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High-speed high-accuracy computation of an infinite integral with unbounded and oscillated integrand

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Abstract We propose an efficient computation method for an infinite integral $\int_0^\infty x dx/(1+x^6 \sin^2 x)$, which has an unbounded integrand with highly oscillated singularity. Computing the value of this integral has been a problem since 1984. We herein demonstrate that the method using the Hilbert transform to change this type of singular function into a smooth function and compute the value of the integral of one million or more significant digits using a superconvergent double exponential quadrature.

MSC Classification Codes: 65D32; 65B99; 65Y20; 68W05

Keywords: Numerical quadrature; Highly oscillated integral; Multiple-precision computation; Double exponential quadrature; Computational complexity

1 Introduction

The convergence of the infinite integral

$$I = \int_0^\infty \frac{x}{1 + x^6 \sin^2 x} dx \tag{1}$$

was demonstrated by Goursat [1] and Hardy [2] early in the 20th century. In the present paper, we refer to this integral as Goursat's integral. In the RIMS conference at Kyoto University in 1986, Toda of Chiba University proposed the problem of calculating not only the convergence of the integral but also the value of the integral. In 1987, Ninomiya of Nagoya University obtained an approximate value for about 20 significant digits by using an automatic quadrature routine and an acceleration method in quadruple-precision computation [3]. In 2009, Hatano, Ninomiya, Sugiura, and Hasegawa obtained an approximate value for about 73 significant digits by using a contour integral in octuple-precision computation [4].

In the present paper, we attempt to improve the algorithm used to further evaluate Goursat's integral from the viewpoint of reducing the computational complexity, and performing thorough high-speed, high-accuracy computation. Furthermore, we consider the relation between the number of digits to be computed and the required number of function evaluations.

Specifically, we consider Goursat's integral as Sato's hyperfunction, change the integral using a type of Hilbert transform, and obtain an easily computable integral using the double exponential quadrature formula (DE quadrature) [5], [7], [8].

Moreover, we propose a superconvergent DE quadrature made to correspond to this integral in particular, in order to enable more advanced high-speed, high-accuracy computation. This quadrature has high performance, and the number of function evaluations to obtain the value of the integral of N significant digits is $O(N)$. Finally, numerical integration of one million or more significant digits is performed, and the relationship between the number of digits and the computational complexity is considered.

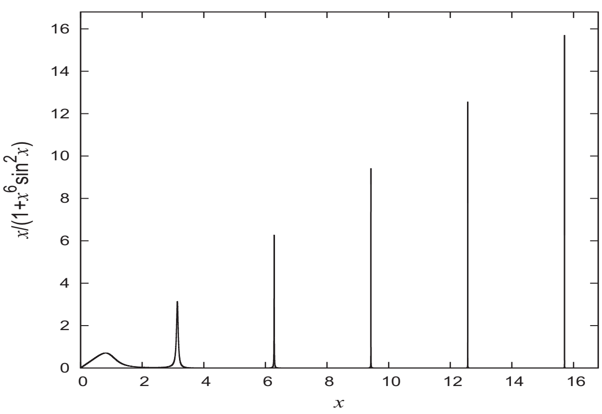


Figure 1: Integrand of Goursat's integral

2 Computational difficulty of Goursat's integral

The integrand of (1) is shown in Figure 1. We consider the case of computing the value (1) of N significant digits using a general quadrature formula. In rough analysis, this integrand has a peak of height $O(k)$ and the half width $O(1/k^3)$ near $x \approx k\pi$ $(k = 1, 2, 3, \cdots)$. The integration of interval $[0, k\pi]$ converges to true value (1) with an error of $O(1/k)$. Therefore, in order to compute N significant digits, the computation of the interval $[0, 10^N\pi]$ is required. Moreover, in this case, the mesh size of the integration must be smaller than the width of the peak. For example, when using a composite Gauss-Legendre quadrature, the endpoint of which corresponds to a peak, each integration of the interval $[(k-1)\pi, k\pi]$ needs $O(Nk^{3/2})$ function evaluations, and must be computed for $k = 1, 2, \cdots, 10^N$. In this case, the number of function evaluations to compute N significant digits is $O(N \cdot 10^{5N/2})$, and this method requires exponential time.

