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BAR CONSTRUCTION AND TANNAKIZATION

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Abstract. We continue our study of tannakizations of symmetric monoidal stable ∞-categories, begun in [17]. The issue treated in this paper is the calculation of tannakizations of examples of symmetric monoidal stable ∞-categories with fiber functors. We consider the case of symmetric monoidal ∞-categories of perfect complexes on perfect derived stacks. The first main result especially says that our tannakization includes the bar construction for an augmented commutative ring spectrum and its equivariant version as a special case. We apply it to the study of the tannakization of the stable infinity-category of mixed Tate motives over a perfect field. We prove that its tannakization can be obtained from the $\mathbb{G}_m$-equivariant bar construction of a commutative differential graded algebra equipped with $\mathbb{G}_m$-action. Moreover, under Beilinson-Soulé vanishing conjecture, we prove that the underlying group scheme of the tannakization is the motivic Galois group for mixed Tate motives, constructed in [4], [21], [22].

1. Introduction

In [17] we have constructed tannakizations of stable symmetric monoidal ∞-categories. Let R be a commutative ring spectrum. Let $\mathcal{C}^{\otimes}$ be an R-linear small symmetric monoidal stable idempotent-complete ∞-category, equipped with an R-linear symmetric monoidal exact functor $F : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ where $\mathrm{PMod}_R^{\otimes}$ denotes the symmetric monoidal ∞-category of compact R-spectra. (Despite we use the machinery of quasi-categories in the text, by an ∞-category we informally mean an $(\infty, 1)$-category in this introduction.) In loc. cit., given $F : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ we construct a derived affine group scheme G over R, which is an analogue of an affine group scheme in derived algebraic geometry [34], [25]. The derived affine group scheme G comes equipped with action on F which is universal among all actions of derived affine group schemes. We call it the tannakization of $F : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$. This construction was applied to the ∞-category of mixed motives to obtain derived motivic Galois group.

The purpose of this paper is to calculate tannakizations of some examples of $F : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$; our principal interest here is the case when $\mathcal{C}^{\otimes}$ is the symmetric monoidal ∞-category $\mathrm{PMod}_Y^{\otimes}$ of perfect complexes on a derived stack Y and F is induced by $\operatorname{Spec} R \to Y$. We will study the tannakization under the assumption of perfectness on derived stacks, introduced in [1], which particularly includes two cases:

- (i) Y is an affine derived scheme over R, that is, $Y = \operatorname{Spec} A$ over $\operatorname{Spec} R$ with A a commutative ring spectrum,
- (ii) Y is the quotient stack $[X/G]$ where X is an affine derived scheme $X = \operatorname{Spec} A$ and G is an algebraic group in characteristic zero.

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We note that for our purpose the assumption of affineness on Y in (i) and X in (ii) is not essential since $\mathrm{PMod}_Y^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ depends only on a Zariski neighborhood of the image of $\mathrm{Spec}\, R \to Y$. Also, we remark that A in (i) and (ii) can be nonconnective. Our result may be expressed as follows (cf. Theorem 4.9, Corollary 4.10):

Theorem 1. *Let Y be a derived stack over R and $\mathrm{Spec}\, R \to Y$ a section of the structure map $Y \to \mathrm{Spec}\, R$. Let $\mathrm{PMod}_Y^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ be the associated pullback symmetric monoidal functor. Suppose that Y is perfect (the cases (i) and (ii) satisfy this property). Let G be the derived affine group scheme arising from Čech nerve associated to $\mathrm{Spec}\, R \to Y$. Then the tannakization of the R-linear symmetric monoidal functor $\mathrm{PMod}_Y^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ is equivalent to G.*

Bar construction and equivariant bar construction. One of our motivations of this paper arises from comparison between derived group schemes obtained by tannakization and bar constructions and its variants. Bar construction has been an important device in various contexts of homotopy theory, mixed Tate motives and non-abelian Hodge theory, etc. In the case (i), Čech nerve in Aff_R associated to $\mathrm{Spec}\, R \to Y = \mathrm{Spec}\, A$, which we can regard as a derived affine group scheme over R, is known as the bar construction of an augmented commutative ring spectrum (or commutative differential graded algebra) whose explicit construction can be given by bar resolutions. In the case (ii), we can think of the Čech nerve as the G-equivariant version of the bar construction. As a matter of fact, our actual aim is to study a relationship between our tannakization and bar constructions and its equivariant versions; Theorem 1 especially means that our method of tannakizations includes bar constructions and the equivariant versions as a special case. This allows one to link bar constructions and the variants to more general method of tannakizations.

Mixed Tate motives. It would be worth mentioning that the equivariant versions are also important to applications to the motivic contexts: for instance, in order to take weight structures into account, one often uses $\mathbb{G}_m$-equivariant version of bar construction. Our results fit very naturally in with the structure of mixed Tate motives. In Section 6 and 7, we will study the applications to mixed Tate motives. Let $\mathsf{DM}^{\otimes} := \mathsf{DM}^{\otimes}(k)$ be the symmetric monoidal stable ∞-category of mixed motives over a base scheme $\mathrm{Spec}\, k$, where k is a perfect field (see Section 6.1 for our convention). We work with coefficients of a field $\mathbf{K}$ of characteristic zero; all stable ∞-categories are $H\mathbf{K}$-linear, where $H\mathbf{K}$ denotes the Eilenberg-MacLane spectrum. Let $\mathsf{DTM}_{\vee}^{\otimes} \subset \mathsf{DM}^{\otimes}$ be the small symmetric monoidal stable ∞-category of mixed Tate motives which admit duals (see Section 6.2). For a mixed Weil cohomology theory (such as étale cohomology, de Rham cohomology), there exists a homological realization functor $\mathsf{R}_T : \mathsf{DTM}_{\vee}^{\otimes} \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$, that is a $H\mathbf{K}$-linear symmetric monoidal exact functor (the field of coefficients $\mathbf{K}$ depends on the choice of a mixed Weil cohomology theory). By applying the above theorem, we deduce Theorem 6.11 which informally says:

Theorem 2. *Let $\mathsf{MTG} = \mathrm{Spec}\, B$ be the tannakization of $\mathsf{R}_T : \mathsf{DTM}_{\vee}^{\otimes} \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$. (Here B is a commutative differential graded $\mathbf{K}$-algebra.) Then MTG is obtained from the $\mathbb{G}_m$-equivariant bar construction of a commutative differential graded $\mathbf{K}$-algebra $\overline{Q}$ equipped with $\mathbb{G}_m$-action. Namely, it is the Čech nerve of a morphism of derived stacks $\mathrm{Spec}\, H\mathbf{K} \to [\mathrm{Spec}\, \overline{Q}/\mathbb{G}_m]$.*

We remark that the underlying complex $\overline{Q}$ can be described in terms of Bloch's cycle complexes. The proof of Theorem 2 consists of two keys; one is Theorem 1, and another is to identify $\mathsf{R}_T : \mathsf{DTM}^{\otimes}_{\vee} \to \mathrm{PMod}^{\otimes}_{H\mathbf{K}}$ with a certain pullback functor between ∞-categories of perfect complexes on derived stacks, which makes use of the module-theoretic (i.e. Morita-theoretic) presentation theorem of the stable ∞-category $\mathsf{DTM}^{\otimes}_{\vee}$, see [31].

If Beilinson-Soulé vanishing conjecture holds for the base field k (e.g. k is a number field), there is a traditional line passing to a group scheme. Under the vanishing conjecture, one can define the motivic t-structure on $\mathsf{DTM}_{\vee}$. The heart of this t-structure is a neutral Tannakian category (cf. [30], [9]), and we can extract an affine group scheme MTG over $\mathbf{K}$ from it. The so-called motivic Galois group for mixed Tate motives MTG is constructed notably by Bloch-Kriz, Kriz-May, Levine [4], [21], [22]. The vanishing conjecture does not imply that the stable ∞-category of complexes of the heart recovers the original ∞-category $\mathsf{DTM}_{\vee}$. However, we can describe a quite nice relation between MTG and MTG:

Theorem 3. *Suppose that Beilinson-Soulé vanishing conjecture holds for k. Then the group scheme MTG is the underlying group scheme (cf. Definition 7.14) of* MTG.

This result is proved in the final Section; Theorem 7.15. Roughly speaking, the underlying group scheme of MTG is obtained by truncating higher homotopy groups of valued points of MTG. In view of Theorem 2 and 3, we can say that the derived motivic Galois group constructed from $\mathsf{DM}^{\otimes}$ in [17] is a natural generalization of MTG to the whole mixed motives.

This paper is organized as follows: In Section 2, we will review some of notions and notation which we need in this paper. In Section 3, after preparing an appropriate setup we clarify the meaning of action of a derived affine group scheme on a symmetric monoidal functor $F : \mathcal{C}^{\otimes} \to \mathrm{PMod}^{\otimes}_R$. More precisely, we show that giving an extension of F to $\mathcal{C}^{\otimes} \to \mathrm{PMod}^{\otimes}_G$ is equivalent to giving an action of G on F, where $\mathrm{PMod}^{\otimes}_G$ is the symmetric monoidal ∞-category of perfect representations of G defined in Section 3. Section 4 contains the proof of Theorem 1. In Section 5, we give a brief exposition of bar constructions from our viewpoint. Sections 6 and 7 are devoted to the study of the tannakization of stable ∞-category of mixed Tate motives; we prove Theorem 2 and 3.

2. Notation and Convention

We fix notation and convention.

∞-categories. In this paper, we use theory of quasi-categories as in [17]. A quasi-category is a simplicial set which satisfies the weak Kan condition of Boardman-Vogt: A quasi-category S is a simplicial set such that for any $0 < i < n$ and any diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & S \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

of solid arrows, there exists a dotted arrow filling the diagram. Here Λ^n_i is the i-th horn and Δ^n is the standard n-simplex. Following [23] we shall refer to quasi-categories as

∞-categories. Our main references are [23] and [24] (see also [18], [25]). We often refer to a map $S \to T$ of ∞-categories as a functor. We call a vertex in an ∞-category S (resp. an edge) an object (resp. a morphism). For the rapid introduction to ∞-categories, we refer to [23, Chapter 1], [12], [11, Section 2]. For the quick survey on various approaches to $(\infty, 1)$-categories and their relations, we refer to [2].

- Δ: the category of linearly ordered finite sets (consisting of $[0], [1], \dots, [n] = \{0, \dots, n\}, \dots$)
- Δ^n: the standard n-simplex
- N: the simplicial nerve functor (cf. [23, 1.1.5])
- $\mathcal{C}^{op}$: the opposite ∞-category of an ∞-category $\mathcal{C}$
- Let $\mathcal{C}$ be an ∞-category and suppose that we are given an object c. Then $\mathcal{C}_{c/}$ and $\mathcal{C}_{/c}$ denote the undercategory and overcategory respectively (cf. [23, 1.2.9]).
- Cat_∞: the ∞-category of small ∞-categories in a fixed universe (cf. [23, 3.0.0.1])
- $\widehat{\mathrm{Cat}}_\infty$: ∞-category of ∞-categories
- $\mathcal{S}$: ∞-category of small spaces (cf. [23, 1.2.16])
- $\mathrm{h}(\mathcal{C})$: homotopy category of an ∞-category (cf. [23, 1.2.3.1])
- $\mathrm{Fun}(A, B)$: the function complex for simplicial sets A and B
- $\mathrm{Fun}_C(A, B)$: the simplicial subset of $\mathrm{Fun}(A, B)$ classifying maps which are compatible with given projections $A \to C$ and $B \to C$.
- $\mathrm{Map}(A, B)$: the largest Kan complex of $\mathrm{Fun}(A, B)$ when A and B are ∞-categories,
- $\mathrm{Map}_{\mathcal{C}}(C, C')$: the mapping space from an object $C \in \mathcal{C}$ to $C' \in \mathcal{C}$ where $\mathcal{C}$ is an ∞-category. We usually view it as an object in $\mathcal{S}$ (cf. [23, 1.2.2]).

Stable ∞-categories, symmetric monoidal ∞-categories and spectra. For the definitions of (symmetric) monoidal ∞-categories and ∞-operads, their algebra objects, we shall refer to [24]. The theory of stable ∞-categories is developed in [24, Chapter 1]. We list some of notation.

- $\mathbb{S}$: the sphere spectrum
- Sp: ∞-category of spectra, we denote the smash product by $\otimes$
- PSp the full subcategory of Sp spanned by compact spectra
- Mod_A: ∞-category of A-module spectra for a commutative ring spectrum A
- PMod_A: the full subcategory of Mod_A spanned by compact objects (in Mod_A, an object is compact if and only if it is dualizable, see [1]) . We refer to objects in PMod_A as perfect A-module (spectra).
- Fin_*: the category of pointed finite sets $\langle 0\rangle_* = \{*\}, \langle 1\rangle_* = \{1, *\}, \dots, \langle n\rangle_* = \{1 \dots, n, *\}, \dots$. A morphism is a map $f : \langle n\rangle_* \to \langle m\rangle_*$ such that $f(*) = *$. Note that f is not assumed to be order-preserving.
- Let $\mathcal{M}^\otimes \to \mathcal{O}^\otimes$ be a fibration of ∞-operads. We denote by $\mathrm{Alg}_{/\mathcal{O}^\otimes}(\mathcal{M}^\otimes)$ the ∞-category of algebra objects (cf. [24, 2.1.3.1]). We often write $\mathrm{Alg}(\mathcal{M}^\otimes)$ or $\mathrm{Alg}(\mathcal{M})$ for $\mathrm{Alg}_{/\mathcal{O}^\otimes}(\mathcal{M}^\otimes)$. Suppose that $\mathcal{P}^\otimes \to \mathcal{O}^\otimes$ is a map of ∞-operads. $\mathrm{Alg}_{\mathcal{P}^\otimes/\mathcal{O}^\otimes}(\mathcal{M}^\otimes)$: ∞-category of $\mathcal{P}$-algebra objects.
- $\mathrm{CAlg}(\mathcal{M}^\otimes)$: ∞-category of commutative algebra objects in a symmetric monoidal ∞-category $\mathcal{M}^\otimes \to \mathrm{N}(\mathrm{Fin}_*)$.
- CAlg_R: ∞-category of commutative algebra objects in the symmetric monoidal ∞-category $\mathrm{Mod}_R^\otimes$ where R is a commutative ring spectrum. When $R = \mathbb{S}$, we set $\mathrm{CAlg} = \mathrm{CAlg}_{\mathbb{S}}$.

- $\mathrm{Mod}_A^{\otimes}(\mathcal{M}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$: symmetric monoidal ∞-category of A-module objects, where $\mathcal{M}^{\otimes}$ is a symmetric monoidal ∞-category such that (1) the underlying ∞-category admits a colimit for any simplicial diagram, and (2) its tensor product functor $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ preserves colimits of simplicial diagrams separately in each variable. Here A belongs to $\mathrm{CAlg}(\mathcal{M}^{\otimes})$. cf. [24, 3.3.3, 4.4.2].

Let $\mathcal{C}^{\otimes}$ be the symmetric monoidal ∞-category. We usually denote, dropping the subscript $\otimes$, by $\mathcal{C}$ its underlying ∞-category. We say that an object X in $\mathcal{C}$ is dualizable if there exist an object $X^{\vee}$ and two morphisms $e : X \otimes X^{\vee} \to \mathbf{1}$ and $c : \mathbf{1} \to X \otimes X^{\vee}$ with $\mathbf{1}$ a unit such that the composition

$$X \stackrel{\mathrm{Id}_X \otimes c}{\longrightarrow} X \otimes X^{\vee} \otimes X \stackrel{e \otimes \mathrm{Id}_X}{\longrightarrow} X$$

is equivalent to the identity, and

$$X^{\vee} \stackrel{c \otimes \mathrm{Id}_{X^{\vee}}}{\longrightarrow} X^{\vee} \otimes X \otimes X^{\vee} \stackrel{\mathrm{Id}_{X^{\vee}} \otimes e}{\longrightarrow} X^{\vee}$$

is equivalent to the identity. The symmetric monoidal structure of $\mathcal{C}$ induces that of the homotopy category $\mathrm{h}(\mathcal{C})$. If we consider X to be an object also in $\mathrm{h}(\mathcal{C})$, then X is dualizable in $\mathcal{C}$ if and only if X is dualizable in $\mathrm{h}(\mathcal{C})$. For example, for $R \in \mathrm{CAlg}$, compact and dualizable objects coincide in the symmetric monoidal ∞-category $\mathrm{Mod}_R^{\otimes}$ (cf. [1]).

Let us recall the symmetric monoidal ∞-categories $\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}}$ and $\mathrm{Cat}_{\infty}^{\mathrm{st}}$ (see [17, Section 3.2], [1], [24] for details). Let $\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}}$ be the subcategory of $\widehat{\mathrm{Cat}}_{\infty}$ spanned by stable presentable ∞-categories, in which morphisms are functors which preserves small colimits. For $\mathcal{C}, \mathcal{D} \in \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}}$, $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ is defined to be the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by functors which preserves small colimits. Then $\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}}$ has a symmetric monoidal structure $\otimes : \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}} \times \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}} \to \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}}$ such that for $\mathcal{C}, \mathcal{D}, \in \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}}$, there exists a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$, which induces an equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ for every $\mathcal{E} \in \widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}}$, where the right hand side indicates the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ spanned by functors which preserves small colimits separately in each variable. A unit is equivalent to Sp. Let $\mathrm{Cat}_{\infty}^{\mathrm{st}}$ denote the subcategory of Cat_{∞} which consists of small stable idempotent-complete ∞-categories. Morphisms in $\mathrm{Cat}_{\infty}^{\mathrm{st}}$ are functors that preserve finite colimits, that is, exact functors. There is a symmetric monoidal structure on $\mathrm{Cat}_{\infty}^{\mathrm{st}}$. For $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_{\infty}^{\mathrm{st}}$ the tensor product $\mathcal{C} \otimes \mathcal{D}$ has the following universality: There is a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ which preserves finite colimits separately in each variable, such that if $\mathcal{E} \in \mathrm{Cat}_{\infty}^{\mathrm{st}}$ and $\mathrm{Fun}_{fc}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ spanned by functors which preserve finite colimits separately in each variable, then the composition induces a categorical equivalence

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \to \mathrm{Fun}_{fc}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

where $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$ is the full subcategory of $\mathrm{Fun}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E})$ spanned by exact functors. A unit is equivalent to PSp. An object (resp. a morphism) in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})$ can be regarded as a symmetric monoidal stable presentable ∞-category whose tensor operation preserves small colimits separately in each variable (resp. a symmetric monoidal functor which preserves small colimits). Similarly, an object (resp. a morphism) in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)$ can be regarded as a symmetric monoidal small stable idempotent-complete ∞-category whose tensor opearation preserves finite colimits separately in each variable (resp. a symmetric monoidal functor which preserves finite colimits). See [17, Section 3.2]. If R is a commutative ring spectrum, we refer to an object in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_\infty)_{\mathrm{Mod}^{\otimes}_R/}$ (resp. $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}})_{\mathrm{PMod}^{\otimes}_R/}$) simply as an R-linear symmetric monoidal stable presentable ∞-category (resp. an R-linear symmetric monoidal small stable idempotent-complete ∞-category) . We refer to morphisms in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_\infty)_{\mathrm{Mod}^{\otimes}_R/}$ (or $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}})_{\mathrm{PMod}^{\otimes}_R/}$) as R-linear symmetric monoidal functors.

3. Derived group schemes and the ∞-categories of representations

In this Section we first recall the definitions of ∞-categories of representations of derived affine group schemes and the tannakization of symmetric monoidal stable idempotent-complete ∞-categories. The aim of this Section is to prove Proposition 3.4 and Corollary 3.7.

3.1. **Derived affine group scheme G and ∞-categories Mod_G and PMod_G.** We refer to [17, Appendix, Section 3.1] for the basic definitions concerning derived group schemes. Let R be a commutative ring spectrum. Let G be a derived affine group scheme over R. This can be viewed as a group object $\psi : \mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R := (\mathrm{CAlg}_R)^{op}$ (see [17, Definition A.2]). In this paper, we refer to an object in Aff_R as an affine (derived) scheme over R and call Aff_R the ∞-category of affine (derived) schemes over R. From Grothendick's viewpoint of "functor of points", a derived affine group scheme over R is a functor $(\mathrm{Aff}_R)^{op} \to \mathrm{Grp}(\mathcal{S})$ such that the composite $(\mathrm{Aff}_R)^{op} \to \mathcal{S}$ with the forgetful functor $\mathrm{Grp}(\mathcal{S}) \to \mathcal{S}$ is represented by an affine scheme, where $\mathrm{Grp}(\mathcal{S})$ is the ∞-category of group objects in $\mathcal{S}$. We will recall the definition of the symmetric monoidal ∞-category $\mathrm{Mod}^{\otimes}_G$. Set $G = \mathrm{Spec}\, B$ so that B is a commutative Hopf ring spectrum over R which is described by a cosimplicial object $\phi := \psi^{op} : \mathrm{N}(\Delta) \to \mathrm{CAlg}_R$. We here abuse notation and B indicates also the the underlying object $\phi([1])$ in CAlg_R. Let

$$\Theta : \mathrm{CAlg} \longrightarrow \mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_\infty)$$

be a functor which carries $A \in \mathrm{CAlg}$ to the symmetric monoidal ∞-category Mod_A and sends a map $A \to A'$ in CAlg to a colimit-preserving symmetric monoidal base change functor $\mathrm{Mod}_A \to \mathrm{Mod}'_A : M \mapsto M \otimes_A A'$ (see [17, section 3.3]). This functor induces

$$\Theta_R : \mathrm{CAlg}_R \simeq \mathrm{CAlg}_{R/} \longrightarrow \mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_\infty)_{\mathrm{Mod}^{\otimes}_R/}.$$

Consider the composition $\mathrm{N}(\Delta) \stackrel{\phi}{\to} \mathrm{CAlg}_R \stackrel{\Theta_R}{\to} \mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_\infty)_{\mathrm{Mod}^{\otimes}_R/}$. We define $\mathrm{Mod}^{\otimes}_G$ to be a limit of this composition. We call it the ∞-category of representations of G. The underlying ∞-category is stable and presentable. Since the forgetful functor $\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_\infty)_{\mathrm{Mod}^{\otimes}_R/} \to \widehat{\mathrm{Cat}}_\infty$ is limit-preserving, we see that the underlying ∞-category of $\mathrm{Mod}^{\otimes}_G$, which we denote by Mod_G, is a limit of the composition $\mathrm{N}(\Delta) \stackrel{\Theta_R \circ \phi}{\longrightarrow} \mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_\infty)_{\mathrm{Mod}^{\otimes}_R/} \to \widehat{\mathrm{Cat}}_\infty$. There is the natural symmetric monoidal functor $\mathrm{Mod}^{\otimes}_G \to$

$\mathrm{Mod}_R^{\otimes}$ and we let $\mathrm{PMod}_G^{\otimes}$ the inverse image of the full subcategory $\mathrm{PMod}_R^{\otimes}$. Alternatively, there is a natural categorical equivalence $\mathrm{PMod}_G \simeq \lim_{[n]\in\Delta}\mathrm{PMod}_{\phi([n])}$ and $\mathrm{PMod}_G^{\otimes}$ is a symmetric monoidal full subcategory of $\mathrm{Mod}_G^{\otimes}$ spanned by dualizable objects. We call it the ∞-category of perfect representations of G.

3.2. **∞-categories of modules over presheaves.** Let $(\mathrm{CAlg}_R)^{op} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ be Yoneda embedding, where $\widehat{\mathcal{S}}$ denotes the ∞-category of (not necessarily small) spaces, i.e. Kan complexes. We shall refer to objects in $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ as presheaves on CAlg_R or simply functors. By left Kan extension of Θ_R, we have a colimit-preserving functor

$$\overline{\Theta}_R : \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}}) \to (\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)_{\mathrm{Mod}_R^{\otimes}/})^{op}.$$

Let $\mathrm{N}(\Delta)^{op} \stackrel{\phi^{op}}{\to} (\mathrm{CAlg}_R)^{op} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ be the composition and let $\mathsf{B}G$ denote the colimit. Remember $\overline{\Theta}_R(\mathsf{B}G) = \mathrm{Mod}_{\mathsf{B}G}^{\otimes} \simeq \mathrm{Mod}_G^{\otimes}$ (we hope that our notation give rise to no confusion). Note that the notation $\mathsf{B}G$ conflicts with the notation $\mathsf{B}G$ in [17]. In [17], we define $\mathsf{B}G$ to be the étale sheafification of the colimit of $\mathrm{N}(\Delta)^{op} \stackrel{\phi^{op}}{\to} (\mathrm{CAlg}_R)^{op} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$. However, this confliction induces no difference on the images of $\overline{\Theta}_R$: By the flat descent theory of modules on CAlg (cf. [25, VII Section 6, VIII 2.7.14]), if $P \to P'$ is a fpqc (or étale) sheafification of $P \in \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ then $\overline{\Theta}_R(P) \to \overline{\Theta}_R(P')$ is an equivalence.

