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INTER-UNIVERSAL TEICHMÜLLER THEORY I: CONSTRUCTION OF HODGE THEATERS

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Abstract. The present paper is the first in a series of four papers, the goal of which is to establish an *arithmetic* version of *Teichmüller theory* for **number fields** equipped with an **elliptic curve** — which we refer to as **"inter-universal Teichmüller theory"** — by applying the theory of *semi-graphs of anabelioids*, *Frobenioids*, the *étale theta function*, and *log-shells* developed in earlier papers by the author. We begin by fixing what we call *"initial Θ-data"*, which consists of an *elliptic curve* E_F over a *number field* F, and a *prime number* $l \geq 5$, as well as some other technical data satisfying certain technical properties. This data determines various *hyperbolic orbicurves* that are related via finite étale coverings to the once-punctured elliptic curve X_F determined by E_F. These finite étale coverings admit various *symmetry properties* arising from the **additive** and **multiplicative** structures on the ring $\mathbb{F}_l = \mathbb{Z}/l\mathbb{Z}$ acting on the *l-torsion points* of the elliptic curve. We then construct *"$\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters"* associated to the given Θ-data. These $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters may be thought of as *miniature models of* **conventional scheme theory** in which the **two underlying combinatorial dimensions** of a number field — which may be thought of as corresponding to the **additive** and **multiplicative** structures of a ring or, alternatively, to the **group of units** and **value group** of a local field associated to the number field — are, in some sense, **"dismantled"** or **"disentangled"** from one another. All $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters are isomorphic to one another, but may also be related to one another by means of a **"Θ-link"**, which relates certain *Frobenioid-theoretic* portions of one $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater to another is a fashion that is **not compatible** *with the respective* **conventional ring/scheme theory structures**. In particular, it is a *highly nontrivial problem to relate the ring structures* on either side of the Θ-link to one another. This will be achieved, up to certain *"relatively mild indeterminacies"*, in future papers in the series by applying the **absolute anabelian geometry** developed in earlier papers by the author. The resulting *description of an* **"alien ring structure"** [associated, say, to the *domain* of the Θ-link] in terms of a given ring structure [associated, say, to the *codomain* of the Θ-link] will be applied in the final paper of the series to obtain results in *diophantine geometry*. Finally, we discuss certain technical results concerning **profinite conjugates** *of decomposition and inertia groups in the* **tempered fundamental group** of a p-adic hyperbolic curve that will be of use in the development of the theory of the present series of papers, but are also of independent interest.

Contents:

Introduction

§I1. Summary of Main Results

The present paper is the first in a series of four papers, the goal of which is to establish an **arithmetic** version of **Teichmüller theory** for **number fields** equipped with an **elliptic curve**, by applying the theory of *semi-graphs of anabelioids*, *Frobenioids*, the *étale theta function*, and *log-shells* developed in [SemiAnbd], [FrdI], [FrdII], [EtTh], and [AbsTopIII]. Unlike many mathematical papers, which are devoted to verifying properties of mathematical objects that are either well-known or easily constructed from well-known mathematical objects, in the present series of papers, most of our efforts will be devoted to **constructing** *new mathematical objects*. It is only in the final portion of the third paper in the series, i.e., [IUTchIII], that we turn to the task of *proving properties of interest* concerning the mathematical objects constructed. In the fourth paper of the series, i.e., [IUTchIV], we show that these properties may be combined with certain elementary computations to obtain *diophantine results* concerning elliptic curves over number fields.

The starting point of our constructions is a collection of **initial Θ-data** [cf. Definition 3.1]. Roughly speaking, this data consists, essentially, of

- an **elliptic curve** E_F over a **number field** F,
- an **algebraic closure** $\overline{F}$ of F,
- a **prime number** $l \geq 5$, and
- a *collection of* **valuations** $\underline{\mathbb{V}}$ of a certain subfield $K \subseteq \overline{F}$

that satisfy certain technical conditions — we refer to Definition 3.1 for more details. Here, we write $F_{\mathrm{mod}} \subseteq F$ for the *field of moduli* of E_F, $K \subseteq \overline{F}$ for the extension field of F determined by the *l-torsion points* of E_F, $X_F \subseteq E_F$ for the *once-punctured elliptic curve* obtained by removing the origin from E_F, and $X_F \rightarrow C_F$ for the *hyperbolic orbicurve* obtained by forming the stack-theoretic quotient of X_F by the natural action of $\{\pm 1\}$. Then F is assumed to be *Galois* over F_{mod}, $\mathrm{Gal}(K/F)$ is assumed to be isomorphic to $GL_2(\mathbb{F}_l)$, E_F is assumed to have *stable reduction* at all of the nonarchimedean valuations of F, $C_K \stackrel{\text{def}}{=} C_F \times_F K$ is assumed to be a K-*core* [cf. [CanLift], Remark 2.1.1], and $\underline{\mathbb{V}}$ is assumed to be a collection of valuations of K such that the natural inclusion $F_{\text{mod}} \subseteq F \subseteq K$ induces a *bijection* $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\text{mod}}$ between $\underline{\mathbb{V}}$ and the set $\mathbb{V}_{\text{mod}}$ of all valuations of the number field F_{mod}. We shall write

$$\mathbb{V}^{\text{bad}}_{\text{mod}} \subseteq \mathbb{V}_{\text{mod}}, \quad \underline{\mathbb{V}}^{\text{bad}} \subseteq \underline{\mathbb{V}}$$

for the set of nonarchimedean valuations of odd residue characteristic over which E_F has *bad [i.e., multiplicative] reduction*; $\mathbb{V}^{\text{good}}_{\text{mod}} \stackrel{\text{def}}{=} \mathbb{V}_{\text{mod}} \setminus \mathbb{V}^{\text{bad}}_{\text{mod}}$, $\underline{\mathbb{V}}^{\text{good}} \stackrel{\text{def}}{=} \underline{\mathbb{V}} \setminus \underline{\mathbb{V}}^{\text{bad}}$. Here, $\mathbb{V}^{\text{bad}}_{\text{mod}}$ is assumed to be *nonempty*. Also, we shall apply the superscripts "non" and "arc" to $\underline{\mathbb{V}}$, $\mathbb{V}_{\text{mod}}$ to denote the subsets of *nonarchimedean* and *archimedean* valuations, respectively.

This data determines a **finite étale covering** $\underline{C}_K \to C_K$ of degree l such that the base-changed covering

$$\underline{X}_K \stackrel{\text{def}}{=} \underline{C}_K \times_{C_F} X_F \quad \to \quad X_K \stackrel{\text{def}}{=} X_F \times_F K$$

arises from a *rank one quotient* $E_K[l] \twoheadrightarrow Q$ ($\cong \mathbb{Z}/l\mathbb{Z}$) of the module $E_K[l]$ of l-torsion points of $E_K(K)$ which, at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, restricts to the quotient arising from *coverings of the dual graph of the special fiber*. Moreover, the above data also determines a **cusp**

$$\underline{\epsilon}$$

of $\underline{C}_K$ which, at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, corresponds to the *canonical generator*, up to ± 1, of Q [i.e., the generator determined by the unique *loop* of the dual graph of the special fiber]. Furthermore, at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, one obtains a natural finite étale covering of degree l

$$\underline{\underline{X}}_{\underline{v}} \quad \to \quad \underline{X}_{\underline{v}} \stackrel{\text{def}}{=} \underline{X}_K \times_K K_{\underline{v}} \quad (\to \quad \underline{C}_{\underline{v}} \stackrel{\text{def}}{=} \underline{C}_K \times_K K_{\underline{v}})$$

by extracting l-th roots of the theta function; at $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$, one obtains a natural finite étale covering of degree l

$$\underline{X}_{\to_{\underline{v}}} \quad \to \quad \underline{X}_{\underline{v}} \stackrel{\text{def}}{=} \underline{X}_K \times_K K_{\underline{v}} \quad (\to \quad \underline{C}_{\underline{v}} \stackrel{\text{def}}{=} \underline{C}_K \times_K K_{\underline{v}})$$

determined by $\underline{\epsilon}$. More details on the structure of the coverings $\underline{C}_K$, $\underline{X}_K$, $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$], $\underline{X}_{\to_{\underline{v}}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$] may be found in [EtTh], §2, as well as in §1 of the present paper.

In this situation, the objects

$$l^{*} \stackrel{\text{def}}{=} (l-1)/2; \quad l^{\pm} \stackrel{\text{def}}{=} (l+1)/2; \quad \mathbb{F}_l^{*} \stackrel{\text{def}}{=} \mathbb{F}_l^{\times}/\{\pm 1\}; \quad \mathbb{F}_l^{\rtimes \pm} \stackrel{\text{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\}$$

[cf. the discussion at the beginning of §4; Definitions 6.1, 6.4] will play an important role in the discussion to follow. The natural action of the stabilizer in $\text{Gal}(K/F)$ of the quotient $E_K[l] \twoheadrightarrow Q$ on Q determines a *natural poly-action* of $\mathbb{F}_l^{*}$ on $\underline{C}_K$, i.e., a natural isomorphism of $\mathbb{F}_l^{*}$ with some *subquotient* of $\text{Aut}(\underline{C}_K)$ [cf. Example 4.3, (iv)]. The $\mathbb{F}_l^{*}$**-symmetry** constituted by this poly-action of $\mathbb{F}_l^{*}$ may be thought of as being essentially **arithmetic** in nature, in the sense that the subquotient of $\mathrm{Aut}(\underline{C}_K)$ that gives rise to this poly-action of $\mathbb{F}_l^{\circledast}$ is induced, via the natural map $\mathrm{Aut}(\underline{C}_K) \to \mathrm{Aut}(K)$, by a subquotient of $\mathrm{Gal}(K/F) \subseteq \mathrm{Aut}(K)$. In a similar vein, the natural action of the automorphisms of the scheme $\underline{X}_K$ on the *cusps* of $\underline{X}_K$ determines a *natural poly-action* of $\mathbb{F}_l^{\rtimes\pm}$ on $\underline{X}_K$, i.e., a natural isomorphism of $\mathbb{F}_l^{\rtimes\pm}$ with some *subquotient* of $\mathrm{Aut}(\underline{X}_K)$ [cf. Definition 6.1, (v)]. The $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** constituted by this poly-action of $\mathbb{F}_l^{\rtimes\pm}$ may be thought of as being essentially **geometric** in nature, in the sense that the subgroup $\mathrm{Aut}_K(\underline{X}_K) \subseteq \mathrm{Aut}(\underline{X}_K)$ [i.e., of K-linear automorphisms] maps *isomorphically* onto the subquotient of $\mathrm{Aut}(\underline{X}_K)$ that gives rise to this poly-action of $\mathbb{F}_l^{\rtimes\pm}$. On the other hand, the **global** $\mathbb{F}_l^{\circledast}$-*symmetry* of $\underline{C}_K$ only extends to a "$\{1\}$-symmetry" [i.e., in essence, fails to extend!] of the *local* coverings $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] and $\underrightarrow{X}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$], while the **global** $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* of $\underline{X}_K$ only extends to a "$\{\pm 1\}$-symmetry" [i.e., in essence, fails to extend!] of the *local* coverings $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] and $\underrightarrow{X}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$] — cf. Fig. I1.1 below.

$$\begin{array}{ccc} & \overset{\{\pm 1\}}{\curvearrowright}\quad \{\underline{\underline{X}}_{\underline{v}} \text{ or } \underrightarrow{X}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} & \\ / & & \backslash \\ \overset{\mathbb{F}_l^{\rtimes\pm}}{\curvearrowright}\quad \underline{X}_K & & \underline{C}_K \quad \overset{\mathbb{F}_l^{\circledast}}{\curvearrowleft} \end{array}$$

Fig. I1.1: Symmetries of coverings of X_F

We shall write $\Pi_{\underline{v}}$ for the *tempered fundamental group* of $\underline{\underline{X}}_{\underline{v}}$, when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [cf. Definition 3.1, (e)]; we shall write $\Pi_{\underline{v}}$ for the *étale fundamental group* of $\underrightarrow{X}_{\underline{v}}$, when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ [cf. Definition 3.1, (f)]. Also, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, we shall write $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$ for the quotient determined by the *absolute Galois group* of the base field $K_{\underline{v}}$. Often, in the present series of papers, we shall consider various types of collections of data — which we shall refer to as **"prime-strips"** — indexed by $\underline{v} \in \underline{\mathbb{V}}$ ($\xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$) that are isomorphic to certain data that arise naturally from $\underline{\underline{X}}_{\underline{v}}$ [when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] or $\underrightarrow{X}_{\underline{v}}$ [when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$]. The main types of prime-strips that will be considered in the present series of papers are summarized in Fig. I1.2 below.

Perhaps the most basic kind of prime-strip is a $\mathcal{D}$**-prime-strip**. When $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the portion of a $\mathcal{D}$-prime-strip labeled by $\underline{v}$ is given by a category equivalent to [the full subcategory determined by the connected objects of] the category of *tempered coverings* of $\underline{\underline{X}}_{\underline{v}}$ [when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$] or *finite étale coverings* of $\underrightarrow{X}_{\underline{v}}$ [when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$]. When $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, an analogous definition may be obtained by applying the theory of Aut-*holomorphic orbispaces* developed in [AbsTopIII], §2. One variant of the notion of a $\mathcal{D}$-prime-strip is the notion of a $\mathcal{D}^{\vdash}$**-prime-strip**. When $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, the portion of a $\mathcal{D}^{\vdash}$-prime-strip labeled by $\underline{v}$ is given by a category equivalent to [the full subcategory determined by the connected objects of] the Galois category associated to $G_{\underline{v}}$; when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, an analogous definition may be given. In some sense, $\mathcal{D}$-prime-strips may be thought of as abstractions of the **"local arithmetic holomorphic structure"** of [copies of] F_{mod} — cf. the discussion of

[AbsTopIII], §I3. On the other hand, $\mathcal{D}^{\vdash}$-prime-strips may be thought of as **"mono-analyticizations"** [i.e., roughly speaking, the arithmetic version of the underlying real analytic structure associated to a holomorphic structure] of $\mathcal{D}$-prime-strips — cf. the discussion of [AbsTopIII], §I3. Throughout the present series of papers, we shall use the notation

$$\vdash$$

to denote *mono-analytic* structures.

<u>*Type of prime-strip*</u>	<u>*Model at* $v \in \mathbb{V}^{\mathrm{bad}}$</u>	<u>*Reference*</u>
$\mathcal{D}$	$\Pi_{\underline{v}}$	I, 4.1, (i)
$\mathcal{D}^{\vdash}$	$G_{\underline{v}}$	I, 4.1, (iii)
$\mathcal{F}$	$\Pi_{\underline{v}} \quad \curvearrowright \quad \mathcal{O}^{\rhd}_{F_{\underline{v}}}$	I, 5.2, (i)
$\mathcal{F}^{\vdash}$	$G_{\underline{v}} \quad \curvearrowright \quad \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \times \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$	I, 5.2, (ii)
$\mathcal{F}^{\vdash\times}$	$G_{\underline{v}} \quad \curvearrowright \quad \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}$	II, 4.9, (vii)
$\mathcal{F}^{\vdash\times\boldsymbol{\mu}}$	$G_{\underline{v}} \quad \curvearrowright \quad \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}/\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$	II, 4.9, (vii)
$\mathcal{F}^{\vdash\blacktriangleright\times\boldsymbol{\mu}}$	$G_{\underline{v}} \quad \curvearrowright \quad \mathcal{O}^{\times\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \times \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$	II, 4.9, (vii)
$\mathcal{F}^{\vdash\blacktriangleright}$	$G_{\underline{v}} \quad \curvearrowright \quad \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$	III, 2.4, (ii)
$\mathcal{F}^{\vdash\perp}$	$G_{\underline{v}} \quad \curvearrowright \quad \mathcal{O}^{\boldsymbol{\mu}_{2l}}_{\overline{F}_{\underline{v}}} \times \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$	III, 2.4, (ii)

$$\mathcal{F}^{\Vdash\cdots} \quad = \quad \mathcal{F}^{\vdash\cdots} \quad + \quad \left\{\text{global realified Frobenioid associated to } F_{\mathrm{mod}}\right\}$$

Fig. I1.2: Types of prime-strips

Next, we recall the notion of a *Frobenioid* over a *base category* [cf. [FrdI] for more details]. Roughly speaking, a **Frobenioid** [typically denoted "$\mathcal{F}$"] may be thought of as a category-theoretic abstraction of the notion of a category of line bundles or monoids of divisors over a **base category** [typically denoted "$\mathcal{D}$"] of topological localizations such as a *Galois category*. In addition to $\mathcal{D}$- and $\mathcal{D}^{\vdash}$-prime-strips, we shall also consider various types of prime-strips that arise from considering various natural Frobenioids — i.e., more concretely, various natural *monoids equipped with a Galois action* — at $\underline{v} \in \underline{\mathbb{V}}$. Perhaps the most basic type of prime-strip arising from such a natural monoid is an $\mathcal{F}$**-prime-strip**. Suppose, for simplicity, that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Then $\underline{v}$ and $\overline{F}$ determine, up to conjugacy, an *algebraic closure* $\overline{F}_{\underline{v}}$ of $K_{\underline{v}}$. Write

- $\mathcal{O}_{\overline{F}_{\underline{v}}}$ for the ring of integers of $\overline{F}_{\underline{v}}$;
- $\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}_{\overline{F}_{\underline{v}}}$ for the multiplicative monoid of nonzero integers;
- $\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}_{\overline{F}_{\underline{v}}}$ for the multiplicative monoid of units;
- $\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}_{\overline{F}_{\underline{v}}}$ for the multiplicative monoid of roots of unity;
- $\mathcal{O}^{\boldsymbol{\mu}_{2l}}_{\overline{F}_{\underline{v}}} \subseteq \mathcal{O}_{\overline{F}_{\underline{v}}}$ for the multiplicative monoid of $2l$-th roots of unity;
- $\underline{\underline{q}}_{\underline{v}} \in \mathcal{O}_{\overline{F}_{\underline{v}}}$ for a $2l$-th root of the q-parameter of E_F at $\underline{v}$.

Thus, $\mathcal{O}_{\overline{F}_{\underline{v}}}$, $\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$, $\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}}$, $\mathcal{O}^{\boldsymbol{\mu}}_{\overline{F}_{\underline{v}}}$, and $\mathcal{O}^{\boldsymbol{\mu}_{2l}}_{\overline{F}_{\underline{v}}}$ are equipped with *natural $G_{\underline{v}}$-actions.* The portion of a $\mathcal{F}$-prime-strip labeled by $\underline{v}$ is given by data isomorphic to the monoid $\mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}$, equipped with its natural $\Pi_{\underline{v}}$ ($\twoheadrightarrow G_{\underline{v}}$)-action [cf. Fig. I1.2]. There are various *mono-analytic* versions of the notion an $\mathcal{F}$-prime-strip; perhaps the most basic is the notion of an $\mathcal{F}^{\vdash}$**-prime-strip**. The portion of a $\mathcal{F}^{\vdash}$-prime-strip labeled by $\underline{v}$ is given by data isomorphic to the monoid $\mathcal{O}^{\times}_{\overline{F}_{\underline{v}}} \times \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$, equipped with its natural $G_{\underline{v}}$-action [cf. Fig. I1.2]. Often we shall regard these various mono-analytic versions of an $\mathcal{F}$-prime-strip as being equipped with an additional **global realified Frobenioid**, which, at a concrete level, corresponds, essentially, to considering various *arithmetic degrees* $\in \mathbb{R}$ at $\underline{v} \in \underline{\mathbb{V}}$ ($\xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$) that are related to one another by means of the *product formula.* Throughout the present series of papers, we shall use the notation

$$\Vdash$$

to denote such prime-strips.

In some sense, the main goal of the present paper may be thought of as the *construction of* $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** [cf. Definition 6.13, (i)]

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

— which may be thought of as *"miniature models of* **conventional scheme theory**" — given, roughly speaking, by **systems of Frobenioids**. To any such $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, one may associate a $\mathcal{D}$**-**$\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** [cf. Definition 6.13, (ii)]

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$$

— i.e., the associated **system of base categories**.

One may think of a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ as the result of **gluing** *together a* $\Theta^{\pm\mathrm{ell}}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$ *to a* ΘNF*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$ [cf. Remark 6.12.2, (ii)]. In a similar vein, one may think of a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ as the result of gluing together a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ to a $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$. A $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ may be thought of as a **bookkeeping device** that allows one to keep track of the action of the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** on the **labels**

$$(-l^* \ < \ \dots \ < \ -1 \ < 0 \ < \ 1 \ < \ \dots \ < \ l^*)$$

— which we think of as elements $\in \mathbb{F}_l$ — in the context of the [orbi]curves $\underline{X}_K$, $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], and $\underline{X}_{\to_{\underline{v}}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$]. The $\mathbb{F}_l^{\rtimes\pm}$-symmetry is represented in a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ by a category equivalent to [the full subcategory determined by the connected objects of] the Galois category of finite étale coverings of $\underline{X}_K$. On the other hand, each of the *labels* referred to above is represented in a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ by a $\mathcal{D}$**-prime-strip**. In a similar vein, a $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ may be thought of as a bookkeeping device that allows one to keep track of the action of the $\mathbb{F}_l^{*}$**-symmetry** on the **labels**

$$(1 \ < \ \dots \ < \ l^*)$$

— which we think of as elements $\in \mathbb{F}_l^{*}$ — in the context of the orbicurves $\underline{C}_K$, $\underline{\underline{X}}_{\underline{v}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], and $\underline{X}_{\to_{\underline{v}}}$ [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$]. The $\mathbb{F}_l^{*}$-symmetry is represented in a $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ by a category equivalent to [the full subcategory determined by the connected objects of] the Galois category of finite étale coverings of $\underline{C}_K$. On the other hand, each of the *labels* referred to above is represented in a $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ by a $\mathcal{D}$**-prime-strip**. The *combinatorial structure* of $\mathcal{D}$-ΘNF- and $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters summarized above [cf. also Fig. I1.3 below] is one of the *main topics* of the present paper and is discussed in detail in §4 and §6. The left-hand portion of Fig. I1.3 corresponds to the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater; the right-hand portion of Fig. I1.3 corresponds to the $\mathcal{D}$-ΘNF-Hodge theater; these left-hand and right-hand portions are glued together along a *single* $\mathcal{D}$*-prime-strip*, depicted as "$[1 \ < \ \dots \ < \ l^*]$", in such a way that the labels $0 \neq \pm t \in \mathbb{F}_l$ on the left are identified with the corresponding label $j \in \mathbb{F}_l^{*}$ on the right.

The $\mathbb{F}_l^{\rtimes\pm}$*-symmetry* has the advantange that, being *geometric* in nature, it allows one to *permute* various copies of "$G_{\underline{v}}$" [where $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] associated to distinct *labels* $\in \mathbb{F}_l$ without inducing *conjugacy indeterminacies*. This phenomenon, which we shall refer to as **conjugate synchronization**, will play a key role in the **Kummer theory** surrounding the *Hodge-Arakelov-theoretic evaluation of the* **theta function** *at* $\boldsymbol{l}$**-torsion points** that is developed in [IUTchII]— cf. the discussion of Remark 6.12.6; [IUTchII], Remark 3.5.2, (ii), (iii); 4.5.3, (i). By contrast, the $\mathbb{F}_l^{*}$*-symmetry* is more suited to situations in which one must *descend* from K to F_{mod}. In the present series of papers, the most important such situation involves the **Kummer theory** surrounding the **reconstruction** of the **number field** F_{mod} from the étale fundamental group of $\underline{C}_K$ — cf. the discussion of Remark 6.12.6;

[IUTchII], Remark 4.7.6. This reconstruction will be discussed in Example 5.1 of the present paper. Here, we note that such situations necessarily induce *global Galois permutations* of the various copies of "$G_{\underline{v}}$" [where $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] associated to distinct *labels* $\in \mathbb{F}_l^{\circledast}$ that are *only well-defined up to conjugacy indeterminacies*. In particular, the $\mathbb{F}_l^{\circledast}$-symmetry is ill-suited to situations, such as those that appear in the theory of *Hodge-Arakelov-theoretic evaluation* that is developed in [IUTchII], that require one to establish *conjugate synchronization*.

$$
\begin{array}{ccccc}
\overset{\{\pm 1\}}{\curvearrowright} \begin{pmatrix} -l^{\circledast} < \ldots < -1 < 0 \\ < 1 < \ldots < l^{\circledast} \end{pmatrix} & \Rightarrow & \begin{bmatrix} 1 < \ldots \\ < l^{\circledast} \end{bmatrix} & \Leftarrow & \begin{pmatrix} 1 < \ldots \\ < l^{\circledast} \end{pmatrix} \\
\Downarrow & & & & \Downarrow \\
\begin{array}{ccc} \pm & \rightarrow & \pm \\ \uparrow & \overset{\mathbb{F}_l^{\rtimes\pm}}{\curvearrowright} & \downarrow \\ \pm & \leftarrow & \pm \end{array} & & & & \begin{array}{ccc} * & \rightarrow & * \\ \uparrow & \overset{\mathbb{F}_l^{\circledast}}{\curvearrowright} & \downarrow \\ * & \leftarrow & * \end{array}
\end{array}
$$

Fig. I1.3: The combinatorial structure of a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater

Ultimately, when, in [IUTchIV], we consider *diophantine applications* of the theory developed in the present series of papers, we will take the prime number l to be *"large"*, i.e., roughly of the order of the *height* of the elliptic curve E_F. When l is regarded as large, the arithmetic of the finite field $\mathbb{F}_l$ "tends to approximate" the arithmetic of the ring of rational integers $\mathbb{Z}$. That is to say, the decomposition that occurs in a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater into the *"additive"* [i.e., $\mathbb{F}_l^{\rtimes\pm}$-] and *"multiplicative"* [i.e., $\mathbb{F}_l^{\circledast}$-] symmetries of the *ring* $\mathbb{F}_l$ may be regarded as a sort of rough, approximate approach to the issue of **"disentangling"** the **multiplicative** and **additive** structures, i.e., "dismantling" the **"two underlying combinatorial dimensions"** [cf. the discussion of [AbsTopIII], §I3], of the **ring** $\mathbb{Z}$ — cf. the discussion of Remarks 6.12.3, 6.12.6.