3 Change by Hilbert transform

First, we analyze the integrand of (1) as a complex function and find that the function has poles $p \approx 0.3 + 0.9i$, $q \approx 0.9 + 0.4i$, $\bar{p}$, $\bar{q}$, and $r_k \approx \pi k + i/(\pi k)^3$, $\bar{r}_k$ $(k = 1, 2, 3, \cdots)$ [4]. Based on the above considerations, the integral of (1) is rewritten as a complex integral:

$$I = \frac{1}{2\pi i} \int_C \varphi(z) dz,$$

$$\varphi(z) = \begin{cases} \dfrac{-\pi i z}{1 + z^6 \sin^2 z}, & \Im z > 0 \\ \dfrac{+\pi i z}{1 + z^6 \sin^2 z}, & \Im z < 0 \end{cases},$$

where the contour C passes through the origin and around $[0, +\infty)$, and no pole exists inside C. The contour C is shown in Figure 2.

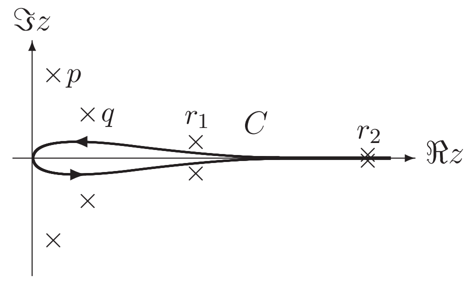


Figure 2: Contour C

Next, in order to eliminate these poles, we consider the Hilbert transform of a modified integrand of (1)

$$\psi(x) = \mathrm{pv} \int_{-\infty}^{\infty} \frac{x}{1 + x^6 \sin^2 y} \cdot \frac{dy}{x - y},$$

where pv indicates the principal value integral. This transform is analytically calculable, and we obtain

$$\psi(z) = \frac{\pi z^6 \sin z \cos z}{\sqrt{1 + z^6}} \frac{z}{1 + z^6 \sin^2 z}.$$

After analyzing the poles, we find that the poles of $\psi(z)$ and $\varphi(z)$ are located at the same position, the signs of the residues of $\psi(z)$ and $\varphi(z)$ are opposite, and $\psi(z)$ is analytic in the neighborhood of the real axis and

$$0 = \frac{1}{2\pi i} \int_C \psi(z) dz.$$

The poles are then canceled using this character:

$$I = \frac{1}{2\pi i} \int_C (\varphi(z) + \psi(z)) dz$$

and the path C can be moved from the real axis. By changing the integral into the integral on the imaginary axis and the branch of $\sqrt{1 + z^6}$, we obtain

$$\begin{aligned} I &= \int_0^{\infty} \left[\frac{t}{1 + t^6 \sinh^2 t} + \Re \frac{2(1 + \sqrt{3}i)t}{2 - t^6 + t^6 \cos((\sqrt{3} + i)t)} \right] dt \\ &+ \int_0^1 \frac{t^7}{\sqrt{1 - t^6}} \left[\frac{\sinh t \cosh t}{1 + t^6 \sinh^2 t} + \Im \frac{(1 + \sqrt{3}i) \sin((\sqrt{3} + i)t)}{2 - t^6 + t^6 \cos((\sqrt{3} + i)t)} \right] dt. \quad (2) \end{aligned}$$

The detailed derivation of (2) is shown in A.1, and the integrands of (2) are shown in Figure 3. The first term of (2) is an infinite integral of function decay $O(t^{-5} e^{-t})$ as $t \to \infty$, and the second term is the integral of a function such as $O((1 - t)^{-1/2})$ as $t \to 1$. These integrals are efficiently computed by the DE quadrature. In this case, the number of function evaluations to compute N significant digits is $O(N \log N)$. Moreover, since the second term of (2) is transformed into the integral of a smooth periodic function by the change of variable $t = \sin \theta$, high-accuracy computation is possible by applying the trapezoidal rule to this periodic integral. The second term of (2) is then calculated as follows:

$$I_2 = \frac{\pi}{2M}\left(\frac{1}{2}G(1) + \sum_{n=1}^{M-1} G\left(\sin\frac{\pi n}{2M}\right)\right) + E_{2,M} + \Delta I_{2,M},$$

where $G(t)$ is the integrand such that

$$G(t) = \frac{t^7}{\sqrt{1+t^2+t^4}}\left[\frac{\sinh t\cosh t}{1+t^6\sinh^2 t} + \Im\frac{(1+\sqrt{3}i)\sin((\sqrt{3}+i)t)}{2-t^6+t^6\cos((\sqrt{3}+i)t)}\right],$$

