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Abstract

In [AKT1], WKB-theoretic transformation was introduced to describe analytic behavior of Borel transformed WKB solutions near a simple turning point. The main purpose of this article is to verify the Borel summability of the transformation series given in [AKT1] on Stokes curves emanating from a simple turning point when all of them run into some poles of order more than two of the potential. We also prove the Borel summability of transformation series for simple pole equations employed in [Ko1, Ko2] under the same assumption.

1 Introduction

From the early days of its development, the turning point problem is one of the central issues in WKB theory. Since the approximation by WKB wave functions breaks down near a turning point, Kramers, in his pioneering work [Kr], replaced the potential by a linear variation (= a simple zero of the potential), and he connected WKB wave function across a turning point by matching WKB wave function to Airy function. The matching method of this kind has been widely used in WKB approximation theory (cf. [BW], [F], [W]).

From a viewpoint of exact WKB analysis, i.e., a WKB theory based on the Borel resummation method initiated by Voros ([V]), Aoki, Kawai and Takei interpreted the matching method as a transformation theory near a simple turning point ([AKT1]); they constructed a transformation series

$$x(\tilde{x},\eta) = x_0(\tilde{x}) + \eta^{-1}x_1(\tilde{x}) + \eta^{-2}x_2(\tilde{x}) + \cdots \tag{1.1}$$

from the stationary Schrödinger equation

$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x})\right)\tilde{\psi} = 0 \tag{1.2}$$

with analytic potential Q and a complex large parameter η to the Airy equation

$$\left(\frac{d^2}{dx^2} - \eta^2 x\right)\psi = 0 \tag{1.3}$$

near a simple turning point of (1.2). By using the transformation series (1.1) WKB solutions of (1.2) can be expressed by those of (1.3) as

$$\tilde{\psi}(\tilde{x},\eta) = \left(\frac{\partial x}{\partial \tilde{x}}(\tilde{x},\eta)\right)^{-1/2} \psi(x(\tilde{x},\eta),\eta). \tag{1.4}$$

Although this relation (1.4) is obtained as a formal relation, it becomes an analytic one after Borel transformation, and they argued that the Voros' connection formula of Borel summed WKB solutions near a simple turning point follows from that of Gauss' hypergeometric functions.

Since Aoki, Kawai and Takei discussed in a general situation, the transformation series $x(\tilde{x},\eta)$ was only obtained near a turning point, and the Borel transform of (1.4) (more precisely an integral representation (2.32) of the Borel transform of ψ) holds only near the reference point of the Borel sum. In this article, by assuming that Q is a rational function and also making some generic assumptions concerning on the Stokes geometry, we will show that the transformation series $x(\tilde{x},\eta)$ is Borel summable near a simple turning point and along Stokes curves emanating from it (Theorem 2.1).

From our results it follows that the relation (1.4) itself is now an exact one if we consider ψ, $\tilde{\psi}$ and $x(\tilde{x},\eta)$ as their Borel sums in appropriate domain. Our result also completes the proof of Voros' connection formula near a simple turning point in the framework of transformation theory since the Borel transform of (1.4) holds near Stokes curves and near the path of integration to define Borel sum.

Our argument in this article is not specific to a simple turning point as the transformation theory is not. To demonstrate it, we also discussed a connection problem near a simple pole of (1.2) through the transformation ([Ko1, Ko2]). (This is also the case for the studies of so-called "fixed singularities". See [AKT2], [KKKoT1], [KKKoT2],

[KKT1] and [KKT2] for details.) In this case (1.2) is transformed to

$$\left(\frac{d^2}{dx^2} - \eta^2\frac{1}{x}\right)\psi = 0, \tag{1.5}$$

and we will show in Section 3 that the transformation series of this case is also Borel summable.

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2 WKB theoretic transformation — a simple turning point case

The main purpose of this section is to verify the Borel summability of transformation series of (2.1) to the WKB theoretic canonical equation (2.16) near Stokes curves emanating from a simple turning point. To make our discussion simple, we assume that all of Stokes curves emanating from a simple turning point in question run into some irregular singular points in our discussion. (See Remark 2.5 and Remark 2.6 in the case that Stokes curves run into a double pole of Q.)

In Section 2.1 we state our main theorem (Theorem 2.1), and review fundamental properties of WKB theoretic transformation to explain the results obtained from Theorem 2.1. In Section 2.2 we show the uniform Borel transformability of transformation series constructed near a simple turning point in question. Finally, in Section 2.3 we prove the Borel summability of the transformation series using its uniform Borel transformability obtained in Section 2.2.

2.1 Fundamental properties of WKB theoretic transformation and its application

We consider the following Schrödinger equation

$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x})\right)\tilde{\psi}(\tilde{x},\eta) = 0 \tag{2.1}$$

with a rational potential $Q(\tilde{x})$ that has a simple turning point at $\tilde{x} = 0$, i.e., $Q(\tilde{x})$ is holomorphic at $\tilde{x} = 0$ and satisfies

$$Q(0) = 0, \quad \frac{dQ}{d\tilde{x}}(0) \neq 0. \tag{2.2}$$

Further we assume the following geometric conditions (2.3) and (2.7); the first assumption is that

(2.3) three Stokes curves $\{T_j\}_{j=1}^3$ emanating from $\tilde{x} = 0$ run into irregular singular points $\{b_j\}_{j=1}^3$ respectively.

Here Stokes curves are integral curves of $\mathrm{Im}\sqrt{Q(\tilde{x})}d\tilde{x} = 0$ emanating from $\tilde{x} = 0$ defined by

$$\mathrm{Im}\int_0^{\tilde{x}} \sqrt{Q(\tilde{x})}d\tilde{x} = 0. \tag{2.4}$$

To give a second assumption we prepare some notation. Let $U^{\tilde{\varepsilon}} = \{\tilde{x} \in \mathbb{C}; |\tilde{x}| < \tilde{\varepsilon}\}$. By taking sufficiently small $\tilde{\varepsilon} > 0$, we may assume that $U^{\tilde{\varepsilon}} \setminus \{T_j\}_{j=1}^3$ is decomposed into three connected components, which we denote them by $\left\{U_j^{\tilde{\varepsilon}}\right\}_{j=1}^3$. We also let $\widehat{U}_{j,\pm}^{\tilde{\varepsilon}}$ be a connected component of

(2.5)

$$\bigcup_{x_0 \in U_j^{\tilde{\varepsilon}}} \left\{x \in \mathbb{C};\ \mathrm{Im}\int_{\tilde{x}_0}^{\tilde{x}} \sqrt{Q(\tilde{x})}d\tilde{x} = 0,\ \pm\mathrm{Re}\int_{\tilde{x}_0}^{\tilde{x}} \sqrt{Q(\tilde{x})}d\tilde{x} \geq 0\right\}$$

which contains $\left\{U_j^{\tilde{\varepsilon}}\right\}_{j=1}^{3}$. For $j_1, j_2 \in \{1,2,3\}$, $j_1 \neq j_2$, we can take a Stokes curve T_j so that $\overline{U_{j_1}^{\tilde{\varepsilon}}} \cap \overline{U_{j_2}^{\tilde{\varepsilon}}} \subset T_j$. We fix the branch of $\sqrt{Q(\tilde{x})}$ on $\left(\overline{U}_{j_1}^{\tilde{\varepsilon}} \cup \overline{U}_{j_2}^{\tilde{\varepsilon}}\right) \setminus \{0\}$ so that

$$\text{Re} \int_0^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x} \geq 0 \tag{2.6}$$

holds for any $\tilde{x} \in T_j$. Our second assumption is, by taking $\tilde{\varepsilon}$ sufficiently small,

(2.7) all of $\widehat{U}_{j_1,+}^{\tilde{\varepsilon}}$ and $\widehat{U}_{j_2,+}^{\tilde{\varepsilon}}$ run into b_j

for any pair of $j_1, j_2 \in \{1,2,3\}$.

Let $\widehat{U}^{\tilde{\varepsilon}}$ be a union of integral curves that through $U^{\tilde{\varepsilon}}$, i.e.,

$$\widehat{U}^{\tilde{\varepsilon}} = \bigcup_{j=1}^{3} \left\{ \bigcup_{*=\pm} \widehat{U}_{j,*}^{\tilde{\varepsilon}} \cup T_j \right\}. \tag{2.8}$$

Then, from the assumptions (2.3) and (2.7), we find that $\widehat{U}^{\tilde{\varepsilon}}$ does not contain any poles nor turning points except for a simple turning point at the origin.

Now we state our main theorem.

Theorem 2.1. *Let $Q(\tilde{x})$ be a meromorphic function that satisfies* (2.2), (2.3) *and* (2.7). *Then there exists a Borel summable series*

$$x(\tilde{x}, \eta) = \sum_{k=0}^{\infty} x_k(\tilde{x}) \eta^{-k} \tag{2.9}$$

on $\widehat{U}^{\tilde{\varepsilon}}$ for which the following conditions (2.10) $\sim$ (2.14) *hold:*

(2.10) $\{x_k(\tilde{x})\}_{k=0}^{\infty}$ *are holomorphic on $\widehat{U}^{\tilde{\varepsilon}}$,*

(2.11) $x_{2k+1}(\tilde{x})$ $(k = 0, 1, 2, \cdots)$ *are identically zero,*

$$(2.12)\qquad x_0(0) = 0,$$

$$(2.13)\qquad \frac{dx_0}{d\tilde{x}} \neq 0 \ \ on\ \widehat{U}^{\tilde{\varepsilon}},$$

$$(2.14)\qquad Q(\tilde{x}) = \left(\frac{dx(\tilde{x},\eta)}{d\tilde{x}}\right)^2 x(\tilde{x},\eta) - \frac{1}{2}\eta^{-2}\left\{x(\tilde{x},\eta);\tilde{x}\right\}.$$

Here $\{x(\tilde{x},\eta);\tilde{x}\}$ *stands for the Schwarzian derivative, i.e.,*

$$(2.15)\qquad \frac{d^3x/d\tilde{x}^3}{dx/d\tilde{x}} - \frac{3}{2}\left(\frac{d^2x/d\tilde{x}^2}{dx/d\tilde{x}}\right)^2.$$

In Section 2.2 and Section 2.3, we will give more detailed properties of $x(\tilde{x},\eta)$ in Theorem 2.1 including growth estimates.

The series $x(\tilde{x},\eta)$ in Theorem 2.1 is the same transformation series as that in [AKT1], which transforms (2.1) to

$$(2.16)\qquad \left(\frac{d^2}{dx^2} - \eta^2 x\right)\psi = 0.$$

Following [KT], we recall the meaning of the transformation. (See [AKT1] for details). We first give the following relations for solutions of Riccati equations associated with (2.1) and (2.16);

Theorem 2.2. ([KT, Theorem 2.16]) *The transformation series* $x(\tilde{x},\eta)$ *in Theorem 2.1 satisfies*

$$(2.17)\qquad \tilde{S}(\tilde{x},\eta) = \left(\frac{dx}{d\tilde{x}}\right) S(x(\tilde{x},\eta),\eta) - \frac{1}{2}\left(\frac{d^2x}{d\tilde{x}^2}\right) \Big/ \left(\frac{dx}{d\tilde{x}}\right).$$

Here formal power series

$$(2.18)\qquad \tilde{S}(\tilde{x},\eta) = \sum_{k=-1}^{\infty} \tilde{S}_k(\tilde{x})\eta^{-k} \quad and \quad S(x,\eta) = \sum_{k=-1}^{\infty} S_k(x)\eta^{-k}$$

are respectively solutions of Riccati equations

$$(2.19)\qquad \tilde{S}^2 + \frac{d\tilde{S}}{d\tilde{x}} = \eta^2 Q(\tilde{x})$$

and

$$S^2 + \frac{dS}{dx} = \eta^2 x \tag{2.20}$$

such that $\tilde{S}_{-1}(\tilde{x})$ *and* $S_{-1}(x)$ *satisfy*

$$\tilde{S}_{-1}(\tilde{x}) = \left(\frac{dx_0}{d\tilde{x}}\right) S_{-1}(x_0(\tilde{x})). \tag{2.21}$$

Let $\tilde{S}^{(\pm)}$ respectively denote the solutions of (2.19) that are determined so that they satisfy $\tilde{S}_{-1}^{(\pm)}(\tilde{x}) = \pm\sqrt{Q(\tilde{x})}$. Then the odd part $\tilde{S}_{\mathrm{odd}}$ of $\tilde{S}$ is defined by

$$\tilde{S}_{\mathrm{odd}} = \frac{1}{2}\left(\tilde{S}^{(+)} - \tilde{S}^{(-)}\right). \tag{2.22}$$

In the same manner, we also define the odd part S_{odd} of S. From Theorem 2.2, we immediately obtain

Corollary 2.3. ([KT, Corollary 2.17]) *If the branches of* $\tilde{S}_{-1}$ *and* S_{-1} *are taken so that they satisfy* (2.21)*, then we have*

$$\tilde{S}_{\mathrm{odd}}(\tilde{x}, \eta) = \left(\frac{dx(\tilde{x}, \eta)}{d\tilde{x}}\right) S_{\mathrm{odd}}(x(\tilde{x}, \eta), \eta). \tag{2.23}$$

Let $\tilde{\psi}_{\pm}(\tilde{x}, \eta)$ denote WKB solutions of (2.1) normalized at a simple turning point $\tilde{x} = 0$, i.e.,

$$\tilde{\psi}_{\pm}(\tilde{x}, \eta) = \frac{1}{\sqrt{\tilde{S}_{\mathrm{odd}}}} \exp\left(\pm \int_0^{\tilde{x}} \tilde{S}_{\mathrm{odd}}(\tilde{x}, \eta) d\tilde{x}\right). \tag{2.24}$$

By the same way, we define WKB solutions $\psi_{\pm}(x, \eta)$ of (2.16) normalized at a simple turning point $x = 0$. The relation for $\tilde{S}_{\mathrm{odd}}$ and S_{odd} in Corollary 2.3 gives

Theorem 2.4. ([KT, Corollary 2.18]) *Let* $\tilde{\psi}_{\pm}(\tilde{x}, \eta)$ *and* $\psi_{\pm}(x, \eta)$ *respectively be WKB solutions of* (2.1) *and* (2.16) *normalized at their* *simple turning points $\tilde{x} = 0$ and $x = 0$. Then they satisfy the following relation;*

$$\tilde{\psi}_{\pm}(\tilde{x},\eta) = \left(\frac{dx(\tilde{x},\eta)}{d\tilde{x}}\right)^{-1/2} \psi_{\pm}(x(\tilde{x},\eta),\eta). \tag{2.25}$$

For simplicity, we take $x_0 = x_0(\tilde{x})$ as a new coordinate variable (cf (2.13)). Then the precise meaning of the right hand side of (2.25) is

$$\left(\frac{d\tilde{x}}{dx_0}\right)^{1/2} \left(1 + \frac{dX(x_0,\eta)}{dx_0}\right)^{-1/2} \sum_{n=0}^{\infty} \frac{(X(x_0,\eta))^n}{n!} \frac{d\psi_{\pm}}{dx_0}(x_0,\eta), \tag{2.26}$$

where $X(x_0,\eta)$ is

$$X(x_0,\eta) = x(\tilde{x}(x_0),\eta) - x_0. \tag{2.27}$$

Let $\tilde{\psi}_{\pm,B}$ and $\psi_{\pm,B}$ respectively denote the Borel transforms of $\tilde{\psi}_{\pm}$ and $\psi_{\pm}$. Through the Borel transformation, (2.26) can be rewritten as $\mathcal{X}\psi_{\pm,B}$, where $\mathcal{X}$ is a microdifferential operator defined by

$$\mathcal{X} =: \left(\frac{\partial\tilde{x}}{\partial x_0}\right)^{1/2} \left(1 + \frac{\partial X}{\partial x_0}\right)^{-1/2} \exp[X(x_0,\eta)\xi] : . \tag{2.28}$$

