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ENRIQUES SURFACES OF HUTCHINSON-GÖPEL TYPE AND MATHIEU AUTOMORPHISMS

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Abstract. We study a class of Enriques surfaces, called of Hutchinson-Göpel type. Starting with the projective geometry of Jacobian Kummer surfaces, we reach the Enriques' sextic expression of these surfaces and their intrinsic symmetry by $G = C_2^3$. We show that this G is of Mathieu type and conversely, that these surfaces are characterized among Enriques surfaces by the group action by C_2^3 with prescribed topological type of fixed point loci. As an application, we construct Mathieu type actions by the groups $C_2 \times \mathfrak{A}_4$ and $C_2 \times C_4$. Two introductory sections are also included.

1. Introduction

From a curve C of genus two and its Göpel subgroup $H \subset (\mathrm{Jac}\, C)_{(2)}$, we can construct an Enriques surface $(\mathrm{Km}\, C)/\varepsilon_H$, which we call *of Hutchinson-Göpel type*. We may say that surfaces of this type among whole Enriques surfaces occupy the analogously important place as Jacobian Kummer surfaces $\mathrm{Km}\, C$, or $\mathrm{Km}(\mathrm{Jac}\, C)$, do among whole $K3$ surfaces. In [9] we characterized these Enriques surfaces as those which have numerically reflective involutions.

In this paper, we will study the group action of Mathieu type on these Enriques surfaces of Hutchinson-Göpel type. In particular we will characterize them by using a special sort of action of Mathieu type by the elementary abelian group C_2^3. As a byproduct, we will also give examples of actions of Mathieu type by the groups $C_2 \times \mathfrak{A}_4$ (of order 24) and $C_2 \times C_4$ (of order 8). These constructions are crucial in the study of automorphisms of Mathieu type on Enriques surfaces; in particular it answers the conjecture we posed in the lecture note [11].

Key words and phrases. Enriques surfaces, Mathieu groups;

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Our starting point is the fact that the Kummer surface $\operatorname{Km} C$ is the $(2,2,2)$-Kummer covering[1] of the projective plane $\mathbb{P}^2$,

$$\operatorname{Km} C \xrightarrow{C_2^3} \mathbb{P}^2,$$

whose equation can be written in the form

$$u^2 = q_1(x,y,z), v^2 = q_2(x,y,z), w^2 = q_3(x,y,z).$$

All branch curves $\{(x,y,z) \in \mathbb{P}^2 \mid q_i(x,y,z) = 0\}$ $(i = 1,2,3)$ are reducible conics and our Enriques surface S of Hutchinson-Göpel type sits in between this covering as the quotient of $\operatorname{Km} C$ by the free involution

$$(u,v,w) \mapsto (-u,-v,-w).$$

By computing invariants, we will see that S is the normalization of the singular sextic surface

$$x^2 + y^2 + z^2 + t^2 + \left(\frac{a}{x^2} + \frac{b}{y^2} + \frac{c}{z^2} + \frac{d}{t^2}\right) xyzt = 0 \tag{1}$$

in $\mathbb{P}^3$, where $a,b,c,d \in \mathbb{C}^*$ are constants. They satisfy the condition $abcd = 1$ corresponding to the Cremona invariance of the six lines $\{q_1q_2q_3 = 0\} \subset \mathbb{P}^2$.

In general, an involution σ acting on an Enriques surface is said to be *Mathieu* or *of Mathieu type* if its Lefschetz number $\chi_{\mathrm{top}}(\operatorname{Fix}\sigma)$ equals four[2], [12]. This is equivalent to saying that the Euler characteristic of the fixed curves $\operatorname{Fix}^-(\sigma)$ is equal to 0 (see the beginning of Section 7 for this notation). We have the following classification of $\operatorname{Fix}^-(\sigma)$ according to its topological types.

- **(M0):** $\operatorname{Fix}^-(\sigma) = \emptyset$, namely σ is a *small* involution.
- **(M1):** $\operatorname{Fix}^-(\sigma)$ is a single elliptic curve.
- **(M2):** $\operatorname{Fix}^-(\sigma)$ is a disjoint union of two elliptic curves.
- **(M3):** $\operatorname{Fix}^-(\sigma)$ is a disjoint union of a genus $g \geq 2$ curve and $(g-1)$ smooth rational curves[3].

Our motivation comes from the following observation.

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[1] This octic model of $\operatorname{Km} C$ is different from the standard nonsingular octic model given by the smooth complete intersection of three diagonal quadrics. See ($\star$2) of Section 5.

[2] This number is exactly the number of fixed points of non-free involutions in the small Mathieu group M_{12}, which implies that the character of Mathieu involutions on $H^*(S,\mathbb{Q})$ coincides with that of involutions in M_{11}. This is the origin of the naming. See also [11].

[3] In fact only $g = 2$ is possible.

Observation 1. The Enriques surface $S = (\mathrm{Km}\, C)/\varepsilon_H$ of Hutchinson-Göpel type has an action of Mathieu type[4] by the elementary abelian group $G = C_2^3$ with the following properties. Let h be the polarization of degree 6 given by (1) above.

(1) The group G preserves the polarization h up to torsion.
(2) There exists a subgroup G_0 of index two, which preserves the polarization h while the coset $G - G_0$ sends h to $h + K_S$.
(3) All involutions in G_0 are of type (M2) above.
(4) All involutions in $G - G_0$ are of type (M0) above.

These are the properties of Mathieu type actions by which we characterize Enriques surfaces of Hutchinson-Göpel type.

Theorem 1. Let S be an Enriques surface with a group action of Mathieu type by $G = C_2^3$ which satisfies the properties (3) and (4) in Observation 1 for a subgroup G_0 of index two. Then S is isomorphic to an Enriques surface of Hutchinson-Göpel type.

Our proof of Theorem 1 (Section 7) exhibits the effective divisor h of Observation 1 in terms of the fixed curves of the group action. In particular we can reconstruct the sextic equation (1) of S. In this way, we see that the group action perfectly characterizes Enriques surfaces of Hutchinson-Göpel type and all parts of the Observation 1 hold true.

The sextic equation (1) has also the following application to our study of Mathieu automorphisms.

Theorem 2. Among those Enriques surfaces of Hutchinson-Göpel type (1), there exists a 1-dimensional subfamily whose members are acted on by the group $C_2 \times \mathfrak{A}_4$ of Mathieu type. Similarly there exists another 1-dimensional subfamily whose members are acted on by the group $C_2 \times C_4$ of Mathieu type.

The paper is organized as follows. Sections 2 and 3 give an introduction to Enriques surfaces. In Section 2 we explain the constructions of Enriques surfaces from rational surfaces, while in Section 3 we focus on the quotients of Kummer surfaces. In Section 4, we introduce a larger family of sextic Enriques surfaces which we call of diagonal type. They contain our Enriques surfaces of Hutchinson-Göpel type as a subfamily of codimension one. We derive the sextic equation by computing the invariants from a $K3$ surface which is a degree 8 cover of the projective plane $\mathbb{P}^2$. In Section 5 we restrict the family to Hutchinson-Göpel case. We give a discussion on the related isogenies between Kummer

[4]This means that every involution is Mathieu.

surfaces and also give the definition of the group action by $G = C_2^3$. In Section 6 we use the sextic equation to study their singularities and give a precise computation for the group actions. Theorem 2 is proved here. In Section 7 we prove Theorem 1.

Throughout the paper, we work over the field $\mathbb{C}$ of complex numbers.

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2. Rational surfaces and Enriques surfaces

An algebraic surface is *rational* if it is birationally equivalent to the projective plane $\mathbb{P}^2$. It is easy to see that a rational surface has the vanishing geometric genus $p_g = 0$ and the irregularity $q = 0$. In the beginning of the history of algebraic surfaces the converse problem was regarded as important.

Problem 1. Is an algebraic surface with $p_g = q = 0$ rational?

Enriques surfaces were discovered by Enriques as the counterexamples to this problem. They have the Kodaira dimension $\kappa = 0$. Nowadays we know that even some of algebraic surfaces of general type have also $p_g = q = 0$, the Godeaux surfaces for example.

Definition 1. An algebraic surface S is an *Enriques surface* if it satisfies $p_g = 0, q = 0$ and $2K_S \sim 0$.

By adjunction formula, a nonsingular rational curve $C \subset S$ satisfies $(C^2) = -2$, hence there are no exceptional curves of the first kind on S. It means that S is *minimal* in its birational equivalence class.

If a $K3$ surface X admits a fixed-point-free involution ε, then the quotient surface X/ε is an Enriques surface. Conversely for an Enriques surface S the canonical double cover

$$X = \mathrm{Spec}_S(\mathcal{O}_S \oplus \mathcal{O}_S(K_S))$$

turns out to be a $K3$ surface and is called the $K3$-*cover* of S. Since a $K3$ surface is simply connected, π is the same as the universal covering of S. In this way, an Enriques surface is nothing but a $K3$ surface mod out by a fixed-point-free involution ε.