Alternatively, this decomposition into additive and multiplicative symmetries in the theory of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters may be compared to groups of **additive** and **multiplicative symmetries** of the **upper half-plane** [cf. Fig. I1.4 below]. Here, the **"cuspidal"** geometry expressed by the additive symmetries of the upper half-plane admits a natural "associated coordinate", namely, the classical ***q*-parameter**, which is reminiscent of the way in which the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is well-adapted to the **Kummer theory** surrounding the *Hodge-Arakelov-theoretic evaluation of the* **theta function** *at* ***l*-torsion points** [cf. the above discussion]. By contrast, the **"toral"**, or **"nodal"** [cf. the classical theory of the structure of *Hecke correspondences* modulo p], geometry expressed by the multiplicative symmetries of the upper half-plane admits a natural "associated coordinate", namely, the classical biholomorphic isomorphism of the upper half-plane with the **unit disc**, which is reminiscent of the way in which the $\mathbb{F}_l^{\circledast}$-symmetry is well-adapted to the

Kummer theory surrounding the **number field** F_{mod} [cf. the above discussion]. For more details, we refer to the discussion of Remark 6.12.3, (iii).

	<u>*Classical upper half-plane*</u>	<u>$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters in inter-universal Teichmüller theory*</u>
Additive symmetry	$z \mapsto z + a$, $z \mapsto -\overline{z} + a \quad (a \in \mathbb{R})$	$\mathbb{F}_l^{\rtimes\pm}$-**symmetry**
"Functions" assoc'd to *add. symm.*	$q \stackrel{\mathrm{def}}{=} e^{2\pi i z}$	**theta fn.** evaluated at ***l*-tors.** [cf. I, 6.12.6, (ii)]
Basepoint assoc'd to *add. symm.*	*single* cusp at infinity	$\underline{\mathbb{V}}^{\pm}$ [cf. I, 6.1, (v)]
Combinatorial prototype assoc'd to *add. symm.*	**cusp**	**cusp**
Multiplicative symmetry	$z \mapsto \frac{z\cdot\cos(t)-\sin(t)}{z\cdot\sin(t)+\cos(t)}$, $z \mapsto \frac{\overline{z}\cdot\cos(t)+\sin(t)}{\overline{z}\cdot\sin(t)-\cos(t)} \quad (t \in \mathbb{R})$	$\mathbb{F}_l^{\circledast}$-**symmetry**
"Functions" assoc'd to *mult. symm.*	$w \stackrel{\mathrm{def}}{=} \frac{z-i}{z+i}$	elements of the **number field** F_{mod} [cf. I, 6.12.6, (iii)]
Basepoints assoc'd to *mult. symm.*	$\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$, $\begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix}$ $\curvearrowright$ $\{$*entire* boundary of $\mathfrak{H}$ $\}$	$\mathbb{F}_l^{\circledast} \curvearrowright \underline{\mathbb{V}}^{\mathrm{Bor}} = \mathbb{F}_l^{\circledast} \cdot \underline{\mathbb{V}}^{\pm\mathrm{un}}$ [cf. I, 4.3, (i)]
Combinatorial prototype assoc'd to *mult. symm.*	**nodes** of mod p Hecke correspondence [cf. II, 4.11.4, (iii), (c)]	**nodes** of mod p Hecke correspondence [cf. II, 4.11.4, (iii), (c)]

Fig. I1.4: Comparison of $\mathbb{F}_l^{\rtimes\pm}$-, $\mathbb{F}_l^{\circledast}$-symmetries with the geometry of the upper half-plane

From the point of view of the *scheme-theoretic* Hodge-Arakelov theory developed in [HASurI], [HASurII], the theory of the *combinatorial structure* of a $\Theta^{\pm\text{ell}}$NF-Hodge theater — and, indeed, the theory of the present series of papers! — may be regarded as a sort of

> **solution** to the problem of constructing **"global multiplicative subspaces"** and **"global canonical generators"** [cf. the *quotient* "Q" and the *cusp* "$\underline{\epsilon}$" that appear in the above discussion!]

— the *nonexistence* of which in a "naive, scheme-theoretic sense" constitutes the *main obstruction* to applying the theory of [HASurI], [HASurII] to diophantine geometry [cf. the discussion of Remark 4.3.1]. Indeed, **prime-strips** may be thought of as **"local analytic sections"** of the natural morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(F_{\mathrm{mod}})$. Thus, it is precisely by working with such "local analytic sections" — i.e., more concretely, by working with the collection of valuations $\underline{\mathbb{V}}$, as opposed to the set of *all* valuations of K — that one can, in some sense, *"simulate"* the notions of a *"global multiplicative subspace"* or a *"global canonical generator"*. On the other hand, such "simulated global objects" may only be achieved at the cost of

> **"dismantling"**, or performing **"surgery"** on, the **global prime structure** *of the number fields* involved [cf. the discussion of Remark 4.3.1]

— a quite *drastic* operation, which has the effect of precipitating *numerous technical difficulties*, whose resolution, via the theory of *semi-graphs of anabelioids*, *Frobenioids*, the *étale theta function*, and *log-shells* developed in [SemiAnbd], [FrdI], [FrdII], [EtTh], and [AbsTopIII], constitutes the bulk of the theory of the present series of papers! From the point of view of "performing surgery on the global prime structure of a number field", the **labels** $\in \mathbb{F}_l^{\circledast}$ that appear in the **"arithmetic"** $\mathbb{F}_l^{\circledast}$**-symmetry** may be thought of as a sort of **"miniature finite approximation"** *of this global prime structure*, in the spirit of the idea of *"Hodge theory at finite resolution"* discussed in [HASurI], §1.3.4. On the other hand, the **labels** $\in \mathbb{F}_l$ that in appear in the **"geometric"** $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** may be thought of as a sort of **"miniature finite approximation"** *of the natural tempered $\mathbb{Z}$-coverings* [i.e., tempered coverings with Galois group $\mathbb{Z}$] of the Tate curves determined by E_F at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, again in the spirit of the idea of "Hodge theory at finite resolution" discussed in [HASurI], §1.3.4.

As discussed above in our explanation of the models at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ for $\mathcal{F}^{\vdash}$-prime-strips, by considering the $2l$-th roots of the ***q*-parameters** of the elliptic curve E_F at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, and, roughly speaking, extending to $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ in such a way as to satisfy the *product formula*, one may construct a natural $\mathcal{F}^{\Vdash}$**-prime-strip** "$\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$" [cf. Example 3.5, (ii); Definition 5.2, (iv)]. This construction admits an *abstract, algorithmic formulation* that allows one to apply it to the underlying *"Θ-Hodge theater"* of an *arbitrary* $\Theta^{\pm\text{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\text{ell}}\text{NF}}$ so as to obtain an $\mathcal{F}^{\Vdash}$-prime-strip

$$
{}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}
$$

[cf. Definitions 3.6, (c); 5.2, (iv)]. On the other hand, by *formally replacing* the $2l$-th roots of the q-parameters that appear in this construction by the *reciprocal* *of the l-th root of the Frobenioid-theoretic* **theta function**, which we shall denote "$\underline{\underline{\Theta}}_{\underline{v}}$" [for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], studied in [EtTh] [cf. also Example 3.2, (ii), of the present paper], one obtains an *abstract, algorithmic formulation* for the construction of an $\mathcal{F}^{\Vdash}$-prime-strip

$$^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$$

[cf. Definitions 3.6, (c); 5.2, (iv)] from [the underlying Θ-Hodge theater of] the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.

Now let $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ be *another* $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater [relative to the given initial Θ-data]. Then we shall refer to the "full poly-isomorphism" of [i.e., the collection of all isomorphisms between] $\mathcal{F}^{\Vdash}$*-prime-strips*

$$^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \quad \overset{\sim}{\to} \quad {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$$

as the **Θ-link** from [the underlying Θ-Hodge theater of] $^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ to [the underlying Θ-Hodge theater of] $^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ [cf. Corollary 3.7, (i); Definition 5.2, (iv)]. One fundamental property of the Θ-link is the property that it induces a collection of isomorphisms [in fact, the full poly-isomorphism] between the $\mathcal{F}^{\vdash\times}$*-prime-strips*

$$^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}} \quad \overset{\sim}{\to} \quad {}^{\ddagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}}$$

associated to $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ and $^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ [cf. Corollary 3.7, (ii), (iii); [IUTchII], Definition 4.9, (vii)].

Now let $\{^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n\in\mathbb{Z}}$ be a *collection of distinct* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theaters* [relative to the given initial Θ-data] indexed by the integers. Thus, by applying the constructions just discussed, we obtain an **infinite chain**

$$\dots \quad \xrightarrow{\Theta} \quad {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta} \quad {}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta} \quad {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta} \quad \dots$$

of **Θ-linked $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** [cf. Corollary 3.8], which will be referred to as the **Frobenius-picture** [associated to the Θ-link]. One fundamental property of this Frobenius-picture is the property that it *fails to admit* **permutation automorphisms** *that switch adjacent indices n, n* + 1, but leave the remaining indices ∈ $\mathbb{Z}$ fixed [cf. Corollary 3.8]. Roughly speaking, the Θ-link $^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \xrightarrow{\Theta} {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ may be thought of as a *formal correspondence*

$$^{n}\underline{\underline{\Theta}}_{\underline{v}} \quad \mapsto \quad {}^{(n+1)}\underline{\underline{q}}_{\underline{v}}$$

[cf. Remark 3.8.1, (i)], which is depicted in Fig. I1.5 below.

In fact, the Θ-link discussed in the present paper is only a **simplified version** of the "Θ-link" that will ultimately play a central role in the present series of papers. The construction of the version of the Θ-link that we shall ultimately be interested in is quite *technically involved* and, indeed, occupies the greater part of the theory to be developed in [IUTchII], [IUTchIII]. On the other hand, the simplified version discussed in the present paper is of interest in that it allows one to give a relatively straightforward introduction to many of the important **qualitative properties** of the Θ-link — such as the *Frobenius-picture* discussed above and the *étale-picture* to be discussed below — that will continue to be of *central importance* in the case of the versions of the Θ-link that will be developed in [IUTchII], [IUTchIII].

$$\dots \quad - - - - \quad \boxed{\begin{array}{c} {}^{n}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \\ \\ {}^{n}\underline{\underline{q}}_{\underline{v}} \quad \rightsquigarrow \quad {}^{n}\underline{\underline{\Theta}}_{\underline{v}} \end{array}} \quad - - - - \quad \boxed{\begin{array}{c} {}^{n+1}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}} \\ \\ {}^{(n+1)}\underline{\underline{q}}_{\underline{v}} \quad \rightsquigarrow \quad {}^{(n+1)}\underline{\underline{\Theta}}_{\underline{v}} \end{array}} \quad - - - - \quad \dots$$

Fig. I1.5: Frobenius-picture associated to the Θ-link

Now let us return to our discussion of the *Frobenius-picture* associated to the Θ-link. The $\mathcal{D}^{\vdash}$-prime-strip associated to the $\mathcal{F}^{\vdash\times}$-prime-strip ${}^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}}$ may, in fact, be naturally identified with the $\mathcal{D}^{\vdash}$-prime-strip ${}^{\dagger}\mathfrak{D}^{\vdash}_{>}$ associated to a certain $\mathcal{F}$-prime-strip ${}^{\dagger}\mathfrak{F}_{>}$ [cf. the discussion preceding Example 5.4] that arises from the Θ-Hodge theater underlying the $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$. The $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}_{>}$ associated to the $\mathcal{F}$-prime-strip ${}^{\dagger}\mathfrak{F}_{>}$ is precisely the $\mathcal{D}$-prime-strip depicted as "$[1 \ < \ \dots \ < \ l^{*}]$" in Fig. I1.3. Thus, the Frobenius-picture discussed above induces an infinite chain of full poly-isomorphisms

$$\dots \quad \overset{\sim}{\to} \quad {}^{(n-1)}\mathfrak{D}^{\vdash}_{>} \quad \overset{\sim}{\to} \quad {}^{n}\mathfrak{D}^{\vdash}_{>} \quad \overset{\sim}{\to} \quad {}^{(n+1)}\mathfrak{D}^{\vdash}_{>} \quad \overset{\sim}{\to} \quad \dots$$

of $\mathcal{D}^{\vdash}$-prime-strips. That is to say, when regarded up to isomorphism, the $\mathcal{D}^{\vdash}$-prime-strip "${}^{(-)}\mathfrak{D}^{\vdash}_{>}$" may be regarded as an **invariant** — i.e., a **"mono-analytic core"** — of the various $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters that occur in the Frobenius-picture [cf. Corollaries 4.12, (ii); 6.10, (ii)]. Unlike the case with the Frobenius-picture, the *relationships* of the various $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters ${}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ to this mono-analytic core — relationships that are depicted by *spokes* in Fig. I1.6 below — are compatible with **arbitrary permutation symmetries** among the spokes [i.e., among the labels $n \in \mathbb{Z}$ of the $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters] — cf. Corollaries 4.12, (iii); 6.10, (iii), (iv). The diagram depicted in Fig. I1.6 below will be referred to as the **étale-picture**.

Thus, the étale-picture may, in some sense, be regarded as a collection of **canonical splittings** of the Frobenius-picture. The existence of such splittings suggests that

> by applying various results from **absolute anabelian geometry** to the various tempered and étale fundamental groups that constitute each $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theater in the étale-picture, one may obtain **algorithmic descriptions** of — i.e., roughly speaking, one may take a **"glimpse"** inside — the **conventional scheme theory** of one $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater ${}^{m}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ in terms of the conventional scheme theory associated to another $\Theta^{\pm \mathrm{ell}}$NF-Hodge theater ${}^{n}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$ [i.e., where $n \neq m$].

Indeed, this point of view constitutes one of the *main themes* of the theory developed in the present series of papers and will be of particular importance in our treatment in [IUTchIII] of the main results of the theory.

$$
\begin{array}{ccccc}
 & & \boxed{{}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}} & & \\
 & \dots & | & \dots & \\
\boxed{{}^{n-1}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}} & — & \boxed{{}^{(-)}\mathfrak{D}^{\vdash}_{>}} & — & \boxed{{}^{n+1}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}} \\
 & \dots & | & \dots & \\
 & & \boxed{{}^{n+2}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}} & &
\end{array}
$$

Fig. I1.6: Étale-picture of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters

Before proceeding, we recall the "heuristic" notions of **Frobenius-like** — i.e., *"order-conscious"* — and **étale-like** — i.e., *"indifferent to order"* — *mathematical structures* discussed in [FrdI], Introduction. These notions will play a *key role* in the theory developed in the present series of papers. In particular, the terms "Frobenius-picture" and "étale-picture" introduced above are motivated by these notions.

The *main result* of the present paper may be summarized as follows.

Theorem A. **($\mathbb{F}_l^{\rtimes\pm}$-/$\mathbb{F}_l^{\circledast}$-Symmetries, Θ-Links, and Frobenius-/Étale-Pictures Associated to $\Theta^{\pm\mathrm{ell}}$NF-Hodge Theaters)** *Fix a collection of* **initial Θ-data**, *which determines, in particular, data* $(E_F,\ \overline{F},\ l,\ \underline{\mathbb{V}})$ *as in the above discussion. Then one may construct a* $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater**

$$
{}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}
$$

— in essence, a system of Frobenioids — associated to this initial Θ-data, as well as an associated $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$**NF-Hodge theater** ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *— in essence, the system of base categories associated to the system of Frobenioids* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$.

(i) **($\mathbb{F}_l^{\rtimes\pm}$- and $\mathbb{F}_l^{\circledast}$-Symmetries)** *The* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *may be obtained as the result of* **gluing** *together a* $\Theta^{\pm\mathrm{ell}}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}}$ *to a* ΘNF*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$ *[cf. Remark 6.12.2, (ii)]; a similar statement holds for the* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$. *The* **global portion** *of a* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ *consists of a category equivalent to [the full subcategory*

*determined by the connected objects of] the Galois category of finite étale coverings of the [orbi]curve $\underline{X}_K$. This global portion is equipped with a $\mathbb{F}_l^{\rtimes\pm}$-***symmetry***, i.e., a poly-action by $\mathbb{F}_l^{\rtimes\pm}$ on the* **labels**

$$(-l^* \ < \ \ldots \ < \ -1 \ < 0 \ < \ 1 \ < \ \ldots \ < \ l^*)$$

*— which we think of as elements $\in \mathbb{F}_l$ — each of which is represented in the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ by a $\mathcal{D}$-***prime-strip** [cf. Fig. I1.3]. The* **global portion** *of a $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ consists of a category equivalent to [the full subcategory determined by the connected objects of] the Galois category of finite étale coverings of the orbicurve $\underline{C}_K$. This global portion is equipped with a $\mathbb{F}_l^{\circledast}$-***symmetry***, i.e., a poly-action by $\mathbb{F}_l^{\circledast}$ on the* **labels**

$$(1 \ < \ \ldots \ < \ l^*)$$

*— which we think of as elements $\in \mathbb{F}_l^{\circledast}$ — each of which is represented in the $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ by a $\mathcal{D}$-***prime-strip** [cf. Fig. I1.3]. The $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ is glued to the $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ along a* **single $\mathcal{D}$-prime-strip** *in such a way that the labels $0 \neq \pm t \in \mathbb{F}_l$ that arise in the $\mathbb{F}_l^{\rtimes\pm}$-symmetry are identified with the corresponding label $j \in \mathbb{F}_l^{\circledast}$ that arises in the $\mathbb{F}_l^{\circledast}$-symmetry.*

(ii) **(Θ-links)** *By considering the $2l$-th roots of the* **q-parameters** *"$\underline{\underline{q}}_{\underline{v}}$" of the elliptic curve E_F at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ and extending to other $\underline{v} \in \underline{\mathbb{V}}$ in such a way as to satisfy the* **product formula***, one may construct a natural* **$\mathcal{F}^{\Vdash}$-prime-strip** *${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ associated to the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$. In a similar vein, by considering the reciprocal of the l-th root of the Frobenioid-theoretic* **theta function** *"$\underline{\underline{\Theta}}_{\underline{v}}$" associated to the elliptic curve E_F at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ and extending to other $\underline{v} \in \underline{\mathbb{V}}$ in such a way as to satisfy the* **product formula***, one may construct a natural* **$\mathcal{F}^{\Vdash}$-prime-strip** *${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$ associated to the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$. Now let ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ be* **another** *$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater [relative to the given initial Θ-data]. Then we shall refer to the "full poly-isomorphism" of [i.e., the collection of all isomorphisms between]* **$\mathcal{F}^{\Vdash}$-prime-strips**

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \quad \stackrel{\sim}{\rightarrow} \quad {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$$

as the **Θ-link** *from [the underlying Θ-Hodge theater of] ${}^{\dagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ to [the underlying Θ-Hodge theater of] ${}^{\ddagger}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$. The Θ-link induces the full poly-isomorphism between the* **$\mathcal{F}^{\vdash\times}$-prime-strips**

$${}^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}} \quad \stackrel{\sim}{\rightarrow} \quad {}^{\ddagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}}$$

associated to ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ and ${}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$.

(iii) **(Frobenius-/Étale-Pictures)** *Let $\{{}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n\in\mathbb{Z}}$ be a* **collection of distinct $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** *[relative to the given initial Θ-data] indexed by the integers. Then the* **infinite chain**

$$\ldots \quad \xrightarrow{\Theta} \quad {}^{(n-1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta} \quad {}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta} \quad {}^{(n+1)}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}} \quad \xrightarrow{\Theta} \quad \ldots$$

of **Θ-linked $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters** *will be referred to as the* **Frobenius-picture** *[associated to the Θ-link] — cf. Fig. I1.5. The Frobenius-picture* **fails** *to admit* **permutation automorphisms** *that switch adjacent indices n, $n+1$, but leave the remaining indices $\in \mathbb{Z}$ fixed. The Frobenius-picture induces an infinite chain of* **full poly-isomorphisms**

$$\dots \quad \xrightarrow{\sim} \quad {}^{(n-1)}\mathfrak{D}^{\vdash}_{>} \quad \xrightarrow{\sim} \quad {}^{n}\mathfrak{D}^{\vdash}_{>} \quad \xrightarrow{\sim} \quad {}^{(n+1)}\mathfrak{D}^{\vdash}_{>} \quad \xrightarrow{\sim} \quad \dots$$

between the various $\mathcal{D}^{\vdash}$-prime-strips ${}^{n}\mathfrak{D}^{\vdash}_{>}$, i.e., in essence, the $\mathcal{D}^{\vdash}$-prime-strips associated to the $\mathcal{F}^{\vdash\times}$-prime-strips ${}^{n}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}}$. The relationships of the various $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters ${}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ to the **"mono-analytic core"** *constituted by the $\mathcal{D}^{\vdash}$-prime-strip "${}^{(-)}\mathfrak{D}^{\vdash}_{>}$" regarded up to isomorphism — relationships that are depicted by* **spokes** *in Fig. I1.6 — are compatible with* **arbitrary permutation symmetries** *among the spokes [i.e., among the labels $n \in \mathbb{Z}$ of the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters]. The diagram depicted in Fig. I1.6 will be referred to as the* **étale-picture**.

In addition to the main result discussed above, we also prove a certain *technical result* concerning **tempered fundamental groups** — cf. Theorem B below — that will be of use in our development of the theory of *Hodge-Arakelov-theoretic evaluation* in [IUTchII]. This result is essentially a routine application of the theory of *maximal compact subgroups* of tempered fundamental groups developed in [SemiAnbd] [cf., especially, [SemiAnbd], Theorems 3.7, 5.4]. Here, we recall that this theory of [SemiAnbd] may be thought of as a sort of **"Combinatorial Section Conjecture"** [cf. Remark 2.5.1 of the present paper; [IUTchII], Remark 1.12.4] — a point of view that is of particular interest in light of the *historical remarks* made in §I5 below. Moreover, Theorem B is of interest *independently of the theory of the present series of papers* in that it yields, for instance, a *new proof* of the *normal terminality* of the tempered fundamental group in its profinite completion, a result originally obtained in [André], Lemma 3.2.1, by means of other techniques [cf. Remark 2.4.1]. This new proof is of interest in that, unlike the techniques of [André], which are only available in the *profinite* case, this new proof [cf. Proposition 2.4, (iii)] holds in the case of **pro-$\widehat{\Sigma}$-completions**, for more general $\widehat{\Sigma}$ [i.e., not just the case of $\widehat{\Sigma} = \mathfrak{Primes}$].

Theorem B. (Profinite Conjugates of Tempered Decomposition and Inertia Groups) *Let k be a* **mixed-characteristic [nonarchimedean] local field**, *X a* **hyperbolic curve** *over k. Write*

$$\Pi_X^{\mathrm{tp}}$$

for the **tempered fundamental group** *$\pi_1^{\mathrm{tp}}(X)$ [relative to a suitable basepoint] of X [cf. [André], §4; [SemiAnbd], Example 3.10]; $\widehat{\Pi}_X$ for the* **étale fundamental group** *[relative to a suitable basepoint] of X. Thus, we have a* **natural inclusion**

$$\Pi_X^{\mathrm{tp}} \quad \hookrightarrow \quad \widehat{\Pi}_X$$

which allows one to identify $\widehat{\Pi}_X$ with the profinite completion of Π_X^{tp}. Then every **decomposition group** *in $\widehat{\Pi}_X$ (respectively,* **inertia group** *in $\widehat{\Pi}_X$) associated to* *a closed point or cusp of X (respectively, to a cusp of X) is contained in Π_X^{tp} if and only if it is a decomposition group in Π_X^{tp} (respectively, inertia group in Π_X^{tp}) associated to a closed point or cusp of X (respectively, to a cusp of X). Moreover, a $\widehat{\Pi}_X$-conjugate of Π_X^{tp} contains a decomposition group in Π_X^{tp} (respectively, inertia group in Π_X^{tp}) associated to a closed point or cusp of X (respectively, to a cusp of X) if and only if it is equal to Π_X^{tp}.*

Theorem B is [essentially] given as Corollary 2.5 in §2. Here, we note that although, in the statement of Corollary 2.5, the hyperbolic curve X is assumed to admit *stable reduction* over the ring of integers $\mathcal{O}_k$ of k, one verifies immediately that this assumption is, in fact, unnecessary.

Finally, we remark that one *important reason* for the need to apply Theorem B in the context of the theory of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters summarized in Theorem A is the following. The $\mathbb{F}_l^{\rtimes\pm}$**-symmetry**, which will play a crucial role in the theory of the present series of papers [cf., especially, [IUTchII], [IUTchIII]], depends, in an essential way, on the *synchronization of the* $\pm$*-indeterminacies* that occur locally at each $\underline{v} \in \underline{\mathbb{V}}$ [cf. Fig. I1.1]. Such a synchronization may only be obtained by making use of the *global portion* of the $\Theta^{\pm\mathrm{ell}}$-Hodge theater under consideration. On the other hand, in order to avail oneself of such **global** $\pm$**-synchronizations** [cf. Remark 6.12.4, (iii)], it is necessary to regard the various **labels** of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry

$$(-l^* \ < \ \dots \ < \ -1 \ < 0 \ < \ 1 \ < \ \dots \ < \ l^*)$$

as conjugacy classes of inertia groups of the [necessarily] *profinite* geometric étale fundamental group of $\underline{X}_K$. That is to say, in order to relate such *global profinite conjugacy classes* to the corresponding *tempered conjugacy classes* [i.e., conjugacy classes with respect to the geometric *tempered* fundamental group] of inertia groups at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [i.e., where the crucial *Hodge-Arakelov-theoretic evaluation* is to be performed!], it is necessary to apply Theorem B — cf. the discussion of Remark 4.5.1; [IUTchII], Remark 2.5.2, for more details.