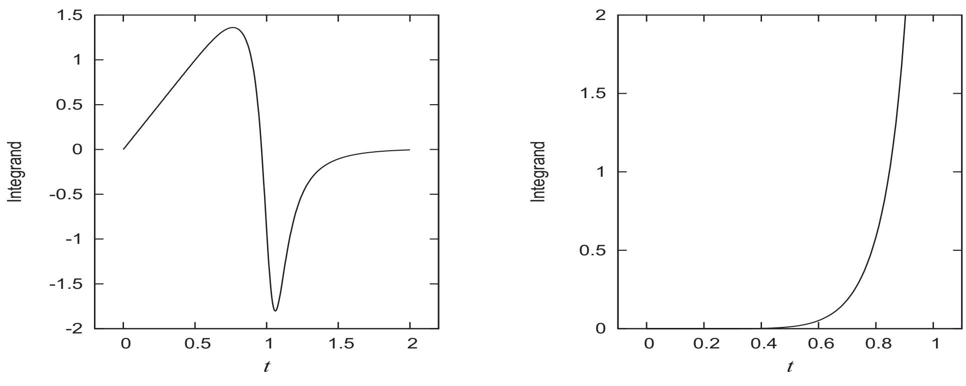
and $E_{2,M}$ is the correction term of the residues at the poles $t = ip, i\bar{p}, iq, i\bar{q}$ using the characteristic function of the error [5], [6],

$$E_{2,M} = \Im\left(\frac{-2\pi p^2 P_M}{3+ip^4\cos p} + \frac{2\pi q^2 Q_M}{3-iq^4\cos q}\right),$$

$$P_M = \frac{1}{1-(p+\sqrt{1+p^2})^{4M}} + \frac{1}{1-(p'+\sqrt{1+p'^2})^{4M}}, \quad p' = -pe^{\pi i/3},$$

$$Q_M = \frac{1}{1-(q'+\sqrt{1+q'^2})^{4M}}, \quad q' = qe^{\pi i/3},$$

and $|\Delta I_{2,M}|$ is the error term bounded by $O(e^{-3.3M})$. In this case, the number of function evaluations needed in order to compute N significant digits is $O(N)$.



First term Second term

Figure 3: Integrands of (2)

4 Superconvergent DE quadrature

Here, we propose the DE quadrature specialized for integrals (2). For simplicity, we consider the first term of (2). By changing the integral path, the first term is

$$I_1 = -\int_0^\infty \frac{t}{1+t^6\sinh^2 t}dt + \Im\frac{2\pi p^2}{3+ip^4\cos p},$$

where p is a pole $p \approx 0.3 + 0.9i$. The above equation is then rewritten as

$$I_1 = -\frac{1}{\pi i}\int_{-\infty}^\infty \frac{t\log(ict)}{1+t^6\sinh^2 t}dt + \Im\frac{2\pi p^2}{3+ip^4\cos p},$$

where $c > 0$ is a constant, and

$$I_1 = \frac{1}{\pi}\int_{-\infty}^\infty \frac{(it+T)\log(ict+cT)}{1+(it+T)^6\cosh^2 t}dt + R_K$$

is obtained by moving the integral path in the direction of $-iT$, $T = \pi(K+1/2)$, where K is a positive integer, and R_K is a term of residues at poles $-ip \approx 0.9-0.3i$, $-i\bar{p}$, $-iq \approx 0.4-0.9i$, $-i\bar{q}$, and $-ir_k, -i\bar{r}_k \approx \pm 1/(\pi k)^3 - \pi k i$ $(k = 1, 2, 3, \ldots, K)$, (see Figure 4). Setting $c = 1/T$, applying the change of variables $t = T'\sinh u$ $(T' = \pi(K-1/2))$ and the trapezoidal rule with mesh size h, we obtain

