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Microlocal analysis of fixed singularities of WKB solutions of a Schrödinger equation with a merging triplet of two simple poles and a simple turning point

In memory of the late Professor Leon Ehrenpreis

By

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Abstract

We first show that the WKB-theoretic canonical form of an M2P1T (merging two poles and one turning point) Schrödinger equation is given by the algebraic Mathieu equation. We further show that, in analyzing the structure of WKB solutions of a Mathieu equation near fixed singular points relevant to simple poles of the equation, we can focus our attention on the pole part of the equation so that we may reduce it to the Legendre equation. The Borel transformation of WKB-theoretic transformations thus obtained gives rise to microdifferential relations, which lead to the microlocal analysis of the Borel transformed WKB solutions of an M2P1T equation near their fixed singular points. The fully detailed account of the results will be given in [9].

0 Introduction

The purpose of this article is to announce the main results of [9] emphasizing the atypical points in its reasoning which cannot be found in earlier papers dealing with seemingly related problems, such as [3] and [8]. As the logical structure of the argument in [9] is intricate, we try to explain the ideas that underlie its formulation of the problem. The target of [9] is the exact WKB analysis of a Schrödinger equation

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x,a)\right)\psi = 0, \tag{0.1}$$

where η is a large parameter and the potential Q contains a triplet of two simple poles and one simple turning point that merge as the parameter a tends to 0. Here "exact WKB analysis" means WKB analysis based on the Borel transformation with respect to the large parameter η; thus our principal aim is to analyze the singularity structure of the Borel transformed WKB solution $\psi_B(x,a,y)$, which solves the Borel transformed Schrödinger equation

$$\left(\frac{\partial^2}{\partial x^2} - Q(x,a)\frac{\partial^2}{\partial y^2}\right)\psi_B(x,a,y) = 0. \tag{0.2}$$

Hence the exact WKB analysis belongs to the most favorite field of the late Professor Ehrenpreis, Fourier analysis in the complex domain.

(Cf. [6]) Our interest in the class of Schrödinger equations with a merging triplet of poles and a turning point originates from our desire to understand the semi-global structure of a Schrödinger equation with two simple poles in its potential. As is now well-known (cf. [12], [13]), a simple pole gives an effect to the Borel transformed WKB solutions that is similar to the effect which a turning point gives. Thus the analysis of the class of Schrödinger equations with two simple poles in their potentials is a natural counterpart of the classes of equations studied in [3] (Schrödinger equations with a merging pair of simple turning points) and in [8] (Schrödinger equations with a merging pair of a simple pole and a simple turning point). One can then easily guess that a WKB-theoretic canonical form of such a Schrödinger equation is the Legendre equation with a large parameter, that is,

$$\left(\frac{d^2}{dx^2} - \eta^2 Q_{\text{Leg}}(x,a)\right)\psi = 0, \tag{0.3}$$

where

$$Q_{\text{Leg}} = \frac{\lambda}{x^2-a^2} + \eta^{-2}\left(\frac{\gamma_+}{(x-a)^2} + \frac{\gamma_-}{(x+a)^2}\right) \tag{0.4}$$

with $\gamma_\pm$ being complex numbers and with λ being an infinite series in η^{-1} with constant coefficients that satisfies an appropriate growth order condition to be discussed later. To emphasize the fact that λ is not a genuine constant but an infinite series we sometimes call such an equation the ∞-Legendre equation. Parenthetically we note that, in what follows, we basically concentrate our attention to the core part of the potential, that is, $\lambda/(x^2-a^2)$ by mainly considering the situation where γ_+ and γ_- are 0; this limitation is helpful in clarifying the logical structure of our reasoning by avoiding technical complexities. By the way, in the exact WKB analysis, an important subject is the analytic structure of the Borel transformed WKB solutions near their fixed singularities (cf. [11, p.112–p.113]. See also [4], [5] and [17]) , that is,

singularities located at
(0.5)
$$y = -\int_{\alpha}^{x} \sqrt{Q(x,a)}\,dx + 2l \int_{\alpha}^{\tilde{\alpha}} \sqrt{Q(x,a)}\,dx \quad (l = \pm 1, \pm 2, \cdots),$$
where α and $\tilde{\alpha}$ are turning points (with a simple pole being regarded as a turning point) of the equation. An important point in [3] and [8] is that the period integral

$$2\int_{\alpha}^{\tilde{\alpha}} \sqrt{Q(x,a)}\,dx \tag{0.6}$$

tends to 0 when we let a tend to 0; hence by showing that the domain of definition of the transformation operator to the canonical form can be chosen to be independent of a, we can analyze the analytic structure of the Borel transformed WKB solution near a fixed singularity with $|l| \gg 1$. But this time we find

$$\int_{-a}^{a} \frac{dx}{\sqrt{x^2 - a^2}} = \pi i \tag{0.7}$$

does not change even when a tends to 0. Thus the strategy in [3] and [8] is not effective in this case. To circumvent the problem we dismantle the potential of its homogeneity and seek for the class of Schrödinger equations which can be transformed to

$$\Big(\frac{d^2}{dx^2} - \eta^2 \frac{aA + xB}{x^2 - a^2}\Big)\psi = 0 \tag{0.8}$$

with A and B being infinite series in η^{-1} that are independent of x, that is, the algebraic ∞-Mathieu equation (if we follow the usage of the terminology of [7, p.98]), which we call the ∞-Mathieu equation for short. In view of the explicit form of the potential in (0.8), we imagine that the class which we now try to analyze would consist of Schrödinger equations with two simple poles and one simple turning point. Fortunately this guess turns out to be correct, as is explained in Section 1 below. Thus widening the target class gives a clean result, but the problem is the fact that the Mathieu equation is a notoriously difficult object to analyze. Hence we next contrive to deduce the analytic properties of Borel transformed WKB solutions near the fixed singularities relevant to the pair of simple poles, which was our original target, by "driving off" the simple turning point. This contrivance will be explained in Section 3, but here we note the following geometric fact that explains why we introduce an auxiliary parameter ρ into our formulation (cf. Definition 1.1 below).

To describe the geometric situation, let A_0 (resp., B_0) denote the degree 0 (in η) part of A (resp., B). Then we can confirm

$$A_0\big|_{a=0} \neq 0 \quad \text{(cf. (1.39) and (1.4))} \tag{0.9}$$

and

$$B_0\big|_{a=0} = Z_0\rho \quad \text{with} \quad Z_0 = \pm 1 \quad \text{(cf. (1.40))}. \tag{0.10}$$

Now, keeping $a/\rho =: \kappa(\neq 0)$ fixed sufficiently small, we let ρ tend to 0. Then, since the turning point t_0 of (0.8) is given by

$$-\frac{aA_0}{B_0} = -\frac{\kappa A_0(0)}{Z_0 + \kappa\beta} + O(\rho) \tag{0.11}$$

with some constant β, it stays away from 0. On the other hand, the simple poles $t = \pm a$ tend to 0. Thus one may expect that the singularity structure of Borel transformed WKB solutions near the fixed singularities relevant to the simple poles can be deduced from that of Borel transformed WKB solutions of the Schrödinger equation whose potential contains two simple poles only, i.e., without a turning point. And this expectation is realized in Section 3. In ending Introduction we note that in deducing the results in the final section (Section 4) from those in Section 1 and Section 3, we make full use of microdifferential relations among objects on the Borel plane which are discussed in Section 2.

1 Definition of an M2P1T equation and its reduction to the Mathieu equation

In what follows, U (resp., V and O) denotes a sufficiently small open neighborhood of the origin $\{t \in \mathbb{C}; t = 0\}$ (resp., $\{a \in \mathbb{C}; a = 0\}$ and $\{\rho \in \mathbb{C}; \rho = 0\}$) and let $f(t, a, \rho)$ denote a holomorphic function that has the following form (1.1) on $U \times V \times O$:

$$f(t, a, \rho) = t\rho g(t, \rho) + \sum_{j \geq 1} a^j f^{(j)}(t, \rho) \tag{1.1}$$

with

$$g(t, \rho) \text{ and } f^{(j)}(t, \rho) \text{ being holomorphic on } U \times O, \tag{1.2}$$

$$g(0, \rho) = 1, \tag{1.3}$$

$$f^{(1)}(0, 0) \neq 0, \tag{1.4}$$

$$\rho^2 \neq \left(f^{(1)}(0, \rho)\right)^2 \quad \text{for } \rho \text{ in } O. \tag{1.5}$$

In what follows we use symbols $f^{(0)}(t, \rho)$ and $\tilde{f}^{(0)}(t, \rho)$ respectively to denote $t\rho g(t, \rho)$ and $\rho g(t, \rho)$.

Definition 1.1. *Let $f(t, a, \rho)$ be as above, let $g_\pm(t)$ be holomorphic functions on U and let Q denote the following potential:*

$$\frac{f(t, a, \rho)}{t^2 - a^2} + \eta^{-2}\left(\frac{g_+(t)}{(t - a)^2} + \frac{g_-(t)}{(t + a)^2}\right) \quad (\eta : \text{a large parameter}). \tag{1.6}$$

Then the Schrödinger operator

$$\frac{d^2}{dt^2} - \eta^2 Q(t, a, \rho, \eta) \tag{1.7}$$

is called an M2P1T, merging two poles and one turning point, operator.