Example 1. Let X be a smooth complete intersection of three quadrics in $\mathbb{P}^5$, defined by the equations

$$q_1(x) + r_1(y) = q_2(x) + r_2(y) = q_3(x) + r_3(y) = 0,$$

where $(x:y)=(x_0:x_1:x_2:y_0:y_1:y_2)\in\mathbb{P}^5$ are homogeneous coordinates of $\mathbb{P}^5$. If the quadratic equations q_i, r_i $(i=1,2,3)$ are general so that the intersections $q_1=q_2=q_3=0$ and $r_1=r_2=r_3=0$ considered in $\mathbb{P}^2$ are both empty, then the involution

$$\varepsilon\colon (x:y)\mapsto(x:-y)$$

is fixed-point-free and we obtain an Enriques surface $S=X/\varepsilon$.

As we mentioned, an Enriques surface appeared as a counterexample to Problem 1. Even though it is not a rational surface, it is closely related to them; a plenty of examples of Enriques surfaces are available by the quadratic twist construction as follows.

Let us consider a rational surface R and a divisor B belonging to the linear system $|-2K_R|$. The double cover of R branched along B,

$$X=\mathrm{Spec}_R(\mathcal{O}_R\oplus\mathcal{O}_R(-K_R))\to R,$$

gives a $K3$ surface if B is nonsingular. More generally if B has at most simple singularities, X has at most rational double points and its minimal desingularization $\tilde{X}$ is a $K3$ surface.

Example 2. The well-known examples are given by sextic curves in $R=\mathbb{P}^2$ or curves of bidegree $(4,4)$ in $R=\mathbb{P}^1\times\mathbb{P}^1$.

Let us assume that the surface R admits an involution $e\colon R\to R$ which is *small*, namely with at most finitely many fixed points over R. Further let us assume that the curve B is invariant under e, $e(B)=B$. Then we can lift e to involutions of X. There are two lifts, one of which acts symplectically on X (namely acts on the space $H^0(\Omega^2_X)$ trivially) and the other anti-symplectically (namely acts by (-1) on the space $H^0(\Omega^2_X)$). We denote the latter by ε. (The former is exactly the composite of ε and the covering transformation.) We can see that ε acts on X freely and the quotient X/ε gives an Enriques surface if B is disjoint from the fixed points of e. We call this Enriques surface the *quadratic twist* of R by (e,B).

Example 3. Let e_0 be an arbitrary involution of $\mathbb{P}^1$ and we consider the small involution $e=(e_0,e_0)$ acting on $R=\mathbb{P}^1\times\mathbb{P}^1$. According to our recipe, we can construct an Enriques surface S which is the quadratic twist of R obtained from e and an e-stable divisor B of bidegree $(4,4)$.

Example 4. We consider the Cremona transformation

$$e\colon (x:y:z)\mapsto(1/x:1/y:1/z)$$

of $\mathbb{P}^2$, where $(x:y:z)$ are the homogeneous coordinates of $\mathbb{P}^2$. Let B be a sextic curve with nodes or cusps at three points $(1:0:0),(0:1:$

0), (0 : 0 : 1) and such that $e(B) = B$. (More generally, the singularities at the three points can be any simple singularities of curves.) Then we can construct the quadratic twist S of $\mathbb{P}^2$ by (e, B).

In this example, it might be easier to consider the surface R obtained by blowing the three points up. The Cremona transformation e induces a biregular automorphism of R and the strict transform $\overline{C}$ of C belongs to the linear system $|-2K_R|$. The Enriques surface S is nothing but the quadratic twist of the surface R by $(e, \overline{C})$.

We borrowed the terminology from the following example.

Example 5. (Kondo [8], Hulek-Schütt [6]) Let $f\colon R \to \mathbb{P}^1$ be a rational elliptic surface with the zero-section and a 2-torsion section. Let e be the translation by the 2-torsion section, which we assume to be small. Let B be a sum of two fibers of f. Then B belongs to $|-2K_R|$ and is obviously stable under e. Thus we obtain an Enriques surface from the quadratic twist construction. In this case the Enriques surface naturally has an elliptic fibration $S \to \mathbb{P}^1$. In the theory of elliptic curves this is called the quadratic twist of f.

We remark that the Enriques surface obtained as the quadratic twist of a rational surface always admits a nontrivial involution. In general any involution σ of an Enriques surface admits two lifts to the $K3$-cover X, one of which is symplectic and the other non-symplectic. We denote the former by σ_K and the latter by σ_R. With one exception, the quotient X/σ_R becomes a rational surface.

This operation can be seen as the converse construction of the quadratic twist. The exception appears in the case where σ_R is also a fixed-point-free involution, in which case the quotient X/σ_R is again an Enriques surface.

3. Abelian surfaces and Enriques surfaces

A two-dimensional torus $T = \mathbb{C}^2/\Gamma$, where $\Gamma \simeq \mathbb{Z}^4$ is a full lattice in $\mathbb{C}^2$, is acted on by the involution $(-1)_T$. It has 16 fixed points which are exactly the 2-torsion points $T_{(2)}$ of T. The *Kummer surface* $\mathrm{Km}\, T$ is obtained as the minimal desingularization of the quotient surface $\overline{\mathrm{Km}} T = T/(-1)_T$. This is known to be a $K3$ surface, equipped with 16 exceptional (-2)-curves.

When T is isomorphic to the direct product $E_1 \times E_2$ of elliptic curves, the Kummer surface $\mathrm{Km}(E_1 \times E_2)$ is the same as the desingularized double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$\overline{\mathrm{Km}}(E_1 \times E_2)\colon w^2 = x(x-1)(x-\lambda)y(y-1)(y-\mu), \tag{2}$$

where $\lambda, \mu \in \mathbb{C} - \{1, 0\}$ are constants and x, y are inhomogeneous coordinates of $\mathbb{P}^1$. The strict transforms of the eight divisors on $\mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$x = 0, 1, \infty, \lambda \text{ and } y = 0, 1, \infty, \mu \tag{3}$$

gives 8 smooth rational curves on $\mathrm{Km}(E_1 \times E_2)$. In this product case together with 16 exceptional curves, it has 24 smooth rational curves with the following configuration (called the *double Kummer configuration*).

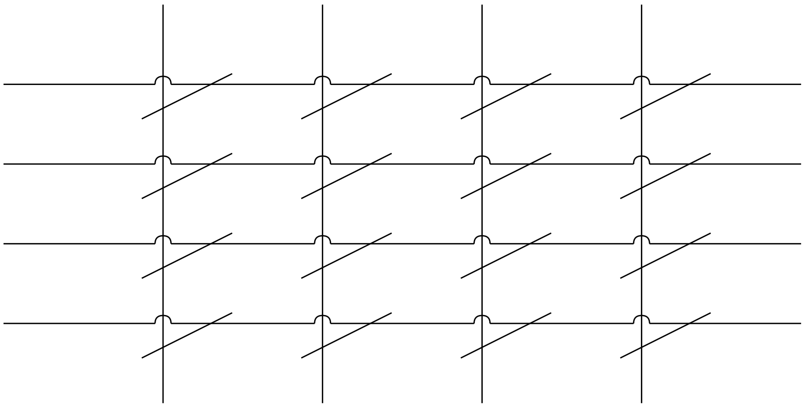

Figure 1: the double Kummer configuration

There are many studies on $\mathrm{Km}\, T$ when T is a principally polarized abelian surface, too. In this case using the theta divisor Θ, the linear system $|2\Theta|$ gives an embedding of the singular surface $T/(-1)_T$ into $\mathbb{P}^3$ as a quartic surface

$$\begin{aligned} x^4 + y^4 + z^4 + t^4 + A(x^2t^2 + y^2z^2) + B(y^2t^2 + x^2z^2) \\ + C(z^2t^2 + x^2y^2) + Dxyzt = 0, \\ A, B, C, D \in \mathbb{C} \end{aligned}$$

which is stable under the Heisenberg group action.

Let us consider the following question: How many Enriques surfaces are there whose universal covering is one of these Kummer surfaces $\mathrm{Km}\, T$? The easiest example is given by the following.

Example 6. (Lieberman) On the Kummer surface $\mathrm{Km}(E_1 \times E_2)$ of product type (2), we have the involutive action

$$\varepsilon\colon (x, y, w) \mapsto \left(\frac{\lambda}{x}, \frac{\mu}{y}, \frac{\lambda\mu w}{x^2y^2}\right).$$

We can see easily that ε is fixed-point-free. Hence $\mathrm{Km}(E_1 \times E_2)/\varepsilon$ is an Enriques surface, which is the quadratic twist of $\mathbb{P}^1 \times \mathbb{P}^1$ by $e\colon (x, y) \mapsto (\lambda/x, \mu/y)$ and the branch divisor (3).

The surface $\mathrm{Km}(E_1 \times E_2)$ is equivalently the desingularized double cover of $\mathbb{P}^2$ branched along 6 lines

$$x = 0, 1, \lambda, \quad y = 0, 1, \mu.$$

(See Figure 2.) The involution above is given as the lift of Cremona involution $(x, y) \mapsto (\frac{\lambda}{x}, \frac{\mu}{y})$, which exhibits the Enriques surface $\mathrm{Km}(E_1 \times E_2)/\varepsilon$ as the quadratic twist of blown up $\mathbb{P}^2$.