Here ξ stands for the symbol of ∂_{x_0} and $: \cdot :$ designates the normal ordered product. (See [A] and [AY] for details.) Since $\tilde{\psi}_{\pm}$ and $\psi_{\pm}$ satisfy (2.25), we can represent $\tilde{\psi}_{\pm,B}$ by $\psi_{\pm,B}$ through the action of $\mathcal{X}$. As we will see in Appendix B, the action of $\mathcal{X}$ can be written as an action of an integro-differential operator and the Borel summability of $X(x_0,\eta)$, more precisely Theorem 2.9, guarantees that this representation of $\tilde{\psi}_{\pm,B}$ holds on $\widehat{V}^{\varepsilon} \times E^{\delta}_{\pm y_0}$ for some $\varepsilon, \delta > 0$, where

$$y_0(x_0) = \int_0^{x_0} \sqrt{x_0}\, dx_0, \tag{2.29}$$

$$\widehat{V}^{\varepsilon} = \{x_0 \in \mathbb{C}; |\mathrm{Im}\ y_0(x_0)| < \varepsilon\}, \tag{2.30}$$

$$E^{\delta}_{\pm y_0} = \bigcup_{s\in\mathbb{R}} \{y \in \mathbb{C}; |y - s \pm y_0(x_0)| < \delta\}. \tag{2.31}$$

Remark 2.1. Since $x_0(\tilde{x})$ maps an integral curve of $\mathrm{Im}\sqrt{Q(\tilde{x})}d\tilde{x} = 0$ that passes through $\overset{\circ}{\tilde{x}}$ to that of $\mathrm{Im}\sqrt{x}dx = 0$ that passes through $x_0(\overset{\circ}{\tilde{x}})$ bijectively, by taking $\varepsilon > 0$ sufficiently small, we can assume that $\widehat{V}^{\varepsilon}$ is contained in $x_0(\widehat{U}^{\tilde{\varepsilon}})$.

Concretely we have

Theorem 2.5. *$\tilde{\psi}_{\pm,B}$ and $\psi_{\pm,B}$ satisfy the following relation on $\widehat{V}^{\varepsilon} \times E^{\delta}_{\pm y_0}$ for sufficiently small $\varepsilon, \delta > 0$;*

$$\begin{aligned}(2.32) \qquad \tilde{\psi}_{\pm,B}(\tilde{x}(x_0), y) = &\left(\frac{\partial \tilde{x}}{\partial x_0}\right)^{1/2} \psi_{\pm,B}(x_0, y) \\ &+ \int_{\mp y_0}^{y} K(x_0, y - y', \partial_{x_0}) \psi_{\pm,B}(x_0, y') dy',\end{aligned}$$

where $K(x, y, \partial_x)$ is a differential operator of infinite order on $\widehat{V}^{\varepsilon} \times E^{\delta}_{\pm y_0}$.

See [SKK] for the notion of a differential operator of infinite order.

2.2 Uniform Borel transformability of transformation series

As a first step to proving the Borel summability of transformation series $x(\tilde{x}, \eta)$ introduced in Theorem 2.1, we show the uniform Borel transformability of $x(\tilde{x}, \eta)$ on $\widehat{U}^{\tilde{\varepsilon}}$ in this subsection. Concretely we prove the following

Proposition 2.6. *Let $Q(\tilde{x})$ be a meromorphic function that satisfies (2.2), (2.3) and (2.7). Then there exist $\tilde{\varepsilon} > 0$ and formal series $x(\tilde{x}, \eta) = x_0(\tilde{x}) + \eta^{-1} x_1(\tilde{x}) + \cdots$ that satisfies (2.10) $\sim$ (2.14) and the following estimates; there exist positive constants C_0 and A such that for all $n \geq 1$ and $\tilde{x} \in \widehat{U}^{\tilde{\varepsilon}}$, $x_n(\tilde{x})$ satisfies*

$$(2.33) \qquad |x_n(\tilde{x})| \leq (|x_0(\tilde{x})| + 1) C_0 n! A^n.$$

Remark 2.2. Since we can take the constant A in (2.33) independent of $\tilde{x}$, we use the phrase "uniform Borel transformable". This uniform Borel transformability guarantees that the Borel transform of $x - x_0$ is holomorphic on $\widehat{U}^{\tilde{\varepsilon}} \times \{y \in \mathbb{C}; |y| < A^{-1}\}$.

Proof. We first remind us the construction of $x(\tilde{x}, \eta)$. We determine $x_k(\tilde{x})$ ($k = 0, 1, 2, \cdots$) inductively by comparing the coefficients of η^{-k} of (2.14). First, by comparing the coefficients of η^0 of (2.14), we find that $x_0(\tilde{x})$ should satisfy

$$Q(\tilde{x}) = \left(\frac{dx_0(\tilde{x})}{d\tilde{x}}\right)^2 x_0(\tilde{x}). \tag{2.34}$$

Therefore we determine $x_0(\tilde{x})$ by

$$x_0(\tilde{x}) = \left(\frac{3}{2}\int_0^{\tilde{x}} \sqrt{Q(\tilde{x})}d\tilde{x}\right)^{2/3}. \tag{2.35}$$

Since $Q(\tilde{x})$ satisfies (2.2), (2.3) and (2.7), we immediately find that $x_0(\tilde{x})$ is holomorphic on $\widehat{U}^{\tilde{\varepsilon}}$ for some $\tilde{\varepsilon} > 0$ and satisfies (2.12). Further, from (2.34), we find the following relation holds;

$$\sqrt{Q(\tilde{x})}d\tilde{x} = \sqrt{x}dx\big|_{x=x_0(\tilde{x})}. \tag{2.36}$$

Therefore x_0 maps the integral curves of $\mathrm{Im}\sqrt{Q(\tilde{x})}d\tilde{x}$ that start from $\mathring{x} \in \widehat{U}^{\tilde{\varepsilon}}$ to those of $\mathrm{Im}\sqrt{x}dx$ that start from $x_0(\mathring{x})$ in a bijective manner. Now it is clear that x_0 maps $\widehat{U}^{\tilde{\varepsilon}}$ to $x_0(\widehat{U}^{\tilde{\varepsilon}})$ bijectively and x_0 satisfies (2.13). We take $z = x_0(\tilde{x})$ as a new coordinate variable on $x_0(\widehat{U}^{\tilde{\varepsilon}})$.

Next we determine x_k ($k \geq 1$). By comparing the coefficients of η^{-k} of (2.14), we find that x_k should satisfy the following relations;

$$2z\frac{dx_k}{dz} + x_k = \Phi_k(z), \tag{2.37}$$

where $\Phi_k(z)$ is

(2.38)

$$
\begin{aligned}
\Phi_k(z) = &- \sum_{\substack{k_1+k_2+k_3=k,\\ k_1,k_2,k_3 \le k-1}} \frac{dx_{k_1}}{dz}\frac{dx_{k_2}}{dz} x_{k_3} \\
&+ \frac{1}{2} \sum_{k_1+k_2=k-2} \left(\frac{d\tilde{x}}{dz}\right)^3 \frac{d^3 x_{k_1}}{d\tilde{x}^3} \\
&\quad \times \sum_{l=\min\{1,k_2\}}^{k_2} (-1)^l \sum^{*}_{\substack{\mu_1+\cdots+\mu_l=k_2,\\ \mu_1,\cdots,\mu_l \ge 1}} \frac{dx_{\mu_1}}{dz} \cdots \frac{dx_{\mu_l}}{dz} \\
&- \frac{3}{4} \sum_{k_1+k_2+k_3=k-2} \left(\frac{d\tilde{x}}{dz}\right)^4 \frac{d^2 x_{k_1}}{d\tilde{x}^2}\frac{d^2 x_{k_2}}{d\tilde{x}^2} \\
&\quad \times \sum_{l=\min\{1,k_3\}}^{k_3} (-1)^l (l+1) \sum^{*}_{\substack{\mu_1+\cdots+\mu_l=k_3,\\ \mu_1,\cdots,\mu_l \ge 1}} \frac{dx_{\mu_1}}{dz} \cdots \frac{dx_{\mu_l}}{dz}.
\end{aligned}
$$

Here we use the following notation;

(2.39)

$$
\sum^{*}_{\substack{\mu_1+\cdots+\mu_l=k,\\ \mu_1,\cdots,\mu_l \ge 1}} \frac{dx_{\mu_1}}{dz} \cdots \frac{dx_{\mu_l}}{dz} = \begin{cases} 1 & (l=0), \\ \displaystyle\sum^{*}_{\substack{\mu_1+\cdots+\mu_l=k,\\ \mu_1,\cdots,\mu_l \ge 1}} \frac{dx_{\mu_1}}{dz} \cdots \frac{dx_{\mu_l}}{dz} & (l \ge 1). \end{cases}
$$

Since Φ_k does not contain x_n $(n \ge k)$, we can inductively determine x_k by (2.37). Concretely we take x_k as

$$
x_k(z) = \frac{z^{-1/2}}{2} \int_0^z z^{-1/2} \Phi_k(z) dz \tag{2.40}
$$

so that x_k is holomorphic at $z=0$ and satisfies (2.37). We can easily find (2.11) since we can inductively check that Φ_{2k+1} $(k \geq 0)$ are identically zero.

Now we confirm the estimation of x_k. First, for sufficiently small $r > 0$, we define D_r^1 and D_r^2 by

$$D_r^1 = \bigcup_{0 \leq s \leq 1} \{z \in \mathbb{C}; |z - s| \leq r\} \tag{2.41}$$

$$D_r^2 = \bigcup_{s \geq 1} \left\{z \in \mathbb{C}; |z - s| \leq \frac{r}{\sqrt{s}}\right\}. \tag{2.42}$$

Since $2\mathrm{Im}z^{3/2}/3$ is expanded to

$$\sqrt{\mathrm{Re}z} \cdot \mathrm{Im}z + \frac{1}{24}(\mathrm{Re}z)^{-3/2}(\mathrm{Im}z)^3 + \cdots \tag{2.43}$$

in D_r^2, we find that $\mathrm{Im} \displaystyle\int_0^z \sqrt{z}dz$ behaves like $\sqrt{\mathrm{Re}z} \cdot \mathrm{Im}z$ in D_r^2 for sufficiently large $\mathrm{Re}z$. Therefore we can take $r > 0$ so that $D_r^1 \cup D_r^2 \subset x_0(\widehat{U}^{\tilde{\varepsilon}})$.

Then we show that $x_k(z)$ $(k \geq 1)$ satisfy the following estimates; there exist positive constants $C_0 < 1$ and $A > 1$ such that for all δ with $0 < \delta < r/3$ and

$\underline{1)\ z \in D_{r-\delta}^1}$

$$|x_k(z)| \leq C_0 k! \delta^{-k} A^k \tag{2.44}$$

$$\left|\frac{dx_k}{dz}(z)\right| \leq C_0 k! \delta^{-k} A^k, \tag{2.45}$$

$\underline{2)\ z \in D_{r-\delta}^2}$

$$|x_k(z)| \leq |z| C_0 k! \delta^{-k} A^k \tag{2.46}$$

$$\left|\frac{dx_k}{dz}(z)\right| \leq C_0 k! \delta^{-k} A^k \tag{2.47}$$

hold. Since (2.44) and (2.45) can be verified by the same discussion as in [AKT1], we only confirm (2.46) and (2.47) here.

Remark 2.3. The condition $0 < \delta < r/3$ is used in the proof of (2.44) and (2.45).

We inductively show that x_k $(k = 1, 2, \cdots)$ satisfy (2.46) and (2.47). First we immediately find that x_1 satisfies (2.46) and (2.47) since x_{2k+1} $(k \geq 0)$ are identically zero. Just to be sure, we check that x_2 satisfies (2.46) and (2.47). From the assumption (2.3), the inverse image $x_0^{-1}(z)$ of z tends to a irregular singular point b_j of (2.1) when z tends to $+\infty$ along positive real axis. Let $Q(\tilde{x})$ have a pole of order $p(\geq 3)$ at $\tilde{x} = b_j$. Then, from (2.34) and (2.35), we find that $x_0(\tilde{x})$ and $dx_0/d\tilde{x}$ behave as

$$x_0(\tilde{x}) = O\left((\tilde{x} - b_j)^{(-p+2)/3}\right), \tag{2.48}$$

$$\frac{dx_0}{d\tilde{x}}(\tilde{x}) = O\left((\tilde{x} - b_j)^{-(p+1)/3}\right) \tag{2.49}$$

when $\tilde{x}$ tends to b_j. Therefore we can take positive constants M_1 and M_2 so that the following holds on D_r^2;

$$M_1 |x_0|^{(p+1)/(p-2)} \leq \left|\frac{dx_0}{d\tilde{x}}\right| \leq M_2 |x_0|^{(p+1)/(p-2)}. \tag{2.50}$$

Now we derive the estimation of x_2 using the representation (2.40). From (2.38), we immediately find that $\Phi_2(z)$ is given by

$$\Phi_2(z) = \frac{1}{2}\left(\frac{d\tilde{x}}{dz}\right)^3 \frac{d^3x_0}{d\tilde{x}^3} - \frac{3}{4}\left(\frac{d\tilde{x}}{dz}\right)^4 \left(\frac{d^2x_0}{d\tilde{x}^2}\right)^2. \tag{2.51}$$

In order to rewrite $\Phi_2(z)$ by $dx_0/d\tilde{x}$ and its derivative in z variable, we use the following relation for a function $f(\tilde{x})$ of $\tilde{x}$;

(2.52)
$$\frac{d^2f}{d\tilde{x}^2}(\tilde{x}(z)) = \left(\frac{d\tilde{x}}{dz}(z)\right)^{-2} \frac{d^2}{dz^2} f(\tilde{x}(z)) + \frac{1}{2}\frac{d}{dz}\left(\frac{d\tilde{x}}{dz}(z)\right)^{-2} \frac{d}{dz} f(\tilde{x}(z))$$

(2.53)

$$\frac{d^3 f}{d\tilde{x}^3}(\tilde{x}(z)) = \left(\frac{d\tilde{x}}{dz}(z)\right)^{-3} \frac{d^3}{dz^3} f(\tilde{x}(z)) + \frac{d}{dz}\left(\frac{d\tilde{x}}{dz}(z)\right)^{-3} \frac{d^2}{dz^2} f(\tilde{x}(z))$$
$$+ \frac{1}{2}\left(\frac{d\tilde{x}}{dz}(z)\right)^{-1} \frac{d^2}{dz^2}\left(\frac{d\tilde{x}}{dz}(z)\right)^{-2} \frac{d}{dz} f(\tilde{x}(z)).$$

Therefore $\Phi_2(z)$ can be rewritten to

$$\Phi_2(z) = \frac{1}{4}\left(\frac{d\tilde{x}}{dz}(z)\right)^{2} \frac{d^2}{dz^2}\left(\frac{d\tilde{x}}{dz}(z)\right)^{-2} - \frac{3}{16}\left[\left(\frac{d\tilde{x}}{dz}\right)^{2} \frac{d}{dz}\left(\frac{d\tilde{x}}{dz}(z)\right)^{-2}\right]^2. \tag{2.54}$$