Remark 1.1. For the sake of simplicity we assume the following condition (1.8) in Section 1:

$$g_+ = g_- = 0. \tag{1.8}$$

Remark 1.2. It immediately follows from (1.3) that (1.7) for $\rho \neq 0$ has a simple turning point when V is chosen sufficiently small.

Remark 1.3. It follows from the trivial relation

$$\frac{t\tilde{f}^{(0)} + af^{(1)}}{t^2 - a^2} = \frac{\tilde{f}^{(0)} + f^{(1)}}{2(t-a)} + \frac{\tilde{f}^{(0)} - f^{(1)}}{2(t+a)} \tag{1.9}$$

that we obtain a sum of simple poles at $a = 0$, not a double pole. Parenthetically we note that the assumption (1.5) guarantees that their residues are different from 0.

Remark 1.4. The reader might wonder why the assumption about the structure of $\tilde{f}^{(0)}(t, \rho)$ is so restrictive. But, since we want to uniformly deal with the problem for an arbitrarily small parameter $\rho(\neq 0)$, some strict restriction on the structure of $\tilde{f}^{(0)}(t, \rho)$ is inevitable. Actually one will be able to find that the function $x_0^{(0)}(t, \rho)$ given by (1.27) below cannot be holomorphic on a fixed neighborhood of the origin $\{t = 0\}$ if we choose, for example,

$$\tilde{f}^{(0)}(t, \rho) = t + \rho, \tag{1.10}$$

although it satisfies

$$\tilde{f}^{(0)}(0, \rho) = \rho, \tag{1.11}$$

the condition we frequently use in our computation.

The purpose of this section is to show that an M2P1T equation is WKB-theoretically transformed to an ∞-Mathieu equation. We refer the reader to [11, Section 2] for the basic properties of "WKB-theoretic transformations", but we note their heuristic explanation as follows: in an intuitive description its core is a formal coordinate transformation from t to $x = x(t, a, \rho, \eta)$ defined by an infinite series

$$x(t, a, \rho, \eta) = \sum_{k \geq 0} x_{2k}(t, a, \rho) \eta^{-2k} \tag{1.12}$$

which satisfies

$$Q(t, a, \rho, \eta) = \Big(\frac{\partial x}{\partial t}\Big)^2 \Big(\frac{aA + xB}{x^2 - a^2}\Big) - \frac{1}{2}\eta^{-2}\{x; t\}, \tag{1.13}$$

for some infinite series

$$A = \sum_{k \geq 0} A_{2k}(a, \rho) \eta^{-2k} \tag{1.14}$$

and

$$B = \sum_{k \geq 0} B_{2k}(a, \rho) \eta^{-2k}, \tag{1.15}$$

where $\{x; t\}$ stands for the Schwarzian derivative

$$-2\Big(\frac{\partial x}{\partial t}\Big)^{1/2} \frac{\partial^2}{\partial t^2} \Big(\frac{\partial x}{\partial t}\Big)^{-1/2}. \tag{1.16}$$

In what follows we call the Schrödinger operator

$$\Big(\frac{d^2}{dx^2} - \eta^2 \frac{aA + xB}{x^2 - a^2}\Big) \tag{1.17}$$

an ∞-Mathieu operator. Using appropriate growth order conditions that $x_{2k}(t, a, \rho)$, $A_{2k}(a, \rho)$ and $B_{2k}(a, \rho)$ satisfy we can construct microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ so that they "intertwine" the Borel transformed M2P1T operator and the Borel transformed ∞-Mathieu operator; we have (Theorem 2.1)

$$N\mathcal{X} = \mathcal{Y} M_\infty, \tag{1.18}$$

where M_∞ denotes the Borel transformed ∞-Mathieu operator and N denotes the Borel transformed M2P1T operator written in (x, y)-variable with the effect of the coordinate change appropriately taken into account (cf. (2.4) for the concrete form of N). See Section 2 for the explicit description of $\mathcal{X}$ in terms of the infinite series x.

In constructing the infinite series x, A and B, we further expand $x_{2k}(t, a, \rho)$ etc. in powers of a; that is, we will seek for x, A and B in the form of double series as follows:

$$x = \sum_{j,k\geq 0} x_{2k}^{(j)}(t, \rho) a^j \eta^{-2k}, \tag{1.19}$$

$$A = \sum_{j,k\geq 0} A_{2k}^{(j)}(\rho) a^j \eta^{-2k}, \tag{1.20}$$

$$B = \sum_{j,k\geq 0} B_{2k}^{(j)}(\rho) a^j \eta^{-2k}. \tag{1.21}$$

Substituting these series into (1.13) and comparing the coefficient of η^0 we find

$$\frac{f(t, a, \rho)}{t^2 - a^2} = \left(\frac{\partial x_0}{\partial t}\right)^2 \frac{aA_0 + x_0 B_0}{x_0^2 - a^2}, \tag{1.22}$$

where

$$x_0(t, a, \rho) = \sum_{j\geq 0} x_0^{(j)}(t, \rho) a^j, \tag{1.23}$$

$$A_0(a, \rho) = \sum_{j\geq 0} A_0^{(j)}(\rho) a^j, \tag{1.24}$$

$$B_0(a, \rho) = \sum_{j\geq 0} B_0^{(j)}(\rho) a^j. \tag{1.25}$$

After multiplying (1.22) by $(t^2-a^2)(x_0^2-a^2)$ we compare the coefficient of a^p to find

(1.26.p)

$$
\begin{aligned}
&- f^{(p-2)} + \sum_{j+k+l=p} x_0^{(j)} x_0^{(k)} f^{(l)} \\
&= t^2 \Big(\sum_{j+k+l=p} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} A_0^{(l-1)} + \sum_{j+k+l+m=p} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} x_0^{(l)} B_0^{(m)} \Big) \\
&- \Big(\sum_{j+k+l=p-2} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} A_0^{(l-1)} + \sum_{j+k+l+m=p-2} \frac{\partial x_0^{(j)}}{\partial t} \frac{\partial x_0^{(k)}}{\partial t} x_0^{(l)} B_0^{(m)} \Big).
\end{aligned}
$$

In (1.26.p) terms whose indices do not meet the requirements should be ignored, as usual. With this convention (1.26.p) with $p = 0$ or 1 is of a peculiar form. For example, we find

(1.26.0) $$t x_0^{(0)2} \tilde{f}^{(0)} = t^2 x_0^{(0)\prime 2} x_0^{(0)} B_0^{(0)}.$$

Here, and in what follows, x' stands for $\partial x / \partial t$. Hence we find

(1.27) $$x_0^{(0)}(t, \rho) = \frac{1}{4B_0^{(0)}} \left(\int_0^t \sqrt{\frac{\tilde{f}^{(0)}(t, \rho)}{t}}\, dt \right)^2,$$

where $B_0^{(0)}$ is a non-zero constant to be fixed later. Then it follows from the assumptions (1.2) and (1.3) that there exists a holomorphic function $\tilde{x}_0^{(0)}(t, \rho)$ that satisfies

(1.28) $$x_0^{(0)}(t, \rho) = t \tilde{x}_0^{(0)}(t, \rho)$$

with

(1.29) $$\tilde{x}_0^{(0)}(0, \rho) = \frac{\rho}{B_0^{(0)}}.$$

Next we consider the case $p = 1$. Then, by using (1.28) we find

(1.26.1)

$$2t\tilde{x}_0^{(0)} x_0^{(1)} t\tilde{f}^{(0)} + t^2 \tilde{x}_0^{(0)2} f^{(1)}$$
$$= t^2 \Big(x_0^{(0)\prime 2} A_0^{(0)} + 2x_0^{(0)\prime} x_0^{(1)\prime} x_0^{(0)} B_0^{(0)} + x_0^{(0)\prime 2} x_0^{(1)} B_0^{(0)} + x_0^{(0)\prime 2} x_0^{(0)} B_0^{(1)} \Big).$$

Hence it suffices to solve

$$2x_0^{(0)\prime} x_0^{(1)\prime} x_0^{(0)} B_0^{(0)} + x_0^{(0)\prime 2} x_0^{(1)} B_0^{(0)} - 2\tilde{x}_0^{(0)} x_0^{(1)} \tilde{f}^{(0)} = -x_0^{(0)\prime 2} A_0^{(0)} - x_0^{(0)\prime 2} x_0^{(0)} B_0^{(1)} + \tilde{x}_0^{(0)2} f^{(1)}. \tag{1.30}$$

Here, and in what follows, we use a new variable s given by

$$s = x_0^{(0)}(t, \rho). \tag{1.31}$$

Using the symbol $\dot{x}$ to denote dx/ds, we then find the following equation (1.32) with the help of (1.27).

$$B_0^{(0)} \Big(2s\frac{d}{ds} - 1 \Big) x_0^{(1)}(s, \rho) = -A_0^{(0)} - sB_0^{(1)} + \big[\big(x_0^{(0)\prime} \big)^{-2} \tilde{x}_0^{(0)2} f^{(1)} \big] \big(t(s, \rho), \rho \big), \tag{1.32}$$

where $t(s, \rho)$ denotes the inverse function of $s = x_0^{(0)}(t, \rho)$. It is clear that (1.32) admits a solution $x_0^{(1)}(s, \rho)$ that is holomorphic near $s = 0$ for arbitrary constants $A_0^{(0)}$ and $B_0^{(1)}$, which are to be fixed later. Furthermore we can immediately see

$$x_0^{(1)}(0, \rho) = \frac{1}{B_0^{(0)}} \big(A_0^{(0)} - f^{(1)}(0, \rho) \big), \tag{1.33}$$