§I2. Gluing Together Models of Conventional Scheme Theory

As discussed in §I1, the system of Frobenioids constituted by a $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater is intended to be a sort of miniature model of **conventional scheme theory**. One then **glues** multiple $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters $\{{}^n\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}\}_{n\in\mathbb{Z}}$ together by means of the full poly-isomorphisms between the *"subsystems of Frobenioids"* constituted by certain $\mathcal{F}^{\Vdash}$*-prime-strips*

$${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \quad \overset{\sim}{\to} \quad {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$$

to form the **Frobenius-picture**. One fundamental observation in this context is the following:

these gluing isomorphisms — i.e., in essence, the correspondences

$${}^n\underline{\underline{\Theta}}_{\underline{v}} \quad \mapsto \quad {}^{(n+1)}\underline{\underline{q}}_{\underline{v}}$$

— and hence the geometry of the resulting Frobenius-picture *lie* **outside** *the framework of* **conventional scheme theory** *in the sense that they do* **not** *arise from* **ring homomorphisms***!*

In particular, although each particular model ${}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ of conventional scheme theory is constructed within the framework of conventional scheme theory, the relationship between the *distinct* [albeit abstractly isomorphic, as $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters!] conventional scheme theories represented by, for instance, neighboring $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters ${}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$, ${}^{n+1}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *cannot be expressed scheme-theoretically.* In this context, it is also important to note that such gluing operations are possible precisely because of the **relatively simple structure** — for instance, by comparison to the structure of a *ring*! — of the **Frobenius-like structures** constituted by the Frobenioids that appear in the various $\mathcal{F}^{\Vdash}$-prime-strips involved, i.e., in essence, collections of **monoids** isomorphic to $\mathbb{N}$ or $\mathbb{R}_{\geq 0}$ [cf. Fig. I1.2].

If one thinks of the geometry of *"conventional scheme theory"* as being analogous to the geometry of *"Euclidean space"*, then the geometry represented by the *Frobenius-picture* corresponds to a *"topological manifold"*, i.e., which is obtained by gluing together various portions of Euclidean space, but which is *not homeomorphic* to Euclidean space. This point of view is illustrated in Fig. I2.1 below, where the various $\Theta^{\pm\mathrm{ell}}$**NF-Hodge theaters** in the *Frobenius-picture* are depicted as [*two-dimensional!* — cf. the discussion of §I1] **twice-punctured topological surfaces of genus one**, glued together along **tubular neighborhoods of cycles**, which correspond to the [*one-dimensional*! — cf. the discussion of §I1] **mono-analytic** data that appears in the isomorphism that constitutes the **Θ-link**. The permutation symmetries in the étale-picture [cf. the discussion of §I1] are depicted in Fig. I2.1 as the **anti-holomorphic reflection** [cf. the discussion of **multiradiality** in [IUTchII], Introduction!] around a gluing cycle between topological surfaces.

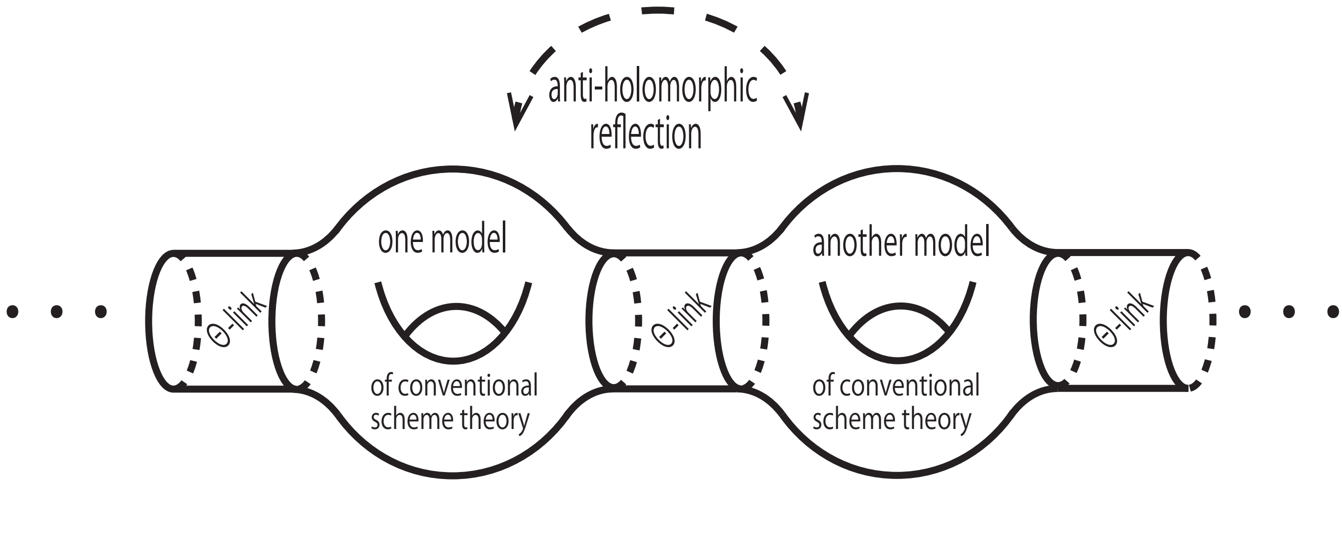

Fig. I2.1: Depiction of Frobenius- and étale-pictures of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters via glued topological surfaces

Another elementary example that illustrates the *spirit* of the gluing operations discussed in the present series of papers is the following. For $i = 0, 1$, let $\mathbb{R}_i$ be a copy of the *real line*; $I_i \subseteq \mathbb{R}_i$ the *closed unit interval* [i.e., corresponding to $[0, 1] \subseteq \mathbb{R}$]. Write $C_0 \subseteq I_0$ for the *Cantor set* and

$$\phi : C_0 \overset{\sim}{\to} I_1$$

for the *bijection* arising from the **Cantor function**. Then if one thinks of $\mathbb{R}_0$ and $\mathbb{R}_1$ as being **glued** to one another by means of ϕ, then it is a *highly nontrivial* problem

> to describe *structures naturally associated to the* **"alien" ring structure** of $\mathbb{R}_0$ — such as, for instance, the subset of **algebraic numbers** $\in \mathbb{R}_0$ — in terms that only require the use of the ring structure of $\mathbb{R}_1$.

A slightly less elementary example that illustrates the *spirit* of the gluing operations discussed in the present series of papers is the following. This example is *technically* much closer to the theory of the present series of papers than the examples involving topological surfaces and Cantor sets given above. For simplicity, let us write

$$G \curvearrowright \mathcal{O}^\times, \quad G \curvearrowright \mathcal{O}^\rhd$$

for the the pairs "$G_{\underline{v}} \curvearrowright \mathcal{O}^\times_{\overline{F}_{\underline{v}}}$", "$G_{\underline{v}} \curvearrowright \mathcal{O}^\rhd_{\overline{F}_{\underline{v}}}$" [cf. the notation of the discussion surrounding Fig. I1.2]. Recall from [AbsTopIII], Proposition 3.2, (iv), that the operation

$$(G \curvearrowright \mathcal{O}^\rhd) \quad \mapsto \quad G$$

of "forgetting $\mathcal{O}^\rhd$" determines a **bijection** from the *group of automorphisms* of the pair $G \curvearrowright \mathcal{O}^\rhd$ — i.e., thought of as an abstract topological monoid equipped with a continuous action by an abstract topological group — to the group of automorphisms of the topological group G. By contrast, we recall from [AbsTopIII], Proposition 3.3, (ii), that the operation

$$(G \curvearrowright \mathcal{O}^\times) \quad \mapsto \quad G$$

of "forgetting $\mathcal{O}^\times$" only determines a **surjection** from the *group of automorphisms* of the pair $G \curvearrowright \mathcal{O}^\times$ — i.e., thought of as an abstract topological monoid equipped with a continuous action by an abstract topological group — to the group of automorphisms of the topological group G; that is to say, the *kernel* of this surjection is given by the **natural action** *of* $\widehat{\mathbb{Z}}^\times$ *on* $\mathcal{O}^\times$. In particular, if one works with *two copies* $G_i \curvearrowright \mathcal{O}^\rhd_i$, where $i = 0, 1$, of $G \curvearrowright \mathcal{O}^\rhd$, which one thinks of as being **glued** to one another by means of an **indeterminate isomorphism**

$$(G_0 \curvearrowright \mathcal{O}^\times_0) \quad \overset{\sim}{\to} \quad (G_1 \curvearrowright \mathcal{O}^\times_1)$$

[i.e., where one thinks of each $(G_i \curvearrowright \mathcal{O}^\times_i)$, for $i = 0, 1$, as an abstract topological monoid equipped with a continuous action by an abstract topological group], then, in general, it is a *highly nontrivial* problem to describe *structures naturally associated to* $(G_0 \curvearrowright \mathcal{O}_0^{\triangleright})$ in terms that only require the use of $(G_1 \curvearrowright \mathcal{O}_1^{\triangleright})$.

One such structure which is of interest in the context of the present series of papers [cf., especially, the theory of [IUTchII], §1] is the natural **cyclotomic rigidity isomorphism** between the group of torsion elements of $\mathcal{O}_0^{\triangleright}$ and an analogous group of torsion elements associated naturally associated to G_0 — i.e., a structure that is manifestly **not preserved** by the natural action of $\widehat{\mathbb{Z}}^{\times}$ on $\mathcal{O}_0^{\times}$!

In the context of the above discussion of Fig. I2.1, it is of interest to note important role played by **Kummer theory** in the present series of papers [cf. the Introductions to [IUTchII], [IUTchIII]]. From the point of view of Fig. I2.1, this role corresponds to the *precise specification* of the gluing cycle within each twice-punctured genus one surface in the illustration. Of course, such a precise specification *depends* on the twice-punctured genus one surface under consideration, i.e., the *same* gluing cycle is subject to *quite different "precise specifications"*, relative to the twice-punctured genus one surface on the *left* and the twice-punctured genus one surface on the *right*. This state of affairs corresponds to the *quite different Kummer theories* to which the monoids/Frobenioids that appear in the Θ-link are subject, relative to the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater in the *domain* of the Θ-link and the $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater in the *codomain* of the Θ-link. At first glance, it might appear that the use of *Kummer theory*, i.e., of the correspondence determined by constructing *Kummer classes*, to achieve this precise specification of the relevant monoids/Frobenioids within each $\Theta^{\pm\mathrm{ell}}$NF-Hodge theater is somewhat *arbitrary*, i.e., that one could perhaps use other correspondences [i.e., correspondences not determined by Kummer classes] to achieve such a precise specification. In fact, however, the **rigidity** of the relevant local and global monoids equipped with Galois actions [cf. Corollary 5.3, (i), (ii), (iv)] implies that, if one imposes the natural condition of **Galois-compatibility**, then

> the correspondence furnished by **Kummer theory** is the only acceptable choice for constructing the required "**precise specification** *of the relevant monoids/Frobenioids within each* $\Theta^{\pm\mathrm{ell}}$*NF-Hodge theater"*

— cf. also the discussion of [IUTchII], Remark 3.6.2, (ii).

The construction of the Frobenius-picture described in §I1 is given in the present paper. More elaborate versions of this Frobenius-picture will be discussed in [IUTchII], [IUTchIII]. Once one constructs the Frobenius-picture, one *natural and fundamental problem*, which will, in fact, be one of the *main themes* of the present series of papers, is the problem of

> *describing an* **alien "arithmetic holomorphic structure"** [i.e., an alien "conventional scheme theory"] corresponding to some ${}^{m}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ *in terms of a "known arithmetic holomorphic structure"* corresponding to ${}^{n}\mathcal{HT}^{\Theta^{\pm\mathrm{ell}}\mathrm{NF}}$ [where $n \neq m$]

— a problem, which, as discussed in §I1, will be approached, in the final portion of [IUTchIII], by applying various results from **absolute anabelian geometry** [i.e., more explicitly, the theory of [SemiAnbd], [EtTh], and [AbsTopIII]] to the various tempered and étale fundamental groups that appear in the **étale-picture**.

The relevance to this problem of the extensive theory of *"reconstruction of ring/scheme structures"* provided by absolute anabelian geometry is evident from the statement of the problem. On the other hand, in this context, it is of interest to note that, unlike conventional anabelian geometry, which typically centers on the goal of *reconstructing a "known scheme-theoretic object"*, in the present series of papers, we wish to apply techniques and results from anabelian geometry in order to analyze the structure of an **unknown***, essentially* **non-scheme-theoretic** *object*, namely, the **Frobenius-picture**, as described above. Put another way, relative to the point of view that *"Galois groups are arithmetic tangent bundles"* [cf. the theory of the *arithmetic Kodaira-Spencer morphism* in [HASurI]], one may think of conventional anabelian geometry as corresponding to the *computation of the automorphisms of a scheme* as

$$H^0(\text{arithmetic tangent bundle})$$

and of the application of absolute anabelian geometry to the analysis of the Frobenius-picture, i.e., to the solution of the problem discussed above, as corresponding to the computation of

$$H^1(\text{arithmetic tangent bundle})$$

— i.e., the *computation of* **"deformations of the arithmetic holomorphic structure"** of a number field equipped with an elliptic curve.

§I3. Basepoints and Inter-universality

As discussed in §I2, the present series of papers is concerned with considering *"deformations of the arithmetic holomorphic structure"* of a number field — i.e., so to speak, with performing **"surgery on the number field"**. At a more concrete level, this means that one must consider situations in which *two distinct "theaters" for conventional ring/scheme theory* — i.e., two distinct $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters — are related to one another by means of a *"correspondence"*, or *"filter"*, that **fails** to be compatible with the respective **ring structures**. In the discussion so far of the portion of the theory developed in the present paper, the main example of such a "filter" is given by the **Θ-link**. As mentioned earlier, more elaborate versions of the Θ-link will be discussed in [IUTchII], [IUTchIII]. The other main example of such a non-ring/scheme-theoretic "filter" in the present series of papers is the **$\mathfrak{log}$-link**, which we shall discuss in [IUTchIII] [cf. also the theory of [AbsTopIII]].

One important aspect of such non-ring/scheme-theoretic filters is the property that they are **incompatible** with various constructions that depend on the **ring structure** of the theaters that constitute the domain and codomain of such a filter. From the point of view of the present series of papers, perhaps the most important example of such a construction is given by the various **étale fundamental groups** — e.g., **Galois groups** — that appear in these theaters. Indeed, these groups are defined, essentially, as **automorphism groups** *of some separably closed* **field**, i.e., the field that arises in the definition of the *fiber functor* associated to the

basepoint determined by a *geometric point* that is used to define the étale fundamental group — cf. the discussion of [IUTchII], Remark 3.6.3, (i); [IUTchIII], Remark 1.2.4, (i); [AbsTopIII], Remark 3.7.7, (i). In particular, unlike the case with ring homomorphisms or morphisms of schemes with respect to which the étale fundamental group satisfies well-known *functoriality* properties, in the case of non-ring/scheme-theoretic filters, the only *"type of mathematical object"* that makes sense *simultaneously* in both the domain and codomain theaters of the filter is the notion of a *topological group.* In particular, the only data that can be considered in relating étale fundamental groups on either side of a filter is the **étale-like structure** constituted by the underlying **abstract topological group** associated to such an étale fundamental group, i.e., devoid of any *auxiliary data* arising from the construction of the group *"as an étale fundamental group associated to a* **basepoint** *determined by a geometric point of a scheme".* It is this fundamental aspect of the theory of the present series of papers — i.e.,

> of relating the distinct *set-theoretic universes* associated to the distinct fiber functors/basepoints on either side of such a non-ring/scheme-theoretic filter

— that we refer to as **inter-universal**. This inter-universal aspect of the theory manifestly leads to the issue of considering

> the extent to which one can understand *various ring/scheme structures* by considering only the underlying **abstract topological group** of some étale fundamental group arising from such a ring/scheme structure

— i.e., in other words, of considering the **absolute anabelian geometry** [cf. the Introductions to [AbsTopI], [AbsTopII], [AbsTopIII]] of the rings/schemes under consideration.

At this point, the careful reader will note that the above discussion of the inter-universal aspects of the theory of the present series of papers depends, in an essential way, on the issue of *distinguishing different* **"types of mathematical object"** and hence, in particular, on the *notion of a "type of mathematical object".* This notion may be formalized via the language of **"species"**, which we develop in the final portion of [IUTchIV].

Another important *"inter-universal"* phenomenon in the present series of papers — i.e., phenomenon which, like the absolute anabelian aspects discussed above, arises from a *"deep sensitivity to particular choices of* **basepoints***"* — is the phenomenon of **conjugate synchronization**, i.e., of synchronization between conjugacy indeterminacies of distinct copies of various local Galois groups, which, as was mentioned in §I1, will play an important role in the theory of [IUTchII], [IUTchIII]. The various **rigidity properties** of the *étale theta function* established in [EtTh] constitute yet another inter-universal phenomenon that will play an important role in theory of [IUTchII], [IUTchIII].

§I4. Relation to Complex and p-adic Teichmüller Theory

In order to understand the sense in which the theory of the present series of papers may be thought of as a sort of *"Teichmüller theory"* of number fields equipped with an elliptic curve, it is useful to recall certain basic, well-known facts concerning the **classical complex Teichmüller theory** of Riemann surfaces of finite type [cf., e.g., [Lehto], Chapter V, §8]. Although such a Riemann surface is **one-dimensional** from a *complex, holomorphic* point of view, this single complex dimension may be thought of consisting of **two** *underlying real analytic dimensions.* Relative to a suitable canonical holomorphic coordinate $z = x + iy$ on the Riemann surface, the **Teichmüller deformation** may be written in the form

$$z \ \mapsto \ \zeta \ = \ \xi + i\eta = \ Kx + iy$$

— where $1 < K < \infty$ is the *dilation* factor associated to the deformation. That is to say, the Teichmüller deformation consists of **dilating one** of the two underlying real analytic dimensions, while keeping the **other dimension fixed**. Moreover, the theory of such Teichmüller deformations may be *summarized* as consisting of

> *the explicit description of a* **varying holomorphic structure** *within a* **fixed real analytic "container"**

— i.e., the underlying real analytic surface associated to the given Riemann surface.

On the other hand, as discussed in [AbsTopIII], §I3, one may think of the **ring structure** of a *number field F* as a **single "arithmetic holomorphic dimension"**, which, in fact, consists of **two** *underlying* **"combinatorial dimensions"**, corresponding to

- its **additive** *structure* "⊞" and its **multiplicative** *structure* "⊠".

When, for simplicity, the number field F is *totally imaginary*, one may think of these two combinatorial dimensions as corresponding to the

- **two cohomological dimensions** of the *absolute Galois group*

G_F of F. A similar statement holds in the case of the absolute Galois group G_k of a **nonarchimedean local field** k. In the case of **complex archimedean fields** k [i.e., topological fields isomorphic to the field of complex numbers equipped with its usual topology], the two combinatorial dimensions of k may also be thought of as corresponding to the

- **two** *underlying* **topological/real dimensions**

of k. Alternatively, in both the nonarchimedean and archimedean cases, one may think of the two underlying combinatorial dimensions of k as corresponding to the

- **group of units** $\mathcal{O}_k^\times$ *and* **value group** $k^\times/\mathcal{O}_k^\times$

of k. Indeed, in the nonarchimedean case, local class field theory implies that this last point of view is consistent with the interpretation of the two underlying combinatorial dimensions via cohomological dimension; in the archimedean case, the consistency of this last point of view with the interpretation of the two underlying combinatorial dimensions via topological/real dimension is immediate from the definitions.

This last interpretation in terms of groups of units and value groups is of particular relevance in the context of the theory of the present series of papers. That is to say, one may think of the **Θ-link**

$$\begin{array}{ccc} {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} & \xrightarrow{\sim} & {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \\ \\ \{\ {}^{\dagger}\underline{\underline{\Theta}}_{\underline{v}} & \mapsto & {}^{\ddagger}\underline{\underline{q}}_{\underline{v}}\ \}_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}} \end{array}$$

— which, as discussed in §I1, induces a *full poly-isomorphism*

$$\begin{array}{ccc} {}^{\dagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}} & \xrightarrow{\sim} & {}^{\ddagger}\mathfrak{F}^{\vdash\times}_{\mathrm{mod}} \\ \\ \{\ \mathcal{O}^{\times}_{F_{\underline{v}}} & \xrightarrow{\sim} & \mathcal{O}^{\times}_{F_{\underline{v}}}\ \}_{\underline{v}\in\underline{\mathbb{V}}^{\mathrm{bad}}} \end{array}$$

— as a sort of *"***Teichmüller deformation** *relative to a* **Θ-dilation***"*, i.e., a deformation of the **ring structure** of the number field equipped with an elliptic curve constituted by the given *initial Θ-data* in which one **dilates** the underlying combinatorial dimension corresponding to the local **value groups** relative to a *"Θ-factor"*, while one leaves **fixed**, up to isomorphism, the underlying combinatorial dimension corresponding to the local **groups of units** [cf. Remark 3.9.3]. This point of view is reminiscent of the discussion in §I1 of **"disentangling/dismantling"** *of various structures associated to number field.*

In [IUTchIII], we shall consider *two-dimensional diagrams* of $\Theta^{\pm\mathrm{ell}}$NF-Hodge theaters which we shall refer to as **log-theta-lattices**. The two dimensions of such diagrams correspond precisely to the *two underlying combinatorial dimensions of a ring.* Of these two dimensions, the "theta dimension" consists of the *Frobenius-picture* associated to [more elaborate versions of] the **Θ-link**. Many of the important properties that involve this "theta dimension" are consequences of the theory of [FrdI], [FrdII], [EtTh]. On the other hand, the "log dimension" consists of iterated copies of the **$\mathfrak{log}$-link**, i.e., diagrams of the sort that are studied in [AbsTopIII]. That is to say, whereas the "theta dimension" corresponds to *"deformations of the arithmetic holomorphic structure"* of the given number field equipped with an elliptic curve, this "log dimension" corresponds to *"rotations of the two underlying combinatorial dimensions"* of a ring that leave the arithmetic holomorphic structure *fixed* — cf. the discussion of the **"juggling of ⊞, ⊠ induced by $\mathfrak{log}$"** in [AbsTopIII], §I3. The *ultimate conclusion* of the theory of [IUTchIII] is that

> the *"a priori unbounded deformations"* of the arithmetic holomorphic structure given by the Θ-link in fact admit **canonical bounds**, which may be thought of as a sort of reflection of the **"hyperbolicity"** of the given number field equipped with an elliptic curve

— cf. [IUTchIII], Corollary 3.12. Such canonical bounds may be thought of as analogues for a number field of canonical bounds that arise from **differentiating Frobenius liftings** in the context of p-adic hyperbolic curves — cf. the discussion in the final portion of [AbsTopIII], §I5. Moreover, such canonical bounds are obtained in [IUTchIII] as a consequence of

> *the explicit description of a* **varying arithmetic holomorphic structure** *within a* **fixed mono-analytic "container"**

— cf. the discussion of §I2! — furnished by [IUTchIII], Corollary 3.11 [cf. also the discussion of [IUTchIII], Remarks 3.12.2, 3.12.3, 3.12.4], i.e., a situation that is *entirely formally analogous* to the summary of complex Teichmüller theory given above.

The significance of the log-theta-lattice lattice is best understood in the context of the analogy between the **inter-universal Teichmüller theory** developed in the present series of papers and the ***p*-adic Teichmüller theory** of [pOrd], [pTeich]. Here, we recall for the convenience of the reader that the p-adic Teichmüller theory of [pOrd], [pTeich] may be summarized, [very!] roughly speaking, as a sort of **generalization**, to the case of **"quite general" *p*-adic hyperbolic curves**, of the classical p-adic theory surrounding the **canonical representation**

$$\pi_1(\ (\mathbb{P}^1 \setminus \{0,1,\infty\})_{\mathbb{Q}_p}\ ) \quad \to \quad \pi_1(\ (\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}_p}\ ) \quad \to \quad PGL_2(\mathbb{Z}_p)$$

— where the "$\pi_1(-)$'s" denote the *étale fundamental group*, relative to a suitable basepoint; $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}_p}$ denotes the *moduli stack of elliptic curves* over $\mathbb{Q}_p$; the first horizontal arrow denotes the morphism induced by the elliptic curve over the projective line minus three points determined by the classical *Legendre form* of the Weierstrass equation; the second horizontal arrow is the representation determined by the *p-power torsion points* of the *tautological elliptic curve* over $(\mathcal{M}_{\mathrm{ell}})_{\mathbb{Q}_p}$. In particular, the reader who is familiar with the theory of the classical representation of the above display, but not with the theory of [pOrd], [pTeich], may nevertheless appreciate, to a substantial degree, the analogy between the *inter-universal Teichmüller theory* developed in the present series of papers and the *p-adic Teichmüller theory* of [pOrd], [pTeich] by

thinking in terms of the
well-known classical properties of this classical representation.

In some sense, the *gap* between the "quite general" p-adic hyperbolic curves that appear in p-adic Teichmüller theory and the classical case of $(\mathbb{P}^1 \setminus \{0,1,\infty\})_{\mathbb{Q}_p}$ may be thought of, roughly speaking, as corresponding, relative to the analogy with the theory of the present series of papers, to the gap between **arbitrary number fields** and the **rational number field** $\mathbb{Q}$. This point of view is especially interesting in the context of the discussion of §I5 below.

