RIMS-1735

Exact WKB analysis of a Schrödinger equation with a merging triplet of two simple poles and one simple turning point — its relevance to the Mathieu equation and the Legendre equation

Dedicated to Professor K. Kataoka on the occasion of his sixtieth birthday

By

Shingo KAMIMOTO, Takahiro KAWAI and Yoshitsugu TAKEI

December 2011

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Exact WKB analysis of a Schrödinger equation with a merging triplet of two simple poles and one simple turning point —— its relevance to the Mathieu equation and the Legendre equation

Dedicated to Professor K. Kataoka on the occasion of his sixtieth birthday

Shingo Kamimoto
Graduate School of Mathematical Sciences
University of Tokyo
Tokyo, 153-8914 JAPAN

Takahiro Kawai
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-8502 JAPAN

and

Yoshitsugu Takei
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-8502 JAPAN

The research of the authors has been supported in part by JSPS grants-in-aid No.22-6971, No.20340028 and No.21340029.

Abstract

We develop the exact WKB analysis of an M2P1T (merging two simple poles and one simple turning point) Schrödinger equation. Our emphasis is put on the analysis of the singularity structure of its Borel transformed WKB solutions near fixed singular points relevant to the two simple poles contained in the potential of the equation. We first show that the WKB-theoretic canonical form of an M2P1T equation is given by an algebraic Mathieu equation, and then we calculate the alien derivative of its Borel transformed WKB solutions at each fixed singular point relevant to the simple poles through the analysis of Borel transformed WKB solutions of the Legendre equations. In the course of the calculation of the alien derivative we make full use of microdifferential operators whose symbols are given by the infinite series that appear in the coefficients of the algebraic Mathieu equation and the Legendre equation.

0 Introduction

The primary aim of this paper is to study the analytic structure of the Borel transform of a WKB solution ψ of the Schrödinger equation

$$\left(\frac{d^2}{dx^2} - \eta^2 Q\right)\psi = 0 \quad (\eta : \text{a large parameter}) \tag{0.1}$$

when the potential Q contains two simple poles. As a simple pole and a simple turning point give similar effects on the analytic structure of Borel transformed WKB solutions ([Ko1] and [Ko2]), the above problem is, in its setting, a natural counterpart of the problems discussed in [AKT2] and [KKKoT], where Q contains two simple turning points (in [AKT2]) and one simple pole and one simple turning point (in [KKKoT]). But we need much deeper insight into the structure of the Schödinger equation in question this time. The difficulty becomes clearly visible if we consider Q_a below as the simplest example of such a potential;

$$Q_a = \frac{1}{a^2 - x^2} \quad (a : \text{a parameter}). \tag{0.2}$$

For this potential Q_a we find the following relation

$$\int_{-a}^{a} \sqrt{Q_a} dx = \pi, \tag{0.3}$$

and this indicates that the distance between two singular points of the Borel transformed WKB solutions whose relative location is independent of x (the so-called "fixed singularities" (cf. [DP], [KT, p.112], [V])) does not diminish when two simple poles in the potential (i.e., $x = \pm a$) coalesce into the origin. In the situation studied in [AKT2] and [KKKoT], integrals corresponding to (0.3) tend to 0 as the relevant turning points (with a simple pole being regarded as a turning point) coalesce, and this fact played a key role in the semi-global study of the problem in [AKT2] and [KKKoT]. To overcome this difficulty we first generalize our target class of Schödinger operators so that each operator in the class contains in its potential Q two simple poles and one simple turning point which merge as a parameter a contained in Q tends to 0. The addition of a simple turning point abates the geometric rigidity which we observed above when two and only two simple poles are relevant. For the sake of brevity and clarity we call such an operator an M2P1T operator, an operator with merging two poles and one turning point. We note that an MTP operator (resp., an MPPT operator) in [AKT2] (resp., [KKKoT]) may be called an M2T operator (resp., an M1P1T operator) if we follow this form of wording. By way of parenthesis we recall that "P" in MTP is the abbreviation of "point" and that "PP" in MPPT is that of "pair of a pole and"; that is, "MTP" means "merging turning points", whereas "MPPT" means "merging pair of a simple pole and a simple turning point". Now, as we will show in Section 1 that a WKB-theoretic canonical form (in the sense of [KT, Chap.2]) of an M2P1T equation is an algebraic Mathieu equation (in the sense of [Er, vol.III, p.98]) with a large parameter η:

(0.4)
$$\left(\frac{d^2}{dx^2}-\eta^2\Big(\frac{aA(a,\eta)+xB(a,\eta)}{x^2-a^2}+\eta^{-2}\Big(\frac{\gamma_+(a)}{(x-a)^2}+\frac{\gamma_-(a)}{(x+a)^2}\Big)\Big)\right)\psi=0,$$

where

(0.5) $A(a,\eta)=\sum_{j,k}A_k^{(j)}a^j\eta^{-k}$ and $B(a,\eta)=\sum_{j,k}B_k^{(j)}a^j\eta^{-k}$ with $A_k^{(j)}$ and $B_k^{(j)}$ satisfying appropriate growth order conditions (cf. Proposition 1.2.1),

(0.6)
$$A_0^{(0)2}\neq B_0^{(0)2},\ A_0^{(0)}B_0^{(0)}\neq 0,$$

and

(0.7) $\gamma_\pm(a)$ are holomorphic near $a=0$.

In what follows, we simply call (0.4) a Mathieu equation. The appearance of infinite series A and B connotes the necessity of employing microdifferential operators whose symbols (in the sense of microlocal analysis (e.g. [K^3])) are $A(a,\eta)$ and $B(a,\eta)$ in our analysis (Section 5), and the growth order conditions on $A_k^{(j)}$ and $B_k^{(j)}$ are intended to guarantee the existence of such microdifferential operators. When we want to emphasize the infinite series character of the constants contained in the Mathieu equation, we call it the ∞-Mathieu equation. This WKB-theoretic reduction of an M2P1T operator to the ∞-Mathieu operator is interesting in its own right, as this is the first example where three turning points (with a simple pole being counted as a turning point) are simultaneously analyzed. But the Mathieu equation is notoriously hard to analyze. Hence to attain our original purpose, that is, to study the analytic structure of Borel transformed WKB solutions near their fixed singularities relevant to the simple poles at $x = \pm a$, we further try to separate out the simple turning point of the Mathieu equation from the simple poles so that we may make use of the results of Koike ([Ko3]) for the Legendre equation. In order to put this idea into practice we further introduce another parameter ρ into an M2P1T operator so that the geometric situation required in Section 2 may be realized. In a word, the role of the parameter ρ in Definition 1.1 is designed to visualize the situation where two simple poles coalesce into the origin with a simple turning point being kept away from the origin; such a situation is realized by letting ρ tend to 0 with keeping ρ/a being a non-zero constant.

The main results in this article were announced in [KKT].

Acknowledgment.

We sincerely thank Professor T. Koike for providing us with his draft concerning the Voros coefficients of the Legendre equation.

1 Reduction of an M2P1T equation to the Mathieu equation

The purpose of this section is to construct a WKB-theoretic transformation that brings an M2P1T equation to its canonical form, i.e., the ∞-Mathieu equation with a large parameter η. As our reasoning is highly intricate, we divide it into several steps to facilitate the understanding of the reader. To begin with let us present the precise definition of an M2P1T operator, i.e., a Schrödinger operator that contains a triplet of two simple poles and one simple turning point which merge as the parameter a tends to 0: Let U (resp., V and O) be a sufficiently small open neighborhood of the origin $\{t=0\}$ (resp., $\{a=0\}$ and $\{\rho=0\}$) and let $f(t,a,\rho)$ be a holomorphic function that has the following form on $U \times V \times O$:

$$f(t,a,\rho) = t\rho g(t,\rho) + \sum_{j\geq 1} a^j f^{(j)}(t,\rho) \tag{1.1}$$

with

(1.2) $g(t,\rho)$ and $f^{(j)}(t,\rho)$ being holomorphic on $U \times O$,

(1.3) $g(0,\rho)=1$,

(1.4) $f^{(1)}(0,0) \neq 0$,

(1.5) $\rho^2 \neq f^{(1)}(0,\rho)^2$ for ρ in O.

In what follows we use symbols $f^{(0)}(t,\rho)$ and $\tilde{f}^{(0)}(t,\rho)$ respectively to denote $t\rho g(t,\rho)$ and $\rho g(t,\rho)$.

Definition 1.1. Let $f(t,a,\rho)$ be as above, let $g_{\pm}(t)$ be holomorphic functions on U and let Q denote the following potential

$$\frac{f(t,a,\rho)}{t^2-a^2} + \eta^{-2}\left(\frac{g_+(t)}{(t-a)^2} + \frac{g_-(t)}{(t+a)^2}\right) \quad (\eta : \text{a large parameter}). \tag{1.6}$$

Then the Schödinger operator

$$\frac{d^2}{dt^2} - \eta^2 Q(t,a,\rho) \tag{1.7}$$

is called an M2P1T operator.

Remark 1.1. It follows from (1.3) and the implicit function theorem that the Schödinger operator (1.7) has a simple turning point for $a \neq 0$ in V if V is sufficiently small, on the condition that ρ is different from 0.

Remark 1.2. To see how and why the numerator f in the potential Q abates the rigidity of the potential Q_a in (0.2) we note the following obvious relation:

$$\frac{t\tilde{f}^{(0)} + af^{(1)}}{t^2 - a^2} = \frac{\tilde{f}^{(0)} + f^{(1)}}{2(t-a)} + \frac{\tilde{f}^{(0)} - f^{(1)}}{2(t+a)}. \tag{1.8}$$

Then the condition (1.5) implies in this situation that the numerators in the right-hand side of (1.8) are different from 0 when evaluated at $t = 0$. Thus two simple poles cross in an additive manner as a passes through 0.

1.1 Formal construction of the transformation that brings an M2P1T equation to the Mathieu equation

Supposing

$$\rho \neq 0 \tag{1.1.1}$$

and

$$\rho^2 \neq f^{(1)}(0,\rho)^2, \tag{1.1.2}$$

we first construct the formal series

$$x = x(t,a,\rho;\eta) = \sum_{j,k\geq 0} x_{2k}^{(j)}(t,\rho)a^j\eta^{-2k}, \tag{1.1.3}$$

$$A = A(a,\rho;\eta) = \sum_{j,k\geq 0} A_{2k}^{(j)}(\rho)a^j\eta^{-2k} \tag{1.1.4}$$

and

$$B = B(a,\rho;\eta) = \sum_{j,k\geq 0} B_{2k}^{(j)}(\rho)a^j\eta^{-2k} \tag{1.1.5}$$

so that they satisfy

(1.1.6) $\quad Q(t,a,\rho;\eta)$

$$= \Big(\frac{\partial x}{\partial t}\Big)^2 \left(\frac{aA + xB}{x^2 - a^2} + \eta^{-2}\Big(\frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2}\Big)\right)$$

$$- \frac{1}{2}\eta^{-2}\{x;t\},$$

where $\{x;t\}$ designates the Schwarzian derivative, i.e.,

$$\{x;t\} = \frac{\partial^3 x/\partial t^3}{\partial x/\partial t} - \frac{3}{2}\left(\frac{\partial^2 x/\partial t^2}{\partial x/\partial t}\right)^2. \tag{1.1.7}$$

It is known (e.g. [KT, Chap.2]) that appropriate growth order conditions on $\{x_{2k}^{(j)}, A_{2k}^{(j)}, B_{2k}^{(j)}\}$ enables these series to relate Borel transformed WKB solutions of an M2P1T equation and those of its canonical form, i.e., the ∞-Mathieu equation. The growth order conditions will be studied later in Section 1.2.

1.1.1 Construction of $\{A_0^{(j)}, B_0^{(j)}, x_0^{(j)}\}$ — the first few terms

Comparing the coefficients of η^0 in (1.1.6) we find

$$\frac{f(t,a,\rho)}{t^2 - a^2} = \Big(\frac{\partial x_0}{\partial t}\Big)^2 \frac{aA_0 + x_0 B_0}{x_0^2 - a^2}, \tag{1.1.1.1}$$

where

$$x_0(t,a,\rho) = \sum_{j\geq 0} x_0^{(j)}(t,\rho)a^j, \tag{1.1.1.2}$$

$$A_0(a,\rho) = \sum_{j\geq 0} A_0^{(j)}(\rho)a^j, \tag{1.1.1.3}$$

$$B_0(a,\rho) = \sum_{j\geq 0} B_0^{(j)}(\rho)a^j. \tag{1.1.1.4}$$

By multiplying (1.1.1.1) by $(t^2-a^2)(x_0^2-a^2)$, we are to find (A_0, B_0, x_0) so that they satisfy

(1.1.1.5)
$$\Big(\sum_{j\geq 0} f^{(j)}(t,\rho)a^j\Big)\Big(\Big(\sum_{j\geq 0} x_0^{(j)}(t,\rho)a^j\Big)^2 - a^2\Big)$$

$$= (t^2-a^2)\Big(\sum_{j\geq 0}\frac{\partial x_0^{(j)}}{\partial t}a^j\Big)^2\Big(\sum_{j\geq 0} A_0^{(j)}(t,\rho)a^{j+1}$$

$$+\Big(\sum_{j\geq 0} x_0^{(j)}(t,\rho)a^j\Big)\Big(\sum_{j\geq 0} B_0^{(j)}(\rho)a^j\Big)\Big).$$

Comparing the coefficients of like powers of a, we find

(1.1.1.5.p) (=[5.p])

$$-f^{(p-2)} + \sum_{j+k+l=p} x_0^{(j)}x_0^{(k)}f^{(l)}$$

$$= t^2\Big(\sum_{j+k+l=p}\frac{\partial x_0^{(j)}}{\partial t}\frac{\partial x_0^{(k)}}{\partial t}A_0^{(l-1)} + \sum_{j+k+l+m=p}\frac{\partial x_0^{(j)}}{\partial t}\frac{\partial x_0^{(k)}}{\partial t}x_0^{(l)}B_0^{(m)}\Big)$$

$$-\Big(\sum_{j+k+l=p-2}\frac{\partial x_0^{(j)}}{\partial t}\frac{\partial x_0^{(k)}}{\partial t}A_0^{(l-1)} + \sum_{j+k+l+m=p-2}\frac{\partial x_0^{(j)}}{\partial t}\frac{\partial x_0^{(k)}}{\partial t}x_0^{(l)}B_0^{(m)}\Big).$$

In what follows we use the symbol [5.p] to denote (1.1.1.5.p) for the brevity of the notation. We also note that terms whose indices do not meet the requirements should be ignored in [5.p]; e.g. for $p=1$, $f^{(p-2)}$, $\displaystyle\sum_{j+k+l=p-2} x_0^{(j)'}x_0^{(k)'}A_0^{(l-1)}$ and $\displaystyle\sum_{j+k+l+m=p-2} x_0^{(j)'}x_0^{(k)'}x_0^{(l)}B_0^{(m)}$ are absent in [5,p] (=[5.1]). Here and also in the following, x' designates $\partial x/\partial t$. With these conventions we find

[5.0]
$$tx_0^{(0)2}\tilde{f}^{(0)} = t^2 x_0^{(0)'2}x_0^{(0)}B_0^{(0)}.$$

Dividing this by $tx_0^{(0)}$, we find

$$[5.0]' \qquad x_0^{(0)}\tilde{f}^{(0)} = tx_0^{(0)\prime 2}B_0^{(0)}.$$

Hence we find

$$(1.1.1.6) \qquad x_0^{(0)}(t,\rho) = \frac{1}{4B_0^{(0)}}\Big(\int_0^t \frac{\sqrt{\tilde{f}^{(0)}(t,\rho)}}{\sqrt{t}}\,dt\Big)^2.$$

Here we assume that $B_0^{(0)}$ can be chosen to be different from 0; we will see later (cf. (1.1.1.22) below) that this is automatically satisfied thanks to the assumption (1.1.1). We note that (1.1.1.6) together with (1.2) and (1.3) entails the existence of holomorphic function $\tilde{x}_0^{(0)}(t,\rho)$ that satisfies

$$(1.1.1.7) \qquad x_0^{(0)}(t,\rho) = t\tilde{x}_0^{(0)}(t,\rho)$$

with

$$(1.1.1.8) \qquad \tilde{x}_0^{(0)}(0,\rho) = \frac{\rho}{B_0^{(0)}}.$$

Although $x_0^{(0)}$ depends on $B_0^{(0)}$ at this stage, $B_0^{(0)}$ will be eventually fixed. Hence we do not make the dependence of $x_0^{(0)}$ on $B_0^{(0)}$ explicit in the above notation. The remark of this sort applies to $x_0^{(p)}$ to be studied below. Next we study

$$[5.1] \qquad \begin{aligned} &2x_0^{(0)}x_0^{(1)}f^{(0)} + x_0^{(0)2}f^{(1)} \\ &= t^2(x_0^{(0)\prime 2}A_0^{(0)} + 2x_0^{(0)\prime}x_0^{(1)\prime}x_0^{(0)}B_0^{(0)} \\ &\qquad + x_0^{(0)\prime 2}x_0^{(1)}B_0^{(0)} + x_0^{(0)\prime 2}x_0^{(0)}B_0^{(1)}). \end{aligned}$$

It then follows from (1.1.1.7) that the left-hand side of [5.1] has the form

$$(1.1.1.9) \qquad t^2\big(2\tilde{x}_0^{(0)}x_0^{(1)}\tilde{f}^{(0)} + \tilde{x}_0^{(0)2}f^{(1)}\big).$$

Thus we are to solve

$$[5.1]' \qquad 2x_0^{(0)'} x_0^{(1)'} x_0^{(0)} B_0^{(0)} + x_0^{(0)'2} x_0^{(1)} B_0^{(0)} - 2\tilde{x}_0^{(0)} x_0^{(1)} \tilde{f}^{(0)}$$
$$= -x_0^{(0)'2} A_0^{(0)} - x_0^{(0)'2} x_0^{(0)} B_0^{(1)} + \tilde{x}_0^{(0)2} f^{(1)}.$$

In view of (1.1.1.7) and (1.1.1.8) we now introduce a new variable

$$(1.1.1.10) \qquad s = x_0^{(0)}(t, \rho);$$

in what follows we use the symbol $\dot{x}(s, \rho)$ to designate dx/ds. Dividing $[5.1]'$ by $x_0^{(0)'2}$ and rewriting the equation in s-variable, we use $[5.0]'$ to find

$$[5.1]'' \qquad 2B_0^{(0)} s \frac{dx_0^{(1)}(s, \rho)}{ds} - B_0^{(0)} x_0^{(1)}(s, \rho)$$
$$= -A_0^{(0)} - sB_0^{(1)} + \left[\left(x_0^{(0)'} \right)^{-2} \tilde{x}_0^{(0)2} f^{(1)} \right] \big(t(s, \rho), \rho \big),$$

where $t(s, \rho)$ designates the inverse function of $s = x_0^{(0)}(t, \rho)$. Then we find that $[5.1]''$ is a differential equation with a regular singularity at $s = 0$ with the characteristic index $1/2$. Hence it has a holomorphic solution $x_0^{(1)}(s, \rho)$ near $s = 0$ for any $A_0^{(0)}$ and $B_0^{(0)}$, which are arbitrary constants at this stage. Furthermore we find

$$(1.1.1.11) \qquad x_0^{(1)}(0, \rho) = \frac{1}{B_0^{(0)}} \big(A_0^{(0)} - f^{(1)}(0, \rho) \big)$$

and

(1.1.1.12)
$$\dot{x}_0^{(1)}(0, \rho) = \frac{1}{B_0^{(0)}} \Big(-B_0^{(1)} + Z_0^{-1} \big(z'(0, \rho) f^{(1)}(0, \rho) + f^{(1)'}(0, \rho) \big) \Big),$$

where

$$(1.1.1.13) \qquad Z_0 = x_0^{(0)'}(0, \rho) \left(= \tilde{x}_0^{(0)}(0, \rho) = \frac{\rho}{B_0^{(0)}} \right)$$

and

$$z(t,\rho) = \big(x_0^{(0)'}(t,\rho)\big)^{-2}\tilde{x}_0^{(0)}(t,\rho)^2. \tag{1.1.1.14}$$

We next consider

$$\begin{aligned}[5.2] \qquad & - f^{(0)} + \big(2x_0^{(2)}x_0^{(0)} + x_0^{(1)2}\big)f^{(0)} + 2x_0^{(1)}x_0^{(0)}f^{(1)} + x_0^{(0)2}f^{(2)} \\ & = t^2\Big[x_0^{(0)'2}A_0^{(1)} + 2x_0^{(0)'}x_0^{(1)'}A_0^{(0)} + x_0^{(0)'2}x_0^{(2)}B_0^{(0)} \\ & \quad + 2x_0^{(0)'}x_0^{(1)'}x_0^{(1)}B_0^{(0)} + x_0^{(0)'2}x_0^{(1)}B_0^{(1)} \\ & \quad + \big(2x_0^{(2)'}x_0^{(0)'} + x_0^{(1)'2}\big)x_0^{(0)}B_0^{(0)} + 2x_0^{(1)'}x_0^{(0)'}x_0^{(0)}B_0^{(1)} \\ & \quad + x_0^{(0)'2}x_0^{(0)}B_0^{(2)}\Big] - x_0^{(0)'2}x_0^{(0)}B_0^{(0)}. \end{aligned}$$

Here we observe a new feature which we did not encounter in the study of [5.p] ($p = 0, 1$): [5.2] is not divisible by t^2 as it stands. Thus the existence of a holomorphic solution $x_0^{(2)}(t,\rho)$ near $t = 0$ requires that the following function $\mathcal{B}^{(1)}(t,\rho)$ given by (1.1.1.15) should vanish at $t = 0$. Note that $t\mathcal{B}^{(1)}(t,\rho)$ is the sum of terms in [5.2] which contain the factor t^1 only, at least explicitly.

$$\mathcal{B}^{(1)}(t,\rho) = \tilde{f}^{(0)} - x_0^{(1)2}\tilde{f}^{(0)} - 2\tilde{x}_0^{(0)}x_0^{(1)}f^{(1)} - x_0^{(0)'2}\tilde{x}_0^{(0)}B_0^{(0)}. \tag{1.1.1.15}$$

Substituting (1.3), (1.1.1.8) and (1.1.1.11) into $\mathcal{B}^{(1)}(t,\rho)$, we find

(1.1.1.16)

$$\begin{aligned} \mathcal{B}^{(1)}(0,\rho) &= \rho - \rho(B_0^{(0)})^{-2}\big(A_0^{(0)} - f^{(1)}(0,\rho)\big)^2 \\ &\quad - 2\rho(B_0^{(0)})^{-1}(B_0^{(0)})^{-1}\big(A_0^{(0)} - f^{(1)}(0,\rho)\big)f^{(1)}(0,\rho) \\ &\quad - \big(\rho(B_0^{(0)})^{-1}\big)^3 B_0^{(0)} \\ &= \rho(B_0^{(0)})^{-2}\Big(B_0^{(0)2} - \big(A_0^{(0)} - f^{(1)}(0,\rho)\big)^2 \end{aligned}$$

$$- 2\big(A_0^{(0)} - f^{(1)}(0,\rho)\big) f^{(1)}(0,\rho) - \rho^2\Big)$$

$$= \rho(B_0^{(0)})^{-2}\Big(B_0^{(0)2} - A_0^{(0)2} + f^{(1)}(0,\rho)^2 - \rho^2\Big).$$

In view of the assumption (1.1.1) we thus require

$$B_0^{(0)2} - A_0^{(0)2} + f^{(1)}(0,\rho)^2 - \rho^2 = 0. \tag{1.1.1.17}$$

Assuming (1.1.1.17) , we can divide [5.2] by $t^2 x_0^{(0)'}(t,\rho)^2$ to find

$$[5.2]' \qquad B_0^{(0)}\Big(2s\frac{d}{ds}x_0^{(2)}(s,\rho) + x_0^{(2)}(s,\rho)\Big)$$

$$- 2\big(x_0^{(0)'}(t,\rho)\big)^{-2}\tilde{x}_0^{(0)}(t,\rho)\tilde{f}^{(0)}(t,\rho)x_0^{(2)}(t,\rho)$$

$$= -A_0^{(1)} - B_0^{(2)}s - 2\dot{x}_0^{(1)}(s,\rho)A_0^{(0)}$$

$$- 2\dot{x}_0^{(1)}(s,\rho)x_0^{(1)}(s,\rho)B_0^{(0)} - x_0^{(1)}(s,\rho)B_0^{(1)}$$

$$- \dot{x}_0^{(1)}(s,\rho)^2 sB_0^{(0)} - 2\dot{x}_0^{(1)}(s,\rho)sB_0^{(1)} + z(t,\rho)f^{(2)}(t,\rho)$$

$$- t^{-1}\big(x_0^{(0)'}(t)\big)^{-2}\big(\mathcal{B}^{(1)}(t,\rho) - \mathcal{B}^{(1)}(0,\rho)\big).$$

Here we note one universal (i.e., common to every p) phenomenon, which was also observed for $p = 1$: $[5.0]'$ entails that the left-hand side of $[5.2]'$ is equal to

$$B_0^{(0)}\big(2s\frac{d}{ds} - 1\big)x_0^{(2)}(s,\rho). \tag{1.1.1.18}$$

This considerably facilitates the computation of $x_0^{(2)}(0,\rho)$ and $\dot{x}_0^{(2)}(0,\rho)$, which are needed in our reasoning. But we postpone their actual computation until the stage where $(A_0^{(0)}, B_0^{(0)})$ is fixed; $(A_0^{(0)}, B_0^{(0)})$ will be fixed without knowing the explicit form of $x_0^{(2)}(0,\rho)$ and $\dot{x}_0^{(2)}(0,\rho)$, whereas their explicit form becomes substantially simplified when $(A_0^{(0)}, B_0^{(0)})$ is fixed. Here we only note that $(\partial\mathcal{B}^{(1)}/\partial t)(0,\rho)$ etc. should be taken into account in the computation of $x_0^{(2)}(0,\rho)$ etc. To fix $(A_0^{(0)}, B_0^{(0)})$, we consider next stage, i.e., [5.3].

[5.3]

$$
\begin{aligned}
&- f^{(1)} + \sum_{j+k+l=3} x_0^{(j)} x_0^{(k)} f^{(l)} \\
&= t^2 \Big(\sum_{j+k+l=2} x_0^{(j)\prime} x_0^{(k)\prime} A_0^{(l)} + \sum_{j+k+l+m=3} x_0^{(j)\prime} x_0^{(k)\prime} x_0^{(l)} B_0^{(m)} \Big) \\
&\quad - \Big[x_0^{(0)\prime 2} A_0^{(0)} + x_0^{(0)\prime 2} x_0^{(0)} B_0^{(1)} + \Big(x_0^{(0)\prime 2} x_0^{(1)} + 2 x_0^{(0)\prime} x_0^{(1)\prime} x_0^{(0)} \Big) B_0^{(0)} \Big].
\end{aligned}
$$

For the existence of a holomorphic solution $x_0^{(3)}(t,\rho)$ of [5.3] near $t = 0$, we clearly need the coincidence of the value of the left-hand side at $t = 0$ and that of the right-hand side. Although one immediately notices another condition is necessary for the existence of $x_0^{(3)}(t,\rho)$, we first concentrate our attention on this coincidence. Then it follows from (1.1.1.8) and (1.1.1.11) that we have

(1.1.1.19)
$$
\begin{aligned}
& f^{(1)}(0,\rho) \Big[-1 + \Big(\frac{1}{B_0^{(0)}} \big(A_0^{(0)} - f^{(1)}(0,\rho) \big) \Big)^2 \Big] \\
&= - \Big(\frac{\rho}{B_0^{(0)}} \Big)^2 \Big[A_0^{(0)} + A_0^{(0)} - f^{(1)}(0,\rho) \Big].
\end{aligned}
$$

Then the substitution of

(1.1.1.17′)
$$
A_0^{(0)2} - B_0^{(0)2} = f^{(1)}(0,\rho)^2 - \rho^2
$$

into (1.1.1.19) entails

(1.1.1.20)
$$
\begin{aligned}
0 &= f^{(1)}(0,\rho) \big(f^{(1)}(0,\rho)^2 - \rho^2 - 2A_0^{(0)} f^{(1)}(0,\rho) \\
&\quad + f^{(1)}(0,\rho)^2 \big) + \rho^2 \big(2A_0^{(0)} - f^{(1)}(0,\rho) \big)
\end{aligned}
$$

$$= 2f^{(1)}(0,\rho)^2\big(f^{(1)}(0,\rho) - A_0^{(0)}\big) - 2\rho^2\big(f^{(1)}(0,\rho) - A_0^{(0)}\big)$$

$$= 2\big(f^{(1)}(0,\rho)^2 - \rho^2\big)\big(f^{(1)}(0,\rho) - A_0^{(0)}\big).$$

Thus the assumption (1.1.2) implies

$$A_0^{(0)} = f^{(1)}(0,\rho). \tag{1.1.1.21}$$

Substituting (1.1.1.21) into (1.1.1.17) we obtain

$$B_0^{(0)2} = \rho^2, \tag{1.1.1.22}$$

that is,

$$B_0^{(0)} = \pm\rho. \tag{1.1.1.22$'$}$$

These results lead to the following important assertions: First (1.1.1.22) together with (1.1.1.13) implies

$$x_0^{(0)'}(0,\rho) = \pm 1, \tag{1.1.1.23}$$

and second, a still more important result follows from (1.1.1.11) and (1.1.1.21):

$$x_0^{(1)}(0,\rho) = 0\,! \tag{1.1.1.24}$$

This result will repeatedly play a decisively important role in our subsequent reasoning.

Before proceeding further, we show how these results are used in the explicit computation of $x_0^{(2)}(0,\rho)$, which will later become necessary to compute $(A_0^{(1)}, B_0^{(1)})$. First, in order to see the explicit form of $[t^{-1}(x_0^{(0)'})^{-2}\ \big(\mathcal{B}^{(1)}(t,\rho) - \mathcal{B}^{(1)}(0,\rho)\big)]$ evaluated at $t = 0$, we calculate $(\partial\mathcal{B}^{(1)}/\partial t)(0,\rho)$:

$$\frac{\partial\mathcal{B}^{(1)}}{\partial t}(0,\rho) = \rho g'(0,\rho) - 2Z_0 f^{(1)}(0,\rho)x_0^{(1)'}(0,\rho) - 2B_0^{(0)}x_0^{(0)''}(0,\rho) - B_0^{(0)}\tilde{x}_0^{(0)'}(0,\rho). \tag{1.1.1.25}$$

In obtaining this result we used (1.1.1.24) at several spots. Replacing $f^{(1)}(0,\rho)$ by $A_0^{(0)}$, we encounter one remarkable cancellation of terms containing $A_0^{(0)}$ in [5.2]′ evaluated at $s=0$:

$$-2\dot{x}_0^{(1)}(0,\rho)A_0^{(0)} + 2Z_0^{-2}\big(Z_0^2 A_0^{(0)}\dot{x}_0^{(1)}(0,\rho)\big) = 0. \tag{1.1.1.26}$$

Cancellation of this sort will play a crucially important role in the construction of $\big(x_0^{(p)}, A_0^{(p)}, B_0^{(p)}\big)$ and their estimation in the subsequent sections. Using (1.1.1.24) again, we thus find

$$B_0^{(0)}x_0^{(2)}(0,\rho) = A_0^{(1)} - f^{(2)}(0,\rho) + \chi_0^{(0)}B_0^{(0)}, \tag{1.1.1.27}$$

where $\chi_0^{(0)}$ is a constant fixed by $g(t,\rho)$ (and $Z_0=\pm 1$). Here we notice no $B_0^{(1)}$-dependent terms remain in the right-hand side of (1.1.1.27).

Now let us return to the study of [5.3]. To find the conditions that guarantee the existence of holomorphic $x_0^{(3)}(t,\rho)$, let us introduce the following functions $\mathcal{B}$, $\mathcal{B}^{(0)}$, $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$:

$$\begin{aligned}\mathcal{B}(t,\rho) = &- f^{(1)}(t,\rho) + \sum_{j+k+l=3} x_0^{(j)}x_0^{(k)}f^{(l)} \\ &+ x_0^{(0)\prime 2}A_0^{(0)} + x_0^{(0)\prime 2}x_0^{(0)}B_0^{(1)} \\ &+ \big(x_0^{(0)\prime 2}x_0^{(1)} + 2x_0^{(0)\prime}x_0^{(1)\prime}x_0^{(0)}\big)B_0^{(0)},\end{aligned} \tag{1.1.1.28}$$

$$\mathcal{B}^{(0)} = -f^{(1)} + x_0^{(0)\prime 2}A_0^{(0)}, \tag{1.1.1.29}$$

$$\begin{aligned}\mathcal{B}^{(1)} = &\, 2\tilde{x}_0^{(0)}x_0^{(2)}f^{(1)} + x_0^{(0)\prime 2}\tilde{x}_0^{(0)}B_0^{(1)} \\ &+ \big(x_0^{(0)\prime 2}\tilde{x}_0^{(1)} + 2x_0^{(0)\prime}x_0^{(1)\prime}\tilde{x}_0^{(0)}\big)B_0^{(0)},\end{aligned} \tag{1.1.1.30}$$

where

$$\tilde{x}_0^{(1)}(t,\rho) = x_0^{(1)}(t,\rho)/t, \tag{1.1.1.31}$$

and

$$\mathcal{B}^{(2)} = 2\big(\tilde{x}_0^{(0)} x_0^{(3)} + \tilde{x}_0^{(1)} x_0^{(2)}\big)\tilde{f}^{(0)} + \tilde{x}_0^{(1)2} f^{(1)} + 2\tilde{x}_0^{(0)}\tilde{x}_0^{(1)} f^{(2)} + \tilde{x}_0^{(0)2} f^{(3)}. \tag{1.1.1.32}$$

It is obvious that we have

$$\mathcal{B} = \mathcal{B}^{(0)} + t\mathcal{B}^{(1)} + t^2\mathcal{B}^{(2)}. \tag{1.1.1.33}$$

One immediately notices that $\mathcal{B}^{(0)}(0,\rho) = 0$ is equivalent to (1.1.1.19) and that "another condition" needed for the existence of holomorphic $x_0^{(3)}(t,\rho)$ is given by

$$\frac{\partial \mathcal{B}^{(0)}}{\partial t}(0,\rho) + \mathcal{B}^{(1)}(0,\rho) = 0. \tag{1.1.1.34}$$

Thus we obtain

$$\begin{aligned} &2Z_0 A_0^{(0)} x_0^{(2)}(0,\rho) + Z_0 B_0^{(1)} \\ &+ \big(\tilde{x}_0^{(1)}(0,\rho) + 2x_0^{(1)'}(0,\rho)\big)B_0^{(0)} \\ &- \frac{\partial f^{(1)}}{\partial t}(0,\rho) + 2A_0^{(0)} Z_0 x_0^{(0)''}(0,\rho) = 0 \end{aligned} \tag{1.1.1.35}$$

with the help of (1.1.1.21), (1.1.1.23) and (1.1.1.24). We now substitute (1.1.1.12) and (1.1.1.27) into (1.1.1.35) to find

$$\begin{aligned} &2Z_0 \frac{A_0^{(0)}}{B_0^{(0)}}\big(A_0^{(1)} - f^{(2)}(0,\rho) + \chi_0^{(0)} B_0^{(0)}\big) \\ &- 2Z_0 B_0^{(1)} + 3z'(0,\rho)A_0^{(0)} + 2f^{(1)'}(0,\rho) \\ &+ 2Z_0 x_0^{(0)''}(0,\rho)A_0^{(0)} = 0. \end{aligned} \tag{1.1.1.36}$$

Dividing (1.1.1.36) by $Z_0(=\pm 1)$, we find

$$2\frac{A_0^{(0)}}{B_0^{(0)}} A_0^{(1)} - 2B_0^{(1)} \tag{1.1.1.37}$$

$$= 2\frac{A_0^{(0)}}{B_0^{(0)}} f^{(2)}(0,\rho) - 2A_0^{(0)}\chi_0^{(0)} - 3Z_0^{-1}z'(0,\rho)A_0^{(0)}$$

$$- 2Z_0^{-1}f^{(1)'}(0,\rho) - 2x_0^{(0)''}(0,\rho)A_0^{(0)}.$$

Thus "another condition" for the existence of holomorphic $x_0^{(3)}(t,\rho)$ gives a constraint on $\big(A_0^{(1)}, B_0^{(1)}\big)$.

Now assumptions (1.1.1.19) and (1.1.1.34) enable us to divide [5.3] by $t^2(x_0^{(0)'})^2$ to obtain

[5.3]′

$$B_0^{(0)}\Big(2s\frac{d}{ds}+1\Big)x_0^{(3)}(s,\rho)$$

$$= -A_0^{(2)} - sB_0^{(3)} - \Big[\sum_{\substack{j+k+l=2\\ l\le 1}} \dot{x}_0^{(j)}\dot{x}_0^{(k)}A_0^{(l)} + \sum_{\substack{j+k+l+m=3\\ j,k,l,m\le 2}} \dot{x}_0^{(j)}\dot{x}_0^{(k)}x_0^{(l)}B_0^{(m)}\Big]$$

$$+ A_0^{(0)} + sB_0^{(1)} + \big(x_0^{(1)} + 2s\dot{x}_0^{(1)}\big)B_0^{(0)}$$

$$+ \Big[\sum_{n=0}^{3}\phi_n(t,\rho)\Big]\Big|_{t=t(s,\rho)}$$

where

$$\phi_0 = 2\big(x_0^{(0)'}\big)^{-2}\,\tilde{x}_0^{(0)}\tilde{f}^{(0)}x_0^{(3)}, \tag{1.1.1.38}$$

$$\phi_1 = \big(x_0^{(0)'}\big)^{-2}\,\big(2\tilde{x}_0^{(1)}x_0^{(2)}\tilde{f}^{(0)} + \tilde{x}_0^{(1)2}f^{(3)} + 2\tilde{x}_0^{(0)}\tilde{x}_0^{(1)}f^{(1)} + \tilde{x}_0^{(0)2}f^{(3)}\big) \tag{1.1.1.39}$$

$$\phi_2 = 2\big(x_0^{(0)'}\big)^{-2}\,\big(\tilde{x}_0^{(0)}f^{(1)}\big)x_0^{(2)'} \tag{1.1.1.40}$$

$$\phi_3 = t^{-2}\big(x_0^{(0)'}\big)^{-2}\big[\mathcal{B}^{(0)}(t,\rho) + t\mathcal{B}^{(1)}(t,\rho) - \mathcal{B}^{(0)}(0,\rho) \tag{1.1.1.41}$$

$$- t\big(\mathcal{B}^{(1)}(0,\rho) + (\partial\mathcal{B}^{(0)}/\partial t)(0,\rho)\big) - t^2\big(x_0^{(0)\prime}\big)^2\phi_2\big].$$

Here we have separated out ϕ_0 (resp., ϕ_2) from ϕ_1 (resp., ϕ_3) to call the attention of the reader to the peculiar roles ϕ_0 and ϕ_2 play in our computation, as has already been noticed when $p = 2$: First, [5.0]$'$ entails ϕ_0 coincides with $2B_0^{(0)}x_0^{(3)}(t,\rho)$; second, we observe $\phi_2(t(0,\rho))$ coincides with $2\dot{x}_0^{(2)}(0,\rho)A_0^{(0)}$.

It is clear that [5.3]$'$ has a holomorphic solution $x_0^{(3)}(t,\rho)$ near $t = 0$ for any $A_0^{(2)}$ and $B_0^{(3)}$, on the condition that $(A_0^{(0)}, B_0^{(0)})$ satisfies (1.1.1.21) and (1.1.1.22) and that $(A_0^{(1)}, B_0^{(1)})$ obeys the constraint (1.1.1.37). Using this holomorphic solution $x_0^{(3)}(t,\rho)$, we can write down [5.4]:

$$[5.4] \qquad - f^{(2)} + \sum_{j+k+l=4} x_0^{(j)}x_0^{(k)}f^{(l)}$$

$$= t^2\Big(\sum_{j+k+l=3} x_0^{(j)\prime}x_0^{(k)\prime}A_0^{(l)} + \sum_{j+k+l+m=4} x_0^{(j)\prime}x_0^{(k)\prime}x_0^{(l)}B_0^{(m)}\Big)$$

$$-\Big(\sum_{j+k+l=1} x_0^{(j)\prime}x_0^{(k)\prime}A_0^{(l)} + \sum_{j+k+l+m=2} x_0^{(j)\prime}x_0^{(k)\prime}x_0^{(l)}B_0^{(m)}\Big).$$

Assuming $x_0^{(4)}(t,\rho)$ is holomorphic near $t = 0$, we set $t = 0$ in [5.4] to obtain

$$(1.1.1.42) \qquad - f^{(2)}(0,\rho) + x_0^{(0)\prime 2}(0,\rho)A_0^{(1)}$$

$$+ 2x_0^{(0)\prime}(0,\rho)x_0^{(1)\prime}(0,\rho)A_0^{(0)} + x_0^{(0)\prime 2}(0,\rho)x_0^{(2)}(0,\rho)B_0^{(0)}$$

$$= 0.$$

Here we have used (1.1.1.7) and (1.1.1.24) to guarantee that there is no contribution from the sum $\sum_{j+k+l=4} x_0^{(j)}x_0^{(k)}f^{(l)}$ and $\Big(\sum_{j+k+l+m=2} x_0^{(j)\prime}x_0^{(k)\prime}$

$x_0^{(l)} B_0^{(m)}\Big) - x_0^{(0)\prime 2} x_0^{(2)} B_0^{(0)}$. Substituting (1.1.1.12) and (1.1.1.27) into (1.1.1.42), we find

(1.1.1.43)

$$
\begin{aligned}
& - f^{(2)}(0,\rho) + A_0^{(1)} + 2A_0^{(0)}\big(B_0^{(0)}\big)^{-1}\Big(- B_0^{(1)} + Z_0^{-1}\big(z'(0,\rho)A_0^{(0)} \\
& + f^{(1)\prime}(0,\rho)\big)\Big) + A_0^{(1)} - f^{(2)}(0,\rho) + \chi_0^{(0)} B_0^{(0)} \\
& = 0.
\end{aligned}
$$

Then the assumption (1.1.2) enables us to solve (1.1.1.37) and (1.1.1.43); $(A_0^{(1)}, B_0^{(1)})$ is fixed in terms of $f^{(2)}(0,\rho)$, $A_0^{(0)}$, $B_0^{(0)}$, $z'(0,\rho)$, $f^{(1)\prime}(0,\rho)$, $\chi_0^{(0)}$, Z_0 and $x^{(0)\prime\prime}(0,\rho)$. We next calculate the coefficient of t^1 in [5.4] to find a constraint on $(A_0^{(2)}, B_0^{(2)})$ which guarantees the existence of holomorphic $x_0^{(4)}(t,\rho)$ near $t=0$. In principle, what we are to do now is to repeat this procedure to find $\big(x_0^{(p)}(t,\rho), A_0^{(p)}, B_0^{(p)}\big)$ for every p and then to estimate them. But the computation becomes more and more complicated as p increases; hence we first describe the core feature of the induction process in Section 1.1.2 and then brush it up in Section 1.1.3, so that the estimation may become smoothly performed with the refined version.

1.1.2 Description of the dependence of $\{x_0^{(p)}(t,\rho)\}_{p\geq 0}$ upon $\{A_0^{(q)}, B_0^{(q)}\}_{q\geq 0}$

As the concrete computation in the preceding subsection indicates, one constraint is placed on $\big(A_0^{(q)}, B_0^{(q)}\big)$ for the existence of holomorphic $x_0^{(q+2)}(t,\rho)$ and another constraint on $\big(A_0^{(q)}, B_0^{(q)}\big)$ is added for the existence of holomorphic $x_0^{(q+3)}(t,\rho)$; these two conditions combined will fix $\big(A_0^{(q)}, B_0^{(q)}\big)$. In order to confirm that this process runs smoothly by the assumption (1.1.2), we like to know the concrete structure of

$x_0^{(p)}(t,\rho)$, or at least its "principal part". For this purpose let us first prepare some notations related to [5.p] (=(1.1.1.5.p)).

Definition 1.1.2.1. Assume $p \geq 4$. Then $\mathcal{B}[p] = \mathcal{B}[p](t,\rho)$, $\mathcal{B}[p]^{(0)}$, $\mathcal{B}[p]^{(1)}$ and $\mathcal{B}[p]^{(2)}$ are respectively defined by the following:

$$(1.1.2.1)\quad \mathcal{B}[p] = \sum_{i+j+k=p-3} \frac{\partial x_0^{(i)}}{\partial t}(t,\rho)\frac{\partial x_0^{(j)}}{\partial t}(t,\rho)A_0^{(k)} + \sum_{i+j+k+l=p-2} x_0^{(k)}(t,\rho)\frac{\partial x_0^{(i)}}{\partial t}(t,\rho)\frac{\partial x_0^{(j)}}{\partial t}(t,\rho)B_0^{(l)} + \sum_{i+j+k=p} x_0^{(i)}(t,\rho)x_0^{(j)}(t,\rho)f^{(k)}(t,\rho) - f^{(p-2)}(t,\rho),$$

$$(1.1.2.2)\quad \mathcal{B}[p]^{(0)} = \sum_{i+j+k=p-3} \frac{\partial x_0^{(i)}}{\partial t}(t,\rho)\frac{\partial x_0^{(j)}}{\partial t}(t,\rho)A_0^{(k)} + \sum_{\substack{i+j+k+l=p-2\\ k\geq 2}} x_0^{(k)}(t,\rho)\frac{\partial x_0^{(i)}}{\partial t}(t,\rho)\frac{\partial x_0^{(j)}}{\partial t}(t,\rho)B_0^{(l)} + \sum_{\substack{i+j+k=p\\ i,j\geq 2,\, k\geq 1}} x_0^{(i)}(t,\rho)x_0^{(j)}(t,\rho)f^{(k)}(t,\rho) - f^{(p-2)}(t,\rho)$$

$$(1.1.2.3)\quad \mathcal{B}[p]^{(1)} = \tilde{x}_0^{(0)}(t,\rho)\Big(\sum_{i+j+l=p-2} \frac{\partial x_0^{(i)}}{\partial t}(t,\rho)\frac{\partial x_0^{(j)}}{\partial t}(t,\rho)B_0^{(l)}\Big) + \tilde{x}_0^{(1)}(t,\rho)\Big(\sum_{i+j+l=p-3} \frac{\partial x_0^{(i)}}{\partial t}(t,\rho)\frac{\partial x_0^{(j)}}{\partial t}(t,\rho)B_0^{(l)}\Big)$$

$$+ 2\tilde{x}_0^{(0)}(t,\rho)\Big(\sum_{\substack{j+k=p\\ j\geq 2,\,k\geq 1}} x_0^{(j)}(t,\rho)f^{(k)}(t,\rho)\Big)$$

$$+ 2\tilde{x}_0^{(1)}(t,\rho)\Big(\sum_{\substack{j+k=p-1\\ j\geq 2,\,k\geq 1}} x_0^{(j)}(t,\rho)f^{(k)}(t,\rho)\Big)$$

$$+ \sum_{\substack{i+j=p\\ i,j\geq 2,}} x_0^{(i)}(t,\rho)x_0^{(j)}(t,\rho)\tilde{f}^{(0)}(t,\rho),$$

(1.1.2.4)

$$\mathcal{B}[p]^{(2)} = \sum_{\substack{i+j+k=p\\ i,j=0,1;\,k\geq 1}} \tilde{x}_0^{(i)}(t,\rho)\tilde{x}_0^{(j)}(t,\rho)f^{(k)}(t,\rho)$$

$$+ 2\Big(\tilde{x}_0^{(0)}(t,\rho)x_0^{(p)}(t,\rho) + \tilde{x}_0^{(1)}(t,\rho)x_0^{(p-1)}(t,\rho)\Big)\tilde{f}^{(0)}(t,\rho).$$

Remark 1.1.2.1. In parallel with (1.1.1.33) we have

(1.1.2.5) $$\mathcal{B}[p] = \mathcal{B}[p]^{(0)} + t\mathcal{B}[p]^{(1)} + t^2\mathcal{B}[p]^{(2)}.$$

To rewrite [5.p] more concretely we further introduce the following symbols. First, we let $E^{(p)}$ denote

(1.1.2.6) $$\sum_{i+j+k=p-1} \frac{\partial x_0^{(i)}}{\partial t}(t,\rho)\frac{\partial x_0^{(j)}}{\partial t}(t,\rho)A_0^{(k)}$$

$$+ \sum_{i+j+k+l=p} \frac{\partial x_0^{(i)}}{\partial t}(t,\rho)\frac{\partial x_0^{(j)}}{\partial t}(t,\rho)x_0^{(k)}(t,\rho)B_0^{(l)}.$$

Second, we define t-independent functions $C_0^{(p)}$ and $D_0^{(p)}$ by the following:

(1.1.2.7) $$C_0^{(p)}(\rho) = \mathcal{B}[p]^{(0)}(0,\rho),$$

(1.1.2.8)
$$D_0^{(p)}(\rho) = \mathcal{B}[p]^{(1)}(0,\rho) + \frac{\partial \mathcal{B}[p]^{(0)}}{\partial t}(0,\rho).$$

Using $C_0^{(p)}$ and $D_0^{(p)}$, we define $\mathcal{F}^{(p)}$ by

(1.1.2.9)
$$\mathcal{B}[p]^{(0)}(t,\rho) + t\mathcal{B}[p]^{(1)}(t,\rho) - \big(C_0^{(p)} + tD_0^{(p)}\big).$$

It is then clear that $\mathcal{F}^{(p)}$ is divisible by t^2 and we use the symbol $\mathcal{E}^{(p)}$ to denote

(1.1.2.10)
$$t^{-2}\mathcal{F}^{(p)}.$$

Having in mind the results in Section 1.1.1, we plan to fix constants $\big(A_0^{(q)}, B_0^{(q)}\big)$ by equations

(1.1.2.11)
$$C_0^{(q+3)} = 0 \quad \text{and} \quad D_0^{(q+2)} = 0,$$

and construct $x_0^{(p)}$ by solving

[5.p]
$$E^{(p)} - \mathcal{E}^{(p)} - \mathcal{B}[p]^{(2)} = 0.$$

As is observed in the preceding subsection, we can rewrite [5.p] using the variable

(1.1.2.12)
$$s = x_0^{(0)}(t,\rho)$$

and its inverse function $t(s,\rho)$ as follows:

[5.p]$'$
$$\begin{aligned}
&B_0^{(0)}\Big(2s\frac{d}{ds} - 1\Big)x_0^{(p)}(s,\rho) \\
&= -A_0^{(p-1)} - B_0^{(p)}s - \sum_{\substack{i+j+k=p-1\\ k\le p-2}} \dot{x}_0^{(i)}\dot{x}_0^{(j)}A_0^{(k)} - \sum_{\substack{i+j+k+l=p\\ i,j,k,l\le p-1}} \dot{x}_0^{(i)}\dot{x}_0^{(j)}x_0^{(k)}B_0^{(l)} \\
&\quad + \Big[\big(x_0^{(0)\prime}(t,\rho)\big)^{-2}\Big(\mathcal{E}^{(p)} + 2\tilde{x}_0^{(1)}(t,\rho)\tilde{f}^{(0)}(t,\rho)x_0^{(p-1)}(t,\rho) \\
&\quad + \sum_{\substack{i+j+k=p\\ i,j=0,1;\, k\ge 1}} \tilde{x}_0^{(i)}(t,\rho)\tilde{x}_0^{(j)}(t,\rho)f^{(k)}(t,\rho)\Big)\Big]\Big|_{t=t(s,\rho)},
\end{aligned}$$

where $\mathcal{E}^{(p)}$ denotes the sum of functions given by (1.1.2.10).

Remark 1.1.2.2. In parallel with (1.1.1.40) we note that the value at $s=0$ of $2\dot{x}_0^{(0)}\dot{x}_0^{(p-1)}A_0^{(0)}$ coincides with that of $\Big[2\big(x_0^{(0)\prime}\big)^{-2}\ \tilde{x}_0^{(0)}f^{(1)}\ x_0^{(p-1)\prime}\Big]\Big|_{t=t(s,\rho)}$, which originates from $\partial\mathcal{B}[p]^{(1)}/\partial t$; through the Taylor expansion this term appears among the terms of $\mathcal{E}^{(p)}$ evaluated at $s=0$. A similar relation holds between $2\dot{x}_0^{(1)}\ \dot{x}_0^{(p-2)}\ A_0^{(0)}$ and $\Big[2\big(x_0^{(0)\prime}\big)^{-2}\ \tilde{x}_0^{(1)}f^{(1)}\ x_0^{(p-2)\prime}\Big]\Big|_{t=t(s,\rho)}$, which also originates from $\partial\mathcal{B}[p]^{(1)}/\partial t$. Furthermore the value at $s=0$ of

$$\sum_{\substack{i+j=p-1\\ i,j\geq 2}} \dot{x}_0^{(i)}\dot{x}_0^{(j)}A_0^{(0)}$$

is also coincident with that of

$$\Big[\big(x_0^{(0)\prime}\big)^{-2}\sum_{\substack{i+j=p-1\\ i,j\geq 2}} x_0^{(i)\prime}x_0^{(j)\prime}f^{(1)}\Big]\Big|_{t=t(s,\rho)},$$

which is a part of the coefficient of t^2 in the Taylor expansion of $\mathcal{B}[p]^{(0)}(t,\rho)$. These coincidences will play important roles in our subsequent reasoning.

In order to facilitate pointing out the core part of our reasoning, we further prepare the following

Definition 1.1.2.2. Let $\big(\vec{A}_0[p],\vec{B}_0[p']\big)$ stand for $\big(A_0^{(0)},\ A_0^{(1)},\ \cdots\ A_0^{(p)},\ B_0^{(0)},\ B_0^{(1)},\ \cdots\ B_0^{(p')}\big)$ and let $X=X\big(\vec{A}_0[p],\ \vec{B}_0[p']\big)$ and $Y=Y\big(\vec{A}_0[p],\vec{B}_0[p']\big)$ be their functions. If X–Y depends only on $\big(\vec{A}_0[q-1],\ \vec{B}_0[q-1]\big)$ for some q, then we say

(1.1.2.13) $$X \underset{(q)}{\equiv} Y.$$

Remark 1.1.2.3. In the above definition we concentrate our attention on the dependence on $\big(\vec{A}_0[p],\, \vec{B}_0[p']\big)$ which are newly introduced to our discussions as parameters in the canonical form of an M2P1T operator. Hence the influence of the quantities contained in the starting operator such as $f^{(k)}(0,\rho)$ are taken into account only through their effects on $\big(\vec{A}_0[p],\, \vec{B}_0[p']\big)$.

As a preparation for Proposition 1.1.2.1 below, we present the following Lemma 1.1.2.1, where we suppose $p \geq 4$ for the sake of the uniformity of expression. (Cf. Remark 1.1.2.4 below.)

Lemma 1.1.2.1. (i) $C_0^{(p+1)}(\rho)$ *has the following structure:*

$$
\begin{aligned}
(1.1.2.14) \quad C_0^{(p+1)}(\rho) = \Big[& \big(x_0^{(0)'}\big)^2 A_0^{(p-2)} \\
& + 2x_0^{(0)'} x_0^{(p-2)'} A_0^{(0)} + \big(x_0^{(0)'}\big)^2 x_0^{(p-1)} B_0^{(0)} \Big] \Big|_{t=0} \\
& + \mathcal{C}^{(p+1)} \Big(x_0^{(i)'} (i \leq p-3), x_0^{(j)} (j \leq p-2), \\
& A_0^{(k)} (k \leq p-3), B_0^{(l)} (l \leq p-3) \Big) \Big|_{t=0},
\end{aligned}
$$

where

$$
\begin{aligned}
(1.1.2.15) \qquad \mathcal{C}^{(p+1)} = & \sum_{\substack{i+j+k=p-2 \\ i,j,k \leq p-3}} x_0^{(i)'} x_0^{(j)'} A_0^{(k)} \\
& + \sum_{\substack{i+j+k+l=p-1 \\ 2 \leq k \leq p-2}} x_0^{(i)'} x_0^{(j)'} x_0^{(k)} B_0^{(l)} \\
& + \sum_{\substack{i+j+k=p+1 \\ i,j \geq 2;\, k \geq 1}} x_0^{(i)} x_0^{(j)} f^{(k)} - f^{(p-1)}.
\end{aligned}
$$

(ii) $D_0^{(p)}(\rho)$ *has the following structure:*

(1.1.2.16)

$$
\begin{aligned}
D_0^{(p)}(\rho) = \Big[& 2\tilde{x}_0^{(0)} x_0^{(0)\prime} x_0^{(p-2)\prime} B_0^{(0)} \\
& + \tilde{x}_0^{(0)} x_0^{(0)\prime 2} B_0^{(p-2)} + 2\tilde{x}_0^{(0)} f^{(1)} x_0^{(p-1)} \\
& + \big(x_0^{(0)\prime}\big)^2 x_0^{(p-2)\prime} B_0^{(0)} \Big]\Big|_{t=0} \\
& + \mathcal{D}^{(p)}\Big(x_0^{(i)\prime}, x_0^{(i')\prime\prime} (i, i' \le p-3), x_0^{(j)} (j \le p-2), \\
& A_0^{(k)} (k \le p-3), B_0^{(l)} (l \le p-3) \Big)\Big|_{t=0},
\end{aligned}
$$

where

(1.1.2.17)

$$
\begin{aligned}
\mathcal{D}^{(p)} = & \, \tilde{x}_0^{(0)} \Big(\sum_{\substack{i+j+l=p-2 \\ i,j,l \le p-3}} x_0^{(i)\prime} x_0^{(j)\prime} B_0^{(l)} \Big) + \tilde{x}_0^{(1)} \Big(\sum_{i+j+l=p-3} x_0^{(i)\prime} x_0^{(j)\prime} B_0^{(l)} \Big) \\
& + 2\tilde{x}_0^{(0)} \Big(\sum_{\substack{j+k=p \\ j,k \ge 2}} x_0^{(j)} f^{(k)} \Big) + 2\tilde{x}_0^{(1)} \Big(\sum_{\substack{j+k=p-1 \\ j \ge 2,\, k \ge 1}} x_0^{(j)} f^{(k)} \Big) \\
& + \sum_{\substack{i+j=p \\ i,j \ge 2}} x_0^{(i)} x_0^{(j)} \tilde{f}^{(0)} + 2 \sum_{i+j+k=p-3} x_0^{(i)\prime\prime} x_0^{(j)\prime} A_0^{(k)} \\
& + \sum_{\substack{i+j+k+l=p-2 \\ 2 \le k \le p-3}} x_0^{(k)\prime} x_0^{(i)\prime} x_0^{(j)\prime} B_0^{(l)} + 2 \sum_{\substack{i+j+k+l=p-2 \\ k \ge 2}} x_0^{(k)} x_0^{(i)\prime\prime} x_0^{(j)\prime} B_0^{(l)} \\
& + 2 \sum_{\substack{i+j+k=p \\ i,j \ge 2;\, k \ge 1}} x_0^{(i)\prime} x_0^{(j)} f^{(k)} + \sum_{\substack{i+j+k=p \\ i,j \ge 2;\, k \ge 1}} x_0^{(i)} x_0^{(j)} f^{(k)\prime} - f^{(p-2)\prime}.
\end{aligned}
$$

Remark 1.1.2.4. In our later reasoning we will basically use the pair of equations $C_0^{(p+1)} = 0$ and $D_0^{(p)} = 0$ to fix $\big(A_0^{(p-2)}, B_0^{(p-2)}\big)$. Hence for the convenience of the future reference we have listed the concrete form of $C_0^{(p+1)}$ and $D_0^{(p)}$, not $C_0^{(p)}$ and $D_0^{(p)}$. We also note that $\mathcal{C}^{(p+1)}$ and $\mathcal{D}^{(p)}$ will turn out to be "non-principal parts" in the computation in what follows, in the sense that only $\left(A_0^{(q)}, B_0^{(q)}\right)$ ($q \leq p-3$) are relevant to these parts. (See Remark 1.1.2.5 after Proposition 1.1.2.1 below.) From the experience in the previous subsection one might find the following term in the "principal parts" of $D_0^{(p)}$

$$(1.1.2.18) \qquad \left(x_0^{(0)'}\right)^2 x_0^{(p-2)'} B_0^{(0)}$$

to be somewhat unexpected. As a matter of fact this term originates from

$$(1.1.2.19) \qquad \frac{\partial}{\partial t}\Big(\sum_{\substack{i+j+k+l=p-2 \\ k \geq 2}} x_0^{(k)} x_0^{(i)'} x_0^{(j)'} B_0^{(l)} \Big),$$

and hence

$$(1.1.2.20) \qquad p-2 \geq 2, \text{ i.e.}, p \geq 4$$

is required for the appearance of this term. This is the reason why we did not encounter this term when $p = 3$. Thus for the sake of the uniformity of presentation we assume $p \geq 4$ in Proposition 1.1.2.1 below. At the same time we note that the term

$$(1.1.2.21) \qquad \tilde{x}_0^{(1)} \Big(\sum_{i+j+l=p-3} x_0^{(i)'} x_0^{(j)'} B_0^{(l)} \Big)\Big|_{t=0}$$

in the "non-principal part" $\mathcal{D}^{(p)}$ coincides with (1.1.2.18) evaluated at $t = 0$ when $p = 3$. Since $x_0^{(1)'}(0, \rho) = \tilde{x}_0^{(1)}(0, \rho)$, the term (1.1.2.21) had better been regarded as one of the principal terms when $p = 3$. This coincidence of terms peculiar to $p = 3$ explains why the "principal part" of (1.1.1.35) assumes the same form as that claimed in $\mathcal{A}_0(p)$ (vi) ($p \geq 4$) in Proposition 1.1.2.1 below; this fact might, at first, look somewhat puzzling in view of the absence of (1.1.2.18) in the "principal part" of (1.1.1.35).

Using these notations we now state the following

Proposition 1.1.2.1. *Let* $x_0^{(p)}(s,\rho)$ *be a solution of the equation* $[5.p]'$ (*listed below* (1.1.2.12)) *with subsidiary conditions*

$$C_0^{(p)} = D_0^{(p)} = 0. \tag{1.1.2.22}$$

Then the following set $\mathcal{A}_0(p)$ *of assertions* $\big(\mathcal{A}_0(p)(\mathrm{i}), \mathcal{A}_0(p)(\mathrm{ii}), \cdots, \mathcal{A}_0(p)(\mathrm{vi})\big)$ *is valid for every* $p \geq 4$.

$$\mathcal{A}_0(p): \begin{cases} \mathcal{A}_0(p)(\mathrm{i}): & x_0^{(p)}(s,\rho) \textit{ is holomorphic near } s=0, \\ \mathcal{A}_0(p)(\mathrm{ii}): & x_0^{(p)}(s,\rho) \textit{ depends on } \big(\vec{A}_0[p-1], \vec{B}_0[p]\big) \\ & = \big(A_0^{(0)}, A_0^{(1)}, \cdots, A_0^{(p-1)}, B_0^{(0)}, B_0^{(1)}, \cdots, B_0^{(p)}\big), \\ \mathcal{A}_0(p)(\mathrm{iii}): & x_0^{(p)}(0,\rho) \underset{(p-1)}{\equiv} A_0^{(p-1)}/B_0^{(0)}, \\ \mathcal{A}_0(p)(\mathrm{iv}): & \dfrac{dx_0^{(p)}}{ds}(0,\rho) \underset{(p)}{\equiv} -B_0^{(p)}/B_0^{(0)}, \\ \mathcal{A}_0(p)(\mathrm{v}): & C_0^{(p)} \underset{(p-3)}{\equiv} 2A_0^{(p-3)} - 2\dfrac{A_0^{(0)}}{B_0^{(0)}} B_0^{(p-3)}, \\ \mathcal{A}_0(p)(\mathrm{vi}): & D_0^{(p)} \underset{(p-2)}{\equiv} 2\dfrac{Z_0 A_0^{(0)}}{B_0^{(0)}} A_0^{(p-2)} - 2Z_0 B_0^{(p-2)} \end{cases}$$

Remark 1.1.2.5. The validity of $\mathcal{A}_0(p)(\mathrm{v})$ and $\mathcal{A}_0(p)(\mathrm{vi})$ justifies calling $\mathcal{C}^{(p)}$ and $\mathcal{D}^{(p)}$ "non-principal parts".