Let $X \in \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$. Let $\mathrm{PMod}_X^{\otimes}$ denote the symmetric monoidal full subcategory of the underlying symmetric monoidal ∞-category $\overline{\Theta}_R$ spanned by dualizable objects. Suppose that $\mathrm{PMod}_X^{\otimes}$ is a small stable idempotent-complete symmetric monoidal ∞-category whose tensor operation $\otimes : \mathrm{PMod}_X \times \mathrm{PMod}_X \to \mathrm{PMod}_X$ preserves finite colimits separately in each variable. Since symmetric monoidal functors carry dualizable objects to dualizable objects, the composition $\mathrm{PMod}_R^{\otimes} \hookrightarrow \mathrm{Mod}_R^{\otimes} \to \mathrm{Mod}_X^{\otimes}$ factors through $\mathrm{PMod}_X^{\otimes} \subset \mathrm{Mod}_X^{\otimes}$, where $\mathrm{Mod}_X^{\otimes}$ is the underlying symmetric monoidal ∞-category of $\overline{\Theta}_R$ and $\mathrm{Mod}_R^{\otimes} \to \mathrm{Mod}_X^{\otimes}$ is the R-linear structure map. Hence we can naturally regard $\mathrm{PMod}_X^{\otimes}$ as an object in $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})_{\mathrm{PMod}_R^{\otimes}/}$. We refer to $\mathrm{PMod}_X^{\otimes}$ as the symmetric monoidal ∞-category of perfect complexes on Y. We here call presheaves enjoying this condition admissible presheaves (functors). For example, affine derived schemes and $\mathsf{B}G$ with G a derived affine group scheme are admissible. Indeed, $\mathsf{B}G$ is described as the colimit of a simplicial affine derived schemes $a : \mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R$ and $\mathrm{Cat}_\infty^{\mathrm{st}} \hookrightarrow \mathrm{Cat}_\infty$ preserves small limits. It follows that $\mathrm{PMod}_{\mathsf{B}G} \simeq \mathrm{PMod}_G \simeq \lim_{[n]}\mathrm{PMod}_{a([n])}$ is stable and idempotent-complete where $\lim_{[n]\in\Delta}\mathrm{PMod}_{a([n])}$ the limit of the cosimplicial diagram of ∞-categories. Let $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})^{\mathrm{adm}}$ be the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ spanned by admissible presheaves. Applying $\overline{\Theta}_R$ and taking full subcategories of $\overline{\Theta}_R(X)$ spanned by dualizable objects we have the functor

$$\overline{\theta}_R : \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})^{\mathrm{adm}} \to (\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})_{\mathrm{PMod}_R^{\otimes}/})^{op}$$

which carries X to $\mathrm{PMod}_X^{\otimes}$ endowed with the R-linear structure map $\mathrm{PMod}_R^{\otimes} \to \mathrm{PMod}_X^{\otimes}$. We remark that by [23, 3.3.3.2, 5.1.2.2] P in $\mathrm{PMod}_X \simeq \lim_{\mathrm{Spec}\, A\to X}\mathrm{PMod}_A$ ($\mathrm{Spec}\, A \to X$ run over $(\mathrm{Aff}_R)_{/X}$) is a finite colimit of a (finite) diagram $I \to \mathrm{PMod}_X$ if and only if for each $\mathrm{Spec}\, A \to X$ the image of P in PMod_A is a finite colimit of the induced diagram.

3.3. **Tannakization.** Let CHopf_R be the ∞-category of commutative Hopf ring spectra over R, that is the full subcategory of $\mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{CAlg}_R)$, spanned by objects satisfying a certain condition (see [17, Appendix]): The opposite ∞-category of CHopf_R is equivalent to the ∞-category of the derived affine group schemes over R. Thus we set $\mathrm{dAffGp}_R := (\mathrm{CHopf}_R)^{op}$, which we shall refer to as the ∞-category of derived affine group schemes over R. Then there is a natural functor

$$\Phi : (\mathrm{dAffGp}_R)^{op} \longrightarrow \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)^{R,\mathrm{aug}} := (\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)_{\mathrm{PMod}_R^\otimes/})_{/\mathrm{PMod}_R^\otimes}$$

which carries G to $\mathrm{PMod}_G^\otimes$ equipped with natural functors $\mathrm{PMod}_R^\otimes \to \mathrm{PMod}_G^\otimes$ (induced by $\mathsf{B}G \to \operatorname{Spec} R$) and $\mathrm{PMod}_G^\otimes \to \mathrm{PMod}_R^\otimes$ (induced by the natural projection $\operatorname{Spec} R \to \mathsf{B}G$). In this paper, we do not need the detail construction of Φ and thus we refer to [17] for the details. We recall the result of [17].

Theorem 3.1. *The functor Φ has a left adjoint functor Ψ, that is, there is an adjunction*

$$\Psi : \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)^{R,\mathrm{aug}} \rightleftarrows (\mathrm{dAffGp}_R)^{op} : \Phi.$$

If $\mathcal{E}$ is an object of $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)^{R,\mathrm{aug}}$, then we refer to $\Psi(\mathcal{E})$ as the tannakization of $\mathcal{E}$. (For this kind of construction for ordinary categories, see [19], [27].)

3.4. **Automorphisms.** Let $\mathcal{C}^\otimes$ denote an R-linear symmetric monoidal small stable idempotent-complete ∞-category, that is, an object in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)_{\mathrm{PMod}_R^\otimes/}$. Namely, if we write $\mathcal{C}$ for the underlying ∞-category, $\mathcal{C}$ is a small stable idempotent-complete ∞-category and the underlying symmetric monoidal ∞-category $\mathcal{C}^\otimes$ is endowed with a symmetric monoidal functor $\mathrm{PMod}_R^\otimes \to \mathcal{C}^\otimes$ which preserves finite colimits. For ease of notation, we usually omit $\mathrm{PMod}_R^\otimes \to \mathcal{C}^\otimes$.

We regard Aff_R as the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$. Let $(\mathrm{Aff}_R)_{/\mathsf{B}G}$ be the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})_{/\mathsf{B}G}$ spanned by objects $X \to \mathsf{B}G$ such that X are affine schemes, that is, objects which belong to the essential image of Yoneda embedding $\mathrm{Aff}_R \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$. There is the natural projection $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathrm{Aff}_R$, that is a right fibration. Let $\pi : \operatorname{Spec} R \to \mathsf{B}G$ be the natural projection. This determines a map between right fibrations

$$\begin{array}{ccc} \mathrm{Aff}_R = (\mathrm{Aff}_R)_{/\operatorname{Spec} R} & \longrightarrow & (\mathrm{Aff}_R)_{/\mathsf{B}G} \\ & \searrow \quad \swarrow & \\ & \mathrm{Aff}_R. & \end{array}$$

Let $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathcal{S}^{op}$ be a functor which assigns $\mathrm{Map}_R^\otimes(\mathcal{C}^\otimes, \mathrm{PMod}_A^\otimes)$ to $\operatorname{Spec} A$ in $(\mathrm{Aff}_R)_{/\mathsf{B}G}$. Here $\mathrm{Map}_R^\otimes(-,-)$ indicates the mapping space in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)_{\mathrm{PMod}_R^\otimes/}$. More precisely, let

$$c : (\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})^{\mathrm{adm}} \stackrel{\overline{\theta}_R}{\to} (\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)_{\mathrm{PMod}_R^\otimes/})^{op} \to \mathcal{S}^{op}$$

be the composition where the first functor is the natural projection, and the third is the image of $\mathcal{C}^\otimes$ by Yoneda embedding $(\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)_{\mathrm{PMod}_R^\otimes/})^{op} \to \mathrm{Fun}(\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)_{\mathrm{PMod}_R^\otimes/}, \widehat{\mathcal{S}})$. By the unstraightening functor [23, 3.2] together with [23, 4.2.4.4] the composition $(\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathcal{S}^{op}$ gives rise to a right fibration $p : \mathcal{M} \to (\mathrm{Aff}_R)_{/\mathsf{B}G}$.

For two objects $\mathcal{C}_1^{\otimes}$, $\mathcal{C}_2^{\otimes}$ in $\mathrm{CAlg}(\mathrm{Cat}_{\infty}^{\mathrm{st}})_{\mathrm{PMod}_R^{\otimes}/}$, we denote by $\mathrm{Map}_R^{\otimes}(\mathcal{C}_1^{\otimes}, \mathcal{C}_2^{\otimes})$ the mapping space. The mapping space $\mathrm{Map}_R^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_{\mathsf{B}G}^{\otimes})$ is homotopy equivalent to the limit of spaces

$$\lim_{\mathrm{Spec}\, A \to \mathsf{B}G} \mathrm{Map}_R^{\otimes}(\mathcal{C}^{\otimes}, \overline{\theta}_R(\mathrm{Spec}\, A))$$

where $\mathrm{Spec}\, A \to \mathsf{B}G$ run over $(\mathrm{Aff}_R)_{/\mathsf{B}G}$ and $\mathrm{PMod}_{\mathsf{B}G}^{\otimes} \simeq \lim_{\mathrm{Spec}\, A \to \mathsf{B}G} \overline{\theta}_R(\mathrm{Spec}\, A)$. Thus according to [23, 3.3.3.2] if we denote by $\mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}((\mathrm{Aff}_R)_{/\mathsf{B}G}, \mathcal{M})$ the simplicial set of the sections of $p : \mathcal{M} \to (\mathrm{Aff}_R)_{/\mathsf{B}G}$ (namely, the set of n-simplexes of $\mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}((\mathrm{Aff}_R)_{/\mathsf{B}G}, \mathcal{M})$ is the set of $(\mathrm{Aff}_R)_{/\mathsf{B}G} \times \Delta^n \to \mathcal{M}$ over $(\mathrm{Aff}_R)_{/\mathsf{B}G}$), then we see that

Lemma 3.2. *There is a categorical equivalence*

$$\mathrm{Map}_R^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_{\mathsf{B}G}^{\otimes}) \simeq \mathrm{Fun}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}((\mathrm{Aff}_R)_{/\mathsf{B}G}, \mathcal{M}).$$

The base change $q : \mathcal{N} := \mathcal{M} \times_{(\mathrm{Aff}_R)_{/\mathsf{B}G}} \mathrm{Aff}_R \stackrel{\mathrm{pr}_2}{\to} \mathrm{Aff}_R$ is also a right fibration since Cartesian fibrations are stable under base changes. Note that this right fibration $q : \mathcal{N} \to \mathrm{Aff}_R$ corresponds to the composition $c' : \mathrm{Aff}_R \to (\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathcal{S}^{op}$. Moreover, $c : (\mathrm{Aff}_R)_{/\mathsf{B}G} \to \mathcal{S}^{op}$ factors through $c' : \mathrm{Aff}_R \to \mathcal{S}^{op}$. Therefore we have a Cartesian equivalence $\mathcal{M} \simeq \mathcal{N} \times_{\mathrm{Aff}_R} (\mathrm{Aff}_R)_{/\mathsf{B}G}$ over $(\mathrm{Aff}_R)_{/\mathsf{B}G}$. Note that $\mathrm{Map}_R^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_R^{\otimes})$ is homotopy equivalent to $\lim_{\mathrm{Spec}\, A \to \mathrm{Spec}\, R} \mathrm{Map}(\mathcal{C}^{\otimes}, \mathrm{PMod}_A^{\otimes})$ where $\mathrm{Spec}\, A \to \mathrm{Spec}\, R$ run over Aff_R. As above, $\mathrm{Map}_R^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_R^{\otimes})$ is homotopy equivalent to $\mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}(\mathrm{Aff}_R, \mathcal{M})$. Moreover, consider the functor $\mathrm{Map}_R^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_{\mathsf{B}G}^{\otimes}) \to \mathrm{Map}_R^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_R^{\otimes})$ induced by the composition with the forgetful functor $\mathrm{PMod}_{\mathsf{B}G}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$. Then it can be viewed as the functor

$$f : \mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}((\mathrm{Aff}_R)_{/\mathsf{B}G}, \mathcal{M}) \to \mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}(\mathrm{Aff}_R, \mathcal{M}) = \mathrm{Map}_{\mathrm{Aff}_R}(\mathrm{Aff}_R, \mathcal{N})$$

induced by the functor $\mathrm{Aff}_R \to (\mathrm{Aff}_R)_{/\mathsf{B}G}$.

We fix a map $F : \mathcal{C}^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ in $\mathrm{CAlg}(\mathrm{Cat}_{\infty}^{\mathrm{st}})_{\mathrm{PMod}_R^{\otimes}/}$. This is equivalent to giving a vertex of $\mathrm{Map}_{\mathrm{Aff}_R}(\mathrm{Aff}_R, \mathcal{N})$. Let $\alpha_* : \mathrm{CAlg}_R \to \mathcal{S}$ be the functor corresponding to the identity right fibration $\mathrm{Aff}_R \to \mathrm{Aff}_R$ via the straightening functor. We may and will assume that α_* is the constant functor whose value is the contractible space. Let $\alpha_{\mathcal{N}} : \mathrm{CAlg}_R \to \mathcal{S}$ be the functor corresponding to the right fibration $\mathcal{N} \to \mathrm{Aff}_R$. The functor F determines a natural transformation $\alpha_* \to \alpha_{\mathcal{N}}$. Thus through the categorical equivalence $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})_{\alpha_*/} \simeq \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_*)$, we regard $\alpha_* \to \alpha_{\mathcal{N}}$ as an object in $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_*)$ where $\mathcal{S}_* = \mathcal{S}_{\Delta^0/}$. We define $\alpha'_{\mathcal{N}} : \mathrm{CAlg}_R \to \mathcal{S}$ so that for any $A \in \mathrm{CAlg}_R$, $\alpha'_{\mathcal{N}}(A)$ is the connected component of $\alpha_{\mathcal{N}}(A)$ on which the image of $\alpha_* \to \alpha_{\mathcal{N}}$ lie. We also regard $\alpha_* \to \alpha'_{\mathcal{N}}$ as an object $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_*)$. Let $\mathcal{S}_{*,\geq 1}$ be the full subcategory of $\mathcal{S}_*$ spanned by pointed spaces $\Delta^0 \to S$ such that S is connected. Notice that $\alpha'_{\mathcal{N}}$ represents the functor

$$\xi : \mathrm{CAlg}_R \to \mathcal{S}_{*,\geq 1}$$

which assigns A to the pointed connected component of $\mathrm{Map}_R^{\otimes}(\mathcal{C}^{\otimes}, \mathrm{PMod}_A^{\otimes})$ which corresponds to the composition $\mathcal{C}^{\otimes} \stackrel{F}{\to} \mathrm{PMod}_R^{\otimes} \to \mathrm{PMod}_A^{\otimes}$. Recall that $\mathrm{Grp}(\mathcal{S})$ is the ∞-category of group objects in $\mathcal{S}$, and the equivalence $\mathcal{S}_{*,\geq 1} \simeq \mathrm{Grp}(\mathcal{S})$ which carries any pointed space $S \in \mathcal{S}_{*,\geq 1}$ to the (based) loop space $\Omega_* S \in \mathrm{Grp}(\mathcal{S})$ (see [17, Appendix]).

Definition 3.3. We write $\mathrm{Aut}(F)$ for $\xi : \mathrm{CAlg}_R \to \mathcal{S}_{*,\geq 1} \simeq \mathrm{Grp}(\mathcal{S})$ and refer to it as the automorphism functor of F.

Consider the diagram in $\mathrm{CAlg}(\mathrm{Cat}_\infty)^{R,\mathrm{aug}}$

$$\begin{array}{ccc} \mathcal{C}^\otimes & & \mathrm{PMod}^\otimes_{\mathsf{B}G} \\ & \searrow \quad \swarrow & \\ & \mathrm{PMod}^\otimes_R . & \end{array}$$

The purpose of this subsection is to prove the following result.

Proposition 3.4. *There is an equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)^{R,\mathrm{aug}}}(\mathcal{C}^\otimes, \mathrm{PMod}^\otimes_{\mathsf{B}G}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\mathcal{S}))}(G, \mathrm{Aut}(F))$$

in $\mathcal{S}$. This equivalence is functorial in the following sense: Let $L : \mathrm{dAffGp}_R \to \mathcal{S}^{op}$ be the functor which assigns G to $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)^{R,\mathrm{aug}}}(\mathcal{C}^\otimes, \mathrm{PMod}^\otimes_{\mathsf{B}G})$. Let $M : \mathrm{dAffGp}_R \to \mathcal{S}^{op}$ be the functor which assigns G to $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\mathcal{S}))}(G, \mathrm{Aut}(F))$. (See the proof below for the formulations of L and M.) Then there exists a natural equivalence from L to M.

Remark 3.5. We would like to remark the intuitive meaning of Proposition 3.4. In the above equivalence, the right hand side is the space (∞-groupoid) of actions of G on F. The left hand side is the space of extensions of F to $\mathcal{C}^\otimes \to \mathrm{PMod}^\otimes_{\mathsf{B}G}$. Hence we can informally say that extending F to $\mathcal{C}^\otimes \to \mathrm{PMod}^\otimes_{\mathsf{B}G}$ is equivalent to giving an action of G on F.

Remark 3.6. The proof below shows that if we replace $\mathrm{PMod}^\otimes_{\mathsf{B}G}$ by $\mathrm{Mod}^\otimes_{\mathsf{B}G}$ the similar assertion also holds. Namely, there is a functorial equivalence

$$\mathrm{Map}_{((\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_\infty)_{\mathrm{Mod}^\otimes_R/})_{/\mathrm{Mod}^\otimes_R}}(\mathcal{C}^\otimes, \mathrm{Mod}^\otimes_{\mathsf{B}G}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\widehat{\mathcal{S}}))}(G, \mathrm{Aut}(F))$$

in $\widehat{\mathcal{S}}$, where $\mathcal{C}^\otimes$ belongs to $(\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_\infty)_{\mathrm{Mod}^\otimes_R/})_{/\mathrm{Mod}^\otimes_R}$. Here $F : \mathcal{C}^\otimes \to \mathrm{Mod}^\otimes_R$ and $\mathrm{Aut}(F)$ is defined in a similar way.

Corollary 3.7. *Suppose that* $\mathrm{Aut}(F)$ *is represented by a derived affine group scheme. Then* $\mathrm{Aut}(F)$ *is equivalent to the tannakization of* $F : \mathcal{C}^\otimes \to \mathrm{PMod}^\otimes_R$.

Proof of Proposition 3.4. In order to make our proof readable we first show the first assertion without defining L and M. The mapping space $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_\infty)^{R,\mathrm{aug}}}(\mathcal{C}^\otimes, \mathrm{PMod}^\otimes_{\mathsf{B}G})$ is the homotopy limit (i.e. the limit in $\mathcal{S}$)

$$\mathrm{Map}^\otimes_R(\mathcal{C}^\otimes, \mathrm{PMod}^\otimes_{\mathsf{B}G}) \times_{\mathrm{Map}^\otimes_R(\mathcal{C}^\otimes, \mathrm{PMod}^\otimes_R)} \{F\}$$

where $\{F\} = \Delta^0 \to \mathrm{Map}^\otimes_R(\mathcal{C}^\otimes, \mathrm{PMod}^\otimes_R)$ is determined by F. The fiber product of Kan complexes

$$P = \mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathrm{B}G}}((\mathrm{Aff}_R)_{/\mathrm{B}G}, \mathcal{M}) \times_{\mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathrm{B}G}}(\mathrm{Aff}_R, \mathcal{M})} \{F\}$$

is a homotopy limit since $\mathrm{Aff}_R \to (\mathrm{Aff}_R)_{/\mathrm{B}G}$ is a monomorphism (that is, a cofibration in the Cartesian simplicial model category of marked simplicial sets $(\mathrm{Set}^+_\Delta)_{/(\mathrm{Aff}_R)_{/\mathrm{B}G}}$, see [23, 3.1.3.7]) and thus f is a Kan fibration. Here $\Delta^0 = \{F\} \to \mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathrm{B}G}}(\mathrm{Aff}_R, \mathcal{M})$

is determined by F. Using the Cartesian equivalence $\mathcal{N}\times_{\mathrm{Aff}_R}(\mathrm{Aff}_R)_{/\mathsf{B}G}\simeq\mathcal{M}$ over $(\mathrm{Aff}_R)_{/\mathsf{B}G}$ we have homotopy equivalences

$$\mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}((\mathrm{Aff}_R)_{/\mathsf{B}G},\mathcal{M})\simeq\mathrm{Map}_{\mathrm{Aff}_R}((\mathrm{Aff}_R)_{/\mathsf{B}G},\mathcal{N})$$

and

$$\mathrm{Map}_{(\mathrm{Aff}_R)_{/\mathsf{B}G}}(\mathrm{Aff}_R,\mathcal{M})\simeq\mathrm{Map}_{\mathrm{Aff}_R}(\mathrm{Aff}_R,\mathcal{N}).$$

Thus P is homotopy equivalent to the fiber product

$$Q=\mathrm{Map}_{\mathrm{Aff}_R}((\mathrm{Aff}_R)_{/\mathsf{B}G},\mathcal{N})\times_{\mathrm{Map}_{\mathrm{Aff}_R}(\mathrm{Aff}_R,\mathcal{N})}\{F\}$$

which is also a homotopy limit, where $\Delta^0=\{F\}\to\mathrm{Map}_{\mathrm{Aff}_R}(\mathrm{Aff}_R,\mathcal{N})$ is determined by the section $\mathrm{Aff}_R\to\mathcal{N}$ corresponding to $F:\mathcal{C}^{\otimes}\to\mathrm{PMod}_R^{\otimes}$. We let $\alpha_{\mathsf{B}G}:\mathrm{CAlg}_R\to\mathcal{S}$ correspoindig to the right fibration $(\mathrm{Aff}_R)_{/\mathsf{B}G}\to\mathrm{Aff}_R$ via the straightening functor. There is the natural transformation $\alpha_*\to\alpha_{\mathsf{B}G}$ determined by $\mathrm{Aff}_R\to(\mathrm{Aff}_R)_{/\mathsf{B}G}$, which we consider to be a functor $\mathrm{CAlg}_R\to\mathcal{S}_{*,\geq 1}$. Observe that $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg},\mathcal{S}_*)}(\alpha_{\mathsf{B}G},\alpha_{\mathcal{N}})$ is homotopy equivalent to Q. By composition with $\mathcal{S}_{*,\geq 1}\simeq\mathrm{Grp}(\mathcal{S})$ we have $G:\mathrm{CAlg}_R\overset{\mathsf{B}G}{\to}\mathcal{S}_{*,\geq 1}\simeq\mathrm{Grp}(\mathcal{S})$ (that is, the composition is the original derived group scheme G). Then we obtain

$$\begin{aligned}Q&\simeq\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R,\mathcal{S}_*)}(\alpha_{\mathsf{B}G},\alpha_{\mathcal{N}})\\&\simeq\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R,\mathcal{S}_{*,\geq 1})}(\alpha_{\mathsf{B}G},\alpha'_{\mathcal{N}})\\&\simeq\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R,\mathrm{Grp}(\mathcal{S}))}(G,\mathrm{Aut}(F)).\end{aligned}$$

Next to see (and formulate) the latter assertion, we will define L and M. Since a derived affine group scheme is a group object in the Cartesian symmetric monoidal ∞-category of Aff_R, thus dAffGp_R is naturally embedded into $\mathrm{Fun}(\mathrm{N}(\Delta)^{op},\mathrm{Fun}(\mathrm{CAlg}_R,\mathcal{S}))$ as a full subcategory. Let $\mathrm{Fun}(\mathrm{N}(\Delta)^{op},\mathrm{Fun}(\mathrm{CAlg}_R,\mathcal{S}))\to\mathrm{Fun}(\mathrm{CAlg}_R,\mathcal{S})$ be the functor taking each simplicial object $\mathrm{N}(\Delta)^{op}\to\mathrm{Fun}(\mathrm{CAlg}_R,\widehat{\mathcal{S}})$ to its colimit. Let $\rho:\mathrm{dAffGp}_R\to\mathrm{Fun}(\mathrm{CAlg}_R,\mathcal{S})$ be the composition. Note that G maps to $\mathsf{B}G$. By the straightening and unstraightening functors [23, 3.2] together with [23, 4.2.4.4], we have the categorical equivalence $\mathrm{Fun}(\mathrm{CAlg}_R,\widehat{\mathrm{Cat}}_\infty)\simeq\mathrm{N}(((\widehat{\mathrm{Set}}_\Delta^+)_{/\mathrm{Aff}_R})^{cf})$ where $(\widehat{\mathrm{Set}}_\Delta^+)_{/\mathrm{Aff}_R}$ is the category of (not necessarily small) marked simplicial sets, which is endowed with the Cartesian model structure in [23, 3.1.3.7] and $(-)^{cf}$ indicates full simplicial subcategory of cofibrant-fibrant objects. In particular, there is the fully faithful functor $\mathrm{Fun}(\mathrm{CAlg}_R,\widehat{\mathcal{S}})\to\mathrm{N}(((\widehat{\mathrm{Set}}_\Delta^+)_{/\mathrm{Aff}_R})^{cf})$ which carries $\mathsf{B}G$ to $(\mathrm{Aff}_R)_{/\mathsf{B}G}\to\mathrm{Aff}_R$. Composing all these functors we have the composition

$$\mathrm{dAffGp}_R\overset{\rho}{\to}\mathrm{Fun}(\mathrm{CAlg}_R,\mathcal{S})\to\mathrm{N}(((\widehat{\mathrm{Set}}_\Delta^+)_{/\mathrm{Aff}_R})^{cf}).$$

Since $\mathrm{dAffGp}_R\simeq(\mathrm{dAffGp}_R)_{\mathrm{Spec}\,R/}$, the composition is extended to $u:\mathrm{dAffGp}_R\to\mathrm{N}(((\widehat{\mathrm{Set}}_\Delta^+)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/}$. Through Yoneda embedding

$$\mathrm{N}(((\widehat{\mathrm{Set}}_\Delta^+)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/}\to\mathrm{Fun}((\mathrm{N}(((\widehat{\mathrm{Set}}_\Delta^+)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/})^{op},\widehat{\mathcal{S}})$$

we define $I:(\mathrm{N}(((\widehat{\mathrm{Set}}_\Delta^+)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/})^{op}\to\widehat{\mathcal{S}}$ to be the functor corresponding to $\mathcal{N}\to\mathrm{Aff}_R$ equipped with the section F. Composing I^{op} with $\mathrm{dAffGp}_R\to\mathrm{N}(((\widehat{\mathrm{Set}}_\Delta^+)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/}$ we have $L:\mathrm{dAffGp}_R\to\widehat{\mathcal{S}}^{op}$. To define M, consider the functor $\mathrm{Fun}(\mathrm{CAlg}_R,\mathrm{Grp}(\mathcal{S}))\to$

$\widehat{\mathcal{S}}^{op}$ determined by $\mathrm{Aut}(F)$ via Yoneda embedding. Then we define M to be the composition

$$\mathrm{dAffGp}_R \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\mathcal{S})) \to \widehat{\mathcal{S}}^{op}.$$

To obtain $L \simeq M$, note that the unstraightening functor induces a fully faithful functor $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}}_*) \subset \mathrm{N}(((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/}$. Let $\mathsf{N} : \mathrm{CAlg}_R \to \mathcal{S}_*$ be a functor corresponding to $\mathcal{N} \to \mathrm{Aff}_R$ equipped with the section F, that is, N corresponds to $\alpha_* \to \alpha_{\mathcal{N}}$. Let $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_*) \to \widehat{\mathcal{S}}^{op}$ be the functor determined by N via Yoneda embedding. The functor L is equivalent to

$$\mathrm{dAffGp}_R \xrightarrow{u} \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}}_*) \subset \mathrm{N}(((\widehat{\mathrm{Set}}^+_\Delta)_{/\mathrm{Aff}_R})^{cf})_{\mathrm{Aff}_R/} \to \widehat{\mathcal{S}}^{op}.$$

Since the essential image of dAffGp_R in $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_*)$ is contained in $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_{*,\geq 1})$, for our purpose we may and will replace $\alpha_{\mathcal{N}}$ by $\alpha'_{\mathcal{N}}$ (in the construction of N) and assume that N belongs to $\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_{*,\geq 1})$. Then we see that L is equivalent to

$$\mathrm{dAffGp}_R \to \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S}_{*,\geq 1}) \simeq \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{Grp}(\mathcal{S})) \to \widehat{\mathcal{S}}^{op}$$

where the first functor is induced by u and the third functor is determined by $\mathrm{Aut}(F)$ via Yoneda embedding. Now the last composition is equivalent to M. □

4. Automorphism of fiber functors

Let Y be a derived stack over R (we fix our convention below) and $\mathrm{PMod}_Y^{\otimes}$ the ∞-category of perfect complexes on Y (Section 3.2), which we regard as an object in $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})_{\mathrm{PMod}_R^{\otimes}/}$. Let $\mathrm{Spec}\, R \to Y$ be a section of the structure morphism $Y \to \mathrm{Spec}\, R$. There is the pullback functor $\mathrm{PMod}_Y^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ in $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})_{\mathrm{PMod}_R^{\otimes}/}$. In this Section, we study the automorphisms of this functor. Our goal is Theorem 4.9 and Corollary 4.10.