In order to derive the estimation of Φ_2 from that of $dx_0/d\tilde{x}$, we use Cauchy's formula as follows; for a holomorphic function $g(z)$ on D_r^2, we have the following representation for $z \in D_{r-\delta}^2$;

$$\frac{d^j}{dz^j} g(z) = \frac{j!}{2\pi i}\int_{|\tilde{z}-z|=d} \frac{g(\tilde{z})}{(\tilde{z}-z)^{j+1}} d\tilde{z}, \tag{2.55}$$

where we take $d > 0$ as

$$d = \delta(2|z|)^{-1/2}. \tag{2.56}$$

Then we immediately find that the integral path of (2.55) is contained in D_r^2. Applying (2.55) to $(d\tilde{x}/dz)^{-2}$, we obtain the following estimates of Φ_2; there exists a positive constant M such that, for $z \in D_{r-\delta}^2$, $\Phi_2(z)$ satisfies

$$|\Phi_2(z)| \leq M|z|\delta^{-2}. \tag{2.57}$$

Actually, for example, the estimation of the first term of (2.54) is given as follows; first, from (2.50), we find that, for $\tilde{z} \in \{\tilde{z}; |\tilde{z} - z| =$

$\delta(2|z|)^{-1/2}\}$, $(d\tilde{x}/dz(\tilde{z}))^{-2}$ is dominated as follows;

$$\left|\left(\frac{d\tilde{x}}{dz}(\tilde{z})\right)^{-2}\right| \leq M_2^2|\tilde{z}|^{2(p+1)/(p-2)} \tag{2.58}$$
$$\leq M_2^2(|z| + \delta(2|z|)^{-1/2})^{2(p+1)/(p-2)}$$
$$\leq M_2^2(2|z|)^{2(p+1)/(p-2)}.$$

Using the representation (2.55) for $j = 2$, we obtain the following estimates from (2.58);

$$\left|\frac{d^2}{dz^2}\left(\frac{d\tilde{x}}{dz}(z)\right)^{-2}\right| \leq \frac{1}{\pi}2|z|\delta^{-2}M_2^2(2|z|)^{2(p+1)/(p-2)}. \tag{2.59}$$

Therefore, using (2.50) again, we immediately find

(2.60)
$$\left|\frac{1}{4}\left(\frac{d\tilde{x}}{dz}(z)\right)^2\frac{d^2}{dz^2}\left(\frac{d\tilde{x}}{dz}(z)\right)^{-2}\right| \leq \frac{1}{\pi}2^{-1+2(p+1)/(p-2)}M_1^{-2}M_2^2|z|\delta^{-2}.$$

In the same way, we have

(2.61)
$$\left|\frac{3}{16}\left[\left(\frac{d\tilde{x}}{dz}\right)^2\frac{d}{dz}\left(\frac{d\tilde{x}}{dz}(z)\right)^{-2}\right]^2\right| \leq \frac{3}{\pi^2}2^{-5+4(p+1)/(p-2)}M_1^{-4}M_2^4|z|\delta^{-2}.$$

Combining (2.60) and (2.61), we arrive at (2.57).

Then, from (2.57), we find that, for arbitrarily small $C_0 > 0$, we can take $A > 0$ so that $\Phi_2(z)$ satisfies

$$|\Phi_2(z)| \leq C_0^2|z|\delta^{-2}A^2 \tag{2.62}$$

on $D^2_{r-\delta}$. Actually it sufficies to set $A = \sqrt{M}C_0^{-1}$. Similarly we can show that

$$|\Phi_2(z)| \leq C_0^2\delta^{-2}A^2 \tag{2.63}$$

holds on $D^1_{r-\delta}$.

Now we divide the integral path of (2.40) as follows;

$$x_2(z) = \frac{z^{-1/2}}{2}\int_0^1 z^{-1/2}\Phi_2(z)dz + \frac{z^{-1/2}}{2}\int_1^z z^{-1/2}\Phi_2(z)dz. \tag{2.64}$$

Here we take the integral path of the first term of (2.64) as a straight line joining 0 and 1 so that the path is contained in $D^1_{r-\delta}$. And we take the path of the second term as a straight line joining 1 and z so that the path is contained in $D^2_{r-\delta}$. Then we obtain the following estimates of the first term of (2.64) for $z \in D^2_{r-\delta}$ from (2.63);

$$\begin{aligned} \left|\frac{z^{-1/2}}{2}\int_0^1 z^{-1/2}\Phi_2(z)dz\right| &\le \frac{|z|^{-1/2}}{2}\int_0^1 |z|^{-1/2}C_0^2\delta^{-2}A^2|dz| \\ &\le |z|^{-1/2}C_0^2\delta^{-2}A^2. \end{aligned} \tag{2.65}$$

Similarly we find the following estimates of the second term of (2.64) for $z \in D^2_{r-\delta}$ from (2.62);

$$\begin{aligned} \left|\frac{z^{-1/2}}{2}\int_1^z z^{-1/2}\Phi_2(z)dz\right| &\le \frac{|z|^{-1/2}}{2}\int_1^z |z|^{1/2}C_0^2\delta^{-2}A^2|dz| \\ &\le \frac{|z|^{-1/2}}{3}\left(|z|^{3/2}+1\right)C_0^2\delta^{-2}A^2. \end{aligned} \tag{2.66}$$

Since $z \in D^2_{r-\delta}$, by taking $r < 1/2$, we find $|z|^{-1/2} \le 2\sqrt{2}|z|$. Therefore, combining (2.65) and (2.66), we obtain

$$|x_2(z)| \le \frac{1+8\sqrt{2}}{3}|z|C_0^2\delta^{-2}A^2 \tag{2.67}$$

for $z \in D^2_{r-\delta}$. Further, from (2.37), we immediately find the following estimates for $z \in D^2_{r-\delta}$;

$$\left|\frac{dx_2}{dz}\right| \le \frac{|x_2| + |\Phi_2(z)|}{2|z|} \tag{2.68}$$

$$\leq \frac{2+4\sqrt{2}}{3} C_0^2 \delta^{-2} A^2.$$

Finally, by taking C_0 so that

$$\frac{1+8\sqrt{2}}{3} C_0 < 1, \tag{2.69}$$

we are convinced that x_2 satisfies (2.46) and (2.47).

Next we show that x_k ($k \geq 2$) satisfies (2.46) and (2.47) under the assumption that x_n ($1 \leq n \leq k-1$) satisfy them. As in the case of the estimation of x_2, we first examine that of Φ_k on $D^2_{r-\delta}$. The first term of (2.38) is directly estimated from the induction hypothesis as follows;

(2.70)
$$\begin{aligned}\sum_{\substack{k_1+k_2+k_3=k,\\ k_1,k_2,k_3 \leq k-1}} \left|\frac{dx_{k_1}}{dz}\right| \left|\frac{dx_{k_2}}{dz}\right| |x_{k_3}| &\leq |z| C_0^2 \delta^{-k} A^k \sum_{\substack{k_1+k_2+k_3=k,\\ k_1,k_2,k_3 \leq k-1}} k_1! k_2! k_3! \\ &\leq |z| C_0^2 \left(\frac{4^2}{k-1} + 12\right) \delta^{-k} A^k (k-1)!.\end{aligned}$$

Here we use the following

Lemma 2.7 ([AKT2]). *For $k, l \in \mathbb{N} = \{1, 2, 3, \cdots\}$ with $l \leq k$, the following inequality holds;*

$$\sum_{\substack{\mu_1+\cdots+\mu_l=k,\\ \mu_1,\cdots,\mu_l \geq 1}} \mu_1! \cdots \mu_l! \leq 4^{l-1}(k-l+1)!. \tag{2.71}$$

In fact, we apply Lemma 2.7 as follows;

(2.72)
$$\begin{aligned}\sum_{\substack{k_1+k_2+k_3=k,\\ k_1,k_2,k_3 \leq k-1}} k_1!k_2!k_3! &= \sum_{\substack{k_1+k_2+k_3=k,\\ 1\leq k_1,k_2,k_3 \leq k-1}} k_1!k_2!k_3! + 3 \sum_{\substack{k_1'+k_2'=k,\\ 1\leq k_1',k_2' \leq k-1}} k_1'!k_2'! \\ &\leq 4^2(k-2)! + 12(k-1)!.\end{aligned}$$

Remark 2.4. We have to care that (2.44) $\sim$ (2.47) hold for $k \geq 1$, on the other hand, x_0 satisfies $|x_0| = |z|$ and $|dx_0/dz| = 1$. Therefore the estimates $|x_0| \leq C_0|z|$ and $|dx_0/dz| \leq C_0$ that is obtained from (2.46) and (2.47) by letting $k = 0$ does not hold for sufficiently small C_0. Hence, to simplify the discussion, when x_0 and x_k $(k \geq 1)$ appear at the same time and the extra factor C_0 is not important in the estimation, we neglect the factor C_0 that appears in (2.44) $\sim$ (2.47).

Then we consider the second term of (2.38), which is the most important term in (2.38) in the sense that $k!$ in the estimation of x_k originates from this term. First we rewrite the third derivative of x_{k_1} in $\tilde{x}$ variable to that of x_{k_1} in z variable using the relation (2.53). And, multiplying $(d\tilde{x}/dz)^3$, we obtain the following relation;

$$\begin{aligned} \left(\frac{d\tilde{x}}{dz}\right)^3 \frac{d^3 x_{k_1}}{d\tilde{x}^3} = \frac{d^3 x_{k_1}}{dz^3} &+ \left(\frac{d\tilde{x}}{dz}\right)^3 \frac{d}{dz}\left(\frac{d\tilde{x}}{dz}\right)^{-3} \frac{d^2 x_{k_1}}{dz^2} \\ &+ \frac{1}{2}\left(\frac{d\tilde{x}}{dz}\right)^2 \frac{d^2}{dz^2}\left(\frac{d\tilde{x}}{dz}\right)^{-2} \frac{dx_{k_1}}{dz}. \end{aligned} \tag{2.73}$$

Since $k_1 \leq k-2$, dx_{k_1}/dz satisfies (2.47) for all δ with $0 < \delta < r/3$ from the induction hypothesis. Now we derive the estimates of the second and the third derivative of x_{k_1} from that of dx_{k_1}/dz through the representation (2.55). In this case, we take

$$d = \frac{\delta}{(k_1+1)\sqrt{2|z|}}. \tag{2.74}$$

Then, for $z \in D^2_{r-\delta}$, if $\tilde{z}$ satisfies $|\tilde{z} - z| \leq \delta/(k_1+1)\sqrt{2|z|}$, we find that $\tilde{z} \in D^2_{r-k_1\delta/(k_1+1)}$. Indeed, since $z \in D^2_{r-\delta}$, we can take $s \geq 1$ so that $|z - s| \leq (r-\delta)/\sqrt{s}$. Therefore $|z| \geq s/2$ holds and $\tilde{z}$ satisfies

$$|\tilde{z} - s| \leq \frac{r-\delta}{\sqrt{s}} + \frac{\delta}{(k_1+1)\sqrt{2|z|}} \tag{2.75}$$

$$\leq \frac{r-\delta}{\sqrt{s}} + \frac{\delta}{(k_1+1)\sqrt{s}}$$
$$= \frac{1}{\sqrt{s}}\left(r - \frac{k_1}{k_1+1}\delta\right).$$

Substituting δ in (2.47) for $k = k_1$ to $k_1\delta/(k_1+1)$, we find that dx_{k_1}/dz satisfies the following estimates for $\tilde{z} \in D^2_{r-k_1\delta/(k_1+1)}$;

$$\left|\frac{dx_{k_1}}{dz}(\tilde{z})\right| \leq k_1!\left(1+\frac{1}{k_1}\right)^{k_1}\delta^{-k_1}A^{k_1} \tag{2.76}$$
$$\leq k_1! e \delta^{-k_1}A^{k_1}$$

Hence, using the representation (2.55), we obtain

$$\left|\frac{d^j}{dz^j}\frac{dx_{k_1}}{dz}(z)\right| \leq \frac{j!}{2\pi}\left(\frac{\delta}{(k_1+1)\sqrt{2|z|}}\right)^{-j} e k_1!\delta^{-k_1}A^{k_1}. \tag{2.77}$$

The estimation of the coefficient of dx_{k_1}/dz is given from (2.60). By the same reasoning, the coefficient of $d^2x_{k_1}/dz^2$ satisfies

(2.78)
$$\left|\left(\frac{d\tilde{x}}{dz}(z)\right)^3 \frac{d}{dz}\left(\frac{d\tilde{x}}{dz}(z)\right)^{-3}\right| \leq \frac{1}{2\pi}2^{3(p+1)/(p-2)}M_1^{-3}M_2^3\sqrt{2|z|}\delta^{-1}.$$

In conclusion, we gain the following estimation; we can take some positive constant M that is independent of z, k_1, C_0, δ and A so that

$$\left|\left(\frac{d\tilde{x}}{dz}\right)^3 \frac{d^3x_{k_1}}{d\tilde{x}^3}\right| \leq |z|M(k_1+2)!\delta^{-k_1-2}A^{k_1} \tag{2.79}$$

holds on $D^2_{r-\delta}$. Actually the estimates of the first term of (2.73) immediately follows from (2.77);

$$\left|\frac{d^3x_{k_1}}{dz^3}\right| \leq \frac{2!}{2\pi}\left(\frac{\delta}{(k_1+1)\sqrt{2|z|}}\right)^{-2} e k_1!\delta^{-k_1}A^{k_1} \tag{2.80}$$

$$\leq |z| 2e\pi^{-1}(k_1+2)!\delta^{-k_1-2}A^{k_1}$$

Similarly the estimates of the second and the third term of (2.73) is obtained from (2.60) and (2.78) as follows;

(2.81)

$$\left| \left(\frac{d\tilde{x}}{dz}\right)^3 \frac{d}{dz}\left(\frac{d\tilde{x}}{dz}\right)^{-3} \frac{d^2 x_{k_1}}{dz^2} \right| \leq \frac{|z|M_2^3}{2\pi^2 M_1^3} 2^{3(p+1)/(p-2)}(k_1+1)!\delta^{-k_1-2}A^{k_1},$$

(2.82)

$$\left| \frac{1}{2}\left(\frac{d\tilde{x}}{dz}\right)^2 \frac{d^2}{dz^2}\left(\frac{d\tilde{x}}{dz}\right)^{-2} \frac{d x_{k_1}}{dz} \right| \leq \frac{|z|M_2^2}{\pi M_1^2} 2^{2(p+1)/(p-2)}k_1!\delta^{-k_1-2}A^{k_1}.$$

Combining (2.80), (2.81) and (2.82), we find that (2.79) holds.