$$\begin{aligned} I_1 &= \frac{2h}{\pi}\sum_{n=1}^{M'}\Re\frac{(T+iT'\sinh nh)\log(1+i(T'/T)\sinh nh)T'\cosh nh}{1+(T+iT'\sinh nh)^6\cosh^2(T'\sinh nh)} \\ &\quad +R_K + E_{h,2K} + \Delta I_{1,h,K,M'}, \end{aligned}$$

where $E_{h,2K}$ is the correction term of the residues at the poles $T'\sinh u = iT - ip$, $iT - i\bar{p}$, $iT - iq$, $iT - i\bar{q}$, and $T'\sinh u = iT - ir_k$, $iT - i\bar{r}_k$ $(k = 1, 2, 3, \ldots, 2K)$ using the characteristic function of the error [5], [6]. $R_K + E_{h,2K}$ is calculated by

$$R_K + E_{h,2K} = \Im\frac{2\pi p^2}{3+ip^4\cos p} + \sum_{k=-1}^{2K}\Re\frac{2r_k^2\log(r_k/T)}{3-i(-1)^k r_k^4\cos r_k}\frac{1}{1-\exp(2\pi i u_k/h)},$$

$$u_k = \log\left(-i(r_k - T)/T' + \sqrt{1-(r_k-T)^2/T'^2}\right), \quad r_0 = q, r_{-1} = p.$$

Taking K, $1/h$, and M' in proportion to the number of significant digits, such that

$$K = \lceil \alpha N \rceil, \quad h = \frac{\pi^2 - 2(\Im\sqrt{p+\pi})\sqrt{2\pi/K}}{N \log 10}, \quad M' = \left\lfloor \frac{1}{h} \sinh^{-1} \frac{N \log 10}{2T'} \right\rfloor,$$

the approximation error $|\Delta I_{1,h,K,M'}|$ is bounded by

$$O(10^{-N}) = O(\exp(-C'M')) = O(\exp(-C''(M'+2K))),$$

where $0 < C' < \pi^2/\sinh^{-1}(\log 10/(2\pi\alpha))$ and $C'' = C'/(1+2\alpha)$, and we obtain the superconvergent DE quadrature and find that the number of function evaluations needed in order to compute N significant digits is $O(N)$ (including evaluation of the residues).

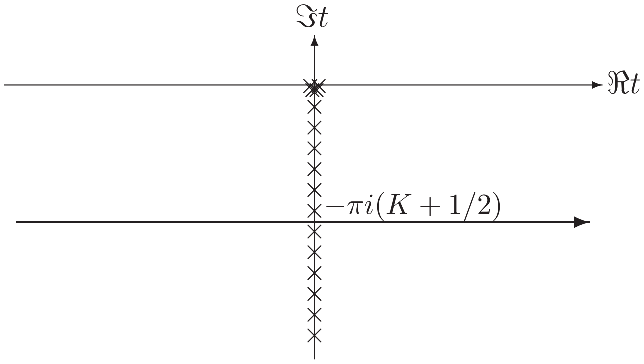


Figure 4: Integration Path

5 Generalized Goursat's integral

With respect to the generalized Goursat's integral $\displaystyle\int_0^\infty \frac{x^m}{1+x^n|\sin x|^l}dx$ derivation of the same transformation, as shown in Section 3, is possible if l is an even number and a number of conditions are satisfied. In this case, concrete $\psi(z)$ is

$$\psi(z) = \frac{\pi z^n \sin z \cos z}{\sqrt{1+z^n}} \frac{z^m}{1+z^n \sin^2 z} \qquad (l=2),$$

$$\psi(z) = \left(\frac{z^{n/2}\sin^2 z + i}{2}\sqrt{1 - iz^{n/2}} + \frac{z^{n/2}\sin^2 z - i}{2}\sqrt{1 + iz^{n/2}} \right)$$
$$\cdot \frac{\pi z^{n/2}\sin z\cos z}{\sqrt{1+z^n}} \frac{z^m}{1+z^n\sin^4 z} \qquad (l = 4), \cdots$$

In this case, the convergence is also shown automatically if a modification such as (1) to (2) is possible.