The analogy between the **inter-universal Teichmüller theory** developed in the present series of papers and the ***p*-adic Teichmüller theory** of [pOrd], [pTeich] is described to a substantial degree in the discussion of [AbsTopIII], §I5, i.e., where the *"future Teichmüller-like extension of the mono-anabelian theory"* may be understood as referring precisely to the inter-universal Teichmüller theory developed in the present series of papers. The starting point of this analogy is the correspondence between a *number field equipped with a [once-punctured] elliptic curve* [in the present series of papers] and a *hyperbolic curve over a positive characteristic perfect field equipped with a nilpotent ordinary indigenous bundle* [in p-adic Teichmüller theory] — cf. Fig. I4.1 below. That is to say, in this analogy, the *number field* — which may be regarded as being equipped with a *finite collection of "exceptional" valuations*, namely, in the notation of §I1, the valuations lying over $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$ — corresponds to the *hyperbolic curve over a positive characteristic perfect field* — which may be thought of as a *one-dimensional function field* over a positive characteristic perfect field, equipped with a *finite collection of "exceptional" valuations*, namely, the valuations corresponding to the cusps of the curve.

On the other hand, the *[once-punctured] elliptic curve* in the present series of papers corresponds to the *nilpotent ordinary indigenous bundle* in p-adic Teichmüller theory. Here, we recall that an indigenous bundle may be thought of as a sort of *"virtual analogue"* of the first cohomology group of the tautological elliptic curve over the moduli stack of elliptic curves. Indeed, the canonical indigenous bundle over the moduli stack of elliptic curves arises precisely as the first de Rham cohomology module of this tautological elliptic curve. Put another way, from the point of view of *fundamental groups*, an indigenous bundle may be thought of as a sort of "virtual analogue" of the **abelianized fundamental group** of the tautological elliptic curve over the moduli stack of elliptic curves. By contrast, in the present series of papers, it is of crucial importance to use the **entire nonabelian profinite étale fundamental group** — i.e., not just its abelizanization! — of the given once-punctured elliptic curve over a number field. Indeed, only by working with the entire profinite étale fundamental group can one avail oneself of the crucial **absolute anabelian theory** developed in [EtTh], [AbsTopIII] [cf. the discussion of §I3]. This state of affairs prompts the following question:

> To what extent can one extend the indigenous bundles that appear in *classical complex* and *p-adic Teichmüller theory* to objects that serve as *"virtual analogues" of the* **entire nonabelian fundamental group** *of the tautological once-punctured elliptic curve* over the moduli stack of [once-punctured] elliptic curves?

Although this question lies beyond the scope of the present series of papers, it is the hope of the author that this question may be addressed in a future paper.

Now let us return to our discussion of the log-theta-lattice, which, as discussed above, consists of two types of arrows, namely, **Θ-link** arrows and **$\mathfrak{log}$-link** arrows. As discussed in [IUTchIII], Remark 1.4.1, (iii) — cf. also Fig. I4.1 below, as well as Remark 3.9.3, (i), of the present paper — the Θ-link arrows correspond to the *"transition from $p^n\mathbb{Z}/p^{n+1}\mathbb{Z}$ to $p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$"*, i.e., the **mixed characteristic extension** *structure of a ring of Witt vectors*, while the $\mathfrak{log}$-link arrows, i.e., the portion of theory that is developed in detail in [AbsTopIII], and which will be incorporated into the theory of the present series of papers in [IUTchIII], correspond to the **Frobenius morphism** *in positive characteristic*. As we shall see in [IUTchIII], these two types of arrows *fail to commute* [cf. [IUTchIII], Remark 1.4.1, (i)]. This noncommutativity, or **"intertwining"**, of the Θ-link and $\mathfrak{log}$-link arrows of the log-theta-lattice may be thought of as the analogue, in the context of the theory of the present series of papers, of the well-known "intertwining between the mixed characteristic extension structure of a ring of Witt vectors and the Frobenius morphism in positive characteristic" that appears in the classical p-adic theory. In particular, taken as a whole, the log-theta-lattice in the theory of the present series of papers may be thought of as an analogue, for number fields equipped with a [once-punctured] elliptic curve, of the **canonical lifting**, *equipped with a* **canonical Frobenius action** — hence also the **canonical Frobenius lifting** over the ordinary locus of the curve — associated to a positive characteristic hyperbolic curve equipped with a nilpotent ordinary indigenous bundle in p-adic Teichmüller theory [cf. Fig. I4.1 below; the discussion of [IUTchIII], Remarks 3.12.3, 3.12.4].

Finally, we observe that it is of particular interest in the context of the present discussion that a theory is developed in [CanLift], §3, that yields an **absolute anabelian reconstruction** *for the canonical liftings of p-adic Teichmüller theory*. That is to say, whereas the *original construction* of such canonical liftings given in [pOrd], §3, is *relatively straightforward*, the *anabelian reconstruction* given in [CanLift], §3, of, for instance, the canonical lifting modulo p^2 of the logarithmic special fiber consists of a *highly nontrivial anabelian argument*. This state of affairs is strongly reminiscent of the stark constrast between the *relatively straightforward* construction of the log-theta-lattice given in the present series of papers and the description of an "alien arithmetic holomorphic structure" given in [IUTchIII], Corollary 3.11 [cf. the discussion in the earlier portion of the present §I4], which is achieved by applying *highly nontrivial results in absolute anabelian geometry* — cf. Fig. I4.1 below. In this context, we observe that the *absolute anabelian theory of [AbsTopIII], §1*, which plays a central role in the theory surrounding [IUTchIII], Corollary 3.11, corresponds, in the theory of [CanLift], §3, to the *absolute anabelian reconstruction of the logarithmic special fiber* given in [AbsAnab], §2 [i.e., in essence, the theory of absolute anabelian geometry over finite fields developed in [Tama1]; cf. also [Cusp], §2]. Moreover, just as the absolute anabelian theory of [AbsTopIII], §1, follows essentially by combining a version of *"Uchida's Lemma"* with the theory of *Belyi cuspidalization* — i.e.,

[AbsTopIII], §1 = Uchida Lem. + Belyi cuspidalization

— the absolute anabelian geometry over finite fields of [Tama1], [Cusp], follows essentially by combining a version of *"Uchida's Lemma"* with an application [to the counting of rational points] of the *Lefschetz trace formula for [powers of] the Frobenius morphism on a curve over a finite field* — i.e.,

[Tama1], [Cusp] = Uchida Lem. + Lefschetz trace formula for Frob.

— cf. the discussion of [AbsTopIII], §I5. That is to say, it is perhaps worthy of note that in the analogy between the inter-universal Teichmüller theory developed in the present series of papers and the p-adic Teichmüller theory of [pOrd], [pTeich], [CanLift], the application of the theory of Belyi cuspidalization over number fields and mixed characteristic local fields may be thought of as corresponding to the Lefschetz trace formula for [powers of] the Frobenius morphism on a curve over a finite field, i.e.,

Belyi cuspidalization $\longleftrightarrow$ Lefschetz trace formula for Frobenius

[Here, we note in passing that the this correspondence may be related to the correspondence discussed in [AbsTopIII], §I5, between Belyi cuspidalization and the

Verschiebung on positive characteristic indigenous bundles by considering the geometry of Hecke correspondences modulo p, i.e., in essence, graphs of the Frobenius morphism in characteristic p!] It is the hope of the author that these analogies and correspondences might serve to stimulate further developments in the theory.

Inter-universal Teichmüller theory	*p-adic Teichmüller theory*
number field F	**hyperbolic curve** C over a *positive characteristic perfect field*
[**once-punctured**] **elliptic curve** X over F	*nilpotent ordinary* **indigenous bundle** P over C
Θ-link arrows of the *log-theta-lattice*	**mixed characteristic extension** structure of a ring of *Witt vectors*
$\mathfrak{log}$-link arrows of the *log-theta-lattice*	the **Frobenius morphism** in *positive characteristic*
the entire **log-theta-lattice**	the resulting **canonical lifting** + **canonical Frobenius action**; **canonical Frobenius lifting** over the ordinary locus
relatively straightforward *original construction of* **log-theta-lattice**	relatively straightforward *original construction of* **canonical liftings**
highly nontrivial *description of* **alien arithmetic** **holomorphic structure** via *absolute anabelian geometry*	highly nontrivial *absolute anabelian* *reconstruction of* **canonical liftings**

Fig. I4.1: Correspondence between inter-universal Teichmüller theory and p-adic Teichmüller theory

§I5. Other Galois-theoretic Approaches to Diophantine Geometry

The notion of **anabelian geometry** dates back to a famous *"letter to Faltings"* [cf. [Groth]], written by Grothendieck in response to Faltings' work on the *Mordell Conjecture* [cf. [Falt]]. Anabelian geometry was apparently originally conceived by Grothendieck as a new approach to obtaining results in **diophantine geometry** such as the Mordell Conjecture. At the time of writing, the author is not aware of any expositions by Grothendieck that expose this approach in detail. Nevertheless, it appears that the thrust of this approach revolves around applying the *Section Conjecture* for hyperbolic curves over number fields to obtain a contradiction by applying this Section Conjecture to the *"limit section"* of the Galois sections associated to any *infinite sequence of rational points* of a proper hyperbolic curve over a number field [cf. [MNT], §4.1(B), for more details]. On the other hand, to the knowledge of the author, at least at the time of writing, it does not appear that any rigorous argument has been obtained either by Grothendieck or by other mathematicians for deriving a new proof of the Mordell Conjecture from the [as yet unproven] Section Conjecture for hyperbolic curves over number fields. Nevertheless, one result that has been obtained is a new proof by M. Kim [cf. [Kim]] of *Siegel's theorem* concerning $\mathbb{Q}$-rational points of the projective line minus three points — a proof which proceeds by obtaining certain bounds on the cardinality of the set of Galois sections, *without* applying the Section Conjecture or any other results from anabelian geometry.

In light of the historical background just discussed, the theory exposed in the present series of papers — which yields, in particular, a method for applying results in **absolute anabelian geometry** to obtain **diophantine results** such as those given in [IUTchIV] — occupies a *somewhat curious position*, relative to the historical development of the mathematical ideas involved. That is to say, at a purely formal level, the implication

$$\textit{anabelian geometry} \quad \Longrightarrow \quad \textit{diophantine results}$$

at first glance looks something like a *"confirmation"* of Grothendieck's original intuition. On the other hand, closer inspection reveals that the approach of the theory of the present series of papers — that is to say, the **precise content** *of the relationship between anabelian geometry and diophantine geometry* established in the present series of papers — *differs quite fundamentally* from the sort of approach that was apparently envisioned by Grothendieck.

Perhaps the most characteristic aspect of this difference lies in the central role played by **anabelian geometry over** $\boldsymbol{p}$**-adic fields** in the present series of papers. That is to say, unlike the case with number fields, one central feature of anabelian geometry over p-adic fields is the *fundamental gap* between **relative** and **absolute** results [cf., e.g., [AbsTopI], Introduction]. This fundamental gap is closely related to the notion of an **"arithmetic Teichmüller theory for number fields"** [cf. the discussion of §I4 of the present paper; [AbsTopIII], §I3, §I5] — i.e., a theory of deformations *not* for the "arithmetic holomorphic structure" of a hyperbolic *curve* over a number field, but rather for the "arithmetic holomorphic structure" of the *number field itself*! To the knowledge of the author, there does not exist any mention of such ideas [i.e., relative vs. absolute p-adic anabelian geometry; the notion of an arithmetic Teichmüller theory for number fields] in the works of Grothendieck.

As discussed in §I4, one fundamental theme of the theory of the present series of papers is the issue of the

> *explicit description of the relationship between the* **additive** *structure and the* **multiplicative** *structure of a ring/number field/local field.*

Relative to the above discussion of the relationship between anabelian geometry and diophantine geometry, it is of interest to note that this issue of understanding/describing the relationship between *addition* and *multiplication* is, on the one hand, a central theme in the *proofs* of various results in *anabelian geometry* [cf., e.g., [Tama1], [pGC], [AbsTopIII]] and, on the other hand, a central aspect of the *diophantine results* obtained in [IUTchIV].

From a historical point of view, it is also of interest to note that results from absolute anabelian geometry are applied in the present series of papers in the context of the **canonical splittings** *of the Frobenius-picture* that arise by considering the *étale-picture* [cf. the discussion in §I1 preceding Theorem A]. This state of affairs is reminiscent — relative to the point of view that the Grothendieck Conjecture constitutes a sort of *"anabelian version"* of the *Tate Conjecture* for abelian varieties [cf. the discussion of [MNT], §1.2] — of the role played by the Tate Conjecture for abelian varieties in obtaining the *diophantine results* of [Falt], namely, by means of the various **semi-simplicity** properties of the Tate module that arise as formal consequences of the Tate Conjecture. That is to say, such semi-simplicity properties may also be thought of as *"canonical splittings"* that arise from Galois-theoretic considerations [cf. the discussion of "canonical splittings" in the final portion of [CombCusp], Introduction].

Certain aspects of the relationship between the inter-universal Teichmüller theory of the present series of papers and other Galois-theoretic approaches to diophantine geometry are best understood in the context of the **analogy**, discussed in §I4, between inter-universal Teichmüller theory and ***p*-adic Teichmüller theory**. One way to think of the starting point of p-adic Teichmüller is as an attempt to construct a *p-adic analogue* of the theory of the action of $SL_2(\mathbb{Z})$ on the *upper half-plane*, i.e., of the natural embedding

$$\rho_{\mathbb{R}}: \quad SL_2(\mathbb{Z}) \quad \hookrightarrow \quad SL_2(\mathbb{R})$$

of $SL_2(\mathbb{Z})$ as a discrete subgroup. This leads naturally to consideration of the *representation*

$$\rho_{\widehat{\mathbb{Z}}} = \prod_{p} \rho_{\mathbb{Z}_p}: \quad SL_2(\mathbb{Z})^\wedge \quad \to \quad SL_2(\widehat{\mathbb{Z}}) = \prod_{p \in \mathfrak{Primes}} SL_2(\mathbb{Z}_p)$$

— where we write $SL_2(\mathbb{Z})^\wedge$ for the *profinite completion* of $SL_2(\mathbb{Z})$. If one thinks of $SL_2(\mathbb{Z})^\wedge$ as the *geometric étale fundamental group* of the *moduli stack of elliptic curves* over a field of characteristic zero, then the p-adic Teichmüller theory of [pOrd], [pTeich] does indeed constitute a generalization of $\rho_{\mathbb{Z}_p}$ to more general p-adic hyperbolic curves.

From a **representation-theoretic** point of view, the next natural direction in which to further develop the theory of [pOrd], [pTeich] consists of attempting to generalize the theory of representations into $SL_2(\mathbb{Z}_p)$ obtained in [pOrd], [pTeich] to a theory concerning representations into $SL_n(\mathbb{Z}_p)$ for *arbitrary* $n \geq 2$. This is precisely the motivation that lies, for instance, behind the work of Joshi and Pauly [cf. [JP]].

On the other hand, unlike the *original motivating representation* $\rho_{\mathbb{R}}$, the representation $\rho_{\widehat{\mathbb{Z}}}$ is *far from injective*, i.e., put another way, the so-called *Congruence Subgroup Problem* fails to hold in the case of SL_2. This failure of injectivity means that working with

> $\rho_{\widehat{\mathbb{Z}}}$ *only allows one to access a* **relatively limited portion** *of* $SL_2(\mathbb{Z})^\wedge$.

From this point of view, a more natural direction in which to further develop the theory of [pOrd], [pTeich] is to consider the **"anabelian version"**

$$\rho_\Delta : \quad SL_2(\mathbb{Z})^\wedge \quad \to \quad \mathrm{Out}(\Delta_{1,1})$$

of $\rho_{\widehat{\mathbb{Z}}}$ — i.e., the natural outer representation on the *geometric étale fundamental group* $\Delta_{1,1}$ of the tautological family of *once-punctured elliptic curves* over the moduli stack of elliptic curves over a field of characteristic zero. Indeed, unlike the case with $\rho_{\widehat{\mathbb{Z}}}$, one knows [cf. [Asada]] that ρ_Δ is **injective**. Thus, the **"arithmetic Teichmüller theory** *for* **number fields** *equipped with a* **[once-punctured] elliptic curve"** constituted by the inter-universal Teichmüller theory developed in the present series of papers may [cf. the discussion of §I4!] be regarded as a realization of this sort of "anabelian" approach to further developing the p-adic Teichmüller theory of [pOrd], [pTeich].

In the context of these two distinct possible directions for the further development of the p-adic Teichmüller theory of [pOrd], [pTeich], it is of interest to recall the following elementary fact:

> If G is a *free pro-p group* of rank ≥ 2, then a [continuous] *representation*
>
> $$\rho_G : \ G \ \to \ GL_n(\mathbb{Q}_p)$$
>
> *can never be injective*!

Indeed, assume that ρ_G is injective and write $\ldots \subseteq H_j \subseteq \ldots \subseteq \mathrm{Im}(\rho_G) \subseteq GL_n(\mathbb{Q}_p)$ for an exhaustive sequence of open normal subgroups of the image of ρ_G. Then since the H_j are closed subgroups $GL_n(\mathbb{Q}_p)$, hence p-adic Lie groups, it follows that the $\mathbb{Q}_p$-*dimension* $\dim(H_j^{\mathrm{ab}} \otimes \mathbb{Q}_p)$ of the tensor product with $\mathbb{Q}_p$ of the abelianization of H_j may be computed at the level of *Lie algebras*, hence is bounded by the $\mathbb{Q}_p$-dimension of the p-adic Lie group $GL_n(\mathbb{Q}_p)$, i.e., we have $\dim(H_j^{\mathrm{ab}} \otimes \mathbb{Q}_p) \leq n^2$, in contradiction to the well-known fact since $G \cong \mathrm{Im}(\rho_G)$ is *free pro-p of rank* ≥ 2, it holds that $\dim(H_j^{\mathrm{ab}} \otimes \mathbb{Q}_p) \to \infty$ as $j \to \infty$. Note, moreover, that

> this sort of argument — i.e., concerning the **asymptotic behavior** *of abelianizations of open subgroups* — is *characteristic* of the sort of proofs

> that typically occur in **anabelian geometry** [cf., e.g., the proofs of [Tama1], [pGC], [CombGC]!].

On the other hand, the fact that ρ_G can never be injective shows that

> so long as one restricts oneself to **representation theory** *into* $GL_n(\mathbb{Q}_p)$ *for a* **fixed** n, one can never access the sort of *asymptotic phenomena* that form the *"technical core"* [cf., e.g., the proofs of [Tama1], [pGC], [CombGC]!] of various important results in anabelian geometry.

Put another way, the two "directions" discussed above — i.e., **representation-theoretic** and **anabelian** — appear to be **essentially mutually alien** to one another.

In this context, it is of interest to observe that the *diophantine results* derived in [IUTchIV] from the inter-universal Teichmüller theory developed in the present series of papers concern essentially **asymptotic behavior**, i.e., they do not concern properties of "a specific rational point over a specific number field", but rather properties of the asymptotic behavior of *"varying rational points over varying number fields"*. One important aspect of this asymptotic nature of the diophantine results derived in [IUTchIV] is that there are **no distinguished number fields** that occur in the theory, i.e., the theory — being *essentially asymptotic* in nature! — is "invariant" with respect to passing to finite extensions of the number field involved [which, from the point of view of the *absolute Galois group* $G_{\mathbb{Q}}$ of $\mathbb{Q}$, corresponds precisely to passing to *smaller and smaller open subgroups*, as in the above discussion!]. This contrasts sharply with the *"representation-theoretic approach to diophantine geometry"* constituted by such works as [Wiles], where *specific rational points over the specific number field* $\mathbb{Q}$ — or, for instance, in generalizations of [Wiles] involving Shimura varieties, over *specific number fields* characteristically associated to the Shimura varieties involved — play a central role.

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Section 0: Notations and Conventions

Monoids and Categories:

We shall use the notation and terminology concerning *monoids* and *categories* of [FrdI], §0.

We shall refer to an isomorphic copy of some object as an *isomorph* of the object.

If $\mathcal{C}$ and $\mathcal{D}$ are *categories*, then we shall refer to as an *isomorphism* $\mathcal{C} \to \mathcal{D}$ any isomorphism class of equivalences of categories $\mathcal{C} \to \mathcal{D}$. [Note that this termniology *differs* from the standard terminology of category theory, but will be *natural in the context of the theory of the present series of papers.*] Thus, from the point of view of *"coarsifications of* 2*-categories of* 1*-categories"* [cf. [FrdI], Appendix, Definition A.1, (ii)], an "isomorphism $\mathcal{C} \to \mathcal{D}$" is precisely an "isomorphism in the usual sense" of the [1-]category constituted by the coarsification of the 2-category of all small 1-categories relative to a suitable universe with respect to which $\mathcal{C}$ and $\mathcal{D}$ are small.

Let $\mathcal{C}$ be a *category*; $A, B \in \mathrm{Ob}(\mathcal{C})$. Then we define a *poly-morphism* $A \to B$ to be a collection of morphisms $A \to B$ [i.e., a subset of the set of morphisms $A \to B$]; if all of the morphisms in the collection are isomorphisms, then we shall refer to the poly-morphism as a *poly-isomorphism*; if $A = B$, then we shall refer to a poly-isomorphism $A \xrightarrow{\sim} B$ as a *poly-automorphism.* We define the *full poly-isomorphism* $A \xrightarrow{\sim} B$ to be the poly-morphism given by the collection of all isomorphisms $A \xrightarrow{\sim} B$.

Let $\mathcal{C}$ be a *category.* We define a *capsule of objects of* $\mathcal{C}$ to be a finite collection $\{A_j\}_{j\in J}$ [i.e., where J is a finite index set] of objects A_j of $\mathcal{C}$; if $|J|$ denotes the cardinality of J, then we shall refer to a capsule with index set J as a $|J|$*-capsule*; also, we shall write $\pi_0(\{A_j\}_{j\in J}) \stackrel{\mathrm{def}}{=} J$. A *morphism of capsules of objects of* $\mathcal{C}$

$$\{A_j\}_{j\in J} \to \{A'_{j'}\}_{j'\in J'}$$

is defined to consist of an injection $\iota : J \hookrightarrow J'$, together with, for each $j \in J$, a morphism $A_j \to A'_{j'}$ of objects of $\mathcal{C}$. Thus, the capsules of objects of $\mathcal{C}$ form a *category* $\mathrm{Capsule}(\mathcal{C})$. A *capsule-full poly-morphism*

$$\{A_j\}_{j\in J} \xrightarrow{\sim} \{A'_{j'}\}_{j'\in J'}$$

between two objects of $\mathrm{Capsule}(\mathcal{C})$ is defined to be a poly-morphism which arises as the poly-morphism of $\mathrm{Capsule}(\mathcal{C})$ determined by the *full poly-isomorphisms* $A_j \xrightarrow{\sim} A'_{\iota(j)}$ [where $j \in J$] between the constituent objects indexed by corresponding indices, relative to some *injection* $\iota : J \hookrightarrow J'$. A *capsule-full poly-isomorphism* is a capsule-full poly-morphism for which the associated injection between index sets is a *bijection.*

If X is a *connected noetherian algebraic stack* which is *generically scheme-like*, then we shall write

$$\mathcal{B}(X)$$

for the *category of finite étale coverings* of X [and morphisms over X]; if A is a *noetherian [commutative] ring [with unity]*, then we shall write $\mathcal{B}(A) \stackrel{\mathrm{def}}{=} \mathcal{B}(\mathrm{Spec}(A))$. Thus, [cf. [FrdI], §0] the subcategory of connected objects $\mathcal{B}(X)^0 \subseteq \mathcal{B}(X)$ may be thought of as the subcategory of *connected finite étale coverings* of X [and morphisms over X].

Let Π be a *topological group*. Then let us write

$$\mathcal{B}^{\mathrm{temp}}(\Pi)$$

for the *category* whose *objects* are *countable* [i.e., of cardinality $\leq$ the cardinality of the set of natural numbers], *discrete* sets equipped with a continuous Π-action and whose *morphisms* are morphisms of Π-sets [cf. [SemiAnbd], §3]. If Π may be written as an inverse limit of an inverse system of surjections of countable discrete topological groups, then we shall say that Π is *tempered* [cf. [SemiAnbd], Definition 3.1, (i)]. A category $\mathcal{C}$ equivalent to a category of the form $\mathcal{B}^{\mathrm{temp}}(\Pi)$, where Π is a tempered topological group, is called a *temperoid* [cf. [SemiAnbd], Definition 3.1, (ii)]. Thus, if $\mathcal{C}$ is a temperoid, then $\mathcal{C}$ is naturally equivalent to $(\mathcal{C}^0)^{\top}$ [cf. [FrdI], §0]. Moreover, one can *reconstruct the topological group* Π*, up to inner automorphism, category-theoretically from* $\mathcal{B}^{\mathrm{temp}}(\Pi)$ *or* $\mathcal{B}^{\mathrm{temp}}(\Pi)^0$ [i.e., the subcategory of connected objects of $\mathcal{B}^{\mathrm{temp}}(\Pi)$]; in particular, for any temperoid $\mathcal{C}$, it makes sense to write

$$\pi_1(\mathcal{C}), \quad \pi_1(\mathcal{C}^0)$$

for the *topological groups*, up to inner automorphism, obtained by applying this reconstruction algorithm [cf. [SemiAnbd], Remark 3.2.1].