We start with our setup of derived stacks. A functor $Y : \mathrm{CAlg}_R \to \widehat{\mathcal{S}}$ is said to be a derived stack (over R) if two condition hold:

(i) there exists a groupoid object $\mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R$ (cf. [17, A.2]) such that Y is equivalent to the colimit of the composite $\mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$,

(ii) Y has affine diagonal, that is, for any two morphisms $\mathrm{Spec}\, A \to Y$ and $\mathrm{Spec}\, B \to Y$, the fiber product $\mathrm{Spec}\, A \times_Y \mathrm{Spec}\, B$ belongs to $\mathrm{Aff}_R \subset \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$.

In this paper, despite Y in the above definition is usually called a pre-stack, we will not equip CAlg_R with Grothendieck topology such as flat, étale topologies since the sheafification Y' of Y by such topologies does induce a categorical equivalence $\mathrm{Mod}_{Y'} \to \mathrm{Mod}_Y$ by the flat descent theory. In addition, such topologies are irrelevant for our argument below. (Conversely, for our purpose one can replace $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ in the above definition by the full subcategory of sheaves with respect to flat topology (see e.g. [34], [25, VII, 5.4] for flat morphisms)). At any rate, we remark that our definition of derived stacks is not standard (compare [34], [25]). We note that our derived stacks are admissible functors.

Example 4.1. We present quotient stacks arising from the action of a derived affine group scheme on an affine scheme as examples of derived stacks. Let $F : \mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R$ be a groupoid object, which we regard as a derived stack. Let $G : \mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R$

be a group object, that is, a derived affine group scheme. Let $F \to G$ be a morphism (i.e., natural transformation) which induces a cartesian diagram

$$\begin{array}{ccc} F([n]) & \longrightarrow & F([m]) \\ \downarrow & & \downarrow \\ G([n]) & \longrightarrow & G([m]) \end{array}$$

in Aff_R for each $[m] \to [n]$. If we write X for $F([0])$, then we can think that the morphism $F \to G$ with the above property means an action of G on X. In this situation, we say that G acts on X and denote by $[X/G]$ the colimit of $\mathrm{N}(\Delta)^{op} \xrightarrow{F} \mathrm{Aff}_R \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$. We refer to $[X/G]$ as the quotient stack. We can think of $\mathsf{B}G$ as the quotient stack $[\operatorname{Spec} R/G]$ where G acts trivially on $\operatorname{Spec} R$.

Let $\pi : \operatorname{Spec} R \to Y$ denote the fixed section and $\pi^* : \mathrm{Mod}_Y^{\otimes} \to \mathrm{Mod}_R^{\otimes}$ the associated symmetric monoidal functor which preserves small colimits. Since Mod_Y and Mod_R are presentable, by adjoint functor theorem (see [23, 5.5.2.9]) there is a right adjoint functor $\pi_* : \mathrm{Mod}_R \to \mathrm{Mod}_Y$. Moreover, according to [24, 8.3.2.6] the right adjoint functor is extended to a right adjoint functor to relative to $\mathrm{N}(\mathrm{Fin}_*)$ (see [24, 8.3.2.2])

$$\begin{array}{ccc} \mathrm{Mod}_R^{\otimes} & \longrightarrow & \mathrm{Mod}_Y^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathrm{N}(\mathrm{Fin}_*). & \end{array}$$

It yields a right adjoint functor

$$\mathrm{CAlg}(\mathrm{Mod}_R^{\otimes}) \to \mathrm{CAlg}(\mathrm{Mod}_Y^{\otimes})$$

of the functor $\mathrm{CAlg}(\mathrm{Mod}_Y^{\otimes}) \to \mathrm{CAlg}(\mathrm{Mod}_R^{\otimes})$ determined by π^*.

Let $\phi : \mathrm{N}(\Delta) \to \mathrm{CAlg}_R$ be a cosimplicial diagram such that the colimit of composition $\mathrm{N}(\Delta)^{op} \xrightarrow{\phi^{op}} \mathrm{Aff}_R \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ is equivalent to Y. Recall from Section 2.1 the functor $\Theta_R : \mathrm{CAlg}_R \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^{\otimes}/}$. Note that by definition $\mathrm{Mod}_Y^{\otimes}$ is a limit of the composition $\phi'' : \mathrm{N}(\Delta) \xrightarrow{\phi} \mathrm{CAlg}_R \xrightarrow{\Theta_R} \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^{\otimes}/} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})$ where the last functor is the forgetful functor. Let $p : \mathcal{M}_{\phi'} \to \mathrm{N}(\Delta)$ be the coCartesian fibration corresponding to the composition $\phi' : \mathrm{N}(\Delta) \xrightarrow{\phi''} \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}}) \to \widehat{\mathrm{Cat}}_\infty$ where the last functor is the forgetful functor. We denote by $\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ spanned by sections $\mathrm{N}(\Delta) \to \mathcal{M}_{\phi'}$ which carries all edges of $\mathrm{N}(\Delta)$ to p-coCartesian edges. Then by [23, 3.3.3.2] Mod_Y is equivalent to $\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ as ∞-categories. Consider the base change of $\mathrm{N}(\Delta)^{op} \xrightarrow{\phi^{op}} \mathrm{Aff}_R \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$, where the second functor is Yoneda embedding, by $\pi : \operatorname{Spec} R \to Y$. Let $Y_n = \phi^{op}([n]) \in \mathrm{Aff}_R$ for each $[n] \in \Delta$. The n-th term of this base change $\tau : \mathrm{N}(\Delta)^{op} \to \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ is equivalent to $Y_n \times_Y \operatorname{Spec} R$ and in particular, it factors through $\mathrm{Aff}_R \subset \mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$. Taking the opposite categories we have $\psi : \mathrm{N}(\Delta) \to \mathrm{CAlg}_R$. Note that $\operatorname{Spec} R$ is a colimit of τ since in the ∞-topos $\mathrm{Fun}(\mathrm{CAlg}_R, \widehat{\mathcal{S}})$ colimits are universal (see [23, Chapter 6]). Thus the natural transformation $\psi^{op} \to \phi^{op}$ induces $\pi : \operatorname{Spec} R \to Y$, and we can informally indicates our situation as follows:

$$\begin{array}{ccccccc}
\cdots \; \substack{\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow} & Y_1 \times_Y \operatorname{Spec} R & \substack{\longrightarrow\\[-1em]\longleftarrow\\[-1em]\longrightarrow} & Y_0 \times_Y \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} R \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} \\
\cdots \; \substack{\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow} & Y_1 & \substack{\longrightarrow\\[-1em]\longleftarrow\\[-1em]\longrightarrow} & Y_0 & \longrightarrow & Y
\end{array}$$

(here $\psi^{op}, \phi^{op} : \mathrm{N}(\Delta)^{op} \to \mathrm{Aff}_R$). We define $\psi' : \mathrm{N}(\Delta) \to \widehat{\mathrm{Cat}}_\infty$ in the same way that we define ϕ', and we let $q : \mathcal{M}_{\psi'} \to \mathrm{N}(\Delta)$ the coCartesian fibration corresponding to ψ'. The natural transformation $\phi \to \psi$ corresponds to a map between coCartesian fibrations $\mathcal{M}_{\phi'} \to \mathcal{M}_{\psi'}$ over $\mathrm{N}(\Delta)$, which carries coCartesian edges to coCartesian edges. Again by [24, 8.3.2.6] there is a right adjoint functor $\mathcal{M}_{\psi'} \to \mathcal{M}_{\phi'}$ of $\mathcal{M}_{\phi'} \to \mathcal{M}_{\psi'}$ relative to $\mathrm{N}(\Delta)$. Let us observe the following:

Lemma 4.2. *The map $\mathcal{M}_{\psi'} \to \mathcal{M}_{\phi'}$ of coCartesian fibrations over $\mathrm{N}(\Delta)$ carries q-coCartesian edges to p-coCartesian edges.*

Proof. It suffices to show that if for any map $r : [m] \to [n]$ in Δ we describe the diagram induced by $\psi^{op} \to \phi^{op}$ as

$$\begin{array}{ccc}
Y_n \times_Y \operatorname{Spec} R & \xrightarrow{a} & Y_m \times_Y \operatorname{Spec} R \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\
Y_n & \xrightarrow{d} & Y_m,
\end{array}$$

then the natural base change morphism $d^* \circ c_* \to b_* \circ a^*$ is an equivalence. It follows from [1, Lemma 3.14]. □

Let

$$\alpha : \mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'}) \rightleftarrows \mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\psi'}) : \beta$$

be functors induced by the adjunction $\mathcal{M}_{\phi'} \rightleftarrows \mathcal{M}_{\psi'}$, where $\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ is the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$, spanned by sections which carries all edges to coCartesian edges and we define $\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})$ in a similar way. Note that by [23, 3.3.3.2]

$$\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'}) \simeq \mathrm{Mod}_Y \quad \text{and} \quad \mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\psi'}) \simeq \mathrm{Mod}_R,$$

and $\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'}) \to \mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\psi'})$ is equivalent to $\pi^* : \mathrm{Mod}_Y \to \mathrm{Mod}_R$ as functors. Then observe that the pair (α, β) forms adjunction. Namely,

$$\begin{array}{rcl}
\mathrm{Map}_{\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\psi'})}(\alpha(a), b) & \simeq & \lim_{[n]\in\Delta} \mathrm{Map}_{\psi'([n])}(\alpha(a_n), b_n) \\
& \to & \lim_{[n]\in\Delta} \mathrm{Map}_{\phi'([n])}(\beta(\alpha(a_n)), \beta(b_n)) \\
& \stackrel{x}{\to} & \lim_{[n]\in\Delta} \mathrm{Map}_{\phi'([n])}(a_n, \beta(b_n)) \\
& \simeq & \mathrm{Map}_{\mathrm{Fun}'_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta), \mathcal{M}_{\phi'})}(a, \beta(b))
\end{array}$$

is equivalence in $\mathcal{S}$, where a_n (resp. b_n) is the projection of a (resp. b) to $\phi'([n])$ (resp. $\psi'([n])$) and x is induced by the unit map of the adjunction $\mathcal{M}_{\phi'} \rightleftarrows \mathcal{M}_{\psi'}$. (The fiber of the adjunction $\mathcal{M}_{\phi'} \rightleftarrows \mathcal{M}_{\psi'}$ over each object of $\mathrm{N}(\Delta)$ forms adjunction.) Notice that $\mathrm{Fun}_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta),\mathcal{M}_{\psi'}) \to \mathrm{Fun}_{\mathrm{N}(\Delta)}(\mathrm{N}(\Delta),\mathcal{M}_{\phi'})$ is equivalent to $\pi_*:\mathrm{Mod}_R\to\mathrm{Mod}_Y$ as functors. Consequently, we have

Lemma 4.3. *Let*

$$\begin{array}{ccc} Y_n\times_Y \mathrm{Spec}\,R & \xrightarrow{s_n} & \mathrm{Spec}\,R \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi} \\ Y_n & \xrightarrow{t_n} & Y \end{array}$$

be the pullback diagram induced by $\psi^{op}([n])\to\phi^{op}([n])$. Then the natural base change morphism $(t_n)^*\circ\pi_*\to(\pi_n)_*\circ(s_n)^*$ is an equivalence of functors from Mod_R to Mod_{Y_n}.

Corollary 4.4. *We abuse notation and we write $(t_n)^*\circ\pi_*\to(\pi_n)_*\circ(s_n)^*$ for the natural base change morphism from $\mathrm{CAlg}(\mathrm{Mod}_R^{\otimes})$ to $\mathrm{CAlg}(\mathrm{Mod}_{Y_n}^{\otimes})$ which is determined by adjunctions (π^*,π_*) and $((\pi_n)^*,(\pi_n)_*)$ relative to $\mathrm{N}(\mathrm{Fin}_*)$. Then $(t_n)^*\circ\pi_*\to(\pi_n)_*\circ(s_n)^*$ is an equivalence of functors.*

Let $\mathbf{1}_R$ be a unit of Mod_R which we here regard as an object in $\mathrm{CAlg}_R=\mathrm{CAlg}(\mathrm{Mod}_R)$. Then there is a lax symmetric monoidal functor $\mathrm{Mod}_R^{\otimes}\to\mathrm{Mod}^{\otimes}_{\pi_*\mathbf{1}_R}(\mathrm{Mod}_Y^{\otimes})$ of symmetric monoidal ∞-categories induced by π_* by the construction of the ∞-operad of module objects [24, 3.3.3.8]. For the notation $\mathrm{Mod}^{\otimes}_{\pi_*\mathbf{1}_R}(\mathrm{Mod}_Y^{\otimes})$, see Section 2.

Lemma 4.5. *The functor $\mathrm{Mod}_R^{\otimes}\to\mathrm{Mod}^{\otimes}_{\pi_*\mathbf{1}_R}(\mathrm{Mod}_Y^{\otimes})$ is a symmetric monoidal equivalence.*

Proof.
We first obeserve that $\mathrm{Mod}_R^{\otimes}\to\mathrm{Mod}^{\otimes}_{\pi_*\mathbf{1}_R}(\mathrm{Mod}_Y^{\otimes})$ is symmteric monoidal. Since it is lax symmetric monoidal, combined with Lemma 4.3 we are reduced to showing the following obvious claim: for a morphism $x:\mathrm{Spec}\,A\to\mathrm{Spec}\,B$ of affine derived schemes and $M,N\in\mathrm{Mod}_A$, the natural map $x_*(M)\otimes_A x_*(N)\to x_*(M\otimes_A N)$ is an equivalence where $x_*:\mathrm{Mod}_A\to\mathrm{Mod}_A(\mathrm{Mod}_B^{\otimes})$ is the natural pushforward functor.

We now adopt notation similar to Lemma 4.3. Since the natural equivalence $(t_n)^*\circ\pi_*\mathbf{1}_R\simeq(\pi_n)_*\circ(s_n)^*\mathbf{1}_R$ by the above result, we have

$$(\pi_n)_*:\mathrm{Mod}_{\psi([n])}=\mathrm{Mod}_{Y_n\times_Y\mathrm{Spec}\,R}\simeq\mathrm{Mod}_{(\pi_n)_*\circ(s_n)^*\mathbf{1}_R}(\mathrm{Mod}^{\otimes}_{\phi([n])})\simeq\mathrm{Mod}_{(t_n)^*\circ\pi_*\mathbf{1}_R}(\mathrm{Mod}^{\otimes}_{\phi([n])})$$

for each n. Then we identify $\mathrm{Mod}_R\to\mathrm{Mod}_{\pi_*\mathbf{1}_R}(\mathrm{Mod}_Y^{\otimes})$ with the limit

$$\lim_{[n]\in\Delta}\mathrm{Mod}_{\psi([n])}\simeq\lim_{[n]\in\Delta}\mathrm{Mod}_{Y_n\times_Y\mathrm{Spec}\,R}\simeq\lim_{[n]\in\Delta}\mathrm{Mod}_{(t_n)^*\circ\pi_*\mathbf{1}_R}(\mathrm{Mod}^{\otimes}_{\phi([n])})$$

which is an equivalence in $\widehat{\mathrm{Cat}}_\infty$. It follows that $\mathrm{Mod}_R^{\otimes}\to\mathrm{Mod}^{\otimes}_{\pi_*\mathbf{1}_R}(\mathrm{Mod}_Y^{\otimes})$ is a symmetric monoidal equivalence. □

Let $\mathrm{Aut}(\pi^*):\mathrm{CAlg}_R\to\mathrm{Grp}(\widehat{\mathcal{S}})$ be the automorphism functor of π^* (defined as in the previous Section, see Remark 3.6), which carries $A\in\mathrm{CAlg}_R$ to the automorphisms of composition $\mathrm{Mod}_Y^{\otimes}\to\mathrm{Mod}_R^{\otimes}\to\mathrm{Mod}_A^{\otimes}$ in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^{\otimes}/}$ where the second functor is the base change by $R\to A$.

Let Δ_+ be the category of finite (possibly empty) linearly ordered sets and we write $[-1]$ for the empty set. Let $\iota:\Delta^1\to\mathrm{N}(\Delta_+)$ be a map which carries $\{0\}$ and $\{1\}$ to $[-1]$ and $[0]$ respectively. It is a fully faithful functor. Let $(\Delta^1)^{op}\to\mathrm{Fun}(\mathrm{CAlg}_R,\widehat{\mathcal{S}})$ be a map corresponding to $\pi:\mathrm{Spec}\,R\to Y$. Let $\rho:\mathrm{N}(\Delta_+)^{op}\to\mathrm{Fun}(\mathrm{CAlg}_R,\widehat{\mathcal{S}})$ be a right Kan extension along $\iota^{op} : (\Delta^1)^{op} \to \mathrm{N}(\Delta_+)^{op}$ which is called Čech nerve (cf. [23, 6.1.2.11]). By our assumption, for each $n \geq 0$, $\rho([n])$ belongs to Aff_R and the restriction of ρ to $\mathrm{N}(\Delta)^{op}$ is a derived affine group scheme which we denote by G_π. By the definition of G_π and $\mathrm{Mod}_{G_\pi}^\otimes$, we see that $\pi^* : \mathrm{Mod}_Y^\otimes \to \mathrm{Mod}_R^\otimes$ factors through the forgetful functor $\mathrm{Mod}_{G_\pi}^\otimes \to \mathrm{Mod}_R^\otimes$. It follows from Remark 3.6 that there exists the natural morphism $G_\pi \to \mathrm{Aut}(\pi^*)$. (Alternatively, we may think that the derived group scheme $G_\pi : (\mathrm{Aff}_R)^{op} \to \mathrm{Grp}(\mathcal{S})$ represents the automorphism group of $\pi : \mathrm{Spec}\, R \to Y$ and thus we have the natural morphism $G_\pi \simeq \mathrm{Aut}(\pi) \to \mathrm{Aut}(\pi^*)$.)

Proposition 4.6. *The natural morphism $G_\pi \to \mathrm{Aut}(\pi^*)$ is an equivalence, that is, $\mathrm{Aut}(\pi^*)$ is represented by G_π.*

Proof. For simplicity, let $G := G_\pi$. Let $G_1 : \mathrm{CAlg}_R \to \widehat{\mathcal{S}}$ and (resp. $\mathrm{Aut}(\pi^*)_1$) be the composite of $G : \mathrm{CAlg}_R \to \mathrm{Grp}(\widehat{\mathcal{S}})$ (resp. $\mathrm{Aut}(\pi^*)$) and the forgetful functor $\mathrm{Grp}(\widehat{\mathcal{S}}) \to \widehat{\mathcal{S}}$. For each $A \in \mathrm{CAlg}_R$, it will suffice to show that the induced map $G_1(A) \to \mathrm{Aut}(\pi^*)_1(A)$ is equivalence in $\widehat{\mathcal{S}}$.

For $A \in \mathrm{CAlg}_R$, let $\pi_A : \mathrm{Spec}\, A \to \mathrm{Spec}\, R \to Y$ denote the composition. Let $\mathbf{1}_A$ be the unit of Mod_A which we here think of as an object of $\mathrm{CAlg}(\mathrm{Mod}_A^\otimes)$. Applying [24, 6.3.5.18] together with Lemma 4.5 and adjunction we deduce

$$\begin{aligned}\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_Y^\otimes /}}(\mathrm{Mod}_A^\otimes, \mathrm{Mod}_A^\otimes) &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_Y^\otimes)}((\pi_A)_*\mathbf{1}_A, (\pi_A)_*\mathbf{1}_A) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_A)}((\pi_A)^*(\pi_A)_*\mathbf{1}_A, \mathbf{1}_A).\end{aligned}$$

Unwinding the definitions we have

$$\begin{aligned}\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_A)}((\pi_A)^*(\pi_A)_*\mathbf{1}_A, \mathbf{1}_A) &\simeq \mathrm{Map}_{(\mathrm{Aff})_{/\mathrm{Spec}\, A}}(\mathrm{Spec}\, A, \mathrm{Spec}\, A \times_Y \mathrm{Spec}\, A) \\ &\simeq \mathrm{Map}_{(\mathrm{Aff})_{/\mathrm{Spec}\, A}}(\mathrm{Spec}\, A, G_1 \times_R A \times_R A) \\ &\simeq \mathrm{Map}_{\mathrm{Aff}_{/Y}}(\mathrm{Spec}\, A, \mathrm{Spec}\, A)\end{aligned}$$

where G_1 is $\mathrm{Spec}\, R \times_Y \mathrm{Spec}\, R \simeq \rho([1])$, and $G_1 \times_R A \times_R A \to \mathrm{Spec}\, A \in (\mathrm{Aff})_{/\mathrm{Spec}\, A}$ is the second projection. Note that through natural equivalences a morphism $\mathrm{Spec}\, A \to \mathrm{Spec}\, A$ over Y, which we regard as an object of $\mathrm{Map}_{\mathrm{Aff}_{/Y}}(\mathrm{Spec}\, A, \mathrm{Spec}\, A)$, induces a symmetric monoidal functor $\mathrm{Mod}_A^\otimes \to \mathrm{Mod}_A^\otimes$ under $\mathrm{Mod}_Y^\otimes$ which we think of as an object of $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_Y^\otimes /}$.

Next using the natural equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_Y^\otimes /}}(\mathrm{Mod}_A^\otimes, \mathrm{Mod}_A^\otimes) \simeq \mathrm{Map}_{\mathrm{Aff}_{/Y}}(\mathrm{Spec}\, A, \mathrm{Spec}\, A)$$

we consider the automorphisms of π^*. To this end let T_A be the fiber product

$$\mathrm{Map}_{\mathrm{Aff}_{/Y}}(\mathrm{Spec}\, A, \mathrm{Spec}\, A) \times_{\mathrm{Map}_{\mathrm{Aff}}(\mathrm{Spec}\, A, \mathrm{Spec}\, A)} \{\mathrm{Id}_{\mathrm{Spec}\, A}\}$$

in $\mathcal{S}$ where the diagram is induced by the forgetful functor $\mathrm{Map}_{\mathrm{Aff}_{/Y}}(\mathrm{Spec}\, A, \mathrm{Spec}\, A) \to \mathrm{Map}_{\mathrm{Aff}}(\mathrm{Spec}\, A, \mathrm{Spec}\, A)$. Similarly, we define S_A to be th fiber product

$$\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_Y^\otimes /}}(\mathrm{Mod}_A^\otimes, \mathrm{Mod}_A^\otimes) \times_{\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})}(\mathrm{Mod}_A^\otimes, \mathrm{Mod}_A^\otimes)} \{\mathrm{Id}\}$$

in $\widehat{\mathcal{S}}$, which is equivalent to T_A. There are natural equivalences

$$\begin{aligned} T_A &\simeq \mathrm{Map}'_{(\mathrm{Aff})_{/\operatorname{Spec} A}}(\operatorname{Spec} A, G_1\times_R A\times_R A)\\ &\simeq \mathrm{Map}_{(\mathrm{Aff})_{/\operatorname{Spec} A}}(\operatorname{Spec} A, G_1\times_R A)\\ &\simeq \mathrm{Map}_{\mathrm{Aff}}(\operatorname{Spec} A, G_1)\end{aligned}$$

in $\widehat{\mathcal{S}}$ where $\mathrm{Map}'_{(\mathrm{Aff})_{/\operatorname{Spec} A}}(\operatorname{Spec} A, G_1\times_R A\times_R A)$ is the fiber product

$$\mathrm{Map}_{(\mathrm{Aff})_{/\operatorname{Spec} A}}(\operatorname{Spec} A, G_1\times_R A\times_R A)\times_{\mathrm{Map}_{\mathrm{Aff}}(\operatorname{Spec} A,\operatorname{Spec} A)}\{\mathrm{Id}_{\operatorname{Spec} A}\}$$

in $\mathcal{S}$ where the diagram is induced by the projection $\mathrm{pr}_3 : G_1\times_R A\times_R A\to \operatorname{Spec} A$. Thus we have an equivalence $\mathrm{Map}_{\mathrm{Aff}}(\operatorname{Spec} A, G_1)\simeq S_A$. When $Y=\operatorname{Spec} R$ and we define S'_A and T'_A in the same way that S_A and T_A are defined, then the assignment $A\mapsto S'_A\simeq T'_A$ is the functor $\mathrm{CAlg}\to\mathcal{S}$ represented by $\operatorname{Spec} R$. Consequently, for $A\in\mathrm{CAlg}_R$, $\mathrm{Aut}(\pi^*)_1(A)$ and $G_1(A)$ are equivalent to the homotopy fiber products $S_A\times_{S'_A}\{A\}$ and $T_A\times_{T'_A}\{A\}$ respectively (here $\{A\}$ means that the vertex corresponding to $\operatorname{Spec} A\to\operatorname{Spec} R$). Hence we have the required equivalence $G_1(A)\simeq\mathrm{Aut}(\pi^*)_1(A)$. □

Let $\mathcal{C}^\otimes,\mathcal{D}^\otimes\in\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})$. Suppose that $\mathcal{C}$ is compactly generated, that is, the natural colimit-preserving functor $\mathrm{Ind}(\mathcal{C}_\circ)\to\mathcal{C}$ is a categorical equivalence, and $\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ induces $\mathcal{C}_\circ\times\mathcal{C}_\circ\to\mathcal{C}_\circ$, which makes $\mathcal{C}_\circ$ a symmetric monoidal ∞-category, where $\mathcal{C}_\circ$ is the full subcategory of compact objects in $\mathcal{C}$ and $\mathrm{Ind}(-)$ indicates the Ind-category (see [23, 5.3.5]). Note that under this assumption, a unit object is compact.

