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On the colored Jones polynomials of ribbon links, boundary links and Brunnian links

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Abstract

Habiro gave principal ideals of $\mathbb{Z}[q, q^{-1}]$ in which certain linear combinations of the colored Jones polynomials of algebraically-split links take values. The author proved that the same linear combinations for ribbon links, boundary links and Brunnian links are contained in smaller ideals of $\mathbb{Z}[q, q^{-1}]$ generated by several elements. In this paper, we prove that these ideals also are principal, each generated by a product of cyclotomic polynomials.

1 Introduction

After the discovery of the Jones polynomial, Reshetikhin and Turaev [7] defined an invariant of framed links whose components are colored by finite dimensional representations of a ribbon Hopf algebra. The *colored Jones polynomial* can be defined as the Reshetikhin-Turaev invariant of links whose components are colored by finite dimensional representations of the quantized enveloping algebra $U_h(sl_2)$.

We are interested in the relationship between *algebraic properties* of the colored Jones polynomial and *topological properties* of links.

In this paper, we consider the following three types of links.

A link is called a *ribbon link* if it bounds the image of an immersion from a disjoint union of disks into S^3 with only ribbon singularities.

An n-component link $L = L_1 \cup \cdots \cup L_n$ is called a *boundary link* if it bounds a disjoint union of n Seifert surfaces $F_1, \ldots, F_n$ in S^3 such that L_i bounds F_i for $i = 1, \ldots, n$.

A link L is called a *Brunnian link* if every proper sublink of L is trivial.

In [4], Habiro used certain linear combinations $J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}}$, $l_1, \ldots, l_n \geq 0$, of the colored Jones polynomials of a link L to construct the unified Witten-Reshetikhin-Turaev invariants for integral homology spheres. He proved that $J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}}$ for an algebraically-split, 0-framed link L is contained in a certain principal ideal of $\mathbb{Z}[q, q^{-1}]$ (Theorem 2.1). This result was improved by the present author [8, 9, 10, 11] in the

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special case of ribbon links, boundary links (Theorem 2.2) and Brunnian links (Theorem 2.4) by using ideals $I_{l_1}, \dots, I_{l_n}$ of $\mathbb{Z}[q, q^{-1}]$, where Theorem 2.2 for boundary links had been conjectured by Habiro [4]. Here, in [8], we gave an alternative proof of the fact that the Jones polynomial of an n-component ribbon link is divisible by the Jones polynomial of the n-component trivial link, which was proved first by Eisermann [1]. The results in [4, 8, 9, 10, 11] are proved by using the *universal sl_2 invariant of bottom tangles* (cf. [3, 4]), which has the universality property for the colored Jones polynomial of links.

In this paper, we prove that the ideal I_l, $l \geq 0$, is a principal ideal generated by a product of cyclotomic polynomials (Theorem 3.1), and rewrite Theorems 2.1, 2.2 and 2.4 by using these generators (Proposition 3.3).

2 Results for the colored Jones polynomial

In this section, we recall results in [4, 9, 10, 11] for the colored Jones polynomial. For the definition of the quantized enveloping algebra $U_h(sl_2)$, see, e.g., [6, 4, 9]. We set $q = \exp h$.

For $m \geq 1$, let V_m denote the m-dimensional irreducible representation of $U_h(sl_2)$. Let $\mathcal{R}$ denote the representation ring of $U_h(sl_2)$ over $\mathbb{Q}(q^{\frac{1}{2}})$, i.e., $\mathcal{R}$ is the $\mathbb{Q}(q^{\frac{1}{2}})$-algebra

$$\mathcal{R} = \mathrm{Span}_{\mathbb{Q}(q^{\frac{1}{2}})}\{V_m \mid m \geq 1\}$$

with the multiplication induced by the tensor product. It is well known that $\mathcal{R} = \mathbb{Q}(q^{\frac{1}{2}})[V_2]$.