In this context, if $\mathcal{C}_1$, $\mathcal{C}_2$ are *temperoids*, then it is natural to define a *morphism*

$$\mathcal{C}_1 \to \mathcal{C}_2$$

to be an isomorphism class of functors $\mathcal{C}_2 \to \mathcal{C}_1$ that preserves finite limits and countable colimits. [Note that this differs — but only slightly! — from the definition given in [SemiAnbd], Definition 3.1, (iii).] In a similar vein, we define a *morphism*

$$\mathcal{C}_1^0 \to \mathcal{C}_2^0$$

to be a morphism $(\mathcal{C}_1^0)^{\top} \to (\mathcal{C}_2^0)^{\top}$ [where we recall that we have natural equivalences of categories $\mathcal{C}_i \stackrel{\sim}{\to} (\mathcal{C}_i^0)^{\top}$ for $i = 1, 2$]. One verifies immediately that an *"isomorphism"* relative to this terminology is equivalent to an "isomorphism of categories" in the sense defined at the beginning of the present discussion of **"Monoids and Categories"**. Finally, if Π_1, Π_2 are tempered topological groups, then we recall that there is a *natural bijective correspondence* between

(a) the set of continuous outer homomorphisms $\Pi_1 \to \Pi_2$,

(b) the set of morphisms $\mathcal{B}^{\mathrm{temp}}(\Pi_1) \to \mathcal{B}^{\mathrm{temp}}(\Pi_2)$, and

(c) the set of morphisms $\mathcal{B}^{\mathrm{temp}}(\Pi_1)^0 \to \mathcal{B}^{\mathrm{temp}}(\Pi_2)^0$

— cf. [SemiAnbd], Proposition 3.2.

Suppose that for $i = 1, 2$, $\mathcal{C}_i$ and $\mathcal{C}'_i$ are *categories*. Then we shall say that two isomorphism classes of functors $\phi : \mathcal{C}_1 \to \mathcal{C}_2$, $\phi' : \mathcal{C}'_1 \to \mathcal{C}'_2$ are *abstractly equivalent* if, for $i = 1, 2$, there exist isomorphisms $\alpha_i : \mathcal{C}_i \xrightarrow{\sim} \mathcal{C}'_i$ such that $\phi' \circ \alpha_1 = \alpha_2 \circ \phi$. We shall also apply this terminology to morphisms between temperoids, as well as to morphisms between subcategories of connected objects of temperoids.

Numbers:

We shall use the abbreviations NF ("number field"), MLF ("mixed-characteristic [nonarchimedean] local field"), CAF ("complex archimedean field"), RAF ("real archimedean field"), AF ("archimedean field") as defined in [AbsTopI], §0; [AbsTopIII], §0. We shall denote the *set of prime numbers* by $\mathfrak{Primes}$.

Let F be a *number field* [i.e., a finite extension of the field of rational numbers]. Then we shall write

$$\mathbb{V}(F) \;=\; \mathbb{V}(F)^{\mathrm{arc}} \bigcup \mathbb{V}(F)^{\mathrm{non}}$$

for the set of *valuations* of F, i.e., the union of the sets of *archimedean* [i.e., $\mathbb{V}(F)^{\mathrm{arc}}$] and *nonarchimedean* valuation [i.e., $\mathbb{V}(F)^{\mathrm{non}}$] of F. Let $v \in \mathbb{V}(F)$. Then we shall write F_v for the *completion* of F at v; if, moreover, L is any [possibly infinite] *Galois extension* of F, then, by a slight abuse of notation, we shall write L_v for the *completion* of L at some valuation $\in \mathbb{V}(L)$ that lies over v. If $v \in \mathbb{V}(F)^{\mathrm{non}}$, then we shall write p_v for the *residue characteristic* of v. If $v \in \mathbb{V}(F)^{\mathrm{arc}}$, then we shall write $p_v \in F_v$ for the unique positive real element of F_v whose *natural logarithm is equal to* 1 [i.e., "$e = 2.71828\ldots$"]. By passing to appropriate *projective* or *inductive* limits, we shall also apply the notation "$\mathbb{V}(F)$", "F_v", "p_v" in situations where "F" is an *infinite extension* of $\mathbb{Q}$.

Curves:

We shall use the terms *hyperbolic curve*, *cusp*, *stable log curve*, and *smooth log curve* as they are defined in [SemiAnbd], §0. We shall use the term *hyperbolic orbicurve* as it is defined in [Cusp], §0.

Section 1: Complements on Coverings of Punctured Elliptic Curves

In the present §1, we discuss certain routine complements — which will be of use in the present series of papers — to the theory of *coverings of once-punctured elliptic curves*, as developed in [EtTh], §2.

Let $l \geq 5$ be an *integer prime to* 6; X a *hyperbolic curve of type* $(1,1)$ over a *field* k *of characteric zero*; $\underline{C}$ a *hyperbolic orbicurve of type* $(1, l\text{-tors})_{\pm}$ [cf. [EtTh], Definition 2.1] over k, whose k*-core* [cf. [CanLift], Remark 2.1.1; [EtTh], the discussion at the beginning of §2] also forms a k-core of X. Thus, $\underline{C}$ determines, up to k-isomorphism, a *hyperbolic orbicurve* $\underline{X}$ *of type* $(1, l\text{-tors})$ [cf. [EtTh], Definition 2.1] over k. Moreover, if we write G_k for the *absolute Galois group of* k [relative to an appropriate choice of basepoint], $\Pi_{(-)}$ for the *arithmetic fundamental group* of a geometrically connected, geometrically normal, generically scheme-like k-algebraic stack of finite type "$(-)$" [i.e., the étale fundamental group $\pi_1((-))$], and $\Delta_{(-)}$ for the *geometric fundamental group* of "$(-)$" [i.e., the kernel of the natural surjection $\Pi_{(-)} \twoheadrightarrow G_k$], then we obtain *natural cartesian diagrams*

$$\begin{array}{ccc} \underline{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \underline{C} & \longrightarrow & C \end{array} \qquad \begin{array}{ccc} \Pi_{\underline{X}} & \longrightarrow & \Pi_X \\ \downarrow & & \downarrow \\ \Pi_{\underline{C}} & \longrightarrow & \Pi_C \end{array} \qquad \begin{array}{ccc} \Delta_{\underline{X}} & \longrightarrow & \Delta_X \\ \downarrow & & \downarrow \\ \Delta_{\underline{C}} & \longrightarrow & \Delta_C \end{array}$$

of finite étale coverings of hyperbolic orbicurves and open immersions of profinite groups. Finally, we let us make the following assumption:

($*$) The natural action of G_k on $\Delta^{\mathrm{ab}}_{\underline{X}} \otimes (\mathbb{Z}/l\mathbb{Z})$ [where the superscript "ab" denotes the abelianization] is *trivial*.

Next, let $\underline{\epsilon}$ be a *nonzero cusp* of $\underline{C}$ — i.e., a cusp that arises from a *nonzero element* of the quotient "Q" that appears in the definition of a "hyperbolic orbicurve of type $(1, l\text{-tors})_{\pm}$" given in [EtTh], Definition 2.1. Write $\underline{\epsilon}^0$ for the unique *"zero cusp"* [i.e., "non-nonzero cusp"] of $\underline{X}$; $\underline{\epsilon}'$, $\underline{\epsilon}''$ for the two cusps of $\underline{X}$ that lie over $\underline{\epsilon}$; and

$$\Delta_{\underline{X}} \twoheadrightarrow \Delta^{\mathrm{ab}}_{\underline{X}} \otimes (\mathbb{Z}/l\mathbb{Z}) \twoheadrightarrow \Delta_{\underline{\epsilon}}$$

for the quotient of $\Delta^{\mathrm{ab}}_{\underline{X}} \otimes (\mathbb{Z}/l\mathbb{Z})$ by the images of the *inertia groups of all nonzero cusps* $\neq \underline{\epsilon}', \underline{\epsilon}''$ *of* $\underline{X}$. Thus, we obtain a *natural exact sequence*

$$0 \longrightarrow I_{\underline{\epsilon}'} \times I_{\underline{\epsilon}''} \longrightarrow \Delta_{\underline{\epsilon}} \longrightarrow \Delta_{\underline{E}} \otimes (\mathbb{Z}/l\mathbb{Z}) \longrightarrow 0$$

— where we write $\underline{E}$ for the genus one compactification of $\underline{X}$, and $I_{\underline{\epsilon}'}$, $I_{\underline{\epsilon}''}$ for the respective images in $\Delta_{\underline{\epsilon}}$ of the inertia groups of the cusps $\underline{\epsilon}'$, $\underline{\epsilon}''$ [so we have *noncanonical* isomorphisms $I_{\underline{\epsilon}'} \cong \mathbb{Z}/l\mathbb{Z} \cong I_{\underline{\epsilon}''}$].

Next, let us observe that G_k, $\mathrm{Gal}(\underline{X}/\underline{C})$ ($\cong \mathbb{Z}/2\mathbb{Z}$) *act naturally on the above exact sequence*. Write $\iota \in \mathrm{Gal}(\underline{C}/\underline{C})$ for the unique nontrivial element. Then ι induces an isomorphism $I_{\underline{\epsilon}'} \cong I_{\underline{\epsilon}''}$; if we use this isomorphism to identify $I_{\underline{\epsilon}'}$, $I_{\underline{\epsilon}''}$, then one verifies immediately that ι acts on the term "$I_{\underline{\epsilon}'} \times I_{\underline{\epsilon}''}$" of the above exact sequence by *switching the two factors*. Moreover, one verifies immediately that ι

acts on $\Delta_{\underline{E}} \otimes (\mathbb{Z}/l\mathbb{Z})$ via multiplication by -1. In particular, since l is *odd*, it follows that the action by ι on $\Delta_{\underline{\epsilon}}$ determines a *decomposition into eigenspaces*

$$\Delta_{\underline{\epsilon}} \xrightarrow{\sim} \Delta^+_{\underline{\epsilon}} \times \Delta^-_{\underline{\epsilon}}$$

— i.e., where ι acts on $\Delta^+_{\underline{\epsilon}}$ (respectively, $\Delta^-_{\underline{\epsilon}}$) by multiplication by $+1$ (respectively, -1). Moreover, the natural composite maps

$$I_{\underline{\epsilon}'} \hookrightarrow \Delta_{\underline{\epsilon}} \twoheadrightarrow \Delta^+_{\underline{\epsilon}}; \quad I_{\underline{\epsilon}''} \hookrightarrow \Delta_{\underline{\epsilon}} \twoheadrightarrow \Delta^+_{\underline{\epsilon}}$$

determine *isomorphisms* $I_{\underline{\epsilon}'} \xrightarrow{\sim} \Delta^+_{\underline{\epsilon}}$, $I_{\underline{\epsilon}''} \xrightarrow{\sim} \Delta^+_{\underline{\epsilon}}$. Since the natural action of G_k on $\Delta_{\underline{\epsilon}}$ clearly *commutes* with the action of ι, we thus conclude that the quotient $\Delta_{\underline{X}} \twoheadrightarrow \Delta_{\underline{\epsilon}} \twoheadrightarrow \Delta^+_{\underline{\epsilon}}$ determines quotients

$$\Pi_{\underline{X}} \twoheadrightarrow J_{\underline{X}}; \quad \Pi_{\underline{C}} \twoheadrightarrow J_{\underline{C}}$$

— where the surjections $\Pi_{\underline{X}} \twoheadrightarrow G_k$, $\Pi_{\underline{C}} \twoheadrightarrow G_k$ induce natural exact sequences $1 \to \Delta^+_{\underline{\epsilon}} \to J_{\underline{X}} \to G_k \to 1$, $1 \to \Delta^+_{\underline{\epsilon}} \times \mathrm{Gal}(\underline{X}/\underline{C}) \to J_{\underline{C}} \to G_k \to 1$; we have a natural inclusion $J_{\underline{X}} \hookrightarrow J_{\underline{C}}$.

Next, let us consider the cusp "$2\underline{\epsilon}$" of $\underline{C}$ — i.e., the cusp whose inverse images in $\underline{X}$ correspond to the points of $\underline{E}$ obtained by multiplying $\underline{\epsilon}'$, $\underline{\epsilon}''$ by 2, relative to the group law of the elliptic curve determined by the pair $(\underline{X}, \epsilon^0)$. Since $2 \neq \pm 1 \pmod{l}$ [a consequence of our assumption that $l \geq 5$], it follows that the *decomposition group* associated to this cusp "$2\underline{\epsilon}$" determines a *section*

$$\sigma : G_k \to J_{\underline{C}}$$

of the natural surjection $J_{\underline{C}} \twoheadrightarrow G_k$. Here, we note that although, a priori, σ is only determined by $2\underline{\epsilon}$ up to composition with an inner automorphism of $J_{\underline{C}}$ determined by an element of $\Delta^+_{\underline{\epsilon}} \times \mathrm{Gal}(\underline{X}/\underline{C})$, in fact, since [in light of the assumption $(*)$!] the natural [outer] action of G_k on $\Delta^+_{\underline{\epsilon}} \times \mathrm{Gal}(\underline{X}/\underline{C})$ is *trivial*, we conclude that σ is *completely determined* by $2\underline{\epsilon}$, and that the subgroup $\mathrm{Im}(\sigma) \subseteq J_{\underline{C}}$ determined by the image of σ is *normal* in $J_{\underline{C}}$. Moreover, by considering the decomposition groups associated to the cusps of $\underline{X}$ lying over $2\underline{\epsilon}$, we conclude that $\mathrm{Im}(\sigma)$ lies inside the subgroup $J_{\underline{X}} \subseteq J_{\underline{C}}$. Thus, the subgroups $\mathrm{Im}(\sigma) \subseteq J_{\underline{X}}$, $\mathrm{Im}(\sigma) \times \mathrm{Gal}(\underline{X}/\underline{C}) \subseteq J_{\underline{C}}$ determine [the horizontal arrows in] *cartesian diagrams*

$$\begin{array}{ccc} \underrightarrow{X} & \longrightarrow & \underline{X} \\ \downarrow & & \downarrow \\ \underrightarrow{C} & \longrightarrow & \underline{C} \end{array} \qquad \begin{array}{ccc} \Pi_{\underrightarrow{X}} & \longrightarrow & \Pi_{\underline{X}} \\ \downarrow & & \downarrow \\ \Pi_{\underrightarrow{C}} & \longrightarrow & \Pi_{\underline{C}} \end{array} \qquad \begin{array}{ccc} \Delta_{\underrightarrow{X}} & \longrightarrow & \Delta_{\underline{X}} \\ \downarrow & & \downarrow \\ \Delta_{\underrightarrow{C}} & \longrightarrow & \Delta_{\underline{C}} \end{array}$$

of *finite étale cyclic coverings* of hyperbolic orbicurves and open immersions [with normal image] of profinite groups; we have $\mathrm{Gal}(\underrightarrow{C}/\underline{C}) \cong \mathbb{Z}/l\mathbb{Z}$, $\mathrm{Gal}(\underline{X}/\underline{C}) \cong \mathbb{Z}/2\mathbb{Z}$, and $\mathrm{Gal}(\underrightarrow{X}/\underline{C}) \xrightarrow{\sim} \mathrm{Gal}(\underline{X}/\underline{C}) \times \mathrm{Gal}(\underrightarrow{C}/\underline{C}) \cong \mathbb{Z}/2l\mathbb{Z}$.

Definition 1.1. We shall refer to a hyperbolic orbicurve over k that arises, up to isomorphism, as the hyperbolic orbicurve $\underrightarrow{X}$ (respectively, $\underrightarrow{C}$) constructed above for some choice of l, $\underline{\epsilon}$ as being *of type* $(1, l\text{-}\underrightarrow{\mathrm{tors}})$ (respectively, $(1, l\text{-}\underrightarrow{\mathrm{tors}})_\pm$).

Remark 1.1.1. The arrow "$\rightarrow$" in the notation "$\underset{\rightarrow}{X}$", "$\underrightarrow{X}$", "$(1, l\text{-}\underrightarrow{\text{tors}})$", "$(1, l\text{-}\underrightarrow{\text{tors}})_{\pm}$" may be thought of as denoting the *"archimedean, ordered labels* $1, 2, \ldots$ *"* — i.e., determined by the *choice of* $\underline{\epsilon}$*!* — on the $\{\pm 1\}$-orbits of elements of the quotient "Q" that appears in the definition of a "hyperbolic orbicurve of type $(1, l\text{-tors})_{\pm}$" given in [EtTh], Definition 2.1.

Remark 1.1.2. We observe that $\underrightarrow{X}$, $\underrightarrow{C}$ are *completely determined, up to* k*-isomorphism, by the data* $(X/k,\ \underline{C},\ \underline{\epsilon})$.

Corollary 1.2. **(Characteristic Nature of Coverings)** *Suppose that* k *is an NF or an MLF. Then there exists a* **functorial group-theoretic algorithm** *[cf. [AbsTopIII], Remark 1.9.8, for more on the meaning of this terminology] to reconstruct*

$$\Pi_{\underline{X}},\ \Pi_{\underrightarrow{C}},\ \Pi_{\underline{C}}\ \textit{(respectively, } \Pi_{\underline{C}}\textit{)}$$

together with the conjugacy classes of the decomposition group(s) determined by the set(s) of cusps $\{\underline{\epsilon}', \underline{\epsilon}''\}$*;* $\{\underline{\epsilon}\}$ *(respectively,* $\{\underline{\epsilon}\}$*) from* $\Pi_{\underrightarrow{X}}$ *(respectively,* $\Pi_{\underrightarrow{C}}$*). Here, the asserted functoriality is with respect to isomorphisms of topological groups; we reconstruct* $\Pi_{\underline{X}}$*,* $\Pi_{\underline{C}}$ *(respectively,* $\Pi_{\underline{C}}$*) as a subgroup of* $\mathrm{Aut}(\Pi_{\underrightarrow{X}})$ *(respectively,* $\mathrm{Aut}(\Pi_{\underrightarrow{C}})$*).*

Proof. For simplicity, we consider the non-resp'd case; the resp'd case is entirely similar [but slightly easier]. The argument is similar to the arguments applied in [EtTh], Proposition 1.8; [EtTh], Proposition 2.4. First, we recall that $\Pi_{\underrightarrow{X}}$, $\Pi_{\underline{X}}$, and $\Pi_{\underline{C}}$ are *slim* [cf., e.g., [AbsTopI], Proposition 2.3, (ii)], hence *embed naturally* into $\mathrm{Aut}(\Pi_{\underrightarrow{X}})$, and that one may recover the subgroup $\Delta_{\underrightarrow{X}} \subseteq \Pi_{\underrightarrow{X}}$ via the algorithms of [AbsTopI], Theorem 2.6, (v), (vi). Next, we recall that the algorithms of [AbsTopII], Corollary 3.3, (i), (ii) — which are applicable in light of [AbsTopI], Example 4.8 — allow one to reconstruct Π_C [together with the natural inclusion $\Pi_{\underline{C}} \hookrightarrow \Pi_C$], as well as the subgroups $\Delta_X \subseteq \Delta_C \subseteq \Pi_C$. In particular, l may be recovered via the formula $l^2 = [\Delta_X : \Delta_{\underline{X}}] \cdot [\Delta_{\underline{X}} : \Delta_{\underrightarrow{X}}] = [\Delta_X : \Delta_{\underrightarrow{X}}] = [\Delta_C : \Delta_{\underrightarrow{X}}]/2$. Next, let us set $H \stackrel{\text{def}}{=} \mathrm{Ker}(\Delta_X \twoheadrightarrow \Delta_X^{\text{ab}} \otimes (\mathbb{Z}/l\mathbb{Z}))$. Then $\Pi_{\underline{X}} \subseteq \Pi_C$ may be recovered via the [easily verified] equality of subgroups $\Pi_{\underline{X}} = \Pi_{\underrightarrow{X}} \cdot H$. The conjugacy classes of the decomposition groups of $\underline{\epsilon}^0$, $\underline{\epsilon}'$, $\underline{\epsilon}''$ in $\Pi_{\underline{X}}$ may be recovered as the *decomposition groups of cusps* [cf. [AbsTopI], Lemma 4.5] whose image in $\mathrm{Gal}(\underrightarrow{X}/\underline{X}) = \Pi_{\underline{X}}/\Pi_{\underrightarrow{X}}$ is *nontrivial*. Next, to reconstruct $\Pi_{\underline{C}} \subseteq \Pi_C$, it suffices to reconstruct the *splitting* of the surjection $\mathrm{Gal}(\underline{X}/C) = \Pi_C/\Pi_{\underline{X}} \twoheadrightarrow \Pi_C/\Pi_X = \mathrm{Gal}(X/C)$ determined by $\mathrm{Gal}(\underline{X}/\underline{C}) = \Pi_{\underline{C}}/\Pi_{\underline{X}}$; but [since l is *prime to* $3!$] this splitting may be characterized [group-theoretically!] as the unique splitting that stabilizes the collection of conjugacy classes of subgroups of $\Pi_{\underline{X}}$ determined by the decomposition groups of $\underline{\epsilon}^0$, $\underline{\epsilon}'$, $\underline{\epsilon}''$. Now $\Pi_{\underrightarrow{C}} \subseteq \Pi_{\underline{C}}$ may be reconstructed by applying the observation that $(\mathbb{Z}/l\mathbb{Z} \cong)\ \mathrm{Gal}(\underrightarrow{X}/\underrightarrow{C}) \subseteq \mathrm{Gal}(\underrightarrow{X}/\underline{C})\ (\cong \mathbb{Z}/2l\mathbb{Z})$ is the *unique maximal subgroup of odd order*. Finally, the conjugacy classes of the decomposition groups of $\underline{\epsilon}'$, $\underline{\epsilon}''$ in $\Pi_{\underline{X}}$ may be recovered as the *decomposition groups of cusps* [cf. [AbsTopI], Lemma 4.5] whose image in $\mathrm{Gal}(\underrightarrow{X}/\underline{X}) = \Pi_{\underline{X}}/\Pi_{\underrightarrow{X}}$ is *nontrivial*, but which are *not fixed* [up to conjugacy] by the outer action of $\mathrm{Gal}(\underline{X}/\underline{C}) = \Pi_{\underline{C}}/\Pi_{\underline{X}}$ on $\Pi_{\underline{X}}$. This completes the proof of Corollary 1.2. ○

Remark 1.2.1. If follows immediately from Corollary 1.2 that

$$\mathrm{Aut}_k(\underset{\rightarrow}{X}) = \mathrm{Gal}(\underset{\rightarrow}{X}/\underline{C}) \ (\cong \mathbb{Z}/2l\mathbb{Z}); \quad \mathrm{Aut}_k(\underset{\rightarrow}{C}) = \mathrm{Gal}(\underset{\rightarrow}{C}/\underline{C}) \ (\cong \mathbb{Z}/l\mathbb{Z})$$

[cf. [EtTh], Remark 2.6.1].

Section 2: Complements on Tempered Coverings

In the present §2, we discuss certain routine complements — which will be of use in the present series of papers — to the theory of *tempered coverings of graphs of anabelioids*, as developed in [SemiAnbd], §3 [cf. also the closely related theory of [CombGC]].

Let Σ, $\widehat{\Sigma}$ be *nonempty sets of prime numbers* such that $\Sigma \subseteq \widehat{\Sigma}$;

$$\mathcal{G}$$

a *semi-graph of anabelioids of pro-*Σ *PSC-type* [cf. [CombGC], Definition 1.1, (i)], whose underlying graph we denote by $\mathbb{G}$. Write $\Pi^{\mathrm{tp}}_{\mathcal{G}}$ for the *tempered fundamental group* of $\mathcal{G}$ [cf. the discussion preceding [SemiAnbd], Proposition 3.6] and $\widehat{\Pi}_{\mathcal{G}}$ for the *pro-*$\widehat{\Sigma}$ [i.e., maximal pro-$\widehat{\Sigma}$ quotient of the profinite] *fundamental group* of $\mathcal{G}$ [cf. the discussion preceding [SemiAnbd], Definition 2.2] — both taken with respect to appropriate choices of basepoints. Thus, since discrete free groups of finite rank inject into their pro-l completions for any prime number l [cf., e.g., [RZ], Proposition 3.3.15], it follows that we have a *natural injection* [cf. [SemiAnbd], Proposition 3.6, (iii), when $\widehat{\Sigma} = \mathfrak{Primes}$; the proof in the case of arbitrary $\widehat{\Sigma}$ is entirely similar]

$$\Pi^{\mathrm{tp}}_{\mathcal{G}} \hookrightarrow \widehat{\Pi}_{\mathcal{G}}$$

that we shall use to regard $\Pi^{\mathrm{tp}}_{\mathcal{G}}$ as a *subgroup* of $\widehat{\Pi}_{\mathcal{G}}$ and $\widehat{\Pi}_{\mathcal{G}}$ as the *pro-*$\widehat{\Sigma}$ *completion* of $\Pi^{\mathrm{tp}}_{\mathcal{G}}$.

Next, let

$$\mathcal{H}$$

be the semi-graph of anabelioids associated to a **connected** *sub-semi-graph* $\mathbb{H} \subseteq \mathbb{G}$. One verifies immediately that the underlying *graph* of anabelioids associated to $\mathcal{H}$ coincides with the underlying graph of anabelioids associated to some semi-graph of anabelioids of pro-Σ PSC-type. That is to say, rough speaking, up to the possible omission of some of the cuspidal edges, $\mathcal{H}$ "is" a semi-graph of anabelioids of pro-Σ PSC-type. In particular, since the omission of cuspidal edges clearly does not affect either the tempered or pro-$\widehat{\Sigma}$ fundamental groups, we shall apply the notation introduced above for "$\mathcal{G}$" to $\mathcal{H}$. We thus obtain a *natural commutative diagram*

$$\begin{array}{ccc} \Pi^{\mathrm{tp}}_{\mathcal{H}} & \longrightarrow & \widehat{\Pi}_{\mathcal{H}} \\ \downarrow & & \downarrow \\ \Pi^{\mathrm{tp}}_{\mathcal{G}} & \longrightarrow & \widehat{\Pi}_{\mathcal{G}} \end{array}$$

of inclusions [cf. [SemiAnbd], Proposition 2.5, (i), when $\widehat{\Sigma} = \mathfrak{Primes}$; the proof in the case of arbitrary $\widehat{\Sigma}$ is entirely similar] *of topological groups*, which we shall use to regard all of the groups in the diagram as subgroups of $\widehat{\Pi}_{\mathcal{G}}$. In particular, one may think of $\Pi^{\mathrm{tp}}_{\mathcal{H}}$ (respectively, $\widehat{\Pi}_{\mathcal{H}}$) as the *decomposition subgroup* in $\Pi^{\mathrm{tp}}_{\mathcal{G}}$ (respectively, $\widehat{\Pi}_{\mathcal{G}}$) associated to the sub-semi-graph $\mathcal{H}$.