Proposition 4.7. *Let* $\mathrm{Map}^{\otimes,\mathrm{L}}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$ *be* $\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$. *Let* $\mathrm{Map}^{\otimes,\mathrm{ex}}(\mathcal{C}_\circ^\otimes,\mathcal{D}^\otimes)$ *be the full subcategory of* $\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(\mathcal{C}_\circ^\otimes,\mathcal{D}^\otimes)$ *spanned by symmetirc monoidal functors which preserves finite colimits. The natural inclusion* $\mathcal{C}_\circ^\otimes\to\mathcal{C}^\otimes$ *induces an equivalence*

$$\mathrm{Map}^{\otimes,\mathrm{L}}(\mathcal{C}^\otimes,\mathcal{D}^\otimes)\to\mathrm{Map}^{\otimes,\mathrm{ex}}(\mathcal{C}_\circ^\otimes,\mathcal{D}^\otimes)$$

in $\widehat{\mathcal{S}}$.

Lemma 4.8. *Let* $\mathcal{C}^{\times n}$ *and* $\mathcal{D}^{\times m}$ *be the* n*-fold product and the* m*-fold product respectively. Let* $\mathrm{Fun}'(\mathcal{C}^{\times n},\mathcal{D}^{\times m})$ *be the full subcategory of* $\mathrm{Fun}'(\mathcal{C}^{\times n},\mathcal{D}^{\times m})$ *spanned by functors which preserves small colimits separately in each variable of* $\mathcal{C}^{\times n}$. $\mathrm{Fun}'_\circ(\mathcal{C}_\circ^{\times n},\mathcal{D}^{\times m})$ *be the full subcategory of* $\mathrm{Fun}(\mathcal{C}_\circ^{\times n},\mathcal{D}^{\times m})$ *spanned by functors which preserves finite colimits separately in each variable of* $\mathcal{C}_\circ^{\times n}$. *Then the natural fully faithful functor* $\mathcal{C}_\circ^{\times n}\to\mathcal{C}^{\times n}$ *induces a categorical equivalence* $\mathrm{Fun}'(\mathcal{C}^{\times n},\mathcal{D}^{\times m})\to\mathrm{Fun}'_\circ(\mathcal{C}_\circ^{\times n},\mathcal{D}^{\times m})$.

Proof. Clearly, we are reduced to the case $m=1$. Thus we will assume that $m=1$. We first consider the case $n=1$. This case is well-known. We show this case for the reader's convenience. By left Kan extension [23, 5.3.5.10] we have a categorical equivalence $\mathrm{Fun}_{\mathrm{cont}}(\mathcal{C},\mathcal{D})\to\mathrm{Fun}(\mathcal{C}_\circ,\mathcal{D})$ induced by $\mathcal{C}_\circ\subset\mathcal{C}$ where $\mathrm{Fun}_{\mathrm{cont}}(\mathcal{C},\mathcal{D})$ is the full subcategory spanned by functors which preserves filtered colimits. The argument of the second paragraph of the proof of [24, 1.1.3.6] says that if $\mathcal{C}_\circ\to\mathcal{D}$ preserves cokernels and kernels, then the corresponding left Kan extension (via the above equivalence) $\mathcal{C}\to\mathcal{D}$ preserves cokernels and kernels, and in particular $\mathcal{C}\to\mathcal{D}$ preserves small colimits by [24, 1.1.4.1] and [23, 4.4.2.7]. Since $\mathcal{C}_\circ\hookrightarrow\mathcal{C}$ preserves finite colimits, we have a categorical equivalence $\mathrm{Fun}'(\mathcal{C},\mathcal{D})\simeq\mathrm{Fun}'_\circ(\mathcal{C}_\circ,\mathcal{D})$, as desired.

Next we consider the case $n = 2$. In this case

$$\begin{aligned}\mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}_\circ)\times\mathrm{Ind}(\mathcal{C}_\circ),\mathcal{D}) &\simeq \mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}_\circ),\mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}_\circ),\mathcal{D}))\\ &\simeq \mathrm{Fun}'_\circ(\mathcal{C}_\circ,\mathrm{Fun}'_\circ(\mathcal{C}_\circ,\mathcal{D}))\\ &\simeq \mathrm{Fun}'_\circ(\mathcal{C}_\circ\times\mathcal{C}_\circ,\mathcal{D}),\end{aligned}$$

where all equivalence follows from the case of $n = 1$ and the fact that $\mathrm{Fun}'(\mathrm{Ind}(\mathcal{C}_\circ),\mathcal{D})$ is presentable [23, 5.5.3.8]. The proof in the case of $n \geq 3$ is similar to $n = 2$ (use induction on n). □

Proof of Proposition 4.7. In virtue of straightening [23, 3.2.0.1, 3.2.5] there exists a map $z : \mathrm{Fin}_* \to (\widehat{\mathrm{Set}}_\Delta)^{cf}$ such that its unstraightening is (coCartesian equivalent to) the symmetric monoidal ∞-category $\mathcal{C}^\otimes \to \mathrm{N}(\mathrm{Fin}_*)$ and $z(\langle n\rangle_*)$ is an ∞-category for each $n \geq 0$. Here $\widehat{\mathrm{Set}}_\Delta$ is the simplicial category of (not necessarily small) ∞-categories (cf. [23, Chapter 3]). For each $n \geq 0$, set $\mathcal{C}^n := z(\langle n\rangle_*)$. Let $\mathcal{C}^n_\circ$ be the full subcategory (simplicial subset) spanned by compact objects. Then the restriction to $\mathcal{C}^n_\circ$ induces $z_\circ : \mathrm{Fin}_* \to (\widehat{\mathrm{Set}}_\Delta)^{cf}$ which carries $\langle n\rangle_*$ to $\mathcal{C}^n_\circ$. Also, there is a natural transformation $z_\circ \to z$ of functors (taking account of [23, 5.3.4.10] and the fact that $\mathcal{C}$ has a final object we see that $\mathcal{C}^{\times n}_\circ$ coincides with the full subcategory of compact objects in $\mathcal{C}^{\times n}$). Similarly, there exists $z' : \mathrm{Fin}_* \to (\widehat{\mathrm{Set}}_\Delta)^{cf}$ such that its unstraightening is (coCartesian equivalent to) a coCartesian fibration $\mathcal{D}^\otimes \to \mathrm{N}(\mathrm{Fin}_*)$ and $\mathcal{D}^n := z'(\langle n\rangle_*)$ is an ∞-category for each $n \geq 0$. Then a natural transformation $\mathrm{N}(z) \to \mathrm{N}(z')$ of functors from $\mathrm{N}(\mathrm{Fin}_*)$ to $\mathrm{N}((\widehat{\mathrm{Set}}_\Delta)^{cf}) = \widehat{\mathrm{Cat}}_\infty$ corresponds to a symmetric monoidal functor $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$. More precisely, there is a homotopy equivalence $\mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*),\widehat{\mathrm{Cat}}_\infty)}(\mathrm{N}(z),\mathrm{N}(z')) \simeq \mathrm{Map}^\otimes(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$, where $\mathrm{Map}^\otimes(\mathcal{C}^\otimes,\mathcal{D}^\otimes)$ is the mapping space of symmetric monoidal functors.

Suppose that $\mathcal{E}$ is either $\mathcal{C}$ or $\mathcal{D}$. Let $\alpha_{n,i} : \langle n\rangle_* \to \langle 1\rangle_*$ be the map which sends i to 1 and sends others to $*$. Let $p_i : \mathcal{E}^n \to \mathcal{E}^1$ be the map of simplicial sets determined by $\alpha_{n,i}$. Let $q^n : (\mathcal{E}^1)^{\times n} \to \mathcal{E}^n$ be a quasi-inverse of the categorical equivalence $p_1 \times \cdots \times p_n : \mathcal{E}^n \to (\mathcal{E}^1)^{\times n}$. Let $r_i : (\mathcal{E}^1)^{\times n} \to \mathcal{E}^1$ be the i-th projection. For $\mathsf{e} = (e_1,\ldots,e_{i-1},e_{i+1},\ldots,e_n) \in (\mathcal{E}^1)^{n-1}$ we let $\iota_i(\mathsf{e})$ be the inclusion $\mathcal{E}^1 \to (\mathcal{E}^1)^{\times n}$ which is informally given by $e \mapsto (e_1,\ldots,e_{i-1},e,e_{i+1},\ldots,e_n)$. We define $\mathrm{Fun}^*(\mathcal{C}^n,\mathcal{E}^m)$ to be the full subcategory (simplicial subset) of $\mathrm{Fun}(\mathcal{C}^n,\mathcal{E}^m)$ by the following condition. A functor $f : \mathcal{C}^n \to \mathcal{E}^m$ belongs to $\mathrm{Fun}^*(\mathcal{C}^n,\mathcal{E}^m)$ if and only if the following two conditions hold:

- $f \circ q^n : (\mathcal{C}^1)^{\times n} \to \mathcal{C}^n \to \mathcal{E}^m$ preserves small colimits separately in each variable,
- for any $1 \leq k \leq n$ and $\mathsf{c} = (c_1,\ldots,c_{k-1},c_{k+1},\ldots,c_n) \in (\mathcal{C}^1)^{n-1}$, there is at most one $1 \leq i \leq m$ such that $r_i \circ (p_1 \times \cdots \times p_m) \circ f \circ q^n \circ \iota_k(\mathsf{c}) : \mathcal{C}^1 \to \mathcal{E}^1$ is not equivalent (as functors) to a constant functor.

Replacing $\mathcal{C}$ by $\mathcal{D}$ in the above condition we define the full subcategory $\mathrm{Fun}^*(\mathcal{D}^n,\mathcal{E}^m)$ of $\mathrm{Fun}(\mathcal{D}^n,\mathcal{E}^m)$ in a similar way.

Let $\mathrm{Fun}^*_\circ(\mathcal{C}^n_\circ,\mathcal{E}^m)$ be the full subcategory (simplicial subset) of $\mathrm{Fun}(\mathcal{C}^n_\circ,\mathcal{E}^m)$, defined as follows. If we use notation similar to above, $f \in \mathrm{Fun}(\mathcal{C}^n_\circ,\mathcal{E}^m)$ belongs to $\mathrm{Fun}^*_\circ(\mathcal{C}^n_\circ,\mathcal{E}^m)$ if and only if the followings hold:

- $f \circ q^n : (\mathcal{C}^1_\circ)^{\times n} \to \mathcal{C}^n_\circ \to \mathcal{E}^m$ preserves finite colimits separately in each variable,

- if $f' : (\mathcal{C}^1)^{\times n} \simeq \mathcal{C}^n \to \mathcal{E}^m$ is an extension of f determined by Lemma 4.8 and the first condition, then f' belongs to $\mathrm{Fun}^*(\mathcal{C}^n, \mathcal{E}^m)$.

Let S be a simplicial category (that is, a category enriched over the monoidal category of simplicial sets) defined as follows: objects are $\mathcal{C}^n$ with $n \geq 0$ and $\mathcal{D}^n$ with $n \geq 0$. Let $\mathrm{Map}^*(\mathcal{C}^n, \mathcal{C}^m)$ be the largest Kan complex of $\mathrm{Fun}^*(\mathcal{C}^n, \mathcal{C}^m)$. The hom simplicial set $\mathrm{Map}_S(\mathcal{C}^n, \mathcal{C}^m)$ is defined to be the full subcategory of $\mathrm{Map}^*(\mathcal{C}^n, \mathcal{C}^m)$ spanned by functors which sends $\mathcal{C}^n_\circ$ to $\mathcal{C}^m_\circ$, and $\mathrm{Map}_S(\mathcal{C}^n, \mathcal{D}^m)$ is the largest Kan complex of $\mathrm{Fun}^*(\mathcal{C}^n, \mathcal{D}^m)$. The simlicial set $\mathrm{Map}_S(\mathcal{D}^n, \mathcal{D}^m)$ is the largest Kan complex of $\mathrm{Fun}^*(\mathcal{D}^n, \mathcal{D}^m)$ for any n and m. The simlicial set $\mathrm{Map}_S(\mathcal{D}^n, \mathcal{C}^m)$ is the empty set for any n and m. These data constitute a simplicial category S. Let $S' = \mathrm{N}(S)$.

Let T be a simplicial category defined as follows: objects are $\mathcal{C}^n_\circ$ with $n \geq 0$ and $\mathcal{D}^n$ with $n \geq 0$. We define subcategory T which satisfies the following properties: let $\mathrm{Map}^*_\circ(\mathcal{C}^n_\circ, \mathcal{C}^m_\circ)$ be the largest Kan complex of $\mathrm{Fun}^*_\circ(\mathcal{C}^n_\circ, \mathcal{C}^m_\circ)$. The mapping space $\mathrm{Map}_T(\mathcal{C}^n_\circ, \mathcal{C}^m_\circ)$ is defined to be $\mathrm{Map}^*_\circ(\mathcal{C}^n_\circ, \mathcal{C}^m_\circ)$, and $\mathrm{Map}_T(\mathcal{C}^n_\circ, \mathcal{D}^m)$ is the largest Kan complex of $\mathrm{Fun}^*_\circ(\mathcal{C}^n_\circ, \mathcal{D}^m)$. The simlicial set $\mathrm{Map}_T(\mathcal{D}^n, \mathcal{D}^m)$ is the largest Kan complex of $\mathrm{Fun}^*(\mathcal{D}^n, \mathcal{D}^m)$ for any n and m. The simplicial set $\mathrm{Map}_T(\mathcal{D}^n, \mathcal{C}^m_\circ)$ is the empty set for any n and m. Let $T' = \mathrm{N}(T)$.

Then there is a natural simplicial functor $S \to T$ which sends $\mathcal{C}^n$ and $\mathcal{D}^n$ to $\mathcal{C}^n_\circ$ and $\mathcal{D}^n$ respectively. The maps of hom simplicial sets

$$\mathrm{Map}_S(\mathcal{C}^n, \mathcal{C}^m) \to \mathrm{Map}_T(\mathcal{C}^n_\circ, \mathcal{C}^m_\circ)$$

and

$$\mathrm{Map}_S(\mathcal{C}^n, \mathcal{D}^m) \to \mathrm{Map}_T(\mathcal{C}^n_\circ, \mathcal{D}^m)$$

are induced by the restriction $\mathcal{C}^n_\circ \subset \mathcal{C}^n$, and in other case, maps of hom simplicial sets are identities. Then by Lemma 4.8 we deduce that the induced functor $S' \to T'$ is a categorical equivalence.

Let $\mathrm{Map}^{\mathrm{L}}_{\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \widehat{\mathrm{Cat}}_\infty)}(\mathrm{N}(z), \mathrm{N}(z'))$ be the full subcategory of

$$\begin{aligned} \mathrm{Map} & {}_{\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \widehat{\mathrm{Cat}}_\infty)}(\mathrm{N}(z), \mathrm{N}(z')) \\ = \quad & \mathrm{Map}(\Delta^1, \mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \widehat{\mathrm{Cat}}_\infty)) \times_{\mathrm{Map}(\partial\Delta^1, \mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \widehat{\mathrm{Cat}}_\infty))} (\mathrm{N}(z), \mathrm{N}(z')) \end{aligned}$$

$((\mathrm{N}(z), \mathrm{N}(z')) = \Delta^0)$ that corresponds to $\mathrm{Map}^{\otimes, \mathrm{L}}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$. The both functors $\mathrm{N}(z)$ and $\mathrm{N}(z')$ factor through $S' \subset \widehat{\mathrm{Cat}}_\infty$. Moreover by the definition of S, we have a categorical equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), S')}(\mathrm{N}(z), \mathrm{N}(z')) \simeq \mathrm{Map}^{\mathrm{L}}_{\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \widehat{\mathrm{Cat}}_\infty)}(\mathrm{N}(z), \mathrm{N}(z')).$$

Similarly, we have

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), T')}(\mathrm{N}(z_\circ), \mathrm{N}(z')) \simeq \mathrm{Map}^{\mathrm{ex}}_{\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \widehat{\mathrm{Cat}}_\infty)}(\mathrm{N}(z_\circ), \mathrm{N}(z')).$$

where $\mathrm{Map}^{\mathrm{ex}}_{\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \widehat{\mathrm{Cat}}_\infty)}(\mathrm{N}(z_\circ), \mathrm{N}(z'))$ is the full subcategory of

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \widehat{\mathrm{Cat}}_\infty)}(\mathrm{N}(z_\circ), \mathrm{N}(z'))$$

that corresponds to $\mathrm{Map}^{\otimes, \mathrm{ex}}(\mathcal{C}^\otimes_\circ, \mathcal{D}^\otimes)$ through the equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\mathrm{Fin}_*), \widehat{\mathrm{Cat}}_\infty)}(\mathrm{N}(z_\circ), \mathrm{N}(z')) \simeq \mathrm{Map}^{\otimes}(\mathcal{C}^\otimes_\circ, \mathcal{D}^\otimes).$$

Now the desired equivalence follows from the categorical equivalence $S' \to T'$. □

Let us recall the definition of perfectness of stacks introduced by Ben-Zvi, Francis, and Nadler in their work on derived Morita theory [1] (this notion is also important to our previous paper [11]). We say that a derived stack Y is perfect if the natural functor $\mathrm{Ind}(\mathrm{PMod}_Y) \to \mathrm{Mod}_Y$ is a categorical equivalence. As a corollary of results of this Section, we have:

Theorem 4.9. *Let Y be a perfect derived stack over R and $\pi : \mathrm{Spec}\, R \to Y$ is a section of the structure morphism $Y \to \mathrm{Spec}\, R$. Let $\pi^* : \mathrm{Mod}_Y^{\otimes} \to \mathrm{Mod}_R^{\otimes}$ be the morphism in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_R^{\otimes}/}$ induced by $\pi : \mathrm{Spec}\, R \to Y$, and let $\pi_\circ^* : \mathrm{PMod}_Y^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ denote its restriction which belongs to $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})_{\mathrm{PMod}_R^{\otimes}/}$. Let $\mathrm{Aut}(\pi_\circ^*) : \mathrm{CAlg}_R \to \mathrm{Grp}(\mathcal{S})$ be the automorphism functor of $\pi_\circ^*$. Then the restriction induces an equivalence of functors $\mathrm{Aut}(\pi^*) \to \mathrm{Aut}(\pi_\circ^*)$. In particular, the tannakization of $\pi_\circ^* : \mathrm{PMod}_Y^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ is equivalent to G_π. (see the setup before Proposition 4.6 for the notation G_π.)*

Proof. Combine Corollary 3.7, Proposition 4.6 and 4.7. □

Corollary 4.10. *Let Y be a derived stack over R equipped with $\pi : \mathrm{Spec}\, R \to Y$ as in Theorem 4.9. Suppose either one of followings:*

(i) *a derived stack Y over R belongs to Aff_R,*
(ii) *let G be an affine group scheme of finite type over a field k of characteristic zero, which we regard as a derived affine group scheme over $R = Hk$. Suppose that G acts on $X \in \mathrm{Aff}_R$ and let $Y = [X/G]$ be the quotient stack (see Example 4.1).*

Then the tannakization of $\pi_\circ^* : \mathrm{PMod}_Y^{\otimes} \to \mathrm{PMod}_R^{\otimes}$ is equivalent to G_π.

Proof. According to Proposition 4.6 and Corollary 3.7 and Theorem 4.9, it will suffices to show that Y is perfect, that is, the natural functor $\mathrm{Ind}(\mathrm{PMod}_Y) \to \mathrm{Mod}_Y$ is a categorical equivalence. Then our claim follows from [1, 3.19, 3.22]. □

Remark 4.11. By this Theorem, for example, we can prove that the tannakization of the ∞-category $\mathsf{Art}(k)^{\otimes}$ of Artin motives endowed with a homological realization functor $\mathsf{Art}(k)^{\otimes} \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$ (cf. [17, Section 6.3]) is equivalent to the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ which we regard as the limit of constant finite derived group schemes over $H\mathbf{K}$.

5. Bar constructions

This Section contains no new result. In this Section, we review the relation between bar constructions and the case (i) of Corollary 4.10. Let $A \in \mathrm{CAlg}_R$ and let $s : R \to A$ be the natural morphism in CAlg_R (note R is an initial object in CAlg_R). Suppose that $t : A \to R$ is a cosection of s, that is, $t \circ s$ is equivalent to the identity of R. Recall that Δ_+ is the category of finite (possibly empty) linearly ordered sets and we write $[-1]$ for the empty set. Let $\iota : \Delta^1 \to \mathrm{N}(\Delta_+)$ be a map which carries $\{0\}$ and $\{1\}$ to $[-1]$ and $[0]$ respectively. It is a fully faithful functor. Let $f : \Delta^1 \to \mathrm{CAlg}_R$ be the map corresponding to $A \to R$. Since CAlg_R admits small colimits, there is a left Kan extension

$$g : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_R$$

of f along ι. We refer to $g^{op} : \mathrm{N}(\Delta_+)^{op} \to \mathrm{Aff}_R$ as the Čech nerve of $f^{op} : (\Delta^1)^{op} \to \mathrm{Aff}_R$. This construction is called the bar construction for $t : A \to R$. The underlying simplicial object $\mathrm{N}(\Delta)^{op} \to \mathrm{N}(\Delta_+)^{op} \to \mathrm{Aff}_R$ is a group object (see [17, Appendix] or [23, 7.2.2.1] for the definition of group objects). Let G be a derived affine group scheme corresponding to the simplicial object.

Let $t_\circ^* : \mathrm{PMod}_A^\otimes \to \mathrm{PMod}_R^\otimes$ be the morphism in $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{st}})_{\mathrm{PMod}_R^\otimes/}$. The case (i) of Corollary 4.10 says:

Theorem 5.1. $\mathrm{Aut}(t_\circ^*)$ *is represented by* G. *In particular, by Corollary 3.7 the tannakization of* $t_\circ^* : \mathrm{PMod}_A^\otimes \to \mathrm{PMod}_R^\otimes$ *is equivalent to* G.

Remark 5.2. For the readers who are familiar with commutative differential graded algebras (dg-algebras for short), we relate the bar construction of commutative dg-algberas with G. Let k be a field of characteristic zero. Let dga_k be the category of commutative dg-algebras over k (cf. [14]). A morphism $P^\bullet \to Q^\bullet$ in dga_k is a weak equivalence (resp. fibration) if it induces a bijection $H^n(P^\bullet) \to H^n(Q^\bullet)$ for each $n \in \mathbb{Z}$ (resp. $P^n \to Q^n$ is a surjective morphism of k-vector spaces for each $n \in \mathbb{Z}$). There is a model category structure on dga_k whose weak equivalences and fibrations are defined in this way (see [14, 2.2.1]). Let $\mathrm{N}(\mathsf{dga}_k^c)_\infty$ be the ∞-category obtained from the full subcategory dga_k^c spanned by cofibrant objects by inverting weak equivalences (see [24, 1.3.4.15]). According to [24, 8.1.4.11], there is a categorical equivalence $\mathrm{N}(\mathsf{dga}_k^c)_\infty \simeq \mathrm{CAlg}_{Hk}$. Let $R = Hk$ and let $t : A \to k$ be an augmentation in dga_k. We abuse notation and we denote by $t : A \to R$ the induced morphism in CAlg_R. The underlying derived scheme of G is the fiber product $\mathrm{Spec}\,R \times_{\mathrm{Spec}\,A} \mathrm{Spec}\,R$ in Aff_R. By this equivalence, the pushout $R \otimes_A R$ in CAlg_R corresponds to a homotopy pushout $k \otimes_A^{\mathbb{L}} k$ in the model category dga_k, which is weak equivalent to a homotopy pushout $A \otimes_{A\otimes_k A}^{\mathbb{L}} k$ of

$$\begin{array}{ccc} A \otimes_k A & \xrightarrow{t\otimes t} & k \\ \downarrow{\scriptstyle m} & & \\ A & & \end{array}$$

where m is the multiplication. We will review the construction of the concrete model of a homotopy pushout $A \otimes_{A\otimes_k A}^{\mathbb{L}} k$ in dga_k, which is known as the bar construction of a commutative dg-algebra (see for example [28], [33]). Consider the adjoint pair

$$T : \mathsf{dga}_{k,A/} \rightleftarrows \mathsf{dga}_{k,A\otimes_k A/} : U$$

where U is the forgetful functor induced by $A \to A \otimes_k A$, $x \mapsto x \otimes 1$, and T is given by formula $M \mapsto M \otimes_A (A \otimes_k A)$. Let $\alpha : \mathrm{Id} \to UT$ and $\beta : TU \to \mathrm{Id}$ be the unit map and counit map respectively. To an object $C \in \mathsf{dga}_{k,A\otimes_k A/}$ one associates a simplicial diagram $(T,U)_\bullet(C)$ in $\mathsf{dga}_{k,A/}$ as follows: Define

$$(T,U)_n(C) = (TU)^{\circ(n+1)}(C) = (TU) \circ \cdots \circ (TU)(C)$$

where the right hand side is the $(n+1)$-fold composition. For $0 \le i \le n+1$,

$$\begin{aligned} d_i : (T,U)_{n+1}(C) &= (TU)^{\circ i} \circ (TU) \circ (TU)^{\circ(n+1-i)}(C) \\ &\to (TU)^{\circ i} \circ \mathrm{Id} \circ (TU)^{\circ(n+1-i)}(C) = (T,U)_n(C) \end{aligned}$$

is induced by β in the middle term. For $0 \leq i \leq n$,

$$\begin{aligned} s_i : (T,U)_n(C) &= (TU)^{\circ i} \circ T \circ \mathrm{Id} \circ U \circ (TU)^{\circ(n-i)}(C) \\ &\to (TU)^{\circ i} \circ T \circ (UT) \circ U \circ (TU)^{\circ(n-i)}(C) = (T,U)_{n+1}(C) \end{aligned}$$

is induced by $\alpha : \mathrm{Id} \to (UT)$ in the middle term. Let us consider A to be an object in $\mathsf{dga}_{k,A\otimes_k A/}$ via $m : A \otimes_k A \to A$. Then by the above construction we obtain the simplicial object $(T,U)_\bullet(A)\otimes_{A\otimes_k A}k$ in dga_k. The totalization $\mathrm{tot}((T,U)_\bullet(A)\otimes_{A\otimes_k A}k) \in \mathsf{dga}_k$, which we call the bar complex, represents the homotopy pushout $A \otimes^{\mathbb{L}}_{A\otimes_k A} k$.