Moreover, the generalized continuous Euler transformation [9], [8] may be applicable to the more complicated oscillatory integral, to which the method of Section 3 cannot be well applied. Naturally, the generalized continuous Euler transformation is applicable to Goursat's integral, and an example is shown in A.2.

6 Computation Example

We use GMP [10] (ver. 5.0.1) & MPFR [11] (ver. 2.4.2) for the multiple-precision computation library of four arithmetic operations, and we construct a computation routine of elementary functions that combines the Taylor series expansion with the CORDIC method of the multi-bit unit. This algorithm is shown in A.3. In order to compute N significant digits using this algorithm, the number of multiplications must be approximately $O(\sqrt{N})$. Although the algorithm requires a far larger number of multiplications than the arithmetic-geometric mean algorithm, this algorithm has the advantage that a proportionality coefficient of the computational complexity can be made small according to the size of the memory.

As another computational complexity reduction method, we use the DE quadrature algorithm, which, if possible, does not include exponential function operation. The above described algorithm has two characteristics. Namely, the mesh size is selected as $h = \log 2/K'$ (where K' is a positive integer), and the step size $K'h$ with respect to the inner loop is computed. Since the $n{+}1$'th value of the double exponential function is $a_{n+1} = \exp(\exp((n+1)K'h)) = \exp(2\exp(nK'h)) = a_n^2$ at this time, high-speed computation is enabled by squaring the n'th value. However, computation of $\log 2$, a small number of computations of the exponential functions, and the memory for the array are needed for initial computation. When applying this algorithm to Goursat's integral, as compared with the usual DE quadrature, the computational complexity decreases by approximately half.

Moreover, an efficient computation method of the pole r_k (which are zeros of $i - (-1)^k z^3 \sin z$) is also required. We use the third-order Newton's method, the accuracy of which is controlled to be optimal in each iteration step. Note that the higher-order Newton's method is effective in multiple-precision computing.

The results in the computation environment for four 2.8-GHz Opteron (K10 six-core) CPUs are shown in Tables 1 and 2. The value up to 100 digits is

$$I = 1.\,16965\ 25542\ 24486\ 47772\ 59225\ 81661\ 19775\ 95884\ 81416\ 66271\\ 46180\ 73171\ 51391\ 33835\ 19905\ 81627\ 12111\ 09181\ 62126\ 67625\\ \cdots.$$

Table 2 shows that approximately 1.54 function evaluations per digit are needed and the number of function evaluations is proportional to the number of significant digits. On the other hand, the computation time is proportional to approximately the 2.6th power of the number of digits N. This reason for this is that approximately twice the number of computations of elementary functions of, e.g., log, exp, atan2, and sincos, and approximately ten multiplication computations are needed for one computation of a function.

In order to analyze the computation time in greater detail, the rate of the computation time of an elementary function and the computation time of multiplication is shown in Table 3. The computation time of built-in elementary functions of the MPFR library is also shown for comparison. This table shows that the computation time of the elementary functions is proportional to the 1.6th power of the digit number N, and the proposed algorithm is approximately 10 times as fast as the MPFR library. Moreover, the multiplication time of the GMP library is the 1.3rd power of N digits.

Next, we predict the computation time for the case in which a more ideal multiple-precision algorithm is used. The computational complexity of the Schönhage-Strassen multiplication algorithm is $O(N \log N \log\log N)$ for N digits. The computational complexity of the algorithm of the elementary function using the arithmetic-geometric mean is $O(N(\log N)^2 \log\log N)$. The computational complexity of Goursat's integral in this case is $O((N \log N)^2$ $\log\log N)$.

Table 1: Example of the Computation of Goursat's Integral

Number of Digits	Execution Time	Error
10,000 (33300 bits)	2.73 s	$3.0 \cdot 10^{-10024}$
100,000 (333,000 bits)	815.44 s (13.6 min)	$4.1 \cdot 10^{-100243}$
1,000,000 (3,330,000 bits)	464,758 s (5.4 days)	$6.0 \cdot 10^{-1002429}$

Table 2: Details of the Computation

Number of Digits	Time (I_1, I_2)	Function Evaluations (I_1, I_2)
10,000	1.36 s, 1.37 s	$8,485$ ($K = 257$), $6,995$
100,000	417.50 s, 397.94 s	$84,400$ ($K = 2,562$), $69,945$
1,000,000	262,130 s, 202,628 s	$842,626$ ($K = 25,616$), $699,449$

7 Conclusion

Goursat's integral is transformed into a smooth integral, various improvements are applied, and the value is evaluated to one million or more significant digits using the superconvergent DE quadrature. The obtained results are presented below.