The following result is the *central technical result* underlying the theory of the present §2.

Proposition 2.1. (Profinite Conjugates of Nontrivial Compact Subgroups) *In the notation of the above discussion, let $\Lambda \subseteq \Pi_{\mathcal{G}}^{\mathrm{tp}}$ be a* **nontrivial compact subgroup**, *$\gamma \in \widehat{\Pi}_{\mathcal{G}}$ an element such that $\gamma \cdot \Lambda \cdot \gamma^{-1} \subseteq \Pi_{\mathcal{G}}^{\mathrm{tp}}$ [or, equivalently, $\Lambda \subseteq \gamma^{-1} \cdot \Pi_{\mathcal{G}}^{\mathrm{tp}} \cdot \gamma$]. Then $\gamma \in \Pi_{\mathcal{G}}^{\mathrm{tp}}$.*

Proof. Write $\widehat{\Gamma}$ for the "pro-$\widehat{\Sigma}$ semi-graph" associated to the *universal pro-$\widehat{\Sigma}$ étale covering* of $\mathcal{G}$ [i.e., the covering corresponding to the subgroup $\{1\} \subseteq \widehat{\Pi}_{\mathcal{G}}$]; Γ^{tp} for the "pro-semi-graph" associated to the *universal tempered covering* of $\mathcal{G}$ [i.e., the covering corresponding to the subgroup $\{1\} \subseteq \Pi_{\mathcal{G}}^{\mathrm{tp}}$]. Thus, we have a natural dense map $\Gamma^{\mathrm{tp}} \to \widehat{\Gamma}$. Let us refer to a ["pro-"]vertex of $\widehat{\Gamma}$ that occurs as the image of a ["pro-"]vertex of Γ^{tp} as *tempered*. Since Λ, $\gamma \cdot \Lambda \cdot \gamma^{-1}$ are *compact* subgroups of $\Pi_{\mathcal{G}}^{\mathrm{tp}}$, it follows from [SemiAnbd], Theorem 3.7, (iii) [cf. also [SemiAnbd], Example 3.10], that there exist *verticial subgroups* $\Lambda', \Lambda'' \subseteq \Pi_{\mathcal{G}}^{\mathrm{tp}}$ such that $\Lambda \subseteq \Lambda'$, $\gamma \cdot \Lambda \cdot \gamma^{-1} \subseteq \Lambda''$. Thus, Λ', Λ'' correspond to *tempered vertices* v', v'' of $\widehat{\Gamma}$; $\{1\} \neq \gamma \cdot \Lambda \cdot \gamma^{-1} \subseteq \gamma \cdot \Lambda' \cdot \gamma^{-1}$, so $(\gamma \cdot \Lambda' \cdot \gamma^{-1}) \bigcap \Lambda'' \neq \{1\}$. Since Λ'', $\gamma \cdot \Lambda' \cdot \gamma^{-1}$ are both verticial subgroups of $\widehat{\Pi}_{\mathcal{G}}$, it thus follows either from [AbsTopII], Proposition 1.3, (iv), or from [NodNon], Proposition 3.9, (i), that the corresponding vertices v'', $(v')^{\gamma}$ of $\widehat{\Gamma}$ are either *equal* or *adjacent*. In particular, since v'' is *tempered*, we thus conclude that $(v')^{\gamma}$ is *tempered*. Thus, v', $(v')^{\gamma}$ are tempered, so $\gamma \in \Pi_{\mathcal{G}}^{\mathrm{tp}}$, as desired. ○

Next, relative to the notation "C", "N" and related terminology concerning *commensurators* and *normalizers* discussed, for instance, in [SemiAnbd], §0; [CombGC], §0, we have the following result.

Proposition 2.2. (Commensurators of Decomposition Subgroups Associated to Sub-semi-graphs) *In the notation of the above discussion, $\widehat{\Pi}_{\mathcal{H}}$ (respectively, $\Pi_{\mathcal{H}}^{\mathrm{tp}}$) is* **commensurably terminal** *in $\widehat{\Pi}_{\mathcal{G}}$ (respectively, $\widehat{\Pi}_{\mathcal{G}}$ [hence, also in $\Pi_{\mathcal{G}}^{\mathrm{tp}}$]). In particular, $\Pi_{\mathcal{G}}^{\mathrm{tp}}$ is commensurably terminal in $\widehat{\Pi}_{\mathcal{G}}$.*

Proof. First, let us observe that by allowing, in Proposition 2.1, Λ to range over the open subgroups of any verticial [hence, in particular, *nontrivial compact*!] subgroup of $\Pi_{\mathcal{G}}^{\mathrm{tp}}$, it follows from Proposition 2.1 that

$$\Pi_{\mathcal{G}}^{\mathrm{tp}} \text{ is } \textit{commensurably terminal} \text{ in } \widehat{\Pi}_{\mathcal{G}}$$

— cf. Remark 2.2.2 below. In particular, by applying this fact to $\mathcal{H}$ [cf. the discussion preceding Proposition 2.1], we conclude that $\Pi_{\mathcal{H}}^{\mathrm{tp}}$ is *commensurably terminal* in $\widehat{\Pi}_{\mathcal{H}}$. Next, let us observe that it is immediate from the definitions that

$$\Pi_{\mathcal{H}}^{\mathrm{tp}} \subseteq C_{\Pi_{\mathcal{G}}^{\mathrm{tp}}}(\Pi_{\mathcal{H}}^{\mathrm{tp}}) \subseteq C_{\widehat{\Pi}_{\mathcal{G}}}(\Pi_{\mathcal{H}}^{\mathrm{tp}}) \subseteq C_{\widehat{\Pi}_{\mathcal{G}}}(\widehat{\Pi}_{\mathcal{H}})$$

[where we think of $\widehat{\Pi}_{\mathcal{H}}$, $\widehat{\Pi}_{\mathcal{G}}$, respectively, as the pro-$\widehat{\Sigma}$ completions of $\Pi_{\mathcal{H}}^{\mathrm{tp}}$, $\Pi_{\mathcal{G}}^{\mathrm{tp}}$]. On the other hand, by the evident pro-$\widehat{\Sigma}$ analogue of [SemiAnbd], Corollary 2.7, (i), we have $C_{\widehat{\Pi}_{\mathcal{G}}}(\widehat{\Pi}_{\mathcal{H}}) = \widehat{\Pi}_{\mathcal{H}}$. Thus, by the *commensurable terminality* of $\Pi^{\mathrm{tp}}_{\mathcal{H}}$ in $\widehat{\Pi}_{\mathcal{H}}$, we conclude that

$$\Pi^{\mathrm{tp}}_{\mathcal{H}} \subseteq C_{\widehat{\Pi}_{\mathcal{G}}}(\Pi^{\mathrm{tp}}_{\mathcal{H}}) \subseteq C_{\widehat{\Pi}_{\mathcal{H}}}(\Pi^{\mathrm{tp}}_{\mathcal{H}}) = \Pi^{\mathrm{tp}}_{\mathcal{H}}$$

— as desired. ○

Remark 2.2.1. It follows immediately from the theory of [SemiAnbd] [cf., e.g., [SemiAnbd], Corollary 2.7, (i)] that, in fact, Propositions 2.1 and 2.2 can be proven for much more general semi-graphs of anabelioids $\mathcal{G}$ than the sort of $\mathcal{G}$ that appears in the above discussion. We leave the routine details of such generalizations to the interested reader.

Remark 2.2.2. Recall that when $\widehat{\Sigma} = \mathfrak{Primes}$, the fact that

$$\Pi^{\mathrm{tp}}_{\mathcal{G}} \text{ is } \textit{normally terminal} \text{ in } \widehat{\Pi}_{\mathcal{G}}$$

may also be derived from the fact that any nonabelian finitely generated free group is *normally terminal* [cf. [André], Lemma 3.2.1; [SemiAnbd], Lemma 6.1, (i)] in its profinite completion. In particular, the proof of the commensurable terminality of $\Pi^{\mathrm{tp}}_{\mathcal{G}}$ in $\widehat{\Pi}_{\mathcal{G}}$ that is given in the proof of Proposition 2.2 may be thought of as a *new proof* of this normal terminality that *does not require one to invoke* [André], Lemma 3.2.1, which is essentially an immediate consequence of the rather difficult *conjugacy separability* result given in [Stb1], Theorem 1. This relation of Proposition 2.1 to the theory of [Stb1] is interesting in light of the *discrete analogue* given in Theorem 2.6 below of [the "tempered version of Theorem 2.6" constituted by] Proposition 2.4 [which is essentially a formal consequence of Proposition 2.1].

Now let k be an *MLF*, $\overline{k}$ an algebraic closure of k, $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$, X a *hyperbolic curve* over k that admits *stable reduction* over the ring of integers $\mathcal{O}_k$ of k. Write

$$\Pi^{\mathrm{tp}}_X, \quad \Delta^{\mathrm{tp}}_X$$

for the respective *"$\widehat{\Sigma}$-tempered"* quotients of the *tempered fundamental groups* $\pi_1^{\mathrm{tp}}(X)$, $\pi_1^{\mathrm{tp}}(X_{\overline{k}})$ [relative to suitable basepoints] of X, $X_{\overline{k}} \stackrel{\mathrm{def}}{=} X \times_k \overline{k}$ [cf. [André], §4; [SemiAnbd], Example 3.10], i.e., the quotients determined by the intersections of the kernels of all continuous surjections of $\pi_1^{\mathrm{tp}}(-)$ onto extensions of a finite group of order a product [possibly with multiplicities] of primes $\in \widehat{\Sigma}$ by a discrete free group of finite rank; write $\widehat{\Pi}_X$, $\widehat{\Delta}_X$ for the respective *pro-*$\widehat{\Sigma}$ [i.e., maximal pro-$\widehat{\Sigma}$ quotients of the profinite] *fundamental groups* of X, $X_{\overline{k}}$. Thus, since discrete free groups of finite rank inject into their pro-l completions for any prime number l [cf., e.g., [RZ], Proposition 3.3.15], we have *natural inclusions*

$$\Pi^{\mathrm{tp}}_X \hookrightarrow \widehat{\Pi}_X, \quad \Delta^{\mathrm{tp}}_X \hookrightarrow \widehat{\Delta}_X$$

[cf., e.g., [SemiAnbd], Proposition 3.6, (iii), when $\widehat{\Sigma} = \mathfrak{Primes}$]; $\widehat{\Pi}_X$, $\widehat{\Delta}_X$ may be identified with the *pro-*$\widehat{\Sigma}$ *completions* of Π^{tp}_X, Δ^{tp}_X.

Now suppose that the **residue characteristic** p of k is **not contained** in Σ; that the semi-graph of anabelioids $\mathcal{G}$ of the above discussion is the *pro-*Σ *semi-graph of anabelioids associated to the geometric special fiber of the stable model* $\mathcal{X}$ *of* X *over* $\mathcal{O}_k$ [cf., e.g., [SemiAnbd], Example 3.10]; and that the *sub-semi-graph* $\mathbb{H} \subseteq \mathbb{G}$ is *stabilized* by the natural action of G_k on $\mathbb{G}$. Thus, we have *natural surjections*

$$\Delta_X^{\mathrm{tp}} \twoheadrightarrow \Pi_{\mathcal{G}}^{\mathrm{tp}}; \quad \widehat{\Delta}_X \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}}$$

of topological groups.

Corollary 2.3. (Subgroups of Tempered Fundamental Groups Associated to Sub-semi-graphs) *In the notation of the above discussion:*

(i) The closed subgroups

$$\Delta_{X,\mathbb{H}}^{\mathrm{tp}} \stackrel{\mathrm{def}}{=} \Delta_X^{\mathrm{tp}} \times_{\Pi_{\mathcal{G}}^{\mathrm{tp}}} \Pi_{\mathcal{H}}^{\mathrm{tp}} \subseteq \Delta_X^{\mathrm{tp}}; \quad \widehat{\Delta}_{X,\mathbb{H}} \stackrel{\mathrm{def}}{=} \widehat{\Delta}_X \times_{\widehat{\Pi}_{\mathcal{G}}} \widehat{\Pi}_{\mathcal{H}} \subseteq \widehat{\Delta}_X$$

are **commensurably terminal**. *In particular, the natural outer actions of* G_k *on* Δ_X^{tp}, $\widehat{\Delta}_X$ *determine* **natural outer actions** *of* G_k *on* $\Delta_{X,\mathbb{H}}^{\mathrm{tp}}$, $\widehat{\Delta}_{X,\mathbb{H}}$.

(ii) The **closure** *of* $\Delta_{X,\mathbb{H}}^{\mathrm{tp}} \subseteq \Delta_X^{\mathrm{tp}} \subseteq \widehat{\Delta}_X$ *in* $\widehat{\Delta}_X$ *is equal to* $\widehat{\Delta}_{X,\mathbb{H}}$.

(iii) Suppose that [at least] one of the following conditions holds: (a) $\widehat{\Sigma}$ *contains a prime number* $l \notin \Sigma \bigcup \{p\}$*; (b)* $\widehat{\Sigma} = \mathfrak{Primes}$. *Then* $\widehat{\Delta}_{X,\mathbb{H}}$ *is* **slim**. *In particular, the natural outer actions of* G_k *on* $\Delta_{X,\mathbb{H}}^{\mathrm{tp}}$, $\widehat{\Delta}_{X,\mathbb{H}}$ *[cf. (i)] determine* **natural exact sequences of center-free topological groups** *[cf. (ii); the slimness of* $\widehat{\Delta}_{X,\mathbb{H}}$*; [AbsAnab], Theorem 1.1.1, (ii)]*

$$1 \to \Delta_{X,\mathbb{H}}^{\mathrm{tp}} \to \Pi_{X,\mathbb{H}}^{\mathrm{tp}} \to G_k \to 1$$

$$1 \to \widehat{\Delta}_{X,\mathbb{H}} \to \widehat{\Pi}_{X,\mathbb{H}} \to G_k \to 1$$

— where $\Pi_{X,\mathbb{H}}^{\mathrm{tp}}$, $\widehat{\Pi}_{X,\mathbb{H}}$ *are* **defined** *so as to render the sequences exact.*

(iv) Suppose that the hypothesis of (iii) holds. Then the images of the natural inclusions $\Pi_{X,\mathbb{H}}^{\mathrm{tp}} \hookrightarrow \Pi_X^{\mathrm{tp}}$, $\widehat{\Pi}_{X,\mathbb{H}} \hookrightarrow \widehat{\Pi}_X$ *are* **commensurably terminal**.

(v) We have: $\widehat{\Delta}_{X,\mathbb{H}} \bigcap \Delta_X^{\mathrm{tp}} = \Delta_{X,\mathbb{H}}^{\mathrm{tp}} \subseteq \widehat{\Delta}_X$.

(vi) Let

$$I_x \subseteq \Delta_X^{\mathrm{tp}} \text{ (respectively, } I_x \subseteq \widehat{\Delta}_X\text{)}$$

be an **inertia group** *associated to a cusp* x *of* X. *Write* ξ *for the cusp of the stable model* $\mathcal{X}$ *corresponding to* x. *Then the following conditions are equivalent:*

(a) I_x *lies in a* Δ_X^{tp}*- (respectively,* $\widehat{\Delta}_X$*-) conjugate of* $\Delta_{X,\mathbb{H}}^{\mathrm{tp}}$ *(respectively,* $\widehat{\Delta}_{X,\mathbb{H}}$*);*

(b) ξ *meets an irreducible component of the special fiber of* $\mathcal{X}$ *that is* **contained** *in* $\mathbb{H}$.

Proof. Assertion (i) follows immediately from Proposition 2.2. Assertion (ii) follows immediately from the definitions of the various tempered fundamental groups involved, together with the following elementary *observation*: If $G \twoheadrightarrow F$ is a surjection of finitely generated free discrete groups, which induces a surjection $\widehat{G} \twoheadrightarrow \widehat{F}$ between the respective profinite completions [so, by applying the well-known residual finiteness of free groups [cf., e.g., [SemiAnbd], Corollary 1.7], we think of G and F as subgroups of $\widehat{G}$ and $\widehat{F}$, respectively], then $H \stackrel{\mathrm{def}}{=} \mathrm{Ker}(G \twoheadrightarrow F)$ is *dense* in $\widehat{H} \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\widehat{G} \twoheadrightarrow \widehat{F})$, relative to the profinite topology of $\widehat{G}$. Indeed, let $\iota : F \hookrightarrow G$ be a *section* of the given surjection $G \twoheadrightarrow F$ [which exists since F is *free*]. Then if $\{g_i\}_{i \in \mathbb{N}}$ is a sequence of elements of G that converges, in the profinite topology of $\widehat{G}$, to a given element $h \in \widehat{H}$, and maps to a sequence of elements $\{f_i\}_{i \in \mathbb{N}}$ of F [which necessarily converges, in the profinite topology of $\widehat{F}$, to the *identity element* $1 \in \widehat{F}$], then one verifies immediately that $\{g_i \cdot \iota(f_i)^{-1}\}_{i \in \mathbb{N}}$ is a sequence of elements of H that converges, in the profinite topology of $\widehat{G}$, to h. This completes the proof of the *observation* and hence of assertion (ii).

Next, we consider assertion (iii). In the following, we give, in effect, *two distinct proofs* of the slimness of $\widehat{\Delta}_{X,\mathbb{H}}$: one is *elementary*, but requires one to assume that *condition (a)* holds; the other *depends on the highly nontrivial theory of [Tama2]* and requires one to assume that *condition (b)* holds. If condition (a) holds, then let us set $\Sigma^* \stackrel{\mathrm{def}}{=} \Sigma \bigcup \{l\}$. If condition (b) holds, but condition (a) does not hold [so $\widehat{\Sigma} = \mathfrak{Primes} = \Sigma \bigcup \{p\}$], then let us set $\Sigma^* \stackrel{\mathrm{def}}{=} \Sigma$. Thus, in either case, $p \notin \Sigma^* \supseteq \Sigma$.

Let $J \subseteq \widehat{\Delta}_X$ be an *open subgroup*. Write $J_{\mathbb{H}} \stackrel{\mathrm{def}}{=} J \bigcap \widehat{\Delta}_{X,\mathbb{H}}$; $J \twoheadrightarrow J^*$ for the *maximal pro-Σ^* quotient*; $J^*_{\mathbb{H}} \subseteq J^*$ for the image of $J_{\mathbb{H}}$ in J^*. Now suppose that $\alpha \in \widehat{\Delta}_{X,\mathbb{H}}$ *commutes* with $J_{\mathbb{H}}$. Let v be a vertex of the dual graph of the geometric special fiber of a stable model $\mathcal{X}_J$ of the covering X_J of $X_{\overline{k}}$ determined by J. Write $J_v \subseteq J$ for the decomposition group [well-defined up to conjugation in J] associated to v; $J^*_v \subseteq J^*$ for the image of J_v in J^*. Then let us observe that

> (†) there exists an open subgroup $J_0 \subseteq \widehat{\Delta}_X$ which is *independent* of J, v, and α such that if $J \subseteq J_0$, then for arbitrary v [and α] as above, it holds that $J^*_v \bigcap J^*_{\mathbb{H}}$ ($\subseteq J^*$) is *infinite* and *nonabelian*.

Indeed, if condition (a) holds, then it follows immediately from the definitions that the image of the homomorphism $J_v \subseteq J \subseteq \widehat{\Delta}_X \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}}$ is *pro-*Σ; in particular, since $l \notin \Sigma$, and $\mathrm{Ker}(J_v \subseteq J \subseteq \widehat{\Delta}_X \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}}) \subseteq J_v \bigcap J_{\mathbb{H}}$, it follows that $J_v \bigcap J_{\mathbb{H}}$, hence also $J^*_v \bigcap J^*_{\mathbb{H}}$, *surjects* onto the *maximal pro-l quotient* of J_v, which is isomorphic to the pro-l completion of the fundamental group of a hyperbolic Riemann surface, hence [as is well-known] is *infinite* and *nonabelian* [so we may take $J_0 \stackrel{\mathrm{def}}{=} \widehat{\Delta}_X$]. Suppose, on the other hand, that condition (b) holds, but condition (a) does *not* hold. Then it follows immediately from [Tama2], Theorem 0.2, (v), that, for an appropriate choice of J_0, if $J \subseteq J_0$, then every v corresponds to an irreducible component that either maps to a point in $\mathcal{X}$ or contains a node that maps to a smooth point of $\mathcal{X}$. In particular, it follows that for every choice of v, there exists at least one *pro-*Σ*, torsion-free, pro-cyclic subgroup* $F \subseteq J_v$ that *lies in* $\mathrm{Ker}(J_v \subseteq J \subseteq \widehat{\Delta}_X \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}}) \subseteq J_v \bigcap J_{\mathbb{H}}$ and, moreover, *maps injectively into* J^*. Thus, we obtain an injection

$F \hookrightarrow J_v^* \bigcap J_{\mathbb{H}}^*$; a similar statement holds when F is replaced by any J_v-conjugate of F. Moreover, it follows from the well-known structure of the pro-Σ completion of the fundamental group of a hyperbolic Riemann surface [such as J_v^*] that the image of such a group F *topologically normally generates* a closed subgroup of $J_v^* \bigcap J_{\mathbb{H}}^*$ which is *infinite* and *nonabelian*. This completes the proof of (†).

Next, let us observe that it follows by applying either [AbsTopII], Proposition 1.3, (iv), or [NodNon], Proposition 3.9, (i), to the various $\widehat{\Delta}_X$-conjugates in J^* of $J_v^* \bigcap J_{\mathbb{H}}^*$ as in (†) that the fact that α *commutes* with $J_v^* \bigcap J_{\mathbb{H}}^*$ implies that α *fixes* v. If condition (a) holds, then the fact that conjugation by α on the *maximal pro-l quotient* of J_v [which, as we saw above, is a quotient of $J_v^* \bigcap J_{\mathbb{H}}^*$] is *trivial* implies [cf. the argument concerning the inertia group "$I_v \subseteq D_v$" in the latter portion of the proof of [SemiAnbd], Corollary 3.11] that α not only fixes v, but also *acts trivially on the irreducible component of the special fiber of* $\mathcal{X}_J$ *determined by* v; since J and v as in (†) are *arbitrary*, we thus conclude that α is the *identity element*, as desired. Suppose, on the other hand, that condition (b) holds, but condition (a) does *not* hold. Then since J and v as in (†) are *arbitrary*, we thus conclude again from [Tama2], Theorem 0.2, (v), that α fixes not only v, but also *every closed point* on the irreducible component of the special fiber of $\mathcal{X}_J$ determined by v, hence that α *acts trivially on this irreducible component*. Again since J and v as in (†) are *arbitrary*, we thus conclude that α is the *identity element*, as desired. This completes the proof of assertion (iii). In light of the exact sequences of assertion (iii), assertion (iv) follows immediately from assertion (i). Assertion (vi) follows immediately from [CombGC], Proposition 1.5, (i), by passing to pro-Σ completions.

Finally, it follows immediately from the definitions of the various tempered fundamental groups involved that to verify assertion (v), it suffices to verify the following analogue of assertion (v) for a *nonabelian finitely generated free discrete group* G: for any finitely generated subgroup $F \subseteq G$, if we use the notation "$\wedge$" to denote the profinite completion, then $\widehat{F} \bigcap G = F$. But to verify this assertion concerning G, it follows immediately from [SemiAnbd], Corollary 1.6, (ii), that we may assume without loss of generality that the inclusion $F \subseteq G$ admits a *splitting* $G \twoheadrightarrow F$ [i.e., such that the composite $F \hookrightarrow G \twoheadrightarrow F$ is the identity on F], in which case the desired equality "$\widehat{F} \bigcap G = F$" follows immediately. This completes the proof of assertion (v), and hence of Corollary 2.3. ◯

Next, we observe the following *arithmetic analogue* of Proposition 2.1.

Proposition 2.4. (Profinite Conjugates of Nontrivial Arithmetic Compact Subgroups) *In the notation of the above discussion:*

(i) Let $\Lambda \subseteq \Delta_X^{\mathrm{tp}}$ *be a* **nontrivial pro-Σ compact subgroup**, $\gamma \in \widehat{\Pi}_X$ *an element such that* $\gamma \cdot \Lambda \cdot \gamma^{-1} \subseteq \Delta_X^{\mathrm{tp}}$ *[or, equivalently,* $\Lambda \subseteq \gamma^{-1} \cdot \Delta_X^{\mathrm{tp}} \cdot \gamma$*]. Then* $\gamma \in \Pi_X^{\mathrm{tp}}$.

(ii) Suppose that $\widehat{\Sigma} = \mathfrak{Primes}$. *Let* $\Lambda \subseteq \Pi_X^{\mathrm{tp}}$ *be a* **[nontrivial] compact subgroup** *whose image in* G_k *is* **open**, $\gamma \in \widehat{\Pi}_X$ *an element such that* $\gamma \cdot \Lambda \cdot \gamma^{-1} \subseteq \Pi_X^{\mathrm{tp}}$ *[or, equivalently,* $\Lambda \subseteq \gamma^{-1} \cdot \Pi_X^{\mathrm{tp}} \cdot \gamma$*]. Then* $\gamma \in \Pi_X^{\mathrm{tp}}$.