6. Mixed Tate motives

In this Section, as an application of the results we have proved; in particular Theorem 4.9 and Corollary 4.10, we will describe the tannakization of the stable ∞-category of mixed Tate motives equipped with the realization functor as the $\mathbb{G}_m$-equivariant bar construction of a commutative dg-algebra. The main goal of this Section is Theorem 6.11. We emphasize that this section works without assuming Beilinson-Soulé vanishing conjecture. In what follows we often use model categories. Our references for them are [16] and [23, Appendix].

6.1. **Review of ∞-category of mixed motives.** Let $\mathbf{K}$ be a field of characteristic zero. Let $\mathcal{A}$ be the abelian category of $\mathbf{K}$-vector spaces. We equip the category of complexes of $\mathbf{K}$-vector spaces, denoted by $\mathrm{Comp}(\mathcal{A})$, with the projective model structure, in which weak equivalences are quasi-isomorphisms, and fibrations are degreewise surjective maps (cf. e.g. [16, Section 2.3], [23, Appendix], [5]).

Let k be a perfect field. Let $\mathrm{DM}^{eff}(k)$ be the category of complexes of $\mathcal{A}$-valued Nisnevich sheaves with transfers (the indroductory references of this notion include [26] and [7]). For a smooth scheme X separated of finite type over k, we denote by $L(X)$ the $\mathcal{A}$-valued Nisnevich sheaves with transfers which is represented by X (cf. [26, page.15]). We equip $\mathrm{DM}^{eff}(k)$ with the symmetric monoidal model structure in [5, Example 4.12]. The triangulated subcategory of the homotopy category of this model category $\mathrm{DM}^{eff}(k)$, spanned by right bounded complexes, is equivalent to the triangulated category $\mathbf{DM}^{\mathrm{eff},-}_{Nis}(k,\mathbf{K})$ constructed in [26, Lecture 14].

The pointed algebraic torus $\mathrm{Spec}(k) \to \mathbb{G}_m$ over k induces a split monomorphism $L(\mathrm{Spec}(k)) \to L(\mathbb{G}_m)$ in $\mathrm{DM}^{eff}(k)$. Then we define $\mathbf{K}(1)$ to be

$$\mathrm{Coker}(L(\mathrm{Spec}(k)) \to L(\mathbb{G}_m))[-1].$$

Let $\mathrm{DM}(k)$ be the category of symmetric $\mathbf{K}(1)$-spectra in $(\mathrm{DM}^{eff}(k))^{\mathfrak{S}}$ (cf. [5, Section 7]) which is endowed with the symmetric monoidal model structure in [5, Example 7.15] (see loc. cit. for details). Then we have a sequence of left Quillen symmetric monoidal functors

$$\mathrm{Comp}(\mathcal{A}) \longrightarrow \mathrm{DM}^{eff}(k) \stackrel{\Sigma^\infty}{\longrightarrow} \mathrm{DM}(k),$$

where the first functor sends the unit to $L(\mathrm{Spec}(k))$, and the second functor is the infinite suspension functor.

Recall the localization method in [24, 1.3.4.1, 1.3.1.15, 4.1.3.4] (see also [10], [17, Section 6] and [17, Proposition 6.8]); it associates to any (symmetric monoidal) model category $\mathbb{M}$ a (symmetric monoidal) ∞-category $\mathrm{N}(\mathbb{M}^c)_\infty$. Here $\mathbb{M}^c$ is the full subcategory spanned by cofibrant objects (this restriction is due to the technical reason for the construction of symmetric monoidal ∞-categories). We shall refer to the associated (symmetric monoidal) ∞-category as the (symmetric monoidal) ∞-category obtained from the model category $\mathbb{M}$ by inverting weak equivalences. Applying this localization, we obtain a symmetric monoidal functors of symmetric monoidal ∞-categories

$$\mathrm{Mod}_{H\mathbf{K}}^{\otimes} \simeq \mathrm{N}(\mathrm{Comp}(\mathcal{A})^c)_\infty \to \mathrm{N}(\mathrm{DM}^{eff}(k)^c)_\infty \to \mathrm{N}(\mathrm{DM}(k)^c)_\infty.$$

where the first equivalence follows from [24, 8.1.2.13]. Here $H\mathbf{K}$ denotes the Eilenberg-MacLane spectrum. We shall write DM and DM^{eff} for $\mathrm{N}(\mathrm{DM}(k)^c)_\infty$ and $\mathrm{N}(\mathrm{DM}^{eff}(k)^c)_\infty$ respectively. When we indicate that DM is the symmetric monoidal ∞-category, we denote it by $\mathsf{DM}^{\otimes}$. In [17, Section 6] we have constructed another symmetric monoidal stable presentable ∞-category $\mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK})$ by using the recipe in [6] and [29]. We do not review the construction; but there is an equivalence $\mathsf{DM}^{\otimes} \simeq \mathsf{Sp}^{\otimes}_{\mathrm{Tate}}(\mathbf{HK})$ (cf. [24, Remark 6.6]).

It should be emphasized that there are several (quite different but equivalent) constructions of the category of mixed motives as differential-graded categories and model categories. One can obtain ∞-categories from differential-graded categories and model categories. In our work, it is important to treat "the category of mixed motives" as a *symmetric monoidal* ∞-category, and therefore we choose the symmeric monoidal model category $\mathrm{DM}(k)$ constructed by Cisinski-Déglise.

6.2. **∞-category of mixed Tate motives.** Let us recall the stable ∞-category of mixed Tate motives. We also denote by $\mathbf{K}(1)$ its image of $\mathbf{K}(1) \in \mathrm{DM}^{eff}(k)$ in $\mathrm{DM}(k)$. It is a cofibrant object and $\mathbf{K}(1)$ can be regard as an object in the ∞-category DM. There exists the dual object of $\mathbf{K}(1)$ in DM, which we will denote by $\mathbf{K}(-1)$. Let DTM be the presentable stable subcategory generated by $\mathbf{K}(1)^{\otimes n} = \mathbf{K}(n)$ for $n \in \mathbb{Z}$, where $\mathbf{K}(1)^{\otimes n}$ is the n-fold tensor product in $\mathsf{DM}^{\otimes}$. Namely, DTM is the smallest stable subcategory in DM, which admits coproducts (thus all small colimits) and consists of $\mathbf{K}(n)$ for all $n \in \mathbb{Z}$. The tensor product functor $\otimes : \mathsf{DM} \times \mathsf{DM} \to \mathsf{DM}$ preserves small colimits and translations (suspensions and loops) separately in each variable, and thus the symmetric monoidal structure of DM induces a symmetric monoidal structure on DTM. We denote by $\mathsf{DTM}^{\otimes}$ the resulting symmetric monoidal stable presentable ∞-category. Note that the inclusion $\mathsf{DTM} \hookrightarrow \mathsf{DM}$ preserves small colimits. Let DTM_{gm} be the smallest stable subcategory consisting of $\mathbf{K}(n)$ for $n \in \mathbb{Z}$. Since $\mathbf{K}(n)$ is compact in DM for every $n \in \mathbb{Z}$, every object in DTM_{gm} is compact in DM. Let $\mathrm{Ind}(\mathsf{DTM}_{gm}) \to \mathsf{DTM}$ be a (colimit-preserving) left Kan extension of $\mathsf{DTM}_{gm} \to \mathsf{DTM}$, which is fully faithful by [23, 5.3.5.11]. Hence it identifies $\mathrm{Ind}(\mathsf{DTM}_{gm})$ with DTM. The symmetric monoidal functor $\mathrm{Mod}_{H\mathbf{K}}^{\otimes} \to \mathsf{DM}^{\otimes}$ factors through $\mathsf{DTM}^{\otimes} \subset \mathsf{DM}^{\otimes}$ since $\mathsf{DTM}^{\otimes} \hookrightarrow \mathsf{DM}^{\otimes}$ preserves small colimits, and DTM contains the unit of DM. The factorization $\mathrm{Mod}_{H\mathbf{K}}^{\otimes} \to \mathsf{DTM}^{\otimes} \hookrightarrow \mathsf{DM}^{\otimes}$ is regarded as a map in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})_{\mathrm{Mod}_{H\mathrm{K}}^{\otimes}/}$ which we also denote by $\mathsf{DTM}^{\otimes} \hookrightarrow \mathsf{DM}^{\otimes}$.

Lemma 6.1. *Let $\mathsf{DTM}_\vee$ be the full subcategory of $\mathsf{DTM}^{\otimes}$ spanned by dualizable objects. Let $\mathsf{DTM}_\circ$ be the full subcategory of DTM spanned by compact objects. Then $\mathsf{DTM}_\circ = \mathsf{DTM}_\vee$.*

Proof. Observe that every object in $\mathsf{DTM}_\vee$ is compact in DTM. To this end, it is enough to show that the unit object of $\mathsf{DTM}^{\otimes}$ is compact (cf. [6, 2.5.1]). This is implied by [6, Theorem 2.7.10]. For any $n \in \mathbb{Z}$, $\mathbf{K}(n)$ belongs to $\mathsf{DTM}_\vee$. Therefore $\mathsf{DTM}_{gm} \subset$

$\mathsf{DTM}_{\vee} \subset \mathsf{DTM}_{\circ}$. Notice that $\mathsf{DTM}_{gm} \subset \mathsf{DTM}_{\circ}$ can be viewed as an idempotent-completion (see e.g. [3, Lemma 2.14]). Moreover DTM is idempotent-complete by [23, 4.4.5.16]. It will suffice to prove that the inclusion $\mathsf{DTM}_{\vee} \subset \mathsf{DTM}$ is closed under retracts. It easily follows from the definition of dualizable objects. □

Let $\prod_S \mathrm{DM}$ be a product of the category DM, indexed by a small set S. There is a combinatorial model structure on $\prod_S \mathrm{DM}$, called projective model structure (cf. [23, A. 2.8.2]), in which weak equivalences (resp. fibrations) are termwise weak equivalences (resp. termwise fibrations) in DM. Notice that cofibrations in $\prod_S \mathrm{DM}$ are termwise cofibrations. When $S = \mathbb{N}$, $\prod_{\mathbb{N}} \mathrm{DM}$ has a symmetric monoidal structure defined as follows: Let $(M_i)_{i\in\mathbb{N}}$ and $(N_j)_{j\in\mathbb{N}}$ be two objects in $\prod_{\mathbb{N}} \mathrm{DM}$. Then $(M_i)_{i\in\mathbb{N}} \otimes (N_j)_{j\in\mathbb{N}}$ is defined to be $(\oplus_{i+j=k} M_i \otimes N_j)_{k\in\mathbb{N}}$.

Lemma 6.2. *With the above symmetric monoidal structure, $\prod_{\mathbb{N}} \mathrm{DM}$ is a symmetric monoidal model category in the sense of* [23, A 3.1.2].

Proof. We must prove that cofibrations $\alpha : (M_i) = (M_i)_{i\in\mathbb{N}} \to (M_i) = (M'_i)_{i\in\mathbb{N}}$ and $\beta : (N_i) = (N_i)_{i\in\mathbb{N}} \to (N_i) = (N'_i)_{i\in\mathbb{N}}$ induce a cofibration

$$\alpha \wedge \beta : (M_i) \otimes (N'_i) \coprod_{(M_i)\otimes(N_i)} (M'_i) \otimes (N_i) \to (M'_i) \otimes (N'_i),$$

and moreover if either α or β is a trivial cofibration, then $\alpha \wedge \beta$ is also a trivial cofibration. Unwinding the definition, we are reduced to showing that

$$\bigoplus_{i+j=k} \big(M_i \otimes N'_j \coprod_{M_i \otimes N_j} M'_i \otimes N_j\big) \to \bigoplus_{i+j=k} M'_i \otimes N'_j$$

is a cofibration in DM, and moreover it is a trivial cofibration if either α or β is a trivial cofibration. This is implied by the left lifting property of (trivial) cofibrations and the fact that DM is a symmetric monoidal model category. □

Consider the symmetric monoidal functor $\xi : \prod_{\mathbb{N}} \mathrm{DM} \to \mathrm{DM}$, which carries (M_i) to $\oplus_i M_i \otimes \mathbf{K}(-i)$. Here $\mathbf{K}(-1)$ is a cofibrant "model" of the dual of $\mathbf{K}(1)$, and $\mathbf{K}(-i)$ is i-fold tensor product of $\mathbf{K}(-1)$ in the symmetric monoidal category DM. Since $\mathbf{K}(-i)$ is cofibrant, we see that ξ is a left Quillen adjoint functor. By the localization, we obtain a symmetric monoidal left adjoint functor

$$f := \mathrm{N}(\xi) : \mathsf{DM}^{\otimes}_{\mathbb{N}} := \mathrm{N}((\prod_{\mathbb{N}} \mathrm{DM})^c)_\infty \to \mathrm{N}(\mathrm{DM}^c)_\infty = \mathsf{DM}^{\otimes}.$$

By the relative version of adjoint functor theorem [24, 8.3.2.6] (see also [25, VIII 3.2.1]), f has a lax symmetric monoidal right adjoint functor which we denote by $g : \mathsf{DM}^{\otimes} \to \mathsf{DM}^{\otimes}_{\mathbb{N}}$. It yields $g : \mathrm{CAlg}(\mathsf{DM}^{\otimes}) \to \mathrm{CAlg}(\mathsf{DM}^{\otimes}_{\mathbb{N}})$. We set $A := g(1_{\mathsf{DM}})$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}_{\mathbb{N}})$, where 1_{DM} is a unit in $\mathsf{DM}^{\otimes}$. The adjoint pair

$$f : \mathsf{DM}_{\mathbb{N}} \rightleftarrows \mathsf{DM} : g$$

induces the adjoint pair

$$f : \mathrm{h}(\mathsf{DM}_{\mathbb{N}}) \rightleftarrows \mathrm{h}(\mathsf{DM}) : g$$

of homotopy categories. Let $\mathsf{Hom}(N, -)$ denote the internal Hom object given by the right adjoint of $(-) \otimes N : \mathsf{DM} \to \mathsf{DM}$. Then g is given by $M \mapsto (\mathsf{Hom}(\mathbf{K}(-i), M))_{i\in\mathbb{N}}$.

Thus the underlying object A in $\mathrm{h}(\mathsf{DM})$ is $(\mathbf{K}(i))_{i\in\mathbb{N}}$, that is, the i-th term is $\mathbf{K}(i)$. Moreover, by the straightforward calculation of adjunction maps, we see that the commutative algebra structure of A in the symmetric monoidal homotopy category $\mathrm{h}(\mathsf{DM})$ is given by

$$(\mathbf{K}(i))_{i\in\mathbb{N}}\otimes(\mathbf{K}(j))_{j\in\mathbb{N}}=(\oplus_{i+j=k}\mathbf{K}(i)\otimes\mathbf{K}(j))_{k\in\mathbb{N}}\to(\mathbf{K}(k))_{k\in\mathbb{N}}$$

where the second map is induced by the identity maps $\mathbf{K}(i)\otimes\mathbf{K}(j)\simeq\mathbf{K}(k)\to\mathbf{K}(k)$.

Now recall from [31] the notion of "periodic" commutative ring object (in loc. cit. the notion of "periodizable" is introduced, and we use this notion in a slightly modified form). Let $\prod_{\mathbb{Z}}\mathrm{DM}$ be the product of DM indexed by $\mathbb{Z}$, which is a combinatorial model category defined as above. By the tensor product $(M_i)_{i\in\mathbb{Z}}\otimes(N_j)_{j\in\mathbb{Z}}=(\oplus_{i+j=k}M_i\otimes N_j)_{k\in\mathbb{Z}}$, $\prod_{\mathbb{Z}}\mathrm{DM}$ is a symmetric monoidal model category in the same way that $\prod_{\mathbb{N}}\mathrm{DM}$ is so. Let $\mathsf{DM}^{\otimes}_{\mathbb{Z}}$ be the symmetric monoidal ∞-category obtained from $(\prod_{\mathbb{Z}}\mathrm{DM})^c$ by inverting weak equivalences. A commutative algebra object X in $\mathsf{DM}^{\otimes}_{\mathbb{Z}}$ is said to be periodic if the underlying object is of the form

$$(\dots,\mathbf{K}(-1),\mathbf{K}(0),\mathbf{K}(1),\dots),$$

that is, $\mathbf{K}(i)$ sits in the i-th degree, and the commutative algebra structure of X in $\mathrm{h}(\mathsf{DM}^{\otimes}_{\mathbb{Z}})$ induced by that in $\mathsf{DM}^{\otimes}_{\mathbb{Z}}$ is determined by the identity maps $\mathbf{K}(i)\otimes\mathbf{K}(j)\to\mathbf{K}(i+j)$.

A periodic commutative algebra object actually exists. To construct it, we let $i:\mathsf{DM}^{\otimes}_{\mathbb{N}}\to\mathsf{DM}^{\otimes}_{\mathbb{Z}}$ be the symmetric monoidal functor informally given by $(M_i)_{i\in\mathbb{N}}\mapsto(\dots,0,0,M_0,M_1,\dots)$. Namely, it is determined by inserting 0 in each negative degree. Then $P_+:=i(A)$ belongs to $\mathrm{CAlg}(\mathsf{DM}^{\otimes}_{\mathbb{Z}})$. According to [31, Proposition 4.2] and its proof, we have:

Proposition 6.3 ([31])**.** *There exists a morphism $P_+\to P$ in $\mathrm{CAlg}(\mathsf{DM}^{\otimes}_{\mathbb{Z}})$ such that P is periodic.*

Remark 6.4. Let $\mathbf{K}(1)_1$ be the object of the form $(\dots,0,\mathbf{K}(1),0,\dots)$ where $\mathbf{K}(1)$ sits in the 1-st degree. Let $\mathrm{Sym}^*_{P_+}:\mathrm{Mod}_{P_+}(\mathsf{DM}^{\otimes}_{\mathbb{Z}})\to\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{P_+}(\mathsf{DM}^{\otimes}_{\mathbb{Z}}))$ be the left adjoint of the forgetful functor. Let

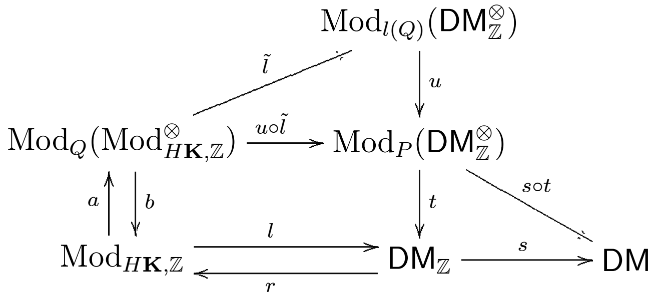
$$\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{P_+}(\mathsf{DM}^{\otimes}_{\mathbb{Z}}))\rightleftarrows\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{P_+}(\mathsf{DM}^{\otimes}_{\mathbb{Z}}))[\mathrm{Sym}^*_{P_+}(\kappa)^{-1}]$$

be the localization adjoint pair (cf. [23, 5.2.7.2, 5.5.4]) which inverts $\mathrm{Sym}^*_{P_+}(\kappa)$, where $\kappa:\mathbf{K}(1)_1\otimes P_+\to P_+$ in $\mathrm{Mod}_{P_+}(\mathsf{DM}^{\otimes}_{\mathbb{Z}})$ induced by the natural embedding $\mathbf{K}(1)_1\to P_+$ in the 1-st degree. The morphism $P_+\to P$ is obtained as the unit map of this adjoint pair.

Let $\prod_{\mathbb{Z}}\mathrm{Comp}(\mathcal{A})$ be the product of the category $\mathrm{Comp}(\mathcal{A})$, that is endowed with the projective model structure. As in Lemma 6.2, we see that $\prod_{\mathbb{Z}}\mathrm{Comp}(\mathcal{A})$ is a symmetric monoidal model category, whose tensor product is given by $(A_i)_{i\in\mathbb{Z}}\otimes(B_j)_{j\in\mathbb{Z}}=(\oplus_{i+j=k}A_i\otimes B_j)_{k\in\mathbb{Z}}$. Then the natural left Quillen adjoint symmetric monoidal functor $\mathrm{Comp}(\mathcal{A})\to\mathrm{DM}$ naturally extends to a left Quillen adjoint symmetric monoidal functor $l:\prod_{\mathbb{Z}}\mathrm{Comp}(\mathcal{A})\to\prod_{\mathbb{Z}}\mathrm{DM}$. It gives rise to the symmetric monoidal left adjoint functor of ∞-categories

$$l:\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}}:=\mathrm{N}(\prod_{\mathbb{Z}}\mathrm{Comp}(\mathcal{A})^c)^{\otimes}_{\infty}\to\mathsf{DM}^{\otimes}_{\mathbb{Z}}.$$

According to the relative version of adjoint functor theorem [24, 8.3.2.6] (see also [25, VIII 3.2.1]), l has a lax symmetric monoidal right adjoint functor r. Let $Q := r(P) \in \mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}})$. Let $\mathrm{DM} \to \prod_{\mathbb{Z}} \mathrm{DM}_{\mathbb{Z}}$ be the left Quillen symmetric monoidal functor which carries M to (M_i) where $M_0 = M$ and $M_i = 0$ if $i \neq 0$. Thus we have a symmetric monoidal functor $\mathsf{DM} \to \mathsf{DM}_{\mathbb{Z}}$, and again by the relative version of adjoint functor theorem we obtain a lax symmetric monoidal functor $s : \mathsf{DM}_{\mathbb{Z}} \to \mathsf{DM}$ as the right adjoint. Therefore there exists a diagram of symmetric monoidal ∞-categories:



such that

- $\tilde{l}$ is a symmetric monoidal functor induced by l,
- u is the symmetric monoidal base change functor induced by the counit map $l(Q) = l(r(P)) \to P$,
- t is the forgetful monoidal functor which is a lax symmetric monoidal functor,
- a is the base change functor, and b is the forgetful functor.

Let $z := s \circ t \circ u \circ \tilde{l}$. We recall the theorem by Spitzweck [31, Theorem 4.3] (see also its proof):

Theorem 6.5 ([31]). *The composite* $z : \mathrm{Mod}_Q(\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}}) \to \mathsf{DM}$ *gives an equivalence* $\mathrm{Mod}_Q(\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}}) \simeq \mathsf{DTM}$ *as symmetric monoidal ∞-categories.*

Furthermore, we can see that z gives an equivalence of them as $H\mathbf{K}$-*linear* symmetric monoidal ∞-categories. To see this, it is enough to show that z is promoted to a $H\mathbf{K}$-linear symmetric monoidal functor. To treat problems of this type, the following Lemma is useful.

Lemma 6.6. *Let $\mathcal{C}^{\otimes}$ be in* $\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})$. *We denote by $\mathcal{C}$ the underlying ∞-category. Suppose that a unit* $\mathbf{1}$ *of $\mathcal{C}^{\otimes}$ is compact in $\mathcal{C}$. Let $\mathcal{C}_{\mathbf{1}} \subset \mathcal{C}$ be the smallest stable subcategory which admits small colimits and contains* $\mathbf{1}$. *The ∞-category $\mathcal{C}_{\mathbf{1}}$ admits a symmetric monoidal structure induced by that of $\mathcal{C}^{\otimes}$. Then there exist A in* CAlg *and an equivalence* $\mathrm{Mod}^{\otimes}_A \simeq \mathcal{C}^{\otimes}$ *of symmetric monoidal ∞-categories. Moreover, if R is a commutative ring spectrum and $p : \mathrm{Mod}^{\otimes}_R \to \mathcal{C}^{\otimes}$ is a symmetric monoidal colimit-preserving functor, then p factors through $\mathcal{C}^{\otimes}_{\mathbf{1}} \subset \mathcal{C}^{\otimes}$ and there exists a morphism $R \to A$ in* CAlg, *up to the contractible space of choice, which induces* $\mathrm{Mod}^{\otimes}_R \to \mathcal{C}^{\otimes}_{\mathbf{1}} \simeq \mathrm{Mod}^{\otimes}_A$ *(as the base change).*

Proof. The first assertion follows from [24, 8.1.2.7]; the characterization of symmetric monoidal stable ∞-categories of module spectra. Since p preserves small colimits, p factors through $\mathcal{C}^{\otimes}_{\mathbf{1}} \subset \mathcal{C}^{\otimes}$. The last assertion is implied by [24, 6.3.5.18]. □

Remark 6.7. Under the assumption of Lemma 6.6, A is considered to be the "endomorphism algebra" of the unit, and we can say that giving a R-linear structure, that is, a symmetric monoidal colimit-preserving functor $\mathrm{Mod}_R^{\otimes} \to \mathcal{C}^{\otimes}$ is equivalent to giving a morphism $R \to A$ in CAlg.

Return to the case of $H\mathbf{K}$-linear symmetric monoidal ∞-category $\mathsf{DTM}^{\otimes}$. The endomorphism algebra of the unit of $\mathsf{DTM}^{\otimes}$ is $H\mathbf{K}$ (i.e. $\mathbf{K}$), and its $H\mathbf{K}$-linear structure is determined by the identity $H\mathbf{K} \to H\mathbf{K}$. Thus, to promote z to a $H\mathbf{K}$-linear symmetric monoidal functor, it is enough to show that $f \circ a \circ q : \mathrm{Mod}_{H\mathbf{K}}^{\otimes} \to \mathsf{DTM}^{\otimes}$ induces the identity morphism of endomorphism algebras of units $H\mathbf{K} \to H\mathbf{K}$, where q is the inclusion $\mathrm{Mod}_{H\mathbf{K}}^{\otimes} \to \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes}$ into the degree zero part. This claim is clear from our construction.

