	2.01996	5.09800	0.69895
40	0.10228	0.45482	0.45482	2.02241	4.53489	0.61676
50	0.10239	0.45520	0.45520	2.02355	4.25499	0.57604
order	-0.02066	-0.01550	-0.01550	-0.01033	0.46917	0.49927

Next, we consider the case of $c = -10$; Table 2 shows its verification results. In this table, "fail" denotes that the invertibility condition failed in Theorem 3.1. The same tendency as Table 1 is seen for this problem.

6.2 Linearized semilinear equation

We show a computational result in the case of linearized equation of the following semilinear PDEs:

$$\begin{cases} -\Delta u = \lambda(1 + u + u^2 - au^3) & \text{in } \Omega, \quad (46a) \\ u = 0, & \text{on } \partial\Omega, \quad (46b) \end{cases}$$

where $\lambda > 0$ and $0 \leq a \leq 1$ are constants. For the constant parameters λ and a, it is known that (46a) and (46b) has at least two positive solutions; we denote them as the upper and lower solutions, respectively. Let u_h be the finite element solutions for (46a) and (46b). This indicates that $u_h \in S_h(\Omega)$ satisfies the following variational equation:

$$(\nabla u_h, \nabla v_h)_{L^2(\Omega)^2} = \lambda \left(1 + u_h + u_h^2 - au_h^3, v_h\right)_{L^2(\Omega)}, \quad \forall v_h \in S_h(\Omega).$$

The finite element solutions u_h were obtained by the Newton-Raphson method using usual floating point arithmetic. Then, the linearized operator at u_h is defined by $\mathscr{L} = -\Delta + \lambda\left(1 + 2u_h - 3au_h^2\right)$. We introduce the operator norm estimates for the inverse linearized operator $\mathscr{L}^{-1}$.

Table 3 shows the verification results of the linearized inverse operator at the upper approximate solution u_h for $\lambda = 4$ and $a = 0.001$. The effectiveness of the method and the validity of our new estimates for this problem were shown.

Table 4 shows the verification results of the linearized inverse operator at the lower approximate solution u_h for $\lambda = 4$ and $a = 0.001$.

Remark 6.1 (Computer environment) *All computations were carried out on a Intel Xeon E5520 2.27GHz (OS: Red Hat Enterprise Linux Server release 5.5) by using INTLAB version 6.0, a toolbox in MATLAB R2010a developed by Rump [10] for self-validating algorithms. Therefore, all numerical values in these tables are verified data in the sense of strictly rounding error control.*

Table 3: Linearized semilinear equation at the upper approximate solution.

$1/h$	$M_\phi^{00}(h)$	$M_\phi^{01}(h)$	$M_\phi^{10}(h)$	$M_\phi^{11}(h)$	(16)	(26)
10	0.07082	0.32622	0.32622	2.19839	10.34011	0.83100
20	0.07297	0.33356	0.33356	2.22458	2.89776	0.40142
30	0.07338	0.33498	0.33498	2.22970	2.50199	0.36286
40	0.07353	0.33547	0.33547	2.23150	2.38044	0.35076
50	0.07360	0.33571	0.33571	2.23234	2.32652	0.34538
order	-0.01708	-0.01272	-0.01272	-0.00681	0.61830	0.37149

Table 4: Linearized semilinear equation at the lower approximate solution.

$1/h$	$M_\phi^{00}(h)$	$M_\phi^{01}(h)$	$M_\phi^{10}(h)$	$M_\phi^{11}(h)$	(16)	(26)
10	0.07255	0.32630	0.32630	1.46826	1.51285	0.34420
20	0.07489	0.33379	0.33379	1.48834	1.49944	0.33828
30	0.07534	0.33523	0.33523	1.49223	1.49716	0.33722
40	0.07550	0.33574	0.33574	1.49360	1.49638	0.33686
50	0.07558	0.33597	0.33597	1.49424	1.49602	0.33669
order	-0.01817	-0.01293	-0.01293	-0.00779	0.00487	0.00969

7 Conclusion

We propose a method for constructive a posteriori estimates of inverse operators for boundary value problems. It is particularly notable that as in (26) and (34), our proposed estimates are expected to converge to the exact operator norm according to Theorem 5.2. By comparing the a posteriori estimates (16), which given by [6] and (26) for some test problem, we show that this holds. Our proposed new estimates (26) are smaller than the previous estimates (16) in the test problems, and more closely reflect the true error.

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