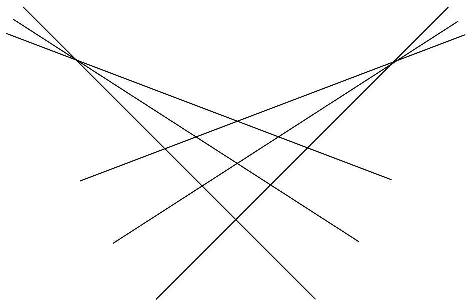

Figure 2

Another Enriques surface can be obtained from the surface $\mathrm{Km}(E_1 \times E_2)$ as follows when $\lambda \neq \mu$. We note that under this condition, the three lines passing through two of the points $(0, 0), (1, 1), (\lambda, \mu)$ can be given by

$$x - y,\ \mu x - \lambda y,\ (\mu - 1)(x - 1) - (\lambda - 1)(y - 1).$$

We make the coordinate change

$$X = \frac{\mu x - \lambda y}{x - y}, Y = \frac{(\mu - 1)(x - 1) - (\lambda - 1)(y - 1)}{x - y}.$$

The six branch lines then become

$$\begin{aligned} X &= \lambda, \mu \\ Y &= \lambda - 1, \mu - 1 \\ X/Y &= \lambda/(\lambda - 1), \mu/(\mu - 1). \end{aligned}$$

These six lines are preserved by the Cremona transformation

$$(X, Y) \mapsto \left(\frac{\lambda\mu}{X}, \frac{(\lambda - 1)(\mu - 1)}{Y} \right).$$

Hence the Kummer surface

$$\begin{aligned}&\mathrm{Km}(E_1 \times E_2)\colon\\ &w^2 = (X-\lambda)(X-\mu)(Y-\lambda+1)(Y-\mu+1)\\ &\qquad \times (\lambda Y-(\lambda-1)X)(\mu Y-(\mu-1)X)\end{aligned}$$

has the automorphism

$$\varepsilon\colon (X,Y,w) \mapsto \left(\frac{\lambda\mu}{X}, \frac{(\lambda-1)(\mu-1)}{Y}, \frac{\lambda(\lambda-1)\mu(\mu-1)w}{X^2Y^2}\right)$$

whenever $\lambda \neq \mu$. Moreover this automorphism has no fixed points; hence we obtain the Enriques surface $\mathrm{Km}(E_1 \times E_2)/\varepsilon$. This Enriques surface with $\lambda = \mu = 1^{1/3}$ was found by Kondo and constructed in full generality by Mukai [9]. It is called an Enriques surface *of Kondo-Mukai type.*

Remark 1. It is interesting to find out the limit of the above Enriques surface $\mathrm{Km}(E_1 \times E_2)/\varepsilon$ when λ goes to μ. The limit is not anymore an Enriques surface but a rational surface with quotient singularities of type $\frac{1}{4}(1,1)$. A more precise description is the following: Let R be the minimal resolution of the double cover of $\mathbb{P}^2$ branched along the union of four tangent lines

$$x = 0,\ x-2y+z = 0,\ x-2\lambda y+\lambda^2 z = 0,\ z = 0$$

of the conic $xz = y^2$. The pullback of the conic splits into two smooth rational curves C_1 and C_2 in R. Let R' be the blow-up of R at the four points $C_1 \cap C_2$. Then the strict transforms of C_1 and C_2 become (-4)-$\mathbb{P}^1$'s. The limit of the Enriques surface $\mathrm{Km}(E_1 \times E_2)/\varepsilon$ is the rational surface R' contracted along these two (-4)-$\mathbb{P}^1$'s.

Remark 2. (Ohashi [13]) When E_1 and E_2 are taken generically, these two surfaces are the only Enriques surfaces (up to isomorphism) whose universal covering is the surface $\mathrm{Km}(E_1 \times E_2)$.

Let us proceed to the study of $\mathrm{Km}(A)$, where (A, Θ) is a principally polarized abelian surface. In this case, there are three Enriques surfaces known whose universal coverings are isomorphic to $\mathrm{Km}\,A$ ([10],[14]). Here we introduce the surface obtained from a Göpel subgroup $H \subset A_{(2)}$. The next observation is fundamental.

Lemma 1. Suppose that we are given six distinct lines $l_1, \cdots, l_6$ in the projective plane, whose three intersection points $p_1 = l_1 \cap l_4, p_2 = l_2 \cap l_5, p_3 = l_3 \cap l_6$ are not collinear and the lines $\overline{p_i p_j}$ are different from l_i. Then the following conditions are equivalent.

(1) A suitable quadratic Cremona transformation with center p_1, p_2, p_3 sends l_1, l_2, l_3 to l_4, l_5, l_6 respectively.
(2) All $l_1, \cdots, l_6$ are tangent to a smooth conic or both l_1, l_2, l_3 and l_4, l_5, l_6 are concurrent (after suitable renumberings $2 \leftrightarrow 5$ or $3 \leftrightarrow 6$).

Proof. This is an extended version of [10, Proposition 5.1]. We sketch the proof. Let us choose linear coordinates $(x : y : z)$ such that p_1, p_2, p_3 are the vertices of the coordinate triangle $xyz = 0$. Then the six lines are given by

$$l_i \colon y = \alpha_i x \ (i = 1, 4), \ l_j \colon z = \alpha_j y \ (j = 2, 5), \ l_k \colon x = \alpha_k z \ (k = 3, 6)$$

for $\alpha_1, \cdots, \alpha_6 \in \mathbb{C}^*$. We easily see that the condition (1) is equivalent to $\prod_{i=1}^6 \alpha_i = 1$. Let us consider a conic in the *dual* projective plane

$$Q \colon ax^2 + by^2 + cz^2 + dyz + ezx + fxy = 0.$$

For Q to contain the six points q_i corresponding to l_i, we have the following conditions

$$\alpha_1\alpha_4 = \frac{b}{a}, \qquad \alpha_2\alpha_5 = \frac{c}{b}, \qquad \alpha_3\alpha_6 = \frac{a}{c},$$
$$\alpha_1 + \alpha_4 = \frac{f}{a}, \ \alpha_2 + \alpha_5 = \frac{d}{b}, \ \alpha_3 + \alpha_6 = \frac{e}{c}.$$

Thus $\prod_{i=1}^6 \alpha_i = 1$ is equivalent to the existence of such Q. If Q is smooth, then the former condition of (2) is satisfied by taking the dual of Q. If Q is a union of two distinct lines, then the points q_i and q_{i+3} must lie on different components for $i = 1, 2, 3$, hence the latter configuration of (2) occurs. (By the same reason, Q cannot be a double line.) □

We have already encountered the latter configuration of lines in Figure 2; in this case the double cover of $\mathbb{P}^2$ branched along $\sum l_i$ is birational to $\mathrm{Km}(E_1 \times E_2)$. Even in the former case of (2) of Lemma 1, the lift of the Cremona involution to double cover gives an automorphism of $\mathrm{Km}(A)$ without fixed points. Hence we obtain an Enriques surface $\mathrm{Km}(A)/\varepsilon$.

This Enriques surface is described in the following way (and characterized by the presence of a numerically reflective involution) by Mukai [10]. Let $H \subset A_{(2)}$ be a Göpel subgroup, namely, H is a subgroup consisting of four elements and the Weil pairing with respect to 2Θ,

$$A_{(2)} \times A_{(2)} \to \mu_2,$$

is trivial on $H \times H$. There are 15 such subgroups. One such H defines four nodes of the Kummer quartic surface in $\mathbb{P}^3$, and if we take the homogeneous coordinates $(x : y : z : t)$ of $\mathbb{P}^3$ so that the coordinate points coincide with the four nodes, then the Kummer quartic surface has the equation

$$q(xt + yz, yt + xz, zt + xy) + (\text{const.})xyzt = 0. \tag{4}$$

(We assume that the four nodes are not coplanar.) This equation is invariant under the standard Cremona transformation

$$(x : y : z : t) \mapsto \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{t}\right).$$

Moreover, this involutive automorphism is free from fixed points over the Kummer quartic surface. Let us denote by ε_H this free involution on $\mathrm{Km}(A)$. The Enriques surface $\mathrm{Km}(A)/\varepsilon_H$ is thus determined by the principally polarized abelian surface (A, Θ) and the Göpel subgroup H. We call this surface *the Enriques surface of Hutchinson-Göpel type* since the expression (4) was first found by Hutchinson [5] using theta functions. (See also Keum [4, §3].)

Remark 3. The limit of the Enriques surface $\mathrm{Km}(A)/\varepsilon_H$ when H becomes coplanar is also a rational surface with two quotient singular points of type $\frac{1}{4}(1, 1)$ as in Remark 1.

4. Sextic Enriques surfaces of diagonal type

Now we consider the Kummer $(2, 2, 2)$-covering of the projective plane $\mathbb{P}^2$ with coordinates $x = (x_1 : x_2 : x_3)$ branched along three conics $q_i(x) = 0,\ i = 1, 2, 3$:

$$\overline{X}\colon w_1^2 = q_1(x),\ w_2^2 = q_2(x),\ w_3^2 = q_3(x).$$

These equations define a $(2, 2, 2)$ complete intersection in $\mathbb{P}^5$ with homogeneous coordinates $(w_1 : w_2 : w_3 : x_1 : x_2 : x_3)$. Hence the minimal desingularization X of $\overline{X}$ is a $K3$ surface if it has at most rational double points. It has the action by C_2^3 arising from covering transformations. Among them, we focus on the involution

$$\varepsilon\colon (w_1 : w_2 : w_3 : x_1 : x_2 : x_3) \mapsto (-w_1 : -w_2 : -w_3 : x_1 : x_2 : x_3).$$

It is free of fixed points on X if and only if the locus $q_1(x) = q_2(x) = q_3(x) = 0$ is empty in $\mathbb{P}^2$. In this way we obtain the Enriques surface $S = X/\varepsilon$.