Habiro [4] studied the following elements in $\mathcal{R}$

$$\tilde{P}'_l = \frac{q^{\frac{1}{2}l}}{\{l\}_q!} \prod_{i=0}^{l-1} (V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}),$$

for $l \geq 0$, which are used in an important technical step in his construction of the unified Witten-Reshetikhin-Turaev invariants for integral homology spheres.

For the definition of the colored Jones polynomial $J_{L;X_1,\dots,X_n}$ of L with ith component L_i colored by $X_i \in \mathcal{R}$, see, e.g., [5, 4, 9].

Set

$$\{i\}_q = q^i - 1, \quad \{i\}_{q,n} = \{i\}_q\{i-1\}_q \cdots \{i-n+1\}_q, \quad \{n\}_q! = \{n\}_{q,n},$$

for $i \in \mathbb{Z}, n \geq 0$.

Habiro [4] proved the following.

Theorem 2.1 (Habiro [4]). *Let L be an n-component, algebraically-split link with 0-framing. We have*

$$J_{L;\tilde{P}'_{l_1},\dots,\tilde{P}'_{l_n}} \in \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q} \mathbb{Z}[q, q^{-1}], \qquad (1)$$

for $l_1, \dots, l_n \geq 0$, where $l_{\max} = \max(l_1 \dots, l_n)$.

Set

$$f_{l,k} = \{l-k\}_q!\{k\}_q!,$$

for $0 \leq k \leq l$. For $l \geq 0$, let I_l be the ideal of $\mathbb{Z}[q, q^{-1}]$ generated by $f_{l,0}, \ldots, f_{l,l}$.

In [9, 10], we proved the following.

Theorem 2.2 ([9, 10])**.** *Let L be an n-component ribbon or boundary link with 0-framing. For $l_1, \ldots, l_n \geq 0$, we have*

$$J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}} \in \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q} \prod_{1\leq i\leq n, i\neq i_M} I_{l_i}, \tag{2}$$

where $l_{\max} = \max(l_1, \ldots, l_n)$ and i_M is an integer such that $l_{i_M} = l_{\max}$.

Remark 2.3. Theorem 2.2 for boundary links had been conjectured by Habiro [4].

In [11], we prove the following.

Theorem 2.4 ([11])**.** *Let L be an n-component Brunnian link with $n \geq 3$. We have*

$$J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}} \in \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q\{l_{\min}\}_q!} \prod_{1\leq i\leq n, i\neq i_M, i_m} I_{l_i}, \tag{3}$$

for $l_1, \ldots, l_n \geq 0$, where $l_{\max} = \max(l_1, \ldots, l_n)$, $l_{\min} = \min(l_1, \ldots, l_n)$ and i_M, i_m, $i_M \neq i_m$, are integers such that $l_{i_M} = l_{\max}$, $l_{i_m} = l_{\min}$, respectively.

Let us compare Theorems 2.1, 2.2 and 2.4. For $l_1, \ldots, l_n \geq 0$, let $Z_a^{(l_1,\ldots,l_n)}$, $Z_{r,b}^{(l_1,\ldots,l_n)}$ and $Z_{Br}^{(l_1,\ldots,l_n)}$ denote the ideals of $\mathbb{Z}[q, q^{-1}]$ at the right hand sides of (1), (2) and (3), respectively, i.e., we set

$$\begin{aligned}
Z_a^{(l_1,\ldots,l_n)} &= \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q}\mathbb{Z}[q, q^{-1}], \\
Z_{r,b}^{(l_1,\ldots,l_n)} &= \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q} \prod_{1\leq i\leq n, i\neq i_M} I_{l_i}, \\
Z_{Br}^{(l_1,\ldots,l_n)} &= \frac{\{2l_{\max}+1\}_{q,l_{\max}+1}}{\{1\}_q\{l_{\min}\}_q!} \prod_{1\leq i\leq n, i\neq i_M, i_m} I_{l_i}.
\end{aligned}$$