Proof of Proposition 1.1.2.1. [I] Let us first confirm $\mathcal{A}_0(4)$. As the argument for this case serves as a good specimen of the reasoning for the general case, we give it in a detailed manner. To begin with we summarize the results obtained in the precedent subsection. First, we know (i) the explicit form of the equation that $x_0^{(0)}(t,\rho)$ satisfies (cf. [5.0]′), (i′) the concrete form of $x_0^{(0)}(t,\rho)$ and (ii) $x_0^{(0)}(0,\rho)$ and $x_0^{(0)\prime}(0,\rho)$ (cf. (1.1.1.7), (1.1.1.8) and (1.1.1.23)); second, we know (i) the concrete form of the equation that $x_0^{(1)}(s,\rho)$ satisfies (cf. [5.1]″) and (ii) $x_0^{(1)}(0,\rho)$ and $\dot{x}_0^{(1)}(0,\rho)$ (cf. (1.1.1.11), (1.1.1.12) and (1.1.1.24)); third, we know (i) the explicit form of the equation that $x_0^{(2)}(s,\rho)$ satisfies (cf. [5.2]′ and (1.1.1.18)) and (ii) $x_0^{(2)}(0,\rho)$ (cf. (1.1.1.27)), and fourthly we present the explicit form of the equation that $x_0^{(3)}(s,\rho)$ satisfies (cf. [5.3]′). These results, among other things, guarantee the validity of $\mathcal{A}_0(q)$(ii) ($q \leq 3$). At the same time we notice that we have so far fixed $\left(A_0^{(1)}, B_0^{(1)}\right)$ (cf. (1.1.1.37) and (1.1.1.43)) to guarantee the holomorphy of $x_0^{(q)}(s,\rho)$ ($q \leq 3$) near $s = 0$. One important observation to be made is that holomorphic $x_0^{(3)}(s,\rho)$ exists for arbitrary constants $\left(A_0^{(2)}, B_0^{(2)}, B_0^{(3)}\right)$ at this stage; any constraints have not yet been imposed upon these constants on which $x_0^{(3)}(s,\rho)$ depends.

Now, to find a holomorphic solution $x_0^{(4)}$ of [5.4]′ we are to suppose $C_0^{(4)} = D_0^{(4)} = 0$. To find the explicit constraints on the parameters $\left(A_0^{(1)}, B_0^{(1)}\right)$ and others, we want to have concrete expressions of $C_0^{(4)}$ and $D_0^{(4)}$ which enable us to see their implications. The explicit computation of all terms in $C_0^{(p)}$ and $D_0^{(p)}$ is a laborious task, but the filtration with respect to p we are using facilitates our computation substantially. For example, the thorough computation of $\dot{x}_0^{(2)}(0,\rho)$ is considerably more arduous than that of $x_0^{(2)}(0,\rho)$, but the confirmation of $\mathcal{A}_0(2)$(iv) is a rather straightforward task; in the right-hand side of [5.2]′ all terms except for $-A_0^{(1)} - B_0^{(2)}s$ are expressed in terms of

(1.1.2.23) $\quad \{x_0^{(q)}(q = 0, 1)$ and their derivatives, $f^{(0)}, f^{(1)}$ and $f^{(2)}$, $A_0^{(0)}, B_0^{(0)}$ and $B_0^{(1)}\}$, and hence, thanks to $\mathcal{A}_0(q)$(ii) ($q = 0, 1$), we may ignore them in confirming $\mathcal{A}_0(2)$(iv). Similarly the confirmation of $\mathcal{A}_0(3)$(iii), which we need in confirming $\mathcal{A}_0(4)$(vi), is not difficult, if we note the following fact (C):

(C) If we set $s = 0$ in the right-hand side of $[5.3]'$, the remaining terms are free from $B_0^{(2)}$.

Clearly $\mathcal{A}_0(3)$(iii) follows from (C), and the fact (C) is a consequence of the following two facts (C.i) and (C.ii):

(C.i) $-2\dot{x}_0^{(2)}(0,\rho)A_0^{(0)}$ and $\phi_2(0,\rho)$ cancel out (cf. Remark 1.1.2.2),

(C.ii)

$$\Big[\sum_{\substack{j+k+l+m=3\\ j,k,l,m\le 2}} \dot{x}_0^{(j)}\dot{x}_0^{(k)}x_0^{(l)}B_0^{(m)}\Big]\Big|_{s=0}$$

$$= \Big[\sum_{\substack{j+k+l+m=3\\ l=2}} \dot{x}_0^{(j)}\dot{x}_0^{(k)}x_0^{(l)}B_0^{(m)}\Big]\Big|_{s=0}$$

$$= \Big[\sum_{j+k+m=1} \dot{x}_0^{(j)}\dot{x}_0^{(k)}B_0^{(0)-1}\big(A_0^{(1)} - f^{(2)}(0,\rho) + \chi_0^{(0)}B_0^{(0)}\big)B_0^{(m)}\Big]\Big|_{s=0},$$

which follows from (1.1.1.24) and (1.1.1.27). Since these terms are the only terms in the right-hand side of $[5.3]'$ that may contain $B_0^{(2)}$, (C.i) and (C.ii) entail (C). The disappearance of $B_0^{(p-1)}$ in the right-hand side of $[5.p]'$ is a universal phenomenon, as we will see below.

Using $\mathcal{A}_0(2)$(iv) and $\mathcal{A}_0(3)$(iii), which we have just confirmed, together with the results obtained in the preceding subsection, we can now confirm $\mathcal{A}_0(4)$(v) and $\mathcal{A}_0(4)$(vi). Let us first compute $C_0^{(4)}$. Then it follows from (1.1.2.14), (1.1.1.12) and (1.1.1.27) that

$$C_0^{(4)}(\rho) \underset{(1)}{\equiv} A_0^{(1)} + 2\dot{x}_0^{(1)}(0,\rho)A_0^{(0)} + x_0^{(2)}(0,\rho)B_0^{(0)} \tag{1.1.2.24}$$

$$\underset{(1)}{\equiv} 2A_0^{(1)} - 2\frac{A_0^{(0)}}{B_0^{(0)}}B_0^{(1)}.$$

This confirms $\mathcal{A}_0(4)(\text{v})$. To compute $D_0^{(4)}$ we apply $\mathcal{A}_0(2)(\text{iv})$ and $\mathcal{A}_0(3)(\text{iii})$ together with $\mathcal{A}_0(q)(\text{ii})$ ($q \leq 2$) to (1.1.2.16) to find

$$D_0^{(4)}(\rho) \underset{(2)}{\equiv} 2Z_0\Big(-\frac{B_0^{(2)}}{B_0^{(0)}}\Big)B_0^{(0)} + Z_0B_0^{(2)} \tag{1.1.2.25}$$

$$+2Z_0A_0^{(0)}\Big(\frac{A_0^{(2)}}{B_0^{(0)}}\Big)+Z_0\Big(-\frac{B_0^{(2)}}{B_0^{(0)}}\Big)B_0^{(0)}$$

$$= 2\frac{Z_0A_0^{(0)}}{B_0^{(0)}}A_0^{(2)} - 2Z_0B_0^{(2)}.$$

This validates $\mathcal{A}_0(4)(\text{vi})$.

These concrete expressions of the "top parts" of $C_0^{(4)}$ and $D_0^{(4)}$ tell us how the subsidiary conditions given by (1.1.2.22) (with $p = 4$) put new constraints on $\big(A_0^{(1)}, B_0^{(1)}, A_0^{(2)}, B_0^{(2)}\big)$. Now we know $\big(A_0^{(1)}, B_0^{(1)}\big)$. obeys the constraint (1.1.1.37) which may be summarized, in our current context, as follows:

$$2\frac{A_0^{(0)}}{B_0^{(0)}}A_0^{(1)} - 2B_0^{(1)} = \text{given data.} \tag{1.1.2.26}$$

Considering this equation simultaneously with $C_0^{(4)}(\rho) = 0$, we find that these two constraints are consistent, i.e., admit a simultaneous (unique) solution $\big(A_0^{(1)}, B_0^{(1)}\big)$ thanks to our assumption (1.1.2) (supplemented with (1.1.1.21) and (1.1.1.22)). Thus $\mathcal{A}_0(4)(\text{i})$ is valid, and then $\mathcal{A}_0(4)$(ii), $\mathcal{A}_0(4)$(iii) and $\mathcal{A}_0(4)$(iv) can be readily confirmed. In order to make our argument as concrete as possible, let us write down [5.4]′ explicitly:

[5.4]′

$$
\begin{aligned}
&B_0^{(0)}\Big(2s\frac{d}{ds}-1\Big)x_0^{(4)}(s,\rho)\\
&=-A_0^{(3)}-B_0^{(4)}s-\sum_{\substack{i+j+k=3\\ k\le 2}}\dot{x}_0^{(i)}\dot{x}_0^{(j)}A_0^{(k)}-\sum_{\substack{i+j+k+l=4\\ i,j,k,l\le 3}}\dot{x}_0^{(i)}\dot{x}_0^{(j)}x_0^{(k)}B_0^{(l)}\\
&\quad+\Big[\big(x_0^{(0)\prime}(t,\rho)\big)^{-2}\Big(\mathcal{E}^{(4)}+2\tilde{x}_0^{(1)}(t,\rho)\tilde{f}^{(0)}(t,\rho)x_0^{(3)}(t,\rho)\\
&\quad+\sum_{\substack{i+j+k=4\\ i,j=0,1\ k\ge 1}}\tilde{x}_0^{(i)}(t,\rho)\tilde{x}_0^{(j)}(t,\rho)f^{(k)}(t,\rho)\Big)\Big]\Big|_{t=t(s,\rho)},
\end{aligned}
$$

where

$$
\begin{aligned}
(1.1.2.27)\qquad \mathcal{E}^{(4)}=&\,t^{-2}\Big[\sum_{i+j+k=1}x_0^{(i)\prime}(t,\rho)x_0^{(j)\prime}(t,\rho)A_0^{(k)}\\
&+x_0^{(2)}(t,\rho)\big(x_0^{(0)\prime}(t,\rho)\big)^2B_0^{(0)}-f^{(2)}\\
&+t\tilde{x}_0^{(0)}(t,\rho)\Big(\sum_{i+j+l=2}x_0^{(i)\prime}(t,\rho)x_0^{(j)\prime}(t,\rho)B_0^{(l)}\Big)\\
&+t\tilde{x}_0^{(1)}(t,\rho)\Big(\sum_{i+j+l=1}x_0^{(i)\prime}(t,\rho)x_0^{(j)\prime}(t,\rho)B_0^{(l)}\Big)\\
&+2t\tilde{x}_0^{(0)}(t,\rho)\Big(\sum_{\substack{j+k=4\\ j\ge 2,\ k\ge 1}}x_0^{(j)}(t,\rho)f^{(k)}(t,\rho)\Big)\\
&+2t\tilde{x}_0^{(1)}(t,\rho)x_0^{(2)}(t,\rho)f^{(1)}(t,\rho)
\end{aligned}
$$

$$+\left(x_0^{(2)}(t,\rho)\right)^2 f^{(0)}(t,\rho)-\left(C_0^{(4)}+tD_0^{(4)}\right)\Big].$$

Since $x_0^{(4)}(s,\rho)$ is a (unique) holomorphic solution of $[5.4]'$, which has a regular singularity at $s=0$ with its characteristic index $1/2$, it suffices to examine the structure of each term in the right-hand side of $[5.4]'$ to find how $x_0^{(4)}(s,\rho)$ depends on the parameters. Since we have validated $\mathcal{A}_0(q)$(ii) ($q\le 3$), the explicit form of the right-hand side of $[5.4]'$ entails that $x_0^{(4)}(s,\rho)$ depends on $\big(\vec{A}_0[3],\vec{B}_0[4]\big)$. This means that $\mathcal{A}_0(4)$(ii) is confirmed. To validate $\mathcal{A}_0(4)$(iii), we need $\mathcal{A}_0(3)$(iv), which we have not yet checked; but its confirmation is a straightforward one, because all terms except for $-B_0^{(3)}s$ in the right-hand side of $[5.3]'$ are free from $B_0^{(3)}$ (and $A_0^{(3)}$). Exactly in parallel with the confirmation of $\mathcal{A}_0(3)$(iii), we then use the cancellation of $-2\dot{x}_0^{(3)}(0,\rho)A_0^{(0)}$ and $2\tilde{x}_0^{(0)}(0,\rho)\ f^{(1)}(0,\rho)\ x_0^{(3)\prime}(0,\rho)$ (cf. Remark 1.1.2.2) and the relation

(1.1.2.28)
$$\Big[\sum_{\substack{i+j+k+l=4\\ i,j,k,l\le 3}}\dot{x}_0^{(i)}\dot{x}_0^{(j)}x_0^{(k)}B_0^{(l)}\Big]\Big|_{s=0} =\Big[\sum_{\substack{i+j+k+l=4\\ k=2,3}}\dot{x}_0^{(i)}\dot{x}_0^{(j)}x_0^{(k)}B_0^{(l)}\Big]\Big|_{s=0},$$

which follows from (1.1.1.24). Here we clearly observe that the right-hand side of (1.1.2.28) is free from $B_0^{(3)}$. Then by checking indices of all terms in the right-hand side of $[5.4]'$ (including terms in $\mathcal{E}^{(4)}$) we use $\mathcal{A}_0(q)$(ii) ($q\le 3$) to conclude that the right-hand side of $[5.4]'$ evaluated at $s=0$ is independent of $B_0^{(3)}$. Thus we have confirmed $\mathcal{A}_0(4)$(iii). The confirmation of $\mathcal{A}_0(4)$(iv) is a straightforward one, because all terms except for $-B_0^{(4)}s$ in the right-hand side of $[5.4]'$ are free from $B_0^{(4)}$ (and $A_0^{(4)}$, which has not yet come into our discussion).

Thus we have confirmed $\big(\mathcal{A}_0(4)(\text{i}),\mathcal{A}_0(4)(\text{ii}),\cdots\mathcal{A}_0(4)(\text{vi})\big)$. In the course of the confirmation new parameters $\left(A_0^{(3)}, B_0^{(4)}\right)$ came into our discussion, whereas $\left(A_0^{(1)}, B_0^{(1)}\right)$ was fixed and one constraint $D_0^{(4)} = 0$ was imposed on $\left(A_0^{(2)}, B_0^{(2)}\right)$. Thus our reasoning enters the next stage with free parameters $\left(A_0^{(3)}, B_0^{(3)}, B_0^{(4)}\right)$, neither free nor fixed parameters $\left(A_0^{(2)}, B_0^{(2)}\right)$ (i.e., constants controlled by $D_0^{(4)} = 0$) and fixed constants $\left(A_0^{(q)}, B_0^{(q)}\right)$ $(q = 0, 1)$.

[II] Let us now suppose that $\mathcal{A}_0(p)$ $(4 \le p \le q)$ has been validated and show that $\mathcal{A}_0(q+1)$ is valid. To begin with, we note that in part [I] of this proof we have confirmed the following statements $(\mathcal{S}1)$, $(\mathcal{S}2)$ and $(\mathcal{S}3)$ besides our real target $\mathcal{A}_0(4)$.

$(\mathcal{S}1)$ $\mathcal{A}_0(p)$(i) and $\mathcal{A}_0(p)$(ii) are valid for $0 \le p \le 3$ (with the conventional understanding that $A_0^{(-1)} = 0$).

$(\mathcal{S}2)$ $\mathcal{A}_0(p)$(iii) is valid for $p = 2, 3$, and $\mathcal{A}_0(p)$(iv) is valid for $1 \le p \le 3$.

$(\mathcal{S}2)$ $\left(A_0^{(1)}, B_0^{(1)}\right)$ is fixed.

We also note that $\left(A_0^{(0)}, B_0^{(0)}\right)$ has been fixed in Section 1.1.1.

It then follows from $(\mathcal{S}1)$ that the right-hand side of $[5.q{+}1]'$ depends on $\left(\vec{A}_0[q], \vec{B}_0[q+1]\right)$. On the other hand, the conditions $C_0^{(q+1)} = D_0^{(q+1)} = 0$ guarantee the unique existence of holomorphic solution $x_0^{(q+1)}(s, \rho)$ of $[5.q+1]'$. Hence $\mathcal{A}_0(q+1)$(i) and $\mathcal{A}_0(q+1)$(ii) are valid on the condition that $C_0^{(q+1)} = D_0^{(q+1)} = 0$ are consistent with previously imposed constraints on $\left(\vec{A}_0[q], \vec{B}_0[q+1]\right)$. In parallel with the reasoning in part [I] it suffices to confirm $\mathcal{A}_0(q+1)$(v) and $\mathcal{A}_0(q+1)$(vi); $\mathcal{A}_0(q+1)$(v) combined with $\mathcal{A}_0(q+1)$(vi) shows the existence of constants $\left(A_0^{(q-2)}, B_0^{(q-2)}\right)$ that satisfy $D_0^{(q)} = C_0^{(q+1)} = 0$, with the help of the assumption (1.1.2). Parenthetically $\mathcal{A}_0(q+1)$(vi) describes the constraint upon $\left(A_0^{(q-1)}, B_0^{(q-1)}\right)$, which will be used to fix them at the next stage. On the other hand, the confirmation of $\mathcal{A}_0(q+1)$(v) and $\mathcal{A}_0(q+1)$(vi) is readily done by

(α) applying $\mathcal{A}_0(p)$(ii) $(0 \leq p \leq q)$, $\mathcal{A}_0(p)$(iii) $(2 \leq p \leq q)$ and $\mathcal{A}_0(p)$(iv) $(1 \leq p \leq q-1)$ to (1.1.2.14), and

(β) applying $\mathcal{A}_0(p)$(ii) $(0 \leq p \leq q)$, $\mathcal{A}_0(p)$(iii) $(2 \leq p \leq q)$ and $\mathcal{A}_0(p)$(iv) $(1 \leq p \leq q-1)$ to (1.1.2.16).

By way of parenthesis the counterpart of $\mathcal{A}_0(p)$(iii) $(p = 0, 1)$ (resp., $\mathcal{A}_0(0)$(iv)) is given by $x_0^{(0)}(0,\rho) = x_0^{(1)}(0,\rho) = 0$ (resp., $\dot{x}_0^{(0)}(0,\rho) = 1$), which are used in the above confirmation.

Thus what remains to be confirmed is $\bigl(\mathcal{A}_0(q+1)(\text{iii}), \mathcal{A}_0(q+1)(\text{iv})\bigr)$. Using the explicit form of $[5.q+1]'$ together with $\mathcal{A}_0(p)$(ii) $(0 \leq p \leq q)$, we immediately find $\mathcal{A}_0(q+1)$(iv). To validate $\mathcal{A}_0(q+1)$(iii), we use the setoff between $-2\dot{x}_0^{(q)}(0,\rho)\, A_0^{(0)}$ and $2\tilde{x}_0^{(0)}(0,\rho)\, f^{(1)}(0,\rho)\, x_0^{(q)\prime}(0,\rho)$ together with the relation

(1.1.2.29)
$$\Bigl[\sum_{\substack{i+j+k+l=q+1\\ i,j,k,l\leq q}} \dot{x}_0^{(i)}\dot{x}_0^{(j)}x_0^{(k)}B_0^{(l)}\Bigr]\Big|_{s=0} = \Bigl[\sum_{\substack{i+j+k+l=q+1\\ 2\leq k\leq q}} \dot{x}_0^{(i)}\dot{x}_0^{(j)}x_0^{(k)}B_0^{(l)}\Bigr]\Big|_{s=0}.$$

Thus exactly the same reasoning used to confirm $\mathcal{A}_0(4)$(iii) shows that $\mathcal{A}_0(q+1)$(iii) is valid. Thus we have confirmed $(\mathcal{A}_0(q+1)(\text{i}), \mathcal{A}_0(q+1)(\text{ii}), \cdots, \mathcal{A}_0(q+1)(\text{vi}))$, and hence the induction proceeds.

□

1.1.3 Formal construction of $\{x_n^{(p)}, A_n^{(p)}, B_n^{(p)}\}_{p,n\geq 0}$ —— the case where $g_\pm(t) = 0$

Although the reasoning in the previous subsection is natural and instructive, the setting employed there is somewhat clumsy, particularly when we want to estimate the growth order of $\{x_0^{(p)}, A_0^{(p)}, B_0^{(p)}\}_{p\geq 0}$.

The primary purpose of this subsection is to present a more refined induction procedure for the construction of $\{x_0^{(p)}, A_0^{(p)}, B_0^{(p)}\}_{p\geq 0}$. We later (in Proposition 1.1.3.2) confirm that the procedure works for the construction of $\{x_n^{(p)}, A_n^{(p)}, B_n^{(p)}\}_{p,n\geq 0}$ which are used to transform an M2P1T equation to its canonical form; for the sake of simplicity of reasoning we assume $g_\pm(t) = 0$ in this subsection. In what follows, $x_0^{(0)}(s,\rho)$ denotes the holomorphic function given by (1.1.1.6) and $x_0^{(1)}(s,\rho)$ is the holomorphic solution of $[5.1]''$ satisfying the condition (1.1.1.24) , that is,

$$x_0^{(1)}(0,\rho) = 0. \tag{1.1.3.1}$$

The constants $A_0^{(0)}$ and $B_0^{(0)}$ are those satisfying (1.1.1.21) and (1.1.1.22), respectively and $(A_0^{(1)}, B_0^{(1)})$ designates a common solution of (1.1.1.37) and (1.1.1.43): In this subsection we conventionally understand that the relations $C_0^{(3)}(A_0^{(0)}, B_0^{(0)}) = D_0^{(2)}(A_0^{(0)}, B_0^{(0)}) = 0$ and $C_0^{(4)}(A_0^{(1)}, B_0^{(1)}) = D_0^{(3)}(A_0^{(1)}, B_0^{(1)}) = 0$ respectively mean the relations that $(A_0^{(0)}, B_0^{(0)})$ and $(A_0^{(1)}, B_0^{(1)})$ satisfy. We also understand $C_0^{(2)}(A_0^{(-1)}, B_0^{(-1)}) = 0$ to be an empty condition, which is a reflection of the fact that [5.2] is free from the constant term. By way of parenthesis we note that $D_0^{(3)}(A_0^{(1)}, B_0^{(1)}) = 0$ is well-defined (i.e., without any extra convention) as is given by (1.1.1.37) despite the seeming ambiguity in separating out its "principal part" (cf. Remark 1.1.2.4). Similarly $C_0^{(p+1)}$ with $p = 3$ given by (1.1.2.14) is coincident with (1.1.1.43).

In order to present the refined induction procedure we prepare some notations and auxiliary results. We use the symbol $\mathfrak{A}_0(p)$ to mean the assertion that a triplet of data $T_0^{(r)} = \{x_0^{(r)}(s,\rho),\ A_0^{(r)}, B_0^{(r)}\}$ is given for $0 \leq r \leq p$ so that they satisfy the following conditions:

(1.1.3.2.r) $x_0^{(r)}(s,\rho)$ is a holomorphic solution of $[5.r]'$ (to be found below (1.1.2.12)) near $s = 0$, (1.1.3.3.r) $x_0^{(r)}(s,\rho)$ depends on $(\vec{A}_0[r-1], \vec{B}_0[r]) = (A_0^{(0)}, A_0^{(1)}, \cdots, A_0^{(r-1)}, B_0^{(0)}, B_0^{(1)}, \cdots, B_0^{(r)})$,

(1.1.3.4.r) $C_0^{(r+3)}(\rho)$ and $D_0^{(r+2)}(\rho)$ depend on $(\vec{A}_0[r], \vec{B}_0[r])$, and $(\vec{A}_0[r], \vec{B}_0[r])$ satisfies the relations $C_0^{(r+3)}(\rho) = D_0^{(r+2)}(\rho) = 0$,

(1.1.3.5.r)
$$C_0^{(r+3)}(\rho) \underset{(r)}{\equiv} 2A_0^{(r)} - 2\frac{A_0^{(0)}}{B_0^{(0)}} B_0^{(r)},$$

(1.1.3.6.r)
$$D_0^{(r+2)}(\rho) \underset{(r)}{\equiv} 2Z_0 A_0^{(r)} \frac{A_0^{(0)}}{B_0^{(0)}} - 2Z_0 B_0^{(r)}.$$

We will show later in Proposition 1.1.3.1 that $\mathfrak{A}_0(p)$ entails $\mathfrak{A}_0(p+1)$.

Remark 1.1.3.1. A main difference of the contents of $\mathcal{A}_0(p)$ and $\mathfrak{A}_0(p)$ is that $\mathfrak{A}_0(p)$ refers to the structure of $C_0^{(r+3)}(\rho)$ and $D_0^{(r+2)}(\rho)$ for $r \le p-1$; in view of Lemma 1.1.2.1 one might be puzzled with the appearance of $x_0^{(p+1)}(0,\rho)$ in the expression of $C_0^{(p+3)}(\rho)$ and $D_0^{(p+2)}(\rho)$. As Lemma 1.1.3.3 and Lemma 1.1.3.4 below show, $x_0^{(p+1)}(0,\rho)$ can be written down in terms of $\{T_0^{(r)}\}_{0\le r\le p}$ and $x_0^{(p+1)}(0,\rho) - A_0^{(p)}/B_0^{(0)}$ is free from $A_0^{(p)}$ and $B_0^{(p)}$. These facts are implicitly woven into conditions (1.1.3.4.r), (1.1.3.5.r) and (1.1.3.6.r). The reader will find the mechanism in the proof of Proposition 1.1.3.1, where conditions (1.1.3.5.r) and (1.1.3.6.r) are confirmed for $r = p+1$.

In proving Lemma 1.1.3.1 $\sim$ Lemma 1.1.3.4 below we assume that $\mathfrak{A}_0(p)$ $(p \ge 1)$ has been validated.

Lemma 1.1.3.1. *The right-hand side of* $[5.p+1]'$ $(p \ge 1)$ *has the following form:*

(1.1.3.7)
$$-A_0^{(p)} - B_0^{(p+1)} s + B_0^{(0)} R_0^{(p+1)}(s,\rho),$$

where $B_0^{(p+1)}$ *is a complex number and*

(1.1.3.8)

$$B_0^{(0)}R_0^{(p+1)}(s,\rho) = -\sum_{\substack{i+j+k=p\\ k\le p-1}} \dot{x}_0^{(i)}\dot{x}_0^{(j)}A_0^{(k)} - \sum_{\substack{i+j+k+l=p+1\\ i,j,k,l\le p}} \dot{x}_0^{(i)}\dot{x}_0^{(j)}x_0^{(k)}B_0^{(l)}$$

$$+\Big[\big(x_0^{(0)\prime}(t,\rho)\big)^{-2}t^{-2}\Big(\sum_{i+j+k=p-2} x_0^{(i)\prime}x_0^{(j)\prime}A_0^{(k)}$$

$$+\sum_{i+j+k+l=p-1} x_0^{(i)\prime}x_0^{(j)\prime}x_0^{(k)}B_0^{(l)} + \sum_{\substack{i+j+k=p+1\\ k\ge 1}} x_0^{(i)}x_0^{(j)}f^{(k)}$$

$$+\sum_{\substack{i+j=p+1\\ i,j\ge 1}} x_0^{(i)}x_0^{(j)}f^{(0)} - f^{(p-1)}\Big)\Big]\Big|_{t=t(s,\rho)}.$$

Remark 1.1.3.2. The factor $B_0^{(0)}$ in front of $R_0^{(p+1)}(s,\rho)$ is a rather conventional one; it will turn out to be notationally convenient when we estimate the growth order of $x_0^{(p)}(s,\rho)$ etc. with an emphasis on their ρ-dependence. Recall that $B_0^{(0)} = \pm\rho$ holds by (1.1.1.22′).

Proof of Lemma 1.1.3.1*.* Since $C_0^{(p+1)}(\rho) = D_0^{(p+1)}(\rho) = 0$ holds by the assumption, we can read off the above result immediately from (1.1.1.5.$p+1$) in view of the definition of [5.$p+1$]′. We only note that we have shifted

$$2\big(x_0^{(0)\prime}(t,\rho)\big)^{-2}\tilde{x}_0^{(0)}(t,\rho)x_0^{(p+1)}(t,\rho)\tilde{f}^{(0)}(t,\rho)\Big|_{t=t(s,\rho)} \tag{1.1.3.9}$$

to the left-hand side of [5.$p+1$]′; we have left

$$\Big[\big(x_0^{(0)\prime}(t,\rho)\big)^{-2}t^{-2}\Big(\sum_{\substack{i+j=p+1\\ i,j\ge 1}} x_0^{(i)}x_0^{(j)}f^{(0)}\Big)\Big]\Big|_{t=t(s,\rho)} \tag{1.1.3.10}$$

in $B_0^{(0)} R_0^{(p+1)}(s,\rho)$ despite the fact that a term similar to (1.1.3.9), i.e.,

$$2\big(x_0^{(0)\prime}\big)^{-2} \tilde{x}_0^{(1)} x_0^{(p)} \tilde{f}^{(0)}\Big|_{t=t(s,\rho)}, \tag{1.1.3.11}$$

is contained in the sum (1.1.3.10); this non-uniformity of treatment is just due to the convention that the left-hand of $[5.p+1]'$ should contain only the (at this level) unknown function $x_0^{(p+1)}(s,\rho)$ and that its right-hand side should consist of given data.

□

Lemma 1.1.3.2. *The function $R_0^{(p+1)}(s,\rho)$ is determined by $\{T_0^{(r)}\}_{0\le r\le p}$, and it is free from $A_0^{(p)}$.*

Proof. This is an immediate consequence of the concrete expression (1.1.3.8) of $R_0^{(p+1)}(s,\rho)$.

□

Lemma 1.1.3.3. (i) *For an arbitrary complex number $B_0^{(p+1)}$ we find a unique holomorphic solution $x_0^{(p+1)}(s,\rho)$ near $s=0$ of the following equation $[5.p+1]'$:*

(1.1.3.12) $(=[5.p+1]')$

$$B_0^{(0)}\big(2s\frac{d}{ds}-1\big)x_0^{(p+1)}(s,\rho) = -A_0^{(p)} - B_0^{(p+1)}s + B_0^{(0)}R_0^{(p+1)}(s,\rho).$$

(ii) *The solution $x_0^{(p+1)}(s,\rho)$ depends on $(\vec{A}_0[p], \vec{B}_0[p+1])$.*

(iii) *For the above solution $x_0^{(p+1)}(s,\rho)$ we find*

$$B_0^{(0)} x_0^{(p+1)}(0,\rho) = A_0^{(p)} - B_0^{(0)} R_0^{(p+1)}(0,\rho) \tag{1.1.3.13}$$

and

$$B_0^{(0)} \dot{x}_0^{(p+1)}(0,\rho) = -B_0^{(p+1)} + B_0^{(0)} \dot{R}_0^{(p+1)}(0,\rho). \tag{1.1.3.14}$$

Proof. (i) Since $C_0^{(p+1)}(\rho) = D_0^{(p+1)}(\rho) = 0$ holds by the assumption, and since $B_0^{(0)}$ is different from 0 by the assumption (1.1.1) together with the relation (1.1.1.22′), the unique existence of a holomorphic solution of $[5.p+1]'$ is evident.
(ii) This immediately follows from Lemma 1.1.3.2 (on the condition that $\mathfrak{A}_0(p)$ is valid).
(iii) By setting $s = 0$ in (1.1.3.12), we readily obtain (1.1.3.13). By first differentiating both sides of (1.1.3.12) and then setting $s = 0$, we obtain (1.1.3.14).

$\square$

Remark 1.1.3.3. It is clear that relations similar to (1.1.3.13) and (1.1.3.14) hold for any holomorphic solution $x_0^{(q)}(s,\rho)$ of the following equation
(1.1.3.15)

$$B_0^{(0)}\big(2s\frac{d}{ds} - 1\big)x_0^{(q)}(s,\rho) = -A_0^{(q-1)} - B_0^{(q)}s + B_0^{(0)}R_0^{(q)}(s,\rho),$$

where $A_0^{(q-1)}$ and $B_0^{(q)}$ are complex numbers and $R_0^{(q)}(s,\rho)$ is holomorphic near $s = 0$; that is, we have

$$B_0^{(0)}x_0^{(q)}(0,\rho) = A_0^{(q-1)} - B_0^{(0)}R_0^{(q)}(0,\rho) \tag{1.1.3.16}$$

and

$$B_0^{(0)}\dot{x}_0^{(q)}(0,\rho) = -B_0^{(q)} + B_0^{(0)}\dot{R}_0^{(q)}(0,\rho). \tag{1.1.3.17}$$

Lemma 1.1.3.4. *The value $B_0^{(0)}R_0^{(p+1)}(0,\rho)$ is free from $B_0^{(p)}$.*

Proof. When $p = 0$, [5,1]″ together with (1.1.1.21) entails that $B_0^{(0)}R_0^{(1)}(0,\rho)$ coincides with $A_0^{(0)}$; thus it is free from $B_0^{(0)}$. Hence we assume $p \geq 1$ in the discussion below. It then follows from (1.1.3.3.r) that $x_0^{(r)}(s,\rho)$ $(0 \leq r \leq p-1)$ is free from $B_0^{(p)}$. Hence the terms in $R_0^{(p+1)}(s,\rho)$ whose relevance we have to check are those containing $B_0^{(p)}$, $x_0^{(p)}$ or $\dot{x}_0^{(p)}$. Furthermore, (1.1.3.16) with $q = p$ guarantees that it suffices to concentrate our attention on terms containing $B_0^{(p)}$ or $\dot{x}_0^{(p)}$. Thus the terms to be checked are the following:

$$-\Big(\sum_{i+j+k=1} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)x_0^{(k)}(0,\rho)\Big)B_0^{(p)}, \tag{1.1.3.18}$$

$$\begin{aligned} &-\Big(\sum_{\substack{i+j=p+1\\ i,j\leq p}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big)x_0^{(0)}(0,\rho)B_0^{(0)} \\ &-\Big(\sum_{i+j=p} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big)\Big(\sum_{k+l=1} x_0^{(k)}(0,\rho)B_0^{(l)}\Big), \end{aligned} \tag{1.1.3.19}$$

$$-2\dot{x}_0^{(0)}(0,\rho)\dot{x}_0^{(p)}(0,\rho)A_0^{(0)} \tag{1.1.3.20}$$

and terms in the coefficients of the Taylor expansion of

$$\Big[\big(x_0^{(0)\prime}\big)^{-2}t^{-2}\big(2x_0^{(0)}x_0^{(p)}f^{(1)} + 2x_0^{(1)}x_0^{(p)}f^{(0)}\big)\Big]\Big|_{t=t(s,\rho)}. \tag{1.1.3.21}$$

Here we encounter a situation essentially the same as that observed in the fact (C) used for the confirmation of $\mathcal{A}_0(3)$(iii) in the proof of Proposition 1.1.2.1. First the important relation (1.1.3.1) together with (1.1.1.7), i.e., $x_0^{(0)}(0,\rho) = 0$, entails the vanishing of each term in the sum (1.1.3.18) and the sum (1.1.3.19); this reasoning corresponds to (C.ii). Second, (1.1.3.20) is cancelled out by the term

$$2\big(x_0^{(0)\prime}\big)^{-2}\tilde{x}_0^{(0)}x_0^{(p)\prime}f^{(1)}\Big|_{t=t(0,\rho)} = 2\dot{x}_0^{(p)}(0,\rho)f^{(1)}(0,\rho) = 2\dot{x}_0^{(p)}(0,\rho)A_0^{(0)}, \tag{1.1.3.22}$$

which originates from

$$\Big[\big(x_0^{(0)\prime}\big)^{-2}t^{-2}\big(2x_0^{(0)}x_0^{(p)}f^{(1)}\big)\Big]\Big|_{t=t(s,\rho)}. \tag{1.1.3.23}$$

This fact corresponds to (C.i). We note that the contribution from

$$\Big[\big(x_0^{(0)'}\big)^{-2} t^{-2}\big(2x_0^{(1)}x_0^{(p)}f^{(0)}\big)\Big]\Big|_{t=t(s,\rho)}. \tag{1.1.3.24}$$

is

$$2\big(x_0^{(0)'}(0,\rho)\big)^{-2}\tilde{x}_0^{(1)}(0,\rho)\tilde{f}^{(0)}(0,\rho)x_0^{(p)}(0,\rho); \tag{1.1.3.25}$$

thus this part is irrelevant to $B_0^{(p)}$. This completes the proof of Lemma 1.1.3.4.

□

So far we have constructed a holomorphic solution $x_0^{(p+1)}(s,\rho)$ of $[5.p+1]'$ by using the data given in $\mathfrak{A}_0(p)$ together with a newly added arbitrary complex number $B_0^{(p+1)}$. Since

$$C_0^{(p+3)}(\rho) = D_0^{(p+2)}(\rho) = 0 \tag{1.1.3.26}$$

is contained in the assertion $\mathfrak{A}_0(p)$, the equation $[5.p+2]'$ is given by

$$B_0^{(0)}\big(2s\frac{d}{ds}-1\big)x_0^{(p+2)}(s,\rho) = -A_0^{(p+1)} - B_0^{(p+2)}s + B_0^{(0)}R_0^{(p+2)}(s,\rho), \tag{1.1.3.27}$$

where $A_0^{(p+1)}$ and $B_0^{(p+2)}$ are newly added arbitrary complex numbers and $R_0^{(p+2)}(s,\rho)$ is given by replacing p with $p+1$ in (1.1.3.8). Note that $x_0^{(p+1)}(s,\rho)$ and $B_0^{(p+1)}$ are available at this stage. Furthermore, by using exactly the same reasoning as in the proof of Lemma 1.1.3.4, we find

(1.1.3.28) $R_0^{(p+2)}(0,\rho)$ is free from ($A_0^{(p+1)}$ and) $B_0^{(p+1)}$.

For the sake of the completeness of the reasoning we note that no condition on $A_0^{(p)}$ and $B_0^{(p)}$ are used in the proof of Lemma 1.1.3.4.

We are now ready to prove the following

Proposition 1.1.3.1. *The assertion $\mathfrak{A}_0(p)$ is valid for every $p \geq 1$.*

Proof. As we have confirmed the validity of $\mathfrak{A}_0(1)$ in previous subsections, it suffices to validate $\mathfrak{A}_0(p+1)$ supposing that $\mathfrak{A}_0(p)$ is valid. (It is possible to start the induction from $p = 0$, but to avoid the use of conventional interpretation of the symbol such as $D_0^{(2)}$ we have started from $p = 1$.) As we have seen above, we have constructed $x_0^{(p+1)}(s,\rho)$ that satisfied (1.1.3.2.$p+1$) and (1.1.3.3.$p+1$) by incorporating an a priori arbitrary complex number $B_0^{(p+1)}$ with the given data. Furthermore the condition (1.1.3.26) contained in $\mathfrak{A}_0(p)$ enables us to find the equation (1.1.3.27) for $x_0^{(p+2)}(s,\rho)$, where $A_0^{(p+1)}$ and $B_0^{(p+2)}$ are a priori arbitrary complex numbers and $B_0^{(p+1)}$ and $x_0^{(p+1)}(s,\rho)$ are used to define $R_0^{(p+2)}(s,\rho)$. Thus what we have to do for confirming $\mathfrak{A}_0(p+1)$ is to show (1.1.3.5.$p+1$) and (1.1.3.6.$p+1$) and to prove that $\big(A_0^{(p+1)}, B_0^{(p+1)}\big)$ can be chosen so that

$$C_0^{(p+4)}(\rho) = D_0^{(p+3)}(\rho) = 0 \tag{1.1.3.29}$$

may be satisfied. Meanwhile, once we confirm (1.1.3.5.$p+1$) and (1.1.3.6.$p+1$), we can readily solve (1.1.3.29) to fix $\big(A_0^{(p+1)}, B_0^{(p+1)}\big)$ thanks to the assumption (1.1.2) combined with (1.1.1.21) and (1.1.1.22). To confirm (1.1.3.5.$p+1$) and (1.1.3.6.$p+1$) we substitute (1.1.3.14) and (1.1.3.16) with $q = p+2$ into (1.1.2.14) and (1.1.2.16) . Then the required results follow from (1.1.3.3.r) ($r \leq p+1$) together with (1.1.3.13). As the reasoning is the same for (1.1.3.5.p+1) and (1.1.3.6.p+1), we show the reasoning for $C_0^{(p+4)}(\rho)$. By substituting (1.1.3.14) and (1.1.3.16) (with $q = p+2$) into (1.1.2.14) we find the following:

$$\begin{aligned} C_0^{(p+4)}(\rho) &= A_0^{(p+1)} + 2\frac{A_0^{(0)}}{B_0^{(0)}}\big(-B_0^{(p+1)} + B_0^{(0)}\dot{R}_0^{(p+1)}(0,\rho)\big) \\ &\quad + \big(A_0^{(p+1)} - B_0^{(0)}R_0^{(p+2)}(0,\rho)\big) \end{aligned} \tag{1.1.3.30}$$

$$+\,\mathcal{C}^{(p+4)}\big(x_0^{(i)\prime}(i\le p), x_0^{(j)}(j\le p+1),$$
$$A_0^{(k)}(k\le p), B_0^{(l)}(l\le p)\big)\Big|_{t=0}.$$

Then it follows from (1.1.3.28) and the structure of $\mathcal{C}^{(p+4)}\big|_{t=0}$ supplemented by (1.1.3.13) that

$$2A_0^{(0)}\dot{R}_0^{(p+1)}(0,\rho) - B_0^{(0)}R_0^{(p+2)}(0,\rho) + \mathcal{C}_0^{(p+4)}\Big|_{t=0} \tag{1.1.3.31}$$

is free from $A_0^{(p+1)}$ and $B_0^{(p+1)}$; it depends on $\big(\vec{A}_0[p], \vec{B}_0[p]\big)$ by (1.1.3.3.r) ($r\le p$). Thus we find

$$C_0^{(p+4)} \underset{(p+1)}{\equiv} 2A_0^{(p+1)} - 2\,\frac{A_0^{(0)}}{B_0^{(0)}}B_0^{(p+1)}. \tag{1.1.3.5.$p+1$}$$

As we have noted earlier, we can readily find $(A_0^{(p+1)}, B_0^{(p+1)})$ that annihilates $C_0^{(p+4)}(\rho)$ and $D_0^{(p+3)}(\rho)$ by their expressions (1.1.3.5.$p+1$) and (1.1.3.6.$p+1$). Thus we obtain the required triplet $T_0^{(p+1)} = \big\{x_0^{(p+1)}(s,\rho),\ A_0^{(p+1)},\ B_0^{(p+1)}\big\}$. Therefore $\mathfrak{A}_0(p+1)$ is validated, and the induction proceeds.

□

Next we study how the construction of triplets $T_l^{(r)} = \big\{x_l^{(r)}(s,\rho),$ $A_l^{(r)},\ B_l^{(r)}\big\}$ $(l, r\ge 0)$ are done. In what follows we use the symbol

$$\{x;t\}_n^{(p)} \tag{1.1.3.32}$$

to denote the coefficient of $a^p\eta^{-n}$ of the expansion of $\{x;t\}$, that is, (1.1.3.33) $$\{x;t\} = \sum_{p,n\geq 0} \{x;t\}_n^{(p)} a^p \eta^{-n}.$$

We eventually need more explicit description of $\{x;t\}$ in terms of the derivatives of $x_l^{(r)}$, but it suffices to use this simplified symbol for the time being.

First we note (1.1.6) with $g_\pm = 0$ entails

(1.1.3.34)
$$(x^2-a^2)f = (t^2-a^2)\Big(\frac{\partial x}{\partial t}\Big)^2(aA+xB) - \frac{1}{2}\eta^{-2}(t^2-a^2)(x^2-a^2)\{x;t\}.$$

Since

(1.1.3.35)
$$x_{2\nu+1}(t,a,\rho) = A_{2\nu+1}(a,\rho) = B_{2\nu+1}(a,\rho) = 0 \quad (\nu = 0,1,2,\cdots)$$

holds by Proposition A.1 in Appendix A, we then find the following relation (1.1.3.36) for $n \geq 1$ by the comparison of the coefficients of η^{-2n} of (1.1.3.34) :

(1.1.3.36)
$$\begin{aligned}&\Big(\sum_{i+j=n} x_{2i}x_{2j}\Big)f \\ &= (t^2-a^2)\Big(\sum_{i+j+k=n} x'_{2i}x'_{2j}aA_{2k} + \sum_{i+j+k+l=n} x'_{2i}x'_{2j}x_{2k}B_{2l}\Big) \\ &\quad - \frac{1}{2}(t^2-a^2)\sum_{\substack{i+j+k=n-1 \\ r\geq 0}} x_{2i}x_{2j}\{x;t\}_{2k}^{(r)}a^r\end{aligned}$$

$$+\frac{1}{2}(t^2-a^2)a^2\Big(\sum_{r\geq 0}\{x;t\}^{(r)}_{2(n-1)}a^r\Big).$$

Expanding (1.1.3.36) in powers of a and comparing the coefficients of a^p, we obtain

(1.1.3.37)

$$\sum_{\substack{q+r+u=p\\ i+j=n}} x^{(q)}_{2i}x^{(r)}_{2j}f^{(u)}$$

$$= t^2\Big[\sum_{\substack{q+r+u=p-1\\ i+j+k=n}} x^{(q)\prime}_{2i}x^{(r)\prime}_{2j}A^{(u)}_{2k} + \sum_{\substack{q+r+u+v=p\\ i+j+k+l=n}} x^{(q)\prime}_{2i}x^{(r)\prime}_{2j}x^{(u)}_{2k}B^{(v)}_{2l}$$

$$-\frac{1}{2}\sum_{\substack{q+r+u=p\\ i+j+k=n-1}} x^{(q)}_{2i}x^{(r)}_{2j}\{x;t\}^{(u)}_{2k} + \frac{1}{2}\{x;t\}^{(p-2)}_{2(n-1)}\Big]$$

$$-\Big[\sum_{\substack{q+r+u=p-3\\ i+j+k=n}} x^{(q)\prime}_{2i}x^{(r)\prime}_{2j}A^{(u)}_{2k} + \sum_{\substack{q+r+u+v=p-2\\ i+j+k+l=n}} x^{(q)\prime}_{2i}x^{(r)\prime}_{2j}x^{(u)}_{2k}B^{(v)}_{2l}$$

$$-\frac{1}{2}\sum_{\substack{q+r+u=p-2\\ i+j+k=n-1}} x^{(q)}_{2i}x^{(r)}_{2j}\{x;t\}^{(u)}_{2k} + \frac{1}{2}\{x;t\}^{(p-4)}_{2(n-1)}\Big].$$

Let us now define $\Phi^{(p)}_{2n}$ and $\Psi^{(p)}_{2n}$ by the following:

(1.1.3.38) $$\Phi^{(p)}_{2n} = \sum_{\substack{q+r+u=p-3\\ i+j+k=n}} x^{(q)\prime}_{2i}x^{(r)\prime}_{2j}A^{(u)}_{2k} + \sum_{\substack{q+r+u+v=p-2\\ i+j+k+l=n}} x^{(q)\prime}_{2i}x^{(r)\prime}_{2j}x^{(u)}_{2k}B^{(v)}_{2l}$$

$$+\Big(\sum_{\substack{q+r+u=p\\ i+j=n}} x^{(q)}_{2i}x^{(r)}_{2j}f^{(u)} - 2x^{(0)}_0x^{(p)}_{2n}f^{(0)}\Big)$$

$$- \frac{1}{2} \sum_{\substack{q+r+u=p-2 \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x;t\}_{2k}^{(u)} + \frac{1}{2}\{x;t\}_{2(n-1)}^{(p-4)},$$

(1.1.3.39)

$$\Psi_{2n}^{(p)} = \sum_{\substack{q+r+u=p-1 \\ i+j+k=n}} x_{2i}^{(q)\prime} x_{2j}^{(r)\prime} A_{2k}^{(u)} + \sum_{\substack{q+r+u+v=p \\ i+j+k+l=n}} x_{2i}^{(q)\prime} x_{2j}^{(r)\prime} x_{2k}^{(u)} B_{2l}^{(v)}$$

$$- 2\tilde{x}_0^{(0)} \tilde{f}^{(0)} x_{2n}^{(p)} - \frac{1}{2} \sum_{\substack{q+r+u=p \\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x;t\}_{2k}^{(u)} + \frac{1}{2}\{x;t\}_{2(n-1)}^{(p-2)}.$$

Remark 1.1.3.4. The separation of terms into $\Phi_{2n}^{(p)}$ and $\Psi_{2n}^{(p)}$ is somewhat loosely done to make the expression simpler in view of our experience in Section 1.1.1. Some terms which evidently contain the factor t^2 remain in $\Phi_{2n}^{(p)}$; a typical example is $\sum_{\substack{i+j=n \\ i,j \le n-1}} x_{2i}^{(0)} x_{2j}^{(p)} f^{(0)}$. Since leaving these terms in $\Phi_{2n}^{(p)}$ does not cause any problems in our induction procedure described below, we have not paid much attention to this point. The term $2x_0^{(0)} x_{2n}^{(p)} f^{(0)}$ plays an exceptional role in our reasoning, and we have separated it from $\Phi_{2n}^{(p)}$ and put $-2t^{-2} x_0^{(0)} f^{(0)} x_{2n}^{(p)}$ into $\Psi_{2n}^{(p)}$.

Thus we are to determine $T_l^{(r)} = \left\{x_l^{(r)}, A_l^{(r)}, B_l^{(r)}\right\}$ $(r, l \ge 0)$ so that they satisfy

(1.1.3.40)
$$\Phi_{2n}^{(p)} - t^2 \Psi_{2n}^{(p)} = 0$$

for every $p, n \ge 0$. Using the variable

(1.1.3.41)
$$s = x_0^{(0)}(t, \rho),$$

we can rewrite (1.1.3.40) as

$$(1.1.3.42)\qquad \big(2s\frac{d}{ds}-1\big)x_{2n}^{(p)} = -\frac{A_{2n}^{(p-1)}}{B_0^{(0)}} - \frac{B_{2n}^{(p)}}{B_0^{(0)}}s + R_{2n}^{(p)}(s,\rho),$$

where

$$(1.1.3.43)\qquad R_{2n}^{(p)}(s,\rho)$$

$$= -\sum_{\substack{q+r+u=p-1\\ i+j+k=n\\ (u,k)\neq(p-1,n)}} \dot{x}_{2i}^{(q)}\dot{x}_{2j}^{(r)}\frac{A_{2k}^{(u)}}{B_0^{(0)}} \qquad (\alpha.\mathrm{i})$$

$$-\sum_{\substack{q+r+u+v=p\\ i+j+k+l=n}}{}^{*}\dot{x}_{2i}^{(q)}\dot{x}_{2j}^{(r)}x_{2k}^{(u)}\frac{B_{2l}^{(v)}}{B_0^{(0)}} \qquad (\alpha.\mathrm{ii})$$

$$+t^{-2}\sum_{\substack{q+r+u=p-3\\ i+j+k=n}} \dot{x}_{2i}^{(q)}\dot{x}_{2j}^{(r)}\frac{A_{2k}^{(u)}}{B_0^{(0)}} \qquad (\alpha.\mathrm{iii})$$

$$+t^{-2}\sum_{\substack{q+r+u+v=p-2\\ i+j+k+l=n}} \dot{x}_{2i}^{(q)}\dot{x}_{2j}^{(r)}x_{2k}^{(u)}\frac{B_{2l}^{(v)}}{B_0^{(0)}} \qquad (\alpha.\mathrm{iv})$$

$$+\frac{t^{-2}}{B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2\sum_{\substack{q+r+u=p\\ i+j=n\\ i,j\leq n-1}} x_{2i}^{(q)}x_{2j}^{(r)}f^{(u)} \qquad (\alpha.\mathrm{v})$$

$$+\frac{2t^{-2}}{B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2\sum_{\substack{q+r+u=p\\ q\leq p-1}} x_{2n}^{(q)}x_0^{(r)}f^{(u)} \qquad (\alpha.\mathrm{vi})$$

$$- \frac{t^{-2}}{2B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 \sum_{\substack{q+r+u=p-2\\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x;t\}_{2k}^{(u)} \qquad (\alpha.\mathrm{vii})$$

$$+ \frac{t^{-2}}{2B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 \{x;t\}_{2(n-1)}^{(p-4)} \qquad (\alpha.\mathrm{viii})$$

$$+ \frac{1}{2B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 \sum_{\substack{q+r+u=p\\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x;t\}_{2k}^{(u)} \qquad (\alpha.\mathrm{ix})$$

$$- \frac{1}{2B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 \{x;t\}_{2(n-1)}^{(p-2)} \qquad (\alpha.\mathrm{x})$$

with $\sum^*$ in (α.ii) meaning the following:

(1.1.3.44) $$\sum_{\substack{q+r+u+v=p\\ i+j+k+l=n}}{}^{*} = \sum_{\substack{q+r+u+v=p\\ i+j+k+l=n\\ (q,i),(r,j),(u,k),(v,l)\neq(p,n)}} .$$

Here the formula number ($\alpha.l$) is put to each sum for the later reference. We note that, as is usual,

(1.1.3.45) $$\frac{2t^{-2}}{B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 x_{2n}^{(p)} x_0^{(0)} f^{(0)}$$

has been shifted to the left-hand side of (1.1.3.42) thanks to [5.0]′; this is the reason why we encounter somewhat puzzling sums (α.v) and (α.vi). Our task is to show a generalization of Proposition 1.1.3.1 that is applicable to $T_l^{(r)} = \big\{x_l^{(r)},\, A_l^{(r)},\, B_l^{(r)}\big\}$ ($l \gneqq 0$). In order to see how we can, and really do, adjust the constants contained in $R_{2n}^{(p)}$ to find a holomorphic solution $x_{2n}^{(p)}(s,\rho)$ of (1.1.3.42) near $s = 0$, we first show a generalization of Proposition 1.1.2.1. To present the generalization we prepare some notations.

Definition 1.1.3.1. (i) The infinite vector $\big(x_l^{(0)}, x_l^{(1)}, \cdots, x_l^{(r)}, \cdots\big)$ (resp., $\big(A_l^{(0)}, A_l^{(1)}, \cdots, A_l^{(r)}, \cdots\big)$ and $\big(B_l^{(0)}, B_l^{(1)}, \cdots, B_l^{(r)}, \cdots\big)$) is denoted by $\vec{x}_l[\infty]$ (resp., $\vec{A}_l[\infty]$ and $\vec{B}_l[\infty]$).
(ii) $\vec{x}_n[p]$ (resp., $\vec{A}_n[p]$ and $\vec{B}_n[p]$) stands for $\big(\vec{x}_0[\infty], \vec{x}_1[\infty], \cdots, \vec{x}_{n-1}[\infty], x_n^{(0)}, x_n^{(1)}, \cdots, x_n^{(p)}\big)$ (resp., $\big(\vec{A}_0[\infty], \vec{A}_1[\infty], \cdots, \vec{A}_{n-1}[\infty], A_n^{(0)}, A_n^{(1)}, \cdots, A_n^{(p)}\big)$ and $\big(\vec{B}_0[\infty], \vec{B}_1[\infty], \cdots, \vec{B}_{n-1}[\infty], B_n^{(0)}, B_n^{(1)}, \cdots, B_n^{(p)}\big)$).
(iii) We say $\vec{x}_l[\infty]$ is holomorphic near $s = 0$ (or $t = 0$) if there exists a neighborhood U (resp., O) of $\{s \in \mathbb{C}; s = 0\}$ (resp., $\{\rho \in \mathbb{C}; \rho = 0\}$) for which $x_l^{(r)}(s, \rho)$ is holomorphic on $U \times (O - \{0\})$ for every $r \geq 0$.
(iv) We say $\vec{x}_n[p]$ is holomorphic near $s = 0$ (or $t = 0$) if there exists a neighborhood U (resp., O) of $\{s \in \mathbb{C}; s = 0\}$ (resp., $\{\rho \in \mathbb{C}; \rho = 0\}$) for which the following holds:

(iv.a) $x_l^{(r)}(s, \rho)$ is holomorphic on $U \times (O - \{0\})$ for $0 \leq l \leq n - 1$ and $r \geq 0$,

and

(iv.b) $x_n^{(r)}(s, \rho)$ is holomorphic on $U \times (O - \{0\})$ for $0 \leq r \leq p$.

(v) Let $\mathcal{X} = \mathcal{X}(\vec{A}_n[p], \vec{B}_n[p'])$ and $\mathcal{Y} = \mathcal{Y}(\vec{A}_n[p], \vec{B}_n[p'])$ be functions of $\vec{A}_n[p]$ and $\vec{B}_n[p']$. If $\mathcal{X} - \mathcal{Y}$ depends only on $(\vec{A}_n[q-1], \vec{B}_n[q-1])$, then we say

(1.1.3.46) $$\mathcal{X} \underset{(n;q)}{\equiv} \mathcal{Y}.$$

If there is no fear of confusion, we abbreviate it as

(1.1.3.47) $$\mathcal{X} \underset{(q)}{\equiv} \mathcal{Y}.$$

Remark 1.1.3.5. As a convention we understand

(1.1.3.48)

$$\big(\vec{A}_n[-1], \vec{B}_n[-1]\big) \\ = \big(\vec{A}_0[\infty], \vec{A}_1[\infty], \cdots, \vec{A}_{n-1}[\infty], \vec{B}_0[\infty], \vec{B}_1[\infty], \cdots, \vec{B}_{n-1}[\infty]\big).$$

Although the following Lemma 1.1.3.5 is an immediate consequence of (1.1.3.38) (together with (1.1.1.23)), it plays an important role in finding the concrete description of the conditions which guarantee the existence of a holomorphic solution $x_{2n}^{(p)}(s,\rho)$ of (1.1.3.42) with $p > r$. (Cf. Proposition 1.1.3.2 and Proposition 1.1.3.3 below.)

Lemma 1.1.3.5. *If* $\vec{x}_{2n}[r]$ *is holomorphic near* $s = 0$*, then* $\Phi_{2n}^{(r)}(t,\rho)$ *is holomorphic near* $t = 0$*.*

As mentioned in the above, this is an immediate consequence of the definition of $\Phi_{2n}^{(r)}$. The importance of Lemma 1.1.3.5 consists in the fact that the holomorphy of $\Phi_{2n}^{(r+1)}(t,\rho)$ near $t = 0$ is needed to describe the conditions which guarantee the existence of a holomorphic solution $x_{2n}^{(r+1)}(t,\rho)$ of (1.1.3.42) with $p = r+1$ on the condition that $\vec{x}_{2n}[r]$ is holomorphic. In what follows we let $[E; r, l]$ designate the following equation:

$$[E; r, l] \qquad \Big(2s\frac{d}{ds} - 1\Big)x_l^{(r)}(s,\rho) = -\frac{A_l^{(r-1)}}{B_0^{(0)}} - \frac{B_l^{(r)}}{B_0^{(0)}}s + R_l^{(r)}(s,\rho),$$

where

(1.1.3.49) $A_l^{(r-1)}$ and $B_l^{(r)}$ are complex numbers,

$$(1.1.3.50) \qquad A_l^{(-1)} = 0,$$

$$(1.1.3.51) \qquad A_{2\nu+1}^{(r-1)} = B_{2\nu+1}^{(r)} = R_{2\nu+1}^{(r)} = 0 \text{ for } r, \nu = 0, 1, 2, \cdots,$$

(1.1.3.52) $R_{2n}^{(p)}$ is given by (1.1.3.43).

Remark 1.1.3.6. In our subsequent discussion, we arrange our reasoning so that each quantity in the definition of $R^{(p)}_{2n}$ has been given by preceding arguments.

Let us begin our discussion by showing the following

Lemma 1.1.3.6. *Suppose that constants* $(\vec{A}_l[\infty], \vec{B}_l[\infty]$ $(l = 0, 1, \cdots, 2n-1)$ *and holomorphic (near* $s = 0$*)* $\vec{x}_l[\infty]$ $(l = 0, 1, \cdots, 2n-1)$ *are given with* $x^{(r)}_l$ *satisfying* $[E; r, l]$*. Suppose further*

$$x^{(0)}_l(0, \rho) = 0 \quad (l = 0, 1, \cdots, 2n-1). \tag{1.1.3.53}$$

Then there exists a holomorphic (in s) solution $x^{(r)}_{2n}(s, \rho)$ *of* $[E; r, 2n]$ *for* $r = 0, 1$ *for any* $\left(A^{(0)}_{2n}, B^{(0)}_{2n}, B^{(1)}_{2n}\right)$*. Furthermore they satisfy the following:*

$$x^{(0)}_{2n}(0, \rho) = 0, \tag{1.1.3.54}$$

$$\dot{x}^{(0)}_{2n}(0, \rho) \underset{(2n;0)}{\equiv} -\frac{B^{(0)}_{2n}}{B^{(0)}_0}, \tag{1.1.3.55}$$

$$x^{(1)}_{2n}(0, \rho) \underset{(2n;0)}{\equiv} \frac{A^{(0)}_{2n}}{B^{(0)}_0}, \tag{1.1.3.56}$$

$$\dot{x}^{(1)}_{2n}(0, \rho) \underset{(2n;1)}{\equiv} -\frac{B^{(1)}_{2n}}{B^{(0)}_0}. \tag{1.1.3.57}$$

Proof. We first show the existence of holomorphic $x^{(0)}_{2n}(s, \rho)$ and confirm its properties (1.1.3.54) and (1.1.3.55). Checking each term in (1.1.3.43), we readily find that the possible singularity of $R^{(0)}_{2n}$ arises from the sum (α.v). On the other hand, (1.1.3.53) and the definition of $f^{(0)}$ entail

$$\sum_{\substack{q+r+u=0\\ i+j=n\\ i,j\le n-1}} x^{(q)}_{2i} x^{(r)}_{2j} f^{(u)} = \Big(\sum_{\substack{i+j=n\\ i,j\le n-1}} x^{(0)}_{2i} x^{(0)}_{2j}\Big) f^{(0)} = O(t^3). \tag{1.1.3.58}$$

Hence the contribution from (α.v) is holomorphic near $t = 0$. Therefore $[E; 0, 2n]$ has a (unique) holomorphic solution $x^{(0)}_{2n}(s, \rho)$ for any complex number $B^{(0)}_{2n}$. Furthermore the contribution from (α.v) depends only on $\big(\vec{A}_0[\infty],\ \vec{A}_1[\infty],\ \cdots,\ \vec{A}_{2n-1}[\infty],\ \vec{B}_0[\infty],\ \vec{B}_1[\infty],\ \cdots,\ \vec{B}_{2n-1}[\infty]\big)$, and it vanishes at $t = 0$. On the other hand it follows from (1.1.3.44) that each term in (α.ii) with $p = 0$ contains a factor $x^{(0)}_{2k}$ with $k \leq n-1$. Hence the value of (α.ii) at $s = 0$ is 0. Clearly (α.ix) with $p = 0$ also vanishes at $s = 0$. Thus we obtain (1.1.3.54) . Since (α.ii) also depends only on $\big(\vec{A}_0[\infty],\ \vec{A}_1[\infty],\ \cdots,\ \vec{A}_{2n-1}[\infty],\ \vec{B}_0[\infty],\ \vec{B}_1[\infty],\ \cdots,\ \vec{B}_{2n-1}[\infty]\big)$, $R^{(0)}_{2n}(s, \rho)$ depends only on these parameters. Therefore we find

$$\Big(2s\frac{d}{ds} - 1\Big)x^{(0)}_{2n}(s, \rho) + \frac{B^{(0)}_{2n}}{B^{(0)}_0}s \underset{(2n;0)}{\equiv} 0, \tag{1.1.3.59}$$

and, in particular, we obtain (1.1.3.55).