(1.34)

$$\dot{x}_0^{(1)}(0, \rho) = \frac{1}{B_0^{(0)}} \Big(- B_0^{(1)} + Z_0^{-1} \big(z'(0, \rho) f^{(1)}(0, \rho) + f^{(1)\prime}(0, \rho) \big) \Big),$$

where

$$Z_0 = x_0^{(0)\prime}(0,\rho) \tag{1.35}$$

and

$$z(t,\rho) = \big(x_0^{(0)\prime}(t,\rho)\big)^{-2}\tilde{x}_0^{(0)}(t,\rho)^2. \tag{1.36}$$

For $p \geq 2$ (1.26.p) assumes the following form:

$$C_0^{(p)}(\rho) + D_0^{(p)}(\rho)t + t^2\mathcal{E}_0^{(p)} = 0, \tag{1.37.p}$$

where $C_0^{(p)}$ and $D_0^{(p)}$ are free from t and $\mathcal{E}_0^{(p)}$ contains in it at least

$$\sum_{j+k+l=p} x_0^{(j)\prime}x_0^{(k)\prime}A_0^{(l-1)} + \sum_{j+k+l+m=p} x_0^{(j)\prime}x_0^{(k)\prime}x_0^{(l)}B_0^{(m)}. \tag{1.38}$$

One can readily find that $C_0^{(2)}$ is absent in (1.37.2) and that $D_0^{(2)} = 0$ gives a quadratic constraint on $\big(A_0^{(0)}, B_0^{(0)}\big)$ (cf. [9, (1.1.1.17)]). Hence, by assuming $D_0^{(2)} = 0$, we can solve the equation $\mathcal{E}_0^{(2)} = 0$ to find $x_0^{(2)}(t,\rho)$ that is holomorphic near $t = 0$. As one of the most exciting points in our computation becomes visible at the next stage, we hasten to study the situation where $p = 3$; we will come back to the explicit computation of $x_0^{(2)}(t,\rho)$ after the study of the case. For this purpose we assume $C_0^{(3)} = 0$. Then a straightforward computation shows that this gives another quadratic constraint on $\big(A_0^{(0)}, B_0^{(0)}\big)$ (cf. [9, (1.1.1.19)]). The equations $D_0^{(2)} = C_0^{(3)} = 0$ lead to

$$A_0^{(0)} = f^{(1)}(0,\rho) \tag{1.39}$$

and

$$B_0^{(0)2} = \rho^2. \tag{1.40}$$

Thus it follows from (1.29) and (1.35) that

$$Z_0^2 = 1. \tag{1.41}$$

And, by (1.33) we find the following amazing result:

$$x_0^{(1)}(0,\rho) = 0! \tag{1.42}$$

The relation (1.42) together with (1.41) plays a crucially important role at several points in the reasoning of [9]. As a typical example of such points we show here how (1.41) and (1.42) effect the computation of $x_0^{(2)}(0,\rho)$. To begin with, we rewrite (1.26.2) explicitly in $s\big(=x_0^{(0)}(t,\rho)\big)$-variable:

(1.26′.2)

$$\begin{aligned} &B_0^{(0)}\Big(2s\frac{d}{ds}-1\Big)x_0^{(2)}(s,\rho) = -A_0^{(1)} - B_0^{(2)}s \\ &- 2\dot{x}_0^{(1)}(s,\rho)A_0^{(0)} - 2\dot{x}_0^{(1)}(s,\rho)x_0^{(1)}(s,\rho)B_0^{(0)} - 2\dot{x}_0^{(1)}(s,\rho)sB_0^{(1)} \\ &- x_0^{(1)}(s,\rho)B_0^{(1)} - \dot{x}_0^{(1)}(s,\rho)^2 sB_0^{(0)} + \big(z(t,\rho)f^{(2)}(t,\rho) \\ &- \big[t^{-1}\big(x_0^{(0)\prime}(t,\rho)\big)^{-2}\big(\mathcal{B}^{(1)}(t,\rho)-\mathcal{B}^{(1)}(0,\rho)\big)\big]\big)\big|_{t=t(s,\rho)}, \end{aligned}$$

where $z(t,\rho)$ is the function given by (1.36) and

$$\mathcal{B}^{(1)}(t,\rho) = \tilde{f}^{(0)} - x_0^{(1)2}\tilde{f}^{(0)} - 2\tilde{x}_0^{(0)}x_0^{(1)}f^{(1)} - x_0^{(0)\prime 2}\tilde{x}_0^{(0)}B_0^{(0)}. \tag{1.43}$$

By way of parenthesis we note that the condition $D_0^{(2)} = 0$ is given by $\mathcal{B}^{(1)}(0,\rho) = 0$. To evaluate the term in the brackets in (1.26′.2) at $s = 0$ we compute $\partial\mathcal{B}^{(1)}/\partial t|_{t=0}$ to find

(1.44)

$$\rho g'(0,\rho) - 2Z_0 f^{(1)}(0,\rho)x_0^{(1)\prime}(0,\rho) - 2B_0^{(0)}x_0^{(0)\prime\prime}(0,\rho) - B_0^{(0)}\tilde{x}_0^{(0)\prime}(0,\rho).$$

In this computation we have repeatedly used (1.41) and (1.42); for example, we have used (1.42) to claim

$$\big(x_0^{(1)2}\tilde{f}^{(0)}\big)'\big|_{t=0} = x_0^{(1)}\big(\tilde{x}_0^{(0)}f^{(1)}\big)'\big|_{t=0} = 0. \tag{1.45}$$

Using (1.41), we further notice a remarkable cancellation of terms in the right-hand side of (1.26′.2) when they are evaluated at $s = 0$; it follows from (1.39) that $-2\dot{x}_0^{(1)}(0,\rho)A_0^{(0)}$ is cancelled by $-\big(x_0^{(0)\prime}(0,\rho)\big)^{-2}\big(-2Z_0 f^{(1)}(0,\rho)x_0^{(1)\prime}(0,\rho)\big)$ in (1.44), i.e.,

$$-2\dot{x}_0^{(1)}(0,\rho)A_0^{(0)} + 2Z_0^{-2}\big(Z_0^2 A_0^{(0)}\dot{x}_0^{(1)}(0,\rho)\big) = 0. \tag{1.46}$$

An important implication of (1.46) is that the cancelling terms originally depended on $B_0^{(1)}$ through $\dot{x}_0^{(1)}(0,\rho)$ (cf. (1.34)). Furthermore other $B_0^{(1)}$-dependent terms in the right-hand side of (1.26′.2), i.e.,

$$\begin{aligned} &-2\dot{x}_0^{(1)}(s,\rho)x_0^{(1)}(s,\rho)B_0^{(0)} - 2\dot{x}_0^{(1)}(s,\rho)sB_0^{(1)} \\ &- x_0^{(1)}(s,\rho)B_0^{(1)} - \dot{x}_0^{(1)}(s,\rho)^2 sB_0^{(0)} \end{aligned} \tag{1.47}$$

also vanish when evaluated at $s=0$, thanks to (1.42). It then follows from (1.26′.2) that

$$B_0^{(0)}x_0^{(2)}(0,\rho) = A_0^{(1)} - f^{(2)}(0,\rho) + \chi_0^{(0)}B_0^{(0)}, \tag{1.48}$$

where $\chi_0^{(0)}$ is a constant fixed by $g(t,\rho)$ (and $Z_0=\pm 1$). Thus $x_0^{(2)}(0,\rho)$ is free from $B_0^{(1)}$, and this fact, together with the explicit form of $\dot{x}_0^{(1)}(0,\rho)$ given by (1.34), enables us to explicitly describe $D_0^{(3)}$ and $C_0^{(4)}$. An important point is that these "cancellations and vanishings" occur for every $p \geq 2$ and that they make the concrete expression of the core parts of $D_0^{(p+1)}$ and $C_0^{(p+2)}$ to be "uniform", as is shown below:

(1.49) $C_0^{(p+2)} - 2\Big(A_0^{(p-1)} - \dfrac{A_0^{(0)}}{B_0^{(0)}}B_0^{(p-1)}\Big)$ depends only on $\big(A_0^{(q)},B_0^{(q)}\big)$ $(q \leq p-2)$ and given data such as $f^{(q)}(0,\rho)$ $(q \leq p-1)$,

and

(1.50) $D_0^{(p+1)} - 2Z_0\Big(\dfrac{A_0^{(0)}}{B_0^{(0)}}A_0^{(p-1)} - B_0^{(p-1)}\Big)$ depends only on $\big(A_0^{(q)},B_0^{(q)}\big)$ $(q \leq p-2)$ and given data.