Now (2.47) and (2.79) enable us to estimate the second term of (2.71) as follows;

(2.83)

$$\begin{aligned}
&\Big|\frac{1}{2}\sum_{k_1+k_2=k-2}\left(\frac{d\tilde{x}}{dz}\right)^3 \frac{d^3 x_{k_1}}{d\tilde{x}^3} \sum_{l=\min\{1,k_2\}}^{k_2} \sideset{}{^*}\sum_{\substack{\mu_1+\cdots+\mu_l=k_2,\\ \mu_1,\cdots,\mu_l\geq 1}} (-1)^l \frac{dx_{\mu_1}}{dz}\cdots\frac{dx_{\mu_l}}{dz}\Big| \\
&\leq |z|\frac{M}{2}\delta^{-k}A^{k-2}\sum_{k_1+k_2=k-2}(k_1+2)!\sum_{l=\min\{1,k_2\}}^{k_2} C_0^l \sideset{}{^*}\sum_{\substack{\mu_1+\cdots+\mu_l=k_2,\\ \mu_1,\cdots,\mu_l\geq 1}} \mu_1!\cdots\mu_l! \\
&\leq |z|\frac{M}{2}\delta^{-k}A^{k-2}\sum_{k_1+k_2=k-2}(k_1+2)!\left(1+\sum_{l=1}^{k_2} C_0^l 4^{l-1}(k_2-l+1)!\right) \\
&\leq |z|\frac{M}{2}\delta^{-k}A^{k-2}\sum_{k_1+k_2=k-2}(k_1+2)!k_2!\left(1+\sum_{l=1}^{\infty} \frac{C_0^l 4^{l-1}}{(l-1)!}\right) \\
&\leq |z|\frac{M(1+C_0 e^{4C_0})}{2}\left(1+\frac{4}{k}\right)k!\delta^{-k}A^{k-2}.
\end{aligned}$$

Here we applied (2.71) to the second line of (2.83). Then, since we can take some positive constant M as (2.79) so that

$$\left|\left(\frac{d\tilde{x}}{dz}\right)^2 \frac{d^2 x_{k_1}}{d\tilde{x}^2}\right| \leq \sqrt{|z|}M(k_1+1)!\delta^{-k_1-1}A^{k_1} \tag{2.84}$$

holds for $k_1 \leq k-1$, by the similar discussion, we find the following estimates for the third term of (2.38);

$$\begin{aligned}
&\Big|\frac{3}{4}\sum_{k_1+k_2+k_3=k-2}\left(\frac{d\tilde{x}}{dz}\right)^4 \frac{d^2 x_{k_1}}{d\tilde{x}^2}\frac{d^2 x_{k_2}}{d\tilde{x}^2} \\
&\qquad\times\sum_{l=\min\{1,k_3\}}^{k_3}\ \sum^{*}_{\substack{\mu_1+\cdots+\mu_l=k_3,\\ \mu_1,\cdots,\mu_l\geq 1}}(-1)^l(l+1)\frac{dx_{\mu_1}}{dz}\cdots\frac{dx_{\mu_l}}{dz}\Big| \\
&\leq 9|z|M^2(1+4C_0^2e^{4C_0})\left(1+\frac{4}{k-1}\right)(k-1)!\delta^{-k}A^{k-2}.
\end{aligned} \tag{2.85}$$

In conclusion, we obtain the following estimates for $\Phi_k(z)$; there exists some positive constant M that is independent of $C_0(<1)$ and A such that, for $k \geq 2$, $0 < \delta < r/3$ and $z \in D^2_{r-\delta}$,

$$|\Phi_k(z)| \leq |z|M(C_0^2 + A^{-2})k!\delta^{-k}A^k \tag{2.86}$$

holds under the assumption that x_n $(1 \leq n \leq k-1)$ satisfy (2.46) and (2.47). Similarly we can show the followings; there exists some positive constant M that is independent of $C_0(<1)$ and A such that, for $k \geq 2$, $0 < \delta < r/3$ and $z \in D^1_{r-\delta}$,

$$|\Phi_k(z)| \leq M(C_0^2 + A^{-2})k!\delta^{-k}A^k \tag{2.87}$$

holds under the assumption that x_n $(1 \leq n \leq k-1)$ satisfy (2.44) and (2.45). Then, by the same discussion with the case of $k = 2$, we find that $x_k(z)$ satisfies

$$|x_k(z)| \leq \frac{1+8\sqrt{2}}{3}|z|M(C_0^2 + A^{-2})k!\delta^{-k}A^k \tag{2.88}$$

$$\left|\frac{dx_k}{dz}(z)\right| \leq \frac{2+4\sqrt{2}}{3}M(C_0^2+A^{-2})k!\delta^{-k}A^k \tag{2.89}$$

for $z \in D^2_{r-\delta}$. Therefore, by taking C_0 sufficiently small so that

$$\frac{1+8\sqrt{2}}{3}MC_0 < \frac{1}{2} \tag{2.90}$$

and then A sufficiently large so that

$$\frac{1+8\sqrt{2}}{3}MA^{-2} < \frac{1}{2}C_0, \tag{2.91}$$

we find that x_k satisfies (2.46) and (2.47). Thus the induction proceeds and (2.46) and (2.47) holds for all $k \geq 1$. By fixing $\delta = r/6$ and combining (2.44) and (2.46), we obtain the following estimates; there exist positive constants r and A such that

$$|x_k(z)| \leq (|z|+1)k!A^k \tag{2.92}$$

for $k \geq 1$ and $z \in D^0_r = D^1_r \cup D^2_r$.

By the same discussion, we can show that (2.92) holds on

$$D^{\pm}_r = \{z \in \mathbb{C}; e^{\mp 2\pi i/3}z \in D^0_r\} \tag{2.93}$$

for some r and A. Bearing in mind that we can take $\varepsilon > 0$ so that

$$\widehat{V}^{\varepsilon} = \left\{z \in \mathbb{C}; \left|\mathrm{Im}\int_0^z \sqrt{z}dz\right| < \varepsilon\right\} \subset \bigcup_{*\in\{0,\pm\}} D^*_r, \tag{2.94}$$

we immediately see that we can take a neighborhood $U^{\tilde{\varepsilon}}$ of $\tilde{x} = 0$ such that (2.33) holds on $\widehat{U}^{\tilde{\varepsilon}}$. □

Remark 2.5. By the same discussion with the proof of Proposition 2.6, we can show that the transformation series $x(\tilde{x}, \eta)$ satisfies (2.92) when $x_0^{-1}(D^0_r)$ runs into some double pole b_1 of $Q(\tilde{x})$, i.e., $Q(\tilde{x})$ has the following expansion at $\tilde{x} = b_1$;

$$Q(\tilde{x}) = \frac{\alpha}{(\tilde{x}-b_1)^2} + \frac{\beta}{\tilde{x}-b_1} + f(\tilde{x}), \tag{2.95}$$

where $\alpha, \beta \in \mathbb{C}$. In fact, under the assumption, we find that

$$\int_0^{\tilde{x}} \sqrt{Q(\tilde{x})} d\tilde{x} = \sqrt{\alpha} \log(\tilde{x} - b_1) + f(\tilde{x}) \tag{2.96}$$

holds around $\tilde{x} = b_1$, where the integral path is taken along the Stokes curve emanating from the simple turning point $\tilde{x} = 0$ and $f(\tilde{x})$ is multi-valued analytic function that is bounded at $\tilde{x} = b_1$. Therefore, from (2.35), we obtain the following estimates of $\sqrt{Q(\tilde{x}(z))}$; there exists positive constants M_1 and M_2 such that

$$M_1 \left| \exp\left[\frac{-2}{3\sqrt{\alpha}} z^{3/2} \right] \right| \leq \left| \sqrt{Q(\tilde{x}(z))} \right| \leq M_2 \left| \exp\left[\frac{-2}{3\sqrt{\alpha}} z^{3/2} \right] \right| \tag{2.97}$$

holds on D_r^2. Then, from (2.34), we find that $dx_0/d\tilde{x}$ satisfies

$$\frac{M_1}{\sqrt{|z|}} \left| \exp\left[\frac{-2}{3\sqrt{\alpha}} z^{3/2} \right] \right| \leq \left| \frac{dx_0}{d\tilde{x}} \right| \leq \frac{M_2}{\sqrt{|z|}} \left| \exp\left[\frac{-2}{3\sqrt{\alpha}} z^{3/2} \right] \right| \tag{2.98}$$

on D_r^2. Since we can take some positive constant M so that

$$\left| \left(z + \frac{e^{i\theta}\delta}{\sqrt{2|z|}} \right)^{3/2} - z^{3/2} \right| \leq M \tag{2.99}$$

holds for $\theta \in \mathbb{R}$, sufficiently small $\delta > 0$ and $z \in D_r^2$, by the same way with the derivation of (2.60) and (2.61), we obtain

$$\left| \left(\frac{d\tilde{x}}{dz}(z) \right)^2 \frac{d^j}{dz^j} \left(\frac{d\tilde{x}}{dz}(z) \right)^{-2} \right| \leq \frac{j!}{2\pi} 2^{-1/2} e^M M_1^{-2} M_2^2 |z|^{j/2} \delta^{-j} \tag{2.100}$$

for $z \in D_{r-\delta}^2$. Therefore the estimates (2.92) follows from exactly the same discussion with the proof of Proposition 2.6.

2.3 Borel summability of transformation series

Now we show the Borel summability of transformation series. For simplicity, we discuss in z variable as in the proof of Proposition 2.6

and we assume that

$$x_0^{-1}(e^{2(j-1)\pi i/3}\mathbb{R}_{\geq 0}) = T_j \quad (j = 1, 2, 3). \tag{2.101}$$

We take $\varepsilon > 0$ so that $\widehat{V}^{\varepsilon}$ is contained in $x_0(\widehat{U}^{\tilde{\varepsilon}})$ and (2.92) holds there. Let $\widehat{V}_j^{\varepsilon}$ $(j = 1, 2, 3)$ denote

$$\widehat{V}_j^{\varepsilon} = \{z \in \mathbb{C}; \mathrm{Re}(e^{-2(j-1)\pi i/3}z) > 0\} \cap \widehat{V}^{\varepsilon} \tag{2.102}$$

and $p_j(\geq 3)$ $(j = 1, 2, 3)$ be the orders of poles of $Q(\tilde{x})$ at b_j. Then, from [Ko4] (and also [DLS]), we immediately find the following

Theorem 2.8 ([Ko4])**.** *There exist some positive constants C_1, C_2 and δ such that*

$$\left|\tilde{R}_B(z, y)\right| \leq C_1 |z|^{-3(p_j-4)/2(p_j-2)} \exp[C_2|y|], \tag{2.103}$$

$$|R_B(z, y)| \leq C_1 |z|^{-5/2} \exp[C_2|y|] \tag{2.104}$$

hold on $\left(\widehat{V}_j^{2\varepsilon/3} \setminus \widehat{V}_j^{\varepsilon/3}\right) \times E_\delta^+$ $(j = 1, 2, 3)$, where $\tilde{R}_B$ and $R_B$are the Borel transform of $\tilde{R} = \eta^{-1}\tilde{S}_{\mathrm{odd}}(\tilde{x}(z), \eta) - S_{-1}(\tilde{x}(z))$ and $R = \eta^{-1}S_{\mathrm{odd}}(z, \eta) - S_{-1}(z)$ respectively and

$$E_\delta^+ = \bigcup_{s \geq 0} \{y \in \mathbb{C}; |y - s| \leq \delta\}. \tag{2.105}$$

Now we apply Theorem A.1 to

(2.106)
$$F(z, X, \eta) = \int_0^{z+X} \eta^{-1} S_{\mathrm{odd}}(z, \eta) dz - \int_0^{\tilde{x}(z)} \eta^{-1} \tilde{S}_{\mathrm{odd}}(\tilde{x}, \eta) d\tilde{x}$$

in X variable. Indeed, (2.23) guarantees that the transformation series $x(\tilde{x}, \eta)$ satisfies

$$\int_0^x \eta^{-1} S_{\mathrm{odd}}(x, \eta) dx \bigg|_{x = x(z,\eta)} - \int_0^{\tilde{x}(z)} \eta^{-1} \tilde{S}_{\mathrm{odd}}(\tilde{x}, \eta) d\tilde{x} = 0 \tag{2.107}$$

and the Borel summability of $X(z,\eta) = x(\tilde{x}(z),\eta) - z$ on a neighborhood W^ε of $\partial \widehat{V}^{\varepsilon/2}$ follows from that of $F(z,X,\eta)$. Hence our task is to confirm that $F(z,X,\eta)$ satisfies the conditions corresponding to (A.1), (A.2) and (A.3). First, bearing the shape of $\widehat{V}^\varepsilon$ in mind, we easily see that we can take some positive constant r so that $z+X \in \widehat{V}^{2\varepsilon/3} \setminus \widehat{V}^{\varepsilon/3}$ for (z,X) in

$$D_r^\varepsilon = \left\{ (z,X) \in W^\varepsilon \times \mathbb{C}; |X| \le \frac{r}{\sqrt{|z|}} \right\} \tag{2.108}$$

and the coefficients of $F(z,X,\eta)$ are holomorphic there. Since $\tilde{S}_{-1}(\tilde{x})$ and $S_{-1}(z) = \sqrt{z}$ satisfy (2.21), we find that

$$F_0(z,X) = \frac{2}{3}\left((z+X)^{3/2} - z^{3/2}\right). \tag{2.109}$$

Therefore we immediately find that $F_0(z,0) = 0$ and, by taking r sufficiently small, we can take some positive constant M so that F_0 satisfies

$$\left| \frac{F_0(z,X)}{X} \right| \ge M\sqrt{|z|} \tag{2.110}$$

for $(z,X) \in D_r^\varepsilon$. Finally, the Borel summability of $\tilde{F} = F - F_0$ on D_r^ε is derived from (2.103) and (2.104) as follows; we first remind us that the integration in (2.106) is defined by a contour integral around $z=0$. Let $z+X$ be in $\partial\widehat{V}^{\varepsilon_0}$ with $\varepsilon/3 < \varepsilon_0 < 2\varepsilon/3$. Since (2.104) guarantees the integrability of R_B at infinity, by deforming the contour along $\partial\widehat{V}^{\varepsilon_0}$, we find that the following estimates holds on $D_r^\varepsilon \times E_\delta^+$; there exists some positive constants C_1 and C_2 such that

$$\left| \int_0^{z+X} R_B(z,y)dz \right| \le C_1 \exp[C_2|y|]. \tag{2.111}$$

Further, since $dx_0/d\tilde{x}$ satisfies (2.50), we find that, for some positive constants C_1 and C_2,

$$\left|\tilde{R}_B(z,y)\frac{d\tilde{x}}{dz}\right| \le C_1|z|^{-5/2}\exp[C_2|y|] \tag{2.112}$$

holds on $W^{\varepsilon} \times E_{\delta}^{+}$ and, by the same discussion as in the derivation of (2.111), we obtain

$$\left|\int_0^{\tilde{x}(z)} \tilde{R}_B(\tilde{x},y)d\tilde{x}\right| \le C_1\exp[C_2|y|] \tag{2.113}$$

for $(z,y) \in W^{\varepsilon} \times E_{\delta}$.