1. A number of integrals with computation difficulty owing to the poles are easily computable by finding a conjugate function using the Hilbert transform.

2. A superconvergent DE quadrature for a certain type of integral is proposed.

3. A fast computation method of elementary functions is proposed that is approximately 10 times as fast as the MPFR, which is an existing high-speed library.

Table 3: Mean Execution Time (CPU Time) of log, exp, atan2, and sincos

Number of Digits	Present Method	MPFR	Multiplication Time μ
10,000	16.8μ	151μ	$\mu = 0.0641$ ms
100,000	25.1μ	243μ	$\mu = 1.71$ ms
1,000,000	65.1μ	543μ	$\mu = 26.8$ ms

4. A DE quadrature algorithm with few exponential function computations is proposed, and the integration speed is increased. This method can be applied to all types of DE quadratures and is especially effective for multiple-precision computation.

A.1 Derivation of the integral (2)

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_C (\varphi(z)+\psi(z))dz \\ &= \frac{1}{2\pi i}\lim_{n\to\infty}\left(-\int_0^{R_n e^{i\theta}}(\varphi(z)+\psi(z))dz + \int_{R_n}^{R_n e^{i\theta}}(\varphi(z)+\psi(z))dz\right) + \mathrm{cc}, \end{aligned}$$

where $0 < \theta < \pi/6$, $R_n = (n+1/2)\pi$, and cc is the complex conjugate. Since $|\varphi(z)+\psi(z)| \sim O(1/|z|^2)$ as $|z| = (n+1/2)\pi \to \infty$, the second term vanishes (see Path 1 of Figure 5), and we have

$$\begin{aligned} I &= -\frac{1}{2\pi i}\int_0^{Re^{\pi i(1-\varepsilon)/6}}(\varphi(z)+\psi(z))dz + \mathrm{cc} \\ &= -\frac{1}{2\pi i}\int_0^{Re^{\pi i(1+\varepsilon)/6}}(\varphi(z)+\psi(z))dz + \frac{1}{\pi i}\int_{e^{\pi i/6}}^{Re^{\pi i(1+\varepsilon)/6}}\psi(z)dz + \mathrm{cc} \\ &= -\frac{1}{2\pi i}\int_0^{R(i+\varepsilon)}(\varphi(z)+\psi(z))dz + \frac{1}{\pi i}\int_{e^{\pi i/6}}^{Re^{\pi i(1+\varepsilon)/6}}\psi(z)dz + \mathrm{cc}, \end{aligned}$$

where the symbol $\lim_{R\to+\infty}\lim_{\varepsilon\to+0}$ is omitted and

$$\begin{aligned} \int_{e^{\pi i/6}}^{Re^{\pi i(1+\varepsilon)/6}}\psi(z)dz &= \int_{e^{\pi i/6}}^{Re^{\pi i(1+\varepsilon)/6}}(\varphi(z)+\psi(z))dz - \int_{e^{\pi i/6}}^{Re^{\pi i/6}}\varphi(z)dz \\ &= \left[\int_{e^{\pi i/6}}^{0} + \int_0^{R(i+\varepsilon)}\right](\varphi(z)+\psi(z))dz - \int_{e^{\pi i/6}}^{Re^{\pi i/6}}\varphi(z)dz \end{aligned}$$

(see Path 2 of Figure 5). Considering that $\frac{1}{2\pi i}\int_i^{R(i+\varepsilon)}\psi(z)dz$ is a pure imaginary number, we have

$$I = \frac{1}{2\pi i}\left[\int_0^{Ri} -2\int_0^{Re^{\pi i/6}}\right]\varphi(z)dz + \frac{1}{2\pi i}\left[\int_0^{i} -2\int_0^{e^{\pi i/6}}\right]\psi(z)dz + \mathrm{cc}.$$