As is clear from (1.49) and (1.50) we can determine $\big(A_0^{(p-1)}, B_0^{(p-1)}\big)$ $(p \geq 2)$ recursively by solving linear equations. (The solvability of the equations is guaranteed by the assumption (1.5) together with the explicit computations (1.39) and (1.40) of $A_0^{(0)}$ and $B_0^{(0)}$.) Here we emphasize the importance of the point that the main parts "$2\big(A_0^{(p-1)} - A_0^{(0)}B_0^{(p-1)}/B_0^{(0)}\big)$ " and "$2Z_0\big(A_0^{(0)}A_0^{(p-1)}/B_0^{(0)} - B_0^{(p-1)}\big)$ " are of the same form for every p. Parenthetically we note that $C_0^{(p+2)}$ (resp., $D_0^{(p+1)}$) read off from (1.26.$p+2$) (resp., (1.26.$p+1$)) at first contains $x_0^{(p)}(0,\rho)$ and $x_0^{(p-1)\prime}(0,\rho)$; their "principal parts", the parts which may be dependent on $A_0^{(p-1)}$ and $B_0^{(p-1)}$, are at first respectively given as follows (cf. [9, Lemma 1.1.2.1]):

$$\big[(x_0^{(0)\prime})^2 A_0^{(p-1)} + 2x_0^{(0)\prime}x_0^{(p-1)\prime}A_0^{(0)} + x_0^{(0)\prime 2}x_0^{(p)}B_0^{(0)}\big]\big|_{t=0}, \tag{1.51}$$

$$\begin{aligned}\big[&2\tilde{x}_0^{(0)}x_0^{(0)\prime}x_0^{(p-1)\prime}B_0^{(0)} + \tilde{x}_0^{(0)}x_0^{(0)\prime 2}B_0^{(p-1)} \\ &+ 2\tilde{x}_0^{(0)}f^{(1)}x_0^{(p)} + \big(x_0^{(0)\prime}\big)^2 x_0^{(p-1)\prime}B_0^{(0)}\big]\big|_{t=0}.\end{aligned} \tag{1.52}$$

Thus the clean and uniform results (1.49) and (1.50) are almost miraculous, and at the same time we believe that, without such uniform expressions, it should be impossible to find conditions that would guarantee the recursive solvability of equations $C_0^{(p+2)} = D_0^{(p+1)} = 0$.

Thus a naive way of inductively determining $(x_0^{(p)}, A_0^{(p)}, B_0^{(p)})$ $(p \geq 1)$ is as follows:

In order to find a holomorphic (in t) solution $x_0^{(p)}(t,\rho)$ of (1.26.p) one first requires $C_0^{(p)} = D_0^{(p)} = 0$; then by rewriting (1.26.p) in $s(= x_0^{(0)}(t,\rho))$-variable we find

$$B_0^{(0)}\Big(2s\frac{d}{ds}-1\Big)x_0^{(p)}(s,\rho) = -A_0^{(p-1)} - B_0^{(p)}s + B_0^{(0)}R_0^{(p)}(s,\rho), \tag{1.26$'$.p}$$

where

(1.53.p)

$$B_0^{(0)} R_0^{(p)}(s,\rho) = -\sum_{\substack{i+j+k=p-1\\ k\le p-2}} \dot{x}_0^{(i)}\dot{x}_0^{(j)} A_0^{(k)} - \sum_{\substack{i+j+k+l=p\\ i,j,k,l\le p-1}} \dot{x}_0^{(i)}\dot{x}_0^{(j)} x_0^{(k)} B_0^{(l)}$$

$$+ \Big[\big(x_0^{(0)\prime}(t,\rho) \big)^{-2} t^{-2}$$

$$\times \Big(\sum_{i+j+k=p-3} x_0^{(i)\prime} x_0^{(j)\prime} A_0^{(k)} + \sum_{i+j+k+l=p-2} x_0^{(i)\prime} x_0^{(j)\prime} x_0^{(k)} B_0^{(l)}$$

$$+ \sum_{\substack{i+j+k=p\\ k\ge 1}} x_0^{(i)} x_0^{(j)} f^{(k)} + \sum_{\substack{i+j=p\\ i,j\ge 1}} x_0^{(i)} x_0^{(j)} f^{(0)} - f^{(p-2)} \Big) \Big] \Big|_{t=t(s,\rho)}.$$

It is then clear that ($1.26'.p$) admits a holomorphic solution $x_0^{(p)}(s,\rho)$ for any complex numbers $A_0^{(p-1)}$ and $B_0^{(p)}$, as we have assumed $C_0^{(p)} = D_0^{(p)} = 0$. On the other hand, if we admit (1.49) and (1.50), the equation $C_0^{(p)} = 0$ combined with $D_0^{(p-1)} = 0$, a relation required in the preceding stage, will fix $A_0^{(p-3)}$ and $B_0^{(p-3)}$ (for $p \ge 4$), which have not yet been completely fixed so far. At the same time, the condition $D_0^{(p)} = 0$ will be used at the next stage to fix $A_0^{(p-2)}$ and $B_0^{(p-2)}$. Thus the reader might find the reasoning to be somewhat clumsy, particularly because of the unevenness of the indices in question. Hence we present here the core of the more refined induction procedure with some comments on its background. We note that the induction scheme we present below is also suited for the growth order estimation of the functions constructed. See [9, Section 1.1.3 and Section 1.2] for the details.

Let us first prepare some notations. We denote a triplet $\{x_0^{(r)}(s,\rho), A_0^{(r)}, B_0^{(r)}\}$ by $T_0^{(r)}$ and use the symbol $\mathfrak{A}_0(p)$ to mean the assertion that $T_0^{(r)}$ is given for $0 \le r \le p$ so that each of them satisfies the following conditions (1.54.r) $\sim$ (1.58.r):

(1.54.r) $x_0^{(r)}(s,\rho)$ is a holomorphic solution of $(1.26'.r)$ near $s=0$,

(1.55.r) $x_0^{(r)}(s,\rho)$ depends on $\big(\overrightarrow{A}_0[r-1], \overrightarrow{B}_0[r]\big) \underset{\text{def}}{=} \big(A_0^{(0)}, A_0^{(1)}, \cdots, A_0^{(r-1)}, B_0^{(0)}, B_0^{(1)}, \cdots, B_0^{(r)}\big)$,

(1.56.r) $C_0^{(r+3)}$ and $D_0^{(r+2)}$ depend on $\big(\overrightarrow{A}_0[r], \overrightarrow{B}_0[r]\big)$, and $\big(\overrightarrow{A}_0[r], \overrightarrow{B}_0[r]\big)$ annihilates them,

(1.57.r) $C_0^{(r+3)} - 2\big(A_0^{(r)} - \dfrac{A_0^{(0)}}{B_0^{(0)}} B_0^{(r)}\big)$ is independent of $\big(A_0^{(r)}, B_0^{(r)}\big)$,

(1.58.r) $D_0^{(r+2)} - 2Z_0\Big(\dfrac{A_0^{(0)}}{B_0^{(0)}} A_0^{(r)} - B_0^{(r)}\Big)$ is independent of $\big(A_0^{(r)}, B_0^{(r)}\big)$.

Then we obtain

Proposition 1.1. *The assertion $\mathfrak{A}(p)$ is valid for every $p \geq 1$.*

The proof of this proposition is done in an inductive manner (cf. [9, Section 1.1.3]). But we imagine that the first reactions to this proposition of the reader might be the following:

[A] Is the claim logically self-contained? For example, the concrete expression (1.51) (resp., (1.52)) of $C_0^{(p+2)}$ (resp., $D_0^{(p+1)}$) indicates that we need $x_0^{(p_0+1)}(0,\rho)$ for the description of $C_0^{(p_0+3)}$ and $D_0^{(p_0+2)}$, but $\mathfrak{A}_0(p_0)$ refers to $T_0^{(r)}$ $(r \leq p_0)$ only.

[B] Well, this may not be a logical question but a rather psychological one. Still, I wonder why (1.56.p_0) is valid despite the presence of $x_0^{(p_0+1)}$ in $C_0^{(p_0+3)}$; in view of (1.55.p_0+1) I think $\overrightarrow{B}_0[r]$ in (1.56.r) might be $\overrightarrow{B}_0[r+1]$.

So let us first dispel potential sources of such uneasiness. Actually both [A] and [B] are reasonable concerns and the core of the proof of

Proposition 1.1 is closely related to them. The answer to [A] is rather easy: although $x_0^{(p_0+1)}(s,\rho)$ is not referred to in $\mathfrak{A}_0(p_0)$, the assertion $\mathfrak{A}_0(p_0)$ trivially entails the vanishing of $C_0^{(p_0+1)}$ and $D_0^{(p_0+1)}$ and hence the existence of a holomorphic solution $x_0^{(p_0+1)}(s,\rho)$ of $(1.26'.p_0+1)$ is guaranteed. Then it follows from $(1.26'.p_0+1)$ that $x_0^{(p_0+1)}(0,\rho)$ is given by

$$x_0^{(p_0+1)}(0,\rho) = \left(B_0^{(0)}\right)^{-1} A_0^{(p_0)} - R_0^{(p_0+1)}(0,\rho). \tag{1.59}$$

Thus $x_0^{(p_0+1)}(0,\rho)$ is described by $T_0^{(r)}$ $(r \le p_0)$. Note that $R_0^{(p_0+1)}(s,\rho)$ is determined by $T_0^{(r)}$ $(r \le p_0)$. (Cf. (1.53.p)) This concrete expression of $x_0^{(p_0+1)}(0,\rho)$ will also alleviate the anxiety [B]. Still, the reader might wonder:

[B′] How can we proceed with a seemingly rather vague expression like (1.59)? For example, how can we find (1.57.p_0+1) and (1.58.p_0+1), which are needed to proceed one step further, that is, to confirm $\mathfrak{A}_0(p_0+1)$ using the data in $\mathfrak{A}_0(p_0)$?