For $l_1, \ldots, l_n \geq 0$, we have

$$Z_{r,b}^{(l_1,\ldots,l_n)} \subset Z_a^{(l_1,\ldots,l_n)}, \quad Z_{r,b}^{(l_1,\ldots,l_n)} \subset Z_{Br}^{(l_1,\ldots,l_n)},$$

since we have

$$\begin{aligned}
Z_{r,b}^{(l_1,\ldots,l_n)} &= \Big(\prod_{1\leq i\leq n, i\neq i_M} I_{l_i}\Big) \cdot Z_a^{(l_1,\ldots,l_n)}, \\
&= \big(\{l_{\min}\}_q! I_{l_{\min}}\big) \cdot Z_{Br}^{(l_1,\ldots,l_n)}.
\end{aligned}$$

On the other hand, there are no inclusion which satisfies for all $l_1, \ldots, l_n \geq 0$ between $Z_a^{(l_1,\ldots,l_n)}$ and $Z_{Br}^{(l_1,\ldots,l_n)}$. For example, we have $Z_a^{(2,2,2,2)} \not\subset Z_{Br}^{(2,2,2,2)}$ and $Z_{Br}^{(2,2,2,2)} \not\subset Z_a^{(2,2,2,2)}$ since

$$\begin{aligned} Z_a^{(2,2,2,2)} &= \frac{\{5\}_{q,3}}{\{1\}_q}\mathbb{Z}[q,q^{-1}] \\ &= (q-1)^2(q+1)(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q^1+1)\mathbb{Z}[q,q^{-1}], \\ Z_{Br}^{(2,2,2,2)} &= \frac{\{5\}_{q,3}}{\{1\}_q\{2\}_q!}\{1\}_q^4\mathbb{Z}[q,q^{-1}] \\ &= (q-1)^4(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q^1+1)\mathbb{Z}[q,q^{-1}]. \end{aligned}$$

Since a Brunnian link with $n \geq 3$ components is algebraically-split with 0-framing, we have the following refinement of Theorem 2.4.

Theorem 2.5. *Let L be an n-component Brunnian link with $n \geq 3$. We have*

$$J_{L;\tilde{P}'_{l_1},\ldots,\tilde{P}'_{l_n}} \in Z_a^{(l_1,\ldots,l_n)} \cap Z_{Br}^{(l_1,\ldots,l_n)},$$

for $l_1, \ldots, l_n \geq 0$.

3 Main result for the ideal I_l

In this section, we state the main result of this paper.

For $l \geq 0$, recall the generators $f_{l,0}, \ldots, f_{l,l}$ of the ideal I_l. Set

$$g_l = GCD(f_{l,0}, \ldots, f_{l,l}).$$

It is clear that $I_l \subset g_l\mathbb{Z}[q,q^{-1}]$. The opposite inclusion follows if and only if I_l is principal. Since $\mathbb{Z}[q,q^{-1}]$ is *not* a principal ideal domain, it had been a problem if I_l is principal or not. The main result in this paper (Theorem 3.1) is that I_l is principal, where we determine g_l explicitly. The proof is in Section 4.

For $m \geq 1$, let $\Phi_m = \prod_{d|m}(q^d-1)^{\mu(\frac{m}{d})} \in \mathbb{Z}[q]$ denote the mth cyclotomic polynomial, where $\prod_{d|m}$ denotes the product over all positive divisors d of m, and μ is the Möbius function. For $r \in \mathbb{Q}$, we denote by $\lfloor r \rfloor$ the largest integer smaller than or equal to r.

Theorem 3.1. *For $l \geq 0$, the ideal I_l is the principal ideal generated by g_l. Moreover, we have*

$$g_l = \prod_{m \geq 1} \Phi_m^{t_{l,m}}, \tag{4}$$

where

$$t_{l,m} = \begin{cases} \lfloor \frac{l+1}{m} \rfloor - 1 & \text{for } 1 \leq m \leq l, \\ 0 & \text{for } l < m. \end{cases}$$

Here is a table of $t_{l,m}$ for $1 \le m \le 4$, $0 \le l \le 16$.