We next investigate the structure of $R^{(1)}_{2n}$. The contribution from (α.v) with $p = 1$ is:

$$\Big(\sum_{\substack{q+r=1\\ i+j=n\\ i,j\leq n-1}} x^{(q)}_{2i}x^{(r)}_{2j}\Big)f^{(0)} + \Big(\sum_{\substack{i+j=n\\ i,j\leq n-1}} x^{(0)}_{2i}x^{(0)}_{2j}\Big)f^{(1)}. \tag{1.1.3.60}$$

Then it follows from (1.1.3.53) that

$$\sum_{\substack{q+r=1\\ i+j=n\\ i,j\leq n-1}} x^{(q)}_{2i}x^{(r)}_{2j} = O(t), \tag{1.1.3.61}$$

$$\sum_{\substack{i+j=n\\ i,j\leq n-1}} x^{(0)}_{2i}x^{(0)}_{2j} = O(t^2). \tag{1.1.3.62}$$

Hence the contribution from (α.v) with $p = 1$ is holomorphic near $t = 0$. Similarly the contribution from (α.vi) with $p = 1$ is holomorphic near $t=0$, because

$$x_{2n}^{(0)}\bigl(x_0^{(1)}f^{(0)}+x_0^{(0)}f^{(1)}\bigr)=O(t^2) \tag{1.1.3.63}$$

by (1.1.3.54). Other terms in $R_{2n}^{(1)}$ are evidently holomorphic near $s=0$, and hence $[E;1,2n]$ has a holomorphic solution $x_{2n}^{(1)}(s,\rho)$ near $s=0$ for any complex numbers $A_{2n}^{(0)}$, $B_{2n}^{(0)}$ and $B_{2n}^{(1)}$. To confirm its property (1.1.3.56) we next show

(1.1.3.64) $R_{2n}^{(1)}(0,\rho)$ is free from $B_{2n}^{(0)}$.

The proof of this fact is basically the same as that of Proposition 1.1.2.1; in parallel with the cancellation (C.i),

$$\frac{2}{B_0^{(0)}}\dot{x}_{2n}^{(0)}(0,\rho)\dot{x}_0^{(0)}(0,\rho)f^{(1)}(0,\rho), \tag{1.1.3.65}$$

which originates from the Taylor expansion of the second term in (1.1.3.63), is cancelled out by the term

$$-2\dot{x}_0^{(0)}\dot{x}_{2n}^{(0)}\frac{A_0^{(0)}}{B_0^{(0)}}\Big|_{s=0} \tag{1.1.3.66}$$

in the sum (α.i) evaluated at $s=0$, whereas, in parallel with (C.ii), the term which contains $\dot{x}_{2n}^{(0)}$ and $B_{2n}^{(0)}$ in the sum (α.ii) evaluated at $s=0$, that is,

(1.1.3.67)

$$-\Bigl(2\dot{x}_{2n}^{(0)}\dot{x}_0^{(1)}x_0^{(0)}+2\dot{x}_{2n}^{(0)}\dot{x}_0^{(0)}x_0^{(1)}+\sum_{q+r+u=1}\dot{x}_0^{(q)}\dot{x}_0^{(r)}x_0^{(u)}\frac{B_{2n}^{(0)}}{B_0^{(0)}}\Bigr)\Big|_{s=0}$$

is equal to

$$-\Bigl(2\dot{x}_{2n}^{(0)}\dot{x}_0^{(1)}x_0^{(0)}+2\dot{x}_{2n}^{(0)}\dot{x}_0^{(0)}x_0^{(1)}+\bigl(x_0^{(1)}+2\dot{x}_0^{(1)}x_0^{(1)}\bigr)\frac{B_{2n}^{(0)}}{B_0^{(0)}}\Bigr)\Big|_{s=0}, \tag{1.1.3.68}$$

which vanishes by (1.1.1.7) and (1.1.1.24). Thus the evaluation of $[E;1,2n]$ at $s=0$ entails

$$x_{2n}^{(1)}(0,\rho) \underset{(2n;0)}{\equiv} \frac{A_{2n}^{(0)}}{B_0^{(0)}}, \tag{1.1.3.69}$$

as is required. Since $R_{2n}^{(1)}(s,\rho)$ is clearly free from $B_{2n}^{(1)}$ (and $A_{2n}^{(1)}$, which has not yet appeared in our discussion), the relation (1.1.3.57) is an immediate consequence of $[E;1,2n]$.

□

An important fact which lies behind the existence of holomorphic $x_{2n}^{(r)}(s,\rho)$ with $r=0,1$ is the validity of the following:

$$\Phi_{2n}^{(0)}\Big|_{t=0} = \frac{d\Phi_{2n}^{(0)}}{dt}\Big|_{t=0} = \Phi_{2n}^{(1)}\Big|_{t=0} = \frac{d\Phi_{2n}^{(1)}}{dt}\Big|_{t=0} = 0. \tag{1.1.3.70}$$

In passing we note

$$\Phi_{2n}^{(2)}\Big|_{t=0} = 0 \tag{1.1.3.71}$$

also follows from (1.1.3.53) and (1.1.3.54), although we cannot expect

$$\frac{d\Phi_{2n}^{(2)}}{dt}\Big|_{t=0} = 0 \tag{1.1.3.72}$$

in general. Actually as we will see below (1.1.3.72) gives a constraint on $A_{2n}^{(0)}$ and $B_{2n}^{(0)}$, which are free parameters in Lemma 1.1.3.6. Now, in parallel with Proposition 1.1.2.1 we find the following

Proposition 1.1.3.2. *Let us suppose the same conditions as in Lemma 1.1.3.6, that is, the existence of constants* $(\vec{A}_l[\infty], \vec{B}_l[\infty])$ $(l=0,1,\cdots,2n-1)$ *and holomorphic* $\vec{x}_l[\infty]$ $(l=0,1,\cdots,2n-1)$ *that satisfies* (1.1.3.53). *Then the following set* $\mathcal{A}_{2n}(p)$ *of assertions* $(\mathcal{A}_{2n}(p)(\mathrm{i}), \mathcal{A}_{2n}(p)(\mathrm{ii}), \cdots, \mathcal{A}_{2n}(p)(\mathrm{vi}))$ *is valid for every* $p \geq 0$ *with* *the proviso that* $\mathcal{A}_{2n}(p)(\mathrm{v})$ $(p = 0, 1, 2)$ *and* $\mathcal{A}_{2n}(p)(\mathrm{vi})$ $(p = 0, 1)$ *are void statements (i.e., trivially correct statements in the sense that both sides are* 0 *under the convention*

$$A_{2n}^{(q)} = B_{2n}^{(q')} = 0 \text{ for } q, q' = -3, -2 \text{ and } q' = -1, \tag{1.1.3.73}$$

which supplements (1.1.3.50)*.)*

$$
\mathcal{A}_{2n}(p):\begin{cases}
\mathcal{A}_{2n}(p)(\mathrm{i}): & \textit{We can find constraints on parameters } \big(A_{2n}^{(p-2)},\ B_{2n}^{(p-2)},\ A_{2n}^{(p-3)},\ B_{2n}^{(p-3)}\big) \textit{ which are consistent with the constraints on } \big(\vec{A}_{2n}[p-3],\ \vec{B}_{2n}[p-3]\big) \textit{ that have been given in previous stages (i,e., in } \mathcal{A}_{2n}(p')(\mathrm{i})\ (0\le p'\le p-1)\textit{), so that a solution } x_{2n}^{(p)}(s,\rho) \textit{ of } [E;p,2n] \textit{ is holomorphic in } s,\\[2ex]
\mathcal{A}_{2n}(p)(\mathrm{ii}): & \textit{The solution } x_{2n}^{(p)}(s,\rho) \textit{ found in } \mathcal{A}_{2n}(p)(\mathrm{i}) \textit{ depends on } \big(\vec{A}_{2n}[p-1],\ \vec{B}_{2n}[p]\big),\\[2ex]
\mathcal{A}_{2n}(p)(\mathrm{iii}): & x_{2n}^{(p)}(0,\rho) \underset{(2n;p-1)}{\equiv} \dfrac{A_{2n}^{(p-1)}}{B_0^{(0)}},\\[2ex]
\mathcal{A}_{2n}(p)(\mathrm{iv}): & \dot{x}_{2n}^{(p)}(0,\rho) \underset{(2n;p)}{\equiv} -\dfrac{B_{2n}^{(p)}}{B_0^{(0)}},\\[2ex]
\mathcal{A}_{2n}(p)(\mathrm{v}): & \Phi_{2n}^{(p)}\big|_{t=0} \underset{(2n;p-3)}{\equiv} A_{2n}^{(p-3)} - 2\dfrac{A_0^{(0)}}{B_0^{(0)}} B_{2n}^{(p-3)},\\[2ex]
\mathcal{A}_{2n}(p)(\mathrm{vi}): & \dfrac{d\Phi_{2n}^{(p)}}{dt}\Big|_{t=0} \underset{(2n;p-2)}{\equiv} 2Z_0\dfrac{A_0^{(0)}}{B_0^{(0)}} A_{2n}^{(p-2)} - 2Z_0 B_{2n}^{(p-2)}.
\end{cases}
$$

Proof. With the convention (1.1.3.73) we find by Lemma 1.1.3.6 and (1.1.3.70) that $\mathcal{A}_{2n}(0)$ and $\mathcal{A}_{2n}(1)$ are valid. To make the induction run smoothly we confirm $\mathcal{A}_{2n}(2)$ separately, although one may build it in the induction procedure. We first note that $\mathcal{A}_{2n}(2)(\mathrm{vi})$ follows from

$\mathcal{A}_{2n}(1)$(iii) and $\mathcal{A}_{2n}(0)$(iv) through the explicit computation of each term in $d\Phi_{2n}^{(2)}/dt\big|_{t=0}$. In the computation we repeatedly use (1.1.1.24); for example, $2f^{(0)\prime}x_{2n}^{(1)}x_0^{(1)}\big|_{t=0}$, which may depend on $A_{2n}^{(0)}$ through $x_{2n}^{(1)}\big|_{t=0}$, actually vanishes thanks to the vanishing factor $x_0^{(1)}\big|_{t=0}$, and so on. Then, as the constraint on $\big(A_{2n}^{(0)}, B_{2n}^{(0)}\big)$ required in $\mathcal{A}_{2n}(2)$(i), we employ

$$\frac{d\Phi_{2n}^{(2)}}{dt}\Big|_{t=0} = 0; \tag{1.1.3.74}$$

the confirmed assertion $\mathcal{A}_{2n}(2)$(vi) guarantees that this gives a linear relation of $\big(A_{2n}^{(0)}, B_{2n}^{(0)}\big)$ whose coefficients are determined by $\big(\vec{A}_{2n}[-1], \vec{B}_{2n}[-1]\big)$ (in the notation of (1.1.3.48)). It is clear from the definition of $R_{2n}^{(2)}$ that (1.1.3.74) together with (1.1.3.71) entails the holomorphy of $R_{2n}^{(2)}(s,\rho)$ near $s=0$ and hence the existence of a holomorphic solution $x_{2n}^{(2)}(s,\rho)$ of $[E;2,2n]$. Thus we have validated $\mathcal{A}_{2n}(2)$(i). The assertion $\mathcal{A}_{2n}(2)$(ii) then immediately follows from the definition of the equation $[E;2,2n]$. To confirm $\mathcal{A}_{2n}(2)$(iii) it suffices to show that $R_{2n}^{(2)}(0,\rho)$ is free from $B_{2n}^{(1)}$. This fact can be verified by a reasoning similar to the proof of Lemma 1.1.3.4; the terms we have to examine are the following:

$$-2\dot{x}_{2n}^{(1)}(0,\rho)\dot{x}_0^{(0)}(0,\rho)A_0^{(0)}/B_0^{(0)}, \tag{1.1.3.75}$$

$$-\Big(\sum_{q+r+u=1}\dot{x}_0^{(q)}(0,\rho)\dot{x}_0^{(r)}(0,\rho)x_0^{(u)}(0,\rho)\Big)B_{2n}^{(1)}/B_0^{(0)}, \tag{1.1.3.76}$$

$$-2\dot{x}_{2n}^{(1)}(0,\rho)\Big(\sum_{r+u+v=1}\dot{x}_0^{(r)}(0,\rho)x_0^{(u)}(0,\rho)B_0^{(v)}/B_0^{(0)}\Big) \tag{1.1.3.77}$$

and

$$2\dot{x}_{2n}^{(1)}(0,\rho)\dot{x}_0^{(0)}(0,\rho)f^{(1)}(0,\rho)/B_0^{(0)}, \tag{1.1.3.78}$$

which originates from the Taylor expansion of

$$2\sum_{r+u=1} x_{2n}^{(1)} x_0^{(r)} f^{(u)}/B_0^{(0)}. \tag{1.1.3.79}$$

Then, as we have often observed (1.1.3.75) and (1.1.3.78) sum up to 0, and (1.1.3.76) and (1.1.3.77) vanish by (1.1.1.24) together with (1.1.1.7). Thus we have validated $\mathcal{A}_{2n}(2)$(iii). The confirmation of $\mathcal{A}_{2n}(2)$(iv) is trivial, as $R_{2n}^{(2)}(s,\rho)$ does not contain $B_{2n}^{(2)}$. Summing up, we have confirmed $\mathcal{A}_{2n}(2)$. Let us now begin the induction argument. Suppose that $\mathcal{A}_{2n}(p)$ is valid for $0 \le p \le p_0 - 1$ with $p_0 \ge 3$. Then, as is in the confirmation of $\mathcal{A}_{2n}(2)$, we see that $\mathcal{A}_{2n}(p_0)$(v) follows from $\mathcal{A}_{2n}(p_0-2)$(iii) and $\mathcal{A}_{2n}(p_0-3)$(iv), and that $\mathcal{A}_{2n}(p_0)$(vi) follows from $\mathcal{A}_{2n}(p_0-1)$(iii) and $\mathcal{A}_{2n}(p_0-2)$(iv). In order to guarantee the existence of a holomorphic solution $x_{2n}^{(p_0)}(s,\rho)$ of $[E; p_0, 2n]$, we require

$$\Phi_{2n}^{(p_0)}\Big|_{t=0} = 0 \tag{1.1.3.80}$$

and

$$\frac{d\Phi_{2n}^{(p_0)}}{dt}\Big|_{t=0} = 0. \tag{1.1.3.81}$$

The condition (1.1.3.81) gives a linear constraint on $\big(A_{2n}^{(p_0-2)}, B_{2n}^{(p_0-2)}\big)$ whose coefficients are described by $\big(\vec{A}_{2n}[p_0-3], \vec{B}_{2n}[p_0-3]\big)$, whereas (1.1.3.80) supplemented by $\mathcal{A}_{2n}(p_0)$(v), together with the constraint on $\big(A_{2n}^{(p_0-3)}, B_{2n}^{(p_0-3)}\big)$ given in the preceding stage, i.e.,

$$\frac{d\Phi_{2n}^{(p_0-1)}}{dt}\Big|_{t=0} = 0, \tag{1.1.3.82}$$

fixes $\big(A_{2n}^{(p_0-3)}, B_{2n}^{(p_0-3)}\big)$ in terms of $\big(\vec{A}_{2n}[p_0-4], \vec{B}_{2n}[p_0-4]\big)$. Here we have used the assumption (1.1.2) together with (1.1.1.21) and (1.1.1.22). Then the validity of $\mathcal{A}_{2n}(p_0)$(i) and $\mathcal{A}_{2n}(p_0)$(ii) is obvious. The confirmation of $\mathcal{A}_{2n}(p_0)$(iii) requires the validation of the fact that $R_{2n}^{(p_0)}(0,\rho)$

is free from $B_{2n}^{(p_0-1)}$; this validation can be done by exactly the same reasoning used when $p_0 = 2$. Thus we have confirmed $\mathcal{A}_{2n}(p_0)$(iii). The validation of $\mathcal{A}_{2n}(p_0)$(iv) is trivial, as $R_{2n}^{(p_0)}(s,\rho)$ is free from $B_{2n}^{(p_0)}$. Hence the induction proceeds, and $\mathcal{A}_{2n}(p)$ is seen to be valid for every $p \geq 0$.

$\square$

Remark 1.1.3.7. As is clear from the above proof, "constraints on parameters $\big(A_{2n}^{(p-2)}, B_{2n}^{(p-2)}\ A_{2n}^{(p-3)}, B_{2n}^{(p-3)}\big)$" to be found in $\mathcal{A}_{2n}(p)$(i) are (1.1.3.80) and (1.1.3.81). These conditions turn out to be consistent with previously imposed constraints on $\big(\vec{A}_{2n}[p-3],\ \vec{B}_{2n}[p-3]\big)$ by $\mathcal{A}_{2n}(p)$(v) and $\mathcal{A}_{2n}(p)$(vi), and hence we have avoided the explicit statement of the conditions in $\mathcal{A}_{2n}(p)$(i).

In Proposition 1.1.3.2 indices of the fixed quantity at the stage $\mathcal{A}_{2n}(p)$ are not uniform; $x_{2n}^{(p)}$ is fixed with free parameters $\big(A_{2n}^{(p-1)}, B_{2n}^{(p-1)}, B_{2n}^{(p)}\big)$ and parameters $\big(A_{2n}^{(p-2)}, B_{2n}^{(p-2)}\big)$ constrained by (1.1.3.81), whereas $\big(\vec{A}_{2n}[p-3],\ \vec{B}_{2n}[p-3]\big)$ is fixed. Hence we rearrange the setting so that $T_l^{(r)} = \{x_l^{(r)},\ A_l^{(r)},\ B_l^{(r)}\}$ $(l, r \geq 0)$, following the way in which Proposition 1.1.3.1 is stated. In what follows we assume the same conditions as in Lemma 1.1.3.6, that is, the existence of constants $\big(\vec{A}_l[\infty],\ \vec{B}_l[\infty]\big)$ $(l = 0, 1, \cdots, 2n-1)$ and holomorphic $\vec{x}_l[\infty]$ $(l = 0, 1, \cdots, 2n-1)$ that satisfies (1.1.3.53). Under this assumption we use the symbol $\mathfrak{A}_{2n}(p-1)$ to mean the assertion that a triplet of data $T_{2n}^{(r)} = \{x_{2n}^{(r)}(s,\rho),\ A_{2n}^{(r)},\ B_{2n}^{(r)}\}$ is given for $0 \leq r \leq p-1$ so that they satisfy the following conditions:

(1.1.3.83.r) $x_{2n}^{(r)}(s,\rho)$ is a holomorphic solution of $[E; r, 2n]$ near $s = 0$, (1.1.3.84.r) $x^{(r)}_{2n}(s,\rho)$ depends on $\big(\vec{A}_{2n}[r-1],\ \vec{B}_{2n}[r]\big)$,

(1.1.3.85.r) $$x^{(0)}_{2n}(0,\rho)=0,$$

(1.1.3.86.r) $\Phi^{(r+3)}_{2n}\big|_{t=0}$ and $\dfrac{d\Phi^{(r+2)}_{2n}}{dt}\Big|_{t=0}$ depend on $\big(\vec{A}_{2n}[r],\ \vec{B}_{2n}[r]\big)$, and $\big(\vec{A}_{2n}[r],\ \vec{B}_{2n}[r]\big)$ satisfies $\Phi^{(r+3)}_{2n}\big|_{t=0}=\dfrac{d\Phi^{(r+2)}_{2n}}{dt}\Big|_{t=0}=0$,

(1.1.3.87.r) $$\Phi^{(r+3)}_{2n}\big|_{t=0}\underset{(2n;r)}{\equiv}2A^{(r)}_{2n}-2\frac{A^{(0)}_0}{B^{(0)}_0}B^{(r)}_{2n},$$

(1.1.3.88.r) $$\frac{d\Phi^{(r+2)}_{2n}}{dt}\Big|_{t=0}\underset{(2n;r)}{\equiv}2Z_0\frac{A^{(0)}_0}{B^{(0)}_0}A^{(r)}_{2n}-2Z_0B^{(r)}_{2n}.$$

Proposition 1.1.3.3. *The assertion* $\mathfrak{A}_{2n}(p)$ *is valid for every* $p\geq 0$.

As the proof is essentially the same as that of Proposition 1.1.3.1, we describe its core part only. In the course of the proof of Proposition 1.1.3.2 we have seen that $\mathfrak{A}_{2n}(p)$ ($p=0,1,2$) are valid. Let us suppose that $\mathfrak{A}_{2n}(p)$ is valid for $0\leq p\leq p_0-1$ with $p_0\geq 3$, and we want to confirm $\mathfrak{A}_{2n}(p_0)$. By adding an arbitrary complex number $B^{(p_0)}_{2n}$ to the given data $T^{(r)}_{2n}$ ($r\leq p_0-1$) we can define the equation $[E;p_0,2n]$. It then follows from (1.1.3.86.(p_0-3)) and (1.1.3.86.(p_0-2)) that

(1.1.3.89) $$\Phi^{(p_0)}_{2n}\big|_{t=0}=\frac{d\Phi^{(p_0)}_{2n}}{dt}\Big|_{t=0}=0$$

holds. Hence $R^{(p_0)}_{2n}$ is holomorphic near $s=0$. Then (1.1.3.83.p_0) and (1.1.3.84.p_0) are immediate consequences of $[E;p_0,2n]$. As the confirmation of (1.1.3.87.p_0) and (1.1.3.88.p_0) requires the description of

$x_{2n}^{(p_0+1)}(0,\rho)$, we further consider $[E; p_0+1, 2n]$; we add arbitrary constants $A_{2n}^{(p_0)}$ and $B_{2n}^{(p_0+1)}$ to the given data to write down $[E; p_0+1, 2n]$ with the aid of $x_{2n}^{(p_0)}(s,\rho)$ we have just constructed. Then (1.1.3.86.(p_0-2)) and (1.1.3.86.(p_0-1)) guarantee

$$\Phi_{2n}^{(p_0+1)}\Big|_{t=0} = \frac{d\Phi_{2n}^{(p_0+1)}}{dt}\Big|_{t=0} = 0. \tag{1.1.3.90}$$

Hence $R_{2n}^{(p_0+1)}(s,\rho)$ is holomorphic near $s=0$. Furthermore, by the same reasoning as in the proof of Lemma 1.1.3.4, we can verify that $R_{2n}^{(p_0+1)}(0,\rho)$ is free from $B_{2n}^{(p_0)}$. Then, evaluating $[E; p_0+1, 2n]$ at $s=0$, we find

$$x_{2n}^{(p_0+1)}(0,\rho) \underset{(2n;p_0)}{\equiv} \frac{A_{2n}^{(p_0)}}{B_0^{(0)}} - R_{2n}^{(p_0+1)}(0,\rho). \tag{1.1.3.91}$$

Using this relation together with

$$\dot{x}_{2n}^{(p_0)}(0,\rho) \underset{(2n;p_0)}{\equiv} -\frac{B_{2n}^{(p_0)}}{B_0^{(0)}}, \tag{1.1.3.92}$$

we find (1.1.3.87.p_0) and (1.1.3.88.p_0). Then we can fix $\left(A_{2n}^{(p_0)}, B_{2n}^{(p_0)}\right)$ so that $\left(\vec{A}_{2n}[p_0], \vec{B}_{2n}[p_0]\right)$ annihilates $\Phi_{2n}^{(p_0+3)}\big|_{t=0}$ and $d\Phi_{2n}^{(p_0+2)}/dt\big|_{t=0}$, as is required in (1.1.3.86.p_0). We note that no constraint is imposed upon the complex number $B_{2n}^{(p_0+1)}$ introduced for defining $[E; p_0+1, 2n]$ at this stage. Hence the induction proceeds, completing the proof.

1.2 Growth order properties of $T_n^{(p)} = \{x_n^{(p)}, A_n^{(p)}, B_n^{(p)}\}$ $(p, n \geq 0)$ — the case where $g_\pm(t) = 0$

The purpose of this section is to estimate the growth order properties of $\{T_n^{(p)}\}_{p,n\geq 0}$ so that the formal transformation of an M2P1T operator to its canonical form (the ∞-Mathieu equation) may acquire the microlocal analytic meaning, as will be explained later in Section 5. For the sake of simplicity of our reasoning we assume $g_\pm(t) = 0$ in this section. The proof of the corresponding result when $g_\pm \neq 0$ is given in Appendix C. Let us first prepare some notations and elementary inequalities which will be frequently used in our computation.

Definition 1.2.1. For l in $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$ in $\mathbb{N}_0^n$, we define

$$C(l) = \frac{3}{2\pi^2(l+1)^2}, \tag{1.2.1}$$

$$C(\vec{\lambda}) = \prod_{j=1}^{n} C(\lambda_j). \tag{1.2.2}$$

An important property they enjoy is described by the following

Lemma 1.2.1. *When $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ ranges over the set of all vectors that satisfy*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = l, \tag{1.2.3}$$

the sum of $C(\vec{\lambda})$ is dominated by $C(l)$, that is,

$$\sum_{\lambda_1+\lambda_2+\cdots+\lambda_n=l} C(\vec{\lambda}) \leq C(l). \tag{1.2.4}$$

See [KKKoT, Lemma B.3] for the proof.

Lemma 1.2.2. *The following inequality* (1.2.5) *holds for any positive integers l and n satisfying $l \geq n$:*

$$\sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=l \\ \lambda_1,\lambda_2,\cdots,\lambda_n \geq 1}} \lambda_1!\lambda_2!\cdots\lambda_n! \leq 4^{n-1}(l-n+1)! \tag{1.2.5}$$

See [AKT2, Lemma A.4] for the proof.

In what follows we use the symbol $\|h\|_{[r]}$ for a holomorphic function $h(s)$ on $\{s \in \mathbb{C}; |s| \le r\}$ $(r > 0)$ to denote its supremum norm on the disc, that is,

$$\|h\|_{[r]} = \sup_{|s|\le r} |h(s)|. \tag{1.2.6}$$

Using these symbols we now give the precise statement on the growth order of $|f^{(j)}(s,\rho)|$:

There exist positive constants σ_0, κ_0 and L_0 for which the following inequality (1.2.7) holds for every j in $\mathbb{N}_0$ and ρ in $\{\rho \in \mathbb{C}; 0 < |\rho| \le \sigma_0\}$:

$$\|f^{(j)}(\cdot,\rho)\|_{[\sigma_0]} \le \kappa_0 C(j) L_0^j. \tag{1.2.7}$$

Here the auxiliary factor $C(j)$ is intended for the convenience in performing the induction procedure in what follows.

We begin our estimation by studying the growth order property of the triplet $T_0^{(p)} = \{x_0^{(p)}(s,\rho),\ A_0^{(p)},\ B_0^{(p)}\}$ $(p \ge 0)$. For the sake of convenience we introduce the following notations:

$$\tilde{A}_0^{(p)} \underset{\text{def}}{=} A_0^{(p)}/B_0^{(0)} \quad \text{and} \quad \tilde{B}_0^{(p)} \underset{\text{def}}{=} B_0^{(p)}/B_0^{(0)}, \tag{1.2.8}$$

$$\tilde{A}_0^{(-1)} = 0, \tag{1.2.9}$$

$$z_0^{(p)}(s,\rho) \underset{\text{def}}{=} x_0^{(p)}(s,\rho) - \tilde{A}_0^{(p-1)} + \tilde{B}_0^{(p)} s. \tag{1.2.10}$$

It then follows from

$$\left(2s\frac{d}{ds} - 1\right) z_0^{(p)}(s,\rho) = R_0^{(p)}(s,\rho), \tag{1.1.3.15$'$}$$

(1.1.3.16) and (1.1.3.17) (with $q = p$) that we find

$$z_0^{(p)}(0,\rho) = x_0^{(p)}(0,\rho) - \tilde{A}_0^{(p-1)} = -R_0^{(p)}(0,\rho) \tag{1.2.11}$$

and

$$\dot{z}_0^{(p)}(0,\rho) = \dot{x}_0^{(p)}(0,\rho) + \tilde{B}_0^{(p)} = \dot{R}_0^{(p)}(0,\rho). \tag{1.2.12}$$

We first prepare the following

Lemma 1.2.3. *There exist positive constants* (r_0, R_0) *and sufficiently small positive constant* C_0 *for which the following estimate* $[G; p, 0]$ *holds for every* $p \geq 1$ *and* ρ *in* $\{\rho \in \mathbb{C}; 0 < |\rho| \leq r_0\}$.

$$[G;p,0]\begin{cases} (p.\mathrm{i}) & |z_0^{(p+1)}(0,\rho)| \leq C_0 C(p)\big(R_0|\rho|^{-1}\big)^p \\ (p.\mathrm{ii}) & |\dot{z}_0^{(p)}(0,\rho)| \leq C_0 C(p)\big(R_0|\rho|^{-1}\big)^p \\ (p.\mathrm{iii}) & \|z_0^{(p)}(\cdot,\rho)\|_{[r_0]} \leq C_0 C(p)\big(R_0|\rho|^{-1}\big)^p \\ (p.\mathrm{iv}) & \|\dot{z}_0^{(p)}(\cdot,\rho)\|_{[r_0]} \leq C_0 C(p)\big(R_0|\rho|^{-1}\big)^p \\ (p.\mathrm{v}) & |\tilde{A}_0^{(p)}(\rho)| \leq C_0 C(p)\big(R_0|\rho|^{-1}\big)^p \\ (p.\mathrm{vi}) & |\tilde{B}_0^{(p)}(\rho)| \leq C_0 C(p)\big(R_0|\rho|^{-1}\big)^p \end{cases}$$

Remark 1.2.1. We may assume that r_0 and R_0^{-1} are sufficiently small, and hence, $\big(R_0|\rho|^{-1}\big)^{-1}$ is also sufficiently small. (In what follows, we consider r_0 and R_0^{-1} as sufficiently small positive constants.) Therefore it is clear that $[G; p, 0]$ entails

$$(p.\widetilde{\mathrm{iii}}) \quad \|x_0^{(p)}(\cdot,\rho)\|_{[r_0]} \leq \Big(1 + r_0 + \big(R_0|\rho|^{-1}\big)^{-1}\Big) C_0 C(p)\big(R_0|\rho|^{-1}\big)^p \leq 2C_0 C(p)\big(R_0|\rho|^{-1}\big)^p$$

and

$$(p.\widetilde{\mathrm{iv}}) \qquad \|\dot{x}_0^{(p)}(\cdot,\rho)\|_{[r_0]} \leq 2C_0 C(p)\big(R_0|\rho|^{-1}\big)^p.$$

Furthermore these estimates hold for $p = 1$ by the concrete computation in Section 1.1.3. We also note that, as the form of the estimates

$[G;p,0]$ for $p \geq 1$ indicates, we can take $C_0 > 0$ arbitrarily small by taking $R_0 > 0$ sufficiently large.

Proof of Lemma 1.2.3. Before embarking on the induction, we check the situation concretely when $p = 0$. When $p = 0$, $\tilde{B}_0^{(0)} = 1$ and $|\tilde{A}_0^{(0)}| = |A_0^{(0)}|/|\rho|$. Thus (0.v) and (0.vi) are violated. Furthermore $x_0^{(1)}(0,\rho) = 0$ entails

$$z_0^{(1)}(0,\rho) = -\tilde{A}_0^{(0)}, \tag{1.2.13}$$

and hence (0.i) is also violated, whereas (0.ii), (0.iii) and (0.iv) trivially hold as $z_0^{(0)}(s,\rho) = x_0^{(0)}(s,\rho) - \tilde{B}_0^{(0)}s = 0$ holds. Since the results in Section 1.1.3 confirm $[G;1,0]$, we assume that $[G;p,0]$ is valid for $1 \leq p \leq p_0 - 1$ and validate $[G;p_0,0]$. As the reasoning is lengthy, we separate it into several parts.

[I] Let us first confirm the most delicate statement (p_0.i). As we will see later, the confirmation of (p_0.ii) can be done in a similar manner (actually simpler because the relevant index is p_0, not p_0+1). To begin with we note that Proposition 1.1.3.1 guarantees that $T_0^{(p)}$ exists for every $p \geq 0$ and that it annihilates $\Phi_0^{(p)}|_{t=0}$ and $d\Phi_0^{(p)}/dt|_{t=0}$ (cf. (1.1.3.38)) for every p. Hence $R_0^{(p)}(s,\rho)$ given by (1.1.3.8) is holomorphic in s if taken as a whole, though each individual term in the sum may be singular at $s = 0$. Therefore we find

$$\begin{aligned} R_0^{(p_0+1)}(0,\rho) &= \frac{1}{2\pi i}\int_{|s|=r_0} R_0^{(p_0+1)}(s,\rho)\frac{ds}{s} \\ &= \frac{1}{2\pi i}\oint R_0^{(p_0+1)}(s,\rho)\frac{ds}{s}. \end{aligned} \tag{1.2.14}$$

In order to clarify our reasoning we label the terms in $R_0^{(p_0+1)}$ as follows:

(1.2.15)

$R_0^{(p_0+1)}(s,\rho)$

$$= -\sum_{\substack{i+j+k=p_0\\ k\le p_0-1}} \dot{x}_0^{(i)}(s,\rho)\dot{x}_0^{(j)}(s,\rho)\tilde{A}_0^{(k)} \tag{β.i}$$

$$- \sum_{\substack{i+j+k+l=p_0+1\\ i,j,k,l\le p_0}} \dot{x}_0^{(i)}(s,\rho)\dot{x}_0^{(j)}(s,\rho)x_0^{(k)}(s,\rho)\tilde{B}_0^{(l)} \tag{β.ii}$$

$$+ t^{-2}\Big(\sum_{i+j+k=p_0-2} \dot{x}_0^{(i)}(s,\rho)\dot{x}_0^{(j)}(s,\rho)\tilde{A}_0^{(k)}\Big) \tag{β.iii}$$

$$+ t^{-2}\Big(\sum_{i+j+k+l=p_0-1} \dot{x}_0^{(i)}(s,\rho)\dot{x}_0^{(j)}(s,\rho)x_0^{(k)}(s,\rho)\tilde{B}_0^{(l)}\Big) \tag{β.iv}$$

$$+ \Big(\frac{dt}{ds}\Big)^2 \frac{t^{-2}}{B_0^{(0)}}\Big(\sum_{\substack{i+j+k=p_0+1\\ k\ge 2}} x_0^{(i)}(s,\rho)x_0^{(j)}(s,\rho)f^{(k)}(t(s,\rho),\rho)\Big) \tag{β.v}$$

$$+ \Big(\frac{dt}{ds}\Big)^2 \frac{t^{-2}}{B_0^{(0)}}\Big(\sum_{i+j=p_0} x_0^{(i)}(s,\rho)x_0^{(j)}(s,\rho)f^{(1)}(t(s,\rho),\rho)\Big) \tag{β.vi}$$

$$+ \Big(\frac{dt}{ds}\Big)^2 \frac{t^{-1}}{B_0^{(0)}}\tilde{f}^{(0)}(t(s,\rho),\rho)\Big(\sum_{\substack{i+j=p_0+1\\ i,j\ge 1}} x_0^{(i)}(s,\rho)x_0^{(j)}(s,\rho)\Big) \tag{β.vii}$$

$$- \Big(\frac{dt}{ds}\Big)^2 \frac{t^{-2}}{B_0^{(0)}} f^{(p_0-1)}(t(s,\rho),\rho). \tag{β.viii}$$

In what follows we use the symbol $(\beta.j)$ ($j = \mathrm{i}, \mathrm{ii}, \cdots, \mathrm{viii}$) to denote the sum labeled by the symbol; for example, we denote Cauchy's integral of the second sum in $R_0^{(p_0+1)}$ as follows:

$$\frac{1}{2\pi i}\oint (\beta.\mathrm{ii})\,\frac{ds}{s}. \tag{1.2.16}$$

Since (β.ii) is holomorphic near $s = 0$, this is equal to

$$-\sum_{\substack{i+j+k+l=p_0+1\\ i,j,k,l\le p_0}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)x_0^{(k)}(0,\rho)\tilde{B}_0^{(l)}. \tag{1.2.17}$$

In using the induction hypothesis we have to take extra care in dealing with $\dot{x}_0^{(0)}$, $x_0^{(0)}$ and $\tilde{B}_0^{(0)}$, and we also use

$$x_0^{(1)}(0,\rho) = 0 \tag{1.2.18}$$

as an excellent substitute of (p.i) with $p = 0$. Thanks to the constraint on the indices in (1.2.17), at most two indices among (i, j, k, l) may become 0. Furthermore (1.2.18) implies the vanishing of annoying terms such as $\dot{x}_0^{(0)}(0,\rho)^2 x_0^{(1)}(0,\rho)\tilde{B}_0^{(p_0)}$ and $\dot{x}_0^{(0)}(0,\rho)\dot{x}_0^{(p_0)}(0,\rho)x_0^{(1)}(0,\rho)\tilde{B}_0^{(0)}$. Among the surviving terms let us consider the estimation of the following terms as an example; this term is one of the terms that give the worst contribution to the estimates of (β.ii):

$$\begin{aligned}&\left|\dot{x}_0^{(0)}(0,\rho)\dot{x}_0^{(1)}(0,\rho)x_0^{(p_0)}(0,\rho)\tilde{B}_0^{(0)}\right|\\ &\quad\le 2^2(C(0))^{-2}C_0^2C(0)C(1)C(p_0-1)C(0)\big(R_0|\rho|^{-1}\big)^{p_0}.\end{aligned} \tag{1.2.19}$$

Since $\dot{x}_0^{(0)}(0,\rho) = \tilde{B}_0^{(0)} = 1$, the estimates (1.2.19) follows from ($p_0 - 1$.i), (1.ii), ($p_0 - 1$.v) and (1.vi). The unnecessary factor $(C(0))^{-2}C(0)^2$ is inserted for the convenience of applying Lemma 1.2.1 to the estimation of the constant (N.ii) used in (1.2.21) below. In this way, we obtain the following estimates from the induction hypothesis and Lemma 1.2.1:

$$\begin{aligned}&\left|-\sum_{\substack{i+j+k+l=p_0+1\\ i,j,k,l\le p_0}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)x_0^{(k)}(0,\rho)\tilde{B}_0^{(l)}\right|\\ &\quad\le \big(2^2(C(0))^{-2} + 2^3(C(0))^{-1}C_0 + 2^4C_0^2\big)C_0^2C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0}.\end{aligned} \tag{1.2.20}$$

(Actually it suffices to use $\big(2^2(C(0))^{-2}+2^3(C(0))^{-1}C_0+2^4C_0^2\big)C_0$ as the extra factor due to the vanishing of $x_0^{(0)}(0,\rho)$.) Hence we obtain

$$\Big|\frac{1}{2\pi i}\oint(\beta.\text{ii})\frac{ds}{s}\Big| \le N(\text{ii})C_0C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0}, \tag{1.2.21}$$

where

$$N(\text{ii}) = \big(2^2(C(0))^{-2}+2^3(C(0))^{-1}C_0+2^4C_0^2\big)C_0. \tag{1.2.22}$$

It is clear that $N(\text{ii})$ has the form γC_0 with a constant γ that is uniformly bounded for $C_0 \le 1$. Otherwise stated, we can choose a sufficiently small constant $N(\text{ii})$ that is independent of p_0 by choosing C_0 sufficiently small. The choice of $N(\text{ii})$ is made in accordance with the number of sums used to compute $z_0^{(p_0+1)}(0,\rho)$, that is, 7 at this stage, although we need to make it smaller to sum up around 20 kinds of such sums in computing $\big(\tilde{A}_0^{(p_0)},\tilde{B}_0^{(p_0)}\big)$. This is the reason why we keep an extra constant N_0 in (1.2.56) below. Thus, logically speaking, we should fix $N(j)$ at the very end of the proof of this lemma. The important point is that we can choose them independent of p_0,

Since the domination of Cauchy's integral of $(\beta.\text{i})$ requires some delicate treatment as we will see below, we next study the contribution from $(\beta.j)$ ($j = \text{iii},\text{iv},\text{v}$). As these terms may contain singularities at $s = 0$ through the factor t^{-2}, we estimate the contour integral for $r_0 \neq 0$. When $p_0 = 2$, $(\beta.\text{iii})$ reduces to $t^{-2}\tilde{A}_0^{(0)}$, and hence we find

$$\frac{1}{2\pi i}\oint(\beta.\text{iii};p_0=2)\frac{ds}{s} = \frac{1}{2\pi i}\int_{|s|=r_0}\frac{s^2}{t^2}\tilde{A}_0^{(0)}\frac{ds}{s^3} \underset{\text{def}}{=} \tilde{A}_0^{(0)}I(r). \tag{1.2.23}$$

Therefore we have the following relation (1.2.24) for $R_0 \ge 1$:

$$\Big|\frac{1}{2\pi i}\int_{|s|=r_0}(\beta.\text{iii};p_0=2)\frac{ds}{s}\Big| \le N_0(\text{iii})C_0C(2)\big(R_0|\rho|^{-1}\big)^2, \tag{1.2.24}$$

where $N_0(\text{iii})$ is a constant which has the form

$$\gamma R_0^{-1} \tag{1.2.25}$$

with γ being given by

(1.2.26) $$\big(C_0C(2)\big)^{-1}|A_0^{(0)}|\,|I(r)|\big(R_0|\rho|^{-1}\big)^{-1}.$$

When $p_0 \geq 3$, the induction hypothesis entails the following:

(1.2.27)
$$\begin{aligned}
&\Big|\frac{1}{2\pi i}\oint(\beta.\mathrm{iii})\frac{ds}{s}\Big| \\
&= \Big|\frac{1}{2\pi i}\int_{|s|=r_0}\frac{s^2}{t^2}\Big[\tilde{A}_0^{(0)}\Big(2\dot{x}_0^{(p_0-2)}(s,\rho)+\sum_{\substack{i+j=p_0-2\\ i,j\geq 1}}\dot{x}_0^{(i)}(s,\rho)\dot{x}_0^{(j)}(s,\rho)\Big) \\
&\quad + \sum_{1\leq k\leq p_0-3}\tilde{A}_0^{(k)}\Big(2\dot{x}_0^{(p_0-2-k)}(s,\rho)+\sum_{\substack{i+j=p_0-2-k\\ i,j\geq 1}}\dot{x}_0^{(i)}(s,\rho)\dot{x}_0^{(j)}(s,\rho)\Big) \\
&\quad + \tilde{A}_0^{(p_0-2)}\Big]\frac{ds}{s^3}\Big| \\
&\leq |I(r)|\Big(4(C(0))^{-1}|\rho|^{-1}|A_0^{(0)}|(1+C_0)+4C_0(1+C_0)+1\Big) \\
&\quad \times C_0C(p_0-2)\big(R_0|\rho|^{-1}\big)^{p_0-2} \\
&\leq N(\mathrm{iii})C_0C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0},
\end{aligned}$$

where

(1.2.28)
$$N(\mathrm{iii}) = 4|I(r)|\Big(4(C(0))^{-1}|A_0^{(0)}|(1+C_0)+4|\rho|C_0(1+C_0)+|\rho|\Big)|\rho|R_0^{-2}.$$

Here the factor 4 dominates $C(p_0-2)/C(p_0)$ for $p_0 \geq 3$. The estimation of the integral of $(\beta.j)$ $(j=\mathrm{iv},\mathrm{v})$ can be done in a similar manner, and we find

(1.2.29) $$\Big|\oint(\beta.\mathrm{iv})\frac{ds}{s}\Big| \leq N(\mathrm{iv})C_0C(p_0-1)\big(R_0|\rho|^{-1}\big)^{p_0}$$

and

$$\text{(1.2.30)} \qquad \Big| \oint (\beta.\text{v}) \frac{ds}{s} \Big| \le N(\text{v}) C_0 C(p_0 - 1) \big(R_0 |\rho|^{-1} \big)^{p_0},$$

where

$$\text{(1.2.31)} \qquad N(\text{iv}), N(\text{v}) = \gamma \big(R_0 |\rho|^{-1} \big)^{-1}$$

with a uniformly bounded constant γ for $R_0|\rho|^{-1} \gg 1$. The domination of the integral of (β.viii) is trivial: by (1.2.7) we have

$$\text{(1.2.32)} \qquad \Big| \oint (\beta.\text{viii}) \frac{ds}{s} \Big| \le N(\text{viii}) C_0 C(p_0 - 1) \big(R_0 |\rho|^{-1} \big)^{p_0}$$

with

$$\text{(1.2.33)} \qquad N(\text{viii}) = \gamma \big(R_0 |\rho|^{-1} \big)^{-1}.$$

Thus what remain to be examined are (β.i), (β.vi) and (β.vii). Interestingly enough, their estimation is closely related to the fact C observed below (1.1.2.23) in the proof of Proposition 1.1.2.1.

We first study (β.vii). By the Taylor expansion we find

$$\text{(1.2.34)} \qquad \sum_{\substack{i+j=p_0+1 \\ i,j \ge 1}} x_0^{(i)}(s,\rho) x_0^{(j)}(s,\rho)$$

$$= \sum_{\substack{i+j=p_0+1 \\ i,j \ge 1}} x_0^{(i)}(0,\rho) x_0^{(j)}(0,\rho)$$

$$+ 2 \sum_{\substack{i+j=p_0+1 \\ i,j \ge 1}} x_0^{(i)}(0,\rho) \dot{x}_0^{(j)}(0,\rho) s + O(s^2).$$

Since $\tilde{f}^{(0)} = \rho g(t,\rho)$ with $g(0,\rho) = 1$, the substitution of (1.2.34) into the integral in the left-hand side of (1.2.35) below entails the following:

$$\text{(1.2.35)} \quad \Big| \frac{1}{2\pi i} \oint \Big(\frac{dt}{ds} \Big)^2 \frac{t^{-1}}{B_0^{(0)}} \tilde{f}^{(0)} \Big(\sum_{\substack{i+j=p_0+1 \\ i,j \ge 1}} x_0^{(i)} x_0^{(j)} \Big) \frac{ds}{s} \Big|$$

$$= \Big| \frac{1}{2\pi i} \oint \Big(\frac{dt}{ds}\Big)^2 \Big(\frac{s}{t}\Big) Z_0 g(t,\rho) \Big\{ \sum_{\substack{i+j=p_0+1\\ i,j\geq 1}} x_0^{(i)}(0,\rho) x_0^{(j)}(0,\rho)$$

$$+ 2s\Big(\sum_{\substack{i+j=p_0+1\\ i,j\geq 1}} x_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big) + O(s^2)\Big\} \frac{ds}{s^2} \Big|,$$

where $Z_0 = \pm 1$ (cf. (1.1.1.13) and (1.1.1.22)). Clearly there is no contribution to the resulting integral from the third term in the braces (i.e., $O(s^2)$), whereas $[G; p, 0]$ $(p \leq p_0 - 1)$ is effectively used to estimate the contribution from the first sum and that from the second one. Let us now recall

$$x_0^{(1)}(0,\rho) = 0. \tag{1.2.36}$$

Hence the indices (i, j) in the first sum and the index i in the second sum may be assumed to be equal to or greater than 2. Hence the induction hypothesis entails

$$\Big| \frac{1}{2\pi i} \oint (\beta.\text{vii}) \frac{ds}{s} \Big| \leq N(\text{vii}) C_0 C(p_0) \big(R_0 |\rho|^{-1}\big)^{p_0}, \tag{1.2.37}$$

where

$$N(\text{vii}) = \gamma C_0 \Big(\big(R_0|\rho|^{-1}\big)^{-1} + 2\Big) \tag{1.2.38}$$

with γ being a uniformly bounded constant for $C_0 \leq 1$.

We next study the contribution from $(\beta.\text{i})$ and $(\beta.\text{vi})$. At first one might be puzzled by the term

$$- \sum_{i+j=p_0} \dot{x}_0^{(i)}(s,\rho)\dot{x}_0^{(j)}(s,\rho)\tilde{A}_0^{(0)}, \tag{1.2.39}$$

which contains

$$-2\dot{x}_0^{(p_0)} \tilde{A}_0^{(0)}. \tag{1.2.40}$$

Fortunately the contribution of this term is cancelled by the contribution from the coefficient of s^2 in the Taylor expansion of

$$\left(\frac{dt}{ds}\right)^2 \frac{t^{-2}}{B_0^{(0)}}\Big(\sum_{i+j=p_0} x_0^{(i)}(s,\rho)x_0^{(j)}(s,\rho)\Big) f^{(1)}(t(s,\rho),\rho) \tag{1.2.41}$$

after the contour integration $\oint ds/s$, as we see below. By expanding

$$\sum_{i+j=p_0} x_0^{(i)}(s,\rho)x_0^{(j)}(s,\rho) \tag{1.2.42}$$

in powers of s as

$$\begin{aligned}
&\sum_{i+j=p_0} x_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho) \\
&+ 2s\sum_{i+j=p_0} x_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho) \\
&+ s^2\Big\{\sum_{i+j=p_0} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho) \\
&\qquad + \sum_{i+j=p_0} x_0^{(i)}(0,\rho)\ddot{x}_0^{(j)}(0,\rho)\Big\} \\
&+ O(s^3),
\end{aligned} \tag{1.2.43}$$

we find

$$\frac{1}{2\pi i}\oint (\beta.\mathrm{vi})\frac{ds}{s} = I_0 + I_1, \tag{1.2.44}$$

where
(1.2.45)

$$I_0 = \frac{1}{2\pi i}\frac{1}{B_0^{(0)}}\oint \frac{s^2}{t^2}\left(\frac{dt}{ds}\right)^2\Big(\sum_{i+j=p_0} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big) f^{(1)}(t,\rho)\frac{ds}{s}$$

and

$$(1.2.46)\qquad I_1 = \frac{1}{2\pi i}\frac{1}{B_0^{(0)}}\oint \frac{s^2}{t^2}\Big(\frac{dt}{ds}\Big)^2\Big\{\sum_{\substack{i+j=p_0\\ i,j\geq 2}} x_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)$$

$$+ 2s\Big(\sum_{\substack{i+j=p_0\\ i\geq 2}} x_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big)$$

$$+ s^2\Big(\sum_{\substack{i+j=p_0\\ i\geq 2}} x_0^{(i)}(0,\rho)\ddot{x}_0^{(j)}(0,\rho)\Big)$$

$$+ O(s^3)\Big\}f^{(1)}(t,\rho)\frac{ds}{s^3}.$$

On the other hand (1.1.1.21) and (1.1.1.23) entail

$$(1.2.47)\qquad I_0 = \frac{1}{B_0^{(0)}}\Big(\sum_{i+j=p_0} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big)A_0^{(0)}.$$

Hence the puzzling part (1.2.39) of the contribution from (β.i) is cancelled out by I_0 ! Therefore

$$(1.2.48)\qquad \frac{1}{2\pi i}\oint\big\{(\beta.\mathrm{i}) + (\beta.\mathrm{vi})\big\}\frac{ds}{s} = I_2 + I_1,$$

where

$$(1.2.49)\qquad I_2 = \frac{-1}{2\pi i}\oint\Big(\sum_{\substack{i+j+k=p_0\\ 1\leq k\leq p_0-1}} \dot{x}_0^{(i)}(s,\rho)\dot{x}_0^{(j)}(s,\rho)\tilde{A}_0^{(k)}\Big)\frac{ds}{s}.$$

Since

$$(1.2.50)\qquad \sum_{\substack{i+j+k=p_0\\ 1\leq k\leq p_0-1}} \dot{x}_0^{(i)}(s,\rho)\dot{x}_0^{(j)}(s,\rho)\tilde{A}_0^{(k)}$$

$$= 2\dot{x}_0^{(0)}(s,\rho)\Big(\sum_{\substack{j+k=p_0\\ j\geq 1,\ p_0-1\geq k\geq 1}} \dot{x}_0^{(j)}(s,\rho)\tilde{A}_0^{(k)}\Big)$$

$$+\sum_{\substack{i+j+k=p_0\\ i,j\geq 1,\ p_0-1\geq k\geq 1}} \dot{x}_0^{(i)}(s,\rho)\dot{x}_0^{(j)}(s,\rho)\tilde{A}_0^{(k)}$$

holds, the induction hypothesis entails the existence of a constant $N(\mathrm{i})$ which satisfy the following:

$$|I_2| \leq N(\mathrm{i}) C_0 C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0}, \tag{1.2.51}$$

$$N(\mathrm{i}) = 4C_0(2+C_0). \tag{1.2.52}$$

To estimate I_1, we note that $\ddot{x}_0^{(0)}(s,\rho)=0$ and

$$\begin{aligned} |\ddot{x}_0^{(j)}(0,\rho)| &= \Big|\frac{1}{2\pi i}\oint \dot{x}_0^{(j)}(s,\rho)\frac{ds}{s^2}\Big| \\ &\leq 2r_0^{-2}C_0C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0} \end{aligned} \tag{1.2.53}$$

holds for $j\geq 1$. Using these facts, we find

$$|I_1| \leq N(\mathrm{vi}) C_0 C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0} \tag{1.2.54}$$

with

$$N(\mathrm{vi}) = \gamma\Big(C_0\big(R_0\big)^{-2}|\rho| + R_0^{-1} + r_0^{-1}C_0R_0^{-1}\Big), \tag{1.2.55}$$

where γ is a constant originating from innocent factors in the integrand (i.e., irrelevant to C_0, R_0 and $|\rho|^{-1}$).

Summing up the estimates of the contributions from $\beta(j)$ ($j=\mathrm{i},\mathrm{ii},\cdots,\mathrm{viii}$) we find that $[G;p,0]$ $(1\leq p\leq p_0-1)$ entails

$$|z_0^{(p_0+1)}(0,\rho)| \leq N_0 C_0 C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0}, \tag{1.2.56}$$

where N_0 is a constant which is independent of p_0 and can be chosen as small as we want if we choose C_0 and R_0^{-1} sufficiently small. Note that each $N(j)$ found in the above contains a factor C_0 or R_0^{-1} or their sum. We also note that the estimate (1.2.56) validates, in particular, $(p_0.\mathrm{i})$.

Let us next confirm (p_0.ii). In view of (1.2.12) we start with (1.2.57) below, whose counterpart in the confirmation of (p_0.i) is (1.2.14).

$$\dot{z}_0^{(p_0)}(0,\rho) = \dot{R}_0^{(p_0)}(0,\rho) = \frac{1}{2\pi i}\oint R_0^{(p_0)}(s,\rho)\frac{ds}{s^2}. \tag{1.2.57}$$

An important difference between (1.2.14) and (1.2.57) is the following point: the index in question in (1.2.14) was (p_0+1), whereas the corresponding index is p_0 in (1.2.57). Thus the domination is easier this time. Actually, as we will note below, even the estimation of the contribution from (β.i) (cf. (1.2.40) and (1.2.60) below) does not require the subtle reasoning related to the fact C observed in the proof of Proposition 1.1.2.1. Hence we avoid the detailed reasoning and content ourselves with locating the points which need some special attention. In what follows we let $I(j)$ $(j=\mathrm{i},\mathrm{ii},\cdots,\mathrm{viii})$ denote

(1.2.58)

$$\frac{1}{2\pi i}\oint \left[(\beta.j) \text{ with the index } (p_0+1) \text{ being replaced by } p_0\right]\frac{ds}{s^2}.$$

(i) Concerning the estimation of I(i): Since we have

$$\begin{aligned}\sum_{\substack{i+j+k=p_0-1\\ k\le p_0-2}} \dot{x}_0^{(i)}\dot{x}_0^{(j)}\tilde{A}_0^{(k)} &= \tilde{A}_0^{(0)}\Big(2\dot{x}_0^{(p_0-1)} + \sum_{\substack{i+j=p_0-1\\ i,j\ge 1}}\dot{x}_0^{(i)}\dot{x}_0^{(j)}\Big)\\ &\quad + \sum_{1\le k\le p_0-2}\tilde{A}_0^{(k)}\Big(2\dot{x}_0^{(p_0-1-k)} + \sum_{\substack{i+j=p_0-1-k\\ i,j\ge 1}}\dot{x}_0^{(i)}\dot{x}_0^{(j)}\Big),\end{aligned} \tag{1.2.59}$$

we find that the most troublesome term may be

$$2\tilde{A}_0^{(0)}\dot{x}_0^{(p_0-1)}. \tag{1.2.60}$$

However, even the contribution from this term is dominated by

$$|A_0^{(0)}||\rho|^{-1}2C_0C(p_0-1)\big(R_0|\rho|^{-1}\big)^{p_0-1} = 2|A_0^{(0)}|\,\frac{C(p_0-1)}{C(p_0)}R_0^{-1}C_0C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0}. \tag{1.2.61}$$

Hence we can readily find

$$|I(\mathrm{i})| \le N(\mathrm{i})C_0C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0} \tag{1.2.62}$$

with a constant $N(\mathrm{i})$ that can be chosen arbitrarily small independently of p_0 by choosing R_0^{-1} sufficiently small. Making a contrast to the earlier estimation of $\oint(\beta.\mathrm{i})ds/s$ with the index (p_0+1), the estimation $I(\mathrm{i})$ does not require the cancellation among terms in $(\beta.\mathrm{i})$ and $(\beta.\mathrm{vi})$.

(ii) Concerning the estimation of $I(\mathrm{ii})$: The integral $I(\mathrm{ii})$ cannot enjoy such a simple form as (1.2.17), because the double pole s^{-2} is contained in the integrand. Still, the restriction on indices (i,j,k,l) again guarantees that at most two of them are allowed to be 0. Hence the induction hypothesis entails

$$|I(\mathrm{ii})| \le N(\mathrm{ii})C_0C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0} \tag{1.2.63}$$

for a constant $N(\mathrm{ii})$ that contains a factor C_0 like the constant $N(\mathrm{ii})$ in (1.2.22).

(iii) Concerning the estimation of $I(\mathrm{iii})$: Since we have (for $p_0 \ge 4$)

(1.2.64)

$$I(\mathrm{iii}) = \frac{1}{2\pi i}\oint \frac{s^2}{t^2}\Big(\sum_{i+j+k=p_0-3} \dot{x}_0^{(i)}\dot{x}_0^{(j)}\tilde{A}_0^{(k)}\Big)\frac{ds}{s^4}$$

$$= \frac{1}{2\pi i}\oint \frac{s^2}{t^2}\Bigg(\tilde{A}_0^{(0)}\Big(2\dot{x}_0^{(p_0-3)} + \sum_{\substack{i+j=p_0-3\\ i,j\ge 1}} \dot{x}_0^{(i)}\dot{x}_0^{(j)}\Big)$$

$$+\sum_{1\le k\le p_0-4}\tilde{A}_0^{(k)}\Big(2\dot{x}_0^{(p_0-3-k)}+\sum_{\substack{i+j=p_0-3-k\\ i,j\ge 1}}\dot{x}_0^{(i)}\dot{x}_0^{(j)}\Big)+\tilde{A}_0^{(p_0-3)}\Bigg)\frac{ds}{s^4},$$

we use the induction hypothesis to find

$$|I(\mathrm{iii})|\le N(\mathrm{iii})C_0C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0-2} \tag{1.2.65}$$

with a constant $N(\mathrm{iii})$ that can be chosen arbitrarily small independently of p_0 by choosing C_0 and R_0^{-1} sufficiently small.

(iv), (v) The estimation of $I(\mathrm{iv})$ and $I(\mathrm{v})$ can be done in the same way as in the estimation of $I(\mathrm{iii})$.

(vi) Concerning the estimation of $I(\mathrm{vi})$: By using the Taylor expansion of

$$\sum_{i+j=p_0-1}x_0^{(i)}(s,\rho)x_0^{(j)}(s,\rho) \tag{1.2.66}$$

in s, we can readily confirm

$$|I(\mathrm{vi})|\le N(\mathrm{vi})C_0C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0} \tag{1.2.67}$$

with a constant $N(\mathrm{vi})$ that can be chosen arbitrarily small independently of p_0 by choosing C_0 and R_0^{-1} sufficiently small. We note that the estimation is uniformly done including the part corresponding to I_0 given by (1.2.45), that is,

(1.2.68)

$$\frac{1}{2\pi i}\frac{1}{B_0^{(0)}}\oint\frac{s^2}{t^2}\Big(\frac{dt}{ds}\Big)^2\Big(\sum_{i+j=p_0-1}\dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\Big)f^{(1)}(t,\rho)\frac{ds}{s^2},$$

just because the required exponent in the right-hand side of (1.2.67) is p_0, not p_0-1.

(vii) The estimation of $I(\mathrm{vii})$ can be done similarly as that of $I(\mathrm{vi})$ with the help of the Taylor expansion of $x_0^{(i)}(s,\rho)$ in s.

(viii) The required estimation of I(viii) is attained by choosing R_0 sufficiently large compared with κ_0 and L_0 in (1.2.7).

Summing up the observations (i), (ii), $\cdots$, (viii) we find that the validity $[G;p,0]$ $(1 \le p \le p_0-1)$ implies that

$$|\dot{z}_0^{(p_0)}(0,\rho)| \le N_1 C_0 C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0} \tag{1.2.69}$$

holds for any given small constant N_1 if we choose C_0 and R_0^{-1} sufficiently small. In particular we have thus confirmed (p_0.ii).