(iii) Δ_X^{tp} *(respectively,* Π_X^{tp}*) is* **commensurably terminal** *in* $\widehat{\Delta}_X$ *(respectively,* $\widehat{\Pi}_X$*).*

Proof. Next, we consider assertion (i). First, let us *observe* that since [as is well-known — cf., e.g., [Config], Remark 1.2.2] $\widehat{\Delta}_X$ is *torsion-free*, it follows that there exists a finite index characteristic open subgroup $J \subseteq \Delta_X^{\mathrm{tp}}$ [i.e., as in the previous paragraph] such that $J \bigcap \Lambda$ has *nontrivial image* in the *pro-*Σ *completion of the abelianization* of J, hence in $\Pi_{\mathcal{G}_J}^{\mathrm{tp}}$ [since, as is well-known, the surjection $J \twoheadrightarrow \Pi_{\mathcal{G}_J}^{\mathrm{tp}}$ induces an isomorphism between the pro-Σ completions of the respective abelianizations]. Since the quotient Π_X^{tp} *surjects* onto G_k, and J is open of finite index in Δ_X^{tp}, we may assume without loss of generality that γ lies in the closure $\widehat{J}$ of J in $\widehat{\Pi}_X$. Since $J \bigcap \Lambda$ has *nontrivial image* in $\Pi_{\mathcal{G}_J}^{\mathrm{tp}}$, it thus follows from Proposition 2.1 [applied to $\mathcal{G}_J$] that the image of γ via the natural surjection $\widehat{J} \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}_J}$ lies in $\Pi_{\mathcal{G}_J}^{\mathrm{tp}}$. Since, by allowing J to *vary*, Π_X^{tp} (respectively, $\widehat{\Pi}_X$) may be written as an inverse limit of the topological groups $\Pi_X^{\mathrm{tp}}/\mathrm{Ker}(J \twoheadrightarrow \Pi_{\mathcal{G}_J}^{\mathrm{tp}})$ (respectively, $\widehat{\Pi}_X/\mathrm{Ker}(\widehat{J} \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}_J})$), we thus conclude that [the original] γ lies in Π_X^{tp}, as desired.

Next, we consider assertion (ii). First, let us *observe* that it follows from a similar argument to the argument applied to prove Proposition 2.1 — where, instead of applying [SemiAnbd], Theorem 3.7, (iii), we apply its *arithmetic analogue*, namely, [SemiAnbd], Theorem 5.4, (ii); [SemiAnbd], Example 5.6 — that the image of γ in $\widehat{\Pi}_X/\mathrm{Ker}(\widehat{\Delta}_X \twoheadrightarrow \widehat{\Pi}_{\mathcal{G}^*})$ lies in $\Pi_X^{\mathrm{tp}}/\mathrm{Ker}(\Delta_X^{\mathrm{tp}} \twoheadrightarrow \Pi_{\mathcal{G}^*}^{\mathrm{tp}})$, where [by invoking the hypothesis that $\widehat{\Sigma} = \mathfrak{Primes}$] we take $\mathcal{G}^*$ to be a *semi-graph of anabelioids* as in [SemiAnbd], Example 5.6, i.e., the semi-graph of anabelioids whose finite étale coverings correspond to *arbitrary admissible coverings* of the geometric special fiber of the stable model $\mathcal{X}$. Here, we note that when one applies either [AbsTopII], Proposition 1.3, (iv), or [NodNon], Proposition 3.9, (i) — after, say, restricting the outer action of G_k on $\Pi_{\mathcal{G}^*}^{\mathrm{tp}}$ to a closed pro-Σ subgroup of the inertia group I_k of G_k that maps isomorphically onto the maximal pro-Σ quotient of I_k — to the vertices "v''", "$(v')^\gamma$", one may only conclude that these two vertices either *coincide*, are *adjacent*, or *admit a common adjacent vertex*; but this is still sufficient to conclude the *temperedness* of "$(v')^\gamma$" from that of "v''". Now [just as in the proof of assertion (i)] by applying [the evident analogue of] this *observation* to the quotients $\Pi_X^{\mathrm{tp}} \twoheadrightarrow \Pi_X^{\mathrm{tp}}/\mathrm{Ker}(J \twoheadrightarrow \Pi_{\mathcal{G}_J^*}^{\mathrm{tp}})$ — where $J \subseteq \Delta_X^{\mathrm{tp}}$ is a finite index characteristic open subgroup, and $\mathcal{G}_J^*$ is the semi-graph of anabelioids whose finite étale coverings correspond to *arbitrary admissible coverings* of the geometric special fiber of any stable model of the covering of X determined by J — we conclude that $\gamma \in \Pi_X^{\mathrm{tp}}$, as desired.

Finally, we consider assertion (iii). Just as in the proof of Proposition 2.2, the commensurable terminality of Δ_X^{tp} in $\widehat{\Delta}_X$ follows immediately from assertion (i), by allowing, in assertion (i), Λ to range over the open subgroups of a pro-Σ Sylow [hence, in particular, nontrivial pro-Σ compact!] subgroup of a verticial subgroup of $\Delta_{\mathcal{G}}^{\mathrm{tp}}$. The commensurable terminality of Π_X^{tp} in $\widehat{\Pi}_X$ then follows immediately from the commensurable terminality of Δ_X^{tp} in $\widehat{\Delta}_X$. ◯

Remark 2.4.1. Thus, when $\widehat{\Sigma} = \mathfrak{Primes}$, the proof given above of Proposition 2.4, (iii), yields a *new proof* of [André], Corollary 6.2.2 [cf. also [SemiAnbd], Lemma 6.1, (ii), (iii)] which is *independent* of [André], Lemma 3.2.1, hence also of [Stb1], Theorem 1 [cf. the discussion of Remark 2.2.2].

Corollary 2.5. (Profinite Conjugates of Tempered Decomposition and Inertia Groups) *In the notation of the above discussion, suppose further that* $\widehat{\Sigma} = \mathfrak{Primes}$. *Then every* **decomposition group** *in* $\widehat{\Pi}_X$ *(respectively,* **inertia group** *in* $\widehat{\Pi}_X$*) associated to a closed point or cusp of* X *(respectively, to a cusp of* X*) is contained in* Π_X^{tp} *if and only if it is a decomposition group in* Π_X^{tp} *(respectively, inertia group in* Π_X^{tp}*) associated to a closed point or cusp of* X *(respectively, to a cusp of* X*). Moreover, a* $\widehat{\Pi}_X$*-conjugate of* Π_X^{tp} *contains a decomposition group in* Π_X^{tp} *(respectively, inertia group in* Π_X^{tp}*) associated to a closed point or cusp of* X *(respectively, to a cusp of* X*) if and only if it is equal to* Π_X^{tp}.

Proof. Let $D_x \subseteq \Pi_X^{\mathrm{tp}}$ be the decomposition group in Π_X^{tp} associated to a closed point or cusp x of X; $I_x \stackrel{\mathrm{def}}{=} D_x \bigcap \Delta_X^{\mathrm{tp}}$. Then the decomposition groups of $\widehat{\Pi}_X$ associated to x are precisely the $\widehat{\Pi}_X$-conjugates of D_x; the decomposition groups of Π_X^{tp} associated to x are precisely the Π_X^{tp}-conjugates of D_x. Since D_x is compact and surjects onto an open subgroup of G_k, it thus follows from Proposition 2.4, (ii), that a $\widehat{\Pi}_X$-conjugate of D_x is contained in Π_X^{tp} if and only if it is, in fact, a Π_X^{tp}-conjugate of D_x, and that a $\widehat{\Pi}_X$-conjugate of Π_X^{tp} contains D_x if and only if it is, in fact, equal to Π_X^{tp}. In a similar vein, when x is a cusp of X [so $I_x \cong \widehat{\mathbb{Z}}$], it follows — i.e., by applying Proposition 2.4, (i), to the unique maximal pro-Σ subgroup of I_x — that a $\widehat{\Pi}_X$-conjugate of I_x is contained in Π_X^{tp} if and only if it is, in fact, a Π_X^{tp}-conjugate of I_x, and that a $\widehat{\Pi}_X$-conjugate of Π_X^{tp} contains I_x if and only if it is, in fact, equal to Π_X^{tp}. This completes the proof of Corollary 2.5. ○

Remark 2.5.1. The content of Corollary 2.5 may be regarded as a sort of [very weak!] version of the *"Section Conjecture"* of anabelian geometry — i.e., as the assertion that certain sections of the tempered fundamental group [namely, those that arise from geometric sections of the profinite fundamental group] are geometric as sections of the tempered fundamental group. This point of view is reminiscent of the point of view of [SemiAnbd], Remark 6.9.1. Perhaps one way of summarizing this circle of ideas is to state that one may think of

(i) the *classification of maximal compact subgroups of tempered fundamental groups* given in [SemiAnbd], Theorem 3.7, (iv); [SemiAnbd], Theorem 5.4, (ii), or, for that matter,

(ii) the more elementary fact that *"any finite group acting on a tree [without inversion] fixes at least one vertex"* [cf. [SemiAnbd], Lemma 1.8, (ii)] from which these results of [SemiAnbd] are derived

as a sort of **combinatorial version of the Section Conjecture**.

Finally, we observe that Proposition 2.4, Corollary 2.5 admit the following *discrete analogues*, which may be regarded as generalizations of [André], Lemma 3.2.1 [cf. Theorem 2.6 below in the case where $H = F = G$ is free]; [EtTh], Lemma 2.17.

Theorem 2.6. **(Profinite Conjugates of Discrete Subgroups)** *Let F be a group that contains a subgroup of finite index $G \subseteq F$ such that G is either a* **free discrete group of finite rank** *or an* **orientable surface group** *[i.e., a fundamental group of a compact orientable topological surface of genus ≥ 2]; $H \subseteq F$ an infinite subgroup. Since F is residually finite [cf., e.g., [SemiAnbd], Corollary 1.7], we shall write $H, G \subseteq F \subseteq \widehat{F}$, where $\widehat{F}$ denotes the* **profinite completion** *of F. Let $\gamma \in \widehat{F}$ be an element such that*

$$\gamma \cdot H \cdot \gamma^{-1} \subseteq F \quad \textit{[or, equivalently, } H \subseteq \gamma^{-1} \cdot F \cdot \gamma\textit{]}.$$

Write $H_G \stackrel{\mathrm{def}}{=} H \bigcap G$. Then $\gamma \in F \cdot N_{\widehat{F}}(H_G)$, i.e., $\gamma \cdot H_G \cdot \gamma^{-1} = \delta \cdot H_G \cdot \delta^{-1}$, for some $\delta \in F$. If, moreover, H_G is **nonabelian**, *then $\gamma \in F$.*

Proof. Let us first consider the case where H_G is *abelian*. In this case, it follows from Lemma 2.7, (iv), below, that H_G is *cyclic*. Thus, by applying Lemma 2.7, (ii), it follows that by replacing G by an appropriate finite index subgroup of G, we may assume that the natural composite homomorphism $H_G \hookrightarrow G \twoheadrightarrow G^{\mathrm{ab}}$ is a *split injection*. In particular, by Lemma 2.7, (v), we conclude that $N_{\widehat{G}}(H_G) = \widehat{H}_G$, where we write $\widehat{H}_G$ for the closure of H_G in the profinite completion $\widehat{G}$ of G. Next, let us observe that by multiplying γ on the left by an appropriate element of F, we may assume that $\gamma \in \widehat{G}$. Thus, we have $\gamma \cdot H_G \cdot \gamma^{-1} \subseteq F \bigcap \widehat{G} = G$. Next, let us recall that G is *conjugacy separable*. Indeed, this is precisely the content of [Stb1], Theorem 1, when G is *free*; [Stb2], Theorem 3.3, when G is an *orientable surface group*. Since G is conjugacy separable, it follows that $\gamma \cdot H_G \cdot \gamma^{-1} = \epsilon \cdot H_G \cdot \epsilon^{-1}$ for some $\epsilon \in G$, so $\gamma \in G \cdot N_{\widehat{G}}(H_G) = G \cdot \widehat{H}_G \subseteq F \cdot N_{\widehat{F}}(H_G)$, as desired. This completes the proof of Theorem 2.6 when H_G is *abelian*.

Thus, let us assume for the remainder of the proof of Theorem 2.6 that H_G is *nonabelian*. Then, by applying Lemma 2.7, (iii), it follows that, after replacing G by an appropriate finite index subgroup of G, we may assume that there exist elements $x, y \in H_G$ that generate a *free abelian subgroup of rank two* $M \subseteq G^{\mathrm{ab}}$ such that the injection $M \hookrightarrow G^{\mathrm{ab}}$ *splits*. Write $H_x, H_y \subseteq H_G$ for the subgroups generated, respectively, by x and y; $\widehat{H}_x, \widehat{H}_y \subseteq \widehat{G}$ for the respective closures of H_x, H_y. Then by Lemma 2.7, (v), we conclude that $N_{\widehat{G}}(H_x) = \widehat{H}_x$, $N_{\widehat{G}}(H_y) = \widehat{H}_y$. Next, let us observe that by multiplying γ on the left by an appropriate element of F, we may assume that $\gamma \in \widehat{G}$. Thus, we have $\gamma \cdot H_G \cdot \gamma^{-1} \subseteq F \bigcap \widehat{G} = G$. In particular, by applying the portion of Theorem 2.6 that has already been proven to the subgroups $H_x, H_y \subseteq G$, we conclude that $\gamma \in G \cdot N_{\widehat{G}}(H_x) = G \cdot \widehat{H}_x$, $\gamma \in G \cdot N_{\widehat{G}}(H_y) = G \cdot \widehat{H}_y$. Thus, by projecting to $\widehat{G}^{\mathrm{ab}}$, and applying the fact that M is of *rank two*, we conclude that $\gamma \in G$, as desired. This completes the proof of Theorem 2.6. ◯

Remark 2.6.1. Note that in the situation of Theorem 2.6, if H_G is *abelian*, then — unlike the tempered case discussed in Proposition 2.4! — it is *not necessarily* the case that $F = \gamma^{-1} \cdot F \cdot \gamma$.

Lemma 2.7. **(Well-known Properties of Free Groups and Orientable Surface Groups)** *Let G be a group as in Theorem 2.6. Write $\widehat{G}$ for the profinite completion of G. Then:*

(i) Any subgroup of G generated by two elements of G is **free**.

(ii) Let $x \in G$ be an element $\neq 1$. Then there exists a finite index subgroup $G_1 \subseteq G$ such that $x \in G_1$, and x has **nontrivial image** *in the abelianization G_1^{ab} of G_1.*

(iii) Let $x, y \in G$ be **noncommuting** *elements of G. Then there exists a finite index subgroup $G_1 \subseteq G$ and a positive integer n such that $x^n, y^n \in G_1$, and the images of x^n and y^n in the abelianization G_1^{ab} of G_1 generate a* **free abelian subgroup of rank two**.

(iv) Any abelian subgroup of G is **cyclic**.

(v) Let $\widehat{T} \subseteq \widehat{G}$ be a closed subgroup such that there exists a continuous surjection of topological groups $\widehat{G} \twoheadrightarrow \widehat{\mathbb{Z}}$ that induces an isomorphism $\widehat{T} \xrightarrow{\sim} \widehat{\mathbb{Z}}$. Then $\widehat{T}$ is **normally terminal** *in $\widehat{G}$.*

Proof. First, we consider assertion (i). If G is *free*, then assertion (i) follows from the well-known fact that any subgroup of a free group is free. If G is an *orientable surface group*, then assertion (i) follows immediately from a classical result concerning the *fundamental group of a noncompact surface* due to *Johansson* [cf. [Stl], p. 142; the discussion preceding [FRS], Theorem A1]. This completes the proof of assertion (i). Next, we consider assertion (ii). Since G is *residually finite* [cf., e.g., [SemiAnbd], Corollary 1.7], it follows that there exists a finite index normal subgroup $G_0 \subseteq G$ such that $x \notin G_0$. Thus, it suffices to take G_1 to be the subgroup of G generated by G_0 and x. This completes the proof of assertion (ii).

Next, we consider assertion (iii). By applying assertion (i) to the subgroup J of G generated by x and y, it follows from the fact that x and y are *noncommuting* elements of G that J is a *free group of rank* 2, hence that $x^a \cdot y^b \neq 1$, for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $(a, b) \neq (0, 0)$. Next, let us recall the well-known fact that the abelianization of any finite index subgroup of G is *torsion-free*. Thus, by applying assertion (ii) to x and y, we conclude that there exists a finite index subgroup $G_0 \subseteq G$ and a positive integer m such that $x^m, y^m \in G_0$, and x^m and y^m have *nontrivial image* in the abelianization G_0^{ab} of G_0. Now suppose that $x^{ma} \cdot y^{mb}$ lies in the *kernel* of the natural surjection $G_0 \twoheadrightarrow G_0^{\mathrm{ab}}$ for some $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $(a, b) \neq (0, 0)$. Since G is *residually finite*, and [as we observed above] $x^{ma} \cdot y^{mb} \neq 1$, it follows, by applying assertion (ii) to G_0, that there exists a finite index subgroup $G_1 \subseteq G_0$ and a positive integer n that is divisible by m such that $x^n, y^n, x^{ma} \cdot y^{mb} \in G_1$, and the image of $x^{ma} \cdot y^{mb}$ in G_1^{ab} is *nontrivial.* Since G_1^{ab} is *torsion-free*, it thus follows that the image of $x^{na} \cdot y^{nb}$ in G_1^{ab} is *nontrivial.* On the other hand, by considering the natural homomorphism $G_1^{\mathrm{ab}} \to G_0^{\mathrm{ab}}$, we thus conclude that the images of x^n and y^n in G_1^{ab} generate a *free abelian subgroup of rank two*, as desired. This completes the proof of assertion (iii).

Next, we consider assertion (iv). By assertion (i), it follows that any abelian subgroup of G generated by two elements is *free*, hence *cyclic.* In particular, we conclude that any abelian subgroup J of G is equal to the union of the groups that appear in some chain $G_1 \subseteq G_2 \subseteq \ldots \subseteq G$ of cyclic subgroups of G. On the other hand, by applying assertion (ii) to some generator of G_1, it follows that there exists a finite index subgroup G_0 and a positive integer n such that $G_j^n \subseteq G_0$ for all $j = 1, 2, \ldots$, and, moreover, G_1^n has *nontrivial image* in G_0^{ab}. Thus, by considering the image in [the finitely generated abelian group] G_0^{ab} of the chain of cyclic subgroups $G_1^n \subseteq G_2^n \subseteq \ldots$, we conclude that this chain, hence also the original chain $G_1 \subseteq G_2 \subseteq \ldots$, must *terminate*. Thus, J is *cyclic*, as desired. This completes the proof of assertion (iv).

Finally, we consider assertion (v). By considering the surjection $\widehat{G} \twoheadrightarrow \widehat{\mathbb{Z}}$, we conclude immediately that the normalizer $N_{\widehat{G}}(\widehat{T})$ of $\widehat{T}$ in $\widehat{G}$ is equal to the centralizer $Z_{\widehat{G}}(\widehat{T})$ of $\widehat{T}$ in $\widehat{G}$. If $Z_{\widehat{G}}(\widehat{T}) \neq \widehat{T}$, then it follows immediately that there exists a closed [abelian] subgroup $\widehat{T}_1 \subseteq Z_{\widehat{G}}(\widehat{T})$ containing $\widehat{T}$ such that, for some prime number l, there exists a continuous surjection $\mathbb{Z}_l \times \mathbb{Z}_l \twoheadrightarrow \widehat{T}_1$ whose kernel lies in $l \cdot (\mathbb{Z}_l \times \mathbb{Z}_l)$. In particular, one computes easily that the *l-cohomological dimension* of $\widehat{T}_1$ is ≥ 2. On the other hand, since $\widehat{T}_1$ is of *infinite index* in $\widehat{G}$, it follows immediately that there exists an open subgroup $\widehat{G}_1 \subseteq \widehat{G}$ of $\widehat{G}$ such that $\widehat{T}_1 \subseteq \widehat{G}_1$, and, moreover, there exists a continuous surjection $\phi : \widehat{G}_1 \twoheadrightarrow \mathbb{Z}_l$ whose kernel $\mathrm{Ker}(\phi)$ contains $\widehat{T}_1$. In particular, since the cohomology of $\widehat{T}_1$ may be computed as the direct limit of the cohomologies of open subgroups of $\widehat{G}$ containing $\widehat{T}_1$, it follows immediately from the existence of ϕ, together with the well-known structure of the cohomology of open subgroups of $\widehat{G}$, that the *l-cohomological dimension* of $\widehat{T}_1$ is 1, a contradiction. This completes the proof of assertion (v). ◯

Corollary 2.8. (Subgroups of Topological Fundamental Groups of Complex Hyperbolic Curves) *Let Z be a* **hyperbolic curve** *over $\mathbb{C}$. Write Π_Z for the usual topological fundamental group of Z; $\widehat{\Pi}_Z$ for the profinite completion of Π_Z. Let $H \subseteq \Pi_Z$ be an infinite subgroup [such as a* **cuspidal inertia group***!]; $\gamma \in \widehat{\Pi}_Z$ an element such that*

$$\gamma \cdot H \cdot \gamma^{-1} \subseteq \Pi_Z \quad \textit{[or, equivalently, } H \subseteq \gamma^{-1} \cdot \Pi_Z \cdot \gamma\textit{].}$$

Then $\gamma \in \Pi_Z \cdot N_{\widehat{\Pi}_Z}(H)$, i.e., $\gamma \cdot H \cdot \gamma^{-1} = \delta \cdot H \cdot \delta^{-1}$, for some $\delta \in \Pi_Z$. If, moreover, H is **nonabelian***, then $\gamma \in \Pi_Z$.*

Remark 2.8.1. Corollary 2.8 is an immediate consequence of Theorem 2.6. In fact, in the present series of papers, we shall only apply Corollary 2.8 in the case where Z is *non-proper*, and H is a *cuspidal inertia group*. In this case, the proof of Theorem 2.6 may be simplified somewhat, but we chose to include the general version given here, for the sake of completeness.

Section 3: Chains of Θ-Hodge Theaters

In the present §3, we construct *chains of "Θ-Hodge theaters"*. Each "Θ-Hodge theater" is to be thought of as a sort of **miniature model of the conventional scheme-theoretic arithmetic geometry** that surrounds the **theta function**. This miniature model is formulated via the theory of *Frobenioids* [cf. [FrdI]; [FrdII]; [EtTh], §3, §4, §5]. On the other hand, the *link* [cf. Corollary 3.7, (i)] between adjacent members of such chains is *purely Frobenioid-theoretic*, i.e., it lies outside the framework of ring theory/scheme theory. It is these chains of Θ-Hodge theaters that form the *starting point* of the theory of the present series of papers.

Definition 3.1. We shall refer to as *initial* Θ*-data* any collection of data

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \underline{\epsilon})$$

that satisfies the following conditions:

(a) F is a *number field* such that $\sqrt{-1} \in F$; $\overline{F}$ is an *algebraic closure* of F. Write $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F)$.

(b) X_F is a *once-punctured elliptic curve* [i.e., a hyperbolic curve of type $(1,1)$] over F that admits *stable reduction* over all $v \in \mathbb{V}(F)^{\mathrm{non}}$. Write E_F for the *elliptic curve* over F determined by X_F [so $X_F \subseteq E_F$];

$$X_F \to C_F$$

for the *hyperbolic orbicurve* [cf. §0] over F obtained by forming the stack-theoretic quotient of X_F by the unique F-involution [i.e., automorphism of order two] "-1" of X_F; $F_{\mathrm{mod}} \subseteq F$ for the *field of moduli* [cf., e.g., [AbsTopIII], Definition 5.1, (ii)] of X_F; $\mathbb{V}_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \mathbb{V}(F_{\mathrm{mod}})$;

$$\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}} \subseteq \mathbb{V}_{\mathrm{mod}}$$

for the set of nonarchimedean valuations of F_{mod} of *odd* residue characteristic over which X_F has *bad [i.e., multiplicative] reduction*; $\mathbb{V}_{\mathrm{mod}}^{\mathrm{good}} \stackrel{\mathrm{def}}{=} \mathbb{V}_{\mathrm{mod}} \setminus \mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}}$; $\mathbb{V}(F)^{\Box} \stackrel{\mathrm{def}}{=} \mathbb{V}_{\mathrm{mod}}^{\Box} \times_{\mathbb{V}_{\mathrm{mod}}} \mathbb{V}(F)$ for $\Box \in \{\mathrm{bad}, \mathrm{good}\}$;

$$\Pi_{X_F} \stackrel{\mathrm{def}}{=} \pi_1(X_F) \subseteq \Pi_{C_F} \stackrel{\mathrm{def}}{=} \pi_1(C_F)$$
$$\Delta_X \stackrel{\mathrm{def}}{=} \pi_1(X_F \times_F \overline{F}) \subseteq \Delta_C \stackrel{\mathrm{def}}{=} \pi_1(C_F \times_F \overline{F})$$

for the *étale fundamental groups* [relative to appropriate choices of basepoints] of X_F, C_F, $X_F \times_F \overline{F}$, $C_F \times_F \overline{F}$. [Thus, we have *natural exact sequences* $1 \to \Delta_{(-)} \to \Pi_{(-)_F} \to G_F \to 1$ for "$(-)$" taken to be either "X" or "C".] Here, we suppose further that $\mathbb{V}_{\mathrm{mod}}^{\mathrm{bad}} \neq \emptyset$, that the extension F/F_{mod} is *Galois*, and that the 2-*torsion points* of E_F are *rational* over F.

(c) l is a *prime number* ≥ 5 such that the outer homomorphism

$$G_F \to GL_2(\mathbb{F}_l)$$

determined by the *l-torsion points* of E_F is a *surjection*; write $K \subseteq \overline{F}$ for the finite Galois extension of F determined by the kernel of this homomorphism. Also, we suppose that l is *prime* to the elements of $\mathbb{V}^{\text{bad}}_{\text{mod}}$, as well as to the *orders of the q-parameters* of E_F [i.e., in the terminology of [GenEll], Definition 3.3, the *"local heights"* of E_F] at the primes of $\mathbb{V}(F)^{\text{bad}}$.