Let us specialize to the case where all $q_i(x)$ are reducible conics. More precisely our assumption is as follows.

($\star$) The conic $\{q_i = 0\}$ is the sum of two lines l_i, l_{i+3} ($i = 1, 2, 3$) for six distinct lines $l_1, \cdots, l_6$. The three points $l_1 \cap l_4, l_2 \cap l_5, l_3 \cap l_6$ are also distinct.

Under assumption ($\star$), the $(2,2,2)$-covering $\overline{X}$ has at most rational double points and we obtain the minimal desingularization X and the quotient Enriques surface S. The singularities of $\overline{X}$ consists of 12 nodes located above the three points $l_1 \cap l_4, l_2 \cap l_5, l_3 \cap l_6$. (It follows that the Enriques surface S contains 6 disjoint smooth rational curves as images of the exceptional curves.)

Remark 4. The quotient surface $\overline{X}/\varepsilon$ is nothing but the normalization of the surface

$$u_1^2 = q_1 q_3, \ u_2^2 = q_2 q_3$$

which is the covering of $\mathbb{P}^2$ of degree 4.

The projection of $\mathbb{P}^2$ from the singular point of q_i defines a rational map to the projective line, which in turn defines an elliptic fibration on X and on S. We denote by G_0 the Galois group of $S \to \mathbb{P}^2$. Each nontrivial element $g \in G_0$ corresponds to and defines the double covering of the rational elliptic surface branched along two smooth fibers. Hence $\mathrm{Fix}(g)$ has two smooth elliptic curves as its 1-dimensional components. This shows

Proposition 1. Under assumption ($\star$), the action of $G_0 \simeq C_2^2$ on the Enriques surface S is of Mathieu type and every nontrivial element has (M2) type.

For later use, we give the sextic equation of the Enriques surface S under the condition ($\star$). Here we additionally assume that the three points $\mathrm{Sing}(q_i)$ ($i = 1, 2, 3$) are not collinear. (See also Remark 7.) Then we can choose homogeneous coordinates of $\mathbb{P}^2$ so that the three points are the coordinate points $(x_1 : x_2 : x_3) = (0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)$. The degree 8 cover X over $\mathbb{P}^2$ has the form

$$w_i^2 = \frac{x_{i+1} - \alpha_i x_{i+2}}{x_{i+1} - \beta_i x_{i+2}}, (i = 1, 2, 3 \in \mathbb{Z}/3), \tag{5}$$

hence it has the following field of rational functions

$$\mathbb{C}\left(\frac{x_1}{x_2}, \frac{x_2}{x_3}, \sqrt{\frac{x_2 - \alpha_1 x_3}{x_2 - \beta_1 x_3}}, \sqrt{\frac{x_3 - \alpha_2 x_1}{x_3 - \beta_2 x_1}}, \sqrt{\frac{x_1 - \alpha_3 x_2}{x_1 - \beta_3 x_2}}\right).$$

Here we put $q_i(x) = (\mathrm{const.})(x_{i+1} - \alpha_i x_{i+2})(x_{i+1} - \beta_i x_{i+2})$. X is exactly the minimal model of this field of algebraic functions in two variables.

Since we have the relations

$$\frac{x_{i+1}}{x_{i+2}} = \frac{\beta_i w_i^2 - \alpha_i}{w_i^2 - 1}, i = 1, 2, 3,$$

by multiplying them, X is also the minimal desingularization of the $(2, 2, 2)$ divisor

$$(\star\star) \quad (\beta_1 w_1^2 - \alpha_1)(\beta_2 w_2^2 - \alpha_2)(\beta_3 w_3^2 - \alpha_3) = (w_1^2 - 1)(w_2^2 - 1)(w_3^2 - 1)$$

in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Here we consider $w_i (i = 1, 2, 3)$ as inhomogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 2. Assume that the three reducible conics $q_1 = 0, q_2 = 0, q_3 = 0$ satisfy $(\star)$ and the three points $\operatorname{Sing} q_1, \operatorname{Sing} q_2, \operatorname{Sing} q_3$ are not collinear. Then the Enriques surface $S \to \mathbb{P}^2$ is isomorphic to the minimal desingularization of the sextic surface in $\mathbb{P}^3$ defined by

$$(\star\star\star) \quad a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = \left(\frac{b_0}{x_0^2} + \frac{b_1}{x_1^2} + \frac{b_2}{x_2^2} + \frac{b_3}{x_3^2} \right) x_0 x_1 x_2 x_3,$$

where we put

$$\begin{aligned} a_0 &= \alpha_1\alpha_2\alpha_3 - 1, a_1 = \alpha_1\beta_2\beta_3 - 1, \\ a_2 &= \beta_1\alpha_2\beta_3 - 1, a_3 = \beta_1\beta_2\alpha_3 - 1, \\ b_0 &= \beta_1\beta_2\beta_3 - 1, b_1 = \beta_1\alpha_2\alpha_3 - 1, \\ b_2 &= \alpha_1\beta_2\alpha_3 - 1, b_3 = \alpha_1\alpha_2\beta_3 - 1 \end{aligned}$$

Proof. The Enriques surface is the quotient of the $(2, 2, 2)$ surface $(\star\star)$ by the involution

$$(w_1, w_2, w_3) \mapsto (-w_1, -w_2, -w_3)$$

followed by the minimal desingularization. We focus on the ambient spaces and construct a birational map between $\mathbb{P}^3$ and the quotient of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ by the involution above.

We consider a rational map

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$$

defined by $(w_1, w_2, w_3) \mapsto (x_0 : x_1 : x_2 : x_3) = (1 : w_2 w_3 : w_1 w_3 : w_1 w_2)$. It has four points of indeterminacy $(\infty, \infty, \infty), (\infty, 0, 0), (0, \infty, 0), (0, 0, \infty)$. In other words, the rational map is the projection of the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ from the 3-space spanned by the 4 points. The indeterminacy is resolved by blowings up and we obtain a morphism

$$\mathrm{Bl}_{\text{4-pts}}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^3.$$

This morphism factors through the double cover

$$Y : w^2 = x_0 x_1 x_2 x_3$$

of $\mathbb{P}^3$ branched along the tetrahedron and $\mathrm{Bl}_{4\text{-pts}}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \to Y$ is a birational morphism which contracts 6 quadric surfaces

$$w_1 = 0, \infty, \ w_2 = 0, \infty, \ w_3 = 0, \infty,$$

into 6 edges. Since $(\star\star)$ is an irreducible surface which does not contain any of these six quadric surfaces, by multiplying $w_1^2 w_2^2 w_3^2$,

$$(\beta_1 x_2 x_3 - \alpha_1 x_0 x_1)(\beta_2 x_1 x_3 - \alpha_2 x_0 x_2)(\beta_3 x_1 x_2 - \alpha_3 x_0 x_3) \\ = (x_2 x_3 - x_0 x_1)(x_1 x_3 - x_0 x_2)(x_1 x_2 - x_0 x_3)$$

defines the sextic surface which is birational to the Enriques surface. By reducing coefficients, we obtain $(\star\star\star)$. □

Remark 5. In the proof we have used the four invariants $1, w_2 w_3, w_1 w_3, w_1 w_2$. Instead, we could use the anti-invariants $w_1 w_2 w_3, w_1, w_2, w_3$ to obtain another sextic equation. In this case the indeterminacies are given by

$$(0,0,0), (0,\infty,\infty), (\infty,0,\infty), (\infty,\infty,0)$$

and the computation results in the sextic surface

$$(\star\star\star') : \sum_{i=0}^{3} b_i x_i^2 = x_0 x_1 x_2 x_3 \sum_{i=0}^{3} \frac{a_i}{x_i^2}.$$

This is nothing but the surface obtained from $(\star\star\star)$ by applying the standard Cremona transformation $(x_i) \mapsto (1/x_i)$.

Remark 6. More generally, a $(2,2,2)$ $K3$ surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ which is invariant under the involution

$$(w_1, w_2, w_3) \mapsto (-w_1, -w_2, -w_3)$$

is mapped to the sextic Enriques surface

$$q(x_0, x_1, x_2, x_3) = x_0 x_1 x_2 x_3 \sum_{i=0}^{3} \frac{b_i}{x_i^2},$$

not necessarily of diagonal type. The proof is the same as above.

As is well-known, these sextic surfaces have double lines along the six edges of the tetrahedron $x_0 x_1 x_2 x_3 = 0$.

5. Action of C_2^3 of Mathieu type on Enriques surfaces of Hutchinson-Göpel type

In this section we study Enriques surfaces of Hutchinson-Göpel type explained in Section 3. We show that they are $(2,2)$-covers of the projective plane $\mathbb{P}^2$ branched along three reducible conics and extend the action of $G_0 \simeq C_2^2$ to an action of C_2^3, which is still of Mathieu type.