$m \setminus l$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	0	0	0	1	1	2	2	3	3	4	4	5	5	6	6	7	7
3	0	0	0	0	0	1	1	1	2	2	2	3	3	3	4	4	4
4	0	0	0	0	0	0	0	1	1	1	1	2	2	2	2	3	3

Remark 3.2. In [11], Theorem 3.1 is used in the proof of Theorem 2.4.

Theorem 3.1 implies that the ideals $Z_{r,b}^{(l_1,\dots,l_n)}$ and $Z_{Br}^{(l_1,\dots,l_n)}$ are principal. Moreover, we can write a generator of each principal ideal $Z_a^{(l_1,\dots,l_n)}$, $Z_{r,b}^{(l_1,\dots,l_n)}$ and $Z_{Br}^{(l_1,\dots,l_n)}$ as a product of cyclotomic polynomials as follows.

Proposition 3.3. *For $l_1,\dots,l_n \ge 0$, the ideals $Z_a^{(l_1,\dots,l_n)}$, $Z_{r,b}^{(l_1,\dots,l_n)}$ and $Z_{Br}^{(l_1,\dots,l_n)}$ are principal. Moreover, we have*

$$\begin{aligned}
Z_a^{(l_1,\dots,l_n)} &= \prod_{m\ge 1} \Phi_m^{\lfloor \frac{2l_{\max}+1}{m}\rfloor - \lfloor \frac{l_{\max}-1}{m}\rfloor - \lfloor \frac{1}{m}\rfloor} \mathbb{Z}[q,q^{-1}],\\
Z_{r,b}^{(l_1,\dots,l_n)} &= \prod_{1\le m\le 2l_{\max}+1} \Phi_m^{\lfloor \frac{2l_{\max}+1}{m}\rfloor - \lfloor \frac{l_{\max}-1}{m}\rfloor - \lfloor \frac{1}{m}\rfloor + \sum_{1\le i\le n, i\ne i_M} t_{l_i,m}} \mathbb{Z}[q,q^{-1}],\\
Z_{Br}^{(l_1,\dots,l_n)} &= \prod_{1\le m\le 2l_{\max}+1} \Phi_m^{\lfloor \frac{2l_{\max}+1}{m}\rfloor - \lfloor \frac{l_{\max}-1}{m}\rfloor - \lfloor \frac{1}{m}\rfloor - \lfloor \frac{l_{\min}}{m}\rfloor + \sum_{1\le i\le n, i\ne i_M, i_m} t_{l_i,m}} \mathbb{Z}[q,q^{-1}].
\end{aligned}$$

Proof. The assertion for $Z_a^{(l_1,\dots,l_n)}$ follows from

$$\{l\}_{q,i} = \prod_{m\ge 1} \Phi_m^{\lfloor \frac{l}{m}\rfloor - \lfloor \frac{l-i}{m}\rfloor}, \tag{5}$$

for $0 \le i \le l$. The assertion for $Z_{r,b}^{(l_1,\dots,l_n)}$ and $Z_{Br}^{(l_1,\dots,l_n)}$ follows from (5) and Theorem 3.1. □

Corollary 3.4. *For $l_1,\dots,l_n \ge 0$, we have*

$$Z_a^{(l_1,\dots,l_n)} \cap Z_{Br}^{(l_1,\dots,l_n)} = \prod_{m\ge 1} \Phi_m^{\lfloor \frac{2l_{\max}+1}{m}\rfloor - \lfloor \frac{l_{\max}-1}{m}\rfloor - \lfloor \frac{1}{m}\rfloor + \max(0, \sum_{1\le i\le n, i\ne i_M, i_m} t_{l_i,m} - \lfloor \frac{l_{\min}}{m}\rfloor)} \mathbb{Z}[q,q^{-1}].$$