In conclusion, applying Theorem A.1 to $F(z,X,\eta)$, we find that $X(z,\eta)$ is Borel summable on W^{ε}. More precisely, from (2.110), (2.111) and (2.113), we obtain the following estimates, which is corresponding to (A.11), of the Borel transform X_B of X on $W^{\varepsilon} \times E_{\delta}^{+}$;

$$|X_B(z,y)| \le \frac{C_1}{M\sqrt{|z|}}\exp\left[\left(\frac{4C_1}{Mr}+C_2\right)|y|\right]. \tag{2.114}$$

Finally, combining (2.92) and (2.114), we validate the Borel summability of $X(z,\eta)$ on $\widehat{V}^{\varepsilon/2}$. First, from Cauchy's formula, we obtain the following integral representation of $X_B(z,y)$ in a neighborhood of $(z,y)=(0,0)$;

$$X_B(z,y) = \frac{(z+1)^2}{2\pi i}\oint_{|\tilde{z}-z|=\varepsilon_0}\frac{1}{(\tilde{z}+1)^2}\frac{X_B(\tilde{z},y)}{\tilde{z}-z}d\tilde{z}. \tag{2.115}$$

Here $\varepsilon_0 > 0$ is taken so that the integral path is contained in $\widehat{V}^{\varepsilon/2}$ and we assume that, by taking ε sufficiently small, $z=-1$ is not contained in $\widehat{V}^{\varepsilon}$. Then we deform the integral path along $\partial\widehat{V}^{\varepsilon/2}$ and find that $X_B(z,y)$ is holomorphic on $\widehat{V}^{\varepsilon/2} \times \{y \in \mathbb{C}; |y| < 1/A\}$. In fact, (2.92) guarantees that the integrand of (2.115) is integrable along $\partial\widehat{V}^{\varepsilon/2}$ and holomorphic in a fixed neighborhood of the origin under the deformation of the integral path, i.e., this integral representation gives the analytic continuation of $X_B(z, y)$. Then this integral representation tells us that $X(z, \eta)$ is Borel summable on $\widehat{V}^{\varepsilon/2}$. Further, since $x(\tilde{x}, \eta)$ satisfies (2.11), we find that $X_B(z, y)$ satisfies

$$X_B(z, -y) = -X_B(z, y). \tag{2.116}$$

In conclusion, we obtain the following

Theorem 2.9. *There exist positive constants C_1, C_2, δ and ε such that $X_B(z, y)$ is holomorphic on $\widehat{V}^{\varepsilon} \times E_\delta$ and satisfies the following estimates there;*

$$|X_B(z, y)| \leq C_1(|z| + 1)^2 \exp\left[C_2|y|\right]. \tag{2.117}$$

Here

$$E_\delta = \bigcup_{s \in \mathbb{R}} \{y \in \mathbb{C}; |y - s| \leq \delta\}. \tag{2.118}$$

Remark 2.6. We can also show the Borel summability of the transformation series with a minor change of discussion when the Stokes curves run into some double poles of Q. For simplicity, we consider the case that $x_0^{-1}(\widehat{V}_1^{\varepsilon})$ runs into a double pole b_1 and $\widehat{V}_j^{\varepsilon}$ $(j = 2, 3)$ run into irregular singular points b_j $(j = 2, 3)$ respectively. Then, instead of (2.113), we find from [Ko4] that

$$\left| \int_0^{\tilde{x}(z)} \tilde{R}_B(\tilde{x}, y) d\tilde{x} \right| \leq C_1 |z|^{3/2} \exp[C_2|y|] \tag{2.119}$$

holds for $(z, y) \in \left(W^{\varepsilon} \cap \widehat{V}_1^{\varepsilon}\right) \times E_\delta^{+}$. Therefore, applying Theorem A.1 to $F(z, X, \eta)$, we obtain the following estimates on $\left(W^{\varepsilon} \cap \widehat{V}_1^{\varepsilon}\right) \times E_\delta^{+}$;

$$|X_B(z, y)| \leq \frac{C_1|z|}{M} \exp\left[\left(\frac{4C_1|z|^{3/2}}{Mr} + C_2\right)|y|\right]. \tag{2.120}$$

Similarly, we find that (2.114) holds on $\left(W^\varepsilon \cap \widehat{V}_j^\varepsilon\right) \times E_\delta^+$ $(j = 2, 3)$. In conclusion, applying the same technique as in the proof of Theorem 2.9 to the integral representation

$$\begin{aligned} X_B(z, y) =& \frac{\exp[C_3(z+1)^{3/2} y]}{2\pi i} \\ & \times \oint_{|\tilde{z}-z|=\varepsilon_0} \frac{\exp[-C_3(\tilde{z}+1)^{3/2} y] X_B(\tilde{z}, y)}{\tilde{z} - z} d\tilde{z}, \end{aligned} \tag{2.121}$$

where a positive constant C_3 is taken as

$$C_3 > \frac{4C_1}{Mr}, \tag{2.122}$$

we find the following

Theorem 2.10. *There exist positive constants C_1, C_2, C_3, δ and ε such that $X_B(z, y)$ is holomorphic on $\widehat{V}^\varepsilon \times E_\delta$ and satisfies the following estimates there;*

$$|X_B(z, y)| \le C_1 \exp\left[\left(C_2 + C_3|z|^{3/2}\right) |y|\right]. \tag{2.123}$$

Remark 2.7. It is difficult to derive the Borel summability of $X(z, \eta)$ from (2.14) directly. Therefore we appealed to the implicit function theorem for Borel summable series. However, since S_{odd} and $\tilde{S}_{\text{odd}}$ are not Borel summable on the Stokes curves, we can not show the Borel summability of $X(z, \eta)$ there only by the implicit function theorem. Hence we used a kind of Hartogs' phenomenon to extend the region where $X(z, \eta)$ is Borel summable. We can find a similar discussion in [D1]. There, the convergence of inverse factorial series solution on a neighborhood of a simple turning point was examined. And he used the maximum modulus theorem on the set like W^ε to avoid a direct discussion at the simple turning point. This similarity of the discussion was suggested by Professor R. Schäfke. In [D2], we can also find a similar discussion used in Section 3.3.

3 WKB theoretic transformation — a simple pole case

The main purpose of this section is to show the Borel summability of transformation series, which is given in [Ko1], of (3.2) to the WKB theoretic canonical equation (3.7) near a Stokes curve emanating from a simple pole of $Q(\tilde{x})$ when it runs into some irregular singular points. (See Remark 3.1 and Remark 3.2 in the case that it runs into a double pole of $Q(\tilde{x})$.) Discussions in this section proceed in the same way as in Section 2.

3.1 Fundamental properties of WKB theoretic transformation and its application

Let $Q(\tilde{x})$ be a meromorphic function that has a simple pole at the origin, i.e., $\tilde{x}Q(\tilde{x})$ is holomorphic at $\tilde{x}=0$ and satisfies

$$\tilde{x}Q(\tilde{x})|_{\tilde{x}=0} \neq 0. \tag{3.1}$$

Then we consider the Schrödinger equation

$$\left(\frac{d^2}{d\tilde{x}^2} - \eta^2 Q(\tilde{x})\right)\tilde{\psi}(\tilde{x},\eta) = 0. \tag{3.2}$$

with the following geometric assumptions (3.3) and (3.4); at first, we assume that

(3.3) a Stokes curve T emanating from $\tilde{x}=0$ runs into an irregular singular point b.

Let $\widehat{U}^{\tilde{\varepsilon}}_{\pm}$ be unions of integral curves of $\mathrm{Im}\sqrt{Q(\tilde{x})}d\tilde{x} = 0$ that pass through some $\tilde{x}_0 \in U^{\tilde{\varepsilon}} \setminus T$ and $\pm\mathrm{Re}\displaystyle\int_{\tilde{x}_0}^{\tilde{x}} \sqrt{Q(\tilde{x})}d\tilde{x} \geq 0$ there. Here $U^{\tilde{\varepsilon}}$ denotes a disk $U^{\tilde{\varepsilon}} = \{\tilde{x} \in \mathbb{C}; |\tilde{x}| < \tilde{\varepsilon}\}$ and $\tilde{\varepsilon}$ is a sufficiently small positive constant. Then we assume that we can take $\tilde{\varepsilon} > 0$ so that

(3.4) $\widehat{U}^{\tilde{\varepsilon}}_{+}$ and $\widehat{U}^{\tilde{\varepsilon}}_{-}$ run into b.

We remark here that (3.3) and (3.4) guarantee that

$$\widehat{U}^{\tilde{\varepsilon}} = \widehat{U}^{\tilde{\varepsilon}}_{+} \cup \widehat{U}^{\tilde{\varepsilon}}_{-} \cup T \tag{3.5}$$

does not contain any poles nor turning points except for a simple pole at the origin.

Now, we consider a transformation series

$$x(\tilde{x}, \eta) = \sum_{k=0}^{\infty} x_k(\tilde{x}) \eta^{-k} \tag{3.6}$$

of (3.1) to the following canonical equation on $\widehat{U}^{\tilde{\varepsilon}}$;

$$\left(\frac{d^2}{dx^2} - \eta^2 \frac{1}{x} \right) \psi = 0. \tag{3.7}$$

In parallel with Theorem 2.1, we have the following

Theorem 3.1. *Let $Q(\tilde{x})$ be a meromorphic function that satisfies (3.1), (3.3) and (3.4). Then there exists a Borel summable series $x(\tilde{x}, \eta)$ on $\widehat{U}^{\tilde{\varepsilon}}$ such that*

(3.8) $\{x_k(\tilde{x})\}_{k=0}^{\infty}$ *are holomorphic on* $\widehat{U}^{\tilde{\varepsilon}}$,

(3.9) $x_{2k+1}(\tilde{x})$ $(k = 0, 1, 2, \cdots)$ *are identically zero,*

$$x_0(0) = 0, \tag{3.10}$$

$$\frac{dx_0}{d\tilde{x}} \neq 0 \ \text{on} \ \widehat{U}^{\tilde{\varepsilon}} \tag{3.11}$$

and satisfies the following relation;

$$Q(\tilde{x}) = \left(\frac{dx(\tilde{x}, \eta)}{d\tilde{x}} \right)^2 \frac{1}{x(\tilde{x}, \eta)} - \frac{1}{2} \eta^{-2} \{x(\tilde{x}, \eta); \tilde{x}\}. \tag{3.12}$$

The proof of Theorem 3.1 is given in Section 3.2 and Section 3.3.

Then $x(\tilde{x}, \eta)$ gives the following relations (See [Ko1]);

Theorem 3.2. *Let $\tilde{S}(\tilde{x},\eta)$ and $S(x,\eta)$ respectively be solutions of Riccati equations*

$$\tilde{S}^2 + \frac{d\tilde{S}}{d\tilde{x}} = \eta^2 Q(\tilde{x}) \tag{3.13}$$

and

$$S^2 + \frac{dS}{dx} = \eta^2 \frac{1}{x}, \tag{3.14}$$

where $\tilde{S}_{-1}(\tilde{x})$ and $S_{-1}(x)$ are taken so that they satisfy

$$\tilde{S}_{-1}(\tilde{x}) = \left(\frac{dx_0}{d\tilde{x}}\right) S_{-1}(x_0(\tilde{x})). \tag{3.15}$$

Then $x(\tilde{x},\eta)$ in Theorem 3.1 satisfies the following relation;

$$\tilde{S}(\tilde{x},\eta) = \left(\frac{dx}{d\tilde{x}}\right) S(x(\tilde{x},\eta),\eta) - \frac{1}{2}\left(\frac{d^2x}{d\tilde{x}^2}\right) \Big/ \left(\frac{dx}{d\tilde{x}}\right). \tag{3.16}$$

Corollary 3.3. *Let $\tilde{S}_{\mathrm{odd}}$ and S_{odd} respectively be the odd part of $\tilde{S}$ and S. And assume that $\tilde{S}_{-1}$ and S_{-1} are taken so that they satisfy* (3.15). *Then the following relation holds;*

$$\tilde{S}_{\mathrm{odd}}(\tilde{x},\eta) = \left(\frac{dx(\tilde{x},\eta)}{d\tilde{x}}\right) S_{\mathrm{odd}}(x(\tilde{x},\eta),\eta). \tag{3.17}$$

Now we consider WKB solutions $\tilde{\psi}_\pm(\tilde{x},\eta)$ of (3.2) normalized at a simple pole at the origin, i.e.,

$$\tilde{\psi}_\pm(\tilde{x},\eta) = \frac{1}{\sqrt{\tilde{S}_{\mathrm{odd}}}} \exp\left(\pm \int_0^{\tilde{x}} \tilde{S}_{\mathrm{odd}}(\tilde{x},\eta)d\tilde{x}\right) \tag{3.18}$$

and WKB solutions $\psi_\pm(x,\eta)$ of (3.7) normalized at a simple pole in the same manner. Then they satisfy the following

Theorem 3.4. *Let $\tilde{\psi}_\pm(\tilde{x},\eta)$ and $\psi_\pm(x,\eta)$ respectively be WKB solutions of* (3.2) *and* (3.7) *normalized at their simple pole at the* *origin. Then the following relation holds;*

$$\tilde{\psi}_{\pm}(\tilde{x},\eta) = \left(\frac{dx(\tilde{x},\eta)}{d\tilde{x}}\right)^{-1/2} \psi_{\pm}(x(\tilde{x},\eta),\eta). \tag{3.19}$$

In order to simplify the discussion, we employ $x_0 = x_0(\tilde{x})$ as a new coordinate variable. Then, applying the Borel transform to the relation (3.19), we find that the Borel transform $\tilde{\psi}_{\pm,B}(\tilde{x},y)$ of $\tilde{\psi}_{\pm}(\tilde{x},\eta)$ can be described by $\psi_{\pm,B}(x,y)$ through the action of the following microdifferential operator $\mathcal{X}$;

$$\mathcal{X} =: \left(\frac{\partial\tilde{x}}{\partial x_0}\right)^{1/2} \left(1+\frac{\partial X}{\partial x_0}\right)^{-1/2} \exp[X(x_0,\eta)\xi] :, \tag{3.20}$$

where $X(x_0,\eta)$ is

$$X(x_0,\eta) = x(\tilde{x}(x_0),\eta) - x_0 \tag{3.21}$$

and ξ designates the symbol of ∂_{x_0}. By the same reasoning as in Section 2.1, we obtain the following

Theorem 3.5. *$\tilde{\psi}_{\pm,B}$ and $\psi_{\pm,B}$ satisfy the following relation on $\widehat{V}^{\varepsilon} \times E^{\delta}_{\pm y_0}$ for sufficiently small $\varepsilon, \delta > 0$;*

$$\begin{aligned}\tilde{\psi}_{\pm,B}(\tilde{x}(x_0),y) = &\left(\frac{\partial\tilde{x}}{\partial x_0}\right)^{1/2} \psi_{\pm,B}(x_0,y) \\ &+ \int_{\mp y_0}^{y} K(x_0, y-y', \partial_{x_0})\psi_{\pm,B}(x_0,y')dy',\end{aligned} \tag{3.22}$$

where

$$y_0(x_0) = \int_0^{x_0} \frac{1}{\sqrt{x_0}} dx_0, \tag{3.23}$$

$$\widehat{V}^{\varepsilon} = \{x_0 \in \mathbb{C}; |\mathrm{Im}\ y_0(x_0)| < \varepsilon\}, \tag{3.24}$$

$$E^{\delta}_{\pm y_0} = \bigcup_{s\in\mathbb{R}} \{y \in \mathbb{C}; |y - s \pm y_0(x_0)| < \delta\}. \tag{3.25}$$

and $K(x, y, \partial_x)$ is a differential operator of infinite order on $\widehat{V}^{\varepsilon} \times E^{\delta}_{\pm y_0}$.