Let us begin with the configuration of six distinct lines $l_1, \cdots, l_6$ in $\mathbb{P}^2$. We recall that there exists uniquely a C_2^5-cover of $\mathbb{P}^2$ branched along these lines; it is represented by the diagonal complete intersection surface in $\mathbb{P}^5$ as

$$W\colon \sum_{i=1}^{6} a_i x_i^2 = \sum_{i=1}^{6} b_i x_i^2 = \sum_{i=1}^{6} c_i x_i^2 = 0,$$

where $(x_1 : \cdots : x_6)$ are the homogeneous coordinates of $\mathbb{P}^5$.

We restrict ourselves to the case

($\star 0$) All $l_1, \cdots, l_6$ are tangent to a smooth conic $Q \subset \mathbb{P}^2$.

More concretely, we have a nonsingular curve B of genus two

$$(\star 1)\ w^2 = \prod_{i=1}^{6} (x - \lambda_i),\ \lambda_i \in \mathbb{C}$$

and the quadratic Veronese embedding $v_2 \colon \mathbb{P}^1 \to \mathbb{P}^2$ whose image is $Q = v_2(\mathbb{P}^1)$ so that the lines $l_1, \cdots, l_6$ are nothing but the tangent lines to Q at $v_2(\lambda_i)$. By an easy computation (e.g. [10, Section 5]), the desingularized double cover of $\mathbb{P}^2$ branched along the sum $\sum_{i=1}^{6} l_i$ is isomorphic to the Jacobian Kummer surface $\mathrm{Km}\, B$ of the curve B. The C_2^5-cover branched along six lines in this case is given by the equation

$$W\colon \sum_{i=1}^{6} x_i^2 = \sum_{i=1}^{6} \lambda_i x_i^2 = \sum_{i=1}^{6} \lambda_i^2 x_i^2 = 0.$$

The morphism from W to the double plane branches only along the 15 exceptional curves of $\mathrm{Km}\, B$ corresponding to 15 nonzero 2-torsions of $J(B)$, hence the induced map $W \dashrightarrow \mathrm{Km}\, B$ is the same as induced from the multiplication morphism $x \mapsto 2x$ of $J(B)$. In particular we see that W is isomorphic to $\mathrm{Km}\, B$. (See [15, Theorem 2.5] for the alternative proof using the traditional quadric line complex.)

We take the subgroup H_0 of $J(B)$ consisting of 2-torsions $p_1 - p_4, p_2 - p_5, p_3 - p_6$ and the zero element. Here p_i are the Weierstrass points corresponding to $\lambda_i \in \mathbb{C}$. This H_0 is a Göpel subgroup of $J(B)$ and the quotient abelian surface $J(B)/H_0$ again has a principal polarization. There are two cases:

(1) The quotient surface $J(B)/H_0$ is isomorphic to the Jacobian $J(C)$ of a curve C of genus two.
(2) The surface $J(B)/H_0$ is isomorphic to a product $E_1 \times E_2$ of two elliptic curves.

The group H_0 acts on the Kummer surface $W \simeq \mathrm{Km}\, B$ by the formulas

$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (-x_1 : x_2 : x_3 : -x_4 : x_5 : x_6), \text{ and}$$
$$(x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \mapsto (x_1 : -x_2 : x_3 : x_4 : -x_5 : x_6).$$

Hence the quotient $\mathrm{Km}\, B/H_0$ is a C_2^3-cover of $\mathbb{P}^2$ branched along the three reducible conics

$$(\star 2) \quad l_1 + l_4 \colon q_1 = 0,\ l_2 + l_5 \colon q_2 = 0,\ l_3 + l_6 \colon q_3 = 0.$$

Proposition 3. Assume in ($\star$2) that the three points $\mathrm{Sing}\, q_i$ ($i = 1, 2, 3$) are not collinear. Then the minimal resolution of the quotient surface $\mathrm{Km}\, B/H_0$ is isomorphic to the Jacobian Kummer surface $\mathrm{Km}\, C$ of C and the involution

$$(w_1, w_2, w_3) \mapsto (-w_1, -w_2, -w_3)$$

of $\mathrm{Km}\, C$ coincides with the Hutchinson-Göpel involution ε_H associated to the Göpel subgroup $H := J(B)_{(2)}/H_0$ of $J(C)$ ([10]). In particular, the Enriques cover $S \to \mathbb{P}^2$ of degree 4 with branch curve ($\star$2) is an Enriques surface of Hutchinson-Göpel type.

Proof. We consider the polar m_i of Q at the point $\mathrm{Sing}\, q_i = l_i \cap l_{i+3}$, namely the line connecting $v_2(\lambda_i)$ and $v_2(\lambda_{i+3})$. Since $\mathrm{Sing}\, q_i$ are not collinear, m_1, m_2, m_3 are not concurrent.

We introduce homogeneous coordinates $(x_1 : x_2 : x_3)$ such that m_1, m_2, m_3 are defined by x_1, x_2, x_3. Let $q(x_1, x_2, x_3)$ be the defining equation of Q. Replacing q, m_1, m_2, m_3 by suitable constant multiplications, we can put the defining equations of the conics $l_i + l_{i+3} : q_i = 0$ as $-q + x_i^2$. Now the $K3$ surface $\overline{X}$ is defined by the equations

$$w_i^2 = -q(x_1, x_2, x_3) + x_i^2 \ (i = 1, 2, 3).$$

In particular we see that $\overline{X}$ is contained in the $(2, 2)$ complete intersection

$$V \colon w_1^2 - x_1^2 = w_2^2 - x_2^2 = w_3^2 - x_3^2$$

in $\mathbb{P}^5$. This (quartic del Pezzo) 3-fold V is nothing but the image of the rational map

$$(\star 3) \quad \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$$
$$(x : y : z : t) \mapsto (x_1 : x_2 : x_3 : w_1 : w_2 : w_3)$$
$$= (xt + yz : yt + xz : zt + xy : xt - yz : yt - xz : zt - xy)$$

More precisely, V is isomorphic to the $\mathbb{P}^3$ first blown up at four coordinate points and then contracted along the six (-2) smooth rational curves which are strict transforms of the six edges of the tetrahedron

$xyzt = 0$. The rational map $(\star 3)$ induces a birational equivalence between $\overline{X}$ and the quartic surface

$$(\star 4) \quad q(xt + yz, yt + xz, zt + xy) = 4xyzt.$$

Under $(\star 3)$, the involution $(\underline{x} : \underline{w}) \mapsto (\underline{x} : -\underline{w})$ corresponds to the Cremona involution

$$(x : y : z : t) \mapsto \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{t} \right).$$

Hence S is of Hutchinson-Göpel type (see Section 3). □

Remark 7. The collinearity property of the three points $\operatorname{Sing} q_i$ $(i = 1, 2, 3)$ is equivalent to that the three quadratic equations $(x - \lambda_i)(x - \lambda_{i+3})$ are linearly dependent. In this case, there exists an involution σ of $\mathbb{P}^1$ which sends λ_i to λ_{i+3} for $i = 1, 2, 3$. This involution σ lifts to an involution $\tilde{\sigma}$ of the curve B in $(\star 1)$ and the quotient $B/\tilde{\sigma}$ becomes an elliptic curve. We call such pair (B, H_0) *bielliptic*. In this case the quotient $J(B)/H_0$ is isomorphic to the product of two elliptic curves as principally polarized abelian surfaces.

Corollary 1. Assume that the pair (C, H) is not bielliptic. Then the Enriques surface $\operatorname{Km} C/\varepsilon_H$ obtained from the curve C of genus two and the Göpel subgroup $H \subset J(C)_{(2)}$ is isomorphic to the desingularization of the $(2, 2)$-cover of the projective plane $\mathbb{P}^2$ branched along three reducible conics $(\star 2)$ satisfying the condition $(\star 0)$.

Proof. The quotient abelian surface $J(C)/H$ has a principal polarization which is not reducible. Hence it is isomorphic to the Jacobian $J(B)$ of some curve B of genus two. Also the quotient $H_0 = J(C)_{(2)}/H$ gives a Göpel subgroup of $J(B)$. The pair (B, H_0) is not bielliptic, hence the three points $\operatorname{Sing} q_i$ $(i = 1, 2, 3)$ are not collinear. By the proposition, $\operatorname{Km} C/\varepsilon_H$ is isomorphic to the Enriques surface which is the $(2, 2)$-covering of the projective plane. □

By Lemma 1, we have a Cremona involution σ which exchanges l_1, l_2, l_3 with l_4, l_5, l_6 respectively. This involution σ lifts to $\operatorname{Km} C$ hence we obtain an action by C_2^4 on $\operatorname{Km} C$ and on the Enriques surface S we get the extention of $G_0 \simeq C_2^2$ to the group $G \simeq C_2^3$. The Cremona involution σ has only four isolated fixed points. Hence the lift of σ as an anti-symplectic involution of $\operatorname{Km} C$ has no fixed points. This together with Proposition 1 prove the following.

Proposition 4. The Enriques surface $\operatorname{Km} C/\varepsilon_H$ of Hutchinson-Göpel type has an action of Mathieu type by the elementary abelian group $G \simeq C_2^3$.

In fact, every involution in the coset $G \setminus G_0$ has type (M0). Although we can prove this from geometric consideration so far, we postpone it until Theorem 3 where a straightforward computation of the fixed locus is given.