Example 3.5. *Let L be an n-component algebraically-split link with 0-framing. By Theorem 2.1 and Proposition 3.3, we have*

$$\begin{aligned}
J_{L;\tilde{P}'_1,\dots,\tilde{P}'_1} &\in \Phi_1\Phi_2\Phi_3\mathbb{Z}[q,q^{-1}],\\
J_{L;\tilde{P}'_2,\dots,\tilde{P}'_2} &\in \Phi_1^2\Phi_2\Phi_3\Phi_4\Phi_5\mathbb{Z}[q,q^{-1}],\\
J_{L;\tilde{P}'_3,\dots,\tilde{P}'_3} &\in \Phi_1^3\Phi_2^2\Phi_3\Phi_4\Phi_5\Phi_6\Phi_7\mathbb{Z}[q,q^{-1}].
\end{aligned}$$

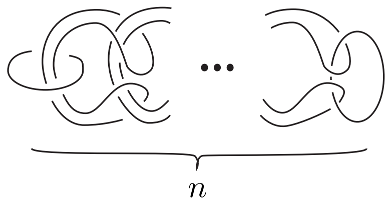

Figure 1: Milnor's link M_n

Let L be an n-component Brunnian link with $n \geq 3$. By Theorem 2.5 and Corollary 3.4, we have

$$\begin{aligned} J_{L;\tilde{P}'_1,\dots,\tilde{P}'_1} &\in \Phi_1^{n-2}\Phi_2\Phi_3\mathbb{Z}[q,q^{-1}],\\ J_{L;\tilde{P}'_2,\dots,\tilde{P}'_2} &\in \Phi_1^{2(n-2)}\Phi_2\Phi_3\Phi_4\Phi_5\mathbb{Z}[q,q^{-1}],\\ J_{L;\tilde{P}'_3,\dots,\tilde{P}'_3} &\in \Phi_1^{3(n-2)}\Phi_2^{n-1}\Phi_3\Phi_4\Phi_5\Phi_6\Phi_7\mathbb{Z}[q,q^{-1}]. \end{aligned}$$

Let L be an n-component ribbon or boundary link with 0-framing. By Theorem 2.2 and Proposition 3.3, we have

$$\begin{aligned} J_{L;\tilde{P}'_1,\dots,\tilde{P}'_1} &\in \Phi_1^{n}\Phi_2\Phi_3\mathbb{Z}[q,q^{-1}],\\ J_{L;\tilde{P}'_2,\dots,\tilde{P}'_2} &\in \Phi_1^{2n}\Phi_2\Phi_3\Phi_4\Phi_5\mathbb{Z}[q,q^{-1}],\\ J_{L;\tilde{P}'_3,\dots,\tilde{P}'_3} &\in \Phi_1^{3n}\Phi_2^{n+1}\Phi_3\Phi_4\Phi_5\Phi_6\Phi_7\mathbb{Z}[q,q^{-1}]. \end{aligned}$$

Example 3.6. *For $n \geq 3$, let M_n be Milnor's n-component Brunnian link depicted in Figure 1. Note that M_3 is the Borromean rings. We have*

$$J_{M_n;\tilde{P}'_1,\dots,\tilde{P}'_1} = (-1)^n q^{-2n+4}\Phi_1^{n-2}\Phi_2^{n-2}\Phi_3\Phi_4^{n-3},$$

which we will prove in a forthcoming paper [12]. This implies that Theorem 2.4 is best possible for the divisibility by Φ_1 and Φ_3 of $J_{L;\tilde{P}'_1,\dots,\tilde{P}'_1}$ with L Brunnian. By Theorem 2.2, this also implies that each M_n is not ribbon or boundary.

4 Proof of Theorem 3.1

We prove Theorem 3.1.

For $a_1,\dots,a_m \in \mathbb{Z}[q,q^{-1}]$, let $(a_1,\dots,a_m)$ denote the ideal in $\mathbb{Z}[q,q^{-1}]$ generated by $a_1,\dots,a_m \in \mathbb{Z}[q,q^{-1}]$.