3.2 Uniform Borel transformability of transformation series

The purpose of Section 3.2 is to verify the following

Proposition 3.6. *Let $Q(\tilde{x})$ be a meromorphic function that satisfies* (3.1), (3.3) *and* (3.4). *Then there exist $\tilde{\varepsilon} > 0$ and formal series $x(\tilde{x}, \eta) = x_0(\tilde{x}) + \eta^{-1}x_1(\tilde{x}) + \cdots$ that satisfies* (3.8) $\sim$ (3.12) *and the following estimates; there exist positive constants C_0 and A such that for all $n \geq 1$ and $\tilde{x} \in \widehat{U}^{\tilde{\varepsilon}}$, $x_n(\tilde{x})$ satisfies*

$$|x_n(\tilde{x})| \leq (|x_0(\tilde{x})| + 1)C_0 n! A^n. \tag{3.26}$$

Proof. Proof of Proposition 3.6 proceeds in the same process as in the proof of Proposition 2.9.

We first review the construction of $x(\tilde{x}, \eta)$. We inductively determine $x_k(\tilde{x})$ ($k = 0, 1, 2, \cdots$) by comparing the coefficients of η^{-k} of (3.12). First, by comparing the coefficients of η^0 of (3.12), we find that $x_0(\tilde{x})$ should satisfy the following relation;

$$Q(\tilde{x}) = \left(\frac{dx_0(\tilde{x})}{d\tilde{x}}\right)^2 \frac{1}{x_0(\tilde{x})}. \tag{3.27}$$

Then we determine $x_0(\tilde{x})$ so that it becomes holomorphic at the origin as follows;

$$x_0(\tilde{x}) = \left(\frac{1}{2}\int_0^{\tilde{x}} \sqrt{Q(\tilde{x})}d\tilde{x}\right)^2. \tag{3.28}$$

From the assumptions (3.1), (3.3) and (3.4), we immediately find that $x_0(\tilde{x})$ satisfies (3.8) and (3.10). Further, by the same reasoning as in the proof of Proposition 2.9, we see that x_0 maps $\widehat{U}^{\tilde{\varepsilon}}$ to $x_0(\widehat{U}^{\tilde{\varepsilon}})$

bijectively and satisfies (3.11). From now, we employ $z = x_0(\tilde{x})$ as a new coordinate variable on $x_0(\widehat{U}^{\tilde{\varepsilon}})$.

Next we determine x_k ($k \geq 1$). By comparing the coefficients of η^{-k} of (3.12), we find that x_k should satisfy the following recurrence relation;

$$2z\frac{dx_k}{dz} - x_k = z\Phi_k(z), \tag{3.29}$$

where $\Phi_k(z)$ is

(3.30)

$$\begin{aligned}
\Phi_k(z) = &-\sum_{l=2}^{k} \frac{(-1)^l}{z^l} \sum^{*}_{\substack{\mu_1+\cdots+\mu_l=k,\\ \mu_1,\cdots,\mu_l\geq 1}} x_{\mu_1}\cdots x_{\mu_l} \\
&- \sum_{\substack{k_1+k_2+k_3=k,\\ k_1,k_2,k_3\leq k-1}} \frac{dx_{k_1}}{dz}\frac{dx_{k_2}}{dz} \sum_{l=\min\{1,k_3\}}^{k_3} \frac{(-1)^l}{z^l} \sum^{*}_{\substack{\mu_1+\cdots+\mu_l=k_3,\\ \mu_1,\cdots,\mu_l\geq 1}} x_{\mu_1}\cdots x_{\mu_l} \\
&+ \frac{z}{2} \sum_{k_1+k_2=k-2} \left(\frac{d\tilde{x}}{dz}\right)^3 \frac{d^3x_{k_1}}{d\tilde{x}^3} \\
&\qquad \times \sum_{l=\min\{1,k_2\}}^{k_2} (-1)^l \sum^{*}_{\substack{\mu_1+\cdots+\mu_l=k_2,\\ \mu_1,\cdots,\mu_l\geq 1}} \frac{dx_{\mu_1}}{dz}\cdots\frac{dx_{\mu_l}}{dz} \\
&- \frac{3z}{4} \sum_{k_1+k_2+k_3=k-2} \left(\frac{d\tilde{x}}{dz}\right)^4 \frac{d^2x_{k_1}}{d\tilde{x}^2}\frac{d^2x_{k_2}}{d\tilde{x}^2} \\
&\qquad \times \sum_{l=\min\{1,k_3\}}^{k_3} (-1)^l(l+1) \sum^{*}_{\substack{\mu_1+\cdots+\mu_l=k_3,\\ \mu_1,\cdots,\mu_l\geq 1}} \frac{dx_{\mu_1}}{dz}\cdots\frac{dx_{\mu_l}}{dz}.
\end{aligned}$$

Since Φ_k does not contain x_n $(n \geq k)$, we inductively determine x_k by

$$x_k(z) = \frac{z^{1/2}}{2} \int_0^z z^{-1/2} \Phi_k(z) dz \tag{3.31}$$

so that x_k is holomorphic at $z = 0$ and satisfies (3.30). We easily see that we can inductively confirm that Φ_{2k+1} $(k \geq 0)$ identically vanish and hence (3.9) holds.

Next we validate the estimates that x_k $(k = 1, 2, \cdots)$ satisfy. We first set

$$D_r^1 = \bigcup_{0 \leq s \leq 1} \{z \in \mathbb{C}; |z - s| \leq r\} \tag{3.32}$$

$$D_r^2 = \bigcup_{s \geq 1} \left\{z \in \mathbb{C}; |z - s| \leq r\sqrt{s}\right\} \tag{3.33}$$

for sufficiently small $r > 0$. Then, taking account of the fact that $2\mathrm{Im} z^{1/2}$ is expanded to

$$\frac{\mathrm{Im} z}{(\mathrm{Re} z)^{1/2}} - \frac{1}{8} \frac{(\mathrm{Im} z)^3}{(\mathrm{Re} z)^{5/2}} + \cdots \tag{3.34}$$

on D_r^2, we find that $\mathrm{Im} \displaystyle\int_0^z z^{-1/2} dz$ behaves as $\mathrm{Im} z \cdot (\mathrm{Re} z)^{-1/2}$ for sufficiently large $\mathrm{Re} z$. Therefore, by taking $r > 0$ sufficiently small, we can assume that $D_r^1 \cup D_r^2$ is included in $x_0(\widehat{U}^{\tilde{\varepsilon}})$.

Now we inductively confirm that $x_k(z)$ $(k \geq 1)$ satisfy the following estimates; there exist positive constants $C_0 < 1$ and $A > 1$ such that, for all δ with $0 < \delta < r/3$ and $z \in D_{r-\delta}^1 \cup D_{r-\delta}^2$,

$$|x_k(z)| \leq |z| C_0 k! \delta^{-k} A^k \tag{3.35}$$

$$\left| \frac{dx_k}{dz}(z) \right| \leq C_0 k! \delta^{-k} A^k \tag{3.36}$$

hold. Since x_1 is identically zero, it is obvious that (3.35) and (3.36) hold when $k = 1$. Therefore our task is to show that x_k satisfies (3.35) and (3.36) under the assumption that x_n $(1 \leq n \leq k-1)$ satisfy them.

We first beware of the behavior of $x_0(\tilde{x})$ when $\tilde{x}$ tends to the irregular singular point b along the Stokes curve T. Let $Q(\tilde{x})$ have a pole of order $p(\geq 3)$ at $\tilde{x} = b$. Then (3.27) and (3.28) tells us that $x_0(\tilde{x})$ and $dx_0/d\tilde{x}$ behave as

$$x_0(\tilde{x}) = O\left((\tilde{x} - b)^{-p+2}\right), \tag{3.37}$$

$$\frac{dx_0}{d\tilde{x}}(\tilde{x}) = O\left((\tilde{x} - b)^{-p+1}\right) \tag{3.38}$$

when $\tilde{x}$ tends to b. Therefore $dx_0/d\tilde{x}$ can be estimated by x_0 as follows; there exist positive constants M_1 and M_2 such that

$$M_1|x_0|^{(p-1)/(p-2)} \leq \left|\frac{dx_0}{d\tilde{x}}\right| \leq M_2|x_0|^{(p-1)/(p-2)} \tag{3.39}$$

holds on D_r^2.

Next we derive the estimates that Φ_k satisfies. Precisely we show the following estimates; there exists some positive constant M that is independent of $C_0(< 1)$ and A such that, for $k \geq 2$, $0 < \delta < r/3$ and $z \in D_{r-\delta}^1 \cup D_{r-\delta}^2$,

$$|\Phi_k(z)| \leq M(C_0^2 + A^{-2})k!\delta^{-k}A^k \tag{3.40}$$

holds. Since (3.40) is obtained for $z \in D_{r-\delta}^1$ by the same discussion given in [Ko4], we prove (3.40) only for $z \in D_{r-\delta}^2$ here. From Lemma 2.7 and (3.35), we find that the first term of (3.30) is estimated as follows;

$$\begin{aligned} &\Big|\sum_{l=2}^{k} \frac{(-1)^l}{z^l} \sideset{}{^*}\sum_{\substack{\mu_1+\cdots+\mu_l=k,\\ \mu_1,\cdots,\mu_l\geq 1}} x_{\mu_1}\cdots x_{\mu_l}\Big| \\ &\leq \sum_{l=2}^{k} \frac{1}{|z|^l} \sideset{}{^*}\sum_{\substack{\mu_1+\cdots+\mu_l=k,\\ \mu_1,\cdots,\mu_l\geq 1}} |z|^l C_0^l \mu_1!\cdots\mu_l!\delta^{-k}A^k \end{aligned} \tag{3.41}$$

$$
\begin{aligned}
&\le 4C_0^2 k!\delta^{-k}A^k \sum_{l=2}^{k} \frac{C_0^{l-2}4^{l-2}}{(l-1)!} \\
&\le 4e^{4C_0}C_0^2 k!\delta^{-k}A^k.
\end{aligned}
$$

Similarly we obtain the following estimates of the second term of (3.30);

$$
\begin{aligned}
&\Big| \sum_{\substack{k_1+k_2+k_3=k,\\ k_1,k_2,k_3\le k-1}} \frac{dx_{k_1}}{dz}\frac{dx_{k_2}}{dz} \sum_{l=\min\{1,k_3\}}^{k_3} \frac{(-1)^l}{z^l} \sum^{*}_{\substack{\mu_1+\cdots+\mu_l=k_3,\\ \mu_1,\cdots,\mu_l\ge 1}} x_{\mu_1}\cdots x_{\mu_l}\Big| \\
&\le C_0^2(1+e^{4C_0})\left(\frac{4^2}{k-1}+12\right)(k-1)!\delta^{-k}A^k.
\end{aligned} \tag{3.42}
$$

Now we examine the third term of (3.30). We first estimate the most important factor

$$
\begin{aligned}
\left(\frac{d\tilde{x}}{dz}\right)^3 \frac{d^3x_{k_1}}{d\tilde{x}^3} &= \frac{d^3x_{k_1}}{dz^3} + \left(\frac{d\tilde{x}}{dz}\right)^3 \frac{d}{dz}\left(\frac{d\tilde{x}}{dz}\right)^{-3}\frac{d^2x_{k_1}}{dz^2} \\
&\quad + \frac{1}{2}\left(\frac{d\tilde{x}}{dz}\right)^2 \frac{d^2}{dz^2}\left(\frac{d\tilde{x}}{dz}\right)^{-2}\frac{dx_{k_1}}{dz}.
\end{aligned} \tag{3.43}
$$

We consider the first term of (3.43). We use the following representation of the derivative of dx_{k_3}/dz on $D^2_{r-\delta}$;

$$
\frac{d^j}{dz^j}\frac{dx_{k_3}}{dz}(z) = \frac{j!}{2\pi i}\int_{|\tilde{z}-z|=d} \frac{1}{(\tilde{z}-z)^{j+1}}\frac{dx_{k_3}}{dz}(\tilde{z})d\tilde{z}, \tag{3.44}
$$

where $d>0$ is taken as

$$
d = \frac{\delta}{\sqrt{2}(k_1+1)}\sqrt{|z|}. \tag{3.45}
$$

Then we find that the integral path of (3.44) is contained in $D^2_{r-k_1\delta/(k_1+1)}$. Actually, if we take $s\ge 1$ so that $|z-s|\le (r-\delta)\sqrt{s}$, we find that $|z|\le 2s$ holds and that $\tilde{z}$ on the integral path satisfies

$$
|\tilde{z}-s| \le (r-\delta)\sqrt{s} + \frac{\delta}{\sqrt{2}(k_1+1)}\sqrt{|z|} \tag{3.46}
$$

$$\leq (r-\delta)\sqrt{s} + \frac{\delta}{\sqrt{2}(k_1+1)}\sqrt{2s}$$
$$= \sqrt{s}\left(r - \frac{k_1}{k_1+1}\delta\right).$$

Therefore, substituting δ in (3.36) for $k = k_1$ to $k_1\delta/(k_1+1)$, we find that dx_{k_1}/dz satisfies the following estimates for $\tilde{z} \in D^2_{r-k_1\delta/(k_1+1)}$;

$$\left|\frac{dx_{k_1}}{dz}(\tilde{z})\right| \leq k_1! e\delta^{-k_1}A^{k_1}. \tag{3.47}$$

Then, from (3.44), we obtain

$$\left|\frac{d^j}{dz^j}\frac{dx_{k_1}}{dz}(z)\right| \leq \frac{j!}{2\pi}\left(\frac{\delta\sqrt{|z|}}{\sqrt{2}(k_1+1)}\right)^{-j} ek_1!\delta^{-k_1}A^{k_1}. \tag{3.48}$$

In the same way, we find from (3.39) that the following estimates hold for $j = 1, 2$;

$$\left|\left(\frac{d\tilde{x}}{dz}(z)\right)^{4-j}\frac{d^j}{dz^j}\left(\frac{d\tilde{x}}{dz}(z)\right)^{-4+j}\right| \tag{3.49}$$
$$\leq \frac{2^{(4-j)(p-1)/(p-2)}}{2\pi}\left(\frac{M_2}{M_1}\right)^{4-j}\left(\frac{\sqrt{2}}{\sqrt{|z|}}\right)^{j}\delta^{-j}.$$

In conclusion, we obtain the following estimates; there exists some positive constant M that is independent of z, k_1, C_0, δ and A such that

$$\left|\left(\frac{d\tilde{x}}{dz}\right)^3\frac{d^3x_{k_1}}{d\tilde{x}^3}\right| \leq \frac{M}{|z|}(k_1+2)!\delta^{-k_1-2}A^{k_1} \tag{3.50}$$

holds on $D^2_{r-\delta}$. Therefore, by the same calculation with (2.83), we gain the following estimates

(3.51)

$$\Big|\frac{z}{2}\sum_{k_1+k_2=k-2}\left(\frac{d\tilde{x}}{dz}\right)^3\frac{d^3x_{k_1}}{d\tilde{x}^3}\sum_{l=\min\{1,k_2\}}^{k_2}(-1)^l\sum^*_{\substack{\mu_1+\cdots+\mu_l=k_2,\\ \mu_1,\cdots,\mu_l\geq 1}}\frac{dx_{\mu_1}}{dz}\cdots\frac{dx_{\mu_l}}{dz}\Big|$$
$$\leq\frac{M(1+C_0e^{4C_0})}{2}\left(1+\frac{4}{k}\right)k!\delta^{-k}A^{k-2}.$$

Similarly, we find that the fourth term of (3.30) is dominated as follows;

$$\Big|\frac{3z}{4}\sum_{k_1+k_2+k_3=k-2}\left(\frac{d\tilde{x}}{dz}\right)^4\frac{d^2x_{k_1}}{d\tilde{x}^2}\frac{d^2x_{k_2}}{d\tilde{x}^2} \tag{3.52}$$
$$\times\sum_{l=\min\{1,k_3\}}^{k_3}\sum^*_{\substack{\mu_1+\cdots+\mu_l=k_3,\\ \mu_1,\cdots,\mu_l\geq 1}}(-1)^l(l+1)\frac{dx_{\mu_1}}{dz}\cdots\frac{dx_{\mu_l}}{dz}\Big|$$
$$\leq 9M^2(1+4C_0^2e^{4C_0})\left(1+\frac{4}{k-1}\right)(k-1)!\delta^{-k}A^{k-2}.$$

Summing up (3.41), (3.42), (3.51) and (3.52), we obtain (3.40).