Remark 8. The image T of the rational map ($\star$3) is the octahedral toric 3-fold and its automorphism group is isomorphic to the semi-direct product $(\mathbb{C}^*)^3.(\mathfrak{S}_4 \times \mathfrak{S}_2)$. The obvious C_2^3 of $\mathrm{Aut}(\mathrm{Km}\, C)$ is induced from the Klein's four-group in $\mathfrak{S}_4$ and the Cremona involution, the generator of $\mathfrak{S}_2$. But any lift of the Cremona involution σ does not come from $\mathrm{Aut}\, T$.

Let us study the symmetry of the sextic surface

$$(\star 5) \qquad \sum_{i=0}^{3} a_i x_i^2 = \left(\sum_{i=0}^{3} \frac{b_i}{x_i^2} \right) x_0 x_1 x_2 x_3.$$

The group $G_0 \simeq C_2^2$ acts by the simultaneous change of signs of two coordinates. The coefficients a_i, b_i $(i = 0, \cdots, 3)$ are given as in Proposition 2. When we are treating Enriques surfaces of Hutchinson-Göpel type, since the six lines satisfy the condition ($\star$0), we have

$$\prod_{i=1}^{3} \alpha_i \prod_{i=1}^{3} \beta_i = 1.$$

By the identity

$$\prod_{i=0}^{3} a_i - \prod_{i=0}^{3} b_i = (\prod_{i=1}^{3} \alpha_i \prod_{i=1}^{3} \beta_i - 1) \prod_{i=1}^{3} (\alpha_i - \beta_i),$$

we obtain $\prod_{i=0}^{3} a_i = \prod_{i=0}^{3} b_i$. By choosing the constants appropriately, the sextic surface ($\star$5) acquires the action of the standard Cremona involution

$$(6) \quad (x_0 : x_1 : x_2 : x_3) \mapsto \left(\frac{(\text{const.})}{x_0} : \frac{(\text{const.})}{x_1} : \frac{(\text{const.})}{x_2} : \frac{(\text{const.})}{x_3} \right).$$

This action together with G_0 gives us the action of $G \simeq C_2^3$.

Remark 9. (1) When a principally polarized abelian surface A is the product $E_1 \times E_2$, then the morphism $\Phi_{|2\Theta|} : A \to \mathbb{P}^3$ is of degree 2 onto a smooth quadric. The limit of Enriques surfaces of Hutchinson-Göpel type, when $(\mathrm{Jac}\, C, H)$ becomes $(E_1 \times E_2, H_0)$, is the Enriques surface $\mathrm{Km}(E_1 \times E_2)/\varepsilon$ of Lieberman type (Example 6) or Kondo-Mukai type according as the Göpel subgroup H_0 is product or not. $\mathrm{Km}(E_1 \times E_2)$ is also a $(2,2,2)$-cover of $\mathbb{P}^2$ with branch along three reducible quadrics ($\star$2). In the latter case they satisfy ($\star$0) and $\mathrm{Sing}(q_i)$ $(i = 1, 2, 3)$ are

collinear.
(2) When the three points $\operatorname{Sing} q_i$ are collinear, there exists an involution of $\mathbb{P}^2$ which exchanges l_i with l_{i+3} for $i = 1, 2, 3$. Thus we have an extention of the group action of G_0 to a group C_2^3 in this case, too. However this action is not of Mathieu type. In this case the Enriques surface coincides with the one in [9] and the coset $C_2^3 \setminus G_0$ contains a numerically trivial involution.

Further discussions on these topics will be pursued elsewhere.

6. Examples of Mathieu actions by large groups

In this section we treat more directly the sextic Enriques surfaces of Hutchinson-Göpel type (Section 5, ($\star$ 5)). We start with studying the singularities of sextic Enriques surfaces and then as an application we give explicit examples of Enriques surfaces of Hutchinson-Göpel type which is acted on by groups $C_2 \times \mathfrak{A}_4$ and $C_2 \times C_4$ of Mathieu type [12].

We recall the sextic equation of Enriques surface of diagonal type from Proposition 2,

(7)

$$F(x_0, x_1, x_2, x_3) =$$

$$(a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2)x_0x_1x_2x_3 + \left(\frac{b_0}{x_0^2} + \frac{b_1}{x_1^2} + \frac{b_2}{x_2^2} + \frac{b_3}{x_3^2}\right) x_0^2x_1^2x_2^2x_3^2,$$

where $\prod_i a_i \prod_i b_i \neq 0$. Let $\overline{S}$ be the singular surface defined by F. By Bertini's theorem every general element in this linear system is smooth outside the coordinate tetrahedron $\Delta = \{x_0x_1x_2x_3 = 0\}$, whereas along the intersection $\Delta \cap \overline{S}$ it always has singularities.

At each coordinate point, say at $P = (0 : 0 : 0 : 1)$, F is expanded to

$$(a_3x_3^3)x_0x_1x_2 + (\text{higher terms in } x_0, x_1, x_2)$$

as a polynomial in variables x_0, x_1, x_2. This shows that $\overline{S}$ has an ordinary triple point at P. It can be resolved by the normalization $\pi\colon S \to \overline{S}$ and $\pi^{-1}(P)$ consists of three points. These three points correspond to the three components $\overline{\{x_i = 0\}}(i = 0, 1, 2)$ of the resolution of the triple point $\{x_0x_1x_2 = 0\}$, so it may be natural to denote them by

$$\pi^{-1}(P) = \{(\overline{0} : 0 : 0 : 1), (0 : \overline{0} : 0 : 1), (0 : 0 : \overline{0} : 1)\}.$$

Along each edges of the tetrahedron Δ, say along $l = \{x_0 = x_1 = 0\}$, F is expanded to

$$x_2x_3((b_0x_2x_3)x_1^2 + (a_2x_2^2 + a_3x_3^2)x_1x_0 + (b_1x_2x_3)x_0^2) + (\text{higher terms in } x_0, x_1)$$

as a polynomial in variables x_0, x_1. Therefore $\overline{S}$ has the singularity of ordinary double lines at $(0:0:x_2:x_3)$ if $g(T) = (b_0x_2x_3)T^2 + (a_2x_2^2 + a_3x_3^2)T + (b_1x_2x_3)$ has only simple roots; if $g(T)$ has multiple roots, it becomes a pinch point (also called a Whitney umbrella singularity). We see that both of these singularities are resolved by the normalization π. The double cover $\tilde{l} := \pi_*^{-1}(l) \to l$ branches at the pinch points. Since the discriminant condition of $g(T)$, $\begin{vmatrix} 2b_0x_2x_3 & a_2x_2^2 + a_3x_3^2 \\ a_2x_2^2 + a_3x_3^2 & 2b_1x_2x_3 \end{vmatrix} = 0$, gives in general four pinch points, the curve $\tilde{l}$ is an elliptic curve. At each coordinate point, say at $(0:0:0:1)$, we see that $\tilde{l}$ contains exactly the two points $(\overline{0}:0:0:1)$ and $(0:\overline{0}:0:1)$. We denote by $\tilde{l}_{ij}$ the strict transform of the edge $l_{ij} = \{x_i = x_j = 0\}$.

As is proved in Section 5, the Enriques surface $\overline{S}$ of Hutchinson-Göpel type satisfies $\prod_i a_i = \prod_i b_i$ in (7). By a suitable scalar multiplication of coordinates, the equation of $\overline{S}$ is normalized into

$$(8)\quad (x_0^2 + x_1^2 + x_2^2 + x_3^2) + \left(\frac{b_0}{x_0^2} + \frac{b_1}{x_1^2} + \frac{b_2}{x_2^2} + \frac{b_3}{x_3^2}\right) x_0x_1x_2x_3, \quad \prod_{i=0}^{3} b_i = 1.$$

To make use of Cremona transformations, we work also with $B_i = \sqrt{b_i}$, $\prod_i B_i = 1$. With the previous notation we can give a full statement of Proposition 4.

Theorem 3. The Enriques surface of Hutchinson-Göpel type (8) has the automorphisms

$$\begin{aligned} s_1 &\colon (x_0 : x_1 : x_2 : x_3) \mapsto (-x_0 : -x_1 : x_2 : x_3), \\ s_2 &\colon (x_0 : x_1 : x_2 : x_3) \mapsto (-x_0 : x_1 : -x_2 : x_3), \\ \sigma &\colon (x_0 : x_1 : x_2 : x_3) \mapsto \left(\frac{B_0}{x_0} : \frac{B_1}{x_1} : \frac{B_2}{x_2} : \frac{B_3}{x_3}\right). \end{aligned}$$

The involutions s_1, s_2 generate the group $G_0 \simeq C_2^2$ and s_1, s_2, σ generate the group $G \simeq C_2^3$. Their types as to the fixed locus are as follows.

(1) Every non-identity element of G_0 has type (M2).
(2) Every element of the coset $G \setminus G_0$ has type (M0).