6.3. **Realization functor and augmentation.** Let E be a mixed Weil theory with $\mathbf{K}$-coefficients (cf. [6, Definition 2.1]). A mixed Weil theory is a presheaf of commutative dg $\mathbf{K}$-algebras on the category of smooth affine schemes over k, which satisfies Nisnevich descent property, $\mathbb{A}^1$-homotopy, Künneth formula and axioms of dimensions, etc (for the precise definition see [6, 2.1.4]). For example, algebraic de Rham cohomology determines a mixed Weil theory with $\mathbf{K} = k$; to any smooth affine scheme X we associates a commutative dg k-algebra $\Gamma(X, \Omega^*_{X/k})$ where $\Omega^*_{X/k}$ is the algebraic de Rham complex arising from the exterior $\mathcal{O}_X$-algebra generated by $\Omega^1_{X/k}$. Another example is l-adic étale cohomology with $\mathbf{K} = \mathbb{Q}_l$ (see [6, Section 3]). To a mixed Weil theory E we can associate

$$\mathsf{R} : \mathsf{DM}^{\otimes} \to \mathrm{Mod}_{H\mathbf{K}}^{\otimes}$$

a morphism in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty}^{\mathrm{L,st}})_{\mathrm{Mod}_{H\mathbf{K}}^{\otimes}/}$ which we call the homological realization functor with respect to E (see [17, Section 6.1, 6.2], [6, 2.6]). Then according to [6, 2.7.14] when E is the mixed Weil theory associated to algebraic de Rham cohomology, for any smooth affine scheme X the image $\mathsf{R}(h(X))$ in $\mathrm{Mod}_{H\mathbf{K}}$ is equivalent to the dual complex of the derived global section $\mathbf{R}\Gamma(X, \Omega^*_{X/k})$ where by [17, 6.8] we identify $\mathrm{Mod}_{H\mathbf{K}}$ with the ∞-category of unbounded complexes of $\mathbf{K}$-vector spaces. We denote by R_T the composition

$$\mathsf{DTM}^{\otimes} \hookrightarrow \mathsf{DM}^{\otimes} \to \mathrm{Mod}_{H\mathbf{K}}^{\otimes}$$

which we call the homological realization of Tate motives (with respect to E). By the restrictions, it gives rise to the morphism $\mathsf{DTM}_{\vee}^{\otimes} \to \mathrm{PMod}_{H\mathbf{K}}^{\otimes}$ in $\mathrm{CAlg}(\mathrm{Cat}_{\infty}^{\mathrm{st}})_{\mathrm{PMod}_{H\mathbf{K}}^{\otimes}/}$ which we denote also by R_T.

Combined with Theorem 6.5 we have the sequence of symmetric monoidal colimit-preserving functors

$$\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes} \xrightarrow{a} \mathrm{Mod}_Q^{\otimes}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}) \simeq \mathsf{DTM}^{\otimes} \xrightarrow{\mathsf{R}_T} \mathrm{Mod}_{H\mathbf{K}}^{\otimes}.$$

By the relative version of adjoint functor theorem, the composition admits a lax symmetric monoidal right adjoint functor ξ. In particular, if we set $R = \xi(1_{H\mathbf{K}})$ with $1_{H\mathbf{K}}$ the unit of $\mathrm{Mod}_{H\mathbf{K}}^{\otimes}$, then R belongs to $\mathrm{CAlg}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes})$. By the functoriality and the construction of Q, we have the natural morphism $Q \to R$ in $\mathrm{CAlg}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes})$. There is a commutative diagram (up to homotopy) of symmetric monoidal ∞-categories

$$\begin{array}{ccccc}
\mathrm{Mod}_Q^{\otimes}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes}) & \xrightarrow[z]{\sim} & \mathsf{DTM}^{\otimes} & \xrightarrow[\mathsf{R}_T]{} & \mathrm{Mod}_{H\mathbf{K}}^{\otimes} \\
\downarrow & & \downarrow & & \uparrow \\
\mathrm{Mod}_R^{\otimes}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes}) & \xrightarrow[\tilde{z}]{} & \mathrm{Mod}_{f(R)}^{\otimes}(\mathsf{DTM}^{\otimes}) & \xrightarrow[\tilde{\mathsf{R}}_T]{} & \mathrm{Mod}_{\mathsf{R}_T(z(R))}^{\otimes}(\mathrm{Mod}_{H\mathbf{K}}^{\otimes})
\end{array}$$

where $\tilde{z}$ and $\tilde{\mathsf{R}}_T$ are induced by z and R_T respectively, the left and central vertical arrows are base change functors, and the right vertical arrow is the counit map $\mathsf{R}_T(z(R)) \to H\mathbf{K}$ in $\mathrm{CAlg}(\mathrm{Mod}_{H\mathbf{K}}^{\otimes})$. Note that all functors in the diagram are $H\mathbf{K}$-linear symmetric monoidal functors. The commutativity of the right square follows from the observation that the counit map $\mathsf{R}_T(z(R)) \to H\mathbf{K}$ is an augmentation of the structure map $H\mathbf{K} \to \mathsf{R}_T(z(R))$.

Lemma 6.8. *The composite $h : \mathcal{C}^{\otimes} := \mathrm{Mod}_R^{\otimes}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes}) \to \mathcal{D}^{\otimes} := \mathrm{Mod}_{H\mathbf{K}}^{\otimes}$ in the above diagram gives an equivalence of $H\mathbf{K}$-linear symmetric monoidal ∞-categories.*

Proof. It will suffice to show that the underlying functor is a categorical equivalence.

The symmetric monoidal functor h is $H\mathbf{K}$-linear. Thus h is essentially surjective.

Next we will show that h is fully faithful. Let $\mathbf{K}_n := (\dots 0, \mathbf{K}, 0 \dots)$ be the object in $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}$ such that $\mathbf{K}$ sits in the n-th degree. Let $R(n)$ be the image of $\mathbf{K}_n$ by the base change functor $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}_R(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes})$. (For any $n \in \mathbb{Z}$, $h(R(n)) \simeq H\mathbf{K}$.) It is enough to prove that

$$\mathrm{Map}_{\mathcal{C}}(R(i), R(j)) \to \mathrm{Map}_{\mathcal{D}}(h(R(i)), h(R(j)))$$

is an equivalence in $\mathcal{S}$. Indeed, $\mathcal{C}$ is generated by the sets $\{R(i)\}_{i\in\mathbb{Z}}$ under finite (co)limits, translations, and filtered colimits. Since $R(i)$ and $h(R(i))$ is compact for each $i \in \mathbb{Z}$ and h is colimit-preserving, we are reduced to showing that the above map is an equivalence in $\mathcal{S}$. (Assuming it to hold, note first that $\mathrm{Map}_{\mathcal{C}}(R(i), N) \to \mathrm{Map}_{\mathcal{D}}(h(R(i)), h(N))$ is an equivalence in $\mathcal{S}$ for N being in the smallest stable subcategory $\mathcal{C}'$ generated by $\{R(i)\}_{i\in\mathbb{Z}}$. Then since $R(i)$ and $h(R(i))$ are compact, $\mathrm{Ind}(\mathcal{C}') \simeq \mathcal{C}$, and h preserves small colimits, thus for any $N \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(R(i), N) \to \mathrm{Map}_{\mathcal{D}}(h(R(i)), h(N))$ is an equivalence. Since $\mathcal{C}$ is generated by $\{R(i)\}_{i\in\mathbb{Z}}$ under finite colimits, translations and filtered colimits, we conclude that for any $M, N \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(M, N) \to \mathrm{Map}_{\mathcal{D}}(h(M), h(N))$ is an equivalence.) Note that $\mathrm{Map}_{\mathcal{C}}(R(i), R(j)) \simeq \mathrm{Map}_{\mathcal{C}}(R(i-j), R)$, and therefore we may and will assume that $j = 0$. Then by using adjunctions we can identify $\mathrm{Map}_{\mathcal{C}}(R(i), R) \to \mathrm{Map}_{\mathcal{D}}(h(R(i)), h(R))$ with the composition

$$\begin{aligned}
\mathrm{Map}_{\mathcal{C}}(R(i), R) &\xrightarrow{\sim} \mathrm{Map}_{\mathrm{Mod}_Q(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes})}(Q(i), R) \\
&\xrightarrow{\sim} \mathrm{Map}_{\mathrm{Mod}_{H\mathbf{K}}}(\mathsf{R}_T(z(Q(i))), H\mathbf{K}) \\
&\xrightarrow{\sim} \mathrm{Map}_{\mathrm{Mod}_{H\mathbf{K}}}(H\mathbf{K}, H\mathbf{K}).
\end{aligned}$$

This proves our Lemma. □

Proposition 6.9. *There exists a $H\mathbf{K}$-linear symmetric monoidal equivalence*

$$\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}^{\otimes} \to \mathrm{Mod}_{\mathrm{B}\mathbb{G}_m}^{\otimes}.$$

Proof. We will construct a symmetric monoidal functor $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}$, which preserves colimits.

For this purpose, we will construct $\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}$ in a somewhat explicit way. Regard the group scheme $\mathbb{G}_m$ over $\mathbf{K}$ as the simplicial scheme, denoted by $G_\bullet$ such that G_i is the i-fold product $\mathbb{G}_m^{\times i}$. This corresponds to the cosimplicial $\mathbf{K}$-algebra $\Gamma(G)^\bullet$ such that $\Gamma(G)^i \simeq \mathbf{K}[t_1^{\pm}, \ldots, t_i^{\pm}]$. The cosimplicial $\mathbf{K}$-algebra $\Gamma(G)^\bullet$ naturally induces the cosimplicial diagram $\rho : \mathrm{N}(\Delta) \to \widehat{\mathrm{Cat}}_\infty$ such that $\rho([i]) = \mathrm{N}(\mathrm{Comp}(\Gamma(G)^i)^c)$. Here $\mathrm{Comp}(\Gamma(G)^i)$ denotes the category of chain complexes of $\Gamma(G)^i$-modules which is endowed with the projective model structure, and $\mathrm{Comp}(\Gamma(G)^i)^c$ is its full subcategoy of cofibrant objects. Each category $\mathrm{Comp}(\Gamma(G)^i)^c$ has the (natural) symmetric monoidal structure, and thus ρ is promoted to $\rho : \mathrm{N}(\Delta) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$, where $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$ is the ∞-category of symmetric monoidal ∞-categories (i.e., commutative algebra objects in the Cartesian symmetric monoidal ∞-category $\widehat{\mathrm{Cat}}_\infty$). The symmetric monoidal category $\mathrm{Comp}(\Gamma(G)^i)^c$ admits the subset of edges of weak equivalences. Inverting weak equivalences in $\mathrm{Comp}(\Gamma(G)^i)^c$, we have $\rho' : \mathrm{N}(\Delta) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$ and the natural transformation $\rho \to \rho'$. such that $\rho'([i])$ is a symmetric monoidal ∞-category obtained from $\mathrm{Comp}(\Gamma(G)^i)^c$ by inverting weak equivalences.

Through the explicit unstraightening functor [23, 3.2.5.2], the maps $\rho, \rho' : \mathrm{N}(\Delta) \rightrightarrows \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$ gives rise to coCartesian fibrations $\mathcal{C}^{\otimes}_{pre} \to \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta)$ and $\mathcal{C}^{\otimes} \to \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta)$. The natural transformation $\rho \to \rho'$ induces a map of coCartesian fibrations

$$\begin{array}{ccc} \mathcal{C}^{\otimes}_{pre} & \xrightarrow{\quad\sigma\quad} & \mathcal{C}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta) & \end{array}$$

which preserves coCartesian edges. Note that for each $[i] \in \Delta$, the fiber $\rho^{-1}([i]) \to \mathrm{N}(\mathrm{Fin}_*) \times \{[i]\} \cong \mathrm{N}(\mathrm{Fin}_*)$ is the symmetric monoidal ∞-category associated to the diagram of $\mathrm{Comp}(\Gamma(G)^i)^c$'s. The fiber $(\rho')^{-1}([i]) \to \mathrm{N}(\mathrm{Fin}_*)$ is the symmetric monoidal ∞-category obtained from $\mathrm{Comp}(\Gamma(G)^i)^c$ by inverting weak equivalences.

Next we define a map of simplicial sets $\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{pre}) \to \mathrm{N}(\mathrm{Fin}_*)$ as follows. For any $a : T \to \mathrm{N}(\mathrm{Fin}_*)$, giving a map $T \to \overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{pre})$ over $\mathrm{N}(\mathrm{Fin}_*)$ amounts to giving $\phi : T \times \mathrm{N}(\Delta) \to \mathcal{C}^{\otimes}_{pre}$ which commutes with $a \times \mathrm{Id} : T \times \mathrm{N}(\Delta) \to \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta)$ and $\mathcal{C}^{\otimes}_{pre} \to \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta)$. Let $\mathrm{Sec}(\mathcal{C}^{\otimes}_{pre})$ be the largest subcomplex of $\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{pre})$, which consists of the following vertexes: a vertex $v \in \overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{pre})$ lying over $\langle i \rangle$ belongs to $\mathrm{Sec}(\mathcal{C}^{\otimes}_{pre})$ exactly when $v : \{\langle i \rangle\} \times \mathrm{N}(\Delta) \to \mathcal{C}^{\otimes}_{pre}$ carries all edges in $\{\langle i \rangle\} \times \mathrm{N}(\Delta)$ to coCartesian edges in $\mathcal{C}^{\otimes}_{pre}$. We define $\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ and $\mathrm{Sec}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ in a similar way. According to [23, 3.1.2.1 (1)], we see that $\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{pre}) \to \mathrm{N}(\mathrm{Fin}_*)$ and $\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ are coCartesian fibrations (notice that $\overline{\mathrm{Sec}}(\mathcal{C}^{\otimes}_{pre}) = \mathrm{N}(\mathrm{Fin}_*) \times_{\mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta))} \mathrm{Fun}(\mathrm{N}(\Delta), \mathcal{C}^{\otimes}_{pre}))$ where $\mathrm{N}(\mathrm{Fin}_*) \to \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta))$ is induced by the identity $\mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta) \to \mathrm{N}(\mathrm{Fin}_*) \times \mathrm{N}(\Delta)$). Moreover, by [23, 3.1.2.1 (2)] we deduce that $\mathrm{Sec}(\mathcal{C}^{\otimes}_{pre}) \to \mathrm{N}(\mathrm{Fin}_*)$ and $\mathrm{Sec}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ are coCartesian fibrations. By construction, furthermore $\mathrm{Sec}(\mathcal{C}^{\otimes}_{pre}) \to \mathrm{N}(\mathrm{Fin}_*)$ is a symmetric monoidal ∞-category. Since the procedure of inverting weak equivalences commutes with finite products [24, 4.1.3.2], we see that $\mathrm{Sec}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ is also a symmetric monoidal ∞-category. We will abuse notation and denote by $\mathrm{Sec}(\mathcal{C}^{\otimes}_{pre})$ and $\mathrm{Sec}(\mathcal{C}^{\otimes})$ the underlying ∞-categories. Note that σ (which preserves coCartesian edges) induces a symmetric monoidal functor $\mathrm{Sec}(\mathcal{C}^{\otimes}_{pre}) \to \mathrm{Sec}(\mathcal{C}^{\otimes})$.

Observe that the symmetric monoidal ∞-category $\mathrm{Sec}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ is equivalent to the symmetric monoidal ∞-category $\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}$. By [23, 3.3.3.2] and [24, 3.2.2.4], the symmetric monoidal ∞-category $\mathrm{Sec}(\mathcal{C}^{\otimes})$ is a limit of the diagram $\rho' : \mathrm{N}(\Delta) \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$. Note that by [24, 8.1.2.13] $\rho'([i])$ is equivalent to $\mathrm{Mod}^{\otimes}_{\Gamma(G)^i}$. Beside, the functor $\Theta : \mathrm{CAlg} \to \mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_\infty)$ which carries A to $\mathrm{Mod}^{\otimes}_A$ (see Section 3.1) is fully faithful [24, 6.3.5.18]. For a symmetric monoidal functor $\phi : \mathrm{Mod}^{\otimes}_A \to \mathrm{Mod}^{\otimes}_B$ in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_\infty)$, one can recover $f : A \to B$ with $\Theta(f) \simeq \phi$ as the induced morphism from the endomorphism spectrum of a unit of $\mathrm{Mod}^{\otimes}_A$ to that of the unit in $\mathrm{Mod}^{\otimes}_B$. Therefore from the construction of ρ' (and ρ) and the definition of $\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}$, we conclude that $\mathrm{Sec}(\mathcal{C}^{\otimes}) \to \mathrm{N}(\mathrm{Fin}_*)$ is equivalent to $\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}$.

Therefore, to construct $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}$, it will suffice to construct a symmetric monoidal functor from $\prod_{\mathbb{Z}} \mathrm{Comp}(\mathcal{A})^c$ to $\mathrm{Sec}(\mathcal{C}^{\otimes}_{pre})$ which carries weak equivalences in $\prod_{\mathbb{Z}} \mathrm{Comp}(\mathcal{A})^c$ to edges in $\mathrm{Sec}(\mathcal{C}^{\otimes}_{pre})$ whose images in $\mathrm{Sec}(\mathcal{C}^{\otimes})$ are equivalences (note the universality of $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}}$ [24, 4.1.3.4]). Let $\mathbf{K}_n$ in $\prod_{\mathbb{Z}} \mathrm{Comp}(\mathcal{A})^c$ be the $\mathbf{K}$ which sits in the n-th degree with respect to $\prod_{\mathbb{Z}}$. To $\mathbf{K}_n$ we attach the weight n representation of $\mathbb{G}_m$ on $\mathbf{K}$. The weight n representation gives rise to an object of $\mathrm{Sec}(\mathcal{C}^{\otimes}_{pre})$ in the obvious way, which we denote by $\mathbf{K}'_n$. For $(M_i)_{i\in\mathbb{Z}} \in \prod_{\mathbb{Z}} \mathrm{Comp}(\mathcal{A})^c$, we attach $\oplus_{i\in\mathbb{Z}} M_i \otimes \mathbf{K}'_i$. Here we consider M_i to be an object in $\mathrm{Sec}(\mathcal{C}^{\otimes}_{pre})$, that is the complex endowed with the trivial action of $\mathbb{G}_m$. This naturally induces a symmetric monoidal functor having the desired property. To prove that the induced functor $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}_{\mathsf{B}\mathbb{G}_m}$ preserves small colimits, it is enough to show that the composite $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}_{\mathsf{B}\mathbb{G}_m} \to \mathrm{Mod}_{H\mathbf{K}}$, where the second functor is forgetful, preserves small colimits since the forgetful functor is conservative and preserves small colimits (an exact functor $p : \mathcal{K} \to \mathcal{L}$ between stable ∞-categories is said to be conservative if for any $K \in \mathcal{K}$, $p(K) \simeq 0$ implies that $K \simeq 0$). The composite carries $(M_i)_{i\in\mathbb{Z}}$ to $\oplus_{i\in\mathbb{Z}} M_i$ and thus we conclude that the composite preserves small colimits. To prove that $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}$ is promoted to a $H\mathbf{K}$-linear symmetric monoidal functor, according to Lemma 6.6 (see also the discussion at the end of 6.3), it suffices to observe that $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}$ induces the identity morphism $H\mathbf{K} \to H\mathbf{K}$ of endomorphism algebras of units. To see this, we are reduced to showing that the composite $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m} \to \mathrm{Mod}^{\otimes}_{H\mathbf{K}}$, where the second functor is the forgetful functor, induces the identity morphism $H\mathbf{K} \to H\mathbf{K}$ of endomorphism algebras of units. This is clear.

We have constructed a symmetric monoidal colimit-preserving functor $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}$ with the (lax symmetric monoidal) right adjoint functor (the existence is assured by the relative version of adjoint functor theorem). To see that $\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}$ is an equivalence of symmetric monoidal ∞-categories, it is enough to show that it induces a categorical equivalence $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}_{\mathsf{B}\mathbb{G}_m}$ of underlying ∞-categories. Moreover, by [17, Lemma 4.11], it suffices to check that it induces an equivalence $\mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}) \to \mathrm{h}(\mathrm{Mod}_{\mathsf{B}\mathbb{G}_m})$ of their homotopy categories. The desired equivalence now follows from [32, Section 8, Theorem 8.5] (see also the strictification theorem [15, 18.7]). ☐

Proposition 6.10. *Let A be an object in* $\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})$. *Let $\overline{A}$ denote the image of A in* $\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{H\mathbf{K}})$ *(via the pullback of* $\mathrm{Spec}\, H\mathbf{K} \to \mathsf{B}\mathbb{G}_m$*). With the notation in the proof of Proposition 6.9, there is the natural augmented simplicial diagram $G_\bullet \to \mathsf{B}\mathbb{G}_m$. This induces the natural functor* $\mathrm{CAlg}(\mathrm{Mod}_{\mathsf{B}\mathbb{G}_m}) \to \lim_{[i]\in\Delta} \mathrm{CAlg}(\mathrm{Mod}_{H\Gamma(G)^i})$. *We denote the image of A in* $\lim_{[i]\in\Delta} \mathrm{CAlg}(\mathrm{Mod}_{H\Gamma(G)^i})$ *by $A^\bullet$. It gives rise to the quotient stack* $[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]$ *(see Example 4.1). Then there exists a natural equivalence*

$$\mathrm{Mod}^{\otimes}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}) \simeq \mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}.$$

Proof. We first construct a symmetric monoidal colimit-preserving functor

$$\mathrm{Mod}^{\otimes}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}) \longrightarrow \mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}.$$

Let $\pi^* : \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m} \to \mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}$ be the symmetric monoidal functor induced by the natural morphism $\pi : [\mathrm{Spec}\,\overline{A}/\mathbb{G}_m] \to \mathsf{B}\mathbb{G}_m$. By the relative version of adjoint functor theorem, there is a lax symmetric monoidal right adjoint fucntor $\pi_* : \mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]} \to \mathrm{Mod}_{\mathsf{B}\mathbb{G}_m}$. If $\mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}$ is a unit of $\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}$, by the definition of $[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]$ and the base-change formula, $\pi_*(\mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]})$ is equivalent to A in $\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})$. Thus we have the composition of symmetric monoidal colimit-preserving functors

$$h : \mathrm{Mod}^{\otimes}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}) \to \mathrm{Mod}_{\pi^*(A)}(\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}) \to \mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}$$

where the second functor is induced by the counit map $\pi^*(A) \simeq \pi^*(\pi_*(\mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]})) \to \mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}$. Note that the composite is naturally a $H\mathbf{K}$-linear symmetric monoidal functor.

Next we will show that h gives an equivalence of symmetric monoidal ∞-categories. It will suffice to prove that the underlying functor of ∞-categories is a categorical equivalence. We first show that h is fully faithful. Let $\mathbf{1}_{\mathsf{B}\mathbb{G}_m}(i) \in \mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m}$ be the object corresponding to $\mathbf{K}_n$ in the proof of Lemma 6.8. Let $A(i)$ be the image of $\mathbf{1}_{\mathsf{B}\mathbb{G}_m}(i)$ under the natural functor $\mathrm{Mod}_{\mathsf{B}\mathbb{G}_m} \to \mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})$. Unwinding the definition of h and using adjunctions, we see that

$$\mathrm{Map}_{\mathrm{Mod}^{\otimes}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})}(A(i), A(j)) \to \mathrm{Map}_{\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}}(h(A(i)), h(A(j)))$$

can be identified with

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})}(A(i), A(j)) &\simeq \mathrm{Map}_{\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})}(A(i-j), A) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathsf{B}\mathbb{G}_m}}(\mathbf{1}_{\mathsf{B}\mathbb{G}_m}(i-j), A) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}}(\pi^*(\mathbf{1}_{\mathsf{B}\mathbb{G}_m}(i-j)), \mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}}(\mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}(i), \mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}(j)).
\end{aligned}$$

Note that $A(i)$ and $h(A(i))$ are compact for each i, and h preserves small colimits. The stable presentable ∞-category $\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})$ is generated by $\{A(i)\}_{i\in\mathbb{Z}}$, that is, $\mathrm{Mod}^{\otimes}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})$ is the smallest stable subcategory which contains the set $\{A(i)\}_{i\in\mathbb{Z}}$ of objects and admits filtered colimits. Therefore for any $N \in \mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})$,

$$\mathrm{Map}_{\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})}(A(i), N) \to \mathrm{Map}_{\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}}(h(A(i)), h(N))$$

is an equivalence in $\mathcal{S}$. Furthermore, it follows from the fact that h is colimit-preserving that for any $M, N \in \mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})$,

$$\mathrm{Map}_{\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})}(M, N) \to \mathrm{Map}_{\mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}}(h(M), h(N))$$

is an equivalence in $\mathcal{S}$. It remains to show that h is essentially surjective. To this end, note that $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]} \simeq \mathrm{Ind}(\mathcal{E})$ where $\mathcal{E}$ is the smallest stable subcategory which contains $\{\mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}(i)\}_{i\in\mathbb{Z}}$. To see this, since $\mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}(i)$ are compact, thus (by [1, Definition 3.7]) it is enough to observe that the right orthogonal of $\{\mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}(i)\}_{i\in\mathbb{Z}}$ is zero, where $\mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}(i) = \pi^*(\mathbf{1}_{\mathsf{B}\mathbb{G}_m}(i))$. The condition that

$$\mathrm{Map}_{\mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}}(\mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}(i), N) \simeq \mathrm{Map}_{\mathrm{Mod}_{\mathsf{B}\mathbb{G}_m}}(\mathbf{1}_{\mathsf{B}\mathbb{G}_m}(i), \pi_*(N)) = 0$$

for any $i \in \mathbb{Z}$ implies that $\pi_*(N) = 0$. Then since π_* is conservative we deduce that $N = 0$, as desired. Since the set $\{\mathbf{1}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}(i)\}_{i\in\mathbb{Z}}$ of compact objects generates $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}$ (in the above sense), thus $\mathrm{Ind}(h(\mathcal{D})) \simeq \mathrm{Mod}_{[\mathrm{Spec}\,\overline{A}/\mathbb{G}_m]}$ (see [23, 5.3.5.11]) where $\mathcal{D}$ is the smallest stable subcategory in $\mathrm{Mod}_A(\mathrm{Mod}^{\otimes}_{\mathsf{B}\mathbb{G}_m})$ which contains $\{A(i)\}_{i\in\mathbb{Z}}$. It follows that h is essentially surjective, noting that h is colimit-preserving and fully faithful. □

6.4. **Tannakization and Derived stack of mixed Tate motives.** Proposition 6.9, 6.10 and Lemma 6.8 allow us to identify the realization functor $\mathsf{R}_T : \mathsf{DTM}^{\otimes} \to \mathrm{Mod}^{\otimes}_{H\mathbf{K}}$ with

$$\rho^* : \mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]} \to \mathrm{Mod}^{\otimes}_{[\mathrm{Spec}\,\overline{R}/\mathbb{G}_m]}$$

induced by the morphism of derived stacks $\rho : [\mathrm{Spec}\,\overline{R}/\mathbb{G}_m] \to [\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]$. Here $\overline{R}$ is the image of R in $\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{H\mathbf{K}})$.