Well, then, we present the core of the proof of Proposition 1.1, which will clarify all these.

Remark 1.5. Here we have tried to follow the late Professor Ehrenpreis in his style of lecturing —— how do you find it, Professor Ehrenpreis?

To perform the induction procedure, let us suppose that $\mathfrak{A}_0(p_0)$ is validated. Then, as we noted to see (1.59), we have

$$C_0^{(p_0+1)} = D_0^{(p_0+1)} = 0, \tag{1.60}$$

and hence we can find a holomorphic solution $x_0^{(p_0+1)}(s,\rho)$ of $(1.26'.p_0+1)$ for any complex number $B_0^{(p_0+1)}$, which meets the requirement (1.54.p_0 +1) and (1.55.p_0+1). Now, the intriguing part of the proof begins here. Since $\mathfrak{A}_0(p_0)$ entails

$$C_0^{(p_0+2)} = D_0^{(p_0+2)} = 0, \tag{1.61}$$

we can further find a holomorphic solution $x_0^{(p_0+2)}(s,\rho)$ of $(1.26'.p_0+2)$ for any complex numbers $A_0^{(p_0+1)}$ and $B_0^{(p_0+2)}$. To confirm $\mathfrak{A}_0(p_0+1)$ we do not make full use of $x_0^{(p_0+2)}(s,\rho)$ but use only $x_0^{(p_0+2)}(0,\rho)$ for the computation of $C_0^{(p_0+4)}$ and $D_0^{(p_0+3)}$. Since it follows from $(1.26'.p_0+2)$ that

$$B_0^{(0)}x_0^{(p_0+2)}(0,\rho) = A_0^{(p_0+1)} - B_0^{(0)}R_0^{(p_0+2)}(0,\rho), \tag{1.62}$$

the following Lemma 1.1 is the key to the proof.

Lemma 1.1. *Let us suppose $\mathfrak{A}_0(p_0)$ is validated. Then we find*

$$B_0^{(0)}R_0^{(p_0+2)}(0,\rho) \textit{ is free from } B_0^{(p_0+1)}. \tag{1.63}$$

Before giving the proof of this lemma, we note the following three facts: first, once the lemma is proved, the confirmation of $\mathfrak{A}_0(p_0+1)$ is an easy task as we will note later. Second, although this is a rather obvious comment, the complex number $B_0^{(p_0+2)}$ introduced to define $x_0^{(p_0+2)}(s,\rho)$ is actually irrelevant to $x_0^{(p_0+2)}(0,\rho)$ and has no relevance to the later argument; in validating $\mathfrak{A}_0(p_0+2)$ we may use another complex number $\tilde{B}_0^{(p_0+2)}$ to construct $\tilde{x}_0^{(p_0+2)}(s,\rho)$ needed there, which may be different from $x_0^{(p_0+2)}(s,\rho)$ constructed above for the auxiliary purpose of finding the constant $x_0^{(p_0+2)}(0,\rho)$, which is irrelevant to $B_0^{(p_0+2)}$. Third, the cancellation among several terms to be observed in the proof of Lemma 1.1 also plays crucially important roles in the estimation of growth orders of $T_0^{(p)}$ etc. (See [C1] and [C2] after Remark 1.7).

Now we give

Proof of Lemma 1.1. In view of $(1.55.r)$ $(r \le p_0)$ we find that the terms in $B_0^{(0)}R_0^{(p_0+2)}(0,\rho)$ which may contain $B_0^{(p_0+1)}$ are those which contain $x_0^{(p_0+1)}$, $\dot{x}_0^{(p_0+1)}$ and $B_0^{(p_0+1)}$ itself. Furthermore we note that $x_0^{(p_0+1)}(0,\rho)$ is seen to be free from $B_0^{(p_0+1)}$ by (1.59) together with the fact that $R_0^{(p_0+1)}(s,\rho)$ is determined by $T_0^{(r)}$ $(r \le p_0)$. Thus we do not worry about $-\Big(\sum_{i+j+k=1}\dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)B_0^{(k)}\Big)x_0^{(p_0+1)}(0,\rho)$ in our computation. Hence it is enough to examine the contribution from the following terms:

$$-\Big(\sum_{i+j+k=1}\dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)x_0^{(k)}(0,\rho)\Big)B_0^{(p_0+1)}, \tag{1.64}$$

$$\begin{aligned} &-\Big(\sum_{\substack{i+j=p_0+2\\ i,j\le p_0+1}}\dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big)x_0^{(0)}(0,\rho)B_0^{(0)}\\ &-\Big(\sum_{i+j=p_0+1}\dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big)\Big(\sum_{k+l=1}x_0^{(k)}(0,\rho)B_0^{(l)}\Big), \end{aligned} \tag{1.65}$$

$$-2\dot{x}_0^{(0)}(0,\rho)\dot{x}_0^{(p_0+1)}(0,\rho)A_0^{(0)} \tag{1.66}$$

and

(1.67) terms that appear in the coefficients of the Taylor expansion in s of $\Big[\big(x_0^{(0)\prime}\big)^{-2}t^{-2}\big(2x_0^{(0)}x_0^{(p_0+1)}f^{(1)}+2x_0^{(1)}x_0^{(p_0+1)}f^{(0)}\big)\Big]\Big|_{t=t(s,\rho)}$.

Here we observe the following two facts:

(1.68) any term that may contain $B_0^{(p_0+1)}$ in (1.64) and (1.65) vanishes because of the vanishing of $x_0^{(i)}(0,\rho)$ $(i=0,1)$,

and

(1.69)
$$-2\dot{x}_0^{(0)}(0,\rho)\dot{x}_0^{(p_0+1)}(0,\rho)A_0^{(0)}+2\big(x_0^{(0)\prime}\big)^{-2}\tilde{x}_0^{(0)}x_0^{(p_0+1)\prime}f^{(1)}\big|_{t=t(0,\rho)}=0,$$

where the second term in (1.69) is the unique relevant term in (1.67). (Cf. Remark 1.6 below.) It is then evident that (1.68) (resp., (1.69)) is a counterpart of (1.45) (resp., (1.46)), which we encountered in the computation of $x_0^{(2)}(0,\rho)$. In any event, (1.68) and (1.69) clearly prove the lemma.

Q.E.D.

Remark 1.6. Since $x_0^{(p_0+1)}(0,\rho)$ is free from $B_0^{(p_0+1)}$ as noted above, $B_0^{(p_0+1)}$ is not contained in

$$2\big(x_0^{(0)\prime}(0,\rho)\big)^{-2}\tilde{x}_0^{(1)}(0,\rho)\tilde{f}^{(0)}(0,\rho)x_0^{(p_0+1)}(0,\rho), \tag{1.70}$$

despite the fact that (1.70) is resembling to the second term in (1.69) in the sense that (1.70) originates from

$$\big[\big(x_0^{(0)\prime}\big)^{-2}t^{-2}\big(2x_0^{(1)}x_0^{(p_0+1)}f^{(0)}\big)\big]\big|_{t=t(s,\rho)}, \tag{1.71}$$

which forms the pair to

$$\big[\big(x_0^{(0)\prime}\big)^{-2}t^{-2}\big(2x_0^{(0)}x_0^{(p_0+1)}f^{(1)}\big)\big]\big|_{t=t(s,\rho)} \tag{1.72}$$

in (1.67), the term which generates the second term in (1.69).

Now Lemma 1.1 and (1.62) imply

$$x_0^{(p_0+2)} - A_0^{(p_0+1)}/B_0^{(0)} \quad \text{depends on only } \big(\overrightarrow{A}_0[p_0], \overrightarrow{B}_0[p_0]\big). \tag{1.73}$$

On the other hand $(1.26'.p_0+1)$ entails

$$B_0^{(0)}\dot{x}_0^{(p_0+1)}(0,\rho) + B_0^{(p_0+1)} = B_0^{(0)}\dot{R}_0^{(p_0+1)}(0,\rho), \tag{1.74}$$

which also depends on only $\big(\overrightarrow{A}_0[p_0], \overrightarrow{B}_0[p_0]\big)$.

Substituting those into (1.51) and (1.52) with $p = p_0+2$, we can validate $(1.57.p_0+1)$ and $(1.58.p_0+1)$. Then we can readily choose $\big(A_0^{(p_0+1)}, B_0^{(p_0+1)}\big)$ so that they satisfy

$$C_0^{(p_0+4)} = D_0^{(p_0+3)} = 0. \tag{1.75}$$

Thus the induction proceeds. This completes the proof of Proposition 1.1.

Remark 1.7. As (1.73) and (1.74) show, expressions like (1.59) nicely fit in with our induction scheme. This is the answer to the query [B′], and the important point in the answer is Lemma 1.1.