[II] Using the results in part [I], together with the induction hypothesis, we next confirm (p_0.v) and (p_0.vi). For this purpose let us write down the conditions $\Phi_0^{(p_0+3)}|_{t=0}$ and $d\Phi_0^{(p_0+2)}/dt|_{t=0}$ using s-variable. For the sake of notational simplicity, in what follows, we keep some t-derivatives as they are; they are denoted as $x_0^{(k)\prime}$ etc. as usual.

$$\begin{aligned}
&\Big(\frac{ds}{dt}\Big)^{-2}\Phi_0^{(p_0+3)}\Big|_{s=0} \\
&= \Big[\sum_{i+j+k=p_0} \dot{x}_0^{(i)}\dot{x}_0^{(j)}A_0^{(k)} \\
&\quad + \Big(\frac{ds}{dt}\Big)^{-2}\Big(\sum_{i+j+k=p_0+3} x_0^{(i)}x_0^{(j)}f^{(k)} - 2x_0^{(0)}x_0^{(p_0+3)}f^{(0)}\Big) \\
&\quad + \sum_{i+j+k+l=p_0+1} \dot{x}_0^{(i)}\dot{x}_0^{(j)}x_0^{(k)}B_0^{(l)}\Big]\Big|_{s=0} \\
&= A_0^{(p_0)} + 2\dot{x}_0^{(p_0)}(0,\rho)A_0^{(0)} \\
&\quad + \sum_{\substack{i+j+k=p_0 \\ i,j,k\le p_0-1}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)A_0^{(k)}
\end{aligned} \tag{1.2.70}$$

$$+\Big(\sum_{\substack{i+j=p_0+2\\ i,j\geq 2}} x_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)\Big)f^{(1)}(0,\rho)$$

$$+\sum_{\substack{i+j+k=p_0+3\\ i,j,k\geq 2}} x_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)f^{(k)}(0,\rho)$$

$$+x_0^{(p_0+1)}(0,\rho)B_0^{(0)}$$

$$+\sum_{\substack{i+j+k+l=p_0+1\\ 2\leq k\leq p_0}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)x_0^{(k)}(0,\rho)B_0^{(l)}$$

$$=A_0^{(p_0)}+2\big(\dot{z}_0^{(p_0)}(0,\rho)-\tilde{B}_0^{(p_0)}\big)A_0^{(0)}$$

$$+\big(z_0^{(p_0+1)}(0,\rho)+\tilde{A}_0^{(p_0)}\big)B_0^{(0)}$$

$$+\sum_{\substack{i+j+k=p_0\\ i,j,k\leq p_0-1}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)A_0^{(k)}$$

$$+\Big(\sum_{\substack{i+j=p_0+2\\ i,j\geq 2}} x_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)\Big)A_0^{(0)}$$

$$+\sum_{\substack{i+j+k=p_0+3\\ i,j,k\geq 2}} x_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)f^{(k)}(0,\rho)$$

$$+\sum_{\substack{i+j+k+l=p_0+1\\ 2\leq k\leq p_0}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)x_0^{(k)}(0,\rho)B_0^{(l)},$$

(1.2.71)

$$\Big(\frac{ds}{dt}\Big)^{-2}\ \frac{d\Phi_0^{(p_0+2)}}{dt}\Big|_{s=0}$$

$$= \Big[2\Big(\frac{ds}{dt}\Big)^{-1} \sum_{i+j+k=p_0-1} x_0^{(i)\prime\prime} \dot{x}_0^{(j)} A_0^{(k)}$$

$$+ \Big(\frac{ds}{dt}\Big)^{-2}\Big(\sum_{i+j+k=p_0+2} x_0^{(i)} x_0^{(j)} f^{(k)\prime} - 2x_0^{(0)} x_0^{(p_0+2)} f^{(0)\prime}\Big)$$

$$+ 2\Big(\frac{ds}{dt}\Big)^{-1}\Big(\sum_{i+j+k=p_0+2} \dot{x}_0^{(i)} x_0^{(j)} f^{(k)} - \Big(\frac{d}{ds}\big(x_0^{(0)} x_0^{(p_0+2)}\big)\Big) f^{(0)}\Big)$$

$$+ 2\Big(\frac{ds}{dt}\Big)^{-1}\Big(\sum_{i+j+k+l=p_0} x_0^{(i)\prime\prime} \dot{x}_0^{(j)} x_0^{(k)} B_0^{(l)}\Big)$$

$$+ \sum_{i+j+k+l=p_0} \dot{x}_0^{(i)} \dot{x}_0^{(j)} x_0^{(k)\prime} B_0^{(l)}\Big]\Big|_{s=0}$$

$$= 2Z_0 \sum_{i+j+k=p_0-1} x_0^{(i)\prime\prime}(0,\rho) \dot{x}_0^{(j)}(0,\rho) A_0^{(k)}$$

$$+ \Big(\sum_{\substack{i+j=p_0+2 \\ i,j\geq 2}} x_0^{(i)}(0,\rho) x_0^{(j)}(0,\rho)\Big)\rho$$

$$+ \sum_{\substack{i+j+k=p_0+2 \\ i,j\geq 2,\, k\geq 1}} x_0^{(i)}(0,\rho) x_0^{(j)}(0,\rho) f^{(k)\prime}(0,\rho)$$

$$+ 2Z_0 x_0^{(p_0+1)}(0,\rho) A_0^{(0)}$$

$$+ 2Z_0 \sum_{\substack{i+j=p_0+1 \\ 2\leq j\leq p_0}} \dot{x}_0^{(i)}(0,\rho) x_0^{(j)}(0,\rho) A_0^{(0)}$$

$$+ 2Z_0 \sum_{\substack{i+j+k=p_0+2 \\ j,k\geq 2}} \dot{x}_0^{(i)}(0,\rho) x_0^{(j)}(0,\rho) f^{(k)}(0,\rho)$$

$$+ 2Z_0 x_0^{(0)\prime\prime}(0,\rho) x_0^{(p_0)}(0,\rho) B_0^{(0)}$$

$$+ 2Z_0 \sum_{\substack{i+j+k+l=p_0 \\ 2\le k\le p_0-1}} x_0^{(i)\prime\prime}(0,\rho)\dot{x}_0^{(j)}(0,\rho) x_0^{(k)}(0,\rho) B_0^{(l)}$$

$$+ 3Z_0 \dot{x}_0^{(p_0)}(0,\rho) B_0^{(0)} + Z_0 B_0^{(p_0)}$$

$$+ Z_0 \sum_{\substack{i+j+k+l=p_0 \\ i,j,k,l\le p_0-1}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\dot{x}_0^{(k)}(0,\rho) B_0^{(l)}$$

$$= 2Z_0 \Big(z_0^{(p_0+1)}(0,\rho) + \tilde{A}_0^{(p_0)} \Big) A_0^{(0)}$$

$$+ 3Z_0 \Big(\dot{z}_0^{(p_0)}(0,\rho) - \tilde{B}_0^{(p_0)} \Big) B_0^{(0)} + Z_0 B_0^{(p_0)}$$

$$+ 2Z_0 \sum_{i+j+k=p_0-1} x_0^{(i)\prime\prime}(0,\rho)\dot{x}_0^{(j)}(0,\rho) A_0^{(k)}$$

$$+ \rho \Big(\sum_{\substack{i+j=p_0+2 \\ i,j\ge 2}} x_0^{(i)}(0,\rho) x_0^{(j)}(0,\rho) \Big)$$

$$+ \sum_{\substack{i+j+k=p_0+2 \\ i,j\ge 2,\ k\ge 1}} x_0^{(i)}(0,\rho) x_0^{(j)}(0,\rho) f^{(k)\prime}(0,\rho)$$

$$+ 2Z_0 \sum_{\substack{i+j=p_0+1 \\ 2\le j\le p_0}} \dot{x}_0^{(i)}(0,\rho) x_0^{(j)}(0,\rho) A_0^{(0)}$$

$$+ 2Z_0 \sum_{\substack{i+j+k=p_0+2 \\ j,k\ge 2}} \dot{x}_0^{(i)}(0,\rho) x_0^{(j)}(0,\rho) f^{(k)}(0,\rho)$$

$$+ 2Z_0 x_0^{(0)\prime\prime}(0,\rho) x_0^{(p_0)}(0,\rho) B_0^{(0)}$$

$$+ 2Z_0 \sum_{\substack{i+j+k+l=p_0 \\ 2\le k\le p_0-1}} x_0^{(i)\prime\prime}(0,\rho)\dot{x}_0^{(j)}(0,\rho)x_0^{(k)}(0,\rho)B_0^{(l)}$$

$$+ Z_0 \sum_{\substack{i+j+k+l=p_0 \\ i,j,k,l\le p_0-1}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\dot{x}_0^{(k)}(0,\rho)B_0^{(l)}.$$

In the above computation we have repeatedly used

$$x_0^{(0)}(s,\rho) = s \quad (\text{cf. } (1.1.1.10)), \tag{1.2.72}$$

$$x_0^{(1)}(0,\rho) = 0 \quad (\text{cf. } (1.1.1.24)), \tag{1.2.73}$$

$$f^{(0)}(t,\rho) = t\rho g(t,\rho) \text{ with } g(0,\rho) = 1 \text{ (cf. (1.1) and (1.3))}, \tag{1.2.74}$$

$$Z_0 = x_0^{(0)\prime}(0,\rho) = \pm 1 \quad (\text{cf. } (1.1.1.13) \text{ and } (1.1.1.23)), \tag{1.2.75}$$

$$f^{(1)}(0,\rho) = A_0^{(0)} \quad (\text{cf. } (1.1.1.21)), \tag{1.2.76}$$

$$B_0^{(0)} = Z_0^{-1}\rho \quad (\text{cf. } (1.1.1.13)), \tag{1.2.77}$$

$$x_0^{(p_0+1)}(0,\rho) = z_0^{(p_0+1)}(0,\rho) + \tilde{A}_0^{(p_0)} \quad (\text{cf. } (1.2.11)) \tag{1.2.78}$$

and

$$\dot{x}_0^{(p_0)}(0,\rho) = \dot{z}_0^{(p_0)}(0,\rho) - \tilde{B}_0^{(p_0)} \quad (\text{cf. } (1.2.12)), \tag{1.2.79}$$

and we have separated out $\big(A_0^{(p_0)}, B_0^{(p_0)}\big)$ from other terms. Here we have used Lemma 1.1.3.4 together with (1.1.3.3.r) that $T_0^{(r)}$ satisfies.

Thus we have found the following relations which determine $\big(\tilde{A}_0^{(p_0)}, \tilde{B}_0^{(p_0)}\big)$:

$$-2\Big(B_0^{(0)}\tilde{A}_0^{(p_0)} - A_0^{(0)}\tilde{B}_0^{(p_0)}\Big) \tag{1.2.70$'$}$$

$$= -2A_0^{(p_0)} + 2\,\frac{A_0^{(0)}}{B_0^{(0)}}\,B_0^{(p_0)}$$

$$= 2\dot{z}_0^{(p_0)}(0,\rho)A_0^{(0)} \qquad (\gamma.\mathrm{i})$$

$$+\, z_0^{(p_0+1)}(0,\rho)B_0^{(0)} \qquad (\gamma.\mathrm{ii})$$

$$+ \sum_{\substack{i+j+k=p_0\\ i,j,k\le p_0-1}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)B_0^{(0)}\tilde{A}_0^{(k)} \qquad (\gamma.\mathrm{iii})$$

$$+ \sum_{\substack{i+j=p_0+2\\ i,j\ge 2}} x_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)B_0^{(0)}\tilde{A}_0^{(0)} \qquad (\gamma.\mathrm{iv})$$

$$+ \sum_{\substack{i+j+k=p_0+3\\ i,j,k\ge 2}} x_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)f^{(k)}(0,\rho) \qquad (\gamma.\mathrm{v})$$

$$+ \sum_{\substack{i+j+k+l=p_0+1\\ 2\le k\le p_0}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)x_0^{(k)}(0,\rho)B_0^{(0)}\tilde{B}_0^{(l)} \qquad (\gamma.\mathrm{vi})$$

$$\underset{\mathrm{def}}{=} \Gamma_0^{(p_0)}$$

$(1.2.71')$

$$-\,2\Big(A_0^{(0)}\tilde{A}_0^{(p_0)} - B_0^{(0)}\tilde{B}_0^{(p_0)}\Big)$$

$$= -2\,\frac{A_0^{(0)}}{B_0^{(0)}}\,A_0^{(p_0)} + 2B_0^{(p_0)}$$

$$= 2z_0^{(p_0+1)}(0,\rho)A_0^{(0)} \qquad (\delta.\mathrm{i})$$

$$+\, 3\dot{z}_0^{(p_0)}(0,\rho)B_0^{(0)} \qquad (\delta.\mathrm{ii})$$

$$+\,2\sum_{i+j+k=p_0-1} x_0^{(i)\prime\prime}(0,\rho)\dot{x}_0^{(j)}(0,\rho)B_0^{(0)}\tilde{A}_0^{(k)} \tag{δ.iii}$$

$$+\,Z_0\rho\Big(\sum_{\substack{i+j=p_0+2\\ i,j\geq 2}} x_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)\Big) \tag{δ.iv}$$

$$+\,Z_0\sum_{\substack{i+j+k=p_0+2\\ i,j\geq 2,\ k\geq 1}} x_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)f^{(k)\prime}(0,\rho) \tag{δ.v}$$

$$+\,2\sum_{\substack{i+j=p_0+1\\ 2\leq j\leq p_0}} \dot{x}_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)A_0^{(0)} \tag{δ.vi}$$

$$+\,2\sum_{\substack{i+j+k=p_0+2\\ j,k\geq 2}} \dot{x}_0^{(i)}(0,\rho)x_0^{(j)}(0,\rho)f^{(k)}(0,\rho) \tag{δ.vii}$$

$$+\,2x_0^{(0)\prime\prime}(0,\rho)x_0^{(p_0)}(0,\rho)B_0^{(0)} \tag{δ.viii}$$

$$+\,2\sum_{\substack{i+j+k+l=p_0\\ 2\leq k\leq p_0-1}} x_0^{(i)\prime\prime}(0,\rho)\dot{x}_0^{(j)}(0,\rho)x_0^{(k)}(0,\rho)B_0^{(0)}\tilde{B}_0^{(l)} \tag{δ.ix}$$

$$+\sum_{\substack{i+j+k+l=p_0\\ i,j,k,l\leq p_0-1}} \dot{x}_0^{(i)}(0,\rho)\dot{x}_0^{(j)}(0,\rho)\dot{x}_0^{(k)}(0,\rho)B_0^{(0)}\tilde{B}_0^{(l)} \tag{δ.x}$$

$$\underset{\text{def}}{=}\Delta_0^{(p_0)}.$$

Then, by using the assumption (1.1.2) together with (1.2.75), (1.2.76) and (1.2.77), we obtain the following relation (1.2.80) from (1.2.70′) and

(1.2.71′):
(1.2.80)

$$\begin{pmatrix} \tilde{A}_0^{(p_0)} \\ \tilde{B}_0^{(p_0)} \end{pmatrix} = \Big(-\frac{1}{2}\Big)\Big(B_0^{(0)2} - A_0^{(0)2}\Big)^{-1} \begin{pmatrix} B_0^{(0)}\Gamma_0^{(p_0)} - A_0^{(0)}\Delta_0^{(p_0)} \\ A_0^{(0)}\Gamma_0^{(p_0)} - B_0^{(0)}\Delta_0^{(p_0)} \end{pmatrix}.$$

Hence it suffices to dominate each of terms $(\gamma.j)$ $(j = \mathrm{i}, \mathrm{ii}, \cdots, \mathrm{vi})$ and $(\delta.j)$ $(j = \mathrm{i}, \mathrm{ii}, \cdots, \mathrm{x})$ by a constant of the form

(1.2.81) $$NC_0C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0}$$

with a constant N which can be chosen sufficiently small and independent of p_0 by letting C_0 and R_0^{-1} sufficiently small. As we have already confirmed the estimate of this sort for $(\gamma.\mathrm{i})$, $(\gamma.\mathrm{ii})$, $(\delta.\mathrm{i})$ and $(\delta.\mathrm{ii})$ it is enough to examine other terms. The reasoning is basically the same as that used in part [I]. For example we find the following estimate (1.2.82) for the sum $(\gamma.\mathrm{iv})$, which one may think to be the most troublesome one in view of the range of indices:

(1.2.82)

$$\begin{aligned} |(\gamma.\mathrm{iv})| &= \Big| \sum_{\substack{i'+j'=p_0 \\ i',j'\geq 1}} x_0^{(i'+1)}(0,\rho)x_0^{(j'+1)}(0,\rho)A_0^{(0)}\Big| \\ &= \Big| \sum_{\substack{i'+j'=p_0 \\ i',j'\geq 1}} \big(z_0^{(i'+1)}(0,\rho) + \tilde{A}_0^{(i')}\big)\big(z_0^{(j'+1)}(0,\rho) + \tilde{A}_0^{(j')}\big)A_0^{(0)}\Big| \\ &\leq 4C_0^2C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0}\big|A_0^{(0)}\big|. \end{aligned}$$

Therefore we find

(1.2.83) $$|(\gamma.\mathrm{iv})| \leq N(\gamma.\mathrm{iv})C_0C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0}$$

with

(1.2.84) $$N(\gamma.\mathrm{iv}) = 4\big|A_0^{(0)}\big|C_0.$$

The same technique that makes full use of the estimate of $|z_0^{(p_0+1)}(0,\rho)|$ also applies to (γ.v), (γ.vi), (δ.iv), (δ.v), (δ.vi), (δ.vii), (δ.viii) and (δ.ix), whereas the rest of terms, i.e., (γ.iii), (δ.iii) and (δ.x) are rather easy to handle. For example we readily find

$$|(\delta.\mathrm{x})| \leq N(\delta.\mathrm{x}) C_0 C(p_0) \big(R_0 |\rho|^{-1}\big)^{p_0} \tag{1.2.85}$$

with

$$N(\delta.\mathrm{x}) = 4\Big((C(0))^{-2} + 2(C(0))^{-1}C_0 + 4C_0^2\Big)|\rho| C_0. \tag{1.2.86}$$

Thus the induction hypothesis together with (1.2.80) entails that

$$|\tilde{A}_0^{(p_0)}| \leq N_2 C_0 C(p_0) \big(R_0 |\rho|^{-1}\big)^{p_0} \tag{1.2.87}$$

and

$$|\tilde{B}_0^{(p_0)}| \leq N_2 C_0 C(p_0) \big(R_0 |\rho|^{-1}\big)^{p_0} \tag{1.2.88}$$

hold, where N_2 is a constant which is independent of p_0 and can be chosen as small as we want if we choose C_0 and R_0^{-1} sufficiently small. In particular we have thus confirmed (p_0.v) and (p_0.vi).

[III] Next we validate (p_0.iii) and (p_0.iv). We first note that, by the same reasoning with the estimation (1.2.69) of $\dot{R}_0^{(p_0)}(0,\rho)$ (cf. (1.2.57)), we find

$$\big\|R_0^{(p_0)}(\cdot,\rho)\big\|_{r_0} \leq N_2 C_0 C(p_0) \big(R_0 |\rho|^{-1}\big)^{p_0} \tag{1.2.89}$$

holds, where N_2 is a constant which is independent of p_0 and can be chosen as small as we want if we choose C_0 and R_0^{-1} sufficiently small. (Since $R_0^{(p_0)}$ is holomorphic at $s = 0$, the estimates (1.2.89) directly follows from the maximum modulus principle and the induction hypothesis.) Then, to obtain (p_0.iii), we use the following integral representation (1.2.91) of the holomorphic solution $x_0^{(p_0)}(s,\rho)$ of the

equation (1.2.90).
(1.2.90) (=$[E; p_0, 0]$)

$$\Big(2s\,\frac{d}{ds}-1\Big)x_0^{(p_0)}(s,\rho) = -\tilde{A}_0^{(p_0-1)} - \tilde{B}_0^{(p_0)}s + R_0^{(p_0)}(s,\rho),$$

(1.2.91)

$$x_0^{(p_0)}(s,\rho) = x_0^{(p_0)}(0,\rho) + \frac{s^{1/2}}{2}\int_0^s u^{-3/2}\Big(R_0^{(p_0)}(u,\rho) - \tilde{A}_0^{(p_0-1)} - \tilde{B}_0^{(p_0)}u + x_0^{(p_0)}(0,\rho)\Big)du.$$

Here we note that the integrand of the integral in the right-hand side of (1.2.91) is integrable near $u=0$, because (1.2.11) entails that it has the form

$$u^{-1/2}\Big(\big(R_0^{(p_0)}(u,\rho) - R_0^{(p_0)}(0,\rho)\big)u^{-1} - \tilde{B}_0^{(p_0)}\Big). \tag{1.2.92}$$

Therefore, combining the results in part [I], [II] and (1.2.89), we obtain the following estimates:

$$\big\|x_0^{(p_0)}(\cdot,\rho)\big\|_{r_0} \le N_2 C_0 C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0}, \tag{1.2.93}$$

where N_2 is a sufficiently small constant. Then (p_0.iii) immediately follows from (1.2.87), (1.2.88) and (1.2.93). Further, to obtain (p_0.iv), we rewrite (1.2.90) as follows:

$$\dot{x}_0^{(p_0)}(s,\rho) = \frac{1}{2s}\Big(x_0^{(p_0)}(s,\rho) - \tilde{A}_0^{(p_0-1)} - \tilde{B}_0^{(p_0)}s + R_0^{(p_0)}(s,\rho)\Big). \tag{1.2.94}$$

Then the following estimates follow from the maximum modulus principle:

$$\big\|\dot{x}_0^{(p_0)}(\cdot,\rho)\big\|_{r_0} \le N_2 C_0 C(p_0)\big(R_0|\rho|^{-1}\big)^{p_0}, \tag{1.2.95}$$

where N_2 is a sufficiently small constant. Thus (p_0.iv) follows from (1.2.88) and (1.2.93).

Summing up the results in part [I], [II] and [III], we conclude that the induction proceeds. This completes the proof of Lemma 1.2.3.

□

We now embark on the proof of Proposition 1.2.1 below. In order to facilitate the concrete expression of the Taylor expansion of the Schwarzian derivative $\{x;t\}$ we prepare the following notations.

Definition 1.2.2. (i) For multi-indices $\vec{\kappa} = (\kappa_1, \kappa_2, \cdots, \kappa_\mu)$ and $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_\mu)$ in $\mathbb{N}_0^\mu$, we define

$$|\vec{\lambda}|_\mu = \sum_{j=1}^{\mu} \lambda_j, \tag{1.2.96}$$

$$\vec{\lambda}! = \prod_{j=1}^{\mu} \lambda_j!. \tag{1.2.97}$$

(ii) For $(\vec{\lambda}, \vec{\kappa})$-dependent quantities $X_{\kappa_j}^{(\lambda_j)}$ (such as $dx_{\kappa_j}^{(\lambda_j)}/dt$) we define

$$X_{\vec{\kappa}}^{(\vec{\lambda})} = \prod_{j=1}^{\mu} X_{\kappa_j}^{(\lambda_j)} \tag{1.2.98}$$

and

$$\sum_{|\vec{\kappa}|_\mu = k}^{*} \sum_{|\vec{\lambda}|_\mu = l} X_{\vec{\kappa}}^{(\vec{\lambda})} = \begin{cases} 1 & \text{for } \mu = 0 \\ \displaystyle\sum_{\substack{|\vec{\kappa}|_\mu = k \\ \kappa_j \geq 1}} \sum_{|\vec{\lambda}|_\mu = l} \prod_{j=1}^{\mu} X_{\kappa_j}^{(\lambda_j)} & \text{for } \mu \geq 1. \end{cases} \tag{1.2.99}$$

For the notational convenience we also introduce the following

Definition 1.2.3. We define $\tilde{A}_{2n}^{(p)}$ and $\tilde{B}_{2n}^{(p)}$ by the following:

$$\tilde{A}_{2n}^{(p)} = A_{2n}^{(p)}/B_0^{(0)}, \quad \tilde{B}_{2n}^{(p)} = B_{2n}^{(p)}/B_0^{(0)}, \tag{1.2.100}$$

$$\tilde{A}_{2n}^{(-1)} = 0. \tag{1.2.101}$$

Proposition 1.2.1. *There exist positive constants* (r_0, R, A) *and a sufficiently small constant* N_0 *for which the following estimate* $[G; p, 2n]$ *holds for every* $p \geq 0$*, every* $n \geq 1$*, every* ρ *in* $\{\rho \in \mathbb{C}; 0 < \rho \leq r_0\}$ *and any positive constant* ε *that is smaller than* $r_0/3$ *:*

$$[G; p, 2n] =$$

$$\begin{cases} (p, 2n)(\mathrm{i}) \ \ |x_{2n}^{(p+1)}(0, \rho)| \leq N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n)! \varepsilon^{-2n} (A|\rho|^{-1})^n, \\ (p, 2n)(\mathrm{ii}) \ \ |\tilde{A}_{2n}^{(p)}(\rho)| \leq N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n)! \varepsilon^{-2n} \big(A|\rho|^{-1}\big)^n, \\ (p, 2n)(\mathrm{iii}) \ |\tilde{B}_{2n}^{(p)}(\rho)| \leq N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n)! \varepsilon^{-2n} \big(A|\rho|^{-1}\big)^n, \\ (p, 2n)(\mathrm{iv}) \ \|x_{2n}^{(p)}(\cdot, \rho)\|_{[r_0-\varepsilon]} \leq N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n)! \varepsilon^{-2n} \big(A|\rho|^{-1}\big)^n, \\ (p, 2n)(\mathrm{v}) \ \ \|\dot{x}_{2n}^{(p)}(\cdot, \rho)\|_{[r_0-\varepsilon]} \leq N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n)! \varepsilon^{-2n} \big(A|\rho|^{-1}\big)^n. \end{cases}$$

In what follows, for the simplicity of the notation, we use the symbol $\|h\|_{[r]}$ to denote $\|h(\cdot, \rho)\|_{[r]}$ even when a holomorphic function $h(s, \rho)$ contains ρ as an auxiliary variables other than s.

Remark 1.2.2. We note that, as the form of the estimates $[G; p, 2n]$ for $n \geq 1$ indicates, we can take $N_0 > 0$ arbitrarily small by taking $A > 0$ sufficiently large.

Remark 1.2.3. As we will see in the proof below, the order of $|\rho|$ relevant to n in $[G; p, 2n]$ is inductively determined by the contribution from (α.ix) in (1.1.3.43). (Cf. (1.2.179) and (1.2.180).)

Remark 1.2.4. In view of Remark 1.2.1, we see that $[G; p, 2n]$ with $n = 0$ coincides with $[G; p, 0]$ in Lemma 1.2.3.

Proof. Aside from the treatment of terms originating from the Schwarzian derivative, the flow of the reasoning is basically the same as that in the proof of Lemma 1.2.3. As the proof is lengthy, we separate it into four parts, part [I] $\sim$ part [IV]. Before beginning the proof we note that the term in the left-hand side of each $(p, 2n)(j)$ $(j = \text{(i)}, \text{(ii)}, \cdots, \text{(v)}$ with $(p, n) = (0, 1)$ vanishes. This fact is implicitly confirmed in what follows, but, in view of its interest, we give a detailed proof in Appendix B.

[I] Let us first study how to dominate the contribution from $\{x; t\}^{(p)}_{2(n-1)}$. Using the Taylor expansion we find

(1.2.102)

$$\{x; t\}^{(p)}_{2(n-1)}$$

$$= \sum_{\substack{k_1+k_2=n-1 \\ l_1+l_2+l_3=p}} \frac{d^3 x^{(l_1)}_{2k_1}}{dt^3} \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^\nu \left(\left(\frac{dx_0}{dt} \right)^{-\nu-1} \right)^{(l_2)} \sum_{|\vec{\kappa}|_\nu = k_2}{}^{*} \sum_{|\vec{\lambda}|_\nu = l_3} \frac{dx^{(\vec{\lambda})}_{2\vec{\kappa}}}{dt}$$

$$- \frac{3}{2} \sum_{\substack{k_1+k_2+k_3=n-1 \\ l_1+l_2+l_3+l_4=p}} \frac{d^2 x^{(l_1)}_{2k_1}}{dt^2} \frac{d^2 x^{(l_2)}_{2k_2}}{dt^2} \sum_{\nu=\min\{1,k_3\}}^{k_3} (-1)^\nu (\nu+1) \left(\left(\frac{dx_0}{dt} \right)^{-\nu-2} \right)^{(l_3)}$$

$$\times \sum_{|\vec{\kappa}|_\nu = k_3}{}^{*} \sum_{|\vec{\lambda}|_\nu = l_4} \frac{dx^{(\vec{\lambda})}_{2\vec{\kappa}}}{dt},$$

where we use the symbol $\left((dx_0/dt)^{-\nu-1}\right)^{(l_2)}$ (resp., $\left((dx_0/dt)^{-\nu-2}\right)^{(l_3)}$ to mean the coefficient of a^{l_2} (resp., a^{l_3}) of the Taylor expansion of $(dx_0/dt)^{-\nu-1}$ (resp., $(dx_0/dt)^{-\nu-2}$) in powers of a. To dominate them we prepare the following

Lemma 1.2.4. *Let $x_0(s, a, \rho)$ denote*

(1.2.103) $$\sum_{p \geq 0} x^{(p)}_0 (s, \rho) a^p.$$

Then Lemma (1.2.3) *entails the existence of some positive constants* r_0, M_0 *and* R *for which the following inequality holds:*

$$\Big\|\Big(\Big(\frac{dx_0}{dt}\Big)^{-\nu}\Big)^{(l)}\Big\|_{[r_0]} \le M_0^\nu C(l)\big(R|\rho|^{-1}\big)^l. \tag{1.2.104}$$

Proof. Since we may assume that

$$\frac{dx_0^{(0)}}{dt}(s,\rho) = \Big(\frac{dt}{ds}\Big)^{-1} \neq 0 \tag{1.2.105}$$

holds on $\{s; |s| \le r_0\}$, it follows from the estimates $(p.\widetilde{\text{iii}})$ of $\dot{x}_0^{(p)}$ in Remark 1.2.1 that there exist some positive constant $\tilde{M}_0$ for which $(dx_0/dt)^{-1}$ is holomorphic on $\Omega = \{(s,a,\rho);\ |s| \le r_0,\ 2R_0|a| \le |\rho|\}$ and

$$\sup_\Omega \Big|\frac{dx_0}{dt}\Big|^{-1} \le \tilde{M}_0 \tag{1.2.106}$$

holds. Hence we find

$$\sup_\Omega \Big|\Big(\frac{dx_0}{dt}\Big)^{-\nu}\Big| \le \tilde{M}_0^\nu. \tag{1.2.107}$$

This then implies

$$\Big\|\Big(\Big(\frac{dx_0}{dt}\Big)^{-\nu}\Big)^{(l)}\Big\|_{[r_0]} \le \tilde{M}_0^\nu\big(2R_0|\rho|^{-1}\big)^l. \tag{1.2.108}$$

On the other hand it immediately follows from the definition (1.2.1) of $C(l)$ that

$$\frac{3}{2\pi^2}\,2^{-l-2} \le C(l) \tag{1.2.109}$$

holds for every l in $\mathbb{N}_0$. Therefore we obtain

$$\Big\|\Big(\Big(\frac{dx_0}{dt}\Big)^{-\nu}\Big)^{(l)}\Big\|_{[r_0]} \le M_0^\nu C(l)\big(R|\rho|^{-1}\big)^l \tag{1.2.110}$$

by setting

$$(1.2.111)\qquad M_0 = \frac{8}{3}\pi^2 \tilde{M}_0 \quad \text{and} \quad R = 4R_0.$$

This completes the proof of Lemma 1.2.4.

□

We now resume the proof of Proposition 1.2.1. Let us begin our reasoning by dominating the first sum in the right-hand side of (1.2.102), namely

$$(1.2.112)\quad S^{(p)}_{2(n-1)} \underset{\text{def}}{=} \sum_{\substack{k_1+k_2=n-1\\ l_1+l_2+l_3=p}} \frac{d^3 x^{(l_1)}_{2k_1}}{dt^3} \sum_{\nu=\min\{1,k_2\}}^{k_2} (-1)^\nu \Big(\Big(\frac{dx_0}{dt}\Big)^{-\nu-1}\Big)^{(l_2)}$$

$$\times \sum^{*}_{|\vec{\kappa}|_\nu=k_2} \sum_{|\vec{\lambda}|_\nu=l_3} \frac{dx^{(\vec{\lambda})}_{2\vec{\kappa}}}{dt}.$$

We first note that $(p.\widetilde{\text{iii}})$ Remark 1.2.1 and Cauchy's integral formula applied to $dx_0^{(l)}/dt$ entail

$$(1.2.113)\qquad \Big\|\frac{d^2 x_0^{(l)}}{dt^2}\Big\|_{[r_0-\varepsilon]} \le M_0 C(l)\big(R|\rho|^{-1}\big)^l \varepsilon^{-1},$$

$$(1.2.114)\qquad \Big\|\frac{d^3 x_0^{(l)}}{dt^3}\Big\|_{[r_0-\varepsilon]} \le 2! M_0 C(l)\big(R|\rho|^{-1}\big)^l \varepsilon^{-2}$$

for $l \geq 0$ and some positive constant M_0. Indeed, (1.2.113) and (1.2.114) follow from (1.2.105) and the following relations for the differentiation of a holomorphic function $f(s)$ with respect to the two variables t and s:

$$(1.2.115)\quad \frac{d^2 f}{dt^2}(s) = \Big(\frac{dt(s)}{ds}\Big)^{-2} \frac{d^2}{ds^2} f(s) + \frac{1}{2}\frac{d}{ds}\Big(\frac{dt(s)}{ds}\Big)^{-2} \frac{d}{ds} f(s),$$

$$(1.2.116)\qquad \frac{d^3 f}{dt^3}(s) = \Big(\frac{dt(s)}{ds}\Big)^{-3}\frac{d^3}{ds^3}f(s) + \frac{d}{ds}\Big(\frac{dt(s)}{ds}\Big)^{-3}\frac{d^2}{ds^2}f(s) + \frac{1}{2}\Big(\frac{dt(s)}{ds}\Big)^{-1}\frac{d^2}{ds^2}\Big(\frac{dt(s)}{ds}\Big)^{-2}\frac{d}{ds}f(s).$$

Remark 1.2.5. Since we can take the constant C_0 in $(p.\widetilde{\text{iii}})$ in Remark 1.2.1 for $p \geq 1$ arbitrarily small by taking R_0 sufficiently large, we can take M_0 in (1.2.113) and (1.2.114) for $l \geq 1$ also arbitrarily small. However this fact does not hold for $l = 0$. Fortunately, our reasoning below does not require M_0 to be arbitrarily small. Hence, for the simplicity of presentation, we use the estimates (1.2.113) and (1.2.114) in the form that is applicable to both cases $l = 0$ and $l \geq 1$, that is, we only assert the existence of some positive constant M_0 there.

Further, the following lemma follows from the induction hypothesis:

Lemma 1.2.5. *For each* (l,k) $(l \geq 0, k \geq 1)$, $[G;l,2k](\mathrm{v})$ *entails the following:*

$$(1.2.117)\qquad \Big\|\frac{d^2 x_{2k}^{(l)}}{ds^2}\Big\|_{[r-\varepsilon]} \leq e^2 N_0 C(l)\big(R|\rho|^{-1}\big)^l (2k+1)!\varepsilon^{-2k-1}\big(A|\rho|^{-1}\big)^k,$$

$$(1.2.118)\qquad \Big\|\frac{d^3 x_{2k}^{(l)}}{ds^3}\Big\|_{[r-\varepsilon]} \leq e^2 N_0 C(l)\big(R|\rho|^{-1}\big)^l (2k+2)!\varepsilon^{-2k-2}\big(A|\rho|^{-1}\big)^k,$$

where $e = 2.718\cdots$.

Proof. Let $\tilde{\varepsilon}$ denote $k\varepsilon/(k+1)$. Then $[G;l,2k](\mathrm{v})$ entails

$$(1.2.119)\qquad \sup_{|s|\leq r-\tilde{\varepsilon}} |\dot{x}_{2k}^{(l)}(s)| \leq N_0 C(l)\big(R|\rho|^{-1}\big)^l (2k)!\tilde{\varepsilon}^{-2k}\big(A|\rho|^{-1}\big)^k$$

$$= N_0 C(l)\big(R|\rho|^{-1}\big)^l (2k)! \left(1+\frac{1}{k}\right)^{2k} \varepsilon^{-2k}\big(A|\rho|^{-1}\big)^k$$
$$\le e^2 N_0 C(l)\big(R|\rho|^{-1}\big)^l (2k)! \varepsilon^{-2k}\big(A|\rho|^{-1}\big)^k.$$

To derive (1.2.117) and (1.2.118), we use (1.2.119) together with the following representation of $d^j x_{2k}^{(l)}/ds^j$ $(j = 2, 3)$:

(1.2.120)
$$\frac{d^j x_{2k}^{(l)}}{ds^j} = \frac{(j-1)!}{2\pi\sqrt{-1}} \int\limits_{|\tilde{s}-s|=(k+1)^{-1}\varepsilon} \frac{\dot{x}_{2k}^{(l)}(\tilde{s})}{(\tilde{s}-s)^{1+j}} d\tilde{s}.$$

Since

(1.2.121)
$$\begin{aligned} |\tilde{s}| &\le |\tilde{s}-s| + |s| \\ &\le (k+1)^{-1}\varepsilon + r - \varepsilon \\ &= r - \tilde{\varepsilon} \end{aligned}$$

holds for s in $\{s; |s| \le r-\varepsilon\}$ and $\tilde{s}$ on the above contour, we obtain (1.2.117) and (1.2.118).

□

We note that Lemma 1.2.5 together with (1.2.115) and (1.2.116) implies the following inequalities (1.2.122) and (1.2.123) for some positive constant M_0:

(1.2.122)
$$\left\|\frac{d^2 x_{2k}^{(l)}}{dt^2}\right\|_{[r_0-\varepsilon]} \le M_0 N_0 C(l)(2k+1)!\big(R|\rho|^{-1}\big)^l \varepsilon^{-2k-1}\big(A|\rho|^{-1}\big)^k,$$

(1.2.123)
$$\left\|\frac{d^3 x_{2k}^{(l)}}{dt^3}\right\|_{[r_0-\varepsilon]} \le M_0 N_0 C(l)(2k+2)!\big(R|\rho|^{-1}\big)^l \varepsilon^{-2k-2}\big(A|\rho|^{-1}\big)^k$$

Let us again return to the proof of Proposition 1.2.1. First we observe that (1.2.114) and Lemma 1.2.3 (via Lemma 1.2.4) entail the following:

(1.2.124)
$$\|S_0^{(p)}\|_{[r_0-\varepsilon]} = \Big\| \sum_{l_1+l_2=p} \frac{d^3x_0^{(l_1)}}{dt^3} \Big(\Big(\frac{dx_0}{dt}\Big)^{-1}\Big)^{(l_2)} \Big\|_{[r_0-\varepsilon]} \le 2M_0^2 C(p) \big(R|\rho|^{-1}\big)^p \varepsilon^{-2}.$$

To dominate $S_{2(n-1)}^{(p)}$ for $n \ge 2$ we assume that $[G; l, 2k]$ for $0 \le l \le p$ and $1 \le k \le n-1$; this assumption is a part of the induction hypothesis to be employed in parts [II], [III] and [V]. We then perform its estimation by separating the situation into the following three cases: (i) $k_1 = 0$, (ii) $k_2 = 0$ and (iii) $k_1, k_2 \ne 0$.

(i) $k_1 = 0$: In this case, applying Lemma 1.2.1 and Lemma 1.2.2, we find

(1.2.125)
$$\begin{aligned}
&\Big\| \sum_{l_1+l_2+l_3=p} \frac{d^3x_0^{(l_1)}}{dt^3} \sum_{\nu=1}^{n-1} (-1)^\nu \Big(\Big(\frac{dx_0}{dt}\Big)^{-\nu-1}\Big)^{(l_2)} \sum_{|\vec\kappa|_\nu=n-1}^{*} \sum_{|\vec\lambda|_\nu=l_3} \frac{dx_{2\vec\kappa}^{(\vec\lambda)}}{dt} \Big\|_{[r_0-\varepsilon]} \\
&\le \sum_{l_1+l_2+l_3=p} 2M_0^2 C(l_1) \big(R|\rho|^{-1}\big)^{l_1} \varepsilon^{-2} \Big(\sum_{\nu=1}^{n-1} M_0^\nu C(l_2) \big(R|\rho|^{-1}\big)^{l_2} \Big) \\
&\qquad \times 4^{-1} (4M_0N_0)^\nu C(l_3) (2(n-1)-\nu+1)! \\
&\qquad \times \big(R|\rho|^{-1}\big)^{l_3} \varepsilon^{-2(n-1)} \big(A|\rho|^{-1}\big)^{n-1} \\
&\le 2M_0^4 N_0 C(p) \big(R|\rho|^{-1}\big)^p (2(n-1))! \varepsilon^{-2n} \\
&\qquad \times \big(A|\rho|^{-1}\big)^{n-1} \sum_{\nu=1}^{n-1} \frac{(4M_0^2N_0)^{\nu-1}}{(\nu-1)!} \\
&\le 2e^{4M_0^2N_0} M_0^4 N_0 C(p) \big(R|\rho|^{-1}\big)^p (2(n-1))!\, \varepsilon^{-2n} \big(A|\rho|^{-1}\big)^{n-1}.
\end{aligned}$$

Here M_0 is taken so that

$$\sup_{|t|\le r_0}\left|\frac{ds}{dt}\right| \le M_0 \tag{1.2.126}$$

holds.

(ii) $k_2 = 0$: In this case we find

$$\begin{aligned}&\Big\|\sum_{l_1+l_2=p}\frac{d^3x^{(l_1)}_{2(n-1)}}{dt^3}\Big(\Big(\frac{dx_0}{dt}\Big)^{-1}\Big)^{(l_2)}\Big\|_{[r_0-\varepsilon]}\\&\le M_0^2N_0C(p)(2n)!\big(R|\rho|^{-1}\big)^p\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n-1}.\end{aligned} \tag{1.2.127}$$

(iii) $k_1, k_2 \ge 1$: We first observe

(1.2.128)

$$\begin{aligned}&\Big\|\sum_{\nu=\min\{1,k_2\}}^{k_2}(-1)^\nu\Big(\Big(\frac{dx_0}{dt}\Big)^{-\nu-1}\Big)^{(l_2)}\sum_{|\vec\kappa|_\nu=k_2}{}^{*}\sum_{|\vec\lambda|_\nu=l_3}\frac{dx^{(\vec\lambda)}_{2\vec\kappa}}{dt}\Big\|_{[r_0-\varepsilon]}\\&\le\sum_{\nu=1}^{k_2}M_0^{\nu+1}C(l_2)\big(R|\rho|^{-1}\big)^{l_2}(M_0N_0)^\nu C(l_3)4^{\nu-1}(2k_2-\nu+1)!\\&\qquad\times\big(R|\rho|^{-1}\big)^{l_3}\,\varepsilon^{-2k_2}\big(A|\rho|^{-1}\big)^{k_2}\\&\le M_0^3e^{4M_0^2N_0}N_0C(l_2)C(l_3)\big(R|\rho|^{-1}\big)^{l_2+l_3}(2k_2)!\varepsilon^{-2k_2}\big(A|\rho|^{-1}\big)^{k_2}.\end{aligned}$$

Hence we obtain (1.2.129) below by (1.2.123) and (1.2.128):

(1.2.129)

$$\begin{aligned}&\Big\|\sum_{\substack{k_1+k_2=n-1\\l_1+l_2+l_3=p\\k_1,k_2\ge1}}\frac{d^3x^{(l_1)}_{2k_1}}{dt^3}\sum_{\nu=\min\{1,k_2\}}^{k_2}(-1)^\nu\Big(\Big(\frac{dx_0}{dt}\Big)^{-\nu-1}\Big)^{(l_2)}\sum_{|\vec\kappa|_\nu=k_2}{}^{*}\sum_{|\vec\lambda|_\nu=l_3}\frac{dx^{(\vec\lambda)}_{2\vec\kappa}}{dt}\Big\|_{[r_0-\varepsilon]}\\&\le M_0^4e^{4M_0^2N_0}N_0^2C(p)\big(|\rho|^{-1}\big)^p(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n-1}.\end{aligned}$$

Thus the following estimate (1.2.130) follows from (1.2.125), (1.2.127) and (1.2.129) for some positive constant M that is independent of N_0, C_0, R and A:

(1.2.130)
$$\big\|S^{(p)}_{2(n-1)}\big\|_{[r_0-\varepsilon]} \le MN_0C(p)\big(R|\rho|^{-1}\big)^p(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n-1}.$$

The reasoning given so far equally applies to the second sum in the right-hand side of (1.2.102), i.e.,

$$(1.2.131)\quad -\frac{3}{2}\sum_{\substack{k_1+k_2+k_3=n-1\\ l_1+l_2+l_3+l_4=p}}\frac{d^2x^{(l_1)}_{2k_1}}{dt^2}\,\frac{d^2x^{(l_2)}_{2k_2}}{dt^2}$$
$$\times\sum_{\nu=\min\{1,k_3\}}^{k_3}(-1)^\nu(\nu+1)\bigg(\Big(\frac{dx_0}{dt}\Big)^{-\nu-2}\bigg)^{(l_3)}\sum_{|\vec\kappa|_\nu=k_3}{}^{*}\sum_{|\vec\lambda|_\nu=l_4}\frac{dx^{(\vec\lambda)}_{2\vec\kappa}}{dt}.$$

Summing up, we have found

$$(1.2.132)\qquad \big\|\{x;t\}^{(p)}_0\big\|_{[r_0-\varepsilon]} \le 2!MC(p)\big(R|\rho|^{-1}\big)^p\varepsilon^{-2},$$

by Lemma 1.2.3 via Lemma 1.2.4, and we have also confirmed, for $n \ge 2$,

(1.2.133)
$$\big\|\{x;t\}^{(p)}_{2(n-1)}\big\|_{[r_0-\varepsilon]} \le MN_0C(p)\big(R|\rho|^{-1}\big)^p(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n-1}$$

for some positive constant M that is independent of N_0, C_0, R and A by assuming the validity of $[G;l,2k]$ for $0 \le l \le p$ and $1 \le k \le n-1$, besides Lemma 1.2.4.

Making use of these results, we now show that the validity of $[G;q,2k]$ (q: arbitrary, $1 \le k \le n-1$) together with the validity of $[G;r,2n]$ ($r \le p_0-1$) entails $[G;p_0,2n]$. In what follows we call these assumptions as the induction hypothesis for short. It is clear that the induction hypothesis is stronger than the assumptions we have used to confirm

(1.2.133) with $p = p_0$. We also remark that the validity of $[G; 0, 2n]$ is guaranteed by the same reasoning if we assume the validity of $[G; q, 2k]$ (q: arbitrary, $1 \leq k \leq n-1$) besides Lemma 1.2.3 (i.e., the validity of $[G; q, 0]$ for $q \geq 1$). Parenthetically we note that it suffices to use only $[G; q, 2k]$ ($1 \leq k \leq n-1$) with $q \leq p_0$ to validate $[G; p_0, 2n]$; the situation is the same when $p_0 = 0$.

[II] Let us first dominate $x_{2n}^{(p_0+1)}(0, \rho) - \tilde{A}_{2n}^{(p_0)}$ on the above induction hypothesis. The reasoning used for the domination is basically the same as that used in the proof of Lemma 1.2.3 reinforced with the results in part [I], which are applied to the estimation of terms $(\alpha.j)$ ($j = \mathrm{vii}, \mathrm{viii}, \mathrm{ix}, \mathrm{x}$) in (1.1.3.43). Hence in what follows we focus our attention on the points which require some special care, and we will try to avoid routine repetitions. As in the proof of Lemma 1.2.3 we use the concrete expression (1.1.3.43) of $R_{2n}^{(p+1)}(s, \rho)$ to dominate

$$- x_{2n}^{(p_0+1)}(0, \rho) + \tilde{A}_{2n}^{(p_0)} = R_{2n}^{(p_0+1)}(0, \rho) = \frac{1}{2\pi i} \int_{|s|=r_0-\varepsilon} R_{2n}^{(p_0+1)}(s, \rho) \frac{ds}{s} = \frac{1}{2\pi i} \oint R_{2n}^{(p_0+1)} \frac{ds}{s}. \tag{1.2.134}$$

As in (1.2.16) we also use the notation

$$\frac{1}{2\pi i} \oint (\alpha.j) \frac{ds}{s} \tag{1.2.135}$$

to denote Cauchy's integral of the term labelled by $(\alpha.j)$ ($j = \mathrm{i}, \mathrm{ii}, \cdots, \mathrm{x}$) in (1.1.3.43). In what follows we use the notation introduced in Definition 1.2.3; $A_{2k}^{(u)}/B_0^{(0)}$ etc. in (1.1.3.43) are respectively denoted by $\tilde{A}_{2k}^{(u)}$ etc.

To begin with, we note that the contribution from the parts

$$- \sum_{\substack{q+r=p_0 \\ i+j=n,\, i,j \leq n-1}} \dot{x}_{2i}^{(q)}(s, \rho) \dot{x}_{2j}^{(r)}(s, \rho) \tilde{A}_0^{(0)} \tag{1.2.136}$$

and

$$-2\sum_{q+r=p_0}\dot{x}_{2n}^{(q)}(s,\rho)\dot{x}_0^{(r)}(s,\rho)\tilde{A}_0^{(0)} \tag{1.2.137}$$

in (α.i) are cancelled out respectively by the worst (in estimating) part of the contribution from (α.v) with $u=1$ and by that from (α.vi) with $u=1$, that is,

$$\frac{1}{2\pi i}\oint\frac{f^{(1)}(t,\rho)}{B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2\frac{s^2}{t^2}\Big(\sum_{\substack{q+r=p_0\\ i+j=n,\, i,j\le n-1}}\dot{x}_{2i}^{(q)}(0,\rho)\dot{x}_{2j}^{(r)}(0,\rho)\Big)\frac{ds}{s} \tag{1.2.138}$$

and

$$\frac{2}{2\pi i}\oint\frac{f^{(1)}(t,\rho)}{B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2\frac{s^2}{t^2}\Big(\sum_{q+r=p_0}\dot{x}_{2n}^{(q)}(0,\rho)\dot{x}_0^{(r)}(0,\rho)\Big)\frac{ds}{s}. \tag{1.2.139}$$

The mechanism of the cancellation is the same for both parts; first we consider the Taylor expansion $x_{2i}^{(q)}(s,\rho)x_{2j}^{(r)}(s,\rho)$ and pick up the coefficient of s^2 and then we use

$$f^{(1)}(t,\rho)\Big(\frac{dt}{ds}\Big)^2\frac{s^2}{t^2}\Big|_{s=0}=A_0^{(0)}. \tag{1.2.140}$$

Once (1.2.138) is set aside, other contributions from (α.v) with $u=1$, i.e.,

$$\begin{aligned}\sum_{\substack{q+r=p_0\\ i+j=n,\, i,j\le n-1}}\frac{1}{2\pi i}\oint\frac{f^{(1)}(t,\rho)}{B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2\frac{1}{t^2}\Big(&x_{2i}^{(q)}(0,\rho)x_{2j}^{(r)}(0,\rho)\\ &+2sx_{2i}^{(q)}(0,\rho)\dot{x}_{2j}^{(r)}(0,\rho)+s^2x_{2i}^{(q)}(0,\rho)\ddot{x}_{2j}^{(r)}(0,\rho)\Big)\frac{ds}{s}\end{aligned} \tag{1.2.141}$$

is seen to be tame. In fact, each integral to be examined contains either $x_{2i}^{(q)}(0,\rho)$ $(1\le i\le n-1)$ in its integrand and hence the integral is dominated by

$$M|\rho|^{-1}N_0^2C(p_0)\big(R|\rho|^{-1}\big)^{p_0-1}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n \tag{1.2.142}$$
$$=MR^{-1}N_0^2C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n,$$

where M is a constant that originates from the innocent part of the integrand such as

$$\Big(\Big(\frac{dt}{ds}\Big)^2\frac{1}{t^2}f^{(1)}(t,\rho)\Big)\frac{ds}{s}. \tag{1.2.143}$$

If we set aside (1.2.139), we use the same reasoning to find the contribution from (α.vi) with $u=1$ is dominated by

$$MR^{-1}N_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n. \tag{1.2.144}$$

Because of the constraint on the indices

$$q+r+u=p_0+1, \tag{1.2.145}$$

contributions from (α.vi) with $u\geq 2$ and (α.v) with $u\geq 2$ are dominated by similar constants, whereas contributions from (α.vi) with $u=0$ and (α.v) with $u=0$ require some special care. To fix the notation we discuss the contribution from (α.vi) with $u=0$; the contribution from (α.v) with $u=0$ is handled in the same manner. We first note that $f^{(0)}$ has the form $\rho t g(t,\rho)$ with $g(0,\rho)=1$. Hence we find

$$\frac{1}{2\pi i}\oint\frac{2t^{-2}}{B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 f^{(0)}\Big(\sum_{\substack{q+r=p_0+1\\ q\leq p_0}}x_{2n}^{(q)}x_0^{(r)}\Big)\frac{ds}{s} \tag{1.2.146}$$
$$=\frac{1}{2\pi i}\oint 2Z_0\,\frac{s}{t}\,g(t,\rho)\Big(\frac{dt}{ds}\Big)^2\Big(\sum_{\substack{q+r=p_0+1\\ q\leq p_0}}x_{2n}^{(q)}x_0^{(r)}\Big)\frac{ds}{s^2}.$$

Thus we observe that the annoying factor $1/B_0^{(0)}$ has disappeared and that it suffices to study the Taylor expansion (in powers of s)

of $\sum x_{2n}^{(q)} x_0^{(r)}$ up to the degree 1 part; each term in the Taylor expansion to be estimated contains $x_{2n}^{(q)}(0,\rho)$ or $x_0^{(r)}(0,\rho)$ as its factor. Since $x_0^{(0)}$ does not appear in the sum, $[G;p,0]$ $(p \geq 1)$ and the induction hypothesis guarantee that each contribution is dominated by

$$MC_0N_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n} \tag{1.2.147}$$

with some positive constant M that is independent of C_0, N_0, R and A. (In what follows, M stands for such a constant.)

Returning to the estimation of (α.i), we find by $[G;p,0]$ $(p \geq 1)$ and the induction hypothesis that each term in (α.i) except for (1.2.137) and (1.2.136) is dominated by a constant of the form

$$M(C_0+N_0)N_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n}. \tag{1.2.148}$$

Next let us study the contribution from (α.ii). This term is basically handled by the application of the induction hypothesis. Since $\dot{x}_0^{(0)}(0,\rho)$ and $\tilde{B}_0^{(0)}$ are not covered by $[G;p,0]$ $(p \geq 1)$, we have to pay attention to them. However, all the terms in (α.ii) contain two factors; one of them has a suffix $(2k_1,(q_1))$ with $k_1 \geq 1$ and the other has a suffix $(2k_2,(q_2))$ with $q_2 \geq 1$. Thus we can dominate the contribution from (α.ii) by

$$MC_0N_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n}. \tag{1.2.149}$$

The succeeding target is (α.iii). In view of the structure of the induction hypotheses, we rewrite

(1.2.150)

$$\sum_{\substack{q+r+u=p_0-2\\ i+j+k=n}} \dot{x}_{2i}^{(q)}\dot{x}_{2j}^{(r)}\tilde{A}_{2k}^{(u)}$$

$$= \tilde{A}_0^{(0)}\left(2\dot{x}_0^{(0)}\dot{x}_{2n}^{(p_0-2)} + \sum_{\substack{q+r=p_0-2\\ i+j=n\\ (q,i),(r,j)\neq(0,0)}} \dot{x}_{2i}^{(q)}\dot{x}_{2j}^{(r)}\right) + \sum_{\substack{q+r+u=p_0-2\\ i+j+k=n\\ (u,k)\neq(0,0)}} \tilde{A}_{2k}^{(u)}\dot{x}_{2i}^{(q)}\dot{x}_{2j}^{(r)}$$

$$= 2\tilde{A}_0^{(0)}\dot{x}_0^{(0)}\dot{x}_{2n}^{(p_0-2)} + \tilde{A}_0^{(0)}\left(\sum_{\substack{q+r=p_0-2\\ i+j=n\\ (q,i),(r,j)\neq(0,0)}} \dot{x}_{2i}^{(q)}\dot{x}_{2j}^{(r)}\right) + \tilde{A}_{2n}^{(p_0-2)}\dot{x}_0^{(0)2}$$

$$+ 2\sum_{\substack{r+u=p_0-2\\ j+k=n\\ (u,k),(r,j)\neq(0,0)}} \tilde{A}_{2k}^{(u)}\dot{x}_0^{(0)}\dot{x}_{2j}^{(r)} + \sum_{\substack{q+r+u=p_0-2\\ i+j+k=n\\ (u,k),(q,i),(r,j)\neq(0,0)}} \tilde{A}_{2k}^{(u)}\dot{x}_{2i}^{(q)}\dot{x}_{2j}^{(r)}.$$

Thus the worst contribution from (α.iii) is dominated as follows:

(1.2.151)
$$\Big|\frac{1}{2\pi i}\oint \frac{s^2}{t^2}\big(2\tilde{A}_0^{(0)}\dot{x}_{2n}^{(p_0-2)}\big)\frac{ds}{s^3}\Big|$$
$$\leq M|\rho|^{-1}N_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0-2}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n$$
$$\leq MR^{-1}N_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0-1}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n.$$

Parenthetically we note that the contribution from $\tilde{A}_{2n}^{(p_0-2)}\dot{x}_0^{(0)2}$ ($= \tilde{A}_{2n}^{(p_0-2)}$) is weaker than (1.2.151) by the factor $|\rho|$.

In parallel with the study of (α.iii) we can readily find that the worst contribution from (α.iv) is

(1.2.152)
$$\Big|\frac{1}{2\pi i}\oint \frac{s^2}{t^2}\big(\dot{x}_0^{(0)2}x_{2n}^{(p_0-1)}\tilde{B}_0^{(0)}\big)\frac{ds}{s^3}\Big|,$$

which is dominated by

(1.2.153)
$$MN_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0-1}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n.$$

The domination of contributions from (α.vii) $\sim$ (α.x) can be done in a similar manner. Since the domination of contributions from (α.viii) and (α.x) are straightforward, we concentrate our attention on (α.vii) and (α.ix). Among the contributions from (α.vii) the worst ones are

$$(1.2.154) \qquad \Big|\frac{1}{2\pi i}\oint \frac{t^{-2}}{2B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 x_0^{(0)2}\{x;t\}_{2(n-1)}^{(p_0-1)}\frac{ds}{s}\Big|$$

and

$$(1.2.155) \qquad \Big|\frac{1}{2\pi i}\oint \frac{t^{-2}}{B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 \big(x_0^{(0)}x_{2(n-1)}^{(p_0-1)}\{x;t\}_0^{(0)}\big)\frac{ds}{s}\Big|,$$

which are respectively dominated by

$$(1.2.156) \qquad \begin{aligned} &|\rho|^{-1}MN_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0-1}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n-1}\\ =&MR^{-1}N_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n-1}\end{aligned}$$

and

$$(1.2.157) \qquad MR^{-1}N_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n-1}.$$

Concerning the contributions of (α.ix) we discuss the case $n = 1$ and the case $n \geq 2$ separately. When $n = 1$, (α.ix) evaluated at $s = 0$ is given by

$$(1.2.158) \qquad \frac{1}{2B_0^{(0)}}\sum_{\substack{q+r+u=p_0+1\\ q,r\geq 2}} x_0^{(q)}(0,\rho)x_0^{(r)}(0,\rho)\{x;t\}_0^{(u)}\Big|_{s=0},$$

which is dominated by

$$(1.2.159) \qquad \begin{aligned} &|\rho|^{-1}MC_0^2C(p_0)\big(R|\rho|^{-1}\big)^{p_0-1}\,\varepsilon^{-2}\\ =&\,MC_0^2(N_0RA)^{-1}N_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0}\,\varepsilon^{-2}A.\end{aligned}$$

When $n \geq 2$, it follows from the results in part [I] together with the induction hypotheses that the sum (α.ix) evaluated at $s = 0$ is dominated by

$$(1.2.160) \qquad MC_0N_0|\rho|^{-1}C(p_0)\big(R|\rho|^{-1}\big)^{p_0-1}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n-1}$$

$$= MC_0R^{-1}N_0C(p_0)\bigl(R|\rho|^{-1}\bigr)^{p_0}(2n)!\,\varepsilon^{-2n}\bigl(A|\rho|^{-1}\bigr)^{n-1}.$$

Summing up the results obtained in this part, we find

$$\begin{aligned}
\text{(1.2.161)}\qquad &\bigl|x_{2n}^{(p_0+1)}(0,\rho)-\tilde{A}_{2n}^{(p_0)}\bigr| = \bigl|R_{2n}^{(p_0+1)}(0,\rho)\bigr| \\
&\le N_2N_0C(p_0)\bigl(R|\rho|^{-1}\bigr)^{p_0}(2n)!\,\varepsilon^{-2n}\bigl(A|\rho|^{-1}\bigr)^{n},
\end{aligned}$$

where

$$\text{(1.2.162)}\qquad N_2 = M\bigl(C_0+N_0+R^{-1}+C_0(N_0RA)^{-1}\bigr).$$

By using the same reasoning as above, we also find

$$\begin{aligned}
\text{(1.2.163)}\qquad &\bigl|\dot{x}_{2n}^{(p_0)}(0,\rho)+\tilde{B}_{2n}^{(p_0)}\bigr| = \bigl|\dot{R}_{2n}^{(p_0)}(0,\rho)\bigr| \\
&\le N_2N_0C(p_0)\bigl(R|\rho|^{-1}\bigr)^{p_0}(2n)!\,\varepsilon^{-2n}\bigl(A|\rho|^{-1}\bigr)^{n}.
\end{aligned}$$

Actually the domination is easier than the confirmation of (1.2.161), because this time we do not need to seek for the cancellation of annoying terms such as (1.2.136), (1.2.137), (1.2.138) and (1.2.139). Hence we omit the proof of (1.2.163).

Remark 1.2.6. By taking C_0 and N_0 sufficiently small and then letting R and A sufficiently large, we may consider the factor N_2 is sufficiently small. Here we note that the factor A is not used essentially in the estimation in part [II] (and also part [III] below), that is, we can obtain (1.2.161) and (1.2.163) with N_2 sufficiently small from the induction hypothesis without taking A sufficiently large. The factor A plays an essential role in part [IV] to make the constant $M(N_0A)^{-1}N_0$ (resp., $MA^{-1}N_0$) in (1.2.179) (resp., (1.2.180)) sufficiently small. Parenthetically we also note that this stage of the reasoning is not an appropriate place to detect the proper order of $|\rho|$ relevant to n; for example the order in question is 0 in (1.2.160), whereas it is -1 in the estimate

(1.2.179) of the corresponding term in part [IV]. Since (1.2.179) is a consequence of Lemma 1.2.3 as we will see later, that is the spot where we find the appropriate order.

[III] Using the results in part [II] we dominate $\tilde{A}_{2n}^{(p_0)}$ and $\tilde{B}_{2n}^{(p_0)}$ by the induction on p_0. The reasoning is basically the same as the reasoning in part [II] of the proof of Lemma 1.2.3 except for the estimation of terms involving the effect of the Schwarzian derivative. By concretely writing down the conditions $(ds/dt)^{-2}\Phi_{2n}^{(p_0+3)}\big|_{s=0} = (ds/dt)^{-2}\big(d\Phi_{2n}^{(p_0+2)}/dt\big)\big|_{s=0}$ $= 0$, we obtain the following relations which determine $\big(\tilde{A}_{2n}^{(p_0)}, \tilde{B}_{2n}^{(p_0)}\big)$, where $z_{2n}^{(p_0)}(s,\rho)$ stands for $x_{2n}^{(p_0)}(s,\rho) - \tilde{A}_{2n}^{(p_0-1)} + \tilde{B}_{2n}^{(p_0)}s$. (Cf. (1.2.70), (1.2.70′), (1.2.71) and (1.2.71′).)

(1.2.164)

$$\begin{aligned}
&-2\Big(B_0^{(0)}\tilde{A}_{2n}^{(p_0)} - A_0^{(0)}\tilde{B}_{2n}^{(p_0)}\Big) \\
&= 2\dot{z}_{2n}^{(p_0)}(0,\rho)A_0^{(0)} && (\tilde{\gamma}.\mathrm{i})\\
&\quad + z_{2n}^{(p_0+1)}(0,\rho)B_0^{(0)} && (\tilde{\gamma}.\mathrm{ii})\\
&\quad + \sum_{\substack{q+r+u=p_0\\ i+j+k=n\\ (q,i),(r,j),(u,k)\neq(p_0,n)}} \dot{x}_{2i}^{(q)}(0,\rho)\dot{x}_{2j}^{(r)}(0,\rho)B_0^{(0)}\tilde{A}_{2k}^{(u)} && (\tilde{\gamma}.\mathrm{iii})\\
&\quad + 2\sum_{\substack{q+r+u=p_0+3\\ q\geq 2,\, r,u\geq 1}} x_0^{(q)}(0,\rho)x_{2n}^{(r)}(0,\rho)f^{(u)}(0,\rho) && (\tilde{\gamma}.\mathrm{iv})\\
&\quad + \sum_{\substack{q+r+u=p_0+3,\, q,r,u\geq 1\\ i+j=n,\, i,j\geq 1}} x_{2i}^{(q)}(0,\rho)x_{2j}^{(r)}(0,\rho)f^{(u)}(0,\rho) && (\tilde{\gamma}.\mathrm{v})
\end{aligned}$$

$$+ \sum_{\substack{q+r+u+v=p_0+1 \\ i+j+k+l=n \\ (u,k)\neq(p_0+1,n)}} \dot{x}_{2i}^{(q)}(0,\rho)\dot{x}_{2j}^{(r)}(0,\rho)x_{2k}^{(u)}(0,\rho)B_0^{(0)}\tilde{B}_{2l}^{(v)} \qquad (\tilde{\gamma}.\mathrm{vi})$$

$$+ \frac{1}{2}\{x;t\}_{2(n-1)}^{(p_0-1)}\big|_{s=0} \qquad (\tilde{\gamma}.\mathrm{vii})$$

$$- \frac{1}{2} \sum_{\substack{q+r+v=p_0+1 \\ i+j+k=n-1}} x_{2i}^{(q)}(0,\rho)x_{2j}^{(r)}(0,\rho)\{x;t\}_{2k}^{(v)}\big|_{s=0} \qquad (\tilde{\gamma}.\mathrm{viii})$$

$$\underset{\mathrm{def}}{=} \Gamma_{2n}^{(p_0)},$$

(1.2.165)

$$-2\Big(A_0^{(0)}\tilde{A}_{2n}^{(p_0)} - B_0^{(0)}\tilde{B}_{2n}^{(p_0)}\Big)$$

$$= 2A_0^{(0)}z_{2n}^{(p_0+1)}(0,\rho) \qquad (\tilde{\delta}.\mathrm{i})$$

$$+ 3B_0^{(0)}\dot{z}_{2n}^{(p_0)}(0,\rho) \qquad (\tilde{\delta}.\mathrm{ii})$$

$$+ 2 \sum_{\substack{q+r+u=p_0-1 \\ i+j+k=n}} x_{2i}^{(q)\prime\prime}\dot{x}_{2j}^{(r)}A_{2k}^{(u)}\big|_{s=0} \qquad (\tilde{\delta}.\mathrm{iii})$$

$$+ \sum_{\substack{q+r=p_0+2 \\ i+j=n}} x_{2i}^{(q)}(0,\rho)x_{2j}^{(r)}(0,\rho)\Big(f^{(0)\prime}\big|_{t=0}\Big) \qquad (\tilde{\delta}.\mathrm{iv})$$

$$+ \sum_{\substack{q+r+u=p_0+2, u\geq 1 \\ i+j=n}} x_{2i}^{(q)}(0,\rho)x_{2j}^{(r)}(0,\rho)\Big(f^{(u)\prime}\big|_{t=0}\Big) \qquad (\tilde{\delta}.\mathrm{v})$$

$$+ \sum_{\substack{q+r=p_0+1 \\ i+j=n, (r,j)\neq(p_0+1,n)}} \dot{x}_{2i}^{(q)}(0,\rho)x_{2j}^{(r)}(0,\rho)f^{(1)}(0,\rho) \qquad (\tilde{\delta}.\mathrm{vi})$$

$$+ \sum_{\substack{q+r+u=p_0+2, u\geq 2\\ i+j=n}} \dot{x}_{2i}^{(q)}(0,\rho) x_{2j}^{(r)}(0,\rho) f^{(u)}(0,\rho) \qquad (\tilde{\delta}.\mathrm{vii})$$

$$+ 2 \sum_{\substack{q+r+u+v=p_0\\ i+j+k+l=n}} \Big(x_{2i}^{(q)\prime\prime} \dot{x}_{2j}^{(r)} \Big)\Big|_{t=0} x_{2k}^{(u)}(0,\rho) B_{2l}^{(v)} \qquad (\tilde{\delta}.\mathrm{viii})$$

$$+ \sum_{\substack{q+r+u+v=p_0\\ i+j+k+l=n\\ (q,i),(r,j),(u,k),(v,l)\neq(p_0,n)}} \dot{x}_{2i}^{(q)}(0,\rho) \dot{x}_{2j}^{(r)}(0,\rho) \dot{x}_{2k}^{(u)}(0,\rho) B_{2l}^{(v)} \qquad (\tilde{\delta}.\mathrm{ix})$$

$$+ \frac{1}{2}\Big(\frac{d}{dt} \{x;t\}_{2(n-1)}^{(p_0-2)} \Big)\Big|_{t=0} \qquad (\tilde{\delta}.\mathrm{x})$$

$$- \frac{1}{2}\Big(\frac{d}{dt} \sum_{\substack{q+r+u=p_0\\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x;t\}_{2k}^{(u)} \Big)\Big|_{t=0} \qquad (\tilde{\delta}.\mathrm{xi})$$

$$\underset{\mathrm{def}}{=} \Delta_{2n}^{(p_0)}.$$

Thus, as in part [II] of the proof of Lemma 1.2.3, it suffices to confirm that

(1.2.166)

$$|\Gamma_{2n}^{(p_0)}|, |\Delta_{2n}^{(p_0)}| \leq N_3 N_0 C(p_0) \big(R|\rho|^{-1}\big)^{p_0} (2n)!\, \varepsilon^{-2n} \big(A|\rho|^{-1}\big)^n$$

holds, where N_3 is a sufficiently small constant given by (1.2.162).