Observe that $[\mathrm{Spec}\,\overline{R}/\mathbb{G}_m] \simeq \mathrm{Spec}\,H\mathbf{K}$. To see this, note that by the property of the realization functor the composite of left adjoint functors

$$\mathrm{Mod}_{\mathsf{B}\mathbb{G}_m} \to \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathrm{Mod}_Q(\mathrm{Mod}^{\otimes}_{H\mathbf{K},\mathbb{Z}}) \simeq \mathsf{DTM} \to \mathrm{Mod}_{H\mathbf{K}}$$

is equivalent to the forgetful functor (since $\mathrm{Mod}_{\mathsf{B}\mathbb{G}_m} \to \mathrm{Mod}_{H\mathbf{K}}$ is $H\mathbf{K}$-linear, the restriction to the full subcategory of the degree zero part of $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}$ is equivalent to the identity functor, and moreover for any $i \in \mathbb{Z}$ the restriction to the degree i part is equivalent to the identity $\mathrm{Mod}_{H\mathbf{K}} \to \mathrm{Mod}_{H\mathbf{K}}$). And its right adjoint functor sends $\mathbf{1}_{H\mathbf{K}}$ to the object R of the form $(\ldots, \mathbf{1}_{H\mathbf{K}}, \mathbf{1}_{H\mathbf{K}}, \mathbf{1}_{H\mathbf{K}}, \ldots)$ which belongs to $\mathrm{CAlg}(\mathrm{Mod}^{\otimes}_{H\mathbf{K}})$. By using adjunction maps and the fact that the above composite is symmetric monoidal, we easily see that R can be viewed as the coordinate ring of $\mathbb{G}_m$ endowed with the action of $\mathbb{G}_m$, determined by the multiplication $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$. Hence $[\mathrm{Spec}\,\overline{R}/\mathbb{G}_m] \simeq \mathrm{Spec}\,H\mathbf{K}$.

We refer to $[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]$ and $\rho : \mathrm{Spec}\,H\mathbf{K} \to [\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]$ as the derived stack of mixed Tate motives and the point determined by the mixed Weil cohomology E respectively.

Theorem 6.11. *Let* MTG *be the derived affine group scheme over* $H\mathbf{K}$ *which is the tannakization of* $\mathsf{R}_T : \mathsf{DTM}^{\otimes}_{\vee} \to \mathrm{PMod}^{\otimes}_{H\mathbf{K}}$. *Then* MTG *is equivalent to the derived affine group scheme arising from the Čech nerve of* $\rho : \mathrm{Spec}\,H\mathbf{K} \to [\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]$.

Proof. Apply Corollary 4.10 to ρ. □

6.5. **Cycle complex and Q.** We describe the ($\mathbb{Z}$-graded) complex Q in terms of Bloch's cycle complexes. We here regard Q as the object in the ∞-category $\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}$.

For this purpose, we need an explicit right adjoint functor $r : \mathsf{DM}_{\mathbb{Z}} \to \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}$ of $l : \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}} \to \mathsf{DM}_{\mathbb{Z}}$. To this end, recall the Quillen adjoit pair

$$\mathbf{1} \otimes (-) : \mathrm{Comp}(\mathcal{A}) \rightleftarrows \mathrm{DM}^{eff}(k) : \Gamma$$

where the right-hand side is the model category in [5, Example 4.12] (cf. Section 6.1) and the left adjoint functor carries a complex M to the tensor product $\mathbf{1} \otimes M$ with the (cofibrant) unit $\mathbf{1}$ of $\mathrm{DM}^{eff}(k)$. Here the tensor product $\mathbf{1} \otimes M$ is considered to be a complex of sheaves with transfers $U \mapsto L(\mathrm{Spec}\, k)(U) \otimes_{\mathbf{K}} M$. The right adjoint functor sends a complex of Nisnevich sheaves with transfers P to the complex $\Gamma(P)$ of sections at $\mathrm{Spec}\, k$. Let F be a Nisnevich sheaf with transfers. Let $\Delta^\bullet$ be the cosimplicial scheme where $\Delta^n = \mathrm{Spec}\, k[x_0, \ldots, x_n]/(\Sigma_{i=0}^n x_i = 0)$ and the j-th face $\Delta^n \hookrightarrow \Delta^{n+1}$ is determined by $x_j = 0$ (see e.g. [26]). We then have the Suslin complex $C_*(F)$ in $\mathrm{DM}^{eff}(k)$, that is the complex of sheaves with transfers, defined by $X \mapsto F(\Delta^\bullet \times_k X)$ (take the Moore complex).

Lemma 6.12. *Let F be a Nisnevich sheaf with transfers. Let F' be the fibrant replacement of F. Then $\Gamma(F')$ is quasi-isomorphic to $C_*(F)(\mathrm{Spec}\, k)$.*

Proof. Fibrant objects in $\mathrm{DM}^{eff}(k)$ are characterized by Nisnevich fibrant complexes whose cohomology sheaves are homotopy invariant (see [5, 4.12] for terminology). Moreover, the canonical morphism $F \to C_*(F)$ is a weak equivalence in $\mathrm{DM}^{eff}(k)$, and cohomology sheaves of $C_*(F)$ are homotopy invariant. The Zariski and Nisnevich hypercohomology of $C_*(F)$ coincide, by [26, 13.10]. Therefore, taking the Zariski topology of $\mathrm{Spec}\, k$ into account, we deduce that $\Gamma(F')$ is quasi-isomorphic to $C_*(F)(\mathrm{Spec}\, k)$. □

For a equidimensional scheme X over k, we denote by $z^n(X, *)$ the Bloch's cycle complex of X (cf. e.g. [26, Lecture 17]).

Corollary 6.13. *Let $n \geq 0$. The total right Quillen derived functor $\mathbb{R}\Gamma$ sends $\mathbf{K}(n)$ to a complex which is quasi-isomorphic to $z^n(\mathrm{Spec}\, k, *)[-2n]$.*

Proof. The comparison theorems [26, 16.7, 19.8] imply that $\mathbb{R}\Gamma(\mathbf{K}(n))$ is quasi-isomorphic to $z^n(\mathbb{A}^n, *)[-2n]$, where $\mathbb{A}^n$ is the n-dimensional affine space. The homotopy invariance of higher Chow groups (see e.g. [26, 17.4 (4)]) shows that $z^n(\mathbb{A}^n, *)[-2n]$ is quasi-isomorphic to $z^n(\mathrm{Spec}\, k, *)[-2n]$. □

Remark 6.14. Let n be a negative integer. Then every morphism from $\mathbf{K}$ to $\mathbf{K}(n)[i]$ in DM is null-homotopic for any $i \in \mathbb{Z}$. Thus by adjunction, the right adjoint functor of the canonical functor $\mathrm{Mod}_{H\mathbf{K}} \to \mathsf{DM}$ carries $\mathbf{K}(n)$ to zero in $\mathrm{Mod}_{H\mathbf{K}}$.

Proposition 6.15. *Let Q_n in $\mathrm{Mod}_{H\mathbf{K}}$ denote the complex of the n-th degree of $Q \in \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}$ (it is not the homological degree). Then Q_n is equivalent to $z^n(\mathrm{Spec}\, k, *)[-2n]$ for any $n \geq 0$, and $Q_n \simeq 0$ for $n < 0$.*

Proof. Recall that Q is the image of

$$\mathbf{K}(*) := (\ldots, \mathbf{K}(-1), \mathbf{K}(0), \mathbf{K}(1), \ldots)$$

by $r : \mathsf{DM}_{\mathbb{Z}} \to \mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}$ (we adopt the notation in Section 6.2). The natural functor $\Sigma^\infty : \mathsf{DM}^{eff} \to \mathsf{DM}$ is fully faithful by the cancellation theorem, and thus the right adjoint $\Omega^\infty : \mathsf{DM} \to \mathsf{DM}^{eff}$ sends $\mathbf{K}(i)$ to $\mathbf{K}(i)$ for $i \ge 0$. Now our claim follows from Corollary 6.13 and Remark 6.14. □

7. Mixed Tate motives assuming Beilinson-Soulé vanishing conjecture

In this Section, we adopt the notation in Section 6. Contrary to the previous Section, in this Section we will assume Beilinson-Soulé vanishing conjecture for the base field k; the motivic cohomology

$$H^{n,i}(\operatorname{Spec} k, \mathbf{K})$$

is zero for $n \le 0$, $i > 0$. Here $H^{n,i}(\operatorname{Spec} k, \mathbf{K})$ denotes the motivic cohomology (following the notation in [26, Definition 3.4]). What we need is that this condition and Proposition 6.15 imply that Q is cohomologically connective, that is, $\pi_n(\overline{Q}) = 0$ for $n > 0$, and $\pi_0(\overline{Q}) = \mathbf{K}$. For example, Beilinson-Soulé vanishing conjecture holds when k is a number field. The goal of this Section is to prove Theorem 7.15 which relates our tannakization MTG of $\mathsf{DTM}_\vee^\otimes$ with the Galois group of mixed Tate motives constructed by Bloch-Kriz [4], Kriz-May [21], Levine [22] (each group scheme is known to be equivalent to one another) under this vanishing conjecture.

7.1. Motivic t-structure on DTM.

Under Beilinson-Soulé vanishing conjecture, one can define motivic t-structure on DTM, as proved by Levine [22] and Kriz-May [21]. We will construct a t-structure in our setting (we do not claim any originality).

We fix our convention on t-structures. Let $\mathcal{C}$ be a stable ∞-category. A t-structure on $\mathcal{C}$ is a t-structure on the triangulated category $\mathrm{h}(\mathcal{C})$ (the homotopy category is naturally endowed with the structure of triangulated category, see [24, Chapter 1]). That is to say, a pair of full subcategories $(\mathcal{C}_{\ge 0}, \mathcal{C}_{\le 0})$ of $\mathcal{C}$ such that

- $\mathcal{C}_{\ge 0}[1] \subset \mathcal{C}_{\ge 0}$ and $\mathcal{C}_{\le 0}[-1] \subset \mathcal{C}_{\le 0}$,
- for $X \in \mathcal{C}_{\ge 0}$ and $Y \in \mathcal{C}_{\le 0}$, the hom group $\mathrm{Hom}_{\mathrm{h}(\mathcal{C})}(X, Y[-1])$ is zero,
- for $X \in \mathcal{C}$, there exists a distinguished triangle

  $$X' \longrightarrow X \longrightarrow X''$$

  in $\mathrm{h}(\mathcal{C})$ such that $X' \in \mathcal{C}_{\ge 0}$ and $X'' \in \mathcal{C}_{\le 0}[-1]$.

We here assume that full subcategories are stable under equivalences. We use homological indexing. Our reference on t-structure is [24] and [20]. We shall write $\mathcal{C}_{\ge n}$ and $\mathcal{C}_{\le n}$ for $\mathcal{C}_{\ge 0}[n]$ and $\mathcal{C}_{\le 0}[n]$ respectively. We denote by $\tau_{\ge n}$ the right adjoint to $\mathcal{C}_{\ge n} \subset \mathcal{C}$. Similarly, we denote by $\tau_{\le n}$ the left adjoint to $\mathcal{C}_{\le n} \subset \mathcal{C}$.

Let $\mathsf{R}_T : \mathsf{DTM} \to \mathrm{Mod}_{H\mathbf{K}}$ be the realization functor of a fixed mixed Weil theory E. Let $(\mathrm{Mod}_{H\mathbf{K},\ge 0}, \mathrm{Mod}_{H\mathbf{K},\le 0})$ be the standard t-structure of $\mathrm{Mod}_{H\mathbf{K}}$ such that X belongs to $\mathrm{Mod}_{H\mathbf{K},\ge 0}$ (resp. $\mathrm{Mod}_{H\mathbf{K},\le 0}$) exactly when the homotopy group $\pi_n(X)$ of the underlying spectra is zero for $n < 0$ (resp. $n > 0$).

Proposition 7.1. *Let*

$$\mathsf{DTM}_{\vee,\ge 0} := \mathsf{R}_T^{-1}(\mathrm{Mod}_{H\mathbf{K},\ge 0}) \cap \mathsf{DTM}_\vee \quad \text{and} \quad \mathsf{DTM}_{\vee,\le 0} := \mathsf{R}_T^{-1}(\mathrm{Mod}_{H\mathbf{K},\le 0}) \cap \mathsf{DTM}_\vee.$$

Then the pair $(\mathsf{DTM}_{\vee,\ge 0}, \mathsf{DTM}_{\vee,\le 0})$ *is a bounded* t*-structure on* $\mathsf{DTM}_\vee$. *(Of course, the realization functor is* t*-exact.)*

Proof. Since R_T is exact, $\mathsf{DTM}_{\vee,\geq 0}[1] \subset \mathsf{DTM}_{\vee,\geq 0}$ and $\mathsf{DTM}_{\vee,\leq 0}[-1] \subset \mathsf{DTM}_{\vee,\leq 0}$.

We next claim that the realization functor induces a conservative functor $\mathsf{DTM}_\vee \to \mathrm{Mod}_{H\mathbf{K}}$. (Recall again that an exact functor $p : \mathcal{K} \to \mathcal{L}$ between stable ∞-categories is said to be conservative if for any $K \in \mathcal{K}$, $p(K) \simeq 0$ implies that $K \simeq 0$.) Note that the realization functor $\mathsf{DTM} \simeq \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]} \stackrel{\rho^*}{\to} \mathrm{Mod}_{H\mathbf{K}}$ is induced by $\rho : \mathrm{Spec}\, H\mathbf{K} \to [\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]$ (see Section 6.4). The morphism ρ extends to $\overline{\rho} : \mathrm{Spec}\, H\mathbf{K} \to \mathrm{Spec}\,\overline{Q}$. Thus the realization functor is decomposed into

$$\mathsf{DTM} \simeq \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]} \to \mathrm{Mod}_{\mathrm{Spec}\,\overline{Q}} \stackrel{\overline{\rho}^*}{\to} \mathrm{Mod}_{H\mathbf{K}}\,.$$

By the definition, the pullback of the projection $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]} \to \mathrm{Mod}_{\mathrm{Spec}\,\overline{Q}}$ is conservative. The stable ∞-category $\mathrm{Mod}_{\overline{Q}}$ admits a t-structure $(\mathrm{Mod}_{\overline{Q},\geq 0}, \mathrm{Mod}_{\overline{Q},\leq 0})$ such that X in $\mathrm{Mod}_{\overline{Q}}$ belongs to $\mathrm{Mod}_{\overline{Q},\leq 0}$ if and only if $\pi_n(X) = 0$ for $n > 0$ (see, [25, VIII, 4.5.4]). According to [25, VIII, 4.1.11], the composite $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n} \to \mathrm{Mod}_{H\mathbf{K}}$ is conservative. Observe that every object $X \in \mathrm{PMod}_{\overline{Q}}$ lies in $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n}$. To see this, note that $\mathrm{PMod}_{\overline{Q}}$ is the smallest stable subcategory which contains $\overline{Q}$ and is closed under retracts. Since $\overline{Q}$ belongs to $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n}$ and $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n}$ is closed under retracts, we see that $\mathrm{PMod}_{\overline{Q}} \subset \bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n}$. Therefore the composite $\mathsf{DTM}_\vee \simeq \mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]} \to \mathrm{Mod}_{H\mathbf{K}}$ is conservative. By using this fact, we verify the second condition of the definition of t-structure.

It remains to show the third condition of t-structure. For this purpose, note first that if $Z \subset \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]}$ denotes the inverse image of $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n}$ and $f : Z \to \mathrm{Mod}_{H\mathbf{K}}$ denotes the restriction of the realization functor, we have $f^{-1}(\mathrm{PMod}_{H\mathbf{K}}) = \mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]}$. Clearly, $f^{-1}(\mathrm{PMod}_{H\mathbf{K}}) \supset \mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]}$ since the realization functor is symmetric monoidal. An object in $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]}$ is dualizable if and only if its image in $\mathrm{Mod}_{\overline{Q}}$ is dualizable. Thus it is enough to show that $g^{-1}(\mathrm{PMod}_{H\mathbf{K}}) = \mathrm{PMod}_{\overline{Q}}$ where $g : \bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\geq n} \to \mathrm{Mod}_{H\mathbf{K}}$. According to [24, VIII 4.5.2 (7)], we have the natural symmetric monoidal fully faithful functor $\bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{\overline{Q},\leq n} \to \lim_{\overline{Q}\to B} \mathrm{Mod}_B$ where B run over connective commutative ring spectra under $\overline{Q}$. An object $M \in \lim_{\overline{Q}\to B} \mathrm{Mod}_B$ belongs to its essential image if and only if the image $M(H\mathbf{K})$ of M in $\mathrm{Mod}_{H\mathbf{K}}$ under the natural projection has trivial homotopy groups $\pi_m(M(H\mathbf{K})) = 0$ for sufficiently large $m >> 0$. Note that every morphism $\overline{Q} \to B$ factors through $\overline{Q} \to H\mathbf{K}$ since $\overline{Q}$ is cohomologically connected. Consequently, we deduce that $g^{-1}(\mathrm{PMod}_{H\mathbf{K}}) \simeq \lim_{\overline{Q}\to B} \mathrm{PMod}_B$. Thus all objects in $g^{-1}(\mathrm{PMod}_{H\mathbf{K}})$ are dualizable. It follows that $g^{-1}(\mathrm{PMod}_{H\mathbf{K}}) = \mathrm{PMod}_{\overline{Q}}$. Next consider

$$\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m],\geq 0} := \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]} \times_{\mathrm{Mod}_{\overline{Q}}} \mathrm{Mod}_{\overline{Q},\geq 0}\,.$$

Then this category is presentable, by [23, 5.5.3.13]. Define $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m],\leq 0}$ by replacing ≥ 0 on the right-hand side by ≤ 0. Then the comonad of $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]} \rightleftarrows \mathrm{Mod}_{\overline{Q}}$ is given by $M \mapsto M \otimes_{H\mathbf{K}} H\mathbf{K}[t^{\pm}]$ (it is checked by using the right adjointability; Lemma 4.3). Therefore we can apply [25, VII 6.20] to deduce that

$$(\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m],\geq 0}, \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m],\leq 0})$$

is a t-structure. Note that since $\mathrm{Mod}_{\overline{Q}} \to \mathrm{Mod}_{H\mathbf{K}}$ is t-exact (it follows from [25, VIII, 4.1.10, 4.5.4 (2)]), $\mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]} \to \mathrm{Mod}_{H\mathbf{K}}$ is also t-exact. We now claim that

$\mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]}$ is stable under the truncations $\tau_{\geq 0}, \tau_{\leq 0}$. Let $M \in \mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]}$. Then $\tau_{\geq 0}M$ and $\tau_{\leq 0}M$ are contained in Z. Thus, to prove that $\tau_{\geq 0}M$ and $\tau_{\leq 0}M$ belong to $\mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]}$, it will suffice to prove that $g(\tau_{\geq 0}M)$ and $g(\tau_{\leq 0}M)$ belong to $\mathrm{PMod}_{H\mathbf{K}}$. Let $H_i = \tau_{\geq i} \circ \tau_{\leq i} = \tau_{\leq i} \circ \tau_{\geq i}$ (this notation slightly differs from the standard one). Using t-exactness, we have

$$\begin{aligned} H_i(g(\tau_{\geq 0}M)) &= g(H_i \circ \tau_{\geq 0}M) \\ &= g(\tau_{\leq i} \circ \tau_{\geq i} \circ \tau_{\geq 0}M) \\ &= g(H_i(M)) \\ &= H_i(g(M)) \end{aligned}$$

for $i \geq 0$. It follows that $H_i(g(\tau_{\geq 0}M))[-i]$ is equivalent to a finite dimensional $\mathbf{K}$-vector space, and the set

$$\{i \in \mathbb{Z} |\ H_i(g(\tau_{\geq 0}M))[-i] \neq 0\}$$

is finite. This implies that $g(\tau_{\geq 0}M)$ lies in $\mathrm{PMod}_{H\mathbf{K}}$. Similarly, $g(\tau_{\leq 0}M)$ lies in $\mathrm{PMod}_{H\mathbf{K}}$. Therefore for any $M \in \mathrm{PMod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]}$ we have the distinguished triangle (in the level of homotopy category)

$$\tau_{\geq 0}M \longrightarrow M \longrightarrow \tau_{\leq -1}M$$

such that $\mathsf{R}_T(\tau_{\geq 0}M) \in \mathrm{Mod}_{H\mathbf{K},\geq 0}$ and $\mathsf{R}_T(\tau_{\leq -1}M) \in \mathrm{Mod}_{H\mathbf{K},\leq 0}[-1]$, as desired.

Finally, this t-structure is clearly bounded. □

Remark 7.2. The definition of t-structure in Proposition 7.1 is compatible with the definition of motivic t-structure on the triangulated category of (all) mixed motives developed by Hanamura [13] (up to an anti-equivalence). In loc. cit., the expected motivic t-structure is constructed using Grothendieck's standard conjectures, Murre conjecture and Beilinson-Soulé vanishing conjecture for smooth projective varieties.

In Proposition 7.1, by the extension of coefficients $\mathbb{Q} \to \mathbf{K}$ we can replace $\mathbf{K}$ by $\mathbb{Q}$.

We refer to $(\mathsf{DTM}_{\vee,\geq 0}, \mathsf{DTM}_{\vee,\leq 0})$ as motivic t-structure on $\mathsf{DTM}_\vee$. We let $\mathsf{DTM}_\vee^\heartsuit := \mathsf{DTM}_{\vee,\geq 0} \cap \mathsf{DTM}_{\vee,\leq 0}$ be the heart. At first sight, it depends on the choice of our realization functor. But the mapping space $\mathrm{Map}(\mathrm{Spec}\,H\mathbf{K}, \mathrm{Spec}\,\overline{Q})$ is connected since $\overline{Q}$ is cohomologically connected (cf. [25, VIII, 4.1.7]). Therefore $\rho^* : \mathrm{Mod}^\otimes_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m]} \to \mathrm{Mod}^\otimes_{H\mathbf{K}}$ is unique up to equivalence.

As a by-product of the proof, we have

Corollary 7.3. *Adopt the notation used in the proof of Proposition 7.1. The realization functor induces a conservative functor* $f : \bigcup_{n\in\mathbb{Z}} \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m],\leq n} \to \mathrm{Mod}_{H\mathbf{K}}$. *In particular,* $\mathsf{DTM}_\vee \to \mathrm{PMod}_{H\mathbf{K}}$ *is conservative. Moreover,* $f^{-1}(\mathrm{PMod}_{H\mathbf{K}})$ *coincides with* $\mathsf{DTM}_\vee$.

Recall that DTM is compactly generated. Namely, we have the natural equivalence $\mathrm{Ind}(\mathsf{DTM}_\circ) \simeq \mathrm{Ind}(\mathsf{DTM}_\vee) \simeq \mathsf{DTM}$.

Corollary 7.4. *Let* $\mathsf{DTM}_{\geq 0} := \mathrm{Ind}(\mathsf{DTM}_{\vee,\geq 0})$ *and* $\mathsf{DTM}_{\leq 0} := \mathrm{Ind}(\mathsf{DTM}_{\vee,\leq 0})$. *Then* $(\mathsf{DTM}_{\geq 0}, \mathsf{DTM}_{\leq 0})$ *is an accessible right complete* t*-structure on* DTM.

Proof. It follows from Proposition 7.1, [25, VIII, 5.4.1] and [24, 1.4.4.13]. □

Let $(\mathrm{Mod}_{H\mathbf{K}}^{\heartsuit})^{\otimes}$ be the symmetric monoidal abelian category such that the underlying category is $\mathrm{Mod}_{H\mathbf{K},\ge 0} \cap \mathrm{Mod}_{H\mathbf{K},\le 0}$ and its symmetric monoidal structure is induced by that of $\mathrm{Mod}_{H\mathbf{K}}^{\otimes}$. It is (the nerve of) the symmetric monoidal category of $\mathbf{K}$-vector spaces. For an affine group scheme G over $\mathbf{K}$ (which can be viewed as a derived affine group scheme over $H\mathbf{K}$), we let $\mathrm{Rep}(G)^{\otimes}$ be the symmetric monoidal full subcategory $z^{-1}((\mathrm{Mod}_{H\mathbf{K}}^{\heartsuit})^{\otimes})$ of $\mathrm{Mod}_{\mathsf{B}G}^{\otimes}$ where $z : \mathrm{Mod}_{\mathsf{B}G}^{\otimes} \to \mathrm{Mod}_{H\mathbf{K}}^{\otimes}$ is the natural projection determined by $\mathrm{Spec}\, H\mathbf{K} \to \mathsf{B}G$. We denote by $\mathrm{Rep}(G)_{\vee}^{\otimes}$ the symmetric monoidal full subcategory of $\mathrm{Rep}(G)^{\otimes}$ which consists of dualizable objects. Apply the classical Tannaka duality by Saavedra, Deligne-Milne, Deligne [30], [9], [8] to the faithful symmetric monoidal exact functor of abelain categories $(\mathsf{DTM}_{\vee}^{\heartsuit})^{\otimes} \to (\mathrm{Mod}_{H\mathbf{K}}^{\heartsuit})^{\otimes}$ induced by the realization functor, we have

Corollary 7.5. *There exist an affine group scheme MTG over $\mathbf{K}$ and an equivalence $(\mathsf{DTM}_{\vee}^{\heartsuit})^{\otimes} \xrightarrow{\sim} \mathrm{Rep}(MTG)_{\vee}^{\otimes}$ of symmetric monoidal ∞-categories.*

We here give a symmetric monoidal equivalence between the abelian category $\mathsf{DTM}_{\vee}^{\heartsuit}$ and the abelian category $\mathbf{TM}_k$ which is constructed via the axiomatic formulation in [22]. Let i be an integer. Let $W_{\ge i}\mathsf{DTM}_{gm} \subset \mathsf{DTM}_{gm}$ (resp. $W_{\le i}\mathsf{DTM}_{gm} \subset \mathsf{DTM}_{gm}$ be the smallest stable subcategory generated by $\mathbf{K}(n)$ for $-2n \ge i$ (resp. $\mathbf{K}(n)$ for $-2n \le i$). Then according to [22, Lemma 1.2], the pair $(W_{\ge i}\mathsf{DTM}_{gm}, W_{\le i}\mathsf{DTM}_{gm})$ is a t-structure. Let $\mathrm{Gr}_i^W : \mathsf{DTM}_{gm} \to W_i\mathsf{DTM}_{gm} := W_{\ge i}\mathsf{DTM}_{gm} \cap W_{\le i}\mathsf{DTM}_{gm}$ be the functor H_0 with respect to this t-structure. When i is even, the ∞-category $W_i\mathsf{DTM}_{gm}$ is equivalent to the full subcategory $\mathrm{h}(\mathrm{PMod}_{H\mathbf{K}})$ of $\mathrm{h}(\mathrm{Mod}_{H\mathbf{K}})$ spanned by bounded complexes of $\mathbf{K}$-vector spaces whose (co)homology are finite dimensional. This equivalence is given by the exact functor $\mathrm{h}(\mathrm{PMod}_{H\mathbf{K}}) \to W_i\mathsf{DTM}_{gm}$ which carries $\mathbf{K}[r]$ to $\mathbf{K}(-i/2)[r]$. If i is odd, $W_i\mathsf{DTM}_{gm}$ is zero. It gives rise to a natural symmetric monoidal exact functor $\mathrm{Gr} : \mathrm{h}(\mathsf{DTM}_{gm}) \to \mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}})$, which sends X to $\{\mathrm{Gr}_i^W(X)\}_{i\in\mathbb{Z}}$, of homotopy categories (which are furthermore triangulated categories). The triangulated category $\mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}}) \simeq \Pi_{\mathbb{Z}}\mathrm{h}(\mathrm{Mod}_{H\mathbf{K}})$ has the standard t-structure determined by the product of pair $(\mathrm{Mod}_{H\mathbf{K},\ge 0}, \mathrm{Mod}_{H\mathbf{K},\le 0})$. We denote it by $(\mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}})_{\ge 0}, \mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}})_{\le 0})$. Let $\mathsf{DTM}_{gm,\ge 0} := \mathrm{Gr}^{-1}(\mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}})_{\ge 0})$ and $\mathsf{DTM}_{gm,\le 0} := \mathrm{Gr}^{-1}(\mathrm{h}(\mathrm{Mod}_{H\mathbf{K},\mathbb{Z}})_{\le 0})$. Then by [22, Theorem 1.4], we have:

Lemma 7.6 ([22]). *The pair $(\mathsf{DTM}_{gm,\ge 0}, \mathsf{DTM}_{gm,\le 0})$ is a bounded t-structure, and* Gr *is t-exact and conservative.*

Let $\mathbf{TM}_k$ be its heart.