Thus we have formally constructed $T_0^{(p)} = \{x_0^{(p)}, A_0^{(p)}, B_0^{(p)}\}$ for every $p \geq 0$. We can further confirm (cf. [9, Lemma 1.2.3]) that they actually define a function

$$x_0(t, a, \rho) = \sum_{p \geq 0} x_0^{(p)}(t, \rho) a^p, \tag{1.76}$$

which is holomorphic on

$$\{(t, a, \rho) \in \mathbb{C}^3; |t| < r_0, \rho \neq 0, |a|, |\rho| < M_0, |a/\rho| < N_0\} \tag{1.77}$$

and constants

$$A_0(a, \rho) = \sum_{p \geq 0} A_0^{(p)}(\rho) a^p \tag{1.78}$$

and

$$B_0(a, \rho) = \sum_{p \geq 0} B_0^{(p)}(\rho) a^p, \tag{1.79}$$

which are convergent on

$$\{(a, \rho) \in \mathbb{C}^2; \rho \neq 0, |a|, |\rho| < M_0, |a/\rho| < N_0\} \tag{1.80}$$

for some positive constants r_0, M_0 and N_0. Although we do not give the details of the proof here, we note the following core facts [C1] and [C2]. Here we use the symbol $(\sigma.j)$ ($j =$ i, ii and iii) to denote the following sums in $R_0^{(p_0+1)}(s, \rho)$ (cf. (1.53.p) with $p = p_0 + 1$):

$$(\sigma.\mathrm{i}) \underset{\mathrm{def}}{=} -\sum_{i+j=p_0} \dot{x}_0^{(i)}(s, \rho) \dot{x}_0^{(j)}(s, \rho) A_0^{(0)} / B_0^{(0)}, \tag{1.81}$$

(cf. the first sum in $(1.53.p_0 + 1)$),

(1.82)

$$(\sigma.\text{ii}) \underset{\text{def}}{=} \left[\left(x_0^{(0)\prime}(t,\rho)\right)^{-2} t^{-2}\Big(\sum_{i+j=p_0} x_0^{(i)}(s,\rho)x_0^{(j)}(s,\rho)f^{(1)}(t,\rho)/B_0^{(0)}\Big)\right]\Big|_{t=t(s,\rho)}$$

(cf. the fifth sum in $(1.53.p_0+1)$),

(1.83)

$$(\sigma.\text{iii}) \underset{\text{def}}{=} \left[\left(x_0^{(0)\prime}(t,\rho)\right)^{-2} t^{-1}\tilde{f}^{(0)}(t)\Big(\sum_{\substack{i+j=p_0+1\\ i,j\geq 1}} x_0^{(i)}(s,\rho)x_0^{(j)}(s,\rho)/B_0^{(0)}\Big)\right]\Big|_{t=t(s,\rho)}$$

(cf. the sixth sum in $(1.53.p_0+1)$).

Now in inductively showing the domination of $\{x_0^{(p)}, A_0^{(p)}, B_0^{(p)}\}$ which guarantees the domains of convergence (1.77) and (1.80) we at first find that each of these three terms might block the induction reasoning from proceeding. But, fortunately we observe

[C1] What we encounter in the induction process is the estimation of the integral of the form, say,

$$I(\text{iii}) = \frac{1}{2\pi i}\oint \frac{(\sigma.\text{iii})}{s}ds; \tag{1.84}$$

then by the Taylor expansion of

$$\sum_{\substack{i+j=p_0+1\\ i,j\geq 1}} x_0^{(i)}(s,\rho)x_0^{(j)}(s,\rho), \tag{1.85}$$

we find the following from the relation $\tilde{f}^{(0)} = \rho g$:

(1.86)

$$|I(\text{iii})| = \Big|\frac{1}{2\pi i}\oint\Big(\frac{dt}{ds}\Big)^2\Big(\frac{s}{t}\Big)Z_0\, g(t,\rho)\Big\{\sum_{\substack{i+j=p_0+1\\ i,j\geq 1}} x_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)$$

$$+ 2s\Big(\sum_{\substack{i+j=p_0+1\\ i,j\geq 1}} x_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big) + O(s^2)\Big\}\frac{ds}{s^2}\Big|.$$

Then in order to make the induction reasoning run smoothly we use (1.42); the second sum in the integrand of the right-hand side gives the contribution of the form

$$\frac{1}{2\pi i}\oint 2\Big(\sum_{\substack{i+j=p_0+1\\ i\geq 2,\, j\geq 1}} x_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big)\frac{ds}{s}. \tag{1.87}$$

See [9] for the details which show how this gain in the margin of indices is important in the induction procedures.

[C2] The integral

$$I(\mathrm{i}) = \frac{1}{2\pi i}\oint \frac{(\sigma.\mathrm{i})}{s}\, ds \tag{1.88}$$

is, notably enough, cancelled by the contribution

(1.89)

$$I_0 = \frac{1}{2\pi i}\frac{1}{B_0^{(0)}}\oint \frac{s^2}{t^2}\Big(\frac{dt}{ds}\Big)^2\Big(\sum_{i+j=p_0} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big) f^{(1)}(t,\rho)\frac{ds}{s},$$

which originates from

$$I(\mathrm{ii}) = \frac{1}{2\pi i}\oint \frac{(\sigma.\mathrm{ii})}{s}\, ds, \tag{1.90}$$

and, furthermore $I(\mathrm{ii}) - I_0$ is amenable to the induction procedure, as is shown in [9].

We readily find [C1] and [C2] are reasonable counterparts of (1.68) and (1.69) respectively.

Thus we have succeeded in constructing $\{x_0(t,a,\rho), A_0(a.\rho), B_0(a,\rho)\}$ which satisfies the highest degree (i.e., degree 0) part in η of the required relation (1.13); hence the reasonable approach to the proof of

(1.13) is to try to construct the perturbation series $\{x = \sum_{k\geq 0} x_{2k}\eta^{-2k}$, $A = \sum_{k\geq 0} A_{2k}\eta^{-2k}$, $B = \sum_{k\geq 0} B_{2k}\eta^{-2k}\}$ so that they satisfy (1.13). As we mentioned earlier we further expand $\{x_{2k}, A_{2k}, B_{2k}\}$ into the power series of a (cf. (1.19), (1.20) and (1.21)), and by comparing the coefficients of a^p in the coefficients of η^{-2n} ($n \geq 1$) of (1.13) multiplied by $(t^2 - a^2)(x^2 - a^2)$ we obtain

$$
\begin{aligned}
&\sum_{\substack{q+r+u=p\\ i+j=n}} x_{2i}^{(q)} x_{2j}^{(r)} f^{(u)} \\
&= t^2\Big[\sum_{\substack{q+r+u=p-1\\ i+j+k=n}} x_{2i}^{(q)\prime} x_{2j}^{(r)\prime} A_{2k}^{(u)} + \sum_{\substack{q+r+u+v=p\\ i+j+k+l=n}} x_{2i}^{(q)\prime} x_{2j}^{(r)\prime} x_{2k}^{(u)} B_{2l}^{(v)} \\
&\quad - \frac{1}{2}\sum_{\substack{q+r+u=p\\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x;t\}_{2k}^{(u)} + \frac{1}{2}\{x;t\}_{2(n-1)}^{(p-2)}\Big] \\
&\quad - \Big[\sum_{\substack{q+r+u=p-3\\ i+j+k=n}} x_{2i}^{(q)\prime} x_{2j}^{(r)\prime} A_{2k}^{(u)} + \sum_{\substack{q+r+u+v=p-2\\ i+j+k+l=n}} x_{2i}^{(q)\prime} x_{2j}^{(r)\prime} x_{2k}^{(u)} B_{2l}^{(v)} \\
&\quad - \frac{1}{2}\sum_{\substack{q+r+u=p-2\\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x;t\}_{2k}^{(u)} + \frac{1}{2}\{x;t\}_{2(n-1)}^{(p-4)}\Big],
\end{aligned}
\tag{1.91}
$$

where $\{x;t\}_{2k}^{(q)}$ designates the coefficient of $a^q\eta^{-2k}$ of $\{x;t\}$, that is,

$$
\{x;t\} = \sum_{q,k\geq 0} \{x;t\}_{2k}^{(q)} a^q \eta^{-2k}. \tag{1.92}
$$

In view of the resemblance between (1.26.p) and (1.91), one expects that the construction and domination of the triplet $T_{2n}^{(r)} = \{x_{2n}^{(r)}(s,\rho)$, $A_{2n}^{(r)}(\rho), B_{2n}^{(r)}(\rho)\}$ ($n \geq 1$) may be performed in parallel with the construction and domination of $T_0^{(r)}$. And, actually this is really the case.

We only note the following facts:

(1.93) in the recursive construction of $x_{2n}^{(p)}(s, \rho)$ $(p = 0, 1, 2, \cdots)$ the relation $x_{2n}^{(0)}(0, \rho) = 0$ plays an important role,

(1.94) assertions similar to [C1] and [C2] (with the appropriate shift of indices) also play important roles,

and

(1.95) in dominating the growth order of $T_{2n}^{(p)}$ we first dominate $\{x; t\}_{2(n-1)}^{(p)}$ using the induction hypothesis and then employ the similar argument used in dominating $T_0^{(p)}$.