Using the induction hypothesis, we readily find that $|(\tilde{\gamma}.j)|$ ($j = \mathrm{i}, \mathrm{ii}, \mathrm{iii}$) is dominated by a constant of the form

$$N_3 N_0 C(p_0) \big(R|\rho|^{-1}\big)^{p_0} (2n)!\, \varepsilon^{-2n} \big(A|\rho|^{-1}\big)^n \qquad (1.2.167)$$

with

$$N_3 = M N_2 \qquad \text{for} \quad (\tilde{\gamma}.\mathrm{i}) \qquad (1.2.168)$$

$$N_3 = |\rho| N_2 \qquad \text{for} \quad (\tilde{\gamma}.\mathrm{ii}) \qquad (1.2.169)$$

$$N_3 = MC_0 \qquad \text{for} \quad (\tilde{\gamma}.\text{iii}). \tag{1.2.170}$$

In view of the wideness of the range of indices we are to pay some attention to ($\tilde{\gamma}$.iv) with $u = 1$. This term is seen to be dominated by a constant of the form (1.2.167) with (1.2.170) if we set

$$\tilde{q} = q - 1, \quad \tilde{r} = r - 1 \tag{1.2.171}$$

and use $[G; \tilde{q}, 0]$ (($\tilde{q}$.i) and ($\tilde{q}$.v)) and $[G; \tilde{r}, 2n]$ (($(\tilde{r}, 2n)$(i)). Parenthetically, here we observe that $\tilde{q}, \tilde{r} \leq p_0$, as we have noted before beginning the discussion of part [II]; this is consistent with our delicate way of constructing $x_{2n}^{(p)}(s, \rho)$. (Cf. Proposition 1.1.3.3.) The same reasoning also applies to ($\tilde{\gamma}$.v) and ($\tilde{\gamma}$.vi). We find they are dominated by a constant of the form (1.2.167) with

$$N_3 = MN_0 \tag{1.2.172}$$

and (1.2.170) respectively. It immediately follows from (1.2.132) and (1.2.133) that $|(\tilde{\gamma}.\text{vii})|$ is dominated by a constant of the form (1.2.167) with

$$N_3 = M|\rho|^2\big(N_0 RA\big)^{-1} \quad \text{for } n = 1 \tag{1.2.173}$$

and

$$N_3 = M|\rho|^2\big(RA\big)^{-1} \quad \text{for } n \geq 2. \tag{1.2.174}$$

To dominate ($\tilde{\gamma}$.viii) we use (1.2.132) and (1.2.133) together with the technique employed in dominating $|(\tilde{\gamma}.\text{iv})|$. Then we find ($\tilde{\gamma}$.viii) satisfies the estimates of the form (1.2.167) with

$$N_3 = M|\rho|^2 C_0\big(RA\big)^{-1}. \tag{1.2.175}$$

Thus we have seen that $|\Gamma_{2n}^{(p_0)}|$ satisfies (1.2.166). The domination of $|\Delta_{2n}^{(p_0)}|$ can be done in the same manner. We only note that, using Cauchy's inequality, the domination of $x_{2i}^{(q)\prime\prime}$ in ($\tilde{\delta}$.iii) and the differentiated terms ($\tilde{\delta}$.x) and ($\tilde{\delta}$.xi) can be done without any trouble, because their value are considered at $s = 0$; the order of ε is not affected by differentiation as in Lemma 1.2.5. Thus by rewriting (1.2.164) and (1.2.165) in the form of (1.2.80) we conclude that $|\tilde{A}_{2n}^{(p_0)}|$ and $|\tilde{B}_{2n}^{(p_0)}|$ are dominated by

$$N_2 N_0 C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n, \tag{1.2.176}$$

where N_2 is a sufficiently small constant of the form (1.2.162).

[IV] Finally let us dominate $\|x_{2n}^{(p_0)}\|_{[r_0-\varepsilon]}$ and $\|\dot{x}_{2n}^{(p_0)}\|_{[r_0-\varepsilon]}$. We first note that, by a straightforward calculation, we find

$$\|R_{2n}^{(p_0)}\|_{[r_0-\varepsilon]} \le N_4 N_0 C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n \tag{1.2.177}$$

with

$$N_4 = M(C_0 + N_0 + R^{-1} + (N_0 A)^{-1}). \tag{1.2.178}$$

(We can not expect the cancellation of terms in $R_{2n}^{(p_0)}(u,\rho)$ which is similar to that observed between (1.2.136) and (1.2.138). However, without the cancellation, we can still confirm (1.2.177), although N_4 contains a term $(N_0 A)^{-1}$; to make this term small we take A sufficiently large.) Here we only mention the estimation of (α.ix), whose contribution determines the order of $|\rho|$ relevant to n in $[G; p, 2n]$. It follows from (1.2.132) and (1.2.133) that

$$\begin{aligned} &\Big\|\frac{1}{2B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 \sum_{q+r+u=p_0} x_0^{(q)} x_0^{(r)} \{x;t\}_0^{(u)}\Big\|_{[r_0-\varepsilon]} \\ &\le M|\rho|^{-1} C(p_0)\big(R|\rho|^{-1}\big)^{p_0} 2!\,\varepsilon^{-2} \\ &\le M(N_0 A)^{-1} N_0 C(p_0)\big(R|\rho|^{-1}\big)^{p_0} 2!\,\varepsilon^{-2} A|\rho|^{-1} \end{aligned} \tag{1.2.179}$$

for $n = 1$ and

$$\Big\|\frac{1}{2B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 \sum_{\substack{q+r+u=p_0\\ i+j+k=n-1}} x_{2i}^{(q)} x_{2j}^{(r)} \{x;t\}_{2k}^{(u)}\Big\|_{[r_0-\varepsilon]} \tag{1.2.180}$$

$$\leq M|\rho|^{-1} N_0 C(p_0) \big(R|\rho|^{-1}\big)^{p_0} (2n)!\, \varepsilon^{-2n} \big(A|\rho|^{-1}\big)^{n-1}$$
$$\leq M A^{-1} N_0 C(p_0) \big(R|\rho|^{-1}\big)^{p_0} (2n)!\, \varepsilon^{-2n} \big(A|\rho|^{-1}\big)^{n}$$

for $n \geq 2$.

Then the domination of $\|x_{2n}^{(p_0)}\|_{[r_0-\varepsilon]}$ and $\|\dot{x}_{2n}^{(p_0)}\|_{[r_0-\varepsilon]}$ can be readily done by the same reasoning as part [III] in the proof of Lemma 1.2.3. Thus the induction proceeds. This completes the proof of Proposition 1.2.1.

□

1.3 Correspondence between a WKB solution of an M2P1T equation and that of the Mathieu equation

The purpose of this section is to show how we can relate a WKB solution of an M2P1T equation to an appropriate WKB solution of an ∞-Mathieu equation. To begin with we summarize the results in Section 1.1, Section 1.2 and Appendix C in the form of Theorem 1.3.1 below. To avoid the notational confusions which we will later explain in Remark 1.3.1, we now assume

$$B_0^{(0)} = \rho. \tag{1.3.1}$$

Theorem 1.3.1. *Let $Q(t, a, \rho)$ be a potential of an M2P1T operator given in Definition 1.1. Then there exist positive constants r and R_0, and holomorphic functions*

$$A_{2n}(a,\rho) = \sum_{j=0}^{\infty} A_{2n}^{(j)}(\rho) a^j, \tag{1.3.2}$$

$$B_{2n}(a,\rho) = \sum_{j=0}^{\infty} B_{2n}^{(j)}(\rho) a^j \tag{1.3.3}$$

and

$$(1.3.4) \qquad x_{2n}(t,a,\rho) = \sum_{j=0}^{\infty} x_{2n}^{(j)}(t,\rho)a^j$$

$(n \geq 0)$ *on*

$$(1.3.5) \quad E^1_{r,R_0} = \{(t,a,\rho) \in \mathbb{C}^3 : |t| \leq r, 0 < |\rho| \leq r, R_0|a| \leq |\rho|\}$$

for which the following conditions are satisfied there:

(1.3.6) $A(a,\rho,\eta), B(a,\rho,\eta)$ *and* $x(t,a,\rho,\eta)$ *satisfy* (1.1.6),

$$(1.3.7) \qquad A_0(0,\rho) = f^{(1)}(0,\rho),$$

$$(1.3.8) \qquad B_0(0,\rho) = \rho,$$

$$(1.3.9) \qquad \frac{\partial x_0}{\partial t}(0,0,\rho) = 1,$$

(1.3.10) *the function* $x_0(t,a,\rho)$ *of* t *is injective for each fixed* a *and* ρ *on* E^1_{r,R_0} ,

$$(1.3.11) \qquad x_0(t,a,\rho)\big|_{t=\pm a} = \pm a.$$

Furthermore there exists a positive constant R_1 *for which the following estimates hold for* $n \geq 1$*:*

$$(1.3.12) \qquad |A_{2n}(a,\rho)| \leq |\rho|(2n)!R_1^n|\rho|^{-n},$$

$$(1.3.13) \qquad |B_{2n}(a,\rho)| \leq |\rho|(2n)!R_1^n|\rho|^{-n},$$

$$(1.3.14) \qquad |x_{2n}(t,a,\rho)| \leq (2n)!R_1^n|\rho|^{-n},$$

$$(1.3.15) \qquad \left|\frac{dx_{2n}}{dt}(t,a,\rho)\right| \leq (2n)!R_1^n|\rho|^{-n}.$$

Remark 1.3.1. Using this occasion we make a correction in our announcement paper [KKT, (1.105) and (1.106)]; the exponent of $|\rho|$ should be $-n$, not $-n+1$. We note that the exponent of $|\rho|$ in [KKT, (1.103) and (1.104)] should be kept intact, i.e., $-n+1$.

Proof. It suffices to show (1.3.10) and (1.3.11), as the other relations have been explicitly stated in Section 1.1 and Section 1.2. In what follows, by taking r sufficiently small, we assume

$$f(\pm a, a, \rho) \neq 0, \tag{1.3.16}$$

which the assumptions (1.3), (1.4) and (1.5) guarantee. Since A_0, B_0 and x_0 satisfy

$$(x_0^2 - a^2) f = (t^2 - a^2)(x_0')^2 (a A_0 + x_0 B_0), \tag{1.3.17}$$

by letting $t = \pm a$ in (1.3.17), we find that

$$x_0^2(\pm a, a, \rho) = a^2 \tag{1.3.18}$$

holds. Since $x_0^{(j)}(0, \rho) = 0$ $(j = 0, 1)$, it follows from (1.1.1.13) that

(1.3.19)

$$\begin{aligned} \frac{x_0(\pm a, a, \rho)}{a} &= \frac{x_0^{(0)}(\pm a, \rho)}{a} + x_0^{(1)}(\pm a, \rho) + a \sum_{j=2}^{\infty} x_0^{(j)}(\pm a, \rho) a^{j-2} \\ &\underset{a \to 0}{\longrightarrow} \pm \frac{\partial x_0^{(0)}}{\partial t}(0, \rho) = \pm \frac{\rho}{B_0^{(0)}}. \end{aligned}$$

Hence (1.3.1) and (1.3.18) entail (1.3.11).

To confirm (1.3.10) we use $s = x_0^{(0)}(t, \rho)$ as a coordinate. Take r_1 and ε be sufficiently small so that $x_0(s, a, \rho)$ is holomorphic on

(1.3.20)

$$\tilde{E}^1_{r_1+2\varepsilon, R_0} = \{(s, a, \rho) \in \mathbb{C}^3 : |s| \le r_1 + 2\varepsilon, 0 < |\rho| \le r_1, R_0|a| \le |\rho|\}.$$

Then, by taking R_0 sufficiently large, we can assume that

$$|x_0(s,a,\rho)-s|<\varepsilon \tag{1.3.21}$$

holds on $\tilde{E}^1_{r_1+2\varepsilon,R_0}$. Therefore, for any $\hat{s}$ in $x_0(\tilde{E}^1_{r_1,R_0})$, we find $|s-\hat{s}|>\varepsilon$ holds on $\{s\in\mathbb{C}:|s|=r_1+2\varepsilon\}$. Appealing to Rouché's theorem, we find that $x_0(s,a,\rho)$ is injective on $\{s\in\mathbb{C}:|s|\le r_1\}$. By taking r so that $x_0^{(0)}(t,\rho)$ is injective and satisfies $|x_0^{(0)}(t,\rho)|\le r_1$ on E^1_{r,R_0}, we obtain (1.3.10).

□

Remark 1.3.2. When $B_0^{(0)}=-\rho$, some minor adjustments of signs etc. are needed at several points in Theorem 1.3.1. For the sake of the reader's convenience, we list up the formulas that require the adjustments below; each formula is appropriately modified and endowed with a new label obtained by adding $'$ to the original number of formulas. In accordance with the adjustments, (1.1.6) is also changed to

$$\begin{aligned} &Q(t,a,\rho;\eta) \\ &=\left(\frac{\partial x}{\partial t}\right)^2\left(\frac{aA+xB}{x^2-a^2}+\eta^{-2}\left(\frac{g_+(-a)}{(x-a)^2}+\frac{g_-(a)}{(x+a)^2}\right)\right) \\ &\quad-\frac{1}{2}\eta^{-2}\{x;t\}. \end{aligned} \tag{1.1.6$'$}$$

(1.3.3′) $A(a,\rho,\eta), B(a,\rho,\eta)$ and $x(t,a,\rho,\eta)$ satisfy (1.1.6′).

$$B_0(0,\rho)=-\rho. \tag{1.3.5$'$}$$

$$\frac{\partial x_0}{\partial t}(0,0,\rho)=-1. \tag{1.3.6$'$}$$

$$x_0(t,a,\rho)\big|_{t=\pm a}=\mp a. \tag{1.3.8$'$}$$

As is shown in [KT], Theorem 1.3.1 entails the following

Theorem 1.3.2. *Let* $\hat{S}$ *and* $\tilde{S}$ *be a solution of*

$$\hat{S}^2 + \frac{\partial \hat{S}}{\partial t} = \eta^2 Q(t, a, \rho, \eta) \tag{1.3.22}$$

and

$$\tilde{S}^2 + \frac{\partial \tilde{S}}{\partial x} = \eta^2 \left(\frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left(\frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2} \right) \right) \tag{1.3.23}$$

respectively, and suppose that

(1.3.24)
$$\arg \hat{S}_{-1}(t, a, \rho) = \arg \left(\frac{\partial x_0}{\partial t} S_{-1}(x_0(t, a, \rho), a, A_0(a, \rho), B_0(a, \rho)) \right)$$

holds. Then they satisfy

(1.3.25)
$$\begin{aligned} &\hat{S}_{\text{odd}}(t, a, \rho, \eta) \\ &= \left(\frac{\partial x}{\partial t} \right) \tilde{S}_{\text{odd}}(x(t, a, \rho, \eta), a, A(a, \rho, \eta), B(a, \rho, \eta), \eta), \end{aligned}$$

where $\hat{S}_{\text{odd}}$ *and* $\tilde{S}_{\text{odd}}$ *respectively be the odd part of* $\hat{S}$ *and* $\tilde{S}$*.*

We also have the following theorem (cf. [AKT1]):

Theorem 1.3.3. *Let* $\hat{\psi}_\pm(t, a, \rho, \eta)$ *be WKB solutions of a generic (i.e.,* $a\rho \neq 0$*) M2P1T equation* (1.7) *that are normalized at a simple pole* $t = a$ *as*

$$\hat{\psi}_\pm(t, a, \rho, \eta) = \frac{1}{\sqrt{\hat{S}_{\text{odd}}}} \exp \left(\pm \int_a^t \hat{S}_{\text{odd}} dt \right), \tag{1.3.26}$$

and let $\tilde{\psi}_\pm(x, a, A, B, \eta)$ *denote WKB solutions of the Mathieu equation*

(1.3.27)
$$\left(\frac{d^2}{dx^2} - \eta^2 \left(\frac{aA + xB}{x^2 - a^2} + \eta^{-2} \left(\frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2} \right) \right) \right) \tilde{\psi} = 0$$

which are normalized at a simple pole $x = a$ as

$$\tilde{\psi}_{\pm}(x, a, A, B, \eta) = \frac{1}{\sqrt{\tilde{S}_{\mathrm{odd}}}} \exp\left(\pm \int_a^x \tilde{S}_{\mathrm{odd}} dx\right). \tag{1.3.28}$$

Then $\hat{\psi}_{\pm}$ and $\tilde{\psi}_{\pm}$ satisfy the following relation on the set E^1_{r,R_0} given by (1.3.5)*:*

$$\begin{aligned}\hat{\psi}_{\pm}(t, a, \rho, \eta) \\ = \left(\frac{\partial x}{\partial t}\right)^{-1/2} \tilde{\psi}_{\pm}(x(t, a, \rho, \eta), a, A(a, \rho, \eta), B(a, \rho, \eta), \eta),\end{aligned} \tag{1.3.29}$$

where $x(t, a, \rho, \eta), A(a, \rho, \eta)$ and $B(a, \rho, \eta)$ are the series given in Theorem 1.3.1.

2 Reduction of the Mathieu equation to the Legendre equation near its simple poles

The main purpose of this section is to construct a transformation that brings the Mathieu equation

$$\left(\frac{d^2}{dx^2} - \eta^2\left(\frac{aA + xB}{x^2 - a^2} + \eta^{-2}\left(\frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2}\right)\right)\right)\tilde{\psi} = 0 \tag{2.1}$$

with genuine constants $A(\neq 0)$ and B to the following Legendre equation

$$\left(\frac{d^2}{dz^2} - \eta^2\left(\frac{a\Lambda^2}{z^2 - a^2} + \eta^{-1}\frac{\sqrt{a}\Lambda}{z^2 - a^2} + \eta^{-2}\frac{az\nu + a^2(\mu^2 - 1)}{(z^2 - a^2)^2}\right)\right)\phi = 0 \tag{2.2}$$

on a neighborhood of the line segment connecting two simple poles at $x = \pm a$.

We note that introducing the large parameter η as in (2.2) to the classical Legendre equation is a natural one from the WKB-theoretic viewpoint; an elementary evidence for the naturalness is given by the fact that the WKB solutions $\psi_\pm$ of (2.2) with $\nu = 0$ and $\mu^2 = 1/4$ is expressed in a closed form, i.e.,

$$(2.3)\qquad \Big(\eta\sqrt{a}\Lambda + \frac{1}{2}\Big)^{-1/2}(z^2-a^2)^{1/4}\Big(\frac{z+\sqrt{z^2-a^2}}{a}\Big)^{\pm(\eta\sqrt{a}\Lambda+1/2)},$$

which forms a counterpart of the interesting formula for $P^{\pm 1/2}_{\sqrt{a}\Lambda}$ and $Q^{\pm 1/2}_{\sqrt{a}\Lambda}$ ([Er, vol.I, p.150])). This naturalness seems to have enabled Koike ([Ko3]) to find the explicit form of the Voros coefficient for (2.2), of which we will make essential use in Section 4. However, there is one technical problem with the equation (2.2); it contains a term with degree 1 in η. Although the appearance of degree 1 (or, more generally, an odd degree part) in η is natural from the viewpoint of the general theory of simple-pole type operators (cf. e.g. [KKoT]), it is somewhat unhandy in this paper; the equations we are dealing with in this paper contain only even degree terms in η. Hence, as an auxiliary equation we consider the following equation:

$$(2.4)\quad \left(\frac{d^2}{dz^2} - \eta^2\left(\frac{a\Gamma}{z^2-a^2} + \eta^{-2}\left(\frac{g_+(a)}{(z-a)^2} + \frac{g_-(-a)}{(z+a)^2}\right)\right)\right)\psi = 0,$$

which can be smoothly related with (2.1). We will show the WKB-theoretic equivalence of (2.2) and (2.4) later in Proposition 2.1. Thus our first task is to construct the transformation series

$$(2.5)\qquad z(x,a,A,B,\eta) = \sum_{n=0}^{\infty} z_{2n}(x,a,A,B)\eta^{-2n}$$

and

$$(2.6)\qquad \Gamma(a,A,B,\eta) = \sum_{n=0}^{\infty} \Gamma_{2n}(a,A,B)\eta^{-2n}$$

so that they satisfy

$$\begin{aligned}\frac{aA + xB}{x^2 - a^2} &+ \eta^{-2}\left(\frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2}\right) \\ &= \left(\frac{\partial z}{\partial x}\right)^2\left(\frac{a\Gamma}{z^2 - a^2} + \eta^{-2}\left(\frac{g_+(a)}{(z-a)^2} + \frac{g_-(-a)}{(z+a)^2}\right)\right) \\ &\quad - \frac{1}{2}\eta^{-2}\{z; x\}.\end{aligned} \tag{2.7}$$

In order to attain the required reduction of the Mathieu equation to the Legendre equation near its simple poles, we need several delicate properties of the series including their domains of definition and estimates. Hence the precise target is to prove the following

Theorem 2.1. *There exist holomorphic functions* $z_{2n}(x, a, A, B)$ *and* $\Gamma_{2n}(a, A, B)$ *on*

(2.8)
$$E^2_{r_1,r_2} = \{(x, a, A, B) \in \mathbb{C}^4 : |x| < r_1|a|, a \neq 0, A \neq 0, |B| < r_2|A|\}$$

for some constants $r_1 > 1$ *and* $r_2 > 0$ *such that* $z(x, a, A, B, \eta)$ *and* $\Gamma(a, A, B, \eta)$ *respectively given by* (2.5) *and* (2.6) *satisfy* (2.7) *and the following conditions there:*

(2.9) *the function* $z_0(x, a, A, B)$ *of* x *is injective on* $D_{r_1|a|} = \{x \in \mathbb{C} : |x| < r_1|a|\}$ *for fixed* a, A *and* B*,*

$$z_0(\pm a, a, A, B) = \pm a, \tag{2.10}$$

$$\frac{\partial z_0}{\partial x}(x, a, A, B) \neq 0. \tag{2.11}$$

Furthermore they satisfy the following estimates: for any $h > 0$ *we can take sufficiently small* $\delta > 0$ *so that*

$$|z_{2n}(x, a, A, B)| \leq (2n)!h^n|aA|^{-n}, \tag{2.12}$$

(2.13) $$|\Gamma_{2n}(a, A, B)| \le (2n)! h^n |aA|^{-n}$$

hold on $E^2_{r_1,\delta}$ *for* $n \ge 1$.

In order to explain the geometric meaning of Theorem 2.1, we give some remarks before beginning its proof.

Remark 2.1. Since the two simple poles of (2.1) are contained in $D_{r_1|a|}$, Theorem 2.1 guarantees that the reduction of (2.1) to (2.4) is successful on a full neighborhood of the line segment joining these two poles. On the other hand, Theorem 2.1 does not say anything about the simple turning point of (2.1).

Remark 2.2. Two simple poles at $x = \pm a$ and the simple turning point at $x = -aA/B$ all merge at the origin when a tends to 0. But, by taking B/A sufficiently small, we can regard that the turning point is sufficiently far away from the two simple poles in the scale of a.

Proof of Theorem 2.1. Let $\tilde{x}, \tilde{z}, \tilde{B}$ and $\tilde{\eta}$ be

(2.14) $$\tilde{x} = x/a,$$

(2.15) $$\tilde{z} = z/a,$$

(2.16) $$\tilde{B} = B/A,$$

(2.17) $$\tilde{\eta} = \sqrt{aA}\eta,$$

then (2.1) is rewritten as follows:

(2.18)
$$\left(\frac{d^2}{d\tilde{x}^2} - \tilde{\eta}^2 \left(\frac{1 + \tilde{x}\tilde{B}}{\tilde{x}^2 - 1} + \tilde{\eta}^{-2}\left(\frac{g_+(a)}{(\tilde{x}-1)^2} + \frac{g_-(-a)}{(\tilde{x}+1)^2}\right)\right)\right)\tilde{\psi} = 0.$$

Hence if we construct

(2.19) $$\tilde{z}(\tilde{x}, \tilde{B}, \tilde{\eta}) = \sum_{n=0}^{\infty} \tilde{z}_{2n}(\tilde{x}, \tilde{B})\tilde{\eta}^{-2n}$$

and

$$\tilde{\Gamma}(\tilde{B},\tilde{\eta}) = \sum_{n=0}^{\infty} \tilde{\Gamma}_{2n}(\tilde{B})\tilde{\eta}^{-2n} \tag{2.20}$$

so that they satisfy

$$\begin{aligned} &\frac{1+\tilde{x}\tilde{B}}{\tilde{x}^2-1} + \tilde{\eta}^{-2}\left(\frac{g_+(a)}{(\tilde{x}-1)^2} + \frac{g_-(-a)}{(\tilde{x}+1)^2}\right) \\ &= \left(\frac{\partial \tilde{z}}{\partial \tilde{x}}\right)^2 \left(\frac{\tilde{\Gamma}}{\tilde{z}^2-1} + \tilde{\eta}^{-2}\left(\frac{g_+(a)}{(\tilde{z}-1)^2} + \frac{g_-(-a)}{(\tilde{z}+1)^2}\right)\right) \\ &\qquad - \frac{1}{2}\tilde{\eta}^{-2}\{\tilde{z};\tilde{x}\}, \end{aligned} \tag{2.21}$$

then we find

$$z(x,a,A,B,\eta) = a\tilde{z}(x/a, B/A, \sqrt{aA}\eta) \tag{2.22}$$

and

$$\Gamma(a,A,B,\eta) = A\tilde{\Gamma}(B/A, \sqrt{aA}\eta) \tag{2.23}$$

satisfy (2.7).

Therefore it suffices to show the following properties of $\tilde{z}$ and $\tilde{\Gamma}$:

(2.24) $\tilde{z}_{2n}(\tilde{x},\tilde{B})$ and $\tilde{\Gamma}_{2n}(\tilde{B})$ are holomorphic on $\tilde{E}^2_{r_1,r_2} = \left\{(\tilde{x},\tilde{B}) \in \mathbb{C}^2 : |\tilde{x}| \le r_1, |\tilde{B}| \le r_2\right\}$ for some positive constants $r_1 > 1$ and $r_2 > 0$,

(2.25) the function $\tilde{z}_0(\tilde{x},\tilde{B})$ of $\tilde{x}$ is injective on $D_{r_1} = \{\tilde{x} \in \mathbb{C} : |\tilde{x}| \le r_1\}$ for fixed $\tilde{B}$ with $|\tilde{B}| \le r_2$,

$$\tilde{z}_0(\pm 1, \tilde{B}) = \pm 1, \tag{2.26}$$

$$\frac{\partial \tilde{z}_0}{\partial \tilde{x}}(\tilde{x},\tilde{B}) \ne 0 \quad \text{on} \quad D_{r_1} \tag{2.27}$$

and they satisfy the following estimates: for any $h > 0$ we can take sufficiently small $\delta > 0$ so that

$$|\tilde{\Gamma}_{2n}(\tilde{B})| \leq (2n)!h^n, \tag{2.28}$$

$$|\tilde{z}_{2n}(\tilde{x}, \tilde{B})| \leq (2n)!h^n \tag{2.29}$$

hold on $\tilde{E}^2_{r_1,\delta}$ for $n \geq 1$.

We first show (2.25), (2.26) and (2.27). Comparing the coefficients of $\tilde{\eta}^0$ of (2.21), we find that $\tilde{z}_0$ and $\tilde{\Gamma}_0$ satisfy

$$\frac{1+\tilde{x}\tilde{B}}{1-\tilde{x}^2} = \left(\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\right)^2 \frac{\tilde{\Gamma}_0}{1-\tilde{z}_0^2}. \tag{2.30}$$

Therefore we take $\tilde{z}_0$ and $\tilde{\Gamma}_0$ as follows:

$$\tilde{z}_0(\tilde{x}, \tilde{B}) = \cos\left(\frac{1}{\sqrt{\tilde{\Gamma}_0}} \int_1^{\tilde{x}} \sqrt{\frac{1+\tilde{x}\tilde{B}}{1-\tilde{x}^2}} d\tilde{x}\right), \tag{2.31}$$

$$\sqrt{\tilde{\Gamma}_0(\tilde{B})} = \frac{-1}{\pi} \int_1^{-1} \sqrt{\frac{1+\tilde{x}\tilde{B}}{1-\tilde{x}^2}} d\tilde{x}. \tag{2.32}$$

From (2.31) and (2.32), we immediately find that

$$\tilde{z}_0(\tilde{x}, 0) = \tilde{x}, \tag{2.33}$$

$$\sqrt{\tilde{\Gamma}_0(0)} = 1 \tag{2.34}$$

and (2.26) hold. Let $r_1 > 1$ be a constant. Then, for any positive constant ε, we can take $\delta > 0$ so that $\tilde{z}_0(\tilde{x}, \tilde{B})$ and $\tilde{\Gamma}_0(\tilde{B})$ are holomorphic on $\tilde{E}^2_{r_1+2\varepsilon,\delta} = \{(\tilde{x}, \tilde{B}) \in \mathbb{C}^2 : |\tilde{x}| \leq r_1 + 2\varepsilon, |\tilde{B}| \leq \delta\}$ and satisfy the following estimates there:

$$\max\left\{|\tilde{z}_0(\tilde{x}, \tilde{B}) - \tilde{x}|, \left|\frac{\partial \tilde{z}_0}{\partial \tilde{x}} - 1\right|, |\tilde{\Gamma}_0(\tilde{B}) - 1|\right\} < \varepsilon. \tag{2.35}$$

Therefore, for any $y \in \tilde{z}_0(\tilde{E}^2_{r_1,\delta})$, we find $|\tilde{x} - y| > \varepsilon$ holds on $\{\tilde{x} \in \mathbb{C} : |\tilde{x}| = r_1 + 2\varepsilon\}$ and hence, appealing to Rouché's theorem, we find that (2.25) holds for $r_2 < \delta$. Note that (2.27) follows also from (2.35).

Next we show (2.24). Comparing the coefficients of $\tilde{\eta}^{-2n}$ $(n \geq 1)$ of (2.21), we obtain the following relations for $\big(\tilde{z}_{2n}, \tilde{\Gamma}_{2n}\big)$ $(n \geq 1)$:

(2.36)

$$\frac{2\tilde{\Gamma}_0}{\tilde{z}_0^2 - 1}\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\frac{\partial \tilde{z}_{2n}}{\partial \tilde{x}} - \left(\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\right)^2 \frac{2\tilde{z}_0\tilde{\Gamma}_0}{(\tilde{z}_0^2 - 1)^2}\tilde{z}_{2n} + \left(\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\right)^2 \frac{\tilde{\Gamma}_{2n}}{\tilde{z}_0^2 - 1} = \tilde{\Phi}_{2n},$$

where $\tilde{\Phi}_{2n}$ $(n \geq 1)$ is a sum of terms that are determined by $\big(\tilde{z}_{2k}, \tilde{\Gamma}_{2k}\big)$ $(0 \leq k \leq n-1)$. Multiplying both sides of (2.36) by $(\tilde{z}_0^2 - 1)/(2\tilde{\Gamma}_0 \partial \tilde{z}_0/\partial \tilde{x})$, we can rewrite (2.36) as follows:

$$\frac{\partial \tilde{z}_{2n}}{\partial \tilde{x}} - \frac{\partial \tilde{z}_0}{\partial \tilde{x}}\frac{\tilde{z}_0}{\tilde{z}_0^2 - 1}\tilde{z}_{2n} + \frac{\partial \tilde{z}_0}{\partial \tilde{x}}\frac{\tilde{\Gamma}_{2n}}{2\tilde{\Gamma}_0} = \Phi_{2n}. \tag{2.37}$$

Now, we give the concrete form of Φ_{2n} $(n \geq 1)$. The concrete form of Φ_2 is rather simple:

(2.38)

$$\Phi_2 = \frac{\tilde{z}_0^2 - 1}{2\tilde{\Gamma}_0}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1}\left\{\frac{1}{2}\{\tilde{z}_0; \tilde{x}\} + \left(\frac{g_+(a)}{(\tilde{x} - 1)^2} + \frac{g_-(-a)}{(\tilde{x} + 1)^2}\right)\right.$$
$$\left. - \left(\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\right)^2 \left(\frac{g_+(a)}{(\tilde{z}_0 - 1)^2} + \frac{g_-(-a)}{(\tilde{z}_0 + 1)^2}\right)\right\}.$$

Then, to simplify the expression of Φ_{2n} $(n \geq 2)$ and also the discussion given below, we introduce $y_{2k}(\tilde{x}, \tilde{B})$ $(k = 0, 1, \cdots)$ by

$$y_0(\tilde{x}, \tilde{B}) = \frac{\tilde{z}_0^2(\tilde{x}, \tilde{B}) - 1}{\tilde{x}^2 - 1}, \tag{2.39}$$

$$y_{2k}(\tilde{x}, \tilde{B}) = \frac{1}{\tilde{x}^2 - 1}\sum_{l=0}^{k} \tilde{z}_{2l}(\tilde{x}, \tilde{B})\tilde{z}_{2(k-l)}(\tilde{x}, \tilde{B}) \quad (k \geq 1). \tag{2.40}$$

We immediately see that they satisfy the following relation:

$$(\tilde{x}^2 - 1)\sum_{n=0}^{\infty} \tilde{\eta}^{-2n} y_{2n}(\tilde{x}, \tilde{B}) = \tilde{z}^2(\tilde{x}, \tilde{B}, \tilde{\eta}) - 1. \tag{2.41}$$

Further we use the following notation: for a multi-index $\vec{\kappa} = (\kappa_1, \kappa_2, \cdots, \kappa_\mu)$ in $\mathbb{N}_0^\mu$ and for κ_j-dependent $(j = 1, 2, \cdots, \mu)$ quantities X_{κ_j}, we define

$$|\vec{\kappa}|_\mu = \sum_{j=1}^{\mu} \kappa_j, \tag{2.42}$$

$$\sum_{|\vec{\kappa}|_\mu = k}^{*} X_{\kappa_1} \cdots X_{\kappa_\mu} = \begin{cases} 1 & \text{for } \mu = 0 \\ \displaystyle\sum_{\substack{|\vec{\kappa}|_\mu = k \\ \kappa_j \geq 1}} X_{\kappa_1} \cdots X_{\kappa_\mu} & \text{for } \mu \geq 1. \end{cases} \tag{2.43}$$

Using these notations, we can describe the concrete form of Φ_{2n} $(n \geq 2)$ as follows:

$$\Phi_{2n} = \Phi_{2n}^{(1)} + \Phi_{2n}^{(2)} + \Phi_{2n}^{(3)}, \tag{2.44}$$

where $\Phi_{2n}^{(1)}$, $\Phi_{2n}^{(2)}$ and $\Phi_{2n}^{(3)}$ are

(2.45)

$$\begin{aligned} \Phi_{2n}^{(1)} = &-\frac{1}{2}\frac{d\tilde{z}_0}{d\tilde{x}} \sum_{\mu=2}^{n} \sum_{|\vec{\kappa}|_\mu = n}^{*} (-1)^\mu \frac{y_{2\kappa_1} \cdots y_{2\kappa_\mu}}{y_0^\mu} \\ &- \frac{1}{2\tilde{\Gamma}_0} \sum_{\substack{k_1 + \cdots + k_4 = n \\ k_1, \cdots, k_4 \leq n-1}} \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1} \frac{d\tilde{z}_{2k_1}}{d\tilde{x}} \frac{d\tilde{z}_{2k_2}}{d\tilde{x}} \tilde{\Gamma}_{2k_3} \\ &\times \sum_{\mu = \min\{1, k_4\}}^{k_4} \sum_{|\vec{\kappa}|_\mu = k_4}^{*} (-1)^\mu \frac{y_{2\kappa_1} \cdots y_{2\kappa_\mu}}{y_0^\mu} \end{aligned}$$

$$+\frac{1}{2}\frac{1}{\tilde{z}_0^2-1}\frac{d\tilde{z}_0}{d\tilde{x}}\sum_{k_1+k_2=n}^{*}\tilde{z}_{2k_1}\tilde{z}_{2k_2},$$

(2.46)

$$\Phi_{2n}^{(2)}=\frac{\tilde{z}_0^2-1}{4\tilde{\Gamma}_0}\sum_{k_1+k_2=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-2}\frac{d^3\tilde{z}_{2k_1}}{d\tilde{x}^3}$$

$$\times\sum_{\mu=\min\{1,k_2\}}^{k_2}\sum_{|\vec{\kappa}|_\mu=k_2}^{*}(-1)^\mu\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}$$

$$-\frac{3(\tilde{z}_0^2-1)}{8\tilde{\Gamma}_0}\sum_{k_1+k_2+k_3=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-3}\frac{d^2\tilde{z}_{2k_1}}{d\tilde{x}^2}\frac{d^2\tilde{z}_{2k_2}}{d\tilde{x}^2}$$

$$\times\sum_{\mu=\min\{1,k_3\}}^{k_3}\sum_{|\vec{\kappa}|_\mu=k_3}^{*}(-1)^\mu(\mu+1)\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}},$$

(2.47)

$$\Phi_{2n}^{(3)}=-\frac{\tilde{z}_0^2-1}{2\tilde{\Gamma}_0}\sum_{k_1+k_2+k_3=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1}\frac{d\tilde{z}_{2k_1}}{d\tilde{x}}\frac{d\tilde{z}_{2k_2}}{d\tilde{x}}\frac{g_+(a)}{(\tilde{z}_0-1)^2}$$

$$\times\sum_{\mu=\min\{1,k_3\}}^{k_3}\sum_{|\vec{\kappa}|_\mu=k_3}^{*}(-1)^\mu(\mu+1)\frac{\tilde{z}_{2\kappa_1}\cdots\tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0-1)^\mu}$$

$$-\frac{\tilde{z}_0^2-1}{2\tilde{\Gamma}_0}\sum_{k_1+k_2+k_3=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1}\frac{d\tilde{z}_{2k_1}}{d\tilde{x}}\frac{d\tilde{z}_{2k_2}}{d\tilde{x}}\frac{g_-(-a)}{(\tilde{z}_0+1)^2}$$

$$\times\sum_{\mu=\min\{1,k_3\}}^{k_3}\sum_{|\vec{\kappa}|_\mu=k_3}^{*}(-1)^\mu(\mu+1)\frac{\tilde{z}_{2\kappa_1}\cdots\tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0+1)^\mu}.$$

Then we recursively determine $(\tilde{z}_{2n}(\tilde{x},\tilde{B}),\tilde{\Gamma}_{2n}(\tilde{B}))$ $(n\geq 1)$ as follows:

(2.48)
$$\tilde{\Gamma}_{2n}(\tilde{B})=\frac{-2\tilde{\Gamma}_0}{\pi}\int_1^{-1}(1-\tilde{z}_0^2)^{-1/2}\Phi_{2n}(\tilde{x},\tilde{B})d\tilde{x},$$

(2.49)
$$\begin{aligned}\tilde{z}_{2n}(\tilde{x},\tilde{B})&=(1-\tilde{z}_0^2)^{1/2}\int_1^{\tilde{x}}(1-\tilde{z}_0^2)^{-1/2}\Big(\Phi_{2n}(\tilde{x},\tilde{B})-\frac{\tilde{\Gamma}_{2n}}{2\tilde{\Gamma}_0}\frac{d\tilde{z}_0}{d\tilde{x}}\Big)d\tilde{x}\\&=(1-\tilde{z}_0^2)^{1/2}\int_{-1}^{\tilde{x}}(1-\tilde{z}_0^2)^{-1/2}\Big(\Phi_{2n}(\tilde{x},\tilde{B})-\frac{\tilde{\Gamma}_{2n}}{2\tilde{\Gamma}_0}\frac{d\tilde{z}_0}{d\tilde{x}}\Big)d\tilde{x}.\end{aligned}$$

Now we inductively confirm that $(\tilde{z}_{2n}(\tilde{x},\tilde{B}),\tilde{\Gamma}_{2n}(\tilde{B}))$ $(n\geq 1)$ satisfy (2.24), (2.37) and

$$\tilde{z}_{2n}(\pm 1,\tilde{B})=0. \tag{2.50}$$

We first confirm that $(\tilde{z}_2(\tilde{x},\tilde{B}),\tilde{\Gamma}_2(\tilde{B}))$ satisfies (2.24) and (2.50). From (2.27) we immediately see that $\{\tilde{z}_0;\tilde{x}\}$ is holomorphic on $\tilde{E}^2_{r_1,r_2}$. Furthermore, using (2.25) and (2.26), we find that

$$\begin{aligned}&(\tilde{z}_0^2-1)\left(\frac{g_+(a)}{(\tilde{x}-1)^2}-\left(\frac{\partial\tilde{z}_0}{\partial\tilde{x}}\right)^2\frac{g_+(a)}{(\tilde{z}_0-1)^2}\right)\\&=g_+(a)\frac{\tilde{z}_0+1}{\tilde{z}_0-1}\left(\left(\frac{\tilde{z}_0-1}{\tilde{x}-1}\right)^2-\left(\frac{\partial\tilde{z}_0}{\partial\tilde{x}}\right)^2\right)\end{aligned} \tag{2.51}$$

is holomorphic at $\tilde{x}=1$ and hence on $\tilde{E}^2_{r_1,r_2}$. By the same reasoning, the counterpart of (2.51) in Φ_2, i.e.,

$$\begin{aligned}&(\tilde{z}_0^2-1)\left(\frac{g_-(a)}{(\tilde{x}+1)^2}-\left(\frac{\partial\tilde{z}_0}{\partial\tilde{x}}\right)^2\frac{g_-(a)}{(\tilde{z}_0+1)^2}\right)\\&=g_-(a)\frac{\tilde{z}_0-1}{\tilde{z}_0+1}\left(\left(\frac{\tilde{z}_0+1}{\tilde{x}+1}\right)^2-\left(\frac{\partial\tilde{z}_0}{\partial\tilde{x}}\right)^2\right)\end{aligned} \tag{2.52}$$

is also holomorphic on $\tilde{E}^2_{r_1,r_2}$. In conclusion Φ_2 is holomorphic on $\tilde{E}^2_{r_1,r_2}$. It is clear from the representation (2.48) (resp. (2.49)) of $\tilde{\Gamma}_2$ (resp. $\tilde{z}_2$) that they are holomorphic and satisfy (2.37) and (2.50) on $\tilde{E}^2_{r_1,r_2}$.

Next we confirm that $(\tilde{z}_{2n}, \tilde{\Gamma}_{2n})$ satisfies (2.24), (2.37) and (2.50) under the assumption that $(\tilde{z}_{2k}, \tilde{\Gamma}_{2k})$ $(1 \leq k \leq n-1)$ satisfy these properties. By the same reasoning as the case of $n = 1$, it suffices to show that Φ_{2n} is holomorphic on $\tilde{E}^2_{r_1,r_2}$. We first note that, since $\tilde{z}_0$ satisfies (2.25) and (2.26), y_0 is holomorphic and satisfies

$$y_0(\tilde{x}, \tilde{B}) \neq 0 \quad \text{on} \quad \tilde{E}^2_{r_1,r_2}. \tag{2.53}$$

Further the holomorphy of y_{2k} $(1 \leq k \leq n-1)$ follows from the induction hypothesis (2.50). Then the holomorphy of $\Phi^{(1)}_{2n}$ and $\Phi^{(2)}_{2n}$ immediately follows from the induction hypothesis also. On the other hand the seeming poles at $\tilde{x} = \pm a$ that appear in (2.47) are cancelled out thanks to Lemma C.1 in Appendix C, and hence $\Phi^{(3)}_{2n}$ is holomorphic on $\tilde{E}^2_{r_1,r_2}$. (Indeed, we can apply Lemma C.1 with $w_0 = \tilde{z}_0 \pm 1$ and $w_k = \tilde{z}_{2k}$ $(k = 1, 2, \cdots)$ in this case.) Thus, we find that Φ_{2n} is holomorphic on $\tilde{E}^2_{r_1,r_2}$. Then the induction proceeds, and hence we obtain (2.24) and (2.50) for $n \geq 1$.

Now we embark on the proof of the estimates (2.28) and (2.29). Let N be an arbitrarily large natural number. In order to derive these estimates, we introduce a new variable ζ given by

$$\zeta = \exp\left[\frac{1}{N}\log\left(\frac{\tilde{x}}{N}\right)\right] \tag{2.54}$$

and we consider a holomorphic function $g(\tilde{x})$ on $D_N = \{\tilde{x} \in \mathbb{C} : |\tilde{x}| \leq N\}$ as a holomorphic function $g(N\zeta^N)$ on $\{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$.

Remark 2.3. As we will see below, to obtain (2.28) and (2.29) for arbitrarily small h, we will let N sufficiently large so that (2.65) holds. Then D_N becomes larger and larger as N increases. Still, the same reasoning as in the proof of (2.24), (2.25) and (2.27) guarantee that,

(2.55) for arbitrary large N, we can take $\delta > 0$ sufficiently small so that $\tilde{z}_{2n}(\tilde{x}, \tilde{B})$ $(n = 0, 1, 2, \cdots)$ are holomorphic on $\tilde{E}^2_{N,\delta}$.

In what follows, we use the following notation: for a holomorphic function $f(\zeta)$ on $\{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$, we define $\|f\|_{\{\varepsilon\}}$ by

$$\|f\|_{\{\varepsilon\}} := \sup_{|\zeta| \leq 1-\varepsilon} |f(\zeta)| \tag{2.56}$$

for $0 < \varepsilon < 1$. Then our task is to show the following

Lemma 2.1. *There exist positive constants $C_0(< 1)$ and C_1 such that, for arbitrarily large natural number N, we can take a sufficiently small positive constant δ (depending on N) so that the following estimates hold for $|\tilde{B}| \leq \delta$ and $0 < \varepsilon \leq (2N)^{-1} \log N$: for $1 \leq k \leq N-1$,*

$$|\tilde{\Gamma}_{2k}| \leq C_0 N^{k-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k, \tag{2.57}$$

$$\|\tilde{z}_{2k}\|_{\{\varepsilon\}} \leq C_0 N^{k+1-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k, \tag{2.58}$$

$$\left\| \frac{d\tilde{z}_{2k}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \leq C_0 N^{k-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k, \tag{2.59}$$

$$\left\| \frac{y_{2k}}{y_0} \right\|_{\{\varepsilon\}} \leq C_0 N^{k-N} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k, \tag{2.60}$$

and for $k \geq N$,

$$|\tilde{\Gamma}_{2k}| \leq C_0 N^{-1} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k, \tag{2.61}$$

$$\|\tilde{z}_{2k}\|_{\{\varepsilon\}} \leq C_0 (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k, \tag{2.62}$$

$$\left\| \frac{d\tilde{z}_{2k}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \leq C_0 N^{-1} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k, \tag{2.63}$$

$$(2.64)\qquad \left\|\frac{y_{2k}}{y_0}\right\|_{\{\varepsilon\}} \le C_0 N^{-1}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k.$$

As the proof of Lemma 2.1 is delicate and lengthy, we describe its role in our whole reasoning before proving it; the proof of Lemma 2.1 will be given after we explain its role. Now a crucial point is that the estimate (2.28) (resp. (2.29)) we want to prove follows from (2.57) and (2.61) (resp. (2.58) and (2.62)). This can be confirmed in the following manner: Let $h > 0$ be an arbitrarily small number. Then we take N so that it satisfies

$$(2.65)\qquad \frac{4C_1}{\log N} < h.$$

By taking $\varepsilon = (2N)^{-1}\log N$, we obtain the following estimates from (2.57) and (2.61) for $n \ge 1$:

$$(2.66)\qquad |\tilde{\Gamma}_{2n}(\tilde{B})| \le C_0 N^{-1}(2n)!\Big(\frac{4C_1}{\log N}\Big)^n$$

for $|\tilde{B}| \le \delta$, where δ is a positive constant appearing in Lemma 2.1. By the same way, we obtain the following estimates from (2.58) and (2.62):

$$(2.67)\qquad |\tilde{z}_{2n}(\tilde{x},\tilde{B})| = |\tilde{z}_{2n}(N\zeta^N,\tilde{B})| \le C_0(2n)!\Big(\frac{4C_1}{\log N}\Big)^n$$

for $|\zeta| \le 1-(2N)^{-1}\log N$ and $|\tilde{B}| \le \delta$. Here we note that, for sufficiently large N,

$$(2.68)\qquad |\tilde{x}| \ge N^{1/2}/2 \text{ holds when } |\zeta| = 1-(2N)^{-1}\log N.$$

Indeed, (2.68) follows from the relation $\tilde{x} = N\zeta^N$ and the following inequality:

$$(2.69)\qquad N\Big(1-\frac{\log N}{2N}\Big)^N \ge \frac{1}{2}N\exp\Big[-\frac{1}{2}\log N\Big] = \frac{1}{2}N^{1/2}$$

holds for sufficiently large N. Thus we can assume that (2.67) holds for $|\tilde{x}| \leq N^{1/2}/2$. Hence, by taking N so that it satisfies $r_1 \leq N^{1/2}/2$ and (2.65), we obtain (2.28) and (2.29). In conclusion, we obtain Theorem 2.1. Thus the proof of Theorem 2.1 will be completed if we verify Lemma 2.1.

Proof of Lemma 2.1. To begin with we confirm that (2.57) $\sim$ (2.60) hold for $k = 1$. We first show that Φ_2 satisfies the following estimates: there exists a positive constant $\tilde{C}_0$ such that, for an arbitrary positive constant $p > 1$, we can take a positive constant δ so that

$$\sup_{|\tilde{x}| \leq N} |\Phi_2(\tilde{x}, \tilde{B})| \leq \tilde{C}_0 N^{-p+1} \tag{2.70}$$

holds for $|\tilde{B}| \leq \delta$.

Remark 2.4. As (2.33) and (2.34) indicate, we readily find

$$\Phi_{2n}(\tilde{x}, 0) = 0 \quad \text{for} \quad n \geq 1. \tag{2.71}$$

Therefore it is natural to expect that (2.70) holds by taking δ sufficiently small depending on N and p.

Indeed, by taking $\delta > 0$ sufficiently small, we may assume that $\tilde{\Gamma}_0$, y_0 and $d\tilde{z}_0/d\tilde{x}$ are holomorphic on $\tilde{E}^2_{N+N^p,\delta}$. Furthermore, since $\tilde{\Gamma}_0(0) = y_0(\tilde{x}, 0) = d\tilde{z}_0/d\tilde{x}(\tilde{x}, 0) = 1$, by letting $\delta > 0$ sufficiently small again, we may also assume that

$$\sup_{\substack{|\tilde{x}| \leq N+N^p \\ |\tilde{B}| \leq \delta}} \left\{ |(\tilde{\Gamma}_0)^{\pm 1}|, |(y_0)^{\pm 1}|, \left| \left(\frac{d\tilde{z}_0}{d\tilde{x}} \right)^{\pm 1} \right| \right\} \leq 2 \tag{2.72}$$

holds. We fix $\tilde{B}$ in the disc $\{\tilde{B} : |\tilde{B}| \leq \delta\}$. Then we obtain the following estimates for $j = 1, 2, \cdots$ from Cauchy's inequality:

$$\sup_{|\tilde{x}| \leq N} \left| \frac{d^j \tilde{z}_0}{d\tilde{x}^j} \right| \leq 2(j-1)! N^{-(j-1)p}. \tag{2.73}$$

Therefore, from (2.73) we obtain

$$|\{\tilde{z}_0;\tilde{x}\}| \leq \left|\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1}\frac{d^3\tilde{z}_0}{d\tilde{x}^3}\right| + \frac{3}{2}\left|\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-2}\left(\frac{d^2\tilde{z}_0}{d\tilde{x}^2}\right)^2\right| \leq 32N^{-2p} \tag{2.74}$$

on D_N. In what follows, we fix p at $N+2$ and take δ sufficiently small so that (2.70) holds.

Next we derive the estimates of (2.51) on D_N. Since (2.51) is holomorphic on D_N, appealing to the maximum modulus principle, it suffices to estimate (2.51) on the boundary ∂D_N of D_N. Further, since $g_\pm(x)$ is holomorphic at the origin, we can assume that

$$|g_\pm(\pm a)| \leq \tilde{C}_1 \tag{2.75}$$

holds for some positive constant $\tilde{C}_1$. From the representation

$$\tilde{z}_0(\tilde{x}) = 1 + (\tilde{x}-1)\frac{\partial\tilde{z}_0}{\partial\tilde{x}} - \int_1^{\tilde{x}}(\tilde{x}-1)\frac{\partial^2\tilde{z}_0}{\partial\tilde{x}^2}d\tilde{x} \tag{2.76}$$

of $\tilde{z}_0(\tilde{x})$, we obtain

$$\left(\frac{\tilde{z}_0-1}{\tilde{x}-1}\right)^2 - \left(\frac{\partial\tilde{z}_0}{\partial\tilde{x}}\right)^2 = -\frac{2}{\tilde{x}-1}\frac{\partial\tilde{z}_0}{\partial\tilde{x}}\int_1^{\tilde{x}}(\tilde{x}-1)\frac{\partial^2\tilde{z}_0}{\partial\tilde{x}^2}d\tilde{x} + \left(\frac{1}{\tilde{x}-1}\int_1^{\tilde{x}}(\tilde{x}-1)\frac{\partial^2\tilde{z}_0}{\partial\tilde{x}^2}d\tilde{x}\right)^2. \tag{2.77}$$

Here we note that it follows from (2.73) that the following estimates hold on ∂D_N:

$$\left|\frac{1}{\tilde{x}-1}\int_1^{\tilde{x}}(\tilde{x}-1)\frac{\partial^2\tilde{z}_0}{\partial\tilde{x}^2}d\tilde{x}\right| \leq \frac{2(N+1)^2}{N-1}N^{-p}. \tag{2.78}$$

Further, by taking δ sufficiently small, we may assume that (2.35) holds with $\varepsilon = 1/2$ on D_N, and hence,

$$N - \frac{3}{2} \leq |\tilde{z}_0 \pm 1| \leq N + \frac{3}{2} \tag{2.79}$$

holds on ∂D_N. Then, combining (2.75), (2.78) and (2.79), we obtain the following estimates of (2.51):

$$\sup_{|\tilde{x}|=N}\left|g_+(a)\frac{\tilde{z}_0+1}{\tilde{z}_0-1}\left(\left(\frac{\tilde{z}_0-1}{\tilde{x}-1}\right)^2-\left(\frac{\partial\tilde{z}_0}{\partial\tilde{x}}\right)^2\right)\right| \tag{2.80}$$
$$\le \tilde{C}_1\frac{N+3/2}{N-3/2}\left(\frac{8(N+1)^2}{N-1}N^{-p}+\left(\frac{2(N+1)^2}{N-1}N^{-p}\right)^2\right)$$
$$\le 320\tilde{C}_1N^{-p+1}.$$

By the same reasoning, we obtain the same estimates with (2.80) for (2.52). Therefore, combining (2.72), (2.74), (2.79) and (2.80), we obtain (2.70).

Now we derive (2.57) $\sim$ (2.60) from (2.70) for $k=1$. Since $0<\varepsilon\le(2N)^{-1}\log N$, (2.57) for $k=1$ immediately follows from the representation (2.48) of $\tilde{\Gamma}_2$, (2.70) for $C_1\ge(C_0)^{-1}\tilde{C}_0$ as follows:

$$|\tilde{\Gamma}_2(\tilde{B})|\le 4\left|\frac{\tilde{\Gamma}_0}{\pi}\right|\sup_{|\tilde{x}|\le N}|\Phi_2|\int_1^{-1}|1-\tilde{z}_0^2|^{-1/2}\left|\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1}\right||d\tilde{z}_0| \tag{2.81}$$
$$\le 8\tilde{C}_0N^{-p+1}$$
$$\le 8\tilde{C}_0N^{-p+1}(\varepsilon N)^{-2}2^{-2}(\log N)^2$$
$$\le C_0N^{-p+2}(\varepsilon N)^{-2}2!(C_0)^{-1}\tilde{C}_0\log N.$$

Next we consider the estimates of $\tilde{z}_2$. Since $\tilde{z}_2$ is holomorphic on D_N, it suffices to estimate it for $\tilde{x}\in\partial D_N$. We obtain the following estimates from (2.49) and (2.70) for $\tilde{x}\in\partial D_N\cap\{\mathrm{Re}\tilde{x}\ge 0\}$:

$$|\tilde{z}_2(\tilde{x},\tilde{B})| \tag{2.82}$$
$$\le|1-\tilde{z}_0^2|^{1/2}\left(2\sup_{|\tilde{x}|\le N}|\Phi_2|+\frac{|\tilde{\Gamma}_2|}{2|\tilde{\Gamma}_0|}\right)\int_1^{\tilde{z}_0(\tilde{x})}|1-\tilde{z}_0^2|^{-1/2}|d\tilde{z}_0|$$

$$\leq 5(2N+3)\tilde{C}_0 N^{-p+1}\tilde{C}_2 \log N$$

$$\leq 20\tilde{C}_0\tilde{C}_2 N^{-p+2}\log N$$

$$\leq C_0 N^{-p+4}(\varepsilon N)^{-2} 2!(C_0)^{-1} 5\tilde{C}_0\tilde{C}_2 \log N,$$

where the integration path is taken as a straight line segment that connects 1 and $\tilde{x}$; thus this choice of the integration path together with the assumption on $\tilde{x}$ enables us to dominate the multivalued integral in the following manner:

$$\sup_{\substack{\pm\mathrm{Re}\tilde{x}\geq 0\\ |\tilde{x}|\leq N}} \int_{\pm 1}^{\tilde{z}_0(\tilde{x})} |1-\tilde{z}_0^2|^{-1/2}|d\tilde{z}_0| \leq \tilde{C}_2 \log N, \tag{2.83}$$

where $\tilde{C}_2$ is a positive constant that is independent of N. Using the second representation of (2.49), we find that $\tilde{z}_2$ satisfies the same estimates with (2.82) for $\tilde{x} \in \partial D_N \cap \{-\mathrm{Re}\tilde{x} \geq 0\}$. Therefore (2.82) holds on D_N. Then, since $|N\zeta^N| \leq N$ for $|\zeta| \leq 1-\varepsilon$, by taking $C_1 \geq (C_0)^{-1}5\tilde{C}_0\tilde{C}_2$, we immediately have (2.58) for $k=1$. Further, from (2.37), (2.70), (2.81) and (2.82), we obtain the following estimates on D_N:

(2.84)

$$\left|\frac{\partial \tilde{z}_2}{\partial \tilde{x}}\right| \leq \left|\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\frac{\tilde{z}_0}{\tilde{z}_0^2-1}\tilde{z}_2\right| + \left|\frac{\partial \tilde{z}_0}{\partial \tilde{x}}\frac{\tilde{\Gamma}_2}{2\tilde{\Gamma}_0}\right| + |\Phi_2|$$

$$\leq \frac{N+1/2}{(N-3/2)^2} 40\tilde{C}_0\tilde{C}_2 N^{-p+2}\log N + 16\tilde{C}_0 N^{-p+1} + \tilde{C}_0 N^{-p+1}$$

$$\leq \tilde{C}_0(320\tilde{C}_2 \log N + 17)N^{-p+1}$$

$$\leq C_0 N^{-p+3}(\varepsilon N)^{-2} 2!(2C_0)^{-1}\tilde{C}_0(320\tilde{C}_2+17)\log N.$$

Therefore, by taking $C_1 \geq (2C_0)^{-1}\tilde{C}_0(320\tilde{C}_2+17)$, we obtain (2.59) for $k=1$. Finally, from (2.40) and (2.82), we obtain the following estimates on D_N:

$$
\begin{aligned}
(2.85) \qquad |y_2| &\leq \left|\frac{2\tilde{z}_0}{\tilde{x}^2-1}\right| |\tilde{z}_2| \\
&\leq \frac{N+1/2}{(N-1)^2} 40\tilde{C}_0\tilde{C}_2 N^{-p+2}\log N \\
&\leq C_0 N^{-p+3}(\varepsilon N)^{-2} 2!(C_0)^{-1}\tilde{C}_0 160\tilde{C}_2 \log N.
\end{aligned}
$$

Hence we obtain (2.60) for $k = 1$ with $C_1 \geq (C_0)^{-1}\tilde{C}_0 160\tilde{C}_2$. In conclusion, we obtain (2.57) $\sim$ (2.60) for $k = 1$. Here we remark that, from the discussion above,

(2.86) we can take $C_0 > 0$ arbitrarily small by taking C_1 sufficiently large.