Lemma 7.7. *The realization functor $\mathsf{R}_{gm} : \mathsf{DTM}_{gm} \to \mathrm{Mod}_{H\mathbf{K}}$ (induced by $\mathsf{R}_T : \mathsf{DTM} \to \mathrm{Mod}_{H\mathbf{K}}$) is t-exact.*

Proof. We will show that the essential image of $\mathsf{DTM}_{gm,\le 0}$ is contained in $\mathrm{Mod}_{H\mathbf{K},\le 0}$. The dual case is similar. Let $X \in \mathsf{DTM}_{gm,\le 0}$. Let m be the cardinal of the set of integers i such that $H_i(X)[-i]$ is not zero (recall our (nonstandard) notation $H_i = \tau_{\le i} \circ \tau_{\ge i}$). We proceed by induction on m. If $m = 0$, we conclude that $X \simeq 0$ (since the t-structure on DTM_{gm} is bounded). Hence this case is clear. By [22, Theorem 1.4 (iii)] we see that the essential image of $\mathbf{TM}_k$ is contained in $\mathrm{Mod}_{H\mathbf{K}}^{\heartsuit}$. Hence the case $m = 1$ follows.

Suppose that our claim holds for $m \le n$. To prove the case when $m = n+1$, consider the distinguished triangle

$$H_i(X) \to X \to \tau_{\le i-1}X$$

where i is the largest number such that $H_i(X)[-i] \neq 0$. Note that the functor $\mathsf{DTM}_{gm} \to \mathrm{Mod}_{H\mathbf{K}}$ is exact, and the images of $H_i(X)$ and $\tau_{\le i-1}X$ is contained in $\mathrm{Mod}_{H\mathbf{K},\le 0}$. Thus we conclude that the image of X is also contained in $\mathrm{Mod}_{H\mathbf{K},\le 0}$. □

Lemma 7.8. *The full subcategory $\mathsf{DTM}_{gm,\ge 0}$ (resp. $\mathsf{DTM}_{gm,\le 0}$) is the inverse image of $\mathrm{Mod}_{H\mathbf{K},\ge 0}$ (resp. $\mathrm{Mod}_{H\mathbf{K},\le 0}$) under $\mathsf{R}_{gm} : \mathsf{DTM}_{gm} \to \mathrm{Mod}_{H\mathbf{K}}$.*

Proof. We will treat the case $\mathsf{DTM}_{gm,\le 0}$. Another case is similar. We have already prove that R_{gm} is t-exact in the previous Lemma. It will suffice to show that if X does not belong to $\mathsf{DTM}_{gm,\le 0}$, then $\mathsf{R}_{gm}(X)$ does not lies in $\mathrm{Mod}_{H\mathbf{K},\le 0}$. For such X, there exists $i \ge 1$ such that $H_i(X) \neq 0$. According to Corollary 7.3, R_{gm} is conservative. Combined with the t-exactness, we deduce that $H_i(\mathsf{R}_{gm}(X))[-i] \neq 0$. This implies that $\mathsf{R}_{gm}(X)$ is not in $\mathrm{Mod}_{H\mathbf{K},\le 0}$, as required. □

By Lemma 7.8, we have a t-exact fully faithful functor $\mathsf{DTM}_{gm} \to \mathsf{DTM}_{\vee}$, and it induces a natural fully faithful functor $\mathbf{TM}_k \to \mathsf{DTM}_{\vee}^{\heartsuit}$ between (nerves of) symmetric monoidal abelian categories.

Proposition 7.9. *The natural inclusion $\mathbf{TM}_k \to \mathsf{DTM}_{\vee}^{\heartsuit}$ is an equivalence.*

Proof. Since $\mathbf{TM}_k$ is (the nerve of) an abelian category, and in particular it is idempotent-complete, thus it is enough to prove that $\mathbf{TM}_k \to \mathsf{DTM}_{\vee}^{\heartsuit}$ is an idempotent-completion. Recall that $\mathsf{DTM}_{gm} \to \mathsf{DTM}_{\vee}$ is an idempotent-completion. Let $X \in \mathbf{TM}_k$. The direct summand of X (which automatically belongs to $\mathsf{DTM}_{\vee}$) lies in $\mathsf{DTM}_{\vee}^{\heartsuit}$ by the definition of t-structure of $\mathsf{DTM}_{\vee}$. Conversely, if $Y \in \mathsf{DTM}_{\vee}^{\heartsuit}$, then there exists $X \in \mathsf{DTM}_{gm}$ such that Y is equivalent to a direct summand of X. Then Y is a direct summand of $H_0(X) \in \mathbf{TM}_k$ (note that we here use the t-exactness of $\mathsf{DTM}_{gm} \to \mathsf{DTM}_{\vee}$). Consequently, $\mathbf{TM}_k \to \mathsf{DTM}_{\vee}^{\heartsuit}$ is an idempotent-completion. □

Corollary 7.10. *The Tannaka dual of $\mathbf{TM}_k$ (endowed with the realization functor) is equivalent to MTG.*

Warning 7.10.1. *In* [22]*, one works over rational coefficients. In this paper, we work over $\mathbf{K}$. Therefore MTG is the base change of the Tannaka dual of the abelian category of mixed Tate motives in* [22] *over $\mathbb{Q}$ to $\mathbf{K}$.*

7.2. **Completion and locally dimensional ∞-category.** Let $\mathsf{DTM}^{\otimes} \to \overline{\mathsf{DTM}}^{\otimes}$ be the left completion of $\mathsf{DTM}^{\otimes}$ with respect to the t-structure $(\mathsf{DTM}_{\ge 0}, \mathsf{DTM}_{\le 0})$ (we refer the reader to [24, 1.2.1.17] and [25, VIII, 4.6.17] for the notions of left completeness and left completion). It is symmetric monoidal, t-exact and colimit-preserving. Here, the ∞-category $\overline{\mathsf{DTM}}$ is the limit of the diagram indexed by $\mathbb{Z}$

$$\cdots \to \mathsf{DTM}_{\le n+1} \stackrel{\tau_{\le n}}{\to} \mathsf{DTM}_{\le n} \stackrel{\tau_{\le n-1}}{\to} \mathsf{DTM}_{\le n-1} \stackrel{\tau_{\le n-2}}{\to} \cdots$$

of ∞-categories. Note that according to [23, 3.3.3] the ∞-category $\overline{\mathsf{DTM}}$ can be identified with the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbb{Z}), \mathsf{DTM})$ spanned by functors $\phi : \mathrm{N}(\mathbb{Z}) \to \mathsf{DTM}$ such that

- for any $n \in \mathbb{Z}$, $\phi([n])$ belongs to $\mathsf{DTM}_{\le -n}$,
- for any $m \le n \in \mathbb{Z}$, the associated map $\phi([m]) \to \phi([n])$ gives an equivalence $\tau_{\le -n}\phi([m]) \to \phi([n])$.

Let $\overline{\mathsf{DTM}}_{\ge 0}$ (resp. $\overline{\mathsf{DTM}}_{\le 0}$) be the full subcategory of $\overline{\mathsf{DTM}}$ spanned by $\phi : \mathrm{N}(\mathbb{Z}) \to \mathsf{DTM}$ such that $\phi([n])$ belongs to $\mathsf{DTM}_{\ge 0}$ (resp. $\mathsf{DTM}_{\le 0}$) for each $n \in \mathbb{Z}$. The functor $\mathsf{DTM} \to \overline{\mathsf{DTM}}$ induces an equivalence $\mathsf{DTM}_{\le 0} \to \overline{\mathsf{DTM}}_{\le 0}$. The pair $(\overline{\mathsf{DTM}}_{\ge 0}, \overline{\mathsf{DTM}}_{\le 0})$ is an accessible, left complete and right complete t-structure of $\overline{\mathsf{DTM}}$.

Proposition 7.11. *The followings hold.*

(i) $\overline{\mathsf{DTM}}_{\le 0}$ *is closed under filtered colimits.*
(ii) *The unit* 1 *belongs to the heart* $\overline{\mathsf{DTM}}^{\heartsuit} := \overline{\mathsf{DTM}}_{\ge 0} \cap \overline{\mathsf{DTM}}_{\le 0}$.
(iii) $\overline{\mathsf{DTM}}_{\ge 0}$ *and* $\overline{\mathsf{DTM}}_{\le 0}$ *are closed under the tensor product* $\overline{\mathsf{DTM}} \times \overline{\mathsf{DTM}} \to \overline{\mathsf{DTM}}$.
(iv) *The unit* 1 *is compact in* $\mathsf{DTM}_{\le n}$ *for each* $n \ge 0$.
(v) *There exists a full subcategory* $\overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$ *of* $\overline{\mathsf{DTM}}^{\heartsuit}$ *such that every object in* $\overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$ *has the dual in* $\overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$, *and* $\overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$ *generates* $\overline{\mathsf{DTM}}^{\heartsuit}$ *under filtered colimits.*
(vi) $\pi_0(\mathrm{Map}_{\overline{\mathsf{DTM}}}(1,1)) = \mathbf{K}$.
(vii) *For any* $X \in \overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$, *the composite*

$$1 \to X \otimes X^{\vee} \to 1$$

of the coevaluation map and the evaluation map corresponds to a nonnegative integer $\dim(X) \in \mathbb{Z} \subset \mathbf{K}$.

Proof. By our construction and $\overline{\mathsf{DTM}}_{\le 0} = \mathsf{DTM}_{\le 0}$, (i) is clear. Since the unit of DTM lies in $\mathsf{DTM}^{\heartsuit} := \mathsf{DTM}_{\ge 0} \cap \mathsf{DTM}_{\le 0}$, (ii) follows.

Next we will prove (iii). By Corollary 7.3 the realization functor induces a conservative functor $\overline{\mathsf{DTM}}_{\le i} = \mathsf{DTM}_{\le i} \to \mathrm{Mod}_{H\mathbf{K}, \le i}$ for each $i \in \mathbb{Z}$ (observe that $\mathsf{DTM}_{\vee, \le i} \subset \mathrm{Mod}_{[\mathrm{Spec}\,\overline{Q}/\mathbb{G}_m], \le i}$). If $X \in \overline{\mathsf{DTM}}$ is not in $\overline{\mathsf{DTM}}_{\le 0}$, there exists $n \ge 1$ such that $H_n(X)$ is not zero. Thus the inverse image of $\mathrm{Mod}_{H\mathbf{K}, \le 0}$ in $\overline{\mathsf{DTM}}$ under the t-exact functor $\overline{\mathsf{DTM}} \to \mathrm{Mod}_{H\mathbf{K}}$ induced by $\mathsf{DTM} \to \mathrm{Mod}_{H\mathbf{K}}$ is $\overline{\mathsf{DTM}}_{\le 0}$. Notice that $\overline{\mathsf{DTM}}_{\ge 0}$ is the full subcategory, spanned by objects X such that $\tau_{\le -1}X \simeq 0$ where $\tau_{\le -1} : \overline{\mathsf{DTM}} \to \overline{\mathsf{DTM}}$, that is, $H_i(X)[-i]$ is zero for $i \le -1$ (since $\overline{\mathsf{DTM}}$ is right t-complete). The condition $H_i(X)[-i]$ is zero for $i \le -1$ is equivalent to the condition that X maps to an object in $\mathrm{Mod}_{H\mathbf{K}, \ge 0}$, again by conservativeness; Corollary 7.3. Namely, the inverse image of $\mathrm{Mod}_{H\mathbf{K}, \ge 0}$ is $\overline{\mathsf{DTM}}_{\ge 0}$. The full subcategories $\mathrm{Mod}_{H\mathbf{K}, \ge 0}$ and $\mathrm{Mod}_{H\mathbf{K}, \le 0}$ are closed under tensor product, and $\overline{\mathsf{DTM}} \to \mathrm{Mod}_{H\mathbf{K}}$ is a symmetric monoidal functor, thus $\overline{\mathsf{DTM}}_{\le 0}$ and $\overline{\mathsf{DTM}}_{\ge 0}$ are closed under tensor product.

The unit 1 is compact in DTM, and so is in $\mathsf{DTM}_{\le n}$ for any $n \in \mathbb{Z}$. Noting that $\overline{\mathsf{DTM}}_{\le n} = \mathsf{DTM}_{\le n}$, we have (iv).

To prove (v), note first that $\mathsf{DTM} \to \overline{\mathsf{DTM}}$ induces equivalences $\bigcup_{n \in \mathbb{Z}} \mathsf{DTM}_{\le n} \to \bigcup_{n \in \mathbb{Z}} \overline{\mathsf{DTM}}_{\le n}$ and $\mathsf{DTM}^{\heartsuit} \to \overline{\mathsf{DTM}}^{\heartsuit}$. In particular, $\mathsf{DTM}_{\vee} \to \overline{\mathsf{DTM}}$ is fully faithful. Let $X \in \overline{\mathsf{DTM}}^{\heartsuit} = \mathsf{DTM}^{\heartsuit}$. Then X is the filtered colimit of a diagram $I \to \mathsf{DTM}_{\vee}$ in DTM (or in $\overline{\mathsf{DTM}}$); $\mathrm{colim}_{\lambda \in I} X_\lambda \simeq X$. Note that $\mathsf{R}_T(H_0(X_\lambda)) \simeq H_0(\mathsf{R}_T(X_\lambda))$ by t-exactness, and it is a dualizable object in $\mathrm{Mod}_{H\mathbf{K}}$, that is, a finite dimensional vector space. It follows from Corollary 7.3 that $H_0(X_\lambda)$ is dualizable, that is, it belongs to $\mathsf{DTM}^{\heartsuit}_{\mathrm{fd}} := \mathsf{DTM}_{\vee} \cap \mathsf{DTM}^{\heartsuit}$. It is obvious that the dual of $H_0(X_\lambda)$ lies in $\overline{\mathsf{DTM}}^{\heartsuit}$. Recall that the realization functor $\mathsf{R}_T : \mathsf{DTM} \to \mathrm{Mod}_{H\mathbf{K}}$ preserves small colimit, which is also t-exact, and H_0 preserves filtered colimits in $\mathrm{Mod}_{H\mathbf{K}}$. Using these facts, we see that the natural map $\mathrm{colim}_\lambda H_0(X_\lambda) \to H_0(\mathrm{colim}_\lambda X_\lambda)$ gives an equivalence $\mathsf{R}_T(\mathrm{colim}_\lambda H_0(X_\lambda)) \to \mathsf{R}_T(H_0(\mathrm{colim}_\lambda X_\lambda))$. The heart $\mathsf{DTM}^{\heartsuit}$ is closed under filtered colimits and thus $\mathrm{colim}_\lambda H_0(X_\lambda)$ is contained in the heart. Hence by Corollary 7.3, $\mathrm{colim}_\lambda H_0(X_\lambda) \to H_0(\mathrm{colim}_\lambda X_\lambda) \simeq X$ is an equivalence. This shows that $\mathsf{DTM}^{\heartsuit}_{\mathrm{fd}}$ generates $\mathsf{DTM}^{\heartsuit} = \overline{\mathsf{DTM}}^{\heartsuit}$ under filtered colimits.

We remark that $H^{0,0}(\mathrm{Spec}\, k, \mathbf{K}) = \mathbf{K}$. Hence (vi) holds. Finally, we will prove (vii). For any $X \in \overline{\mathsf{DTM}}^{\heartsuit}_{\mathrm{fd}}$, the element in $\mathbf{K}$ corresponding to the composite $\mathsf{1} \to X \otimes X^\vee \to \mathsf{1}$ is equal to the element in $\mathbf{K}$ corresponding to $\mathsf{R}_T(\mathsf{1}) \to \mathsf{R}_T(X) \otimes \mathsf{R}_T(X)^\vee \to \mathsf{R}_T(\mathsf{1})$. The latter element is nothing but the dimension of $\mathsf{R}_T(X)$, which lies in $\mathbb{Z}$. □

Corollary 7.12. *The symmetric monoidal ∞-category $\overline{\mathsf{DTM}}^{\otimes}$ endowed with the t-structure $(\overline{\mathsf{DTM}}_{\geq 0}, \overline{\mathsf{DTM}}_{\leq 0})$ is a locally dimensional ∞-category in the sense of* [25, VIII, 5.6].

To state the next result, we prepare some notation. We say that a commutative ring spectrum S is discrete if $\pi_i(S) = 0$ for $i \neq 0$. This property is equivalent to the property that there exists a (usual) commutative ring R such that $HR \simeq S$ in CAlg. Let $\mathrm{CAlg}^{\mathrm{dis}}$ be the ∞-category of discrete commutative ring spectra. The ∞-category $\mathrm{CAlg}^{\mathrm{dis}}$ is equivalent to the nerve of the category of (usual) commutative rings (via Eilenberg-MacLane spectra). Let $\mathfrak{G} : \mathrm{CAlg}^{\mathrm{dis}} \to \widehat{\mathcal{S}}$ be the functor which carries $A \in \mathrm{CAlg}^{\mathrm{dis}}$ to the space $\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})}(\overline{\mathsf{DTM}}^{\otimes}, \mathrm{Mod}_A^{\otimes})$ (which can be constructed by Θ in Section 3.1 and Yoneda embedding). Let $\xi : \mathrm{CAlg}^{\mathrm{dis}} \to \widehat{\mathcal{S}}$ be the functor which carries $A \in \mathrm{CAlg}^{\mathrm{dis}}$ to the space $\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})}(\mathrm{Mod}_{H\mathbf{K}}^{\otimes}, \mathrm{Mod}_A^{\otimes})$. Since there exists a natural equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty^{\mathrm{L,st}})}(\mathrm{Mod}_{H\mathbf{K}}^{\otimes}, \mathrm{Mod}_A^{\otimes}) \simeq \mathrm{Map}_{\mathrm{CAlg}}(H\mathbf{K}, A)$$

(cf. [11, Section 5], [24, 6.3.5.18]), ξ is corepresented by $H\mathbf{K}$. We here write $\mathrm{Spec}\, H\mathbf{K}$ for ξ. There is a sequence of functors $\mathrm{Mod}_{H\mathbf{K}}^{\otimes} \to \overline{\mathsf{DTM}}^{\otimes} \to \mathrm{Mod}_{H\mathbf{K}}^{\otimes}$ whose composite is equivalent to the identity. Therefore we have $\mathrm{Spec}\, H\mathbf{K} \xrightarrow{\eta} \mathfrak{G} \to \mathrm{Spec}\, H\mathbf{K}$ whose composite is the identity. Let $V : \mathrm{CAlg}^{\mathrm{dis}} \to \widehat{\mathcal{S}}$ be a functor equipped with $V \to \mathrm{Spec}\, H\mathbf{K}$. To $f : H\mathbf{K} \to A$ in $\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} := (\mathrm{CAlg}^{\mathrm{dis}})_{H\mathbf{K}/}$ we associate $\{f\} \times_{\mathrm{Spec}\, H\mathbf{K}(A)} V(A)$. It yields the functor $V_0 : \mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \widehat{\mathcal{S}}$. The morphism $\eta : \mathrm{Spec}\, H\mathbf{K} \to \mathfrak{G}$ induces $\eta_0 : (\mathrm{Spec}\, H\mathbf{K})_0 \to \mathfrak{G}_0$. Note that $(\mathrm{Spec}\, H\mathbf{K})_0$ is equivalent to the constant functor taking the value Δ^0, that is, the final object.

The following result is proved by Lurie in the theory of locally dimensional ∞-categories (see [25, VIII, 5.2.12, 5.6.1, 5.6.19 and their proofs]). We here state only the version in view of Corollary 7.12, which fits in with our need.

Proposition 7.13 ([25]). *Let $\mathrm{Grp}^{\mathrm{dis}}$ be the nerve of the category of (usual) groups. Consider the functor $\pi_1(\mathfrak{G}_0, \eta_0) : \mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \to \mathrm{Grp}^{\mathrm{dis}}$ which is given by $A \mapsto \pi_1(\mathfrak{G}_0(A), \eta_0)$. Then $\pi_1(\mathfrak{G}_0, \eta_0)$ is represented by MTG, that is, the Tannaka dual of $(\mathsf{DTM}^{\heartsuit}_{\vee})^{\otimes}$.*

7.3. **Comparison theorem.**

Definition 7.14. Let $G : \mathrm{CAlg}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S})$ be a derived group scheme over $H\mathbf{K}$. Let $\pi_0 : \mathrm{Grp}(\mathcal{S}) \to \mathrm{Grp}^{\mathrm{dis}}$ be the truncation functor given by $G \mapsto \pi_0(G)$. If the composition

$$\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \hookrightarrow \mathrm{CAlg}_{H\mathbf{K}} \stackrel{G}{\to} \mathrm{Grp}(\mathcal{S}) \stackrel{\pi_0}{\to} \mathrm{Grp}^{\mathrm{dis}}$$

is represented by a group scheme G_0 over $\mathbf{K}$, we say that G_0 is the underlying group scheme of G.

Theorem 7.15. *Let* MTG *denote the tannakization of* $\mathsf{R}_T : \mathsf{DTM}^{\otimes}_{\vee} \to \mathrm{PMod}^{\otimes}_{H\mathbf{K}}$ *(cf. Theorem 6.11). Then MTG is the underlying group scheme of* MTG.

Proof. For $A \in \mathrm{CAlg}^{\mathrm{dis}}$, we set $\mathrm{Mod}_{A,\geq 0} = \{X \in \mathrm{Mod}_A \mid \pi_i(X) = 0 \text{ for } i < 0\}$ and $\mathrm{Mod}_{A,\leq 0} = \{X \in \mathrm{Mod}_A \mid \pi_i(X) = 0 \text{ for } i > 0\}$. Then the pair $(\mathrm{Mod}_{A,\geq 0}, \mathrm{Mod}_{A,\leq 0})$ is an accessible, left and right complete t-structure. Thus we have

$$\begin{aligned} \mathrm{Map}^{\mathrm{rex}}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})}(\overline{\mathsf{DTM}}^{\otimes}, \mathrm{Mod}^{\otimes}_A) &\simeq \mathrm{Map}^{\mathrm{rex}}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})}(\mathsf{DTM}^{\otimes}, \mathrm{Mod}^{\otimes}_A) \\ &\hookrightarrow \mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_{\infty})}(\mathsf{DTM}^{\otimes}_{\vee}, \mathrm{Mod}^{\otimes}_A) \end{aligned}$$

where $\mathrm{Map}^{\mathrm{rex}}$ indicates the full subcategory spanned by right t-exact functors, and the second arrow is fully faithful by Proposition 4.7 and the construction of t-structure on DTM. (The essential image consists of symmetric monoidal exact functors which are right t-exact.) Note that $\mathsf{R}_T : \mathsf{DTM}^{\otimes} \to \mathrm{Mod}^{\otimes}_{H\mathbf{K}}$ is t-exact, and it belongs to $\mathrm{Map}^{\mathrm{rex}}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}^{\mathrm{L,st}}_{\infty})}(\mathsf{DTM}^{\otimes}, \mathrm{Mod}^{\otimes}_A)$.

Consider the automorphism functor $\mathrm{Aut}(\mathsf{R}_T) : \mathrm{CAlg}_{H\mathbf{K}} \to \mathrm{Grp}(\mathcal{S})$ of $\mathsf{R}_T : \mathsf{DTM}^{\otimes}_{\vee} \to \mathrm{PMod}^{\otimes}_{H\mathbf{K}}$ in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{st}}_{\infty})_{\mathrm{PMod}^{\otimes}_{H\mathbf{K}}/}$, cf. Definition 3.3 (we abuse notation for R_T). According to Theorem 4.10 and 6.11, $\mathrm{Aut}(\mathsf{R}_T)$ is represented by MTG. On the other hand, using the above equivalence and unfolding the definition of $\pi_1(\mathfrak{S}_0, \eta_0)$ and $\mathrm{Aut}(\mathsf{R}_T)$, we see that the composite

$$\mathrm{CAlg}^{\mathrm{dis}}_{H\mathbf{K}} \hookrightarrow \mathrm{CAlg}_{H\mathbf{K}} \stackrel{\mathrm{Aut}(\mathsf{R}_T)}{\longrightarrow} \mathrm{Grp}(\mathcal{S}) \stackrel{\pi_0}{\to} \mathrm{Grp}^{\mathrm{dis}}$$

is equivalent to $\pi_1(\mathfrak{S}_0, \eta_0)$. Combined with Proposition 7.13 we complete the proof. □

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