For $l \geq 0$, recall that $I_l = (f_{l,0}, f_{l,1},\dots,f_{l,l})$ with $f_{l,i} = \{l-i\}_q!\{i\}_q!$ for $0 \leq i \leq l$.

For $0 \leq k \leq l$, we have

$$(f_{l,0}, f_{l,1},\dots,f_{l,k}) = \{l-k\}_q!(h_{l,k,0}, h_{l,k,1},\dots,h_{l,k,k})$$

with

$$\begin{aligned} h_{l,k,i} &= f_{l,i}/\{l-k\}_q!\\ &= \{l-i\}_{q,k-i}\{i\}_q!, \end{aligned}$$

for $1 \le i \le k$.

Set

$$I_{l,k} = (h_{l,k,0}, h_{l,k,1}, \ldots, h_{l,k,k}),$$
$$g_{l,k} = GCD(h_{l,k,0}, h_{l,k,1}, \ldots, h_{l,k,k}).$$

Note that $I_{l,l} = I_l$ and $g_{l,l} = g_l$.

In what follows, for $a \in \mathbb{Z}[q, q^{-1}] \setminus \{0\}$ and $m \ge 1$, let $d_m(a)$ denote the largest integer i such that $a \in \Phi_m^i \mathbb{Z}[q, q^{-1}]$. For $0 \le k \le l$, we can write

$$g_{l,k} = \prod_{m \ge 1} \Phi_m^{d_m(g_{l,k})},$$

since each $h_{l,k,i}$ is a product of cyclotomic polynomials.

Lemma 4.1. *For $0 \le k \le l$, we have*

$$d_m(g_{l,k}) = \begin{cases} \lfloor \frac{l+1}{m} \rfloor - 1 - \lfloor \frac{l-k}{m} \rfloor & \text{for } 1 \le m \le k, \\ 0 & \text{for } k < m. \end{cases}$$

Proof. We have

$$\begin{aligned} d_m(g_{l,k}) &= \min\{d_m\big(h_{l,k,i}\big) | 0 \le i \le k\} \\ &= \min\{d_m\big(\{l-i\}_{q,k-i}\{i\}_q!\big) | 0 \le i \le k\} \\ &= \min\{\lfloor \frac{l-i}{m} \rfloor - \lfloor \frac{l-k}{m} \rfloor + \lfloor \frac{i}{m} \rfloor | 0 \le i \le k\} \\ &= \min\{\lfloor \frac{l-i}{m} \rfloor + \lfloor \frac{i}{m} \rfloor | 0 \le i \le k\} - \lfloor \frac{l-k}{m} \rfloor. \end{aligned}$$

If $k < m$, then we have $d_m(g_{l,k}) = 0$ since $d_m\big(h_{l,k,k}\big) = d_m\big(\{k\}_q!\big) = 0$.

Let $1 \le m \le k$. Since we have

$$\lfloor \frac{l-(i+am)}{m} \rfloor + \lfloor \frac{i+am}{m} \rfloor = \lfloor \frac{l-i}{m} \rfloor + \lfloor \frac{i}{m} \rfloor,$$

for $0 \le i \le k$ and $a \in \mathbb{Z}$, we have

$$\min\{\lfloor \frac{l-i}{m} \rfloor + \lfloor \frac{i}{m} \rfloor | 0 \le i \le k\} = \min\{\lfloor \frac{l-i}{m} \rfloor + \lfloor \frac{i}{m} \rfloor | 0 \le i \le m-1\}.$$

Here, for $0 \le i \le m-1$, we have $\lfloor \frac{i}{m} \rfloor = 0$ and $\lfloor \frac{l-i}{m} \rfloor$ takes the minimum with $i = m-1$. Thus we have

$$\begin{aligned} \min\{\lfloor \frac{l-i}{m} \rfloor + \lfloor \frac{i}{m} \rfloor | 0 \le i \le m-1\} &= \lfloor \frac{l-(m-1)}{m} \rfloor \\ &= \lfloor \frac{l+1}{m} \rfloor - 1. \end{aligned}$$

This implies

$$d_m(g_{l,k}) = \lfloor \frac{l+1}{m} \rfloor - 1 - \lfloor \frac{l-k}{m} \rfloor.$$

Hence we have the assertion. □

Note that we have the latter part (4) of Theorem 3.1 as follows.