Next we show (2.57) $\sim$ (2.60) for $k = n$ $(2 \leq n \leq N-1)$ under the assumption that these estimates hold for $1 \leq k \leq n-1$. We first confirm the following estimates:

$$
(2.87) \qquad \|\Phi_{2n}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0^2 N^{n-N}(\varepsilon N)^{-2n}(2n)!C_1^n(\log N)^{n-1},
$$

where $\tilde{C}_0$ is a positive constant that is independent of n, N and ε. Let us consider the first term of $\Phi_{2n}^{(1)}$. From Lemma 1.2.2 and the induction hypothesis, we obtain the following estimates:

$$
\begin{aligned}
(2.88) \quad & \left\|\frac{1}{2}\frac{d\tilde{z}_0}{d\tilde{x}}\sum_{\mu=2}^{n}\sum_{|\vec{\kappa}|_\mu=n}^{*}(-1)^\mu\,\frac{y_{2\kappa_1}\cdots y_{2\kappa_\mu}}{y_0^\mu}\right\|_{\{\varepsilon\}} \\
&\leq \frac{1}{2}\left\|\frac{d\tilde{z}_0}{d\tilde{x}}\right\|_{\{\varepsilon\}}\sum_{\mu=2}^{n}\sum_{|\vec{\kappa}|_\mu=n}^{*}\left\|\frac{y_{2\kappa_1}}{y_0}\right\|_{\{\varepsilon\}}\cdots\left\|\frac{y_{2\kappa_\mu}}{y_0}\right\|_{\{\varepsilon\}} \\
&\leq \sum_{\mu=2}^{n}\sum_{|\vec{\kappa}|_\mu=n}^{*} C_0^\mu N^{n-\mu N}(\varepsilon N)^{-2n}(2\kappa_1)!\cdots(2\kappa_\mu)!(C_1\log N)^n
\end{aligned}
$$

$$\leq \sum_{\mu=2}^{n}(4C_0)^{\mu}(2n-\mu+1)!N^{n-\mu N}(\varepsilon N)^{-2n}(C_1\log N)^n$$

$$\leq N^n(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n\sum_{\mu=2}^{n}\frac{(4C_0N^{-N})^{\mu}}{(\mu-2)!}$$

$$\leq 16e^{4C_0}C_0^2N^{n-2N}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n.$$

Here we note that the same reasoning as in (2.88) entails the following estimates for $1\leq k\leq n-1$:

$$\left\|\sum_{\mu=1}^{k}\sum_{|\vec{\kappa}|_{\mu}=k}^{*}(-1)^{\mu}\frac{y_{2\kappa_1}\cdots y_{2\kappa_{\mu}}}{y_0^{\mu}}\right\|_{\{\varepsilon\}} \leq 4e^{4C_0}C_0N^{k-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k. \tag{2.89}$$

Next we consider the second term of $\Phi_{2n}^{(1)}$. Since at least two of k_j's are non-zero, the factor C_0N^{-N} appears at least twice in the estimation of the term. For example, the following part of the term with $k_2=k_3=0$ is one of the essential terms in the estimation:

$$\left\|\frac{1}{2}\sum_{\substack{k_1+k_4=n\\1\leq k_1,k_4\leq n-1}}\frac{d\tilde{z}_{2k_1}}{d\tilde{x}}\sum_{\mu=1}^{k_4}\sum_{|\vec{\kappa}|_{\mu}=k_4}^{*}(-1)^{\mu}\frac{y_{2\kappa_1}\cdots y_{2\kappa_{\mu}}}{y_0^{\mu}}\right\|_{\{\varepsilon\}} \leq 8e^{4C_0}C_0^2N^{n-2N}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n. \tag{2.90}$$

On the other hand, when at least three of $k_1,\cdots,k_4$ are non-zero, the factor C_0N^{-N} appears at least three times. Then, since $C_0N^{-N}\ll 1$, we obtain better estimates than (2.90) for these terms. Therefore the second term of $\Phi_{2n}^{(1)}$ satisfies the following estimates for some positive constant $\tilde{C}_0$:

$$(2.91)\qquad \Bigg\|\frac{1}{2\tilde{\Gamma}_0}\sum_{\substack{k_1+\cdots+k_4=n\\ k_1,\cdots,k_4\le n-1}}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-1}\frac{d\tilde{z}_{2k_1}}{d\tilde{x}}\frac{d\tilde{z}_{2k_2}}{d\tilde{x}}\tilde{\Gamma}_{2k_3} \times\sum_{\mu=\min\{1,k_4\}}^{k_4}\sum\nolimits^*_{|\vec{\kappa}|_\mu=k_4}(-1)^\mu\,\frac{y_{2\kappa_1}\cdots y_{2\kappa_\mu}}{y_0^\mu}\Bigg\|_{\{\varepsilon\}} \le \tilde{C}_0C_0^2N^{n-2N}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n.$$

Finally, since $|\tilde{z}_0^2-1|\ge(|\tilde{x}|-3/2)^2\ge N/8$ holds for $|\zeta|=1-\varepsilon$ $(0<\varepsilon\le(2N)^{-1}\log N)$ (cf. (2.68) and (2.79)), the estimates of the third term of $\Phi_{2n}^{(1)}$ follows from the maximum modulus principle and the induction hypothesis as follows:

$$(2.92)\qquad \Bigg\|\frac{1}{2}\frac{1}{\tilde{z}_0^2-1}\frac{d\tilde{z}_0}{d\tilde{x}}\sum\nolimits^*_{k_1+k_2=n}\tilde{z}_{2k_1}\tilde{z}_{2k_2}\Bigg\|_{\{\varepsilon\}} \le 8C_0^2N^{n+1-2N}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n.$$

We thus obtain the following estimates of $\Phi_{2n}^{(1)}$ from (2.88), (2.91) and (2.92) for some positive constant $\tilde{C}_0$:

$$(2.93)\qquad \|\Phi_{2n}^{(1)}\|_{\{\varepsilon\}}\le\tilde{C}_0C_0^2N^{n+1-2N}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n \le\tilde{C}_0C_0^2N^{n-N}(\varepsilon N)^{-2n}(2n-1)!C_1^n(\log N)^{n-1}$$

Now we consider the estimation of $\Phi_{2n}^{(2)}$. We first show the following

Lemma 2.2. *Let $d\tilde{z}_{2k}/d\tilde{x}$ satisfy* (2.59) *for $0<\varepsilon\le(2N)^{-1}\log N$. Then the following inequalities hold:*

(2.94)

$$\left\|\frac{d^2\tilde{z}_{2k}}{d\tilde{x}^2}\right\|_{\{\varepsilon\}}\le e^2(1-\varepsilon)^{-N}C_0N^{k-1-N}(\varepsilon N)^{-2k-1}(2k+1)!(C_1\log N)^k,$$

(2.95)
$$\left\|\frac{d^3\tilde{z}_{2k}}{d\tilde{x}^3}\right\|_{\{\varepsilon\}} \le e^2(1-\varepsilon)^{-2N}C_0N^{k-2-N} \times \left(1+\frac{\log N}{2k+2}\right)(\varepsilon N)^{-2k-2}(2k+2)!(C_1\log N)^k.$$

Proof. Appealing to the maximum modulus principle, it is enough to show (2.94) and (2.95) for $|\zeta| = 1-\varepsilon$. We first note the following relation:

$$\frac{d^2\tilde{z}_{2k}}{d\tilde{x}^2} = \frac{\zeta^{-N+1}}{N^2}\frac{d}{d\zeta}\frac{d\tilde{z}_{2k}}{d\tilde{x}}, \tag{2.96}$$

$$\frac{d^3\tilde{z}_{2k}}{d\tilde{x}^3} = \left(\frac{\zeta^{-2N+2}}{N^4}\frac{d^2}{d\zeta^2} + \frac{-N+1}{N^4}\zeta^{-2N+1}\frac{d}{d\zeta}\right)\frac{d\tilde{z}_{2k}}{d\tilde{x}}. \tag{2.97}$$

We use the following representation:

$$\frac{d^j}{d\zeta^j}\frac{d\tilde{z}_{2k}}{d\tilde{x}} = \frac{j!}{2\pi\sqrt{-1}}\int_{|\tilde{\zeta}-\zeta|=(k+1)^{-1}\varepsilon}\frac{d\tilde{z}_{2k}}{d\tilde{x}}\frac{d\tilde{\zeta}}{(\tilde{\zeta}-\zeta)^{j+1}}. \tag{2.98}$$

We immediately find that the integral path of (2.98) is contained in $|\tilde{\zeta}| \le 1-\tilde{\varepsilon}$ with $\tilde{\varepsilon} = k\varepsilon/(k+1)$. Since

$$\begin{aligned}\left\|\frac{d\tilde{z}_{2k}}{d\tilde{x}}\right\|_{\{\tilde{\varepsilon}\}} &\le C_0N^{k-N}(\tilde{\varepsilon}N)^{-2k}(2k)!(C_1\log N)^k \\ &= C_0N^{k-N}\left(1+\frac{1}{k}\right)^{2k}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k \\ &\le e^2C_0N^{k-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k\end{aligned} \tag{2.99}$$

follows from (2.59), we obtain the following estimates from (2.98):

(2.100)
$$\left\|\frac{d^j}{d\zeta^j}\frac{d\tilde{z}_{2k}}{d\tilde{x}}\right\|_{\{\varepsilon\}} \le j!(k+1)^j\varepsilon^{-j}e^2C_0N^{k-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k.$$

Then, from (2.100), we obtain the estimates of (2.96) and (2.97) as follows:

$$(2.101)\quad \left\| \frac{\zeta^{-N+1}}{N^2}\frac{d}{d\zeta}\frac{d\tilde{z}_{2k}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \\ \le e^2(1-\varepsilon)^{-N} C_0 N^{k-1-N}(\varepsilon N)^{-2k-1}(2k+1)!(C_1\log N)^k,$$

$$(2.102)\quad \left\| \left(\frac{\zeta^{-2N+2}}{N^4}\frac{d^2}{d\zeta^2} + \frac{-N+1}{N^4}\zeta^{-2N+1}\frac{d}{d\zeta} \right) \frac{d\tilde{z}_{2k}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \\ \le e^2(1-\varepsilon)^{-2N} C_0 N^{k-2-N}(\varepsilon N)^{-2k-2}(2k+2)!(C_1\log N)^k \\ + 2e^2(1-\varepsilon)^{-2N} C_0 N^{k-2-N}(\varepsilon N)^{-2k-1}(2k+1)!(C_1\log N)^k.$$

Since $\varepsilon N \le 2^{-1}\log N$, (2.94) and (2.95) immediately follow from (2.101) and (2.102).

□

We return to the estimation of $\Phi_{2n}^{(2)}$. Let us consider the first term of $\Phi_{2n}^{(2)}$. By the same reasoning as the estimation of (2.89), the following holds for $k \ge 1$:

$$(2.103)\quad \left\| \sum_{\mu=1}^{k} {\sum_{|\vec{\kappa}|_\mu = k}}^{*} (-1)^\mu \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu} \frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}} \cdots \frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}} \right\|_{\{\varepsilon\}} \\ \le 8e^{8C_0} C_0 N^{k-N}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^k.$$

By the discussion similar to the estimation of (2.91), we find that the terms with $k_1 = 0$ or $k_2 = 0$ are essential in the estimation. In particular, since (2.73) holds, we see that the following term with $k_2 = 0$ is the worst contribution:

$$(2.104)\quad \left\| \frac{\tilde{z}_0^2 - 1}{4\tilde{\Gamma}_0} \left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-2} \frac{d^3\tilde{z}_{2(n-1)}}{d\tilde{x}^3} \right\|_{\{\varepsilon\}}$$

$$\leq 2(N^2(1-\varepsilon)^{2N}+3)e^2(1-\varepsilon)^{-2N}C_0N^{n-3-N} \times \left(1+\frac{\log N}{2n}\right)(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}$$
$$\leq e^2C_0N^{n-N}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}.$$

Therefore, having (2.73) in mind, we obtain the following estimates for some positive constant $\tilde{C}_0$:

$$\Big\|\frac{\tilde{z}_0^2-1}{4\tilde{\Gamma}_0}\sum_{k_1+k_2=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-2}\frac{d^3\tilde{z}_{2k_1}}{d\tilde{x}^3} \times \sum_{\mu=\min\{1,k_2\}}^{k_2}\sum_{|\vec{\kappa}|_\mu=k_2}^{*}(-1)^\mu\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}\Big\|_{\{\varepsilon\}} \leq \tilde{C}_0C_0N^{n-N}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}. \tag{2.105}$$

By the same reasoning, we find that the following estimates for the second term of $\Phi_{2n}^{(2)}$ follows from (2.94) and Lemma 1.2.2:

$$\Big\|\frac{3(\tilde{z}_0^2-1)}{8\tilde{\Gamma}_0}\sum_{k_1+k_2+k_3=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-3}\frac{d^2\tilde{z}_{2k_1}}{d\tilde{x}^2}\frac{d^2\tilde{z}_{2k_2}}{d\tilde{x}^2} \times \sum_{\mu=\min\{1,k_3\}}^{k_3}\sum_{|\vec{\kappa}|_\mu=k_3}^{*}(-1)^\mu(\mu+1)\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}\Big\|_{\{\varepsilon\}} \leq \tilde{C}_0C_0N^{n-1-2N}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^{n-1}. \tag{2.106}$$

Thus we see that the following estimates hold for some positive constant $\tilde{C}_0$:

$$\|\Phi_{2n}^{(2)}\|_{\{\varepsilon\}} \leq \tilde{C}_0C_0N^{n-N}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}. \tag{2.107}$$

Finally we consider the estimation of $\Phi_{2n}^{(3)}$. Let us consider the first term of $\Phi_{2n}^{(3)}$. Since it is holomorphic on $|\zeta| < 1$, it suffices to estimate it for $|\zeta| = 1 - \varepsilon$. We first note that, since $|\tilde{z}_0 - 1| \geq 4^{-1}\sqrt{N}$ holds on $|\zeta| = 1 - \varepsilon$, we find the following estimates:

$$\Big\| \sum_{\mu=1}^{k} \sum_{|\vec{\kappa}|_\mu = k}^{*} (-1)^\mu (\mu+1) \frac{\tilde{z}_{2\kappa_1} \cdots \tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0 - 1)^\mu} \Big\|_{\{\varepsilon\}} \leq 32 e^{32C_0} C_0 N^{k-N+1/2} (\varepsilon N)^{-2k} (2k)! (C_1 \log N)^k. \tag{2.108}$$

By the same discussion as the estimation of (2.91), we find that the terms with one of k_j's being $n-1$ are essential in the estimation. In particular, since $32e^{32C_0}N^{-1/2} \ll 1$, comparison of (2.108) and (2.59) entails that the worst is the term with $k_3 = n-1$, which can be estimated as follows:

(2.109)
$$\begin{aligned}
&\Big\| \frac{\tilde{z}_0 + 1}{\tilde{z}_0 - 1} \frac{d\tilde{z}_0}{d\tilde{x}} \frac{g_+(a)}{2\tilde{\Gamma}_0} \sum_{\mu=1}^{n-1} \sum_{|\vec{\kappa}|_\mu = n-1}^{*} (-1)^\mu (\mu+1) \frac{\tilde{z}_{2\kappa_1} \cdots \tilde{z}_{2\kappa_\mu}}{(\tilde{z}_0 - 1)^\mu} \Big\|_{\{\varepsilon\}} \\
&\leq \frac{N(1-\varepsilon)^N + 3/2}{N(1-\varepsilon)^N - 3/2} 4^3 \tilde{C}_1 e^{32C_0} C_0 \\
&\quad \times N^{n-N-1/2} (\varepsilon N)^{-2(n-1)} (2n-2)! (C_1 \log N)^{n-1} \\
&\leq 4^3 \tilde{C}_1 e^{32C_0} C_0 N^{n-N-1/2} \frac{(\log N)^2}{n^2} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1} \\
&\leq 4^5 \tilde{C}_1 e^{32C_0} C_0 N^{n-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1},
\end{aligned}$$

where $\tilde{C}_1$ is a positive constant that satisfies (2.75). In this way, we can obtain the estimates of the first term of $\Phi_{2n}^{(3)}$. On the other hand, we immediately find that the second term also satisfies the same estimates with the first term. Therefore we find that the following estimates hold for $\Phi_{2n}^{(3)}$ with a positive constant $\tilde{C}_0$:

$$\|\Phi_{2n}^{(3)}\|_{\{\varepsilon\}} \le \tilde{C}_0 C_0 N^{n-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^{n-1}. \tag{2.110}$$

By taking $C_1^{-1} \le C_0$ and summing up (2.93), (2.107) and (2.110), we obtain (2.87).

Now we confirm (2.57) $\sim$ (2.60) for $k = n$. We first note that the following estimates follow from (2.48) and (2.87):

$$\begin{aligned} |\tilde{\Gamma}_{2n}(\tilde{B})| &\le \frac{4|\tilde{\Gamma}_0|}{\pi} \int_1^{-1} |1 - \tilde{z}_0^2|^{-1/2} |d\tilde{z}_0| \|\Phi_{2n}\|_{\{\varepsilon\}} \\ &\le 8\tilde{C}_0 C_0^2 N^{n-N} (\varepsilon N)^{-2n} (2n)! C_1^n (\log N)^{n-1}. \end{aligned} \tag{2.111}$$

Then, by taking C_0 sufficiently small so that they satisfy $8\tilde{C}_0 C_0 < 1$, we obtain (2.57). Next, from (2.49), (2.87) and (2.111), we obtain the following estimates on $\{|\tilde{x}| = N(1-\varepsilon)^N\} \cap \{\mathrm{Re}\tilde{x} \ge 0\}$:

(2.112)

$$\begin{aligned} |\tilde{z}_{2n}(\tilde{x}, \tilde{B})| &\le |1 - \tilde{z}_0^2|^{1/2} \int_1^{\tilde{x}} |1 - \tilde{z}_0^2|^{-1/2} |d\tilde{z}_0| \Big(2\|\Phi_{2n}\|_{\{\varepsilon\}} + \frac{|\tilde{\Gamma}_{2n}|}{2|\tilde{\Gamma}_0|} \Big) \\ &\le 20(1-\varepsilon)^N \tilde{C}_0 \tilde{C}_2 C_0^2 N^{n+1-N} (\varepsilon N)^{-2n} (2n)! (C_1 \log N)^n, \end{aligned}$$

where $\tilde{C}_2$ is a positive constant appearing in (2.83). By the same discussion, we find from the second representation of (2.49) that (2.112) also holds on $\{|\tilde{x}| = N(1-\varepsilon)^N\} \cap \{-\mathrm{Re}\tilde{x} \ge 0\}$. Since $\tilde{z}_{2n}$ is holomorphic on $\{|\tilde{x}| \le N(1-\varepsilon)^N\}$, we see that (2.112) holds there. Hence, by taking C_0 so that $20\tilde{C}_0\tilde{C}_2 C_0 < 1$ holds, we obtain (2.58) for $k = n$. Then, using the relation (2.37), we obtain the following estimates from (2.87), (2.111) and (2.112):

(2.113)

$$\Big\| \frac{\partial \tilde{z}_{2n}}{\partial \tilde{x}} \Big\|_{\{\varepsilon\}} \le 2\frac{N(1-\varepsilon)^N + 1/2}{(N(1-\varepsilon)^N - 3/2)^2} \|\tilde{z}_{2n}\|_{\{\varepsilon\}} + 2|\tilde{\Gamma}_{2n}| + \|\Phi_{2n}\|_{\{\varepsilon\}}$$

$$\leq (320\tilde{C}_0\tilde{C}_2 + 9\tilde{C}_0)C_0^2 N^{n-N}(\varepsilon N)^{-2n}(2n)!(C_1 \log N)^n.$$

Therefore, by taking C_0 so that $(320\tilde{C}_0\tilde{C}_2 + 9\tilde{C}_0)C_0 < 1$ holds, we find that $\tilde{z}_{2n}$ satisfies (2.59). Furthermore, by the maximum modulus principle, we obtain the following estimates from (2.40), (2.58), (2.69) and (2.112):

(2.114)

$$\begin{aligned}
&\|y_{2n}\|_{\{\varepsilon\}} \\
&\leq \frac{1}{N^2(1-\varepsilon)^{2N} - 1}\Big(2\|\tilde{z}_0\|_{\{\varepsilon\}}\|\tilde{z}_{2n}\|_{\{\varepsilon\}} + \sum_{k=1}^{n-1} \|\tilde{z}_{2k}\|_{\{\varepsilon\}}\|\tilde{z}_{2(n-k)}\|_{\{\varepsilon\}}\Big) \\
&\leq \frac{4}{N^2(1-\varepsilon)^{2N}} \left(80N(1-\varepsilon)^{2N}\tilde{C}_0\tilde{C}_2 + 2n^{-1}N^{1-N}\right) \\
&\quad \times C_0^2 N^{n+1-N}(\varepsilon N)^{-2n}(2n)!(C_1 \log N)^n \\
&\leq 4(80\tilde{C}_0\tilde{C}_2 + 1)C_0^2 N^{n-N}(\varepsilon N)^{-2n}(2n)!(C_1 \log N)^n.
\end{aligned}$$

Therefore, by taking C_0 so that it satisfies $4(80\tilde{C}_0\tilde{C}_2 + 1)C_0 < 1$, we obtain (2.60) for $k = n$. Thus the induction proceeds and we obtain (2.57) $\sim$ (2.60) for $1 \leq k \leq N-1$.

Now, we confirm (2.61) $\sim$ (2.64) for $k \geq N$. We first remark that, from (2.57) $\sim$ (2.60), we find that (2.61) $\sim$ (2.64) also hold for $1 \leq k \leq N-1$. Hence we show (2.61) $\sim$ (2.64) for $k = n$ $(n \geq N)$ under the assumption that these estimates hold for $1 \leq k \leq n-1$. By the same discussion with the derivation of (2.57) $\sim$ (2.60) from (2.87), it suffices to show the following estimates:

(2.115) $$\|\Phi_{2n}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0^2 N^{-1}(\varepsilon N)^{-2n}(2n)!C_1^n(\log N)^{n-1},$$

where $\tilde{C}_0$ is some positive constant. We first confirm the following estimates:

(2.116) $$\|\Phi_{2n}^{(1)}\|_{\{\varepsilon\}} \leq \tilde{C}_0 C_0^2 N^{-1}(\varepsilon N)^{-2n}(2n-1)!(C_1 \log N)^n.$$

Then, since $\log N \le n$, we find that $\Phi_{2n}^{(1)}$ satisfies (2.115). As in the derivation of (2.93), the following term is essential in the estimation of $\Phi_{2n}^{(1)}$:

$$(2.117)\qquad \begin{aligned} &\Big\| \frac{1}{2}\frac{1}{\tilde{z}_0^2-1}\frac{d\tilde{z}_0}{d\tilde{x}} \sum_{k_1+k_2=n}^{*} \tilde{z}_{2k_1}\tilde{z}_{2k_2} \Big\|_{\{\varepsilon\}} \\ &\le 8N^{-1}C_0^2(\varepsilon N)^{-2n}(C_1\log N)^n \sum_{k_1+k_2=n}^{*}(2k_1)!(2k_2)! \\ &\le 32C_0^2N^{-1}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^n. \end{aligned}$$

Here we used the fact that $|\tilde{z}_0^2-1| \ge N/8$ holds for $|\zeta| = 1-\varepsilon$. In this way, we can show that the first and the second term of $\Phi_{2n}^{(1)}$ also satisfy (2.116) by the same discussion with the estimation of (2.88) and (2.91).

Next, we show the following estimates:

$$(2.118)\qquad \|\Phi_{2n}^{(2)}\|_{\{\varepsilon\}} \le \tilde{C}_0C_0N^{-1}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}.$$

We first note that, by the same discussion with the proof of Lemma 2.2, we obtain the following estimates for $k = 1, 2, \cdots$ from (2.63):

(2.119)

$$\Big\| \frac{d^2\tilde{z}_{2k}}{d\tilde{x}^2} \Big\|_{\{\varepsilon\}} \le e^2(1-\varepsilon)^{-N}C_0N^{-2}(\varepsilon N)^{-2k-1}(2k+1)!(C_1\log N)^k,$$

(2.120)

$$\begin{aligned} \Big\| \frac{d^3\tilde{z}_{2k}}{d\tilde{x}^3} \Big\|_{\{\varepsilon\}} &\le e^2(1-\varepsilon)^{-2N}C_0N^{-3} \\ &\qquad \times \left(1+\frac{\log N}{2k+2}\right)(\varepsilon N)^{-2k-2}(2k+2)!(C_1\log N)^k. \end{aligned}$$

Let us consider the first term of $\Phi_{2n}^{(2)}$, which is essential in the estimation of $\Phi_{2n}^{(2)}$. Since $\log N \leq n$, we find the following estimates from (2.120):

(2.121)

$$\left\|\frac{d^3\tilde{z}_{2(n-1)}}{d\tilde{x}^3}\right\|_{\{\varepsilon\}} \leq \frac{3e^2}{2}(1-\varepsilon)^{-2N}C_0N^{-3}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}.$$

And, since $\log N \leq N$, we find the following estimates from (2.120) for $1 \leq k \leq n-2$:

(2.122)

$$\left\|\frac{d^3\tilde{z}_{2(k-1)}}{d\tilde{x}^3}\right\|_{\{\varepsilon\}} \leq e^2(1-\varepsilon)^{-2N}C_0N^{-2}(\varepsilon N)^{-2k}(2k)!(C_1\log N)^{k-1}.$$

Then, by the same reasoning with the estimates of (2.105), we obtain the following estimates for the first term of $\Phi_{2n}^{(2)}$:

(2.123)

$$\begin{aligned}
&\Bigg\|\frac{\tilde{z}_0^2-1}{4\tilde{\Gamma}_0}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-2}\Bigg\{\frac{d^3\tilde{z}_{2(n-1)}}{d\tilde{x}^3}\\
&\quad+\sum_{k_1+k_2=n-1}^{*}\frac{d^3\tilde{z}_{2k_1}}{d\tilde{x}^3}\sum_{\mu=\min\{1,k_2\}}^{k_2}\sum_{|\vec{\kappa}|_\mu=k_2}^{*}(-1)^\mu\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}\\
&\quad+\frac{d^3\tilde{z}_0}{d\tilde{x}^3}\sum_{\mu=1}^{n-1}\sum_{|\vec{\kappa}|_\mu=k_2}^{*}(-1)^\mu\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}\Bigg\}\Bigg\|_{\{\varepsilon\}}\\
&\leq 8N^2(1-\varepsilon)^{2N}\{e^2(1-\varepsilon)^{-2N}C_0N^{-3}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}\\
&\quad+8e^{8C_0+2}(1-\varepsilon)^{-2N}C_0^2N^{-3}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}\\
&\quad+8e^{8C_0}C_0N^{-1-p}(\varepsilon N)^{-2(n-1)}(2n-2)!(C_1\log N)^{n-1}\}\\
&\leq 8C_0N^{-1}(\varepsilon N)^{-2n}(2n)!(C_1\log N)^{n-1}\\
&\qquad\times\{e^2+8e^{8C_0+2}C_0+e^{8C_0}N^{-p+2}n^{-2}(\log N)^2\}.
\end{aligned}$$

Since $p \geq N+2$, we immediately find that the first term of $\Phi_{2n}^{(2)}$ satisfies (2.118). In the same way, from (2.119), we can show the following estimates:

(2.124)

$$\Big\|\frac{3(\tilde{z}_0^2-1)}{8\tilde{\Gamma}_0}\sum_{k_1+k_2+k_3=n-1}\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-3}\frac{d^2\tilde{z}_{2k_1}}{d\tilde{x}^2}\frac{d^2\tilde{z}_{2k_2}}{d\tilde{x}^2}$$
$$\times\sum_{\mu=\min\{1,k_3\}}^{k_3}\sum_{|\vec{\kappa}|_\mu=k_3}^{*}(-1)^\mu(\mu+1)\left(\frac{d\tilde{z}_0}{d\tilde{x}}\right)^{-\mu}\frac{d\tilde{z}_{2\kappa_1}}{d\tilde{x}}\cdots\frac{d\tilde{z}_{2\kappa_\mu}}{d\tilde{x}}\Big\|_{\{\varepsilon\}}$$
$$\leq \tilde{C}_0C_0N^{-2}(\varepsilon N)^{-2n}(2n-1)!(C_1\log N)^{n-1}.$$

Hence, from (2.123) and (2.124), we obtain (2.118).

Finally, by the same discussion with the estimation of (2.109), we obtain the following estimates:

$$(2.125)\quad \|\Phi_{2n}^{(3)}\|_{\{\varepsilon\}} \leq \tilde{C}_0C_0N^{-3/2}(\varepsilon N)^{-2n+2}(2n-2)!(C_1\log N)^{n-1}.$$

Then, since $N^{1/2} \leq n$ and $(\varepsilon N)^2 \leq n$, we find that $\Phi_{2n}^{(3)}$ satisfies (2.115).

Summing up, we have confirmed (2.61) $\sim$ (2.64) for $k=n$. Thus the induction proceeds. This completes the proof of Lemma 2.1, completing the proof of Theorem 2.1.

□

As is shown in [KT], we can deduce the following Theorem 2.2 from Theorem 2.1:

Theorem 2.2. *Let $\tilde{S}$ and S respectively be a solution of*

$$(2.126)\quad \tilde{S}^2+\frac{\partial\tilde{S}}{\partial x}=\eta^2\left(\frac{aA+xB}{x^2-a^2}+\eta^{-2}\left(\frac{g_+(a)}{(x-a)^2}+\frac{g_-(-a)}{(x+a)^2}\right)\right)$$

and

$$(2.127)\qquad S^2+\frac{\partial S}{\partial z}=\eta^2\left(\frac{a\Gamma}{z^2-a^2}+\eta^{-2}\left(\frac{g_+(a)}{(z-a)^2}+\frac{g_-(-a)}{(z+a)^2}\right)\right),$$

and suppose that

(2.128)

$$\arg\tilde{S}_{-1}(x,a,A,B)=\arg\left(\frac{\partial z_0}{\partial x}S_{-1}(z_0(x,a,A,B),a,\Gamma_0(a,A,B))\right)$$

holds. Then they satisfy

$$(2.129)\qquad \tilde{S}_{\mathrm{odd}}(x,a,A,B,\eta)$$

$$=\left(\frac{\partial z}{\partial x}\right)S_{\mathrm{odd}}(z(x,a,A,B,\eta),a,\Gamma(a,A,B,\eta),\eta)$$

on $E^2_{r_1,r_2}$*, where* $\tilde{S}_{\mathrm{odd}}$ *and* S_{odd} *respectively denote the odd part of* $\tilde{S}$ *and* S*.*

We also have the following

Theorem 2.3. *Let* $\tilde{\psi}_\pm(x,a,A,B,\eta)$ *be WKB solutions of the generic (i.e.,* $a\neq 0$*) Mathieu equation* (2.1) *that are normalized at a simple pole* $x=a$ *as*

$$(2.130)\qquad \tilde{\psi}_\pm(x,a,A,B,\eta)=\frac{1}{\sqrt{\tilde{S}_{\mathrm{odd}}}}\exp\left(\pm\int_a^x\tilde{S}_{\mathrm{odd}}dx\right),$$

and $\psi_\pm(z,a,\Gamma,\eta)$ *be WKB solutions of the Legendre equation* (2.4) *that is normalized at a simple pole* $z=a$ *as*

$$(2.131)\qquad \psi_\pm(z,a,\Gamma,\eta)=\frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_a^z S_{\mathrm{odd}}dz\right).$$

Then they satisfy the following relation (2.132) *on an open set* $E^2_{r_1,r_2}$ *given by* (2.8)*:*

$$(2.132)\qquad \tilde{\psi}_\pm(x,a,A,B,\eta)$$

$$= \left(\frac{\partial z}{\partial x}\right)^{-1/2} \psi_{\pm}(z(x,a,A,B,\eta), a, \Gamma(a,A,B,\eta), \eta),$$

where $z(x,a,A,B,\eta)$ *and* $\Gamma(a,A,B,\eta)$ *are the series constructed in Theorem 2.1.*

We have so far discussed how WKB solutions of (2.1) are related to WKB solutions of (2.4). But we need in Section 3 the Legendre equation in the form (2.2). Here we discuss how WKB solutions of (2.4) and those of (2.2) are related; as we will see below the relation can be found in a straightforward manner. For the sake of simplicity of description we consider the situation when the parameter Γ in (2.4) is a genuine constant; this restriction does not cause any problems in our later discussion, as appropriate use of microdifferential operators will enable us to relate (2.4) with Γ being a genuine constant and (2.4) with Γ being infinite series. (See Proposition 4.3.) To relate (2.4) and (2.2) we define an infinite series

$$\Lambda(a,\Gamma,\eta) = \sum_{n=0}^{\infty} \Lambda_n(a,\Gamma)\eta^{-n} \tag{2.133}$$

and functions $\mu(a)$ and $\nu(a)$ of a by

$$\Lambda = \sqrt{\Gamma + (\sqrt{a}\eta)^{-2}\left(g_+(a) + g_-(-a) + \frac{1}{4}\right)} - \frac{(\sqrt{a}\eta)^{-1}}{2}, \tag{2.134}$$

$$\mu = \sqrt{1 + 2(g_+(a) + g_-(-a))}, \tag{2.135}$$

$$\nu = 2(g_+(a) - g_-(-a)). \tag{2.136}$$

Since $\Lambda(a,\Gamma,\eta)$ satisfy

$$a\Gamma = a\Lambda^2 + \eta^{-1}\sqrt{a}\Lambda - \eta^{-2}(g_+(a) + g_-(-a)), \tag{2.137}$$

we immediately obtain (2.4) from (2.2) by choosing Λ, μ and ν in

(2.2) respectively by (2.134),(2.135) and (2.136). Therefore we find the following

Proposition 2.1. *Let* $T_{\text{odd}}(z, a, \Lambda, \mu, \nu, \eta)$ *and* $\phi_{\pm}(z, a, \Lambda, \mu, \nu, \eta)$ *respectively be the odd part of the solution of the Riccati equation*

(2.138)
$$T^2 + \frac{\partial T}{\partial z} = \eta^2 \left(\frac{a\Lambda^2}{z^2 - a^2} + \eta^{-1} \frac{\sqrt{a}\Lambda}{z^2 - a^2} + \eta^{-2} \frac{az\nu + a^2(\mu^2 - 1)}{(z^2 - a^2)^2} \right)$$

and WKB solutions of (2.2) *that are normalized at a simple pole* $z = a$ *as*

$$\phi_{\pm}(z, a, \Lambda, \mu, \nu, \eta) = \frac{1}{\sqrt{T_{\text{odd}}}} \exp\left(\pm \int_a^z T_{\text{odd}} dz \right). \tag{2.139}$$

Then the following relations hold:

$$S_{\text{odd}}(z, a, \Gamma, \eta) = T_{\text{odd}}(z, a, \Lambda(a, \Gamma, \eta), \mu(a), \nu(a), \eta), \tag{2.140}$$

$$\psi_{\pm}(z, a, \Gamma, \eta) = \phi_{\pm}(z, a, \Lambda(a, \Gamma, \eta), \mu(a), \nu(a), \eta), \tag{2.141}$$

where the infinite series $\Lambda(a, \Gamma, \eta)$ *and the functions* $\mu(a)$ *and* $\nu(a)$ *are those given by* (2.134), (2.135) *and* (2.136) *respectively.*

Remark 2.5. Since $\Lambda(a, \Gamma, \eta)$ given by (2.134) is a convergent power series in η, $\Lambda_n(a, \Gamma)$ $(n \geq 1)$ satisfy the following estimates: There exists a positive constant C such that

$$|\Lambda_n(a, \Gamma)| \leq \sqrt{|\Gamma|} \left(\frac{C}{\sqrt{|a\Gamma|}} \right)^n \tag{2.142}$$

holds for $a\Gamma \neq 0$ and $n \geq 1$.

3 Analytic properties of Borel transformed WKB solutions of the Legendre equation with a large parameter

The main purpose of this section is to present analytic properties of Borel transformed WKB solutions of (2.2) with genuine constants

a, Λ, μ and ν. To begin with, we show the following important

Proposition 3.1. *Let $T_{\text{odd}}(z, a, \Lambda, \eta)$ be the odd part of the solution of* (2.138) *whose top degree part $T_{-1}(z, a, \Lambda)$ is chosen so that it is positive for positive a, $z(> a)$ and Λ. Then we have*

$$\oint_{\gamma} T_{\text{odd}}(z, a, \Lambda, \eta) dz = 2\pi i \sqrt{a} \Lambda \eta + \pi i, \tag{3.1}$$

where γ is a closed curve that encircles two simple poles $z = \pm a$ counterclockwise.

Proof. Let

$$T^{(\pm)}(z, a, \Lambda, \eta) = \sum_{n=-1}^{\infty} T_n^{(\pm)}(z, a, \Lambda) \eta^{-n} \tag{3.2}$$

be the solutions of (2.138) whose top degree parts $T_{-1}^{(\pm)}(z, a, \Lambda)$ are respectively given by

$$T_{-1}^{(\pm)}(z, a, \Lambda) = \pm \sqrt{\frac{a\Lambda^2}{z^2 - a^2}}. \tag{3.3}$$

Then $T_0^{(\pm)}$ and $T_1^{(\pm)}$ are respectively given by

$$T_0^{(\pm)} = \frac{1}{2} \frac{z}{z^2 - a^2} \pm \frac{1}{2} \frac{1}{\sqrt{z^2 - a^2}} \tag{3.4}$$

and

$$T_1^{(\pm)} = \pm \frac{4a\nu z + a^2(4\mu^2 - 1)}{8\sqrt{a}\Lambda (z^2 - a^2)^{3/2}}. \tag{3.5}$$

Further we can inductively confirm that $T_n^{(\pm)}$ $(n \geq 2)$ have the following form:

$$T_n^{(\pm)} = \sum_{2 \leq p \leq n+2} c_{p,n}^{(\pm)} (z^2 - a^2)^{-p/2} \tag{3.6}$$

$$+\sum_{3\le p\le n+2} d_{p,n}^{(\pm)} z(z^2-a^2)^{-p/2},$$

where $c_{p,n}^{(\pm)}$ and $d_{p,n}^{(\pm)}$ are constants. Hence, by noting that

$$\oint_\gamma \frac{dz}{\sqrt{z^2-a^2}} = 2\pi i \tag{3.7}$$

and

$$\oint_\gamma T_n^{(\pm)} dz = 0 \tag{3.8}$$

hold for $n \ge 1$, we immediately obtain (3.1).

□

Now we consider the Voros coefficient

$$V(a,\Lambda,\eta) = \sum_{n=1}^{\infty} V_n \eta^{-n} \tag{3.9}$$

of (2.2), which is, by definition, given by

$$\int_a^\infty \left(T_{\rm odd} - \eta T_{-1} - \frac{1}{2z} \right) dz \tag{3.10}$$

(cf. [DP], [AKT2]). Let $\phi_\pm^{(\infty)}$ be WKB solutions of (2.2) that are normalized at infinity as

$$\phi_\pm^{(\infty)} = \frac{z^{\pm 1/2}}{\sqrt{T_{\rm odd}}} e^{\pm\eta y_+} \exp\left[\pm \int_\infty^z \left(T_{\rm odd} - \eta T_{-1} - \frac{1}{2z} \right) dz \right], \tag{3.11}$$

where

$$y_+(z,a,\Lambda) = \int_a^z \sqrt{\frac{a\Lambda^2}{z^2-a^2}} dz. \tag{3.12}$$

Then WKB solutions (2.139) of (2.2) that are normalized at $z=a$ as (2.139) are written by V and $\phi_\pm^{(\infty)}$ as follows:

$$\phi_\pm = a^{\mp 1/2} \exp\left(\pm V\right) \phi_\pm^{(\infty)}. \tag{3.13}$$

An important property of $\phi_{\pm}^{(\infty)}$ is that they are Borel summable when

(3.14) the path of integration of (3.11) from ∞ to z can be deformed so that it does not intersect Stokes curves of (2.2).

See [KoS] for the proof of the Borel summability of $\phi_{\pm}^{(\infty)}$. Hence the representation (3.13) of $\phi_{\pm}$ entails that the calculation of the alien derivative of $\phi_{\pm}$ is reduced to that of V. Fortunately the explicit form of V has been given by T. Koike ([Ko3]) as follows:

$$(3.15)\quad V_n = \frac{1}{n(n+1)(\sqrt{a}\Lambda)^n} \times \Big[B_{n+1} + \sum_{\substack{k+2l=n+1\\ k,l\geq 0}} \frac{(n+1)!}{k!(2l)!} B_k \Big\{\Big(\frac{1}{2}\Big)^{2l} - \theta_{+}^{2l} - \theta_{-}^{2l}\Big\}\Big]$$

for $n \geq 1$, where B_n $(n = 0, 1, 2, \cdots)$ are Bernoulli numbers defined by

$$(3.16)\qquad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$$

and

$$(3.17)\qquad \theta_{\pm}(\mu, \nu) = \sqrt{\frac{\mu^2 \pm \sqrt{\mu^4 - \nu^2}}{2}}.$$

In [Ko3] the derivation of (3.15) is done in a parallel way to the computation of the Voros coefficient of the Weber equation and the Whittaker equation. See [SS] and [T] (resp., [KoT]) for the computation of the Voros coefficient of the Weber equation (resp., the Whittaker equation). Hence the Borel transform $V_B(a, \Lambda, y)$ of V is concretely given by

(3.18)

$$V_B = \frac{1}{y(\exp(y/\sqrt{a}\Lambda) - 1)}$$

$$\times \left\{ 1 + \cosh\left(\frac{y}{2\sqrt{a}\Lambda}\right) - \cosh\left(\frac{\theta_+ y}{\sqrt{a}\Lambda}\right) - \cosh\left(\frac{\theta_- y}{\sqrt{a}\Lambda}\right) \right\}.$$

It immediately follows from (3.18) that V_B behaves as

$$V_B = \frac{1}{2\sqrt{a}\Lambda}\left(\frac{1}{4} - (\theta_+^2 + \theta_-^2)\right) + O(y) \tag{3.19}$$

near $y = 0$ and

$$V_B = \frac{1 + (-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{2m\pi i(y - 2m\pi i\sqrt{a}\Lambda)} + O(1) \tag{3.20}$$

near $y = 2m\pi i\sqrt{a}\Lambda$ for $m \in \mathbb{Z}\setminus\{0\}$. Therefore V_B is singular at $y = 2m\pi i\sqrt{a}\Lambda$ ($m \in \mathbb{Z}\setminus\{0\}$) and it has simple poles there.

Now let us compute the alien derivative

$$\Delta V = \sum_{m\geq 1} \Delta_{y=2m\pi i\sqrt{a}\Lambda} V \tag{3.21}$$

of the Voros coefficient V by using the alien calculus initiated by [Ec] and developed by [P], [DP] and [Sa]. Since V_B is single-valued and only has simple pole singularities, $\Delta_{y=2m\pi i\sqrt{a}\Lambda} V$ is given by the residue of V_B at $y = 2m\pi i\sqrt{a}\Lambda$, i.e.,

$$\Delta_{y=2m\pi i\sqrt{a}\Lambda} V = \frac{1 + (-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{m}. \tag{3.22}$$

Then, by employing the alien calculus, we find

$$\begin{aligned} &\Delta_{y=2m\pi i\sqrt{a}\Lambda} \exp(\pm V) \\ &\quad = \pm\frac{1 + (-1)^m - \cosh(2m\pi i\theta_+) - \cosh(2m\pi i\theta_-)}{m} \exp(\pm V). \end{aligned} \tag{3.23}$$

Noting the fact that

$$\Delta_{y=2m\pi i\sqrt{a}\Lambda}\left(e^{\mp\eta y_+} a^{\mp 1/2} \phi_\pm^{(\infty)}\right) = 0 \tag{3.24}$$

hold for $m \geq 1$ under the condition (3.14), we find (3.13) entails the following relations when (3.14) is satisfied:

(3.25)
$$\begin{aligned}
&\Delta_{y=2m\pi i\sqrt{a}\Lambda}\left(e^{\mp\eta y_+}\phi_\pm\right)\\
&= \Delta_{y=2m\pi i\sqrt{a}\Lambda}\left(e^{\mp\eta y_+}a^{\mp 1/2}\exp(\pm V)\phi_\pm^{(\infty)}\right)\\
&= e^{\mp\eta y_+}a^{\mp 1/2}\phi_\pm^{(\infty)}\Delta_{y=2m\pi i\sqrt{a}\Lambda}\left(\exp(\pm V)\right)\\
&= \pm\frac{1+(-1)^m-\cosh\left(2m\pi i\theta_+\right)-\cosh\left(2m\pi i\theta_-\right)}{m}\\
&\quad\times e^{\mp\eta y_+}a^{\mp 1/2}\phi_\pm^{(\infty)}\exp(\pm V)\\
&= \pm\frac{1+(-1)^m-\cosh\left(2m\pi i\theta_+\right)-\cosh\left(2m\pi i\theta_-\right)}{m}e^{\mp\eta y_+}\phi_\pm.
\end{aligned}$$

Summing up all these, we obtain the following

Theorem 3.1. *Let $\phi_\pm(z,\eta)$ denote the WKB solutions of the Legendre equation* (2.2) *that are normalized at a simple pole $z=a$ as in* (2.139). *Then their Borel transform $\phi_{\pm,B}(z,y)$ are singular at*

(3.26)
$$y = \mp y_+(z) + 2m\pi i\sqrt{a}\Lambda \quad (m = 0, \pm 1, \pm 2, \cdots),$$

where $y_+(z)$ is the function given by (3.12), *and its alien derivative there satisfies the following relation* (3.27) *for z that can be connected with $z=\infty$ by a path that is contained in the interior of a Stokes region of the Legendre equation.*

(3.27)
$$\left(\Delta_{y=\mp y_+ + 2m\pi i\sqrt{a}\Lambda}\phi_\pm\right)_B(z,y) = \pm\,\Xi_m(\mu,\nu)\phi_{\pm,B}(z,y-2m\pi i\sqrt{a}\Lambda),$$

where

(3.28)
$$\Xi_m(\mu,\nu) = \frac{1}{m}\left\{1+(-1)^m-\cosh\left(2\pi i m\sqrt{\frac{\mu^2+\sqrt{\mu^4-\nu^2}}{2}}\right)\right.$$

$$- \cosh\left(2\pi i m \sqrt{\frac{\mu^2 - \sqrt{\mu^4 - \nu^2}}{2}}\right)\Bigg\}.$$

4 Analytic properties of Borel transformed WKB solutions of the Mathieu equation — properties relevant to simple poles

The principal aim of this section is to deduce analytic properties of Borel transformed WKB solutions of the Mathieu equation (2.1) for $a \neq 0$ and $A \neq 0$ that are relevant to its two simple poles from those of the Legendre equation (2.2) through the transformation obtained in Section 2. To begin with, we show a result corresponding to Proposition 3.1 for the Mathieu equation. First, combining Proposition 2.1 and Proposition 3.1 we immediately find

$$\oint_\gamma S_{\text{odd}}(z, a, \Gamma, \eta) dz = 2\pi i \sqrt{a} \Lambda(a, \Gamma, \eta)\eta + \pi i, \tag{4.1}$$

where γ is the path given in Proposition 3.1. Therefore Proposition 4.1 below follows from Theorem 2.2.

Proposition 4.1. *Let* $\tilde{S}_{\text{odd}}(x, a, A, B, \eta)$ *be the odd part of the solution of* (2.126) *whose top degree part* $\tilde{S}_{-1}(x, a, A, B)$ *is chosen so that it satisfies* (2.128). *Then we have*

$$\oint_\gamma \tilde{S}_{\text{odd}}(x, a, A, B, \eta) dx = 2\pi i \sqrt{a} \Lambda(a, \Gamma(a, A, B, \eta), \eta)\eta + \pi i, \tag{4.2}$$

where the infinite series $\Lambda(a, \Gamma, \eta)$ *and* $\Gamma(a, A, B, \eta)$ *are those given in Proposition 2.1 and Theorem 2.2 respectively and* γ *is a closed curve that encircles two simple poles counterclockwise.*

Let us now employ the relation (2.141) between $\phi_\pm$ and $\psi_\pm$ to deduce analytic properties of $\psi_{\pm,B}$ from those of $\phi_{\pm,B}$. Here we make full use of microlocal analysis, which has been made possible by the estimation (2.142) that Λ_n satisfies. The concrete procedure is as follows: first, by the Taylor expansion, the right-hand side of (2.141), can be written as

$$\sum_{k=0}^{\infty} \frac{\tilde{\Lambda}^k(a,\Gamma,\eta)}{k!} \frac{\partial^k}{\partial \Lambda^k} \phi_\pm(z,a,\Lambda_0(a,\Gamma),\mu(a),\nu(a),\eta), \tag{4.3}$$

where

$$\tilde{\Lambda}(a,\Gamma,\eta) = \Lambda(a,\Gamma,\eta) - \Lambda_0(a,\Gamma). \tag{4.4}$$

Then, taking into account the estimates (2.142) of Λ_n, we can rewrite (4.3) in the form of an action of a microdifferential operator

$$\mathcal{L} =: \exp\left(\tilde{\Lambda}\theta_\Lambda\right) : \tag{4.5}$$

upon $\phi_{\pm,B}$ through the Borel transformation. Here $: \cdot :$ designates the normal ordered product (cf.[A1]) and θ_Λ is the symbol of ∂_Λ, i.e., $: \theta_\Lambda := \partial_\Lambda$. More concretely, we can write the action of $\mathcal{L}$ as an action of an integro-differential operator so that (4.3) can be rewritten as follows:

Proposition 4.2. *Suppose that the constants $a \neq 0$ and Λ in* (2.2) *are different from* 0. *Let $\phi_{\pm,B}$ (resp., $\psi_{\pm,B}$) be the Borel transformed WKB solutions of* (2.2) *(resp.,* (2.4)*) and suppose that they are both normalized at a simple pole $z = a$. Then they satisfy the following relation:*

(4.6)

$$\psi_{\pm,B}(z,a,\Gamma,y)$$
$$= \int_{\mp y_+}^{y} K_\Lambda(a,\Gamma,y-y',\partial_\Lambda)\phi_{\pm,B}(z,a,\Lambda,\mu(a),\nu(a),y')dy'\Big|_{\Lambda=\Lambda_0(a,\Gamma)},$$

where $K_\Lambda(a,\Gamma,y,\partial_\Lambda)$ is a differential operator of infinite order that is defined on $\{(\Lambda,y) \in \mathbb{C}^2\}$, which analytically depends on a and Γ

with the exception $a\Gamma = 0$, *and*

$$y_+(z, a, \Gamma) = \int_a^z \sqrt{\frac{a\Gamma}{z^2 - a^2}} dz, \tag{4.7}$$

$$\Lambda_0(a, \Gamma) = \sqrt{\Gamma}. \tag{4.8}$$

Here $\mu(a)$ *and* $\nu(a)$ *are functions that are respectively given by* (2.135) *and* (2.136).

See [K] and [SKK] for the notion of differential operators of infinite order.

Remark 4.1. The differential operator K_Λ is locally defined for $a, \Gamma \neq 0$. However, as (2.134) implies, K_Λ is multivalued on $\{(a, \Gamma, \Lambda, y) \in \mathbb{C}^4 : a, \Gamma \neq 0\}$.

Remark 4.2. It immediately follows from (4.7) and (4.8) that

$$y_+(z, a, \Gamma) = \int_a^z \sqrt{\frac{a\Lambda_0^2(a, \Gamma)}{z^2 - a^2}} dz. \tag{4.9}$$

Therefore, comparing (3.12) and (4.9), we find that y_+ is preserved by a change of parameters from (Λ, μ, ν) to $(\Gamma, g_+(a), g_-(-a))$.

Combining Theorem 3.1 and Proposition 4.2, we obtain the following

Lemma 4.1. *Let* $\psi_\pm(z, a, \Gamma, \eta)$ *denote the WKB solutions of the Legendre equation* (2.4) *that are normalized at a simple pole* $z = a$ *as in* (2.131). *Then their Borel transform* $\psi_{\pm,B}(z, a, \Gamma, y)$ *are singular at*

$$y = \mp y_+(z, a, \Gamma) + 2m\pi i\sqrt{a\Gamma} \quad (m = 0, \pm 1, \pm 2, \cdots), \tag{4.10}$$

where $y_+(z)$ *is the function given by* (4.7). *Furthermore their alien derivatives there satisfy the following relation* (4.11) *on the condition that* z *can be connected with* $z = \infty$ *by a path that is contained* *in the interior of a Stokes region of the Legendre equation* (2.4)*:*

(4.11)

$$\left(\Delta_{y=\mp y_+ + 2m\pi i\sqrt{a}\Gamma}\ \psi_\pm\right)_B (z, a, \Gamma, y)$$
$$= \pm\ \Xi_m(\mu, \nu)\big(\exp(-2m\pi i\sqrt{a}\tilde{\Lambda}\eta)\psi_\pm\big)_B(z, a, \Gamma, y - 2m\pi i\sqrt{a}\Gamma),$$

where $\mu = \mu(a)$ *and* $\nu = \nu(a)$ *are functions that are given by* (2.135) *and* (2.136) *respectively and* $\tilde{\Lambda}(a, \Gamma, \eta)$ *is a formal power series given by* (2.134) *and* (4.4)*.*

Proof. From the representation (4.6) of $\psi_{\pm,B}$ and the definition of the alien derivative, we find

(4.12)
$$\left(\Delta_{y=2m\pi i\sqrt{a}\Gamma}\ e^{\mp\eta y_+}\psi_\pm\right)_B (z, a, \Gamma, y)$$
$$= \mathcal{L}_{2m\pi i\sqrt{a}\Lambda}\big(\Delta_{y=2m\pi i\sqrt{a}\Lambda}\ \phi^{(0)}_\pm\big)_B(z, a, \Lambda, y)\big|_{\Lambda=\Lambda_0(a,\Gamma)}$$

holds, where $\mathcal{L}_{y_0}$ is the integro-differential operator obtained by taking $y = y_0$ as the end point of integration instead of $y = \mp y_+$ in (4.6) and $\phi^{(0)}_\pm = e^{\mp\eta y_+}\phi_\pm$. Therefore it follows from Theorem 3.1 that the right hand side of (4.12) is equal to

(4.13)
$$\pm\ \Xi_m(\mu(a), \nu(a))\mathcal{L}_{2m\pi i\sqrt{a}\Lambda}\big(\phi^{(0)}_{\pm,B}(z, a, \Lambda, y - 2m\pi i\sqrt{a}\Lambda)\big)\big|_{\Lambda=\Lambda_0(a,\Gamma)}.$$

Let us introduce the following coordinate transformation from (y, Λ) to (y', Λ'):

(4.14)
$$\begin{cases} y' = y - 2m\pi i\sqrt{a}\Lambda \\ \Lambda' = \Lambda. \end{cases}$$

We now prepare the following general lemma:

Lemma 4.2. *Let $F : (y, \Lambda_1, \cdots, \Lambda_p) \to (y', \Lambda'_1, \cdots, \Lambda'_p)$ be a coordinate transformation given by*

$$\begin{cases} y' = y + f(\Lambda_1, \cdots, \Lambda_p) \\ \Lambda'_1 = \Lambda_1 \\ \quad \vdots \\ \Lambda'_p = \Lambda_p, \end{cases} \tag{4.15}$$

where $f(\Lambda_1, \cdots, \Lambda_p)$ is a holomorphic function of $\Lambda = (\Lambda_1, \cdots, \Lambda_p) \in \mathbb{C}^p$ at $\Lambda = \overset{\circ}{\Lambda}$. Let $\tilde{\Lambda}_1, \cdots, \tilde{\Lambda}_p$ be symbols of microdifferential operators of the following form:

(4.16)

$$\tilde{\Lambda}_j(\Lambda_1, \cdots, \Lambda_p, \eta) = \sum_{n=1}^{\infty} \eta^{-n} \Lambda_{j,n}(\Lambda_1, \cdots, \Lambda_p) \quad (j = 1, \cdots, p).$$

Then the following relation holds:

$$\begin{aligned} &: \exp\big(\tilde{\Lambda}(\Lambda, \eta) \cdot \theta_\Lambda\big) : \\ &=: \exp\big[\eta'\big(f(\Lambda' + \tilde{\Lambda}) - f(\Lambda')\big)\big] :: \exp\big(\tilde{\Lambda}(\Lambda', \eta') \cdot \theta_{\Lambda'}\big) :, \end{aligned} \tag{4.17}$$

where $\eta' = \sigma(\partial/\partial y')$, $\theta_{\Lambda_1} = \sigma(\partial/\partial \Lambda_1), \cdots$, $\theta_\Lambda = (\theta_{\Lambda_1}, \cdots, \theta_{\Lambda_p})$, etc., and $\cdot$ is the inner product.

Proof. Let $P(\Lambda, \theta_\Lambda, \eta)$ denote the symbol of the left-hand side of (4.17), i.e.,

$$P(\Lambda, \theta_\Lambda, \eta) = \exp\big(\tilde{\Lambda}(\Lambda, \eta) \cdot \theta_\Lambda\big). \tag{4.18}$$

We first note the following equality:

$$: P(\Lambda, \theta_\Lambda, \eta) :=: P'(\Lambda', \theta_{\Lambda'}, \eta') :, \tag{4.19}$$

where P' is given by

(4.20)

$$P'(\Lambda', \theta_{\Lambda'}, \eta')$$

$$= \exp\big[- F(y,\Lambda)\cdot(\eta',\theta_{\Lambda'})\big] \exp(\partial_{\hat\eta}\cdot\partial_{\hat y} + \partial_{\hat\theta_\Lambda}\cdot\partial_{\hat\Lambda})$$
$$\times P(\Lambda,\hat\theta_\Lambda,\hat\eta)\exp\big[F(\hat y,\hat\Lambda)\cdot(\eta',\theta_{\Lambda'})\big]\Big|_{\substack{(y,\Lambda)=(\hat y,\hat\Lambda)=F^{-1}(y',\Lambda')\\ \hat\theta_\Lambda=\hat\eta=0}}$$
$$= \exp(\partial_{\hat\eta}\cdot\partial_{\hat y} + \partial_{\hat\theta_\Lambda}\cdot\partial_{\hat\Lambda})P(\Lambda,\hat\theta_\Lambda,\hat\eta)$$
$$\times \exp\big[(F(y+\hat y,\Lambda+\hat\Lambda) - F(y,\Lambda))\cdot(\eta',\theta_{\Lambda'})\big]\Big|_{\substack{(y,\Lambda)=F^{-1}(y',\Lambda')\\ \hat\Lambda=\hat y=\hat\theta_\Lambda=\hat\eta=0}}.$$

(Cf. [SKK, Chapter 2, Theorem 1.5.5]. See also the proof of [AKY, Proposition 1.2.13].) Since

$$(F(y+\hat y,\Lambda+\hat\Lambda) - F(y,\Lambda))\cdot(\eta',\theta_{\Lambda'}) \tag{4.21}$$
$$= \hat y\eta' + (f(\Lambda+\hat\Lambda) - f(\Lambda))\eta' + \hat\Lambda\cdot\theta_{\Lambda'}$$

and

$$e^{-\hat z\zeta}\exp(\partial_{\hat\zeta}\cdot\partial_{\hat z})e^{\hat z\zeta}f(\hat\zeta) = f(\hat\zeta+\zeta) \tag{4.22}$$

holds for a holomorphic function $f(\zeta)$, we find

$$P'(\Lambda',\theta_{\Lambda'},\eta') \tag{4.23}$$
$$= \exp(\partial_{\hat\theta_\Lambda}\cdot\partial_{\hat\Lambda})P(\Lambda,\hat\theta_\Lambda,\hat\eta+\eta')$$
$$\times \exp\big[(f(\Lambda+\hat\Lambda) - f(\Lambda))\eta' + \hat\Lambda\cdot\theta_{\Lambda'}\big]\Big|_{\substack{(y,\Lambda)=F^{-1}(y',\Lambda')\\ \hat\Lambda=\hat y=\hat\theta_\Lambda=\hat\eta=0}}$$
$$= \exp(\partial_{\hat\theta_\Lambda}\cdot\partial_{\hat\Lambda})\exp\big(\tilde\Lambda_1(\Lambda,\eta')\hat\theta_{\Lambda_1}\big)\cdots\exp\big(\tilde\Lambda_p(\Lambda,\eta')\hat\theta_{\Lambda_p}\big)$$
$$\times \exp\big[(f(\Lambda+\hat\Lambda) - f(\Lambda))\eta' + \hat\Lambda\cdot\theta_{\Lambda'}\big]\Big|_{\substack{\Lambda=\Lambda'\\ \hat\Lambda=\hat\theta_\Lambda=0}}$$
$$= \exp\big[\eta'\big(f(\Lambda'+\tilde\Lambda) - f(\Lambda')\big) + \tilde\Lambda\cdot\theta_{\Lambda'}\big].$$

Thus we obtain (4.17) from (4.19).

□

We resume the proof of Lemma 4.1. It follows from (4.17) that

$$\boldsymbol{\mathcal{L}} =: \exp\big(\tilde\Lambda(a,\Gamma,\eta)\theta_\Lambda\big) : \tag{4.24}$$

$$=: \exp\big(-2m\pi i\sqrt{a}\eta'\tilde{\Lambda}(a,\Gamma,\eta')\big) :: \exp\big(\tilde{\Lambda}(a,\Gamma,\eta')\theta_{\Lambda'}\big) : .$$

Therefore we find

(4.25)

$$\mathcal{L}_{2m\pi i\sqrt{a}\Lambda}\big(\phi^{(0)}_{\pm,B}(z,a,\Lambda,y-2m\pi i\sqrt{a}\Lambda)\big)$$
$$=: \exp\big(-2m\pi i\sqrt{a}(\Lambda_1+\Lambda_2\eta'^{-1}+\cdots)\big) : \big(\mathcal{L}_0\phi^{(0)}_{\pm,B}\big)(z,a,\Lambda',y'),$$

where the action of $:\eta'^{-1}:$ is fixed by taking $y'=0$ as the end point of integration. Here we note that, from (4.6) and (4.8), we obtain

$$\big(\mathcal{L}_0\phi^{(0)}_{\pm,B}\big)(z,a,\Lambda,y-2m\pi i\sqrt{a}\Lambda)\big|_{\Lambda=\Lambda_0(a,\Gamma)} \tag{4.26}$$
$$=\big(e^{\mp\eta y_\pm}\psi_\pm\big)_B(z,a,\Gamma,y-2m\pi i\sqrt{a}\Gamma).$$

Then (4.11) follows from (4.12), (4.13), (4.25) and (4.26).

□

From (4.1) and (4.8), we find that (4.11) can be rewritten as follows:

$$\Big(\Delta_{y=\mp y_+ + 2m\pi i\sqrt{a}\Gamma}\ \psi_\pm\Big)_B(z,a,\Gamma,y) \tag{4.27}$$
$$=\pm(-1)^m\Xi_m(\mu,\nu)\big(\exp(-m\oint_\gamma S_{\mathrm{odd}}dx)\psi_\pm\big)_B(z,a,\Gamma,y).$$

Now, we will study the singularity structure of Borel transformed WKB solutions of the Mathieu equation (2.1) using the transformation obtained in Theorem 2.1. To begin with, to simplify the notation, we restate the estimates (2.12) and (2.13) in the following form: there exists

(4.28) a continuous increasing function $h:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ that satisfies $h(\delta)\to 0$ when $\delta\to 0$

such that z_{2n} and Γ_{2n} $(n\geq 1)$ given in Theorem 2.1 satisfy the following estimates on $E^2_{r_1,\delta}$ for $0<\delta<r_2$:

$$|z_{2n}(x,a,A,B)|\leq (2n)!h^n(\delta)|aA|^{-n}, \tag{4.29}$$

(4.30) $$|\Gamma_{2n}(a, A, B)| \leq (2n)! h^n(\delta) |aA|^{-n}.$$

Let us consider the following ∞-Legendre equation:

(4.31)
$$\left(\frac{d^2}{dz^2} - \eta^2 \left(\frac{a\Gamma(a, A, B, \eta)}{z^2 - a^2} + \eta^{-2} \left(\frac{g_+(a)}{(z-a)^2} + \frac{g_-(-a)}{(z+a)^2}\right)\right)\right) \psi^\dagger = 0.$$

We immediately see that WKB solutions $\psi^\dagger_\pm(z, a, A, B, \eta)$ of (4.31) that are normalized at its simple pole $z = a$ are given by

(4.32) $$\psi^\dagger_\pm(z, a, A, B, \eta) = \psi_\pm(z, a, \Gamma(a, A, B, \eta), \eta).$$

Similarly to the relation between $\phi_{\pm,B}$ and $\psi_{\pm,B}$ discussed in Proposition 4.2, by applying the Taylor expansion and the Borel transformation successively to (4.32), we can relate the Borel transform of $\psi^\dagger_\pm$ with that of $\psi_\pm$ through the action of a microdifferential operator defined by

(4.33) $$\mathcal{G} =: \exp(\tilde{\Gamma}\theta_\Gamma) :,$$

where

(4.34) $$\tilde{\Gamma}(a, A, B, \eta) = \Gamma(a, A, B, \eta) - \Gamma_0(a, A, B)$$

and θ_Γ is the symbol of ∂_Γ. To be more specific, we find the following thanks to (4.30):

Proposition 4.3. *Let $\psi_{\pm,B}$ (resp. $\psi^\dagger_{\pm,B}$) be the Borel transformed WKB solutions of* (2.4) *(resp.* (4.31)*) for $a \neq 0$ (resp. $A \neq 0$) that are normalized at a simple pole $z = a$. Then $\psi_{\pm,B}$ and $\psi^\dagger_{\pm,B}$ satisfy the following relation:*

(4.35)
$$\psi^\dagger_{\pm,B}(z, a, A, B, y)$$

$$= \int_{\mp y_+}^{y} K_\Gamma(a, A, B, y - y', \partial_\Gamma)\psi_{\pm,B}(z, a, \Gamma, y')dy'\Big|_{\Gamma=\Gamma_0(a,A,B)},$$

where $K_\Gamma(a, A, B, y, \partial_\Gamma)$ is a differential operator of infinite order that is defined on

(4.36)

$$\{(a, A, B, \Gamma, y) \in \mathbb{C}^5 : a, A \neq 0, |B/A| < r_2, h(|B/A|)|y| < \sqrt{|aA|}\},$$

$$y_+(z, a, A, B) = \int_a^z \sqrt{\frac{a\Gamma_0(a, A, B)}{z^2 - a^2}}dz, \tag{4.37}$$

and

$$\sqrt{\Gamma_0(a, A, B)} = \frac{1}{2\pi i\sqrt{a}} \int_\gamma \sqrt{\frac{aA + xB}{x^2 - a^2}}dx. \tag{4.38}$$

In view of Lemma 4.1, we expect that $\psi^\dagger_{\pm,B}$ have singularities at $y = \mp y_+ + 2m\pi i\sqrt{a\Gamma_0}$ $(m = 0, \pm1, \pm2, \cdots)$. This is the case if the representation (4.35) holds there, that is, they actually have the singularities there that correspond to those of $\psi_{\pm,B}$. Let us confirm this fact when these singularities are contained in the domain of definition of the integro-differential operator given in Proposition 4.3. We first note that Γ_0 is independent of a. Indeed, by taking $\tilde{x} = x/a$ as a new variable, we obtain

$$\sqrt{\Gamma_0(a, A, B)} = \frac{1}{2\pi i} \int_\gamma \sqrt{\frac{A + \tilde{x}B}{\tilde{x}^2 - 1}}d\tilde{x}. \tag{4.39}$$

Therefore, by taking r_2 sufficiently small, we can assume that

$$\frac{1}{2}|A| < \big|\Gamma_0(a, A, B)\big| < 2|A| \tag{4.40}$$

holds on $\{|B| < r_2|A|\}$. Hence, if $m \in \mathbb{Z}$, A and B satisfy

$$2\sqrt{2}|m|\pi h(|B/A|) < 1, \tag{4.41}$$

the m-th singular point is in the domain of definition of the integro-differential operator. For each $m \in \mathbb{Z}$, this condition is satisfied by taking $|B/A|$ sufficiently small. Further, through the representation (4.35), we can derive from Lemma 4.1 the following

Lemma 4.3. *Let $\psi^{\dagger}_{\pm}(z, a, A, B, \eta)$ denote the WKB solutions of the ∞-Legendre equation (4.31) that are normalized at a simple pole $z = a$. Then, when (4.41) holds, its Borel transform $\psi^{\dagger}_{\pm,B}(z, a, A, B, y)$ is singular at*

$$y = \mp y_+(z, a, A, B) + 2m\pi i\sqrt{a\Gamma_0(a, A, B)} \tag{4.42}$$

and its alien derivative there satisfies

$$\begin{aligned}
&\Big(\Delta_{y=\mp y_+ + 2m\pi i\sqrt{a\Gamma_0}}\ \psi^{\dagger}_{\pm}\Big)_B(z, a, A, B, y) \\
&= \pm(-1)^m \Xi_m(\mu, \nu)\big(\exp(-m\oint_\gamma S^{\dagger}_{\mathrm{odd}}dx)\psi^{\dagger}_{\pm}\big)_B(z, a, A, B, y),
\end{aligned} \tag{4.43}$$

where $\mu = \mu(a)$ and $\nu = \nu(a)$ are functions that are given by (2.135) and (2.136) respectively and $S^{\dagger}_{\mathrm{odd}}$ is the odd part of the solutions of the Riccati equation associated with (4.31).