It is now clear from (3.31) and (3.40) that $x_k(z)$ satisfies

$$|x_k(z)|\leq|z|M(C_0^2+A^{-2})k!\delta^{-k}A^k \tag{3.53}$$

on $D^1_{r-\delta}\cup D^2_{r-\delta}$. Further, combining (3.40) and (3.53), we find from (3.29) that

$$\left|\frac{dx_k}{dz}(z)\right|\leq\frac{3M}{2}(C_0^2+A^{-2})k!\delta^{-k}A^k \tag{3.54}$$

also holds there. Therefore, by taking C_0 sufficiently small so that

$$\frac{3M}{2}C_0<\frac{1}{2} \tag{3.55}$$

and then A sufficiently large so that

$$\frac{3M}{2}A^{-2} < \frac{1}{2}C_0, \tag{3.56}$$

we find that x_k satisfies (3.35) and (3.36), and hence the induction proceeds. Now, fixing $\delta = r/6$, we obtain the following estimates from (3.35); there exist positive constants r and A such that

$$|x_k(z)| \leq |z|k!A^k \tag{3.57}$$

for $k \geq 1$ and $z \in D_r^1 \cup D_r^2$.

In conclusion, by taking $\varepsilon, \tilde{\varepsilon} > 0$ so that

$$\widehat{V}^{\varepsilon} = \left\{ z \in \mathbb{C}; \left| \operatorname{Im} \int_0^z \frac{1}{\sqrt{z}} dz \right| < \varepsilon \right\} \subset D_r^1 \cup D_r^2 \tag{3.58}$$

and $x_0(U^{\tilde{\varepsilon}}) \subset \widehat{V}^{\varepsilon}$ are satisfied, we obtain (3.26). □

Remark 3.1. As in Section 2.2, we can show that the transformation series $x(\tilde{x}, \eta)$ satisfies (3.57) when $x_0^{-1}(D_r^2)$ runs into some double pole b_1 of $Q(\tilde{x})$. We assume that $Q(\tilde{x})$ is expanded as (2.95). Then, from (2.96), (3.27) and (3.28), we find the following estimates holds on D_r^2; there exists positive constants M_1 and M_2 such that

$$M_1\sqrt{|z|}\left|\exp\left[\frac{-2}{\sqrt{\alpha}}z^{1/2}\right]\right| \leq \left|\frac{dx_0}{d\tilde{x}}\right| \leq M_2\sqrt{|z|}\left|\exp\left[\frac{-2}{\sqrt{\alpha}}z^{1/2}\right]\right|. \tag{3.59}$$

Bearing in mind that we can take some positive constant M so that

$$\left|\left(z + e^{i\theta}\delta\sqrt{|z|}\right)^{1/2} - z^{1/2}\right| \leq M \tag{3.60}$$

holds for $\theta \in \mathbb{R}$, sufficiently small $\delta > 0$ and $z \in D_r^2$, we find from the same discussion with the proof of Proposition 3.6 that $x(\tilde{x}, \eta)$ satisfies (3.57).

3.3 Borel summability of transformation series

As in the proof of Theorem 2.9, we first show the Borel summability of $X(z,\eta) = x(z,\eta) - z$ on a neighborhood W^ε of $\partial\widehat{V}^{\varepsilon/2}$ by applying Theorem A.1 to

(3.61)
$$F(z,X,\eta) = \int_0^{z+X} \eta^{-1} S_{\mathrm{odd}}(z,\eta)dz - \int_0^{\tilde{x}(z)} \eta^{-1}\tilde{S}_{\mathrm{odd}}(\tilde{x},\eta)d\tilde{x}$$

in X variable. We note the following

Theorem 3.7 ([Ko4])**.** *There exist some positive constants C_1, C_2 and δ such that*

$$\left|\tilde{R}_B(z,y)\right| \leq C_1|z|^{-(p-4)/2(p-2)}\exp[C_2|y|], \tag{3.62}$$
$$|R_B(z,y)| \leq C_1|z|^{-3/2}\exp[C_2|y|] \tag{3.63}$$

hold on $\left(\widehat{V}^{2\varepsilon/3} \setminus \widehat{V}^{\varepsilon/3}\right) \times E_\delta^+$, where $\tilde{R}_B$ and R_B are the Borel transform of $\tilde{R} = \eta^{-1}\tilde{S}_{\mathrm{odd}}(\tilde{x}(z),\eta) - S_{-1}(\tilde{x}(z))$ and $R = \eta^{-1}S_{\mathrm{odd}}(z,\eta) - S_{-1}(z)$ respectively and

$$E_\delta^+ = \bigcup_{s\geq 0}\{y \in \mathbb{C}; |y-s| \leq \delta\}. \tag{3.64}$$

We take $r>0$ so that $z+X \in \widehat{V}^{2\varepsilon/3} \setminus \widehat{V}^{\varepsilon/3}$ for (z,X) in

$$D_r^\varepsilon = \left\{(z,X) \in W^\varepsilon \times \mathbb{C}; |X| \leq r\sqrt{|z|}\right\}. \tag{3.65}$$

Since $\tilde{S}_{-1}(\tilde{x})$ and $S_{-1}(z) = z^{-1/2}$ satisfy (3.15), we find that

$$F_0(z,X) = 2\left((z+X)^{1/2} - z^{1/2}\right). \tag{3.66}$$

Therefore we can take some positive constant M so that

$$\left|\frac{F_0(z,X)}{X}\right| \geq \frac{M}{\sqrt{|z|}} \tag{3.67}$$

holds on D_r^ε. Then, by the same discussion with Section 2.3, we find that the Borel transform X_B of X is holomorphic on $W^\varepsilon \times E_\delta^+$ and satisfies the following estimates; there exist some positive constants C_1 and C_2 such that

$$|X_B(z,y)| \le C_1 \sqrt{|z|} \exp\left[C_2|y|\right] \tag{3.68}$$

holds on $W^\varepsilon \times E_\delta^+$. Hence, applying the same technique used in Section 2.3 to the integral representation

$$X_B(z,y) = \frac{(z+1)^2}{2\pi i} \oint_{|\tilde{z}-z|=\varepsilon_0} \frac{1}{(\tilde{z}+1)^2} \frac{X_B(\tilde{z},y)}{\tilde{z}-z} d\tilde{z}, \tag{3.69}$$

we finally obtain the following

Theorem 3.8. *There exist positive constants C_1, C_2, δ and ε such that $X_B(z,y)$ is holomorphic on $\widehat{V}^\varepsilon \times E_\delta$ and satisfies the following estimates there;*

$$|X_B(z,y)| \le C_1(|z|+1)^2 \exp\left[C_2|y|\right]. \tag{3.70}$$

Here

$$E_\delta = \bigcup_{s\in\mathbb{R}} \left\{y \in \mathbb{C}; |y-s| \le \delta\right\}. \tag{3.71}$$

Remark 3.2. As in Section 2.3, we can also show the Borel summability of the transformation series when $x_0^{-1}(\widehat{V}^\varepsilon)$ runs into a double pole b of Q. In this case, we find that

$$\left| \int_0^{\tilde{x}(z)} \tilde{R}_B(\tilde{x},y) d\tilde{x} \right| \le C_1 |z|^{1/2} \exp[C_2|y|] \tag{3.72}$$

holds for $(z,y) \in W^\varepsilon \times E_\delta^+$. (See [Ko4].) Then, applying Theorem A.1 to $F(z,X,\eta)$, we obtain the following estimates on $(z,y) \in W^\varepsilon \times E_\delta^+$;

$$|X_B(z,y)| \le \frac{C_1|z|}{M} \exp\left[\left(\frac{4C_1|z|^{1/2}}{Mr} + C_2\right)|y|\right]. \tag{3.73}$$

Therefore, by the same discussion as in the proof of Theorem 2.9 to the integral representation

$$X_B(z,y) = \frac{\exp[C_3(z+1)^{1/2}y]}{2\pi i} \times \oint_{|\tilde{z}-z|=\varepsilon_0} \frac{\exp[-C_3(\tilde{z}+1)^{1/2}y]X_B(\tilde{z},y)}{\tilde{z}-z} d\tilde{z}, \tag{3.74}$$

where a positive constant C_3 is taken as (2.122), we find the following

Theorem 3.9. *There exist positive constants C_1, C_2, C_3, δ and ε such that $X_B(z,y)$ is holomorphic on $\widehat{V}^{\varepsilon} \times E_\delta$ and satisfies the following estimates there;*

$$|X_B(z,y)| \le C_1 \exp\left[\left(C_2 + C_3|z|^{1/2}\right)|y|\right]. \tag{3.75}$$

A Implicit function theorem for Borel summable series

The purpose of this appendix is to show the implicit function theorem for Borel summable series. Concretely we prove the following

Theorem A.1. *Let $F(\alpha,\eta) = \sum_{n=0}^{\infty} \eta^{-n} F_n(\alpha)$ be formal series in η that satisfies*

(A.1) $F_k(\alpha)$ $(k=0,1,2,\cdots)$ *are holomorphic on* D_r,

(A.2) $F_0(\alpha_0) = 0,\ \dfrac{\partial F_0}{\partial \alpha}(\alpha_0) \neq 0,$

(A.3) $\tilde{F}(\alpha,\eta) := F(\alpha,\eta) - F_0(\alpha)$ *is Borel summable on* D_r

where $\alpha_0 \in \mathbb{C}$ and D_r is

$$D_r = \{\alpha; |\alpha - \alpha_0| \le r\}. \tag{A.4}$$

Then the formal solution $\alpha(\eta) = \alpha_0 + \eta^{-1}\alpha_1 + \eta^{-2}\alpha_2 + \cdots$ of

$$F(\alpha(\eta),\eta) = 0 \tag{A.5}$$

that starts from α_0 uniquely exists and $\tilde{\alpha}(\eta) := \alpha(\eta) - \alpha_0$ is Borel summable.

Proof. First, by expanding $F(\alpha(\eta), \eta)$ at $\alpha = \alpha_0$, we can rewrite (A.5) to the following equality;

(A.6)
$$\sum_{n=0}^{\infty} \eta^{-n} \Big[F_n(\alpha_0) + \sum_{\substack{\mu+k+l=n,\\ k\geq 0, \mu\geq l\geq 1}} \sum_{\substack{\mu_1+\cdots+\mu_l=\mu,\\ \mu_1,\cdots,\mu_l\geq 1}} \frac{\alpha_{\mu_1}\cdots\alpha_{\mu_l}}{l!} \frac{\partial^l F_k}{\partial \alpha^l}(\alpha_0) \Big] = 0.$$

Therefore, by comparing the coefficients of η^{-n} of (A.6), we find the following equalities hold;

$$\text{(A.7)} \qquad \alpha_n \frac{\partial F_0}{\partial \alpha}(\alpha_0) = R_n(\alpha_0, \cdots, \alpha_{n-1}) \ \ (n \geq 1).$$

Here R_n are the remainder terms of the coefficients of η^{-n} of (A.6) that are determined only by $\alpha_0, \cdots, \alpha_{n-1}$ and F. Since $\partial F_0/\partial\alpha$ does not vanish at $\alpha = \alpha_0$, we can inductively determine α_n and hence the uniqueness of $\alpha(\eta)$ immediately follows.

Now we show the Borel summability of the solution $\alpha(\eta)$ of (A.5). At first, we rewrite the assumptions on F more concrete form as follows; first, from (A.2), we can take $M > 0$ such that

$$\text{(A.8)} \qquad \inf_{\alpha \in D_r} \left| \frac{F_0(\alpha)}{\alpha - \alpha_0} \right| \geq M.$$

Next the Borel summability of $\tilde{F}$ guarantees that the Borel transform $\tilde{F}_B$ of $\tilde{F}$ satisfies

$$\text{(A.9)} \qquad \left| \tilde{F}_B(\alpha, y) \right| \leq C_1 \exp\left[C_2 |y|\right]$$

for $(\alpha, y) \in D_r \times E_\delta$ where C_1 and C_2 are positive constants and

$$\text{(A.10)} \qquad E_\delta = \bigcup_{s \geq 0} \{ y \in \mathbb{C}; |y - s| \leq \delta \}.$$

Now our task is to prove that $\tilde{\alpha}_B(y)$ is holomorphic on E_δ and satisfy the following estimates there;

$$|\tilde{\alpha}_B(y)| \leq \frac{C_1}{M} \exp\left[\left(\frac{4C_1}{Mr} + C_2\right)|y|\right]. \tag{A.11}$$

Since $F_0(\alpha_0) = 0$, F_0 can be written as $F_0(\alpha_0 + \tilde{\alpha}) = \tilde{\alpha}\tilde{F}_0(\tilde{\alpha})$ where $\tilde{F}_0(\tilde{\alpha})$ is a holomorphic function on $\tilde{D}_r = \{\tilde{\alpha} \in \mathbb{C}; |\tilde{\alpha}| \leq r\}$ and satisfies

$$\inf_{\alpha \in \tilde{D}_r} |\tilde{F}_0(\tilde{\alpha})| \geq M. \tag{A.12}$$

Therefore (A.5) can be rewritten as follows;

$$\tilde{\alpha}(\eta) = -\frac{1}{\tilde{F}_0(\tilde{\alpha}(\eta))}\tilde{F}(\alpha_0 + \tilde{\alpha}(\eta), \eta). \tag{A.13}$$

Let $G(\tilde{\alpha}, \eta)$ denote the right hand side of (A.13). We expand $G(\tilde{\alpha}, \eta)$ in $\tilde{\alpha}$;

$$G(\tilde{\alpha}, \eta) = \sum_{l=0}^{\infty} G^{(l)}(\eta)\tilde{\alpha}^l. \tag{A.14}$$

In order to obtain the estimation of $\tilde{\alpha}_B(y)$, we rewrite $\tilde{\alpha}(\eta)$ as follows;

$$\tilde{\alpha}(\eta) = \sum_{l=0}^{\infty} \tilde{\alpha}^{(l)}(\eta), \tag{A.15}$$

where $\tilde{\alpha}^{(l)}(\eta)$ $(l = 0, 1, 2, \cdots)$ are formal series that are inductively determined by

$$\tilde{\alpha}^{(0)}(\eta) = G^{(0)}(\eta), \tag{A.16.0}$$

(A.16.l)

$$\tilde{\alpha}^{(l)}(\eta) = \sum_{\substack{\mu_1+\cdots+\mu_j+j=l,\\ \mu_1,\cdots,\mu_j\geq 0, j\geq 1}} G^{(j)}(\eta)\cdot\tilde{\alpha}^{(\mu_1)}(\eta)\cdots\tilde{\alpha}^{(\mu_j)}(\eta) \quad (l \geq 1).$$