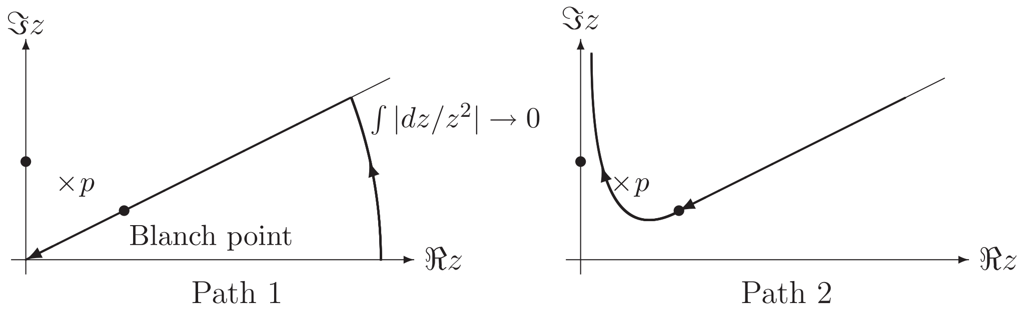


Figure 5: Integration Path

A.2 Computation of Goursat's integral using generalized continuous Euler transformation

We introduce a computation method for complicated oscillating integrals using generalized continuous Euler transformation [9]. An oscillating and logarithm converging integral

$$I = \int_0^\infty f(x)dx$$

is transformed by generalized continuous Euler transformation into

$$I_L = \int_0^L f(x)w_L(x)dx, \quad L > 0,$$

where w_L is a weight function written by the Hermite function

$$h_n(x) = (-1)^n \frac{d^n}{dx^n} e^{-x^2/2}, \quad h_{-1}(x) = \int_x^\infty e^{-t^2/2}dt = \sqrt{\frac{\pi}{2}}\mathrm{erfc}(x/\sqrt{2})$$

as

$$w_L(x) = \sum_{n=0}^{M} \frac{2^n(x+\alpha)^n}{\sqrt{2\pi}n!(\sigma^2 L)^{n/2}} h_{n-1}((2x-L)/(\sigma L^{1/2})), \qquad \alpha, \sigma > 0.$$

At this time, the highly accurate approximation $I_L \approx I$ is obtained for a very small value of L by choosing L and M according to the characteristics of f.

Since this transformation is a simple operation of multiplying a weight function and truncating in the finite interval, the transformation is effective for various oscillating integrals. When the parameters are chosen appropriately, the approximation error is $O(\exp(-CL))$, $C > 0$ as $L \to \infty$, and the slowly converging integral can be regarded as an exponential converging integral. However, high-precision approximation of I_L cannot be expected if the characteristics of f, such as the oscillating cycle, and the order of convergence are not reflected by w_L through a parameter.

We present a computation example of Goursat's integral (1) using this transformation. We set

$$g(x) = \sum_{k=0}^{\lfloor L/\pi \rfloor} f(x+\pi k) w_L(x+\pi k), \quad f(x) = \frac{x}{1 + x^6 \sin^2 x}$$

and integrate $g(x)$ over the interval $[\,0, \pi\,]$ using the DE quadrature and obtain I_L. When the required number of digits is N, choosing the parameters

$$L = 9N, \quad M = \lfloor 3N/(2 + \log N) \rfloor, \quad \sigma = 1, \quad \alpha = 1/2$$

is easily computable. However, since w_L will oscillate greatly if M is enlarged, the cancellation of significant digits should be performed carefully. The number of function evaluations for this method is $O((N \log N)^2)$.

A.3 Fast computation method of elementary functions in multiple-precision computation

We present an example that combines the CORDIC method of an R-bit unit K step and Taylor series expansion. First, we present the computation method for the exponential and logarithmic function.