Proof. As in the proof of Lemma 4.1, it suffices to show

$$\begin{aligned}
&\mathcal{G}_{2m\pi i\sqrt{a\Gamma}}\big(\big(\exp(-2m\pi i\sqrt{a}\tilde{\Lambda}\eta)\psi^{(0)}_{\pm}\big)_B(z, a, \Gamma, y - 2m\pi i\sqrt{a\Gamma})\big)\big|_{\Gamma=\Gamma_0} \\
&= \big(\exp(-m\oint_\gamma S^{\dagger}_{\mathrm{odd}}dx)e^{\mp\eta y_+}\psi^{\dagger}_{\pm}\big)_B(z, a, A, B, y),
\end{aligned} \tag{4.44}$$

where $\mathcal{G}_{y_0}$ is the integro-differential operator obtained by taking $y = y_0$ as the end point of integration instead of $y = \mp y_+$ in (4.35) and $\psi^{(0)}_{\pm} =$

$e^{\mp\eta y_+}\psi_\pm$. Let us introduce the following coordinate transformation from (y,Γ) to (y',Γ'):

(4.45)
$$\begin{cases} y' = y - 2m\pi i\sqrt{a\Gamma} \\ \Gamma' = \Gamma. \end{cases}$$

Then, from Lemma 4.2, we obtain

(4.46)
$$\mathcal{G} =: \exp\left(\tilde{\Gamma}(a,A,B,\eta)\theta_\Gamma\right) :$$
$$=: \exp\left[-2m\pi i\sqrt{a}\eta'\left(\sqrt{\Gamma'+\tilde{\Gamma}}-\sqrt{\Gamma'}\right)\right] :: \exp\left(\tilde{\Gamma}(a,A,B,\eta')\theta_{\Gamma'}\right) : .$$

Therefore we find

(4.47)
$$\mathcal{G}_{2m\pi i\sqrt{a\Gamma}}\left(\left(\exp(-2m\pi i\sqrt{a}\tilde{\Lambda}\eta)\psi_\pm^{(0)}\right)_B(z,a,\Gamma,y-2m\pi i\sqrt{a\Gamma})\right)$$
$$=: \exp\left[-2m\pi i\sqrt{a}\eta'\left(\sqrt{\Gamma'+\tilde{\Gamma}}-\sqrt{\Gamma'}\right)\right] :$$
$$\times : \exp\left(\tilde{\Gamma}(a,A,B,\eta')\theta_{\Gamma'}\right) : \left(\exp(-2m\pi i\sqrt{a}\tilde{\Lambda}\eta')\psi_\pm^{(0)}\right)_B(z,a,\Gamma',y')$$
$$=: \exp\left[-2m\pi i\sqrt{a}\eta'\left(\sqrt{\Gamma'+\tilde{\Gamma}}-\sqrt{\Gamma'}\right)\right] :$$
$$\times \left(\exp(-2m\pi i\sqrt{a}\eta'\tilde{\Lambda}(a,\Gamma'+\tilde{\Gamma},\eta'))\psi_\pm^{(0)}(z,a,\Gamma'+\tilde{\Gamma},\eta')\right)_B$$

where the action of $:\eta'^{-1}:$ is fixed by taking $y'=0$ as the end point of integration. From (4.8) and (4.32), we find that, by replacing Γ with $\Gamma_0(a,A,B)$, the rightmost term of (4.47) equals to

(4.48)
$$: \exp\left[-2m\pi i\sqrt{a}\eta\left(\Lambda(a,\Gamma(a,A,B,\eta),\eta)-\sqrt{\Gamma_0}(a,A,B)\right)\right] :$$
$$\times \left(e^{\mp\eta y_+}\psi_\pm^\dagger\right)_B\left(z,a,A,B,y-2m\pi i\sqrt{a\Gamma_0}\right).$$

Then (4.43) follows from the following equality:

(4.49)

$$\oint_\gamma S^\dagger_{\mathrm{odd}}(z,a,A,B,\eta)dz = 2\pi i\sqrt{a}\Lambda(a,\Gamma(a,A,B,\eta),\eta)\eta + \pi i.$$

□

Now, we derive the singularity structure of Borel transformed WKB solutions of the Mathieu equation (2.1) from Lemma 4.3. We first remark that the Mathieu equation has two simple poles and one simple turning point. On the other hand, the (∞-)Legendre equation has only two simple poles. Therefore, if we want to relate the Mathieu equation with the Legendre equation, in other words, if we want to focus our attention on the two simple poles of the Mathieu equation, we have to remove the effect of the simple turning point. This can be attained by controlling the merging velocity of the turning point, that is, $|A/B|$. Indeed, since the turning point is located at $x=-aA/B$, it is distant enough from the poles located at $x=\pm a$ if $|A/B|$ is large. The existence of the function $h(\delta)$ that satisfies (4.28) $\sim$ (4.30) enables us to ignore the effect of the simple turning point and to derive the structure of Borel transformed WKB solutions of the Mathieu equation at the (fixed) singularities related only to the two simple poles from that of the Legendre equation as is discussed below (especially in Theorem 4.2).

Let $\tilde{\psi}_\pm$ be WKB solutions of the Mathieu equation (2.1). Then, from (2.132), we obtain the following relation:

(4.50)

$$\tilde{\psi}_\pm(x,a,A,B,\eta) = \left(\frac{\partial z}{\partial x}\right)^{-1/2}\psi^\dagger_\pm(z(x,a,A,B,\eta),a,A,B,\eta).$$

For the simplicity of discussion, we take $z_0(x,a,A,B)$ as a new coordinate variable instead of x. This is guaranteed by Theorem 2.1.

Let M and L_∞ respectively be the Borel transformed Mathieu operator expressed in (z_0, a, A, B, y)-coordinate and the Borel transformed ∞-Legendre operator, i.e.,

$$
\begin{aligned}
(4.51)\quad M =& \Big(\frac{\partial x}{\partial z_0}\Big)^{-2}\frac{\partial^2}{\partial z_0^2} - \frac{\partial^2 x}{\partial z_0^2}\Big(\frac{\partial x}{\partial z_0}\Big)^{-3}\frac{\partial}{\partial z_0} \\
& - \frac{aA + xB}{x^2 - a^2}\frac{\partial^2}{\partial y^2} - \frac{g_+(a)}{(x-a)^2} - \frac{g_-(-a)}{(x+a)^2},
\end{aligned}
$$

$$
(4.52)\quad L_\infty = \frac{\partial^2}{\partial z_0^2} - \frac{a\Gamma(a, A, B, \partial/\partial y)}{z_0^2 - a^2}\frac{\partial^2}{\partial y^2} - \frac{g_+(a)}{(z_0 - a)^2} - \frac{g_-(-a)}{(z_0 + a)^2}.
$$

Then, we find the following

Theorem 4.1. *There exist invertible microdifferential operators* $\mathcal{Z}$ *and* $\mathcal{W}$ *that satisfy*

$$
(4.53)\qquad M\mathcal{Z} = \mathcal{W}L_\infty
$$

on

$$
\begin{aligned}
(4.54)\qquad & \big\{(z_0, y, a, A, B; \zeta_0, \eta) \in T^*\mathbb{C}_{z_0} \times \dot{T}^*\mathbb{C}_y \times \mathbb{C}^3 : \\
& \qquad |z_0| < r_1|a|, a \neq 0, A \neq 0, |B| < r_2|A|\big\}
\end{aligned}
$$

for some positive constants r_1 *and* r_2 *with the exception of* $z_0^2 - a^2 = 0$*. The concrete form of* $\mathcal{Z}$ *and* $\mathcal{W}$ *are as follows:*

$$
(4.55)\qquad \mathcal{Z} =: \Big(\frac{\partial x}{\partial z_0}\Big)^{1/2}\Big(1 + \frac{\partial \tilde{z}}{\partial z_0}\Big)^{-1/2}\exp\big(\tilde{z}(z_0, a, A, B, \eta)\zeta_0\big) :,
$$

$$
(4.56)\qquad \mathcal{W} =: \Big(\frac{\partial x}{\partial z_0}\Big)^{-3/2}\Big(1 + \frac{\partial \tilde{z}}{\partial z_0}\Big)^{3/2}\exp\big(\tilde{z}(z_0, a, A, B, \eta)\zeta_0\big) :,
$$

where

$$
(4.57)\qquad \tilde{z}(z_0, a, A, B, \eta) = z(x(z_0, a, A, B), a, A, B, \eta) - z_0.
$$

Theorem 4.1 follows from the following proposition (cf. [AY]):

Proposition 4.4. *Let $x(t)$ be a holomorphic change of variables at the origin from $\mathbb{C}_t$ to $\mathbb{C}_x$ satisfying*

$$x(0) = 0 \text{ and } \frac{dx}{dt}(0) \neq 0 \tag{4.58}$$

and suppose that the following microdifferential operators $\mathcal{P}$ and $\mathcal{Q}$ are given:

$$\mathcal{P} = \frac{\partial^2}{\partial t^2} - p\Big(t, \frac{\partial}{\partial y}\Big)\frac{\partial^2}{\partial y^2}, \tag{4.59}$$

$$\mathcal{Q} = \frac{\partial^2}{\partial x^2} - q\Big(x, \frac{\partial}{\partial y}\Big)\frac{\partial^2}{\partial y^2}, \tag{4.60}$$

where p (resp., q) are microdifferential operators of order 0 defined near $t = 0$ (resp., $x = 0$) except for $\eta = 0$. Furthermore let $r(x, \eta)$ be the symbol of a microdifferential operator of order -1 and suppose that the total symbols $p(t, \eta) := \sigma\big(p(t, \partial/\partial y)\big)$, $q(x, \eta) := \sigma\big(q(x, \partial/\partial y)\big)$ and $z(x, \eta) = x + r(x, \eta)$ satisfy the following relation:

$$p(t, \eta) = \Big(\frac{dz(x(t), \eta)}{dt}\Big)^2 q(z(x(t), \eta), \eta) - \frac{1}{2}\eta^{-2}\{z(x(t), \eta); t\}. \tag{4.61}$$

Then the following relation holds:

$$\mathcal{P}\mathcal{X} = \mathcal{Y}\mathcal{Q}, \tag{4.62}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are microdifferential operators defined by

$$\mathcal{X} =: \Big(\frac{dz}{dt}\Big)^{-1/2} \exp\big(r(x(t), \eta)\xi\big) :, \tag{4.63}$$

$$\mathcal{Y} =: \Big(\frac{dz}{dt}\Big)^{3/2} \exp\big(r(x, \eta)\xi\big) : \tag{4.64}$$

and $\xi = \sigma(\partial/\partial x)$.

Proof. Let $P(x,\xi,\eta)$, $Q(x,\xi,\eta)$, $X(x,\xi,\eta)$ and $Y(x,\xi,\eta)$ respectively be total symbols of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{X}$ and $\mathcal{Y}$ in (x,y)-coordinate. For example, $P(x,\xi,\eta)$ and $Q(x,\xi,\eta)$ are respectively given by

$$P(x,\xi,\eta) = \left(\frac{dt}{dx}\right)^{-2}\xi^2 - \left(\frac{dt}{dx}\right)^{-3}\frac{d^2t}{dx^2}\xi - \eta^2 p(t(x),\eta) \tag{4.65}$$

and

$$Q(x,\xi,\eta) = \xi^2 - \eta^2 q(x,\eta), \tag{4.66}$$

where $t(x)$ is the inverse function of $x(t)$. Then it suffices to show

$$P \circ X(x,\xi,\eta) = Y \circ Q(x,\xi,\eta), \tag{4.67}$$

where the composition $\circ$ is defined by

$$P \circ X(x,\xi,\eta) = \exp\left(\partial_{\hat{\xi}}\partial_{\hat{x}}\right) P(x,\hat{\xi},\eta) X(\hat{x},\xi,\eta)\Big|_{\substack{\hat{x}=x\\ \hat{\xi}=\xi}}. \tag{4.68}$$

(Cf. [A2, Proposition 2.5].) We first note that $P(x,\xi,\eta)$ is expressed in terms of the total symbol

$$\tilde{P}(t,\tau,\eta) = \tau^2 - \eta^2 p(t,\eta) \tag{4.69}$$

of $\mathcal{P}$ in (t,y)-coordinate, where $\tau = \sigma(\partial/\partial t)$, as follows:

$$P(x,\xi,\eta) = e^{-x\xi}\exp\left(\partial_{\hat{\tau}}\partial_{\hat{t}}\right)\tilde{P}(t,\hat{\tau},\eta)e^{x(\hat{t})\xi}\Big|_{\substack{\hat{t}=t(x)\\ \hat{\tau}=0}}. \tag{4.70}$$

Combining (4.68) and (4.70), we find

$$\begin{aligned} &P \circ X(x,\xi,\eta) \\ &\quad= \exp\left(\partial_{\hat{\tau}}\partial_{\hat{t}}\right)\tilde{P}(t,\hat{\tau},\eta)\exp\left(\partial_{\hat{\xi}}\partial_{\hat{x}}\right)e^{(x(\hat{t})-x)\hat{\xi}}X(\hat{x},\xi,\eta)\Big|_{\substack{\hat{t}=t(x),\hat{\tau}=0\\ \hat{x}=x,\hat{\xi}=\xi}} \\ &\quad= \exp\left(\partial_{\hat{\tau}}\partial_{\hat{t}}\right)\tilde{P}(t,\hat{\tau},\eta)e^{(x(\hat{t})-x)\xi}X(x(\hat{t}),\xi,\eta)\Big|_{\substack{\hat{t}=t(x)\\ \hat{\tau}=0}}. \end{aligned} \tag{4.71}$$

Therefore it follows from the concrete form of $\tilde{P}(t,\tau,\eta)$ and (4.71) that

$$P\circ X(x,\xi,\eta) = -\eta^2 p(t(x),\eta)X(x,\xi,\eta) + \frac{\partial^2}{\partial\hat{t}^2}\Big(e^{(x(\hat{t})-x)\xi}X(x(\hat{t}),\xi,\eta)\Big)\Big|_{\hat{t}=t(x)}. \tag{4.72}$$

On the other hand, since Y satisfies $\partial_\xi^k Y = r^k(x,\eta)Y$, we find

$$\begin{aligned}
&Y\circ Q(x,\xi,\eta)\\
&= Y(x,\xi,\eta)Q(x,\xi,\eta) - \eta^2\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\partial^k}{\partial\xi^k}Y(x,\xi,\eta)\frac{\partial^k}{\partial x^k}q(x,\eta)\\
&= Y(x,\xi,\eta)Q(x,\xi,\eta) + \eta^2 Y(x,\xi,\eta)q(x,\eta)\\
&\quad - \eta^2 Y(x,\xi,\eta)q(x+r(x,\eta),\eta)\\
&= Y(x,\xi,\eta)\xi^2 - \eta^2 Y(x,\xi,\eta)q(z(x,\eta),\eta).
\end{aligned} \tag{4.73}$$

Then, since $Y=(dz/dt)^2X$, it follows from (4.61) and (4.73) that

$$Y\circ Q(x,\xi,\eta) = Y(x,\xi,\eta)\xi^2 - \eta^2 X(x,\xi,\eta)p(t(x),\eta) - \frac{1}{2}\{z;t\}X(x,\xi,\eta). \tag{4.74}$$

Thus, comparing (4.72) and (4.74), we find that (4.67) immediately follows if the following relation is confirmed:

$$\frac{\partial^2}{\partial\hat{t}^2}\Big(e^{(x(\hat{t})-x)\xi}X(x(\hat{t}),\xi,\eta)\Big)\Big|_{\hat{t}=t(x)} = Y(x,\xi,\eta)\xi^2 - \frac{1}{2}\{z;t\}X(x,\xi,\eta). \tag{4.75}$$

Since the left hand side of (4.75) is equal to

$$\begin{aligned}
&e^{-x\xi}\frac{\partial^2}{\partial\hat{t}^2}\Big(\Big(\frac{dz(x(\hat{t}),\eta)}{d\hat{t}}\Big)^{-1/2}\exp\big(z(x(\hat{t}),\eta)\xi\big)\Big)\Big|_{\hat{t}=t(x)}\\
&= \exp\big(r(x,\eta)\xi\big)\frac{\partial^2}{\partial t^2}\Big(\frac{dz}{dt}\Big)^{-1/2} + \Big(\frac{dz}{dt}\Big)^{3/2}\xi^2\exp\big(r(x,\eta)\xi\big),
\end{aligned} \tag{4.76}$$

we find that (4.75) is an immediate consequence of

$$\{z;t\} = -2\Big(\frac{dz}{dt}\Big)^{1/2}\frac{d^2}{dt^2}\Big(\frac{dz}{dt}\Big)^{-1/2}. \tag{4.77}$$

This completes the proof.

□

Remark 4.3. In the situation of Theorem 4.1, $\mathcal{P}$ and $\mathcal{Q}$ correspond to M and L_∞ respectively.

In view of (4.29), we obtain the following

Proposition 4.5. *Let* $\psi_{\pm,B}$ *and* $\tilde{\psi}_{\pm,B}$ *respectively be the Borel transformed WKB solutions of* (2.4) *and* (2.1) *for* $a \neq 0$ *and* $A \neq 0$ *that are normalized at their simple poles as* (2.131) *and* (2.130). *Then they satisfy the following relation:*

$$\begin{aligned}(4.78)\quad &\tilde{\psi}_{\pm,B}(z_0,a,A,B,y)\\ &= \int_{\mp y_+}^{y} K_z(z_0,a,A,B,y-y',\partial_{z_0})\psi_{\pm,B}(z_0,a,A,B,y')dy',\end{aligned}$$

where $K_z(z_0,a,A,B,y,\partial_{z_0})$ *is a differential operator of infinite order that is defined on*

(4.79)

$$\begin{aligned}\tilde{E}^2_{r_1,h} = \{(z_0,a,A,B,y) \in \mathbb{C}^5 : &a, A \neq 0, |x| < r_1|a|, |B/A| < r_2,\\ &h(|B/A|)|y| < \sqrt{|aA|}\}\end{aligned}$$

with some positive constants $r_1 > 1$ *and* $r_2 > 0$ *and*

$$y_+(z_0,a,A,B) = \int_a^{z_0}\sqrt{\frac{a\Gamma_0(a,A,B)}{z_0^2-a^2}}dz_0. \tag{4.80}$$

In conclusion, by employing similar discussions to Lemma 4.3 and Proposition 4.1, we obtain

Theorem 4.2. *Let $\tilde{\psi}_\pm(x, a, A, B, \eta)$ be WKB solutions of the Mathieu equation* (2.1) *with $a \neq 0$ and $A \neq 0$ that is normalized at a simple pole $x = a$. Then, for each integer m we can take some positive constant δ so that the following holds when $|B/A| < \delta$ is satisfied: The Borel transform $\tilde{\psi}_{\pm,B}(x, a, A, B, y)$ of $\tilde{\psi}_\pm(x, a, A, B, \eta)$ is singular at*

(4.81) $$y = \mp y_+(x, a, A, B) + 2m\pi i\sqrt{a\Gamma_0(a, A, B)}$$

and its alien derivative there satisfies

(4.82)
$$\Big(\Delta_{y=\mp y_+ + 2m\pi i\sqrt{a\Gamma_0}}\ \tilde{\psi}_\pm\Big)_B (x, a, A, B, y)$$
$$= \pm(-1)^m \Xi_m(\mu, \nu)\big(\exp(-m\oint_\gamma \tilde{S}_{\mathrm{odd}}dx)\tilde{\psi}_\pm\big)_B(x, a, A, B, y),$$

where

(4.83)
$$\Xi_m(\mu, \nu) = \frac{1}{m}\left\{1 + (-1)^m - \cosh\left(2\pi i m\sqrt{\frac{\mu^2 + \sqrt{\mu^4 - \nu^2}}{2}}\right)\right.$$
$$\left. - \cosh\left(2\pi i m\sqrt{\frac{\mu^2 - \sqrt{\mu^4 - \nu^2}}{2}}\right)\right\},$$

(4.84) $$\mu = \mu(a) = \sqrt{1 + 2(g_+(a) + g_-(-a))},$$

(4.85) $$\nu = \nu(a) = 2(g_+(a) - g_-(-a)),$$

(4.86) $$y_+(x, a, A, B) = \int_a^x \sqrt{\frac{aA + xB}{x^2 - a^2}}dx$$

and

$$(4.87) \qquad \sqrt{\Gamma_0(a, A, B)} = \frac{1}{2\pi i\sqrt{a}} \int_\gamma \sqrt{\frac{aA + xB}{x^2 - a^2}} dx.$$

Here γ is a closed curve that encircles two simple poles counter-clockwise.

Remark 4.4. In Theorem 4.2, the positive constant δ should be taken so small that (4.41) is satisfied for $|B/A| < \delta$ for an arbitrarily given $m \in \mathbb{Z}$.

Remark 4.5. In (x, a, A, B)-coordinate, $y_+(x, a, A, B)$ is given by (4.86). However, since

$$(4.88) \qquad z_0(x, a, A, B) = a \cos\left(\frac{1}{\sqrt{a\Gamma_0}} \int_a^x \sqrt{\frac{aA + xB}{x^2 - a^2}} dx\right)$$

satisfies

$$(4.89) \qquad \frac{aA + xB}{x^2 - a^2} = \left(\frac{\partial z_0}{\partial x}\right)^2 \frac{a\Gamma_0}{z_0^2 - a^2},$$

we find (4.80) is equivalent to (4.86).

5 Analytic properties of Borel transformed WKB solutions of an M2P1T equation

In this section, we study WKB theoretic structure of an M2P1T equation

$$(5.1) \qquad \left(\frac{d^2}{dt^2} - \eta^2 Q(t, a, \rho, \eta)\right)\hat{\psi} = 0,$$

where the potential $Q(t, a, \rho)$ is given in Definition 1.1. We constructed transformation series $x(t, a, \rho, \eta)$, $A(a, \rho, \eta)$ and $B(a, \rho, \eta)$ in Section

1 that give equivalence between an M2P1T equation and the following ∞-Mathieu equation:

$$\begin{aligned}(5.2)\qquad &\Big(\frac{d^2}{dx^2} - \eta^2\Big(\frac{aA(a,\rho,\eta) + xB(a,\rho,\eta)}{x^2 - a^2} \\ &\qquad + \eta^{-2}\Big(\frac{g_+(a)}{(x-a)^2} + \frac{g_-(-a)}{(x+a)^2}\Big)\Big)\Big)\tilde{\psi}^\dagger = 0.\end{aligned}$$

As the following discussion shows, (5.2) behaves as the WKB theoretic canonical form of an M2P1T equation.

Let $\hat{\psi}_\pm$ and $\tilde{\psi}^\dagger_\pm$ respectively be WKB solutions of (5.1) and (5.2) that are normalized at their simple poles $t = a$ and $x = a$. Then, from Theorem 1.3.3, we find the following relation holds:

$$(5.3)\qquad \hat{\psi}_\pm(t,a,\rho,\eta) = \Big(\frac{\partial x}{\partial t}\Big)^{-1/2}\tilde{\psi}^\dagger_\pm(x(t,a,\rho,\eta),a,\rho,\eta).$$

For the simplicity of discussion, we take $x_0(t,a,\rho)$ as a new coordinate variable instead of t. This is guaranteed by Theorem 1.3.1. Let N and M_∞ respectively be the Borel transformed M2P1T operator expressed in (x_0,a,ρ,y)-coordinate and the Borel transformed ∞-Mathieu operator, i.e.,

$$(5.4)\quad N = \Big(\frac{\partial t}{\partial x_0}\Big)^{-2}\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2 t}{\partial x_0^2}\Big(\frac{\partial t}{\partial x_0}\Big)^{-3}\frac{\partial}{\partial x_0} - Q(t,a,\rho,\partial/\partial y)\frac{\partial^2}{\partial y^2}$$

(5.5)

$$\begin{aligned}M_\infty =&\frac{\partial^2}{\partial x_0^2} - \frac{aA(a,\rho,\partial/\partial y) + x_0B(a,\rho,\partial/\partial y)}{x_0^2 - a^2}\frac{\partial^2}{\partial y^2} \\ &- \frac{g_+(a)}{(x_0-a)^2} - \frac{g_-(-a)}{(x_0+a)^2}.\end{aligned}$$

Then, from Theorem 1.3.1 and Proposition 4.4, we obtain the following

Theorem 5.1. *There exist invertible microdifferential operators $\mathcal{X}$ and $\mathcal{Y}$ that satisfy*

$$N\mathcal{X} = \mathcal{Y}M_\infty \tag{5.6}$$

on

$$\Big\{(x_0, y, a, \rho; \xi_0, \eta) \in T^*\mathbb{C}_{x_0} \times \dot{T}^*\mathbb{C}_y \times \mathbb{C}^2 : |x_0| < r, 0 < |\rho| < r, R_0|a| < |\rho|\Big\} \tag{5.7}$$

for some positive constants r *and* R_0 *with the exception of* $x_0^2 - a^2 = 0$*. The concrete form of* $\mathcal{X}$ *and* $\mathcal{Y}$ *are as follows:*

$$\mathcal{Z} =: \Big(\frac{\partial t}{\partial x_0}\Big)^{1/2}\Big(1 + \frac{\partial \tilde{x}}{\partial x_0}\Big)^{-1/2} \exp\big(\tilde{x}(x_0, a, \rho, \eta)\xi_0\big) :, \tag{5.8}$$

$$\mathcal{W} =: \Big(\frac{\partial t}{\partial x_0}\Big)^{-3/2}\Big(1 + \frac{\partial \tilde{x}}{\partial x_0}\Big)^{3/2} \exp\big(\tilde{x}(x_0, a, \rho, \eta)\xi_0\big) :, \tag{5.9}$$

where

$$\tilde{x}(x_0, a, \rho, \eta) = x(t(x_0, a, \rho), a, \rho, \eta) - x_0. \tag{5.10}$$

For the correspondence of Borel transformed WKB solutions, we have the following

Proposition 5.1. *Let* $\hat{\psi}_{\pm,B}$ *and* $\tilde{\psi}^{\dagger}_{\pm,B}$ *respectively be Borel transformed WKB solutions of a generic M2P1T equation (i.e.* $a, \rho \neq 0$*) and the* ∞*-Mathieu equation that are normalized at their simple poles* $t = a$ *and* $x = a$*. Then they satisfy the following relation:*

(5.11)

$$\hat{\psi}_{\pm,B}(x_0, a, \rho, y) = \int_{\mp y_+}^{y} K_x(x_0, a, \rho, y - y', \partial_{x_0})\tilde{\psi}^{\dagger}_{\pm,B}(x_0, a, \rho, y')dy',$$

where $K_x(x_0, a, \rho, y, \partial_{x_0})$ *is a differential operator of infinite order that is defined on*

$$\tilde{E}^1_{r,R_0,R_1} = \{(x_0, a, \rho, y) \in \mathbb{C}^4 : |x_0| < r, 0 < |\rho| < r, \tag{5.12}$$

$$R_0|a| < |\rho|, R_1|y| < \sqrt{|\rho|}\},$$

and

$$y_+(x_0, a, \rho) = \int_a^{x_0} \sqrt{\frac{aA(a,\rho) + x_0 B(a,\rho)}{x_0^2 - a^2}} dx_0. \tag{5.13}$$

Thus, the analysis of the singularity structure of Borel transformed WKB solutions of an M2P1T equation should be reduced to that of the ∞-Mathieu equation. However the complete singularity structure of Borel transformed WKB solutions of the (∞-)Mathieu equation is too complicated to be analyzed directly. Fortunately, as the discussion in Section 4 shows, the singularity structure of Borel transformed WKB solutions of the Mathieu equation that is relevant to its two simple poles is now clarified. Using this knowledge for the Mathieu equation, we discuss the singularity structure of Borel transformed WKB solutions of an M2P1T equation in what follows.

We first relate the ∞-Mathieu equation with the Mathieu equation. To this end we use the following relation:

$$\tilde{\psi}^{\dagger}_{\pm}(x, a, \rho, \eta) = \tilde{\psi}_{\pm}(x, a, A(a,\rho,\eta), B(a,\rho,\eta), \eta). \tag{5.14}$$

Applying the Borel transformation to (5.14), we can relate the Borel transform $\tilde{\psi}^{\dagger}_{\pm,B}$ of $\tilde{\psi}^{\dagger}_{\pm}$ with $\tilde{\psi}_{\pm,B}$ through the action of a microdifferential operator

$$\mathcal{AB} =: \exp\left(\tilde{A}\theta_A + \tilde{B}\theta_B\right) :, \tag{5.15}$$

where

$$\tilde{A}(a,\rho,\eta) = A(a,\rho,\eta) - A_0(a,\rho), \tag{5.16}$$

$$\tilde{B}(a,\rho,\eta) = B(a,\rho,\eta) - B_0(a,\rho) \tag{5.17}$$

and θ_A (resp. θ_B) is the symbol of ∂_A (resp. ∂_B). Thanks to the estimates (1.3.12) and (1.3.13), we obtain the following

Proposition 5.2. *Let $\tilde{\psi}^{\dagger}_{\pm,B}$ and $\tilde{\psi}_{\pm,B}$ respectively be Borel transformed WKB solutions of the ∞-Mathieu equation and the Mathieu equation that are normalized at their simple poles $x = a$. Then they satisfy the following relation:*

(5.18)

$$\begin{aligned}&\tilde{\psi}^{\dagger}_{\pm,B}(x,a,\rho,y)\\&\quad = \int_{\mp y_+}^{y} K_{A,B}(a,\rho,y-y',\partial_A,\partial_B)\tilde{\psi}_{\pm,B}(x,a,A,B,y')dy'\Big|_{\substack{A=A_0(a,\rho)\\ B=B_0(a,\rho)}},\end{aligned}$$

where $K_{A,B}(a,\rho,y-y',\partial_A,\partial_B)$ is a differential operator of infinite order that is defined on

(5.19)

$$\{(a,\rho,A,B,y)\in\mathbb{C}^5 : 0<|\rho|<r, R_0|a|<|\rho|, R_1|y|<\sqrt{|\rho|}\}$$

with some positive constants r, R_0 and R_1 and

$$y_+(x,a,\rho) = \int_a^x \sqrt{\frac{aA(a,\rho)+xB(a,\rho)}{x^2-a^2}}dx. \tag{5.20}$$

Now we study the singularity structure of $\tilde{\psi}^{\dagger}_{\pm,B}$ using Theorem 4.2. Let us focus on the m-th singular point of $\tilde{\psi}_{\pm,B}$ located at (4.81). Evidently, from (4.41), the following condition should be satisfied:

$$2\sqrt{2}|m|\pi h(|B_0(a,\rho)/A_0(a,\rho)|) < 1, \tag{5.21}$$

where $h(\delta)$ is a function that satisfies (4.28) $\sim$ (4.30). Since $A_0(0,0) = f^{(1)}(0,0) \neq 0$ and $B_0(0,\rho) = \rho$, Lemma 1.2.3 tells us that, by taking R_0 sufficiently large, we can assume that $A_0(a,\rho)$ and $B_0(a,\rho)$ satisfy

$$\frac{1}{2}\big|f^{(1)}(0,0)\big| \leq \big|A_0(a,\rho)\big| \leq \frac{3}{2}\big|f^{(1)}(0,0)\big|, \tag{5.22}$$

$$\frac{1}{2}|\rho| \leq \big|B_0(a,\rho)\big| \leq \frac{3}{2}|\rho| \tag{5.23}$$

on $\{R_0|a| < |\rho|\}$. Since $h(\delta)$ is an increasing function, we find that (5.21) follows from

$$2\sqrt{2}|m|\pi h(3|\rho|/|f^{(1)}(0,0)|) < 1. \tag{5.24}$$

Therefore, by taking ρ sufficiently small with keeping the relation $R_0|a| < |\rho|$, we can make $|B_0(a,\rho)/A_0(a,\rho)|$ arbitrary small so that (5.24) holds. On the other hand, when

$$2|m|\pi R_1\sqrt{|a\Gamma_0(a, A_0(a,\rho), B_0(a,\rho))|} < \sqrt{|\rho|} \tag{5.25}$$

is satisfied, the m-th singular point is contained in the domain of definition of the integro-differential operator in (5.18). Hence, in view of (4.40) and (5.22), it suffices to take a sufficiently small relative to ρ so that

$$2\sqrt{3}|m|\pi R_1\sqrt{|f^{(1)}(0,0)|}\sqrt{|a|} < \sqrt{|\rho|} \tag{5.26}$$

holds. Then, using Theorem 4.2 and Proposition 5.2, we can show the following

Lemma 5.1. *Let $\tilde{\psi}^{\dagger}_{\pm}$ denote the WKB solutions of the ∞-Mathieu equation* (5.2) *that are normalized at a simple pole $x = a$. Then, when* (5.24) *and* (5.26) *hold, its Borel transform $\tilde{\psi}^{\dagger}_{\pm,B}(x,a,A,B,y)$ is singular at*

$$y = \mp y_+(x,a,\rho) + mp(a,\rho) \tag{5.27}$$

and its alien derivative there satisfies

$$\begin{aligned}(5.28)\quad &\left(\Delta_{y=\mp y_+ + mp}\ \tilde{\psi}^{\dagger}_{\pm}\right)_B(x,a,\rho,y)\\ &= \pm(-1)^m \Xi_m(\mu,\nu)\big(\exp(-m\oint_{\gamma}\tilde{S}^{\dagger}_{\mathrm{odd}}dx)\tilde{\psi}^{\dagger}_{\pm}\big)_B(x,a,\rho,y),\end{aligned}$$

where

$$p(a,\rho) = \int_{\gamma}\sqrt{\frac{f(t,a,\rho)}{t^2-a^2}}dt \tag{5.29}$$

and $\Xi_m(\mu,\nu)$, $\mu(a)$ *and* $\nu(a)$ *are functions that are given by* (4.83), (4.84) *and* (4.85) *respectively.*

Proof. We first note the following relation, which is an immediate consequence of Proposition 4.1:

$$\begin{aligned}(5.30)\qquad &\oint_\gamma \tilde{S}^\dagger_{\text{odd}}(x,a,\rho,\eta)dx \\ &= 2\pi i\sqrt{a}\Lambda(a,\Gamma(a,A(a,\rho,\eta),B(a,\rho,\eta),\eta),\eta)\eta+\pi i,\end{aligned}$$

where Λ and Γ are formal power series given in Section 2, $\tilde{S}^\dagger_{\text{odd}}$ is the odd part of a solution of the Riccati equation associated with the ∞-Mathieu equation and γ is a contour that encircles two simple poles of the ∞-Mathieu equation counterclockwise avoiding its simple turning point. Especially, we find

$$(5.31)\qquad p(a,\rho) = 2\pi i\sqrt{a\Gamma_0(a,A_0(a,\rho),B_0(a,\rho))}.$$

Then, in a way parallel to the proof of Lemma 4.3, applying Lemma 4.2 to the symbol of $\mathcal{AB}$ and using a coordinate transformation F : $(y,A,B)\to(y',A',B')$ defined by

$$(5.32)\qquad \begin{cases} y' = y - 2m\pi i\sqrt{a\Gamma_0(a,A,B)} \\ A' = A \\ B' = B \end{cases}$$

instead of (4.45), we obtain Lemma 5.1.

□

Remark 5.1. From (4.40), (5.22) and (5.31), we find

$$(5.33)\qquad |p(a,\rho)| = O(\sqrt{|a|})$$

when a tends to 0.

Now, from Theorem 1.3.2 and (5.30), we obtain the following

Proposition 5.3. *Let $\hat{S}_{\rm odd}$ be the odd part of a solution of the Riccati equation associated with an M2P1T equation. Then*

$$\begin{aligned}(5.34)\qquad &\oint_\gamma \hat{S}_{\rm odd}(t,a,\rho,\eta)dt \\ &= 2\pi i\sqrt{a}\Lambda(a,\Gamma(a,A(a,\rho,\eta),B(a,\rho,\eta),\eta),\eta)\eta+\pi i\end{aligned}$$

holds, where γ is a contour that encircles two simple poles of the M2P1T equation counterclockwise avoiding its simple turning point.

In conclusion, combining Proposition 5.1, Lemma 5.1 and Proposition 5.3, we obtain

Theorem 5.2. *Let $\hat{\psi}_\pm(t,a,\rho,\eta)$ be WKB solutions of a generic (i.e. $a\neq 0$, $\rho\neq 0$) M2P1T equation that is normalized at a simple pole $t=a$. Then, for each integer m we can take some positive constants δ_1 and δ_2 so that the following holds when $|\rho|<\delta_1$ and $0<|a|<\delta_2|\rho|$ are satisfied: The Borel transform $\hat{\psi}_{\pm,B}(t,a,\rho,y)$ of $\hat{\psi}_\pm(t,a,\rho,\eta)$ is singular at*

$$(5.35)\qquad y=\mp y_+(t,a,\rho)+mp(a,\rho)$$

and its alien derivative there satisfies

$$\begin{aligned}(5.36)\qquad &\left(\Delta_{y=\mp y_+ + mp}\ \hat{\psi}_\pm\right)_B(t,a,\rho,y) \\ &= \pm(-1)^m\Xi_m(\mu,\nu)\big(\exp(-m\oint_\gamma \tilde{S}_{\rm odd}dx)\hat{\psi}_\pm\big)_B(t,a,\rho,y),\end{aligned}$$

where

(5.37)

$$\Xi_m(\mu,\nu)=\frac{1}{m}\left\{1+(-1)^m-\cosh\left(2\pi i m\sqrt{\frac{\mu^2+\sqrt{\mu^4-\nu^2}}{2}}\right)\right.$$

$$- \cosh\left(2\pi i m\sqrt{\frac{\mu^2-\sqrt{\mu^4-\nu^2}}{2}}\right)\Bigg\},$$

(5.38) $$\mu = \mu(a) = \sqrt{1+2(g_+(a)+g_-(-a))},$$

(5.39) $$\nu = \nu(a) = 2(g_+(a)-g_-(-a)),$$

(5.40) $$y_+(t,a,\rho) = \int_a^t \sqrt{\frac{f(t,a,\rho)}{t^2-a^2}}dt$$

and

(5.41) $$p(a,\rho) = \int_\gamma \sqrt{\frac{f(t,a,\rho)}{t^2-a^2}}dt.$$

Here γ is a contour that encircles two simple poles of the M2P1T equation counterclockwise avoiding its simple turning point.

Remark 5.2. In Theorem 5.2, the positive constants δ_1 and δ_2 should be taken so small that (5.24) and (5.26) are satisfied for $|\rho| < \delta_1$ and $0 < |a| < \delta_2|\rho|$ for an arbitrarily given $m \in \mathbb{Z}$.

Remark 5.3. In (t,a,ρ)-coordinate, $y_+(t,a,\rho)$ is given by (5.40). However, since $x_0(t,a,\rho)$ satisfies

(5.42) $$\frac{f(t,a,\rho}{t^2-a^2} = \left(\frac{\partial x_0}{\partial t}\right)^2 \frac{aA_0+x_0B_0}{x_0^2-a^2}$$

we find (5.40) is equivalent to (5.13).

A The vanishing of the odd degree (in η^{-1}) part of the transformation $x(t, a, \eta)$

The purpose of this section is to prove Proposition A.1 below. From the logical viewpoint this result should be placed in Section 1.1.3. But in order not to divert the reader's attention from the main stream of the reasoning we separately show this result here. We also note that one can bypass this reasoning by first constructing $x(t, a, \eta)$ that consists of even degree part and then proving its convergence. We hope the proof of Proposition A.1 will give some insight into the structure of $x(t, a, \eta)$. For the sake of simplicity we assume

$$g_{\pm} = 0 \tag{A.1}$$

in this section.

Proposition A.1. *The transformation x and constants A and B respectively have the form* (1.1.3), (1.1.4) *and* (1.1.5), *that is, their odd degree parts in η^{-1} vanish.*

Proof. Let us begin our discussion by studying the structure of

$$x_1(t, a, \rho) = \sum_{p \geq 0} x_1^{(p)}(t, \rho) a^p \tag{A.2}$$

and

$$A_1(a) = \sum_{p \geq 0} A_1^{(p)} a^p \quad \text{and} \quad B_1(a) = \sum_{p \geq 0} B_1^{(p)} a^p. \tag{A.3}$$

It then follows from (1.1.6) that we have

(A.4)

$$2x_0 x_1 f = (t^2 - a^2)\big[(x_0')^2(aA_1 + x_0 B_1 + x_1 B_0) + 2x_0' x_1'(aA_0 + x_0 B_0)\big],$$

where

$$x_0 = \sum_{p \geq 0} x_0^{(p)}(t, \rho) a^p \tag{A.5}$$

and

$$A_0 = \sum_{p \geq 0} A_0^{(p)} a^p \quad \text{and} \quad B_0 = \sum_{p \geq 0} B_0^{(p)} a^p. \tag{A.6}$$

Comparing the coefficients of a^0 in (A.4), we find

(A.7)

$$2x_0^{(0)} x_1^{(0)} f^{(0)} = t^2 \big[\big(x_0^{(0)\prime}\big)^2 \big(x_0^{(0)} B_1^{(0)} + x_1^{(0)} B_0^{(0)}\big) + 2x_0^{(0)\prime} x_1^{(0)\prime} x_0^{(0)} B_0^{(0)} \big].$$

Then we obtain

(A.8)

$$2t^2 \tilde{x}_0^{(0)} \tilde{f}^{(0)} x_1^{(0)}$$
$$= t^2 \big(x_0^{(0)\prime}\big)^2 \Big[\Big(s B_1^{(0)} + B_0^{(0)} x_1^{(0)}(s, \rho) + 2B_0^{(0)} s \frac{d}{ds} x_1^{(0)}(s, \rho) \Big) \Big] \Big|_{s = x_0^{(0)}(t, \rho)}.$$

Dividing both sides of (A.8) by $t^2 (x_0^{(0)\prime})^2$, we use $[5.0]'$ divided by t, i.e.,

$$\tilde{x}_0^{(0)} \tilde{f}^{(0)} = \big(x_0^{(0)\prime}\big)^2 B_0^{(0)} \tag{A.9}$$

to find

$$B_0^{(0)} \Big(2s \frac{d}{ds} - 1 \Big) x_1^{(0)}(s, \rho) = -s B_1^{(0)}. \tag{A.10}$$

Therefore we obtain

$$x_1^{(0)}(s, \rho) = -\frac{B_1^{(0)}}{B_0^{(0)}} s. \tag{A.11}$$

In particular, we have

$$x_1^{(0)}(0, \rho) = 0, \tag{A.12}$$

$$\dot{x}_1^{(0)}(0,\rho) = -\frac{B_1^{(0)}}{B_0^{(0)}}. \tag{A.13}$$

Similarly comparison of the coefficients of a^1 in (A.4) entails

(A.14)

$$\begin{aligned}
&2\big(x_0^{(0)}x_1^{(1)} + x_0^{(1)}x_1^{(0)}\big)f^{(0)} + 2x_0^{(0)}x_1^{(0)}f^{(1)} \\
&\quad = t^2\Big[\big(x_0^{(0)\prime}\big)^2\big(A_1^{(0)} + x_0^{(1)}B_1^{(0)} + x_0^{(0)}B_1^{(1)} + x_1^{(0)}B_0^{(1)} + x_1^{(1)}B_0^{(0)}\big) \\
&\qquad + 2x_0^{(0)\prime}x_0^{(1)\prime}\big(x_0^{(0)}B_1^{(0)} + x_1^{(0)}B_0^{(0)}\big) + 2x_0^{(0)\prime}x_1^{(0)\prime}A_0^{(0)} \\
&\qquad + 2x_0^{(1)\prime}x_1^{(0)\prime}x_0^{(0)}B_0^{(0)} + 2x_0^{(0)\prime}x_1^{(1)\prime}x_0^{(0)}B_0^{(0)} \\
&\qquad + 2x_0^{(0)\prime}x_1^{(0)\prime}x_0^{(1)}B_0^{(0)} + 2x_0^{(0)\prime}x_1^{(0)\prime}x_0^{(0)}B_0^{(1)}\Big].
\end{aligned}$$

It then follows from (1.1.1.7), (1.1.1.24) and (A.12) that the left-hand side of (A.14) has the form

$$2t^2\big(\tilde{x}_0^{(0)}x_1^{(1)} + t\tilde{x}_0^{(1)}\tilde{x}_1^{(0)}\big)\tilde{f}^{(0)} + 2t^2\tilde{x}_0^{(0)}\tilde{x}_1^{(0)}f^{(1)}, \tag{A.15}$$

where

$$\tilde{x}_1^{(0)} = t^{-1}x_1^{(0)}. \tag{A.16}$$

Hence by dividing both sides of (A.14) by $t^2\big(x_0^{(0)\prime}\big)^2$, we readily find

$$\begin{aligned}
&B_0^{(0)}\Big(2s\frac{d}{ds} - 1\Big)x_1^{(1)}(s,\rho) \\
&\quad = -A_1^{(0)} - sB_1^{(1)} + 2\big(x_0^{(0)\prime}\big)^{-2}\tilde{x}_0^{(0)}\tilde{x}_1^{(0)}f^{(1)} - 2\dot{x}_1^{(0)}A_0^{(0)} + V,
\end{aligned} \tag{A.17}$$

where

$$\begin{aligned}
V =& 2\big(x_0^{(0)\prime}\big)^{-2}\tilde{x}_0^{(1)}x_1^{(0)}\tilde{f}^{(0)}\Big|_{t=t(s,\rho)} \\
&- \big(x_0^{(1)}B_1^{(0)} + x_1^{(0)}B_0^{(1)} + 2\dot{x}_0^{(1)}sB_1^{(0)} + 2\dot{x}_0^{(1)}x_1^{(0)}B_0^{(0)} \\
&+ 2\dot{x}_0^{(1)}\dot{x}_1^{(0)}sB_0^{(0)} + 2\dot{x}_1^{(0)}x_0^{(1)}B_0^{(0)} + 2\dot{x}_1^{(0)}sB_0^{(1)}\big).
\end{aligned} \tag{A.18}$$

Here we note that V vanishes at $s = 0$, and furthermore (1.1.1.21) entails

(A.19)

$$2\big(x_0^{(0)\prime}\big)^{-2}\tilde{x}_0^{(0)}\tilde{x}_1^{(0)}f^{(1)}\Big|_{t=0} = 2Z_0^2\dot{x}_1^{(0)}(0,\rho)A_0^{(0)} = 2A_0^{(0)}\dot{x}_1^{(0)}(0,\rho),$$

where Z_0 stands for $x_0^{(0)\prime}(0,\rho) = \pm 1$ (cf.(1.1.1.13) and (1.1.1.23)). Therefore we obtain

$$x_1^{(1)}(0,\rho) = \frac{A_1^{(0)}}{B_0^{(0)}}. \tag{A.20}$$

Next, by comparing the coefficients of a^2 in (A.4), we encounter terms which do not have factor t^2 explicitly, that is,

$$2f^{(1)}x_0^{(0)}x_1^{(1)} \tag{A.21}$$

in the left-hand side of (A.4) and

$$-\Big[\big(x_0^{(0)\prime}\big)^2\big(x_0^{(0)}B_1^{(0)} + x_1^{(0)}B_0^{(0)}\big) + 2x_0^{(0)\prime}x_1^{(0)\prime}x_0^{(0)}B_0^{(0)}\Big] \tag{A.22}$$

in the right-hand side. It is clear that each term in (A.21) and (A.22) is divisible by t^1. Hence the existence of a holomorphic solution $x_1^{(2)}(s,\rho)$ requires

(A.23)

$$\Big[2f^{(1)}\tilde{x}_0^{(0)}x_1^{(1)}+\big(x_0^{(0)\prime}\big)^2\big(\tilde{x}_0^{(0)}B_1^{(0)}+\tilde{x}_1^{(0)}B_0^{(0)}\big)+2x_0^{(0)\prime}x_1^{(0)\prime}\tilde{x}_0^{(0)}B_0^{(0)}\Big]\Big|_{t=0}= 0.$$

Then by using (A.12), (A.13) and (A.20) we find that

(A.24)

$$2A_0^{(0)}Z_0\frac{A_1^{(0)}}{B_0^{(0)}} + Z_0B_1^{(0)} + Z_0\Big(-\frac{B_1^{(0)}}{B_0^{(0)}}\Big)B_0^{(0)} + 2Z_0^3\Big(-\frac{B_1^{(0)}}{B_0^{(0)}}\Big)B_0^{(0)}$$

$$= 2Z_0\Big(\frac{A_0^{(0)}}{B_0^{(0)}}A_1^{(0)} - B_1^{(0)}\Big) = 0$$

should hold. Similar computation of constant terms in the coefficients of a^3 in (A.4) shows the vanishing of the following sum is required for the existence of $x_1^{(3)}$:

(A.25)

$$\begin{aligned}
&\Big[2\Big(\sum_{j+k+l=3} x_0^{(j)}x_1^{(k)}f^{(l)}\Big) \\
&\quad + \big(x_0^{(0)\prime}\big)^2\big(A_1^{(0)} + x_0^{(0)}B_1^{(1)} + x_0^{(1)}B_1^{(0)} + x_1^{(0)}B_0^{(1)} + x_1^{(1)}B_0^{(0)}\big) \\
&\quad + 2x_0^{(0)\prime}x_0^{(1)\prime}\big(x_0^{(0)}B_1^{(0)} + x_1^{(0)}B_0^{(0)}\big) \\
&\quad + 2x_0^{(0)\prime}x_1^{(0)\prime}\big(A_0^{(0)} + x_0^{(0)}B_0^{(1)} + x_0^{(1)}B_0^{(0)}\big) \\
&\quad + \big(2x_0^{(0)\prime}x_1^{(1)\prime} + 2x_0^{(1)\prime}x_1^{(0)\prime}\big)x_0^{(0)}B_0^{(0)}\Big]\Big|_{t=0} \\
&= A_1^{(0)} + x_1^{(1)}(0,\rho)B_0^{(0)} + 2Z_0^2\dot{x}_1^{(0)}(0,\rho)A_0^{(0)} \\
&= 2A_1^{(0)} - 2\frac{A_0^{(0)}}{B_0^{(0)}}B_1^{(0)}.
\end{aligned}$$

The vanishing of (A.25) together with (A.24) entails the vanishing of $\big(A_1^{(0)}, B_1^{(0)}\big)$ by the assumption (1.1.2) combined with (1.1.1.21) and (1.1.1.22). Then it follows from (A.11) that

$$x_1^{(0)}(s,\rho) = 0. \tag{A.26}$$

Thus we can define

$$\hat{x}_1(t,a,\rho) = a^{-1}x_1(t,a,\rho) \tag{A.27}$$

and

$$\hat{A}_1 = a^{-1}A_1 \quad \text{and} \quad \hat{B}_1 = a^{-1}B_1. \tag{A.28}$$

On the other hand, dividing both sides of (A.4) by a, we find

(A.29)

$$2x_0\hat{x}_1 f = (t^2-a^2)\big[(x_0')^2(a\hat{A}_1+x_0\hat{B}_1+\hat{x}_1B_0)+2x_0'\hat{x}_1'(aA_0+x_0B_0)\big].$$

Hence by repeating the reasoning which guaranteed the vanishing of $\big(x_1^{(0)}(s,\rho), A_1^{(0)}, B_1^{(0)}\big)$, we find the vanishing of $\big(\hat{x}_1^{(0)}(s,\rho), \hat{A}_1^{(0)}, \hat{B}_1^{(0)}\big)$ $= \big(x_1^{(1)}(s,\rho), A_1^{(1)}, B_1^{(1)}\big)$. By repeating this reasoning we find

$$x_1(s,a,\rho) = 0 \tag{A.30}$$

and

$$A_1(a,\rho) = B_1(a,\rho) = 0. \tag{A.31}$$

To prove the required result we use the induction: let us assume

(A.32.ν) $x_{2n-1}(s,a,\rho) = 0$ and $A_{2n-1}(a,\rho) = B_{2n-1}(a,\rho) = 0$ hold for $n \le \nu$,

and show (A.32.$\nu+1$) is valid. First we multiply (1.1.6) (with $g_\pm = 0$) by (x^2-a^2) (t^2-a^2) to find

(A.33)

$$(x^2-a^2)f = (x')^2(aA+xB)(t^2-a^2) - \frac{1}{2}\eta^{-2}(x^2-a^2)(t^2-a^2)\{x;t\}.$$

Comparing the coefficients of $\eta^{-2\nu-1}$ in (A.33) we find

$$\begin{aligned}2x_0x_{2\nu+1}f = (t^2-a^2)\big[(x_0')^2\big(aA_{2\nu+1}+x_0B_{2\nu+1}+x_{2\nu+1}B_0\big)\\ + 2x_0'x_{2\nu+1}'(aA_0+x_0B_0)\big].\end{aligned} \tag{A.34}$$

This has the same form as (A.4); only the suffix 1 in (A.4) is replaced by $\nu+1$. Hence the same reasoning used to show $x_1 = A_1 = B_1 = 0$ applies to (A.34). Then we find (A.32.$\nu+1$) is valid. Therefore the induction proceeds, completing the proof of Proposition A.1.

□

B The vanishing of $x_{2n}^{(1)}(0,\rho)$, $\tilde{A}_{2n}^{(0)}$ and $\tilde{B}_{2n}^{(0)}$ for $n \geq 1$ when $g_{\pm}(t) = 0$.

In Section 1.1 and Section 1.2, the vanishing of $x_0^{(1)}(0,\rho)$ repeatedly played an important role in our reasoning. Hence it is reasonable for the reader to wonder how is the situation for the higher order terms. The answer is that a similar vanishing is observed if $g_{\pm}(t) = 0$ but that it does not hold in general when $g_{\pm}(t) \neq 0$. Hence we content ourselves with a rather weak statement given in Lemma 1.1.3 so that the reasoning in Section 1.1.3 may be applicable to the case where $g_{\pm}(t) \neq 0$. (See the reasoning in Appendix C.) It may be, however, of some interest to see how the actual situation is when $g_{\pm} = 0$. Accordingly, we show the following

Proposition B.1. *Assume $g_{\pm}(t) = 0$. Then we find the following properties for the triplet $T_{2n}^{(p)} = \{x_{2n}^{(p)}, A_{2n}^{(p)}, B_{2n}^{(p)}\}$ constructed in Section 1.1.3:*

(B.1) $$x_{2n}^{(1)}(0,\rho) = 0 \quad \textit{for} \quad n \geq 0,$$

(B.2) $$\dot{x}_{2n}^{(0)}(0,\rho) = 0 \quad \textit{for} \quad n \geq 1,$$

(B.3) $$A_{2n}^{(0)} = B_{2n}^{(0)} = 0 \quad \textit{for} \quad n \geq 1.$$

Proof. Let us first recall

(B.4) $$x_{2j}^{(0)}(0,\rho) = 0 \quad \text{for} \quad j = 0, 1, 2, \cdots .$$

(Cf. (1.1.3.54).) Then, by using (B.4), we validate by the induction on k the following statement $\mathcal{V}(k)$ ($k \geq 1$):

(B.5) $$\mathcal{V}(k) : \begin{cases} \text{(i)} \quad \dot{x}_{2i}^{(0)}(0,\rho) = 0, \ i = 1, 2, \cdots, k, \\ \text{(ii)} \quad x_{2i}^{(1)}(0,\rho) = 0, \ i = 0, 1, \cdots, k, \\ \text{(iii)} \ A_{2i}^{(0)} = B_{2i}^{(0)} = 0, \ i = 1, 2, \cdots, k. \end{cases}$$

Let us first prove $\mathcal{V}(1)$. To begin with, we note

$$R_2^{(0)}(s,\rho) = \frac{1}{2B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 s^2\{x;t\}_0^{(0)}; \tag{B.6}$$

other terms in $R_2^{(0)}(s,\rho)$ do not exist because of the constraints on the indices. Since $\tilde{A}_2^{(-1)} = 0$ by the assumption (1.1.3.50), (B.6) entails

$$x_2^{(0)}(s,\rho) = -\tilde{B}_2^{(0)}s + O(s^2). \tag{B.7}$$

We also note that $z_2^{(0)}(s,\rho)$, which is, by definition, $x_2^{(0)}(s,\rho) - \tilde{A}_2^{(-1)} + \tilde{B}_2^{(0)}s$, satisfies

$$\dot{z}_2^{(0)}(0,\rho) = 0. \tag{B.8}$$

We next show

$$R_2^{(1)}(0,\rho) = 0. \tag{B.9}$$

In what follows (untill (B.18)), we use the symbol $(\alpha.j)$ ($j = \mathrm{i}, \mathrm{ii}, \cdots, \mathrm{x}$) to denote the term labelled by $(\alpha.j)$ in (1.1.3.43) with $(p,n) = (1,1)$; for example, $(\alpha.\mathrm{i})|_{s=0}$ means

$$-\sum_{\substack{q+r+u=0\\ i+j+k=1\\ (u,k)\neq(0,1)}} \dot{x}_{2i}^{(q)}(0,\rho)\dot{x}_{2j}^{(r)}(0,\rho)\tilde{A}_{2k}^{(u)}. \tag{B.10}$$

Using this expression together with (B.7), we find

$$\begin{aligned}(\alpha.\mathrm{i})|_{s=0} &= -2\dot{x}_2^{(0)}(0,\rho)\dot{x}_0^{(0)}(0,\rho)\tilde{A}_0^{(0)}\\ &= 2\tilde{B}_2^{(0)}\tilde{A}_0^{(0)}.\end{aligned} \tag{B.11}$$

Seemingly the ρ-dependence of this term is wilder than that one might expect at this stage. But fortunately, as we will see below (cf. (B.16)), it is cancelled out by $(\alpha.\mathrm{vi})|_{s=0}$, which is equal to

$$\frac{2t^{-2}}{B_0^{(0)}}\dot{t}^2 \sum_{r+u=1} x_2^{(0)}(s,\rho)x_0^{(r)}(s,\rho)f^{(u)}(t,\rho)\Big|_{s=0}. \tag{B.12}$$

Since we have

(B.13) $$x_{2n}^{(0)}(s,\rho)x_0^{(1)}(s,\rho)f^{(0)}(t,\rho) = O(s^3)$$

thanks to the relation

(B.14) $$x_0^{(1)}(0,\rho) = 0,$$

it suffices to study the contribution from $x_2^{(0)}(s,\rho)x_0^{(0)}(s,\rho)f^{(1)}(t,\rho)$. Then it follows from (B.7) and the relation $\big(ds/dt\big|_{t=0}\big)^2 = 1$ that

(B.15) $$\begin{aligned}&\frac{2t^{-2}}{B_0^{(0)}}\dot{t}^2 x_2^{(0)}(s,\rho)x_0^{(0)}(s,\rho)f^{(1)}(t,\rho)\Big|_{s=0}\\ &= \frac{2}{B_0^{(0)}}\big(-\tilde{B}_2^{(0)}\big)A_0^{(0)} = -2\tilde{B}_2^{(0)}\tilde{A}_0^{(0)}.\end{aligned}$$

Thus we find

(B.16) $$(\alpha.\mathrm{i})|_{s=0} + (\alpha.\mathrm{vi})|_{s=0} = 0.$$

In view of the constraint on the indices, we find that $(\alpha.j)$ ($j = \mathrm{iii}, \mathrm{iv}, \mathrm{v}, \mathrm{vii}, \mathrm{viii}, \mathrm{x}$) contains no term. It is clear that $(\alpha.\mathrm{ix})|_{s=0}$ vanishes. Thus the remaining term to be studied is only $(\alpha.\mathrm{ii})|_{s=0}$; because of the constraint on the indices, we find either (i) $q + r + v = 1$ or (ii) $q+r+v = 0$. In case (i) u is 0, and hence $x_{2k}^{(u)}(0,\rho)$ vanishes by (B.4). In case (ii), u should be 1 and hence the constraint $(u,k) \neq (1,1)$ entails $k = 0$, leading to the vanishing of this term by (B.14).

Summing up all these, we thus find

(B.17) $$R_2^{(1)}(0,\rho) = 0.$$

This implies

(B.18) $$z_2^{(1)}(0,\rho) = 0.$$

By using this and (B.8), we next show $\Gamma_2^{(0)} = \Delta_2^{(0)} = 0$ by examining each term in (1.2.164) and (1.2.165). We use symbols $(\tilde{\gamma}.j)$ and $(\tilde{\delta}.k)$ to mean terms labelled by them there.

The vanishing of ($\tilde{\gamma}$.i) and ($\tilde{\gamma}$.ii) immediately follows from (B.8) and (B.18). For $(p_0, n) = (0, 1)$, one of (i, j, k) in ($\tilde{\gamma}$.iii) should be 1, which is forbidden in ($\tilde{\gamma}$.iii). Thus ($\tilde{\gamma}$.iii) contains no term.