Corollary 4.2. *For $l \geq 0$, we have*

$$g_l = g_{l,l} = \prod_{m \geq 1} \Phi_m^{t_{l,m}}.$$

From now, we prove the following generalization of Theorem 3.1.

Proposition 4.3. *For $0 \leq k \leq l$, the ideal $I_{l,k}$ is the principal ideal generated by $g_{l,k}$.*

For $1 \leq k \leq l$, set

$$\tilde{g}_{l,k} = g_{l,k}/g_{l,k-1}.$$

We have

$$\begin{aligned}
\tilde{g}_{l,k} &= \prod_{1 \leq m \leq k} \Phi_m^{\lfloor \frac{l+1}{m} \rfloor - 1 - \lfloor \frac{l-k}{m} \rfloor - \left(\lfloor \frac{l+1}{m} \rfloor - 1 - \lfloor \frac{l-k+1}{m} \rfloor \right)} \\
&= \prod_{1 \leq m \leq k} \Phi_m^{\lfloor \frac{l-k+1}{m} \rfloor - \lfloor \frac{l-k}{m} \rfloor} \\
&= \prod_{\substack{m \mid l-k+1 \\ 1 \leq m \leq k}} \Phi_m.
\end{aligned}$$

We use the following technical lemma.

Lemma 4.4. *For $1 \leq k \leq l$, we have*

$$(\{l-k+1\}_q, \{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) = (\tilde{g}_{l,k}).$$

(Note that $g_{l,k-1} = GCD(\{l\}_{q,k-1}, \{l-1\}_{q,k-2}\{1\}_q, \ldots, \{k-1\}_q!)$ divides $\{k-1\}_q!$.)

Proof of Proposition 4.3 by assuming Lemma 4.4 . We use induction on k. For $k = 0$, clearly we have

$$I_{l,0} = (g_{l,0}) = (\{l\}_q!).$$

For $k \geq 1$, we have

$$\begin{aligned}
I_{l,k} &= (h_{l,k,0}, h_{l,k,1}, \ldots, h_{l,k,k}) \\
&= (\{l\}_{q,k}, \{l-1\}_{q,k-1}\{1\}_q, \ldots, \{l-k+1\}_q\{k-1\}_q!, \{k\}_q!) \\
&= \big(\{l-k+1\}_q(\{l\}_{q,k-1}, \{l-1\}_{q,k-2}\{1\}_q, \ldots, \{k-1\}_q!), \{k\}_q!\big) \\
&= (\{l-k+1\}_q g_{l,k-1}, \{k\}_q!) \\
&= g_{l,k-1}(\{l-k+1\}_q, \{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) \\
&= (g_{l,k-1}\tilde{g}_{l,k}) \\
&= (g_{l,k}),
\end{aligned}$$

where the second equality is given by

$$h_{l,k,0} = \{l-k+1\}_q \cdot \{l-i\}_{q,k-i-1}\{i\}_q,$$

for $0 \leq i \leq k-1$, and the third equality is given by the assumption of the induction.

Hence we have the assertion. □

In what follows, we prove Lemma 4.4. We use the following two lemmas, which are well-known.