We refer the reader to [9, Section 1.2] for the details. Here we content ourselves by quoting the final result which will be used later.

Theorem 1.1. *Let $Q(t, a, \rho, \eta)$ be a potential of an M2P1T operator given by* (1.6)*. Then there exist positive constants r_0, M_0, N_0, R_0 and holomorphic functions $A_{2n}(a, \rho)$,$B_{2n}(a, \rho)$ and $x_{2n}(t, a, \rho)$ $(n \geq 0)$ on*

(1.96) $$\{(t, a, \rho) \in \mathbb{C}^3; |t| < r_0, \rho \neq 0, |a|, |\rho| < M_0, |a/\rho| < N_0\}$$

for which the following conditions are satisfied there;

(1.97) *$A(a, \rho, \eta), B(a, \rho, \eta)$ and $x(t, a, \rho, \eta)$ satisfy* (1.13),

(1.98) $$\frac{1}{2}|f^{(1)}(0, 0)| \leq |A_0(a, \rho)| \leq 2|f^{(1)}(0, 0)|,$$

(1.99) $$|B_0(a, \rho)| \leq 2|\rho|,$$

(1.100) $$\frac{\partial x_0}{\partial t}(t, a, \rho) \neq 0,$$

(1.101) $$x_0^2(\pm a, a, \rho) = a^2,$$

(1.102) *if* $t = t_0(a,\rho)$ *satisfies* $f(t_0,a,\rho) = 0$ *then* $aA_0(a,\rho) + x_0(t_0,a,\rho)B_0(a,\rho) = 0$ *holds,*

the following estimates hold for $n \geq 1$*;*

$$|A_{2n}(a,\rho)| \leq |\rho|(2n)!R_0^n|\rho|^{-n}, \tag{1.103}$$

$$|B_{2n}(a,\rho)| \leq |\rho|(2n)!R_0^n|\rho|^{-n}, \tag{1.104}$$

$$|x_{2n}(t,a,\rho)| \leq |\rho|(2n)!R_0^n|\rho|^{-n}, \tag{1.105}$$

$$\left|\frac{dx_{2n}}{dt}(t,a,\rho)\right| \leq |\rho|(2n)!R_0^n|\rho|^{-n}. \tag{1.106}$$

Remark 1.8. Although we have presented the results assuming (1.8), the construction and the domination of $\{x, A, B\}$ can be done without the assumption. In this case the potential of the canonical form of an M2P1T equation is

$$\frac{aA + xB}{x^2 - a^2} + \eta^{-2}\Big(\frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2}\Big). \tag{1.107}$$

2 Intertwining the Borel transformed Schrödinger operators

As was first observed in [2], the analytic meaning of the formal coordinate transformation becomes most transparent with the help of the Borel transformation. To describe the situation concretely, let us first introduce the inverse function $h(x,a,\rho)$ of $x = x_0(t,a,\rho)$, that is,

$$x = x_0\big(h(x,a,\rho),a,\rho\big),\ t = h\big(x_0(t,a,\rho),a,\rho\big). \tag{2.1}$$

Since we formally find

(2.2)

$$\psi(x_0+\eta^{-2}x_2+\eta^{-4}x_4+\cdots,\eta) = \sum_{n\geq 0}\frac{1}{n!}\Big(\sum_{k\geq 1}x_{2k}\eta^{-2k}\Big)^n\frac{\partial^n}{\partial x^n}\psi(x,\eta)\big|_{x=x_0},$$

its Borel transform has the form

$$\left(\sum_{n\geq 0}\frac{1}{n!}\Big(\sum_{k\geq 1}x_{2k}\big(h(x,a,\rho),a,\rho\big)\Big(\frac{\partial}{\partial y}\Big)^{-2k}\Big)^{n}\frac{\partial^n}{\partial x^n}\right)\psi_B(x,y) \tag{2.3}$$
$$=: \exp\left(\Big(\sum_{k\geq 1}x_{2k}\big(h(x,a,\rho),a,\rho\big)\eta^{-2k}\Big)\xi\right) : \psi_B(x,y).$$

In the right-hand side of (2.3), and also in what follows, we denote by ξ the symbol of $\partial/\partial x$ and use the ideograms in the symbol calculus of microdifferential operators; in particular the ideogram $:\sigma:$ designates the normal ordered product determined by a symbol σ. We note that $:\sigma:$ makes sense as a microdifferential operator when the formal series σ satisfies some growth order conditions like those we discussed in Theorem 1.1. (Cf. Theorem 2.1 below.) See [1] for the details of the symbol calculus. The relation (2.3) indicates that the structure of Schrödinger equations should be most clearly understood when they are Borel transformed. Actually we find Theorem 2.1 below by making use of the formal series constructed in Section 1.

To state the theorem let us prepare some notations.

Let N denote the Borel transform of an M2P1T operator written in (x,y)-coordinate, that is,

(2.4)
$$N=\Big(\frac{\partial h}{\partial x}\Big)^{-2}\frac{\partial^2}{\partial x^2}-\frac{\partial^2 h}{\partial x^2}\Big(\frac{\partial h}{\partial x}\Big)^{-3}\frac{\partial}{\partial x}-Q\left(h(x,a,\rho),a,\rho,\frac{\partial}{\partial y}\right)\frac{\partial^2}{\partial y^2}.$$

We also denote the Borel transform of the ∞-Mathieu equation by M_∞. Using $\{x_{2n}\}_{n\geq 0}$ and the function h in (2.1), we define

$$r_{2k}=x_{2k}\big(h(x,a,\rho),a,\rho\big) \tag{2.5}$$

$$r=\sum_{k\geq 1}r_{2k}\eta^{-2k}, \tag{2.6}$$

$$s = x + r, \tag{2.7}$$

$$\mathcal{X} =: \left(\frac{\partial h}{\partial x}\right)^{1/2} \left(\frac{\partial s}{\partial x}\right)^{-1/2} \exp(r\xi) :, \tag{2.8}$$

$$\mathcal{Y} =: \left(\frac{\partial h}{\partial x}\right)^{-3/2} \left(\frac{\partial s}{\partial x}\right)^{3/2} \exp(r\xi) : . \tag{2.9}$$

To describe the geometric situation we introduce the following set W where C_0, δ_0 and δ_1 are some positive constants:

$$W = \left\{(a, \rho) \in \mathbb{C}^2;\ |a| \leq C_0|\rho|,\ 0 < |\rho| < \delta_0,\ |a| < \delta_1\right\}. \tag{2.10}$$

With these notations we can deduce Theorem 2.1 below from the results in Section 1 by using the same reasoning as in the proof of Theorem 2.6 of [8].

Theorem 2.1. *Let U be a sufficiently small open neighborhood of the closed interval $[-a, a]$. Then, for sufficiently small constants C_0, δ_0 and δ_1, microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ intertwine N and M_∞ on $U \times W_0$ with the exception of $(x^2 - a^2)\eta = 0$, that is, we have*

$$N\mathcal{X} = \mathcal{Y}M_\infty \tag{2.11}$$

with $\mathcal{X}$ and $\mathcal{Y}$ being invertible there.

Although the ∞-Mathieu equation contains infinite series A and B, they satisfy the growth order conditions stated in Theorem 1.1. The growth order conditions enable us to relate, by microdifferential operators, the Borel transformed ∞-Mathieu operator and the Borel transformed Mathieu operator $M = M\big(A, B, c_+, c_-\big)$, that is,

$$M\big(A, B, c_+, c_-\big) = \frac{\partial^2}{\partial x^2} - \frac{aA + xB}{x^2 - a^2}\frac{\partial^2}{\partial y^2} - \frac{c_+}{(x-a)^2} - \frac{c_-}{(x+a)^2} \tag{2.12}$$

with A, B and $c_\pm$ being genuine constants, as the following Theorem 2.2 shows.

Theorem 2.2. *There exist microdifferential operators $\mathcal{A}$ and $\mathcal{B}$ for which the following relation holds:*

$$\mathcal{A}\mathcal{B}M = M_\infty \mathcal{A}\mathcal{B}. \tag{2.13}$$

The proof is essentially the same as the proof of Theorem 4.1 of [10]; it suffices to define

$$\mathcal{A} =: \exp\Big(\sum_{k\geq 1} A_{2k}\eta^{-2k}\Big)a\alpha_0 : \tag{2.14}$$

and

$$\mathcal{B} =: \exp\Big(\sum_{d\geq 1} B_{2k}\eta^{-2k}\Big)\beta_0 :, \tag{2.15}$$

where α_0 (resp., β_0) stands for the symbol of $\partial/\partial(aA_0)$ (resp., $\partial/\partial B_0$).