Initial computation:
$L(j,k) := \log(1 + j/2^{Rk}) \quad (j = 1, 2, \cdots, 2^R - 1, k = 1, 2, \cdots K)$

Algorithm for the exponential and logarithmic function:

Computation of $y = \exp x$
$(0 \leq x < \log 2)$:

for $k = 1, 2, \cdots, K$, do
  $m_k := 0$
  for $r = R-1, R-2, \cdots, 0$, do
    $j := m_k + 2^r$
    if $x \geq L(j,k)$ then $m_k := j$
  end for
  if $m_k > 0$ then $x := x - L(m_k, k)$
end for
$y := \exp x$ (*1)
for $k = 1, 2, \cdots, K$, do
  if $m_k > 0$ then $y := y + y * m_k/2^{Rk}$
end for

Computation of $y = \log x$
$(1/2 < x \leq 1)$:

$t := 1 - x$, $y := 0$
for $k = 1, 2, \cdots, K$, do
  $x := 1 - t$, $m := 0$
  for $r = R-1, R-2, \cdots, 0$, do
    if $x/2^{Rk-r} \leq t$ then   (*2)
      $t := t - x/2^{Rk-r}$
      $m := m + 2^r$
    end if
  end for
  if $m > 0$ then $y := y - L(m,k)$
end for
$y := y + \log(1-t)$ (*3)

(*1) and (*3) are computed by the Taylor series considering $0 \leq x < 2^{-RK}$. The computation of $y * m_k/2^{Rk}$ is accelerated using bit operations and integer multiplication. In order to accelerate the inner loop (*2), low-accuracy comparison is required, except for border values. Furthermore, in order to accelerate Taylor series of log, an expansion $\log(1-t) = -2\tanh^{-1}(t/(2-t))$, $\tanh^{-1} x = x + x^3/3 + x^5/5 + x^7/7 + \cdots$ is used.

Next, we present the computation method of the trigonometric and inverse trigonometric function.

Initial computation:

$T(j,k) := \tan^{-1}(j/2^{Rk}) \quad (j = 1, 2, \cdots, 2^R - 1, k = 1, 2, \cdots K)$

Algorithm for the trigonometric and inverse trigonometric function:

Computation of $y = \sin x$, $z = \cos x$ $(0 \le x < \pi/4)$:
$x := x/2$
for $k = 1, 2, \cdots, K$, do
$m_k := 0$
for $r = R-1, R-2, \cdots, 0$, do
$j := m_k + 2^r$
if $x \ge T(j,k)$ then $m_k := j$
end for
if $m_k > 0$ then $x := x - T(m_k, k)$
end for
$y := \sin x$, $z := \cos x$
for $k = 1, 2, \cdots, K$, do
if $m_k > 0$ then
$t := z * m_k/2^{Rk}$, $u := y * m_k/2^{Rk}$
$y := y + t$, $z := z - u$
end if
end for
$t := 2 * y/(y^2 + z^2)$, $u := y$
$y := z * t$, $z := 1 - u * t$

Computation of $z = \tan^{-1}(x/y)$ $(0 \le \arg(y + ix) < \pi/4)$:
$z := 0$
for $k = 1, 2, \cdots, K$, do
$t := x$, $m := 0$
for $r = R-1, R-2, \cdots, 0$, do
if $y/2^{Rk-r} \le t$ then
$t := t - y/2^{Rk-r}$, $m := m + 2^r$
end if
end for
if $m > 0$ then
$t := y * m/2^{Rk}$, $u := x * m/2^{Rk}$
$x := x - t$, $y := y + u$
$z := z + T(m, k)$
end if
end for
$z := z + \tan^{-1}(x/y)$

Finally, we present a technique for high-speed Taylor series computation for multiple-precision computation. For simplicity, a series by which to compute is $y = \sum_{n=0}^{M} x^n/a_n$, where a_n are integers, is assumed. Then, we choose positive integers M' and P, such that $M < M'2^P$.

High-speed Taylor series computation in multi-precision computation:

Computation of $y = \sum_{n=0}^{M'2^P} x^n/a_n$:
for $k = 0, 1, \cdots, 2^P - 1$, do $s_k := 1/a_k$
$t := 1$, $u := x^{2^P}$, $y := 0$
for $m = 1, 2, \cdots, M' - 1$, do
$t := t * u$
for $k = 0, 1, \cdots, 2^P - 1$, do $s_k := s_k + t/a_{k+m2^P}$
end for
for $k = 2^P - 1, 2^P - 2, \cdots, 0$, do $y := y * x + s_k$

This algorithm contains operations of $M' + 2^P + P - 3$ multiplications, $M'2^P$ integer divisions and a number of additions. If we choose $M' \approx \sqrt{M}$ approximately, then the number of multiplications is reduced to $O(\sqrt{M})$ operations.

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