Proof. Let $\pi \colon S \to \overline{S}$ be the normalization. Then S is the smooth minimal model and the actions extend. It is easy to see that

$$\mathrm{Fix}(s_1) = \tilde{l}_{01} \cup \tilde{l}_{23} \cup \{(0:0:\overline{0}:1), (0:0:1:\overline{0}), (\overline{0}:1:0:0), (1:\overline{0}:0:0)\}$$

and similarly for $s_2, s_3 = s_1s_2$. This shows the assertion about fixed points of s_i. As for σ and σs_i, first we note that they exchange the three pairs of opposite edges of the tetrahedron Δ. Hence their fixed loci exist only in the complement of the coordinate hyperplanes, on which the whole group acts biregularly. In fact we see that, for example, the fixed points of σ consists of the four points of the form

$$(e_0\sqrt{B_0} : e_1\sqrt{B_1} : e_2\sqrt{B_2} : e_3\sqrt{B_3}),$$

where $e_i \in \{\pm 1\}$ satisfy $\prod e_i = -1$ and we fix once for all $\sqrt{B_i}$ for which $\prod \sqrt{B_i} = 1$. Thus the fixed points of Cremona transformations are of type (M0). $\square$

We remark that if the pair (C, H) (or (B, H_0)) admits a special automorphism, it induces a further automorphism of the Enriques surface $S = (\mathrm{Km}\, C)/\varepsilon_H$. Theorem 4 below is an example of this general idea. Recall that a finite group of semi-symplectic automorphisms[5] of an Enriques surface is called *of Mathieu type* if every element g of order 2 or 4 acts with $\chi_{\mathrm{top}}(\mathrm{Fix}(g)) = 4$ (see [12]). Our theorem provides two examples of large group actions of Mathieu type on Enriques surfaces.

Theorem 4. Let S be the Enriques surface of Hutchinson-Göpel type in Section 5.

(1) If B admits an automorphism ψ of order 3, then B has a Göpel subgroup H_0 preserved by ψ, and S acquires an action of Mathieu type by the group $C_2 \times \mathfrak{A}_4$.

(2) If B has an action by the dihedral group D_8 of order 8, then B has a Göpel subgroup H_0 preserved by D_8, and S acquires an action by the group $C_2^3 \rtimes C_2^2$. The restriction to a certain subgroup isomorphic to $C_2 \times C_4$ is of Mathieu type.

Proof. Consult, *e.g.*, [2] or [7, Section 8] for automorphisms of curves of genus two.

(1) We may assume that B is defined by

$$w^2 = (x^3 - \lambda^3)(x^3 - \lambda^{-3}), \lambda \in \mathbb{C}^*.$$

Then the curve B has the automorphism $\psi\colon (x, w) \mapsto (\zeta_3 x, w)$ where ζ_3 is the primitive 3rd root of unity. We label the six branch points as

$$\lambda_i = \zeta_3^{i-1}\lambda \ (i = 1, 2, 3), \quad \lambda_i = \zeta_3^{i-1}\lambda^{-1} \ (i = 4, 5, 6)$$

Then the automorphism ψ acts on the Göpel subgroup $H_0 \subset J(B)_{(2)}$ of Section 5 as follows.

$$p_1 - p_4 \mapsto p_2 - p_5 \mapsto p_3 - p_6 \mapsto p_1 - p_4.$$

The induced automorphism on S is denoted by the same letter ψ. More explicitly, by (5) and Proposition 2, we see that ψ permutes the

[5] An automorphism is semi-symplectic if it acts on the space $H^0(S, \mathcal{O}_S(2K_S))$ trivially.

coordinates w_1, w_2, w_3 and x_1, x_2, x_3. This symmetry has the effect on the equation (8) of $\overline{S}$ that $b_1 = b_2 = b_3 =: A^2$, hence we get the family

$$(9) \quad (x_0^2 + x_1^2 + x_2^2 + x_3^2) + \left(\frac{1}{A^6 x_0^2} + \frac{A^2}{x_1^2} + \frac{A^2}{x_2^2} + \frac{A^2}{x_3^2} \right) x_0 x_1 x_2 x_3 = 0.$$

The action of ψ is given by $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : x_3 : x_1 : x_2)$ and it extends the group $G = C_2^3$ to $C_2 \times \mathfrak{A}_4$. Theorem 3 shows that its unique 2-Sylow subgroup acts with Mathieu character, hence this is an example of a family with a group action of Mathieu type.

(2) We may assume that B is defined by

$$w^2 = x(x^2 - \lambda^2) \left(x^2 - \frac{1}{\lambda^2} \right), \ \lambda \in \mathbb{C}^*,$$

and has the automorphisms

$$\psi \colon (x, w) \mapsto (-x, \sqrt{-1} w), \quad \varphi \colon (x, w) \mapsto \left(\frac{1}{x}, \frac{w}{x^3} \right).$$

These ψ, φ generate the group isomorphic to D_8. Here we label the six branch points as

$$\lambda_1 = 0, \ \lambda_4 = \infty; \ \lambda_2 = \lambda, \ \lambda_5 = -\lambda; \ \lambda_3 = 1/\lambda, \ \lambda_6 = -1/\lambda.$$

Then the Göpel subgroup H_0 is preserved by ψ, φ. This extends the group G of automorphisms of Enriques surface S in Theorem 3 to $C_2^3 \rtimes C_2^2$ (the index 4 of the extention from G corresponds to the order of the reduced automorphism group $\langle \psi, \varphi \rangle / \psi^2$ of B).

Their action on the equation is as follows. As before, we use the notation of (5) and Proposition 2. First the action of φ on w_i is given by

$$w_1 \mapsto w_1^{-1}, \ w_2 \leftrightarrow w_3$$

and we have $(1 : w_2 w_3 : w_3 w_1 : w_1 w_2) \mapsto (w_1 : w_1 w_2 w_3 : w_2 : w_3)$. In view of Remark 5, this is the same as a Cremona transformation followed by a permutation. Next, the action of ψ on w_i is given by

$$w_1 \mapsto w_1, \ w_2 \mapsto w_2^{-1}, \ w_3 \mapsto w_3^{-1}$$

and we have $(x_0 : x_1 : x_2 : x_3) \mapsto (x_1 : x_0 : x_3 : x_2)$. This implies that our Enriques surface has the sextic equation

(10)

$$(x_0^2 + x_1^2 + x_2^2 + x_3^2) + \sqrt{-1} \left(\frac{A^2}{x_0^2} + \frac{A^2}{x_1^2} + \frac{1}{A^2 x_2^2} + \frac{1}{A^2 x_3^2} \right) x_0 x_1 x_2 x_3 = 0.$$

(See Remark 10.) Although S has a group of automorphisms of order 32, most of them does not satisfy the Mathieu condition. However we claim that there are some.

Claim. The action

$$g\colon (x_0 : x_1 : x_2 : x_3) \mapsto \left(\frac{A}{x_1} : \frac{A}{x_0} : \frac{1}{Ax_3} : -\frac{1}{Ax_2} \right)$$

is of Mathieu type of order 4.

Proof. We show by the computation of the fixed locus. The action of g on the edges of the tetrahedron Δ is as follows: it exchanges l_{01} and l_{23}, while it stabilizes the other four edges. For each intersecting pair of stable edges, we have an isolated fixed points. Hence there are four isolated fixed points. These are exactly the isolated points of $\mathrm{Fix}(s_1)$, in view of the relation $g^2 = s_1$. ☐

In this way, we find that the family (10) has the Mathieu type actions generated by g and

$$h\colon (x_0 : x_1 : x_2 : x_3) \mapsto \left(\frac{A}{x_1} : -\frac{A}{x_0} : \frac{1}{Ax_3} : \frac{1}{Ax_2} \right).$$

The relations $g^2 = h^2 = s_1, gh = hg$ show that they in fact generate the group $C_2 \times C_4$. ☐

Remark 10. The coefficient $\sqrt{-1}$ is a kind of subtlety of Enriques surfaces, which makes the equation (10) irreducible. Note that without this adjustment, we obtain the reducible equation

$$\begin{aligned}(x_0^2 + x_1^2 + x_2^2 + x_3^2) &+ \left(\frac{A^2}{x_0^2} + \frac{A^2}{x_1^2} + \frac{1}{A^2x_2^2} + \frac{1}{A^2x_3^2} \right) x_0x_1x_2x_3 \\ = \left(x_0^2 + x_1^2 + \frac{x_0x_1}{A^2x_2x_3}(x_2^2 + x_3^2) \right) &\left(1 + A^2\frac{x_2x_3}{x_0x_1} \right).\end{aligned}$$

The octahedral Enriques surface. A careful look at the equations (9) and (10) shows that they have a member $\overline{S}_{\mathrm{oct}}$ in common,

$$\overline{S}_{\mathrm{oct}}\colon (x_0^2 + x_1^2 + x_2^2 + x_3^2) + \sqrt{-1}\left(\frac{1}{x_0^2} + \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} \right) x_0x_1x_2x_3 = 0.$$

This surface is associated to the curve B (($\star$1), Section 5) which is ramified over the six vertices of the regular octahedron inscribed in $\mathbb{P}^1$. Hence we call the desingularization S_{oct} of this surface the *octahedral Enriques surface.*

The additional automorphisms are quite visible on $\overline{S}_{\mathrm{oct}}$; it is generated by the symmetric group $\mathfrak{S}_4$ acting on the coordinates $\{x_i\}$ and three involutions $\beta_j\colon (x_i) \mapsto (\epsilon_{ij}/x_i)$ $(j = 1, 2, 3)$ where ϵ_{ij} takes value

-1 if $i=j$ and 1 otherwise. Thus S_{oct} is acted on by the group $C_2^3\mathfrak{S}_4$ of order 192. For the convenience, we give here a table of topological structures of the fixed loci of these automorphisms, sorted by the conjugacy classes in $C_2^3\mathfrak{S}_4$.