We immediately find that $\tilde{\alpha}^{(l)}(\eta)$ has the following shape;

$$\tilde{\alpha}^{(l)}(\eta) = \tilde{\alpha}^{(l)}_{l+1}\eta^{-l-1} + \tilde{\alpha}^{(l)}_{l+2}\eta^{-l-2} + \cdots . \tag{A.17}$$

Therefore $\tilde{\alpha}^{(0)}(\eta) + \tilde{\alpha}^{(1)}(\eta) + \cdots$ actually defines a formal series that satisfies (A.12) and this gives another representation of $\tilde{\alpha}(\eta)$. Applying the Borel transformation to (A.16), we find that the Borel transform $\tilde{\alpha}^{(l)}_B(y)$ of $\tilde{\alpha}^{(l)}(\eta)$ satisfies

(A.18.0)

$$\tilde{\alpha}^{(0)}_B(y) = G^{(0)}_B(y),$$

$$\text{(A.18.}l\text{)}\quad \tilde{\alpha}^{(l)}_B(y) = \sum_{\substack{\mu_1+\cdots+\mu_j+j=l,\\ \mu_1,\cdots,\mu_j\geq 0, j\geq 1}} G^{(j)}_B * \tilde{\alpha}^{(\mu_1)}_B * \cdots * \tilde{\alpha}^{(\mu_j)}_B(y) \quad (l \geq 1),$$

where $f * g(y)$ denotes the convolution of $f(y)$ and $g(y)$, i.e.,

$$f * g(y) = \int_0^y f(y-y')g(y')dy'. \tag{A.19}$$

Now we confirm that $\tilde{\alpha}^{(l)}_B(y) \ \ (l = 0, 1, 2, \cdots)$ are holomorphic on E_δ and derive the estimation that they satisfy by using (A.18). First, from Cauchy's integral formula, we find that $G^{(l)}_B(y)$ has the following representation;

$$G^{(l)}_B(y) = \frac{-1}{2\pi i}\oint_{|\tilde{\alpha}|=r} \frac{\tilde{F}_B(\alpha_0 + \tilde{\alpha}, y)}{\tilde{F}_0(\tilde{\alpha})}\frac{d\tilde{\alpha}}{\tilde{\alpha}^{l+1}}. \tag{A.20}$$

Since $\tilde{F}_0(\tilde{\alpha})$ satisfies (A.12), we immediately find from (A.9) and (A.20) that $G^{(l)}_B(y)$ is holomorphic on E_δ and satisfies the following estimates there;

$$\left|G^{(l)}_B(y)\right| \leq \frac{C_1}{Mr^l}\exp\left[C_2|y|\right]. \tag{A.21}$$

Then it is clear from (A.18.0) that $\tilde{\alpha}_B^{(0)}(y)$ is holomorphic on E_δ and we can inductively confirm from the recurrence relation (A.18.l) that $\tilde{\alpha}_B^{(l)}(y)$ $(l = 1, 2, \cdots)$ are also holomorphic on E_δ.

Next, we determine positive constants B_l $(l = 0, 1, 2, \cdots)$ so that they satisfy

$$\left|\tilde{\alpha}_B^{(l)}(y)\right| \leq B_l \frac{|y|^l}{l!} \exp\left[C_2|y|\right]. \tag{A.22.l}$$

on E_δ. Actually, since $\tilde{\alpha}_B^{(0)}(y) = G_B^{(0)}(y)$ satisfies

$$\left|\tilde{\alpha}_B^{(0)}(y)\right| \leq \frac{C_1}{M} \exp\left[C_2|y|\right], \tag{A.23}$$

we can take B_0 as

$$B_0 = \frac{C_1}{M}. \tag{A.24}$$

Further, when (A.22.m) holds for $0 \leq m \leq l-1$, applying these estimates to (A.18.l), we obtain the following estimates for $\tilde{\alpha}_B^{(l)}(y)$;

(A.25)

$$\begin{aligned}
\left|\tilde{\alpha}_B^{(l)}(y)\right| &\leq \sum_{\substack{\mu_1+\cdots+\mu_j+j=l,\\ \mu_1,\cdots,\mu_j\geq 0, j\geq 1}} \frac{C_1}{Mr^j} B_{\mu_1}\cdots B_{\mu_j} \frac{|y|^{\mu_1+\cdots+\mu_j+j}}{(\mu_1+\cdots+\mu_j+j)!} \exp\left[C_2|y|\right] \\
&= \sum_{\substack{\mu_1+\cdots+\mu_j+j=l,\\ \mu_1,\cdots,\mu_j\geq 0, j\geq 1}} \frac{C_1}{Mr^j} B_{\mu_1}\cdots B_{\mu_j} \frac{|y|^l}{l!} \exp\left[C_2|y|\right].
\end{aligned}$$

Here we repeatedly used the following estimation;

$$\begin{aligned}
&\int_0^{|y|} \frac{|y-y'|^{\mu_1}|y'|^{\mu_2}}{\mu_1!\mu_2!} \exp\left[C_2(|y-y'|+|y'|)\right]|dy'| \\
&\leq \frac{|y|^{\mu_1+\mu_2+1}}{(\mu_1+\mu_2+1)!} \exp\left[C_2|y|\right].
\end{aligned} \tag{A.26}$$

Hence we recursively determine B_l $(l = 1, 2, \cdots)$ by

$$B_l = \sum_{\substack{\mu_1+\cdots+\mu_j+j=l,\\ \mu_1,\cdots,\mu_j\geq 0, j\geq 1}} \frac{C_1}{Mr^j} B_{\mu_1} \cdots B_{\mu_j}. \tag{A.27.l}$$

Then we find that B_l satisfies (A.22.l).

Now we derive explicit form of B_l from (A.27.l). Let b_l $(l = 0, 1, 2, \cdots)$ be taken so that they satisfy

$$B_l = \left(\frac{C_1}{M}\right)^{l+1} \frac{1}{r^l} b_l. \tag{A.28}$$

Then the recurrence relation (A.27.l) can be rewritten to that for b_l $(l = 1, 2, \cdots)$ as follows;

$$b_l = \sum_{\substack{\mu_1+\cdots+\mu_j+j=l,\\ \mu_1,\cdots,\mu_j\geq 0, j\geq 1}} b_{\mu_1} \cdots b_{\mu_j}. \tag{A.29.l}$$

We define $b(t)$ by

$$b(t) = \sum_{l=0}^{\infty} b_l t^l. \tag{A.30}$$

Multiplying both hand side of (A.29.l) by t^l for $l \geq 1$ and summing up all of them, we obtain

$$\begin{aligned} \sum_{l=1}^{\infty} b_l t^l &= \sum_{l=1}^{\infty} \sum_{\substack{\mu_1+\cdots+\mu_j+j=l,\\ \mu_1,\cdots,\mu_j\geq 0, j\geq 1}} \left(b_{\mu_1} t^{\mu_1+1}\right) \cdots \left(b_{\mu_j} t^{\mu_j+1}\right) \\ &= \sum_{j=1}^{\infty} \sum_{\mu=0}^{\infty} \sum_{\substack{\mu_1+\cdots+\mu_j=\mu,\\ \mu_1,\cdots,\mu_j\geq 0}} \left(b_{\mu_1} t^{\mu_1+1}\right) \cdots \left(b_{\mu_j} t^{\mu_j+1}\right) \\ &= \sum_{j=1}^{\infty} (tb(t))^j. \end{aligned} \tag{A.31}$$

Since $b_0 = 1$, we find from (A.31) that $b(t)$ satisfies

$$b(t) - 1 = \frac{tb(t)}{1 - tb(t)}. \tag{A.32}$$

Therefore $b(t)$ is explicitly given by

$$b(t) = \frac{1 - \sqrt{1 - 4t}}{2t} = \frac{1}{\Gamma(1/2)} \sum_{l=0}^{\infty} \frac{\Gamma(l + 1/2)}{(n+1)!} 4^l t^l. \tag{A.33}$$

And, from the definition of b_l, we find that b_l ($l = 0, 1, 2, \cdots$) are given by

$$b_l = \frac{\Gamma(l + 1/2)}{\Gamma(1/2)(n+1)!} 4^l. \tag{A.34}$$

Obviously b_l satisfies $b_l \leq 4^l$. Hence, in conclusion, we obtain the following estimates for $\tilde{\alpha}_B^{(l)}(y)$ on E_δ;

$$\left| \tilde{\alpha}_B^{(l)}(y) \right| \leq \frac{C_1}{M} \frac{1}{l!} \left(\frac{4C_1|y|}{rM} \right)^l \exp\left[C_2 |y| \right]. \tag{A.35}$$

Then (A.11) immediately follows from (A.35). Actually

$$|\tilde{\alpha}_B(y)| \leq \sum_{l=0}^{\infty} \left| \tilde{\alpha}_B^{(l)}(y) \right| \leq \frac{C_1}{M} \exp\left[\left(\frac{4C_1|y|}{rM} + C_2 \right) |y| \right]. \tag{A.36}$$

This is the end of the proof. □

Remark A.1. In Section 2.3 and Section 3.3, we use the concrete form of estimation (A.11) of $\tilde{\alpha}_B$ to prove the Borel summability of transformation series.

B Representation of the action of transformation as an integro-differential operator

The main purpose of this appendix is to derive the properties of microdifferential operators $\mathcal{X}$ suggested in Theorem 2.5 and Theorem 3.5 from the Borel summability of the transformation series $x(\tilde{x}, \eta)$. Although the situations considered in Section 2 and Section 3 are different, in the both cases, the microdifferential operators $\mathcal{X}$ have the following form

(B.1) $$\mathcal{X} =: \left(\frac{\partial \tilde{x}}{\partial x_0}\right)^{1/2} \left(1 + \frac{\partial X}{\partial x_0}\right)^{-1/2} \exp(X\xi) : .$$

Here x_0 is the top order term of x that is taken as a new coordinate variable, X denotes $x - x_0$ and ξ stands for the symbol of ∂_{x_0}. Therefore it suffices to show the following proposition in order to attain our aim;

Proposition B.1. *Let $f(x, \eta)$ and $g(x, \eta)$ be formal series in η that have the following shape*

(B.2) $$f(x, \eta) = \sum_{n=1}^{\infty} f_n(x) \eta^{-n},$$

(B.3) $$g(x, \eta) = \sum_{n=0}^{\infty} g_n(x) \eta^{-n},$$

where $f_n(x)$ $(n = 1, 2, \cdots)$ and $g_n(x)$ $(n = 0, 1, \cdots)$ are holomorphic on a domain U of $\mathbb{C}_x$. Further we assume that f and $\tilde{g} = g - g_0$ are uniformly Borel summable on U, i.e., the Borel transform of them $f_B(x, y)$ and $\tilde{g}_B(x, y)$ are holomorphic on $U \times E_\varepsilon$ and satisfy the following estimates there;

(B.4) $$\max \{|f_B(x, y)|, |\tilde{g}_B(x, y)|\} \le C_1 \exp[C_2 |y|],$$

where E_ε is

(B.5) $$E_\varepsilon = \bigcup_{s \ge 0} \{y \in \mathbb{C}; |y - s| \le \varepsilon\}$$

and C_1, C_2 and ε are positive constants. We consider a microdifferential operator $\mathcal{P} = \mathcal{P}(x, \partial_x, \partial_y)$ on

(B.6) $$\Omega = \{(x, y; \xi, \eta) \in T^* (U \times \mathbb{C}_y) ; \eta \ne 0\}$$

defined by

$$\mathcal{P} =: g\exp[\xi f] : . \tag{B.7}$$

Then the action of $\mathcal{P}$ upon a (multi-valued) analytic function $\phi(x,y)$ has the following representation;

$$(\mathcal{P}\phi)(x,y) = g_0(x)\phi(x,y) + \int_{y_0}^{y} K(x, y-y', \partial_x)\phi(x,y')dy', \tag{B.8}$$

where $K(x,y,\partial_x)$ is a differential operator of infinite order on $U \times E_\varepsilon$.

Proof. Let $P_n(x,\eta)$ be the coefficients of ξ^n of $g\exp[\xi f]$, i.e.,

$$P_n(x,\eta) = g(x,\eta)\frac{(f(x,\eta))^n}{n!}. \tag{B.9}$$

Then the action of $\mathcal{P}$ can be written as follows;

$$(\mathcal{P}\phi)(x,y) = \int_{y_0}^{y} \sum_{n=0}^{\infty} P_{n,B}(x,y-y')\frac{\partial^n \phi}{\partial x^n}(x,y')dy'. \tag{B.10}$$

From (B.9), we immediately find that $P_{n,B}$ $(n = 0,1,2,\cdots)$ are given by

$$P_{0,B}(x,y) = g_0(x)\delta(y) + \tilde{g}_B(x,y), \tag{B.11}$$

$$P_{n,B}(x,y) = \frac{1}{n!}\left(g_0(x)f_B^{*n}(x,y) + \tilde{g}_B * f_B^{*n}(x,y)\right) \quad (n \geq 1) \tag{B.12}$$

where f_B^{*n} is

$$f_B^{*n} = \overbrace{f_B * \cdots * f_B}^{n}. \tag{B.13}$$

Therefore, when we write the action of $\mathcal{P}$ in the form (B.8), $K(x,y,\partial_x)$ is given by

$$K(x,y,\partial_x) = \tilde{g}_B(x,y) + \sum_{n=1}^{\infty} P_{n,B}(x,y)\partial_x^n. \tag{B.14}$$

Now we confirm that $K(x, y, \partial_x)$ given by (B.14) defines a differential operator of infinite order on $U \times E_\varepsilon$. First, by repeated use of (A.26), we obtain the following estimates from (B.4) on $U \times E_\varepsilon$ for $n \geq 1$;

(B.15)

$$|P_{n,B}(x, y)| \leq \frac{1}{n!}\left(|g_0(x)|C_1^n \frac{|y|^{n-1}}{(n-1)!} + C_1^{n+1}\frac{|y|^n}{n!}\right)\exp[C_2|y|].$$

Hence the symbol $\sigma(K)(x, y, \xi)$ of K satisfies the following estimates for $(x, y, \xi) \in U \times E_\varepsilon \times \mathbb{C}_\xi$;

(B.16)

$$\begin{aligned}
&|\sigma(K)(x, y, \xi)| \\
&\leq |\tilde{g}_B(x, y)| + \sum_{n=1}^{\infty} |\xi|^n |P_{n,B}(x, y)| \\
&\leq \sum_{n=0}^{\infty}\left(|g_0(x)|C_1^{n+1}\frac{|\xi|^{n+1}|y|^n}{(n+1)!n!} + C_1^{n+1}\frac{|\xi|^n|y|^n}{(n!)^2}\right)\exp[C_2|y|] \\
&\leq C_1\left(|g_0(x)||\xi| + 1\right)\exp\left[2\sqrt{C_1|y||\xi|} + C_2|y|\right].
\end{aligned}$$

This estimates (B.16) guarantees that K defines a differential operator of infinite order on $U \times E_\varepsilon$. □

In conclusion, if

(B.17) $\quad \partial\tilde{x}/\partial x_0 \neq 0$ and X is Borel summable

in a neiborhood $\mathring{x}_0 \in \mathbb{C}$, then we find that $(\partial\tilde{x}/\partial x_0)^{1/2}(1 + \partial X/\partial x_0)^{-1/2}$ and X respectively satisfy the assumption on f and g in Theorem B.1. Therefore we can apply Theorem B.1 to a microdifferential operator $\mathcal{X}$ defined by (B.1) and we attain our purpose.

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