Lemma 4.5 (cf. Habiro [2, Lemma 4.1])**.** *For $a, b \geq 0$, the following conditions are equivalent.*

(i) $(\Phi_a, \Phi_b) = \mathbb{Z}[q, q^{-1}]$

(ii) $\frac{a}{b} \neq p^i$ *for any prime number* $p \geq 0$ *and* $i \in \mathbb{Z}$

Lemma 4.6. *Let $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{Z}[q, q^{-1}]$ such that $(a_i, b_j) = \mathbb{Z}[q, q^{-1}]$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. We have*

$$(a_1 a_2 \cdots a_m, b_1 b_2 \cdots b_n) = \mathbb{Z}[q, q^{-1}].$$

Proof of Lemma 4.4. It is enough to prove the following two equalities.

$$GCD(\{l-k+1\}_q, \{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) = \tilde{g}_{l,k}, \tag{6}$$

$$(\{l-k+1\}_q/\tilde{g}_{l,k}, \{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}/\tilde{g}_{l,k}) = \mathbb{Z}[q, q^{-1}], \tag{7}$$

for $1 \leq k \leq l$.

First, we prove (6). Recall that

$$\tilde{g}_{l,k} = \prod_{\substack{m|l-k+1 \\ 1 \leq m \leq k}} \Phi_m. \tag{8}$$

Since $\{l-k+1\}_q = \prod_{m|l-k+1} \Phi_m$ and $d_m(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) = 0$ for $m > k$, it is enough to check

$$d_m(\{k\}_q \frac{\{k-1\}_q!}{g_{l,k-1}}) \geq 1,$$

for $m|l-k+1$, $1 \leq m \leq k$. Indeed, we have

$$d_m(\{k\}_q) = \begin{cases} 1 & \text{for } m|k, \\ 0 & \text{for } m \not| k, \end{cases} \tag{9}$$

$$d_m(\frac{\{k-1\}_q!}{g_{l,k-1}}) = \begin{cases} 0 & \text{for } m|k, \\ 1 & \text{for } m \not| k. \end{cases} \tag{10}$$

Here, (9) is clear and (10) follows from

$$\begin{aligned}
d_m(\frac{\{k-1\}_q!}{g_{l,k-1}}) &= \lfloor\frac{k-1}{m}\rfloor - t_{l,k-1,m} \\
&= \lfloor\frac{k-1}{m}\rfloor - \lfloor\frac{l+1}{m}\rfloor + 1 + \lfloor\frac{l-k+1}{m}\rfloor \\
&= \lfloor\frac{pm+r-1}{m}\rfloor - \lfloor\frac{(p+p')m+r}{m}\rfloor + 1 + \lfloor\frac{p'm}{m}\rfloor \\
&= \lfloor\frac{r-1}{m}\rfloor + 1 \\
&= \begin{cases} 0 & \text{for } r=0, \\ 1 & \text{for } 1 \le r \le m-1, \end{cases}
\end{aligned}$$

where, we write $k = mp + r$ and $l - k + 1 = mp'$ with $p, p' \ge 1$ and $0 \le r \le m-1$.

We prove (7). By Lemmas 4.5 and 4.6, it is enough to prove that there are no pair of integers $m, n \ge 1$ such that

- $\frac{n}{m} = p^i$ for a prime p and $i \in \mathbb{Z}$,
- $d_m(\{l-k+1\}_q/\tilde{g}_{l,k}) \ge 1$, and
- $d_n(\{k\}_q\frac{\{k-1\}_q!}{g_{l,k-1}}/\tilde{g}_{l,k}) \ge 1$.

Note that

$$\{l-k+1\}_q/\tilde{g}_{l,k} = \prod_{\substack{m|l-k+1 \\ m>k}} \Phi_m.$$

Let $m|l-k+1$, $m > k$. Recall that for $n > k$, we have $d_n(\{k\}_q\frac{\{k-1\}_q!}{g_{l,k-1}}) = 0$. Assume that $1 \le n \le k$ and $n|m$, which implies $n|l-k+1$. The conditions $1 \le n \le k$ and $n|l-k+1$ imply $d_n(\tilde{g}_{l,k-1}) = 1$ by (8). By (9) and (10), we have $d_n(\{k\}_q\frac{\{k-1\}_q!}{g_{l,k-1}}) = 1$. Thus we have $d_n(\{k\}_q\frac{\{k-1\}_q!}{g_{l,k-1}}/\tilde{g_{l,k}}) = 0$, which completes the proof. □

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