Representative	length	order	fixed loci
id	1	1	S_{oct}
β_1	4	2	$\{$4pts.$\}$
$\beta_1\beta_2$	3	2	(two elliptic curves) + $\{$4pts.$\}$
$\beta_1(x_0x_1)(x_2x_3)$	12	4	$\{$4pts.$\}$
$\beta_1\beta_2(x_0x_1)(x_2x_3)$	6	2	(two rational curves) + $\{$4pts.$\}$
$(x_0x_1)(x_2x_3)$	6	2	$\{$4pts.$\}$
$\beta_2(x_0x_1)$	12	2	(a rational curve) + $\{$4pts.$\}$
(x_0x_1)	12	2	(a genus-two curve) + $\{$4pts.$\}$
$\beta_1(x_0x_1)$	12	4	$\{$2pts.$\}$
$\beta_1\beta_2(x_0x_1)$	12	4	$\{$2pts.$\}$
$\beta_1(x_1x_2x_3)$	32	6	$\{$1pt.$\}$
$(x_1x_2x_3)$	32	3	$\{$3pts.$\}$
$\beta_1(x_0x_1x_2x_3)$	24	4	$\{$4pts.$\}$
$(x_0x_1x_2x_3)$	24	4	$\{$2pts.$\}$

We remark that, as the specialization of the families (9) and (10), the group $C_2\times\mathfrak{A}_4$ is generated by $\beta_2\beta_3, \beta_3\beta_1, \beta_1\beta_2, (x_1x_2x_3)$ and the group $C_2\times C_4$ is generated by $\beta_1(x_0x_1)(x_2x_3), \beta_3(x_0x_1)(x_2x_3)$.

7. The characterization

In this section we prove a converse of Theorem 3 (resp. Proposition 1), stating that sextic Enriques surfaces of Hutchinson-Göpel type (resp. of diagonal type) are characterized by the group actions by G (resp. G_0).

To begin with, let us recall the study of involutions of Mathieu type. Every involution s on an Enriques surface S acts on the space $H^0(S,\mathcal{O}_S(2K_S))$ trivially. This means that at a fixed point P of s, the derivative of s satisfies $\det(ds)_P=\pm1$. The fixed point P is called *symplectic* (or *anti-symplectic*) according to the value $\det(ds)_P=+1$ (or -1). The set of symplectic (resp. anti-symplectic) fixed points is denoted by $\mathrm{Fix}^+(s)$ (resp. $\mathrm{Fix}^-(s)$). Geometrically, $\mathrm{Fix}^+(s)$ is exactly the set of isolated fixed points and $\mathrm{Fix}^-(s)$ is the set of fixed curves since s has order two. By topological and holomorphic Lefschetz formulas, we see always $\#\,\mathrm{Fix}^+(s)=4$ and the Mathieu condition is equivalent to $\chi_{\mathrm{top}}(\mathrm{Fix}^-(s))=0$. A more precise argument shows that there are only four types (M0)-(M3) mentioned in the introduction.

Lemma 2. Let s be an involution of (M2) type on an Enriques surface S; we denote the two elliptic curves of $\mathrm{Fix}^-(s)$ by E, F. Then there exists an elliptic fibration $S \to \mathbb{P}^1$ in which $2E$ and $2F$ are multiple fibers.

Proof. It is well-known that the linear system of some multiple of E gives an elliptic fibration $f \colon S \to \mathbb{P}^1$, [1, Chap. VIII]. Since s fixes fibers E, F and those which contain the four points of $\mathrm{Fix}^+(s)$, s acts on the base trivially. Thus s is induced from an automorphism s_0 of the Jacobian fibration $J(f)$. Since s does not have fixed horizontal curves, s_0 acts as a fiberwise translations. Hence E and F must be multiple fibers. □

Here we first give the characterization of Enriques surfaces of diagonal type.

Proposition 5. Let S be an Enriques surface with an action of Mathieu type by the group $G_0 := C_2^2$ such that every nontrivial element is of (M2) type. Then S is birationally equivalent to the sextic Enriques surface of diagonal type, Proposition 2, $(\star\star\star)$.

Proof. We let the group $G_0 = \{1, s_1, s_2, s_3\}$ and let E_i, F_i be the two elliptic curves in the fixed locus $\mathrm{Fix}(s_i)$ respectively for $i = 1, 2, 3$. Lemma 2 shows that the divisor class of $2E_i \sim 2F_i$ defines an elliptic fibration $f_i \colon S \to \mathbb{P}^1$. Moreover, since f_i has exactly two multiple fibers, for $j \neq i$ the curves E_j and F_j are horizontal in the fibration f_i. In particular the intersections

$$E_i \cap E_j, E_i \cap F_j, F_i \cap E_j, F_i \cap F_j \tag{11}$$

are all nonempty.

On the other hand, each of the four intersections of (11) defines an isolated fixed point of s_k because $s_i s_j = s_k$, where k is taken as the element in $\{1, 2, 3\} \setminus \{i, j\}$. These isolated fixed points belong to the set $\mathrm{Fix}^+(s_k)$, which consists of four points. Hence we see that the intersections of (11) all are transversal and consists of one point.

Next let us consider the linear system $\mathcal{L} := |E_1 + E_2 + E_3|$ with $\mathcal{L}^2 = 6$. This is a nef and big divisor, hence it maps S into $\mathbb{P}^3$. Note that the relation

$$E_1 + E_2 + E_3 \sim E_1 + F_2 + F_3 \sim F_1 + E_2 + F_3 \sim F_1 + F_2 + E_3 \tag{12}$$

shows that $\mathcal{L}$ is base-point-free. Then we can use [3, Remark 7.9] to see that at least either $\mathcal{L}$ or $\mathcal{L} + K_S$ gives a birational morphism onto a sextic surface. Noting that $\mathcal{L} + K_S$ is nothing but the system $|E_1 + E_2 + F_3|$, exchanging E_3 and F_3 if necessary, we can assume that

$\mathcal{L}$ gives a birational morphism φ onto a sextic surface $\overline{S} \subset \mathbb{P}^3$. As is known, $\overline{S}$ becomes a sextic surface with double lines along edges of a tetrahedron Δ. In our case the edges of Δ consist of the images of the six elliptic curves $E_1, \cdots, F_3$.

We denote by x_0, x_1, x_2, x_3 the respective global sections of $\mathcal{O}_S(E_1 + E_2 + E_3)$ corresponding to the divisors (12). In these coordinates Δ is nothing but the coordinate tetrahedron $\Delta = \{x_0x_1x_2x_3 = 0\}$. Thus our surface $\overline{S}$ belongs to the following linear system of sextics,

$$q(x_0, x_1, x_2, x_3) + x_0x_1x_2x_3(b_0/x_0^2 + b_1/x_1^2 + b_2/x_2^2 + b_3/x_3^2),$$

where q is a quadric and $b_0, \cdots, b_3 \in \mathbb{C}$ are constants.

The involution s_i induces a linear transformation of the ambient $\mathbb{P}^3$. More precisely, since s_i stabilizes each divisor in (12), $s_i(x_j)$ is just a scalar multiple of x_j for any i, j. By considering their fixed locus, we easily deduce that this action is given by changing the signs of two coordinates. Since $\overline{S}$ is invariant under this change of signs, we have $q(x, y, z, t) = a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2$. Therefore, S is birationally equivalent to a sextic Enriques surface of diagonal type. $\square$

Proof of Theorem 1. We identify the subgroup G_0 with the one in the previous proposition and keep the same notation. Recall from Section 5 that the sextic surface ($\star 5$) is an Enriques surface of Hutchinson-Göpel type exactly when $\prod_i a_i = \prod_i b_i$. This is the case when there exists an action of standard Cremona transformation (6). Let $\sigma \in G - G_0$. We claim that σ exchanges E_i and F_i for any i.

Suppose σ preserved E_1 and F_1. Then we would obtain an effective action of $\langle \sigma, s_2 \rangle \simeq C_2^2$ on both E_1 and F_1. Since s_2 has fixed points on them, it negates the periods. It follows that the elements σ and σs_2, both in the set $G \setminus G_0$, cannot act on E_1 freely, so that for example it would happen that σ has four fixed points on E_1 and σs_2 has four fixed points on F_1. But this is not possible, since on the $K3$ cover X, the symplectic lift $\tilde{\sigma}$ has eight fixed points inside the inverse image of E_1 which is an irreducible elliptic curve (since E_1 is a double fiber). Thus we have proved that σ exchanges E_1 and F_1. The same applies to E_i and F_i for $i = 2, 3$.

Thus σ sends $\sum E_i$ to $\sum F_i$. It follows that σ transforms the sextic model defined by $\mathcal{L}$ to a sextic model defined by $\mathcal{L} + K_S$. As was noticed in Remark 5 (or [11, Remark 4.2]), these two models are related via the Cremona transformation. Thus the Enriques surface is of Hutchinson-Göpel type.

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