Concerning ($\tilde{\gamma}$.iv), we first consider the case where $u = 1$, i.e., $q+r = 2$. If $q = 2$, then $r = 0$; thus $x_2^{(r)}(0, \rho) = 0$ by (B.4). If $q = 0$ or 1 $x_0^{(q)}(0, \rho)$ vanishes by (B.4) or (B.14). Thus every term with $u = 1$ in ($\tilde{\gamma}$.iv) is 0. The same reasoning applies to terms with $u = 2$. Thus ($\tilde{\gamma}$.iv) is 0. It is clear that ($\tilde{\gamma}$.v) contains no term when $n = 1$. To study ($\tilde{\gamma}$.vi), we first consider the case where $q + r + v = 1$. Then $u = 0$, and hence $x_{2k}^{(u)}(0, \rho)$ vanishes by (B.4). When $q + r + v = 0$, $u = 1$; then the constraint $(u, k) \neq (1, 1)$ forces k to be 0. Hence $x_{2k}^{(u)}(0, \rho)$ is 0 by (B.14). Thus ($\tilde{\gamma}$.vi) is 0. The term ($\tilde{\gamma}$.vii) does not exist for $(p_0, n) = (0, 1)$. The vanishing of ($\tilde{\gamma}$.viii) is an immediate consequence of (B.4) (with $j = 0$). Thus we have confirmed

$$\Gamma_2^{(0)} = 0. \tag{B.19}$$

We next study $\Delta_2^{(0)}$. Again by (B.18) and (B.8) we find that ($\tilde{\delta}$.i) and ($\tilde{\delta}$.ii) are 0. When $(p_0, n) = (0, 1)$, ($\tilde{\delta}$.iii) contains no term. To study ($\tilde{\delta}$.iv), we may assume $(i, j) = (0, 1)$ without loss of generality. Then $x_{2i}^{(q)}(0, \rho)$ vanishes for $q = 0$ or 1 by (B.4) or (B.14), whereas $x_2^{(r)}(0, \rho)$ vanishes for $q = 2$ (and hence $r = 0$). Thus ($\tilde{\delta}$.iv) is 0 in our case. The vanishing of ($\tilde{\delta}$.v) can be confirmed in the same manner. Concerning ($\tilde{\delta}$.vi), $x_{2j}^{(r)}(0, \rho) = 0$ for $r = 0$ by (B.4). On the other hand, $r = 1$ forces j to be 0, and hence the vanishing of $x_{2j}^{(r)}(0, \rho)$ follows from (B.14). Thus ($\tilde{\delta}$.vi) is also 0. The vanishing of ($\tilde{\delta}$.vii) is clear for $(p_0, n) = (0, 1)$. For each term in ($\tilde{\delta}$.viii), $u = 0$ in our case, and hence the vanishing of ($\tilde{\delta}$.viii) follows from (B.4). When $(p_0, n) = (0, 1)$, ($\tilde{\delta}$.x) contains no term because of the constraint on the indices, and $(\tilde{\delta}.\mathrm{x})$ does not exist. Finally in $(\tilde{\delta}.\mathrm{xi})$ we find

$$x_{2i}^{(q)} x_{2j}^{(r)} = \big(x_0^{(0)}\big)^2 = s^2. \tag{B.20}$$

Hence

$$\frac{d}{dt}\Big(\sum_{\substack{q+r+u=0 \\ i+j+k=0}} x_{2i}^{(q)} x_{2j}^{(r)} \{x; t\}_{2k}^{(u)} \Big)\Big|_{t=0} = 0. \tag{B.21}$$

Thus we have confirmed

$$\Delta_2^{(0)} = 0. \tag{B.22}$$

Therefore (B.19) and (B.22) imply

$$A_2^{(0)} = B_2^{(0)} = 0. \tag{B.23}$$

Then

$$\dot{x}_2^{(0)}(0, \rho) = 0 \tag{B.24}$$

follows from (B.7) and (B.23), and

$$x_2^{(1)}(0, \rho) = \tilde{A}_2^{(0)} + z_2^{(1)}(0, \rho) = 0 \tag{B.25}$$

follows from (B.18) and (B.23). Thus we have validated $\mathcal{V}(1)$.

Let us now validate $\mathcal{V}(n)$ $(n \geq 2)$ by assuming that $\mathcal{V}(k)$ $(1 \leq k \leq n-1)$ have been validated. To validate $\mathcal{V}(n)$ we first prove

$$R_{2n}^{(0)}(s, \rho) = O(s^2). \tag{B.26}$$

From this point to (B.34), $(\alpha.j)$ $(j = \mathrm{i}, \mathrm{ii}, \cdots, \mathrm{x})$ means the term labelled by $(\alpha.j)$ in (1.1.3.43) with $(p, n) = (0, n)$. As $(\alpha.j)$ $(j = \mathrm{i}, \mathrm{iii}, \mathrm{iv}, \mathrm{vi}, \mathrm{vii}, \mathrm{viii}, \mathrm{x})$ contains no term, we concentrate our attention on other terms.

To study $(\alpha.\mathrm{ii})$, $p = 0$ implies $q = r = u = v = 0$. Then the convention (1.1.3.44) entails

$$i, j, k, l \leq n - 1. \tag{B.27}$$

Hence at most two of (i, j, k, l) are allowed to be 0; otherwise stated, at least two of them are equal to or greater than 1. Therefore it follows from $\mathcal{V}(n-1)$(i), (iii) that

$$(\alpha.\text{ii}) = O(s^2) \tag{B.28}$$

(including the possibility of its vanishing).

Concerning $(\alpha.\text{v})$ with $n \geq 2$, the constraint on the indices entails

$$i, j \geq 1. \tag{B.29}$$

Hence $\mathcal{V}(n-1)$(i) implies

$$x_{2i}^{(0)} x_{2j}^{(0)} f^{(0)} = O(s^5), \tag{B.30}$$

that is,

$$(\alpha.\text{v}) = O(s^3). \tag{B.31}$$

As to $(\alpha.\text{ix})$ we divide the situation into two cases: (i) $k \leq n-2$, (ii) $k = n-1$. In case (i), $i + j = n - 1 - k \geq 1$ and hence (B.4) and $\mathcal{V}(n-1)$(i) entail

$$x_{2i}^{(0)} x_{2j}^{(0)} \{x; t\}_{2k}^{(0)} = O(s^3), \tag{B.32}$$

whereas in case (ii) we find $i = j = 0$ and thus

$$x_0^{(0)} x_0^{(0)} \{x; t\}_{2(n-1)}^{(0)} = O(s^2). \tag{B.33}$$

In any event, we obtain

$$(\alpha.\text{ix}) = O(s^2). \tag{B.34}$$

Summing up all these, we find

$$R_{2n}^{(0)}(s, \rho) = O(s^2). \tag{B.35}$$

Next we study $R_{2n}^{(1)}(0, \rho)$. From this point to (B.44), $(\alpha.j)$ stands for the term labelled by it in (1.1.3.43) with $(p, n) = (1, n)$, where $n \geq 2$.

Let us first examine $(\alpha.\mathrm{i})|_{s=0}$. It follows from the definition that

(B.36)
$$(\alpha.\mathrm{i})|_{s=0} = -\sum_{i+j=n} (\dot{x}_{2i}^{(0)}\dot{x}_{2j}^{(0)}\tilde{A}_0^{(0)})\Big|_{s=0} - \sum_{\substack{i+j+k=n \\ 1\le k\le n-1}} (\dot{x}_{2i}^{(0)}\dot{x}_{2j}^{(0)}\tilde{A}_{2k}^{(0)})\Big|_{s=0}.$$

Then the second sum vanishes by $\mathcal{V}(n-1)(\mathrm{iii})$. On the other hand, all terms except $-(\dot{x}_0^{(0)}\dot{x}_{2n}^{(0)} + \dot{x}_{2n}^{(0)}\dot{x}_0^{(0)})\tilde{A}_0^{(0)}\Big|_{s=0}$ in the first sum vanish by $\mathcal{V}(n-1)(\mathrm{i})$. Thus we find

(B.37)
$$(\alpha.\mathrm{i})|_{s=0} = -2\dot{x}_{2n}^{(0)}(0,\rho)\tilde{A}_0^{(0)}.$$

As one expects, this term is cancelled out by $(\alpha.\mathrm{vi})$; let us confirm it first, by setting aside the study of other terms. Since (B.4) and $\mathcal{V}(n-1)(\mathrm{ii})$ guarantee $x_{2n}^{(0)}x_0^{(1)}f^{(0)} = O(s^3)$, what we have to worry about in $(\alpha.\mathrm{vi})$ is the term

(B.38)
$$\frac{2t^{-2}}{B_0^{(0)}}\Big(\frac{dt}{ds}\Big)^2 x_{2n}^{(0)} s f^{(1)}(t,\rho)\Big|_{s=0},$$

which cancels $(\alpha.\mathrm{i})|_{s=0}$ by (B.4). Let us now return to the study of $(\alpha.\mathrm{ii})|_{s=0}$, following the numbering. To study $(\alpha.\mathrm{ii})|_{s=0}$, we first note that $x_{2k}^{(u)}(0,\rho)$ with $u = 0$ vanishes by (B.4), and hence we suppose $u = 1$. But then, the condition $(u,k) \neq (1,n)$ forces $k \le n-1$. This means that one of (i,j,l) is equal to or greater than 1. Hence $\mathcal{V}(n-1)(\mathrm{i})$, (iii) guarantees

(B.39)
$$\dot{x}_{2i}^{(0)}\dot{x}_{2j}^{(0)}\tilde{B}_{2l}^{(0)}\big|_{s=0} = 0.$$

Thus we find

(B.40)
$$(\alpha.\mathrm{ii})|_{s=0} = 0.$$

It is clear that $(\alpha.\mathrm{iii})$ and $(\alpha.\mathrm{iv})$ contain no term when $p = 1$. As to

(α.v) we rewrite

$$\text{(B.41)}\qquad \sum_{\substack{q+r+u=1\\ i+j=n\\ i,j\le n-1}} x_{2i}^{(q)} x_{2j}^{(r)} f^{(u)} = \sum_{\substack{q+r=1\\ i+j=n\\ i,j\le n-1}} x_{2i}^{(q)} x_{2j}^{(r)} f^{(0)} + \sum_{\substack{i+j=n\\ i,j\le n-1}} x_{2i}^{(0)} x_{2j}^{(0)} f^{(1)}.$$

Then, in the first sum either q or r is 0 and both i and j is equal to or greater than 1. Hence $\mathcal{V}(n-1)$(i), (ii) implies

$$\text{(B.42)}\qquad x_{2i}^{(q)} x_{2j}^{(r)} f^{(0)} = O(s^4).$$

It is also clear from $\mathcal{V}(n-1)$(i) that each term in the second sum is of $O(s^4)$. Thus we obtain

$$\text{(B.43)}\qquad (\alpha.\text{v})|_{s=0} = 0.$$

Since (α.vii), (α.viii) and (α.x) contain no term and since (α.vi) has already been examined, what remains to be studied is (α.ix). But, either q or r is equal to 0 in each term in (α.ix). Hence (B.4) guarantees

$$\text{(B.44)}\qquad (\alpha.\text{ix})|_{s=0} = 0.$$

Thus we have confirmed

$$\text{(B.45)}\qquad R_{2n}^{(1)}(0,\rho) = 0.$$

We now show

$$\text{(B.46)}\qquad \Gamma_{2n}^{(0)} = \Delta_{2n}^{(0)} = 0.$$

To begin with we note

$$\text{(B.47)}\qquad \dot{z}_{2n}^{(0)}(0,\rho) = z_{2n}^{(1)}(0,\rho) = 0$$

follows from (B.35) and (B.45). Then, using the symbol ($\tilde{\gamma}$.i) etc. to denote the corresponding term in (1.2.164) and (1.2.165) with $p_0 = 0$, we find by (B.47) that

$$\text{(B.48)}\qquad (\tilde{\gamma}.\text{i}) = (\tilde{\gamma}.\text{ii}) = 0.$$

Concerning ($\tilde{\gamma}$.iii), we first note

$$q = r = u = 0 \tag{B.49}$$

and hence the constraint on the indices entails

$$i, j, k \leq n - 1. \tag{B.50}$$

Therefore at least one of (i, j, k) is equal to or greater than 1. Then $\mathcal{V}(n-1)$(i), (iii) guarantees

$$(\tilde{\gamma}.\mathrm{iii}) = 0. \tag{B.51}$$

It is clear that ($\tilde{\gamma}$.iv) contains no term when $p_0 = 0$. As to ($\tilde{\gamma}$.v) each term in the sum has the form

$$x_{2i}^{(1)}(0, \rho) x_{2j}^{(1)}(0, \rho) f^{(1)}(0, \rho) \tag{B.52}$$

with $i, j \geq 1$. Hence $\mathcal{V}(n-1)$(ii) implies it vanishes, and thus we have

$$(\tilde{\gamma}.\mathrm{v}) = 0. \tag{B.53}$$

Regarding ($\tilde{\gamma}$.vi) we divide the situation into two cases: (i) $u = 1$ and (ii) $u = 0$. In case (i), $(u, k) \neq (1, n)$ entails $k \leq n - 1$, and hence at least one of (i, j, l) is equal to or greater than 1. Therefore $\mathcal{V}(n-1)$(i), (iii) implies that the term in question is 0. In case (ii), (B.4) applies to the term. Thus

$$(\tilde{\gamma}.\mathrm{vi}) = 0. \tag{B.54}$$

Clearly ($\tilde{\gamma}$.vii) does not exist. Concerning the sum ($\tilde{\gamma}$.viii) either q or r is equal to 0 in each summand and hence (B.4) entails its vanishing. Thus we find

$$\Gamma_{2n}^{(0)} = 0. \tag{B.55}$$

As to $\Delta_{2n}^{(0)}$, ($\tilde{\delta}$.i) and ($\tilde{\delta}$.ii) vanish by (B.47), and ($\tilde{\delta}$.iii) contains no term for $p_0 = 0$. Concerning ($\tilde{\delta}$.iv), we rewrite $\sum_{\substack{q+r=2 \\ i+j=n}} x_{2i}^{(q)}(0, \rho) x_{2j}^{(r)}(0, \rho)$

as follows:

$$2\sum_{i+j=n} x_{2i}^{(0)}(0,\rho)x_{2j}^{(2)}(0,\rho) + \sum_{i+j=n} x_{2j}^{(1)}(0,\rho)x_{2j}^{(1)}(0,\rho). \tag{B.56}$$

Then (B.4) implies the vanishing of each term in the first sum. On the other hand, if one of (i, j) is n then the other is 0 in each term in the second sum. Hence $\mathcal{V}(n-1)$(ii) entails the vanishing of the second sum. Thus we find

$$(\tilde{\delta}.\text{iv}) = 0. \tag{B.57}$$

Similarly (B.4) guarantees

$$(\tilde{\delta}.\text{v}) = 0, \tag{B.58}$$

and we can readily confirm the vanishing of $(\tilde{\delta}.\text{vi})$ in the same way as that used for the confirmation of (B.57). The vanishing of $(\tilde{\delta}.\text{vii})$ and $(\tilde{\delta}.\text{viii})$ is an immediate consequence of (B.4). Concerning $(\tilde{\delta}.\text{ix})$ with $p_0 = 0$, the constraints on the indices entail

$$i, j, k, l \leq n-1, \tag{B.59}$$

and hence at least two of (i, j, k, l) are equal to or greater than 1. Hence $\mathcal{V}(n-1)$(i), (iii) guarantees that every term in $(\tilde{\delta}.\text{ix})$ should be 0. As $(\tilde{\delta}.\text{x})$ does not exist for $p_0 = 0$, the last term to be examined for the confirmation of the vanishing of $\Delta_{2n}^{(0)}$ is $(\tilde{\delta}.\text{xi})$: each term in $(\tilde{\delta}.\text{x})$ for $p_0 = 0$ contains the factor $x_{2i}^{(0)}x_{2j}^{(0)}$. Thus we find

$$(\tilde{\delta}.\text{xi}) = 0. \tag{B.60}$$

Summing up all these we have confirmed

$$R_{2n}^{(0)}(s,\rho) = O(s^2), \tag{B.61}$$

and

$$R_{2n}^{(1)}(0,\rho) = 0 \tag{B.62}$$

together with

$$\Gamma_{2n}^{(0)} = \Delta_{2n}^{(0)} = 0, \tag{B.63}$$

which implies

$$A_{2n}^{(0)} = B_{2n}^{(0)} = 0. \tag{B.64}$$

Therefore we find

$$\dot{x}_{2n}^{(0)}(0,\rho) = -\tilde{B}_{2n}^{(0)} + \dot{R}_{2n}^{(0)}(0,\rho) = 0, \tag{B.65}$$

$$x_{2n}^{(1)}(0,\rho) = \tilde{A}_{2n}^{(0)} + R_{2n}^{(1)}(0,\rho) = 0. \tag{B.66}$$

As (B.64), (B.65) and (B.66), together with $\mathcal{V}(n-1)$, imply that $\mathcal{V}(n)$ is validated. Thus the induction proceeds, and the proof of the proposition is completed.

Remark B.1. By following the reasoning in Appendix C, one can confirm that, if $g_\pm(t) \neq 0$, $x_2^{(1)}(0,\rho)$, together with $(\tilde{A}_2^{(0)}, \tilde{B}_2^{(0)})$, is different from 0 in general.

□

C Construction and estimation of the transformation series that brings an M2P1T equation to the Mathieu equation when $\eta^{-2}\Big(\dfrac{g_+(t)}{(t-a)^2} + \dfrac{g_-(t)}{(t+a)^2}\Big)$ is not 0.

The purpose of this appendix is to confirm the results in Section 1.1.3 and Section 1.2 without assuming $g_\pm(t) = 0$. For the sake of definiteness of the description we assume $B_0^{(0)} = \rho$ (and hence $x_0^{(0)\prime}(0,\rho) = 1$).

For the sake of computation of terms of the form $\Big(\sum_{l\geq 0} z_l(t)\eta^{-l}\Big)^{-p}$ ($p = 1, 2$) we first prepare the following Lemma C.1. The computation of the above series with $p = 2$ is not used in this appendix but used in

Section 1.3. As the reasoning for the case $p = 2$ is basically the same as that for the case $p = 1$ we bring them together here.

Lemma C.1. *Let $w_k(t)$ $(k = 0, 1, 2, \cdots, n)$ be holomorphic functions at $t = t_0$ and satisfy*

$$\frac{dw_0}{dt}(t_0) \neq 0 \tag{C.1}$$

and

$$w_k(t_0) = 0 \quad (k = 0, 1, 2, \cdots). \tag{C.2}$$

Then $f_n(t)$ and $g_n(t)$ $(n = 1, 2, \cdots)$ defined by

(C.3)

$$f_n(t) = \sum_{k_1+k_2+k_3=n} \frac{dw_{k_1}}{dt}\frac{dw_{k_2}}{dt} \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\vec{\kappa}|_\mu = k_3}^{*} (-1)^\mu \frac{w_{\kappa_1} \cdots w_{\kappa_\mu}}{w_0^\mu},$$

(C.4)

$$g_n(t) = \sum_{k_1+k_2+k_3=n} \frac{dw_{k_1}}{dt}\frac{dw_{k_2}}{dt} \sum_{\mu=\min\{1,k_3\}}^{k_3} \sum_{|\vec{\kappa}|_\mu = k_3}^{*} (-1)^\mu (\mu+1) \frac{w_{\kappa_1} \cdots w_{\kappa_\mu}}{w_0^\mu}$$

satisfy the following relations:

$$f_n(t_0) = \frac{dw_0}{dt}\frac{dw_n}{dt}\bigg|_{t=t_0}, \tag{C.5}$$

$$g_n(t_0) = 0. \tag{C.6}$$

In particuler, $w_0^{-1} g_n(t)$ is holomorphic at $t = t_0$.

Proof. Using the assumption (C.2) we define α_k by

$$\alpha_k = \frac{w_k}{w_0}\bigg|_{t=t_0} = \left(\frac{dw_0}{dt}\right)^{-1} \frac{dw_k}{dt}\bigg|_{t=t_0}. \tag{C.7}$$

In order to obtain (C.5), it suffices to show

$$\alpha_n = \sum_{k_1+k_2+k_3=n} \alpha_{k_1}\alpha_{k_2} \sum_{\mu=\min\{1,k_3\}}^{k_3} (-1)^\mu \beta_{k_3}^{(\mu)} \tag{C.8}$$

for $n \geq 1$, where $\beta_k^{(\mu)}$ is a constant defined by

$$\beta_k^{(\mu)} = \sum_{|\vec{\kappa}|_\mu=k}^{*} \alpha_{\kappa_1}\cdots\alpha_{\kappa_\mu}. \tag{C.9}$$

Since $\alpha_0 = \beta_0^{(0)} = 1$, $\beta_n^{(k)} = 0$ for $k \geq n+1$ and

$$\sum_{k_1+k_2=n}^{*} \alpha_{k_1}\beta_{k_2}^{(\mu)} = \beta_n^{(\mu+1)}, \tag{C.10}$$

we find

(C.11)

$$\begin{aligned}
&\sum_{k_1+k_2+k_3=n} \alpha_{k_1}\alpha_{k_2} \sum_{\mu=\min\{1,k_3\}}^{k_3} (-1)^\mu \beta_{k_3}^{(\mu)} \\
&= 2\alpha_0\beta_0^{(0)}\alpha_n + \beta_0^{(0)} \sum_{k_1+k_2=n}^{*} \alpha_{k_1}\alpha_{k_2} + \alpha_0^2 \sum_{\mu=1}^{n} (-1)^\mu \beta_n^{(\mu)} \\
&\quad + 2\alpha_0 \sum_{k_1+k_2=n}^{*} \alpha_{k_1} \sum_{\mu=1}^{k_2} (-1)^\mu \beta_{k_2}^{(\mu)} + \sum_{k_1+k_2+k_3=n}^{*} \alpha_{k_1}\alpha_{k_2} \sum_{\mu=1}^{k_3} (-1)^\mu \beta_{k_3}^{(\mu)} \\
&= 2\alpha_n + \beta_n^{(2)} + \sum_{\mu=1}^{n} (-1)^\mu \beta_n^{(\mu)} + 2\sum_{\mu=1}^{n-1} (-1)^\mu \beta_n^{(\mu+1)} + \sum_{\mu=1}^{n-2} (-1)^\mu \beta_n^{(\mu+2)} \\
&= 2\alpha_n - \beta_n^{(1)}.
\end{aligned}$$

Since $\beta_k^{(1)} = \alpha_k$, we obtain (C.8).

Next, we show

$$\text{(C.12)} \qquad \sum_{k_1+k_2+k_3=n} \alpha_{k_1}\alpha_{k_2} \sum_{\mu=\min\{1,k_3\}}^{k_3} (-1)^\mu(\mu+1)\beta_{k_3}^{(\mu)} = 0.$$

By using the same result as above, we can rewrite the left-hand side of (C.12) as follows:

$$\begin{aligned}
\text{(C.13)} \qquad & \beta_0^{(0)} \sum_{k_1+k_2=n} \alpha_{k_1}\alpha_{k_2} + \alpha_0^2 \sum_{\mu=1}^{n} (-1)^\mu(\mu+1)\beta_n^{(\mu)} \\
& + 2\alpha_0 \sum_{k_1+k_2=n}^{*} \alpha_{k_1} \sum_{\mu=1}^{k_2} (-1)^\mu(\mu+1)\beta_{k_2}^{(\mu)} \\
& + \sum_{k_1+k_2+k_3=n}^{*} \alpha_{k_1}\alpha_{k_2} \sum_{\mu=1}^{k_3} (-1)^\mu(\mu+1)\beta_{k_3}^{(\mu)} \\
& = \sum_{k_1+k_2=n} \alpha_{k_1}\alpha_{k_2} + \sum_{\mu=1}^{n} (-1)^\mu(\mu+1)\beta_n^{(\mu)} \\
& \quad + 2\sum_{\mu=1}^{n-1} (-1)^\mu(\mu+1)\beta_n^{(\mu+1)} + \sum_{\mu=1}^{n-2} (-1)^\mu(\mu+1)\beta_n^{(\mu+2)} \\
& = 2\alpha_n + \beta_n^{(2)} - 2\beta_n^{(1)} - \beta_n^{(2)}.
\end{aligned}$$

Since $\beta_n^{(1)} = \alpha_n$, we obtain (C.12); thus we have confirmed (C.6).

□

Let us now confirm Proposition 1.1.3.2 together with the estimate $[G'; p, 2n]$ given in Proposition C.1 below, which is totally the same estimates with $[G; p, 2n]$ in Proposition 1.2.1, when the lower order term

$$\text{(C.14)} \qquad \eta^{-2}\Big(\frac{g_+(t)}{(t-a)^2} + \frac{g_-(t)}{(t+a)^2}\Big)$$

is not assumed to be 0. In what follows we sometimes refer to this lower order term as the additional term so that the background of our reasoning may become apparent. The main reason why we perform the construction and the estimation simultaneously is that we want to use the analyticity of $\big\{x_{2k}(t,a,\rho)\big\}_{0\le k\le n-1}$ on a sufficiently large set, say on E^1_{r,R_0} in constructing $x_{2k}(t,a,\rho)$; the analyticity of $\big\{x_{2k}\big\}_{0\le k\le n-1}$ enables us to find Lemma C.2. The relation (C.18) leads us to introduce the auxiliary functions $\big\{y_{\pm,2k}(t,a,\rho)\big\}_{0\le k\le n-1}$, which facilitates the manipulation of the singularities at $t=\pm a$ contained in the additional terms, as we will see below.

Proposition C.1. *There exist positive constants* (r_0, R, A) *and a sufficiently small constant* N_0 *for which the following estimate* $[G'; p, 2n]$ *holds for every* $p \ge 0$, *every* $n \ge 1$, *every* ρ *in* $\{\rho \in \mathbb{C}; 0 < |\rho| \le r_0\}$ *and any positive constant* ε *that is smaller than* $r_0/3$:

$$[G'; p, 2n] =$$
$$\begin{cases} (p,2n)(\mathrm{i}) & |x_{2n}^{(p+1)}(0,\rho)| \le N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n)!\varepsilon^{-2n}(A|\rho|^{-1})^n, \\ (p,2n)(\mathrm{ii}) & |\tilde{A}_{2n}^{(p)}| \le N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n)!\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n, \\ (p,2n)(\mathrm{iii}) & |\tilde{B}_{2n}^{(p)}| \le N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n)!\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n, \\ (p,2n)(\mathrm{iv}) & \|x_{2n}^{(p)}\|_{[r_0-\varepsilon]} \le N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n)!\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n, \\ (p,2n)(\mathrm{v}) & \|\dot{x}_{2n}^{(p)}\|_{[r_0-\varepsilon]} \le N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n)!\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n. \end{cases}$$

To confirm Proposition 1.1.3.2 and Proposition C.1 when the potential Q contains the additional terms, we first note that (1.1.6) requires $x=\sum_{n\ge0} x_{2n}(t,a,\rho)\eta^{-2n}$, $A=\sum_{n\ge0} A_{2n}(a,\rho)\eta^{-2n}$, and

$B=\sum_{n\geq 0} B_{2n}(a,\rho)\eta^{-2n}$, should satisfy

(C.15)

$$
\begin{aligned}
&(x^2-a^2)\Big\{f+\eta^{-2}\Big(\frac{t+a}{t-a}g_+(t)+\frac{t-a}{t+a}g_-(t)\Big)\Big\}\\
&=(t^2-a^2)\Big(\frac{\partial x}{\partial t}\Big)^2\Big\{aA+xB+\eta^{-2}\Big(\frac{x+a}{x-a}g_+(a)+\frac{x-a}{x+a}g_-(-a)\Big)\Big\}\\
&\quad-\frac{1}{2}\eta^{-2}(t^2-a^2)(x^2-a^2)\{x;t\}.
\end{aligned}
$$

Since the additional terms do not affect the relation that $A_0(a,\rho)$, $B_0(a,\rho)$ and $x_0(t,a,\rho)$ should satisfy, Proposition 1.1.2.1 and Lemma 1.2.3 apply to the case where Q contains the additional terms.

In parallel with (1.1.3.36), the comparison of the coefficients of η^{-2n} $(n=1,2,\cdots)$ of (C.15) leads us to the following relation:

(C.16)

$$
\begin{aligned}
&\Big(\sum_{k_1+k_2=n} x_{2k_1}x_{2k_2}\Big)f\\
&+\Big(\sum_{k_1+k_2=n-1} x_{2k_1}x_{2k_2}-\delta_{n,1}a^2\Big)\Big(\frac{t+a}{t-a}g_+(t)+\frac{t-a}{t+a}g_-(t)\Big)\\
&=(t^2-a^2)\Big(\sum_{k_1+k_2+k_3=n} x'_{2k_1}x'_{2k_2}aA_{2k_3}+\sum_{k_1+\cdots+k_4=n} x'_{2k_1}x'_{2k_2}x_{2k_3}B_{2k_4}\Big)\\
&\quad+(t^2-a^2)\sum_{k_1+\cdots+k_4=n-1} x'_{2k_1}x'_{2k_2}x_{2k_3}\\
&\qquad\times\Big(\frac{g_+(a)}{x_0-a}\sum_{\mu=\min\{1,k_4\}}^{k_4}\sum_{|\vec{\kappa}|_\mu=k_4}{}^{*}(-1)^\mu\frac{x_{2\kappa_1}\cdots x_{2\kappa_\mu}}{(x_0-a)^\mu}
\end{aligned}
$$

$$+\frac{g_-(-a)}{x_0+a}\sum_{\mu=\min\{1,k_4\}}^{k_4}\sum_{|\vec{\kappa}|_\mu=k_4}{}^{*}(-1)^\mu\frac{x_{2\kappa_1}\cdots x_{2\kappa_\mu}}{(x_0+a)^\mu}\Big)$$

$$+(t^2-a^2)a\sum_{k_1+k_2+k_3=n-1}x'_{2k_1}x'_{2k_2}$$

$$\times\Big(\frac{g_+(a)}{x_0-a}\sum_{\mu=\min\{1,k_3\}}^{k_3}\sum_{|\vec{\kappa}|_\mu=k_3}{}^{*}(-1)^\mu\frac{x_{2\kappa_1}\cdots x_{2\kappa_\mu}}{(x_0-a)^\mu}$$

$$-\frac{g_-(-a)}{x_0+a}\sum_{\mu=\min\{1,k_3\}}^{k_3}\sum_{|\vec{\kappa}|_\mu=k_3}{}^{*}(-1)^\mu\frac{x_{2\kappa_1}\cdots x_{2\kappa_\mu}}{(x_0+a)^\mu}\Big)$$

$$-\frac{1}{2}(t^2-a^2)\sum_{k_1+k_2+k_3=n-1}x_{2k_1}x_{2k_2}\{x;t\}_{2k_3}$$

$$+\frac{1}{2}(t^2-a^2)a^2\{x;t\}_{2(n-1)},$$

where $\delta_{n,1}$ is the Kronecker delta and $\{x;t\}_{2k}$ is the coefficient of η^{-2k} of $\{x;t\}$.

Let us now confirm that $\mathcal{A}_{2n}(p)$ $(p\geq 0)$ hold under the assumption that $\mathcal{A}_{2k}(p)$ and $[G';p,2k]$ hold for $0\leq k\leq n-1$ and $p\geq 0$. It follows from $[G';p,2k]$ $(p\geq 0)$ that $x_{2k}(s,a,\rho)$ is holomorphic on

(C.17)

$$\tilde{E}^1_{r_0,2R}=\{(s,a,\rho)\in\mathbb{C}^3:|s|\leq r_0,0<|\rho|\leq r_0,|a|\leq(2R)^{-1}|\rho|\}.$$

Using this analyticity we first show the following

Lemma C.2.

(C.18) $$x_{2k}(t,a,\rho)|_{t=\pm a}=0$$

holds for $1\leq k\leq n-1$.

Proof. It follows from (C.16) with $n=1$ that x_2 satisfies the following relation:

(C.19)

$$2x_0x_2f + (x_0^2 - a^2)\Big(\frac{t+a}{t-a}g_+(t) + \frac{t-a}{t+a}g_-(t)\Big)$$

$$= (t^2 - a^2)\Big(\sum_{k_1+k_2+k_3=1} x'_{2k_1}x'_{2k_2}aA_{2k_3} + \sum_{k_1+\cdots+k_4=1} x'_{2k_1}x'_{2k_2}x_{2k_3}B_{2k_4}\Big)$$

$$+ (t^2 - a^2)(x'_0)^2\Big(\frac{x_0+a}{x_0-a}g_+(a) + \frac{x_0-a}{x_0+a}g_-(-a)\Big)$$

$$- \frac{1}{2}(t^2 - a^2)(x_0^2 - a^2)\{x;t\}_0.$$

Since x_0 satisfies (1.3.11), by setting $t = \pm a$ in (C.19), we obtain

(C.20) $$2ax_2(t,a,\rho)f(t,a,\rho)|_{t=\pm a} = 0.$$

Hence (C.18) for $k = 1$ follows from (1.3.16). Next we show (C.18) for $k = l$ $(2 \le l \le n-1)$ under the assumption that (C.18) holds for $1 \le k \le l-1$. By setting $t = a$ in (C.16) with $n = l$, we obtain

(C.21)

$$(2ax_{2l}f + 4a^2g_+x'_{2(l-1)})\big|_{t=a}$$

$$= (t^2 - a^2)g_+(a)\frac{x_0+a}{x_0-a}$$

$$\times \sum_{k_1+k_2+k_4=l-1} x'_{2k_1}x'_{2k_2} \sum_{\mu=\min\{1,k_4\}}^{k_4} \sum\nolimits^*_{|\vec{\kappa}|_\mu=k_4} (-1)^\mu \left.\frac{x_{2\kappa_1}\cdots x_{2\kappa_\mu}}{(x_0-a)^\mu}\right|_{t=a}.$$

Since $w_0 = x_0 - a$ and $w_k = x_{2k}$ satisfy (C.1) and (C.2) at $t_0 = a$, (C.5) implies that the right-hand side of (C.21) is equal to $4a^2g_+x'_{2(l-1)}|_{t=a}$. Hence we obtain

(C.22) $$2ax_{2l}f\big|_{t=a} = 0.$$

We then see that $x_{2l}|_{t=a} = 0$. Using the same reasoning as above, we find $x_{2l}|_{t=-a} = 0$ holds. Hence we obtain (C.18) for $k = l$.

□

Let us now define $y_{\pm,2k}(t, a, \rho)$ $(k = 0, 1, 2, \cdots)$ by

$$y_{\pm,0} = \frac{x_0 \mp a}{t \mp a} \tag{C.23}$$

$$y_{\pm,2k} = \frac{x_{2k}}{t \mp a}. \tag{C.24}$$

Then, from Theorem 1.3.1 and (C.18), we find that $(y_{\pm,0})^{-1}$ and $y_{\pm,2k}$ $(k = 0, 1, 2, \cdots)$ are holomorphic on $\tilde{E}^1_{r_0,2R}$. We denote the coefficients of a^p of $(y_{\pm,0})^{-\mu}$ $(\mu = 1, 2, \cdots)$ and $y_{\pm,2k}$ by $w_{\pm}^{\mu,(p)}$ and $y_{\pm,2k}^{(p)}$ respectively as follows:

$$(y_{\pm,0})^{-\mu}(t, a, \rho) = \sum_{p=0}^{\infty} w_{\pm}^{\mu,(p)}(t, \rho) a^p, \tag{C.25}$$

$$y_{\pm,2k}(t, a, \rho) = \sum_{p=0}^{\infty} y_{\pm,2k}^{(p)}(t, \rho) a^p. \tag{C.26}$$

We also denote the coefficients of t^p of $g_{\pm}$ by $g_{\pm}^{(p)}$, i.e.,

$$g_{\pm}(t) = \sum_{p=0}^{\infty} g_{\pm}^{(p)} t^p. \tag{C.27}$$

In parallel with (1.1.3.37), comparison of the coefficients of a^p in (C.16) leads us to the following relation:

(C.28)

$$\sum_{\substack{l_1+l_2+l_3=p \\ k_1+k_2=n}} x_{2k_1}^{(l_1)} x_{2k_2}^{(l_2)} f^{(l_3)} + \mathcal{F}_{2n}^{(p)}$$

$$= t^2\Big[\sum_{\substack{l_1+l_2+l_3=p-1\\ k_1+k_2+k_3=n}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} A_{2k_3}^{(l_3)} + \sum_{\substack{l_1+\cdots+l_4=p\\ k_1+\cdots+k_4=n}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} x_{2k_3}^{(l_3)} B_{2k_4}^{(l_4)}$$

$$- \frac{1}{2} \sum_{\substack{l_1+l_2+l_3=p\\ k_1+k_2+k_3=n-1}} x_{2k_1}^{(l_1)} x_{2k_2}^{(l_2)} \{x;t\}_{2k_3}^{(l_3)} + \frac{1}{2}\{x;t\}_{2(n-1)}^{(p-2)}\Big]$$

$$- \Big[\sum_{\substack{l_1+l_2+l_3=p-3\\ k_1+k_2+k_3=n}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} A_{2k_3}^{(l_3)} + \sum_{\substack{l_1+\cdots+l_4=p-2\\ k_1+\cdots+k_4=n}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} x_{2k_3}^{(l_3)} B_{2k_4}^{(l_4)}$$

$$- \frac{1}{2} \sum_{\substack{l_1+l_2+l_3=p-2\\ k_1+k_2+k_3=n-1}} x_{2k_1}^{(l_1)} x_{2k_2}^{(l_2)} \{x;t\}_{2k_3}^{(l_3)} + \frac{1}{2}\{x;t\}_{2(n-1)}^{(p-4)}\Big] + \mathcal{G}_{2n}^{(p)},$$

where $\mathcal{F}_{2n}^{(p)}$ and $\mathcal{G}_{2n}^{(p)}$ are functions that depend only on $x_{2k}^{(l)}$, $y_{\pm,2k}^{(l)}$ $(0 \le k \le n-1, l \ge 0)$ and $g_\pm$. The concrete forms of $\mathcal{F}_{2n}^{(p)}$ and $\mathcal{G}_{2n}^{(p)}$ are given as follows:

$$\mathcal{F}_{2}^{(p)} = tg_+(t) \sum_{l_1+l_2=p} x_0^{(l_1)} y_{+,0}^{(l_2)} + g_+(t) \sum_{l_1+l_2=p-1} x_0^{(l_1)} y_{+,0}^{(l_2)}$$
$$+ tg_+(t) y_{+,0}^{(p-1)} + g_+(t) y_{+,0}^{(p-2)}$$
$$+ tg_-(t) \sum_{l_1+l_2=p} x_0^{(l_1)} y_{-,0}^{(l_2)} - g_-(t) \sum_{l_1+l_2=p-1} x_0^{(l_1)} y_{-,0}^{(l_2)}$$
$$- tg_-(t) y_{-,0}^{(p-1)} + g_-(t) y_{-,0}^{(p-2)}, \tag{C.29}$$

$$\mathcal{F}_{2n}^{(p)} = 2tg_+(t) \sum_{l_1+l_2=p} x_0^{(l_1)} y_{+,2(n-1)}^{(l_2)} \tag{C.30}$$
$$+ tg_+(t) \sum_{k_1+k_2=n-1}^{*} \sum_{l_1+l_2=p} x_{2k_1}^{(l_1)} y_{+,2k_2}^{(l_2)}$$

$$+ 2g_+(t) \sum_{l_1+l_2=p-1} x_0^{(l_1)} y_{+,2(n-1)}^{(l_2)}$$

$$+ g_+(t) \sum^*_{k_1+k_2=n-1} \sum_{l_1+l_2=p-1} x_{2k_1}^{(l_1)} y_{+,2k_2}^{(l_2)}$$

$$+ 2tg_-(t) \sum_{l_1+l_2=p} x_0^{(l_1)} y_{-,2(n-1)}^{(l_2)}$$

$$+ tg_-(t) \sum^*_{k_1+k_2=n-1} \sum_{l_1+l_2=p} x_{2k_1}^{(l_1)} y_{-,2k_2}^{(l_2)}$$

$$- 2g_-(t) \sum_{l_1+l_2=p-1} x_0^{(l_1)} y_{-,2(n-1)}^{(l_2)}$$

$$- g_-(t) \sum^*_{k_1+k_2=n-1} \sum_{l_1+l_2=p-1} x_{2k_1}^{(l_1)} y_{-,2k_2}^{(l_2)}$$

for $n \geq 2$ and

(C.31)

$$\mathcal{G}_{2n}^{(p)} = \Bigg(t \sum_{\substack{k_1+\cdots+k_4=n-1 \\ l_1+\cdots+l_6=p}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} x_{2k_3}^{(l_3)} g_+^{(l_4)}$$

$$+ \sum_{\substack{k_1+\cdots+k_4=n-1 \\ l_1+\cdots+l_6=p-1}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} x_{2k_3}^{(l_3)} g_+^{(l_4)} \Bigg)$$

$$\times \sum_{\mu=\min\{1,k_4\}}^{k_4} w_+^{\mu+1,(l_5)} \sum^*_{|\vec{\kappa}|_\mu=k_4} \sum_{|\vec{\lambda}|_\mu=l_6} (-1)^\mu y_{+,2\kappa_1}^{(\lambda_{\mu+1})} \cdots y_{+,2\kappa_\mu}^{(\lambda_{2\mu})}$$

$$+ \Bigg(t \sum_{\substack{k_1+\cdots+k_4=n-1 \\ l_1+\cdots+l_6=p}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} x_{2k_3}^{(l_3)} (-1)^{l_4} g_-^{(l_4)}$$

$$
\begin{aligned}
&- \sum_{\substack{k_1+\cdots+k_4=n-1\\ l_1+\cdots+l_6=p-1}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} x_{2k_3}^{(l_3)} (-1)^{l_4} g_-^{(l_4)} \Big) \\
&\times \sum_{\mu=\min\{1,k_4\}}^{k_4} w_-^{\mu+1,(l_5)} \sum_{|\vec{\kappa}|_\mu=k_4}^{*} \sum_{|\vec{\lambda}|_\mu=l_6} (-1)^\mu y_{-,2\kappa_1}^{(\lambda_{\mu+1})} \cdots y_{-,2\kappa_\mu}^{(\lambda_{2\mu})} \\
&+ \Big(t \sum_{\substack{k_1+\cdots+k_4=n-1\\ l_1+\cdots+l_5=p-1}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} g_+^{(l_3)} \\
&\quad + \sum_{\substack{k_1+\cdots+k_4=n-1\\ l_1+\cdots+l_5=p-2}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} g_+^{(l_3)} \Big) \\
&\times \sum_{\mu=\min\{1,k_4\}}^{k_4} w_+^{\mu+1,(l_4)} \sum_{|\vec{\kappa}|_\mu=k_4}^{*} \sum_{|\vec{\lambda}|_\mu=l_5} (-1)^\mu y_{+,2\kappa_1}^{(\lambda_{\mu+1})} \cdots y_{+,2\kappa_\mu}^{(\lambda_{2\mu})} \\
&- \Big(t \sum_{\substack{k_1+\cdots+k_4=n-1\\ l_1+\cdots+l_5=p-1}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} (-1)^{l_3} g_-^{(l_3)} \\
&\quad - \sum_{\substack{k_1+\cdots+k_4=n-1\\ l_1+\cdots+l_5=p-2}} x_{2k_1}^{(l_1)\prime} x_{2k_2}^{(l_2)\prime} (-1)^{l_3} g_-^{(l_3)} \Big) \\
&\times \sum_{\mu=\min\{1,k_4\}}^{k_4} w_-^{\mu+1,(l_4)} \sum_{|\vec{\kappa}|_\mu=k_4}^{*} \sum_{|\vec{\lambda}|_\mu=l_5} (-1)^\mu y_{-,2\kappa_1}^{(\lambda_{\mu+1})} \cdots y_{-,2\kappa_\mu}^{(\lambda_{2\mu})}
\end{aligned}
$$

for $n \geq 1$.

Now, we define $\tilde{\Phi}_{2n}^{(p)}$ and $\tilde{R}_{2n}^{(p)}$ by

(C.32) $$\tilde{\Phi}_{2n}^{(p)} = \Phi_{2n}^{(p)} + \mathcal{F}_{2n}^{(p)} - \mathcal{G}_{2n}^{(p)},$$

(C.33) $$\tilde{R}^{(p)}_{2n} = R^{(p)}_{2n} + \frac{t^{-2}}{B^{(0)}_0}\Big(\frac{dt}{ds}\Big)^2(\mathcal{F}^{(p)}_{2n} - \mathcal{G}^{(p)}_{2n}),$$

where $\Phi^{(p)}_{2n}$ and $R^{(p)}_{2n}$ are respectively given by (1.1.3.38) and (1.1.3.43). It is evident from (C.28) that, if we want to construct $\{x, A, B\}$ when $g_\pm \neq 0$, $\tilde{\Phi}^{(p)}_{2n}$ (resp., $\tilde{R}^{(p)}_{2n}$) is the required substitute of $\Phi^{(p)}_{2n}$ (resp., $R^{(p)}_{2n}$) used in Section 1.1.3. Since $\mathcal{F}^{(p)}_{2n} \underset{(2n;q)}{\equiv} 0$ and $\mathcal{G}^{(p)}_{2n} \underset{(2n;q)}{\equiv} 0$ for any $p \geq 0$ and $q \geq 0$, by the same reasoning as that in Section 1.1.3, we find $\mathcal{A}_{2n}(p)$ $(p \geq 0)$ is also valid in this case.

Next we estimate the constructed series as Proposition C.1 requires. For this purpose we prepare the following

Lemma C.3. *The series* $\mathcal{F}^{(p)}_{2n}$ *and* $\mathcal{G}^{(p)}_{2n}$ $(p \geq 0)$ *satisfy the following estimates for some positive constant* M_0 *under the assumption that* $[G; p, 0]$ *and* $[G'; p, 2k]$ $(1 \leq k \leq n-1, p \geq 0)$ *hold*:

(C.34)
$$\big|\mathcal{F}^{(p+3)}_{2n}\big|_{t=0}\big| \leq M_0 A^{-1} N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n-2)!\varepsilon^{-2n+2}\big(A|\rho|^{-1}\big)^n,$$

(C.35)
$$\big|\mathcal{F}^{(p+2)\prime}_{2n}\big|_{t=0}\big| \leq M_0 A^{-1} N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n-2)!\varepsilon^{-2n+2}\big(A|\rho|^{-1}\big)^n,$$

(C.36)
$$\big\|\mathcal{F}^{(p)}_{2n}\big\|_{[r_0-\varepsilon]} \leq M_0 N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n-2)!\varepsilon^{-2n+2}\big(A|\rho|^{-1}\big)^{n-1},$$

(C.37)
$$\big|\mathcal{G}^{(p+3)}_{2n}\big|_{t=0}\big| \leq M_0 A^{-1} N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n-2)!\varepsilon^{-2n+2}\big(A|\rho|^{-1}\big)^n,$$

(C.38)
$$\big|\mathcal{G}^{(p+2)\prime}_{2n}\big|_{t=0}\big| \leq M_0 A^{-1} N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n-2)!\varepsilon^{-2n+2}\big(A|\rho|^{-1}\big)^n,$$

(C.39)
$$\big\|\mathcal{G}^{(p)}_{2n}\big\|_{[r_0-\varepsilon]} \leq M_0 N_0 C(p)\big(R|\rho|^{-1}\big)^p (2n-2)!\varepsilon^{-2n+2}\big(A|\rho|^{-1}\big)^{n-1}.$$

Proof. To begin with, we derive the estimates of $y^{(p)}_{\pm,2k}$ and $w^{(p)}_{\pm}$ from those of $x^{(p)}_{\pm,2k}$ as follows:

Lemma C.4. *The functions* $y^{(p)}_{\pm,2k}$ *and* $w^{\mu,(p)}_{\pm}$ $(0 \le k \le n-1, p \ge 0, \mu \ge 1)$ *satisfy the following estimates for some positive constant* M *under the assumption that* $[G; p, 0]$ *and* $[G'; p, 2k]$ $(1 \le k \le n-1, p \ge 0)$ *hold:*

$$\big\|y^{(p)}_{\pm,0}\big\|_{[r_0]} \le MC(p)\big(R|\rho|^{-1}\big)^p, \tag{C.40}$$

$$\big\|w^{\mu,(p)}_{\pm}\big\|_{[r_0]} \le M^{\mu}C(p)\big(R|\rho|^{-1}\big)^p, \tag{C.41}$$

$$\big\|y^{(p)}_{\pm,2k}\big\|_{[r_0-\varepsilon]} \le MN_0C(p)\big(R|\rho|^{-1}\big)^p(2k)!\varepsilon^{-2k}\big(A|\rho|^{-1}\big)^k. \tag{C.42}$$

Proof. Since $(y_{\pm,0})^{\pm 1}$ are holomorphic on $\tilde{E}^1_{r,2R}$ and bounded by some positive constant M there, we find that, by taking R sufficiently large if necessary, $y^{(p)}_{\pm,0}$ and $w^{\mu,(p)}_{\pm}$ $(p \ge 0, \mu \ge 1)$ satisfy (C.40) and (C.41). Further, it follows from the definition of $y_{\pm,2k}$ that $y^{(p)}_{\pm,2k}$ satisfy the following relation:

$$y^{(p)}_{\pm,2k} = t^{-1}(x^{(p)}_{2k} \pm y^{(p-1)}_{\pm,2k}), \tag{C.43}$$

where we conventionally regard $y^{(-1)}_{\pm,2k}$ as 0. It is then evident that we can estimate $\big\|y^{(p)}_{\pm,2k}\big\|_{[r_0-\varepsilon]}$ in an inductive manner with the help of $[G'; p, 2k]$. Actually, with an appropriate choice of constants M and R that is specified below, the maximum modulus principle enables to find the following:

$$\begin{aligned}\big\|y^{(p)}_{\pm,2k}\big\|_{[r_0-\varepsilon]} \le& \frac{M}{2}\big(\big\|x^{(p)}_{2k}\big\|_{[r_0-\varepsilon]} + \big\|y^{(p-1)}_{\pm,2k}\big\|_{[r_0-\varepsilon]}\big) \\ \le& \frac{M}{2}\big(1 + MR^{-1}|\rho|C(p-1)C(p)^{-1}\big)\end{aligned} \tag{C.44}$$

$$\times N_0 C(p)\big(R|\rho|^{-1}\big)^p (2k)!\varepsilon^{-2k}\big(A|\rho|^{-1}\big)^k$$

$$\leq M N_0 C(p)\big(R|\rho|^{-1}\big)^p (2k)!\varepsilon^{-2k}\big(A|\rho|^{-1}\big)^k.$$

Here we take $M > 0$ so that

(C.45) $$\sup_{|s|=r_0-\varepsilon} \left|t^{-1}(s)\right| \leq M/2$$

holds for $0 < \varepsilon < r_0/3$ and assume that, by taking R sufficiently large,

(C.46) $$M R^{-1}|\rho| C(p-1) C(p)^{-1} \leq 1$$

holds.

□

Remark C.1. As the recursive relation (C.43) for $y^{(p)}_{\pm,2k}$ ($k \geq 1$) implies, if we write $y^{(p)}_{\pm,2k}$ in terms of $x^{(p)}_{2k}$, it looks as if it had a pole at $t = 0$ whose order became higher and higher with increasing p. However (C.18) guarantees that the pole actually does not appear. This is also the case for $y^{(p)}_{\pm,0}$ and $w^{\mu,(p)}_{\pm}$.

Now let us return to the proof of Lemma C.3. Suppose $g_\pm$ is bounded by some positive constant M as follows:

(C.47) $$\big\|g_\pm\big\|_{[r_0]} \leq M.$$

Then, for example, the second term of $\mathcal{F}_2^{(p+3)}$, i.e., $g_+ \sum_{l_1+l_2=p+2} x_0^{(l_1)} y_{+,0}^{(l_2)}$ is estimated as follows for $t = 0$:

(C.48) $$\big\|g_+\big\|_{[r_0]} \sum_{l_1+l_2=p+2} \big|x_0^{(l_1)}(0,\rho)\big| \big\|y_{+,0}^{(l_2)}\big\|_{[r_0]}$$

$$\leq M^2 C_0 C(p+2)\big(R|\rho|^{-1}\big)^{p+1}.$$

In this way, we can readily confirm that the following estimate holds for $p \geq 0$:

(C.49) $$\big|\mathcal{F}_2^{(p+3)}\big|_{t=0}\big| \leq 4M^2 C_0 C(p+2)\big(R|\rho|^{-1}\big)^{p+1}.$$

Therefore, by taking M_0 sufficiently large so that $4M^2C_0R \le M_0N_0$ holds, we obtain (C.34) for $n = 1$. In the same way, we easily find that $\mathcal{F}_{2n}^{(p+3)}\big|_{t=0}$ $(n = 2, 3, \cdots)$ satisfy (C.34). The estimation of $\mathcal{F}_{2n}^{(p+2)\prime}\big|_{t=0}$ required in (C.35) can be also done in a similar manner; by using Cauchy's inequality we can estimate, for example, the derivative of the third term of $\mathcal{F}_{2n}^{(p+2)}\big|_{t=0}$ evaluated at $t = 0$, i.e., $\Big(2g_+(t)\sum_{l_1+l_2=p+1} x_0^{(l_1)}y_{+,2(n-1)}^{(l_2)}\Big)'\Big|_{t=0}$ as follows:

(C.50)

$$\Big|\Big(2g_+(t)\sum_{l_1+l_2=p+1} x_0^{(l_1)}y_{+,2(n-1)}^{(l_2)}\Big)'\Big|_{t=0}\Big|$$
$$\le \frac{2}{r_0-\varepsilon}\big\|g_+\big\|_{[r_0]}\sum_{l_1+l_2=p+1}\big\|x_0^{(l_1)}\big\|_{[r_0]}\big\|y_{+,2(n-1)}^{(l_2)}\big\|_{[r_0-\varepsilon]}$$
$$\le \frac{3M^2N_0}{r_0}C(p+2)\big(R|\rho|^{-1}\big)^{p+1}(2n-2)!\varepsilon^{-2n+2}\big(A|\rho|^{-1}\big)^{n-1}.$$

In this way, we find the estimation (C.35). The estimate (C.36) is an immediate consequence of the induction hypothesis.

Next, we confirm (C.37). Since

$$\big|x_{2k}^{(l)\prime}(0,\rho)\big| \le \frac{M}{r_0-\varepsilon}\big\|x_{2k}^{(l)}\big\|_{[r_0-\varepsilon]} \tag{C.51}$$

holds for some positive constant M and since we may assume that $g_\pm^{(l)}$ satisfy

$$\big|g_\pm^{(l)}\big| \le MC(l)r_0^{-l}, \tag{C.52}$$

the first term of $\mathcal{G}_{2n}^{(p+3)}\big|_{t=0}$ can be estimated as follows:

(C.53)

$$
\begin{aligned}
&\sum_{\substack{k_1+\cdots+k_4=n-1\\ l_1+\cdots+l_6=p+2}} \big|x_{2k_1}^{(l_1)\prime}(0,\rho)\big|\big|x_{2k_2}^{(l_2)\prime}(0,\rho)\big|\big|x_{2k_3}^{(l_3)}(0,\rho)\big|\big|g_+^{(l_4)}\big| \\
&\quad\times \sum_{\mu=\min\{1,k_4\}}^{k_4} \big\|w_+^{\mu+1,(l_5)}\big\|_{[r_0]} \sum_{|\vec{\kappa}|_\mu=k_4}^{*} \sum_{|\vec{\lambda}|_\mu=l_6} \big\|y_{+,2\kappa_1}^{(\lambda_{\mu+1})}\big\|_{[r_0-\varepsilon]} \cdots \big\|y_{+,2\kappa_\mu}^{(\lambda_{2\mu})}\big\|_{[r_0-\varepsilon]} \\
&\le \frac{M^3}{(r_0-\varepsilon)^2}\big(R|\rho|^{-1}\big)^{p+1}\varepsilon^{-2n+2}\big(A|\rho|^{-1}\big)^{n-1} \\
&\quad\times \sum_{\substack{k_1+\cdots+k_4=n-1\\ l_1+\cdots+l_6=p+2}} C(l_1)C(l_2)C(l_3)C(l_4)(2k_1)!(2k_2)!(2k_3)!\Big(\frac{|\rho|}{r_0R}\Big)^{l_4} \\
&\quad\times \sum_{\mu=\min\{1,k_4\}}^{k_4} M^{2\mu+1}N_0^{\mu}C(l_5)C(l_6)\sum_{|\vec{\kappa}|_\mu=k_4}^{*}(2\kappa_1)!\cdots(2\kappa_\mu)! \\
&\le 9r_0^{-2}M^3e^{4M^2N_0}C(p+2)\big(R|\rho|^{-1}\big)^{p+1}(2n-2)!\varepsilon^{-2n+2}\big(A|\rho|^{-1}\big)^{n-1}.
\end{aligned}
$$

Similar estimation is validated for other terms in $\mathcal{G}_{2n}^{(p+3)}\big|_{t=0}$. Hence, by taking M_0 so that $36r_0^{-2}M^3e^{4M^2N_0}R \le N_0M_0$ holds, we obtain (C.37). We can confirm (C.38) in a similar manner. The validation of (C.39) is a straightforward task.

□

Finally let us discuss how to deduce $[G'; p_0, 2n]$ from Lemma C.3. Since the estimates (1.2.161) still holds, we can deduce the following estimates for $\tilde{R}_{2n}^{(p_0+1)}(0,\rho)$ from (C.34) and (C.37) with $p = p_0 - 2$:

(C.54) $$\big|\tilde{R}_{2n}^{(p_0+1)}(0,\rho)\big| \le N_1N_0C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^{n},$$

where

$$N_1 = M(C_0 + N_0 + R^{-1} + (N_0 A)^{-1}) \tag{C.55}$$

with a positive constant M that is independent of C_0, N_0, R and A. Since (1.2.163) and (1.2.177) also hold, we obtain the following estimates from (C.36) and (C.39) with $p = p_0$:

$$\big|\dot{\tilde{R}}_{2n}^{(p_0)}(0,\rho)\big| \le N_1 N_0 C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n, \tag{C.56}$$

$$\big\|\tilde{R}_{2n}^{(p_0)}\big\|_{[r_0-\varepsilon]} \le N_1 N_0 C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n. \tag{C.57}$$

Now let us define $\tilde{\Gamma}_{2n}^{(p)}$ and $\tilde{\Delta}_{2n}^{(p)}$ by

$$\tilde{\Gamma}_{2n}^{(p)} = \Gamma_{2n}^{(p)} + \big(\mathcal{F}_{2n}^{(p+3)} - \mathcal{G}_{2n}^{(p+3)}\big)\big|_{t=0}, \tag{C.58}$$

$$\tilde{\Delta}_{2n}^{(p)} = \Delta_{2n}^{(p)} + \big(\mathcal{F}_{2n}^{(p+2)\prime} - \mathcal{G}_{2n}^{(p+2)\prime}\big)\big|_{t=0}. \tag{C.59}$$

(Here we note that $\Gamma_{2n}^{(p)}$ and $\Delta_{2n}^{(p)}$ are obtained from $\Phi_{2n}^{(p+3)}\big|_{t=0}$ and $\Phi_{2n}^{(p+2)\prime}\big|_{t=0}$ respectively.) Hence, in view of (C.32), we find that what plays the role of $\Gamma_{2n}^{(p)}$ (resp., $\Delta_{2n}^{(p)}$) in this case is $\tilde{\Gamma}_{2n}^{(p)}$ (resp., $\tilde{\Delta}_{2n}^{(p)}$). Then, combining (1.2.166), we obtain the following estimates from (C.34), (C.35), (C.37) and (C.38) with $p = p_0$:

$$|\tilde{\Gamma}_{2n}^{(p_0)}|,\ |\tilde{\Delta}_{2n}^{(p_0)}| \le N_1 N_0 C(p_0)\big(R|\rho|^{-1}\big)^{p_0}(2n)!\,\varepsilon^{-2n}\big(A|\rho|^{-1}\big)^n. \tag{C.60}$$

Thus, by the same reasoning with part [III] and [IV] in the proof of Proposition 1.2.1, we find that $[G'; p, 2n]$ follows from (C.54), (C.56), (C.57) and (C.60). Therefore, the induction proceeds, and hence, we obtain Proposition C.1.

References

[A1] T. Aoki: Symbols and formal symbols of pseudodifferential operators, Advanced Studies in Pure Mathematics, **4**, Kinokuniya, 1984, pp.181–208.

[A2] ______: Calcul exponentiel des opérateurs microdifférentiels d'ordre infini. I, Ann. Inst. Fourier, Grenoble **33** (1983), 227-250.

[AKY] T. Aoki, K. Kataoka and S. Yamazaki: Hyperfunctions, FBI transformations and pseudo-differential operators of infinite order, (in Japanese) Kyoritsu-Shuppan CO., LTD, 2004.

[AKT1] T. Aoki, T. Kawai and T. Takei: The Bender-Wu analysis and the Voros theory, Special Functions, Springer-Verlag, 1991, pp.1–29.

[AKT2] ______: The Bender-Wu analysis and the Voros theory. II, Advanced Studies in Pure Mathematics, **54**, Math. Soc. Japan, 2009, pp.19–94.

[AY] T. Aoki and J. Yoshida: Microlocal reduction of ordinary differential operators with a large parameter, Publ. RIMS, Kyoto Univ., **29**(1993), 959–975.

[DP] E. Delabaere and F. Pham: Resurgent methods in semi-classical asymptotics, Ann. Inst. Henri Poincaré, **71**(1999), 1–94.

[Ec] J. Ecalle: Les fonctions résurgentes, I, II, III, Publ. Math. d'Orsay, Univ. Paris-Sud, 1981 (Tome I, II), 1985 (Tome III).

[Er] A. Erdélyi: Higher Transcendental Functions, I, II, III, McGraw-Hill, 1955; reprinted in 1981 by Robert E. Krieger Publishing Company, Malabar, Florida.

[KKKoT] S. Kamimoto, T. Kawai, T. Koike and Y. Takei: On the WKB theoretic structure of the Schrödinger operators with a merging pair of a simple pole and a simple turning point, Kyoto J. Math., **50** (2010), 101–164.

[KKT] S. Kamimoto, T. Kawai and Y. Takei: Microlocal analysis of fixed singularities of WKB solutions of a Schrödinger equation with a merging triplet of two simple poles and a simple turning point, to appear in The Mathematical Legacy of Leon Ehrenpreis, 1930 - 2010.

[K^3] M. Kashiwara, T. Kawai, T. Kimura: Foundations of Algebraic Analysis, Princeton University Press, Princeton, 1986.

[K] T. Kawai: Systems of linear differential equations of infinite order — an aspect of infinite analysis, Proc. Symp. in Pure Math., **49**, Part 1, Amer. Math. Soc., 1989, pp.3–17.

[KKoT] T. Kawai, T. Koike and Y. Takei: On the exact WKB analysis of higher order simple-pole type operators, Adv. in Math., **228** (2011), 63–96.

[KT] ______: Algebraic Analysis of Singular Perturbation Theory, Amer. Math. Soc., 2005.

[Ko1] T. Koike: On a regular singular point in the exact WKB analysis, Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear, Kyoto Univ. Press, 2000, pp.39–54.

[Ko2] ______: On the exact WKB analysis of second order linear ordinary differential equations with simple poles, Publ. RIMS, Kyoto Univ., **36** (2000), 297–319.

[Ko3] ______: in preparation.

[KoS] T. Koike and R. Schäfke: in preparation.

[KoT] T. Koike and Y. Takei: On the Voros coefficient for the Whittaker equation with a large parameter — Some progress around Sato's conjecture in exact WKB analysis, Publ. RIMS, Kyoto Univ., **47** (2011), 375–395.

[P] F. Pham: Resurgence, quantized canonical transformations, and multi-instanton expansion, Algebraic Analysis, Vol. II, Academic Press, 1988, pp.699–726.

[SKK] M. Sato, T. Kawai and M. Kashiwara: Microfunctions and pseudo-differential equations, Lect. Notes in Math., **287**, Springer, 1973, pp.265–529.

[Sa] D. Sauzin: Resurgent functions and splitting problems, RIMS Kôkyûroku, **1493**, 2006, pp.48–117.

[SS] H. Shen and H. J. Silverstone: Observations on the JWKB treatment of the quadratic barrier, Algebraic Analysis of Differential Equations, Springer, 2008, pp.307–319.

[T] Y. Takei: Sato's conjecture for the Weber equation and transformation theory for Schrödinger equations with a merging pair of turning points, RIMS Kôkyûroku Bessatsu, **B10** (2008), 205–224.

[V] A. Voros: The return of the quartic oscillator — The complex WKB method. Ann. Inst. Henri Poincaré, **39** (1983), 211–338.