These theorems assert that the microlocal structure of Borel transformed WKB solutions of an M2P1T equation coincides with that of a Mathieu equation. By appropriately representing the action of the microdifferential operator in question as an integro-differential operator acting on multi-valued analytic functions, we can deduce informations on the alien derivatives of WKB solutions of an M2P1T equation from those of its canonical equation. To attain this goal, we first show the following

Theorem 2.3. *The action of the microdifferential operator $\mathcal{X}$ (given by (2.8)) upon the Borel-transformed WKB solution $\psi_{+,B}$ of the ∞-Mathieu equation is expressed as an integro-differential operator of the form*

$$\mathcal{X}\psi_{+,B} = \int_{-y_+}^{y} K(x,a,\rho,y-y',\partial/\partial x)\psi_{+,B}(x,a,\rho,y')dy', \tag{2.16}$$

where

$$y_+(x,a,\rho) = \int_a^x \sqrt{\frac{aA_0(a,\rho)+xB_0(a,\rho)}{x^2-a^2}}dx \tag{2.17}$$

and $K(x, a, \rho, y, \partial/\partial x)$ is a differential operator of infinite order (in the sense of [15]*) which is defined on $\{(x, a, \rho, y) \in \mathbb{C}^4;\ (x, a, \rho) \in U \times W,\ |y| < C|\rho|^{1/2}\}$ for some positive constant C. Similar expressions are also available for the action of $\mathcal{A}$ and $\mathcal{B}$ on the Borel transformed WKB solutions of a Mathieu equation.*

3 Can we focus our attention on the simple poles of the Mathieu equation?

As we emphasized in Introduction, our original problem was to analyze the singularity structure of Borel transformed WKB solutions near fixed singularities determined by a pair of simple poles contained in the potential. But the canonical equation of an M2P1T equation, i.e., the Mathieu equation contains a simple turning point besides two simple poles. Unfortunately no effective WKB-theoretic results are known for the Mathieu equation, but T. Koike has succeeded in computing the Voros coefficient for the Legendre equation. (Private communication. See also [14].) Hence, if we can somehow focus our attention on the simple poles of the Mathieu equation, we will be able to make use of the results of Koike. And, actually this expectation is realized in Section 4. The problem is what we mean by saying "focus our attention on the pole part". The answer is given by Theorem 3.1 below. In what follows, $Q_L(z, C, \gamma_+, \gamma_-)$ denotes

$$\frac{aC}{z^2 - a^2} + \eta^{-2}\Big(\frac{\gamma_+}{(z-a)^2} + \frac{\gamma_-}{(z+a)^2}\Big), \tag{3.1}$$

and $Q_M(x, A, B, c_+, c_-)$ denotes

$$\frac{aA + xB}{x^2 - a^2} + \eta^{-2}\Big(\frac{c_+}{(x-a)^2} + \frac{c_-}{(x+a)^2}\Big). \tag{3.2}$$

Theorem 3.1. *Let $r_1(>1)$ and r_2 be positive constants with r_2 sufficiently small and denote by Ω_{r_1,r_2} the following set:*

$$\left\{(x,a,A,B)\in\mathbb{C}^4;\ |x|<r_1|a|,\ a\neq 0,\ A\neq 0,\ |B|<r_2|A|\right\}. \tag{3.3}$$

Then we can construct infinite series

$$z(x,a,A,B,\eta)=\sum_{k\geq 0} z_{2k}(x,a,A,B)\eta^{-2k} \tag{3.4}$$

and

$$C(a,A,B,\eta)=\sum_{k\geq 0} C_{2k}(a,A,B)\eta^{-2k} \tag{3.5}$$

so that they satisfy the following conditions (3.6) $\sim$ (3.10)*.*

(3.6) *z_{2k} and C_{2k} are holomorphic on Ω_{r_1,r_2},*

(3.7) *for each fixed constants a, A and B the function $z_0(x,a,A,B)$ of x is injective on $\{x\in\mathbb{C};\ |x|<r_1|a|\}$,*

$$(z_0(\pm a,a,A,B))^2=a^2, \tag{3.8}$$

$$\frac{\partial z_0}{\partial x}(x,a,A,B)\neq 0 \quad \text{on} \quad \Omega_{r_1,r_2}, \tag{3.9}$$

$$\begin{aligned} &Q_M(x,A,B,c_+,c_-)\\ &\quad=\Big(\frac{\partial z}{\partial x}\Big)^2 Q_L(z(x,a,A,B,\eta),C,c_+,c_-)-\frac{1}{2}\eta^{-2}\{z;x\}. \end{aligned} \tag{3.10}$$

Further the constructed series z and C satisfy the following estimates:

(3.11) *for any $\varepsilon>0$ we can find sufficiently small r_2 for which*

$$|z_{2k}(x,a,A,B)|\leq (2k)!\varepsilon^k|aA|^{-k} \tag{3.11.i}$$

and

$$|C_{2k}(a, A, B)| \leq (2k)!\varepsilon^k |aA|^{-k} \tag{3.11.ii}$$

hold on Ω_{r_1,r_2} *for every* $k \geq 1$.

In parallel with the reasoning in Section 2 the relation (3.10) together with the estimates (3.11.i) and (3.11.ii) entails that the Borel transformed Mathieu operator and the Borel transformed Legendre operator are intertwined on Ω_{r_1,r_2} by microdifferential operators and that the microdifferential operators enjoy the integral representation similar to (2.16). The point is that the simple turning point of the Mathieu equation, i.e., $-aA/B$, is necessitated to be outside Ω_{r_1,r_2} for sufficiently small r_2. We refer the reader to [9] for the proof of Theorem 3.1; the formal construction of the series z and C is rather straightforward, but their estimation is quite intricate.

As Koike has explicitly written down the Voros coefficient for the Legendre-type equation with a large parameter that has the form

$$\left(\frac{d^2}{dz^2} - \eta^2\Big(\frac{a\Lambda^2}{z^2-a^2} + \eta^{-1}\frac{\sqrt{a}\Lambda}{z^2-a^2} + \eta^{-2}\,\frac{az\nu + a^2(\mu^2-1)}{(z^2-a^2)^2}\Big)\right)\phi = 0, \tag{3.12}$$

we prepare Lemma 3.1 below so that we may make use of Koike's results in Section 4.

Lemma 3.1. *We can rewrite*

$$\Big(\frac{d^2}{dz^2} - \eta^2 Q_L(z, C, c_+, c_-)\Big)\psi = 0 \tag{3.13}$$

in the form (3.12) *if we choose* μ, ν *and*

$$\Lambda(a, C, \eta) = \sum_{k\geq 0} \Lambda_k(a, C)\eta^{-k}, \tag{3.14}$$

by

$$\mu^2 = 1 + 2(c_+ + c_-), \tag{3.15}$$

$$\nu = 2(c_+ - c_-), \tag{3.16}$$

$$\Lambda = \sqrt{C - \left(\sqrt{a}\eta\right)^{-2}\left(c_+ + c_- - \frac{1}{4}\right)} - \frac{\left(\sqrt{a}\eta\right)^{-1}}{2}. \tag{3.17}$$

The proof is straightforward.

4 Singularity structure of the Borel transformed WKB solutions of an M2P1T equation

As stated in Section 3, we can focus our attention on the pole part of the Mathieu equation so that the part may be analyzed with the help of the results for the Legendre equation. Hence by the same reasoning as in [8, Section 5] (cf. [4] and [16] for the basic properties of the alien derivative) we obtain the following

Theorem 4.1. *Let $\tilde{\psi}_+(t, a, \rho, \eta)$ be a WKB solution of a generic (i.e., $a \neq 0$, $\rho \neq 0$) M2P1T equation that is normalized at a simple pole $\{t = a\}$. Then for every positive integer l we can find positive constants δ_1 and δ_2 so that the following relation* (4.1) *holds, where $\Delta_{y=-y_+(t,a,\rho)+l\varpi}$ designates the alien derivative at the fixed singularity $-y_+(t, a, \rho) + l\varpi$ and the suffix B indicates the Borel transform in the parentheses:*

$$\begin{aligned}
&\left(\Delta_{y=-y_+(t,a,\rho)+l\varpi}\tilde{\psi}_+\right)_B(t, a, \rho, y) \\
&\quad = \frac{(-1)^l}{l}\left\{1 + (-1)^l - \cosh\left(2\pi i l\sqrt{\frac{\mu^2 + \sqrt{\mu^4 - \nu^2}}{2}}\right)\right. \\
&\qquad\qquad \left. - \cosh\left(2\pi i l\sqrt{\frac{\mu^2 - \sqrt{\mu^4 - \nu^2}}{2}}\right)\right\} \\
&\qquad \times \left(\exp\left(-l\oint_\gamma \tilde{S}_{\mathrm{odd}}\,dt\right)\tilde{\psi}_+\right)_B(t, a, \rho, y),
\end{aligned} \tag{4.1}$$

where $\tilde{S}_{\mathrm{odd}}$ denotes the odd part of the solution $\tilde{S}$ of the Riccati equation associated with the M2P1T equation and γ is a closed curve that encircles two simple poles counterclockwise, and

$$\mu^2 = 1 + 2\big(g_+(a) + g_-(-a)\big), \tag{4.2}$$

$$\nu = 2\big(g_+(a) - g_-(-a)\big), \tag{4.3}$$

$$y_+(t, a, \rho) = \int_a^t \sqrt{\frac{f(t, a, \rho)}{t^2 - a^2}}\, dt, \tag{4.4}$$

$$\varpi(a, \rho) = \oint_\gamma \sqrt{\frac{f(t, a, \rho)}{t^2 - a^2}}\, dt. \tag{4.5}$$

Remark 4.1. The highest degree part in η of $\displaystyle\oint_\gamma \tilde{S}_{\mathrm{odd}}\, dt$ is $\eta\varpi(a, \rho)$.

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