(d) $\underline{C}_K$ is a *hyperbolic orbicurve of type* $(1, l\text{-tors})_\pm$ [cf. [EtTh], Definition 2.1] over K, with K*-core* [cf. [CanLift], Remark 2.1.1; [EtTh], the discussion at the beginning of §2] given by $C_K \stackrel{\text{def}}{=} C_F \times_F K$. [Thus, by the *surjectivity* of (c), it follows that $\underline{C}_K$ is *completely determined*, up to isomorphism over F, by C_F.] In particular, $\underline{C}_K$ determines, up to K-isomorphism, a *hyperbolic orbicurve* $\underline{X}_K$ *of type* $(1, l\text{-tors})$ [cf. [EtTh], Definition 2.1] over K, together with *natural cartesian diagrams*

$$\begin{array}{ccc} \underline{X}_K & \longrightarrow & X_F \\ \downarrow & & \downarrow \\ \underline{C}_K & \longrightarrow & C_F \end{array} \qquad \begin{array}{ccc} \Pi_{\underline{X}_K} & \longrightarrow & \Pi_{X_F} \\ \downarrow & & \downarrow \\ \Pi_{\underline{C}_K} & \longrightarrow & \Pi_{C_F} \end{array} \qquad \begin{array}{ccc} \Delta_{\underline{X}} & \longrightarrow & \Delta_X \\ \downarrow & & \downarrow \\ \Delta_{\underline{C}} & \longrightarrow & \Delta_C \end{array}$$

of finite étale coverings of hyperbolic orbicurves and open immersions of profinite groups. Finally, we recall from [EtTh], Proposition 2.2, that $\Delta_{\underline{C}}$ admits *uniquely determined* open subgroups $\Delta_{\underline{\underline{X}}} \subseteq \Delta_{\underline{\underline{C}}} \subseteq \Delta_{\underline{C}}$, which may be thought of as corresponding to finite étale coverings of $\underline{C}_{\overline{F}} \stackrel{\text{def}}{=} \underline{C} \times_F \overline{F}$ by *hyperbolic orbicurves* $\underline{\underline{X}}_{\overline{F}}$, $\underline{\underline{C}}_{\overline{F}}$ *of type* $(1, l\text{-tors}^\Theta)$, $(1, l\text{-tors}^\Theta)_\pm$, respectively [cf. [EtTh], Definition 2.3].

(e) $\underline{\mathbb{V}} \subseteq \mathbb{V}(K)$ is a subset that induces a *natural bijection*

$$\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\text{mod}}$$

— i.e., a *section* of the natural surjection $\mathbb{V}(K) \twoheadrightarrow \mathbb{V}_{\text{mod}}$. Write $\underline{\mathbb{V}}^{\text{non}} \stackrel{\text{def}}{=} \underline{\mathbb{V}} \bigcap \mathbb{V}(K)^{\text{non}}$, $\underline{\mathbb{V}}^{\text{arc}} \stackrel{\text{def}}{=} \underline{\mathbb{V}} \bigcap \mathbb{V}(K)^{\text{arc}}$, $\underline{\mathbb{V}}^{\text{good}} \stackrel{\text{def}}{=} \underline{\mathbb{V}} \bigcap \mathbb{V}(K)^{\text{good}}$, $\underline{\mathbb{V}}^{\text{bad}} \stackrel{\text{def}}{=} \underline{\mathbb{V}} \bigcap \mathbb{V}(K)^{\text{bad}}$. For each $\underline{v} \in \mathbb{V}(K)$, we shall use the *subscript* $\underline{v}$ to denote the result of base-changing hyperbolic orbicurves over F or K to $K_{\underline{v}}$. Thus, for each $\underline{v} \in \mathbb{V}(K)$ lying under a $\overline{v} \in \mathbb{V}(\overline{F})$, we have *natural cartesian diagrams*

$$\begin{array}{ccccc} \underline{\underline{X}}_{\overline{v}} & \longrightarrow & \underline{X}_{\underline{v}} & \longrightarrow & X_{\underline{v}} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\underline{C}}_{\overline{v}} & \longrightarrow & \underline{C}_{\underline{v}} & \longrightarrow & C_{\underline{v}} \end{array} \qquad \begin{array}{ccccc} \Delta_{\underline{\underline{X}}} & \longrightarrow & \Pi_{\underline{X}_{\underline{v}}} & \longrightarrow & \Pi_{X_{\underline{v}}} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_{\underline{\underline{C}}} & \longrightarrow & \Pi_{\underline{C}_{\underline{v}}} & \longrightarrow & \Pi_{C_{\underline{v}}} \end{array}$$

of finite étale coverings of hyperbolic orbicurves and injections of profinite groups. Here, the *subscript* $\overline{v}$ denotes base-change with respect to $\overline{F} \hookrightarrow \overline{F}_{\overline{v}}$; the various profinite groups "$\Pi_{(-)}$" admit *natural outer surjections* onto the decomposition group $G_{\underline{v}} \subseteq G_K \stackrel{\text{def}}{=} \text{Gal}(\overline{F}/K)$ determined, up to G_K-conjugacy, by $\underline{v}$. If $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, then we assume further that the hyperbolic orbicurve $\underline{C}_{\underline{v}}$ is *of type* $(1, \mathbb{Z}/l\mathbb{Z})_{\pm}$ [cf. [EtTh], Definition 2.5, (i)]. [Here, we note that it follows from the portion of (b) concerning 2-*torsion points* that the base field $K_{\underline{v}}$ satisfies the assumption "$K = \ddot{K}$" of [EtTh], Definition 2.5, (i).] Finally, we observe that when $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, it follows from the theory of [EtTh], §2 — i.e., roughly speaking, *"by taking an l-th root of the theta function"* — that $\underline{X}_{\underline{v}}$, $\underline{C}_{\underline{v}}$ admit *natural models*

$$\underline{\underline{X}}_{\underline{v}}, \quad \underline{\underline{C}}_{\underline{v}}$$

over $K_{\underline{v}}$, which are hyperbolic orbicurves *of type* $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})$, $(1, (\mathbb{Z}/l\mathbb{Z})^{\Theta})_{\pm}$, respectively [cf. [EtTh], Definition 2.5, (i)]; these models determine open subgroups $\Pi_{\underline{\underline{X}}_{\underline{v}}} \subseteq \Pi_{\underline{\underline{C}}_{\underline{v}}} \subseteq \Pi_{\underline{C}_{\underline{v}}}$. If $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, then, relative to the notation of Remark 3.1.1 below, we shall write $\Pi_{\underline{v}} \stackrel{\text{def}}{=} \Pi^{\text{tp}}_{\underline{\underline{X}}_{\underline{v}}}$.

(f) $\underline{\epsilon}$ is a *cusp* of the hyperbolic orbicurve $\underline{C}_K$ [cf. (d)] that arises from a *nonzero element* of the quotient "Q" that appears in the definition of a "hyperbolic orbicurve of type $(1, l\text{-tors})_{\pm}$" given in [EtTh], Definition 2.1. If $\underline{v} \in \underline{\mathbb{V}}$, then let us write $\underline{\epsilon}_{\underline{v}}$ for the cusp of $\underline{C}_{\underline{v}}$ determined by $\underline{\epsilon}$. If $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$, then we assume that $\underline{\epsilon}_{\underline{v}}$ is the cusp that arises from the *canonical generator* [up to sign] "± 1" of the quotient "$\widehat{\mathbb{Z}}$" that appears in the definition of a "hyperbolic orbicurve of type $(1, \mathbb{Z}/l\mathbb{Z})_{\pm}$" given in [EtTh], Definition 2.5, (i). Thus, the data $(X_K \stackrel{\text{def}}{=} X_F \times_F K, \underline{C}_K, \underline{\epsilon})$ determines *hyperbolic orbicurves*

$$\underrightarrow{X}_K, \quad \underrightarrow{C}_K$$

of type $(1, l\text{-}\underrightarrow{\text{tors}})$, $(1, l\text{-}\underrightarrow{\text{tors}})_{\pm}$, respectively [cf. Definition 1.1, Remark 1.1.2], as well as open subgroups $\Pi_{\underrightarrow{X}_K} \subseteq \Pi_{\underrightarrow{C}_K} \subseteq \Pi_{C_F}$, $\Delta_{\underrightarrow{X}} \subseteq \Delta_{\underrightarrow{C}} \subseteq \Delta_C$. If $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$, then we shall write $\Pi_{\underline{v}} \stackrel{\text{def}}{=} \Pi_{\underrightarrow{X}_{\underline{v}}}$.

Remark 3.1.1. Relative to the notation of Definition 3.1, (e), suppose that $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$. Then in addition to the various profinite groups $\Pi_{(-)_{\underline{v}}}$, $\Delta_{(-)}$, one also has corresponding *tempered fundamental groups*

$$\Pi^{\text{tp}}_{(-)_{\underline{v}}}; \quad \Delta^{\text{tp}}_{(-)_{\underline{v}}}$$

[cf. [André], §4; [SemiAnbd], Example 3.10], whose profinite completions may be identified with $\Pi_{(-)_{\underline{v}}}$, $\Delta_{(-)}$. Here, we note that unlike "$\Delta_{(-)}$", the topological group $\Delta^{\text{tp}}_{(-)_{\underline{v}}}$ *depends, a priori, on* $\underline{v}$.

Remark 3.1.2.

(i) Observe that the open subgroup $\Pi_{\underline{X}_K} \subseteq \Pi_{\underline{C}_K}$ may be *constructed group-theoretically from the topological group* $\Pi_{\underline{C}_K}$. Indeed, it follows immediately from the construction of the coverings "$\underline{X}$", "$\underline{C}$" in the discussion at the beginning of [EtTh], §2 [cf. also [AbsAnab], Lemma 1.1.4, (i)], that the closed subgroup $\Delta_{\underline{X}} \subseteq \Pi_{\underline{C}_K}$ may be characterized by a rather simple explicit algorithm. Since the decomposition groups of $\Pi_{\underline{C}_K}$ at the *nonzero cusps* — i.e., the cusps whose inertia groups are contained in $\Delta_{\underline{X}}$ [cf. the discussion at the beginning of §1] — are also *group-theoretic* [cf., e.g., [AbsTopI], Lemma 4.5], the above observation follows immediately from the easily verified fact that the image of any of these decomposition groups associated to nonzero cusps coincides with the image of $\Pi_{\underline{X}_K}$ in $\Pi_{\underline{C}_K}/\Delta_{\underline{X}}$.

(ii) In light of the observation of (i), it makes sense to adopt the following convention:

> Instead of applying the *group-theoretic reconstruction algorithm* of [AbsTopIII], Theorem 1.9, directly to $\Pi_{\underline{C}_K}$ [or topological groups isomorphic to $\Pi_{\underline{C}_K}$], we shall apply this reconstruction algorithm to the *open subgroup* $\Pi_{\underline{X}_K} \subseteq \Pi_{\underline{C}_K}$ to reconstruct the *function field of* $\underline{X}_K$, equipped with its *natural* $\mathrm{Gal}(\underline{X}_K/\underline{C}_K) \cong \Pi_{\underline{C}_K}/\Pi_{\underline{X}_K}$*-action.*

In this context, we shall refer to this approach of applying [AbsTopIII], Theorem 1.9, as the **Θ-approach** to [AbsTopIII], Theorem 1.9. Note that, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ (respectively, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$), one may also adopt a "Θ-approach" to applying [AbsTopIII], Theorem 1.9, to $\Pi_{\underline{C}_{\underline{v}}}$ or [by applying Corollary 1.2] $\Pi_{\underrightarrow{X}_{\underline{v}}}$, $\Pi_{\underrightarrow{C}_{\underline{v}}}$ (respectively, to $\Pi^{\mathrm{tp}}_{\underline{C}_{\underline{v}}}$ or [by applying [EtTh], Proposition 2.4] $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}$). In the present series of papers, we shall always think of [AbsTopIII], Theorem 1.9 [as well as the other results of [AbsTopIII] that arise as consequences of [AbsTopIII], Theorem 1.9] as being applied to [isomorphs of] $\Pi_{\underline{C}_K}$ or, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ (respectively, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$), $\Pi_{\underline{C}_{\underline{v}}}$, $\Pi_{\underrightarrow{X}_{\underline{v}}}$, $\Pi_{\underrightarrow{C}_{\underline{v}}}$ (respectively, $\Pi^{\mathrm{tp}}_{\underline{C}_{\underline{v}}}$, $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}$) via the "$\Theta$-approach".

(iii) Recall from the discussion at the beginning of [EtTh], §2, the *tautological extension*

$$1 \to \Delta_\Theta \to \Delta_X^\Theta \to \Delta_X^{\mathrm{ell}} \to 1$$

— where $\Delta_\Theta \stackrel{\mathrm{def}}{=} [\Delta_X, \Delta_X]/[\Delta_X, [\Delta_X, \Delta_X]]$; $\Delta_X^\Theta \stackrel{\mathrm{def}}{=} \Delta_X/[\Delta_X, [\Delta_X, \Delta_X]]$; $\Delta_X^{\mathrm{ell}} \stackrel{\mathrm{def}}{=} \Delta_X^{\mathrm{ab}}$. The extension class $\in H^2(\Delta_X^{\mathrm{ell}}, \Delta_\Theta)$ of this extension determines a *tautological isomorphism*

$$M_X \stackrel{\sim}{\to} \Delta_\Theta$$

— where we recall from [AbsTopIII], Theorem 1.9, (b), that the module "M_X" of [AbsTopIII], Theorem 1.9, (b) [cf. also [AbsTopIII], Proposition 1.4, (ii)], may be *naturally identified* with $\mathrm{Hom}(H^2(\Delta_X^{\mathrm{ell}}, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}})$. In particular, we obtain a *tautological isomorphism*

$$M_{\underline{X}} \stackrel{\sim}{\to} (l \cdot \Delta_\Theta)$$

[i.e., since $[\Delta_X : \Delta_{\underline{X}}] = l$]. From the point of view of the theory of the present series of papers, the **significance** *of the "Θ-approach"* lies precisely in the existence of this tautological isomorphism $M_{\underline{X}} \stackrel{\sim}{\to} (l \cdot \Delta_\Theta)$, which will be applied in [IUTchII] at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. That is to say, the Θ-approach involves applying the reconstruction algorithm of [AbsTopIII], Theorem 1.9, via the cyclotome $M_{\underline{X}}$, which may be identified, via the above tautological isomorphism, with the cyclotome $(l \cdot \Delta_\Theta)$, which plays a central role in the theory of [EtTh] — cf., especially, the discussion of *"cyclotomic rigidity"* in [EtTh], Corollary 2.19, (i).

(iv) If one thinks of the prime number l as being *"large"*, then the role played by the covering $\underline{X}$ in the above discussion of the *"Θ-approach"* is reminiscent of the role played by the *universal covering of a complex elliptic curve by the complex plane* in the holomorphic reconstruction theory of [AbsTopIII], §2 [cf., e.g., [AbsTopIII], Propositions 2.5, 2.6].

Remark 3.1.3. Since $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \neq \emptyset$ [cf. Definition 3.1, (b)], it follows immediately from Definition 3.1, (d), (e), (f), that the data $(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \underline{\epsilon})$ is, in fact, *completely determined* by the data $(\overline{F}/F,\ X_F,\ \underline{C}_K,\ \underline{\mathbb{V}})$, and that $\underline{C}_K$ is *completely determined up to K-isomorphism* by the data $(\overline{F}/F,\ X_F,\ l,\ \underline{\mathbb{V}})$. Finally, we remark that for given data $(X_F,\ l)$, distinct choices of "$\underline{\mathbb{V}}$" will not affect the theory in any significant way.

Remark 3.1.4. It follows immediately from the definitions that at each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ [which is necessarily *prime to l* — cf. Definition 3.1, (c)] (respectively, each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ which is *prime to l*; each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$), $\underline{\underline{X}}_{\underline{v}}$ (respectively, $\underline{\overrightarrow{X}}_{\underline{v}}$; $\underline{X}_{\underline{v}}$) admits a *stable model* over the ring of integers of $K_{\underline{v}}$.

Remark 3.1.5. In addition to working with the *field* F_{mod} and various extensions of F_{mod} contained in $\overline{F}$, we shall also have occasion to work with the *algebraic stack*

$$S_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathcal{O}_K) \ // \ \mathrm{Gal}(K/F_{\mathrm{mod}})$$

obtained by forming the *stack-theoretic quotient* [i.e., "//"] of the spectrum of the ring of integers $\mathcal{O}_K$ of K by the Galois group $\mathrm{Gal}(K/F_{\mathrm{mod}})$. Thus, any finite extension $L \subseteq \overline{F}$ of F_{mod} in $\overline{F}$ determines, by forming the integral closure of S_{mod} in L, an algebraic stack $S_{\mathrm{mod},L}$ over S_{mod}. In particular, by considering *arithmetic line bundles* over such $S_{\mathrm{mod},L}$, one may associate to any quotient $\mathrm{Gal}(\overline{F}/F_{\mathrm{mod}}) \twoheadrightarrow Q$ a *Frobenioid* via [the easily verified *"stack-theoretic version"* of] the construction of [FrdI], Example 6.3. One verifies immediately that an appropriate analogue of [FrdI], Theorem 6.4, holds for such stack-theoretic versions of the Frobenioids constructed in [FrdI], Example 6.3. Also, we observe that upon passing to either the *perfection* or the *realification*, such stack-theoretic versions become *naturally isomorphic* to the non-stack-theoretic versions [i.e., of [FrdI], Example 6.3, as stated].

Remark 3.1.6. In light of the important role played by the various orbicurves constructed in [EtTh], §2, in the present series of papers, we take the opportunity to correct an unfortunate — albeit in fact *irrelevant*! — error in [EtTh]. In the discussion preceding [EtTh], Definition 2.1, one must in fact assume that the integer l is *odd* in order for the quotient $\overline{\Delta}_X$ to be *well-defined*. Since, ultimately, in [EtTh] [cf. the discussion following [EtTh], Remark 5.7.1], as well as in the present series of papers, this is the only case that is of interest, this oversight does not affect either the present series of papers or the bulk of the remainder of [EtTh]. Indeed, the only places in [EtTh] where the case of *even* l is used are [EtTh], Remark 2.2.1, and the application of [EtTh], Remark 2.2.1, in the proof of [EtTh], Proposition 2.12, for the orbicurves "$\underline{\underline{\dot{C}}}$". Thus, [EtTh], Remark 2.2.1, must be *deleted*; in [EtTh], Proposition 2.12, one must in fact *exclude* the case where the orbicurve under consideration is "$\underline{\underline{\dot{C}}}$". On the other hand, this theory involving [EtTh], Proposition 2.12 [cf., especially, [EtTh], Corollaries 2.18, 2.19] is only applied *after* the discussion following [EtTh], Remark 5.7.1, i.e., which only treats the curves "$\underline{\underline{X}}$". That is to say, ultimately, in [EtTh], as well as in the present series of papers, one is only interested in the curves "$\underline{\underline{X}}$", whose treatment only requires the case of *odd* l.

Given *initial* Θ*-data* as in Definition 3.1, the theory of Frobenioids given in [FrdI], [FrdII], [EtTh] allows one to construct various *associated Frobenioids*, as follows.

Example 3.2. Frobenioids at Bad Nonarchimedean Primes. Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}} = \underline{\mathbb{V}} \bigcap \mathbb{V}(K)^{\mathrm{bad}}$. Then let us recall the theory of the *"Frobenioid-theoretic theta function"* discussed in [EtTh], §5:

(i) By the theory of [EtTh], the hyperbolic curve $\underline{\underline{X}}_{\underline{v}}$ determines a *tempered Frobenioid*

$$\underline{\underline{\mathcal{F}}}_{\underline{v}}$$

[i.e., the Frobenioid denoted "$\mathcal{C}$" in the discussion at the beginning of [EtTh], §5; cf. also the discussion of Remark 3.2.4 below] over a *base category*

$$\mathcal{D}_{\underline{v}}$$

[i.e., the category denoted "$\mathcal{D}$" in the discussion at the beginning of [EtTh], §5]. We recall from the theory of [EtTh] that $\mathcal{D}_{\underline{v}}$ may be thought of as the *category of connected tempered coverings* — i.e., "$\mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})^0$" in the notation of [EtTh], Example 3.9 — of the hyperbolic curve $\underline{\underline{X}}_{\underline{v}}$. In the following, we shall write

$$\mathcal{D}^{\vdash}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{B}(K_{\underline{v}})^0$$

[cf. the notational conventions concerning categories discussed in §0]. Also, we observe that $\mathcal{D}^{\vdash}_{\underline{v}}$ may be naturally regarded [by pulling back finite étale coverings via the structure morphism $\underline{\underline{X}}_{\underline{v}} \to \mathrm{Spec}(K_{\underline{v}})$] as a *full subcategory*

$$\mathcal{D}^{\vdash}_{\underline{v}} \subseteq \mathcal{D}_{\underline{v}}$$

of $\mathcal{D}_{\underline{v}}$, and that we have a *natural functor* $\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\vdash}_{\underline{v}}$, which is *left-adjoint* to the natural inclusion functor $\mathcal{D}^{\vdash}_{\underline{v}} \hookrightarrow \mathcal{D}_{\underline{v}}$ [cf. [FrdII], Example 1.3, (ii)]. If $(-)$ is an object of $\mathcal{D}_{\underline{v}}$, then we shall denote by "$\mathbb{T}_{(-)}$" the *Frobenius-trivial object* of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [which is completely determined up to isomorphism] that lies over "$(-)$".

(ii) Next, let us recall [cf. [EtTh], Proposition 5.1; [FrdI], Corollary 4.10] that the *birationalization*

$$\underline{\underline{\mathcal{F}}}_{\underline{v}}^{\div} \stackrel{\mathrm{def}}{=} \underline{\underline{\mathcal{F}}}_{\underline{v}}^{\mathrm{birat}}$$

may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. Remark 3.2.1 below]. Write

$$\underline{\underline{\ddot{Y}}}_{\underline{v}} \to \underline{\underline{X}}_{\underline{v}}$$

for the tempered covering determined by the object "$\underline{\underline{\ddot{Y}}}^{\log}$" in the discussion at the beginning of [EtTh], §5. Thus, we may think of $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ as an object of $\mathcal{D}_{\underline{v}}$ [cf. the object "$A_{\circledcirc}$" of [EtTh], §5, in the "double underline case"]. Then let us recall the *"Frobenioid-theoretic l-th root of the theta function"*, which is *normalized* so as to attain the value 1 at the point "$\sqrt{-1}$" [cf. [EtTh], Theorem 5.7]; we shall denote the *reciprocal* of [i.e., "1 over"] this theta function by

$$\underline{\underline{\Theta}}_{\underline{v}} \in \mathcal{O}^{\times}(\mathbb{T}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}^{\div})$$

— where we use the superscript "$\div$" to denote the image in $\underline{\underline{\mathcal{F}}}_{\underline{v}}^{\div}$ of an object of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. Here, we recall that $\underline{\underline{\Theta}}_{\underline{v}}$ is *completely determined* up to *multiplication by a* $2l$*-th root of unity* [i.e., an element of $\boldsymbol{\mu}_{2l}(\mathbb{T}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}^{\div})$] and the *action of the group of automorphisms* $l \cdot \underline{\mathbb{Z}}_{\underline{v}} \subseteq \mathrm{Aut}(\mathbb{T}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}})$ [i.e., we write $\underline{\mathbb{Z}}_{\underline{v}}$ for the group denoted "$\underline{\mathbb{Z}}$" in [EtTh], Theorem 5.7]. Moreover, we recall from the theory of [EtTh], §5 [cf. the discussion at the beginning of [EtTh], §5; [EtTh], Theorem 5.7] that

$$\mathbb{T}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}} \quad \text{[regarded up to isomorphism] and}$$

$$\underline{\underline{\Theta}}_{\underline{v}} \quad \text{[regarded up to the } \boldsymbol{\mu}_{2l}(\mathbb{T}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}^{\div}),\ l \cdot \underline{\mathbb{Z}} \text{ indeterminacies discussed above]}$$

may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. Remark 3.2.1 below].

(iii) Next, we recall from [EtTh], Corollary 3.8, (ii) [cf. also [EtTh], Proposition 5.1], that the $p_{\underline{v}}$*-adic Frobenioid* constituted by the *"base-theoretic hull"* [cf. [EtTh], Remark 3.6.2]

$$\mathcal{C}_{\underline{v}} \subseteq \underline{\underline{\mathcal{F}}}_{\underline{v}}$$

[i.e., we write $\mathcal{C}_{\underline{v}}$ for the subcategory "$\mathcal{C}^{\text{bs-fld}}$" of [EtTh], Definition 3.6, (iv)] may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. Remark 3.2.1 below].

(iv) Write $q_{\underline{v}}$ for the *q-parameter* of the elliptic curve $E_{\underline{v}}$ over $K_{\underline{v}}$. Thus, we may think of $q_{\underline{v}}$ as an element $q_{\underline{v}} \in \mathcal{O}^{\triangleright}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$ $(\cong \mathcal{O}^{\triangleright}_{K_{\underline{v}}})$. Note that it follows from our assumption concerning 2*-torsion* [cf. Definition 3.1, (b)], together with the definition of "K" [cf. Definition 3.1, (c)], that $q_{\underline{v}}$ admits a $2l$*-th root* in $\mathcal{O}^{\triangleright}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$ $(\cong \mathcal{O}^{\triangleright}_{K_{\underline{v}}})$. Then one computes immediately from the final formula of [EtTh], Proposition 1.4, (ii), that the *value of* $\underline{\underline{\Theta}}_{\underline{v}}$ *at* $\sqrt{-q_{\underline{v}}}$ is equal to

$$\underline{\underline{q}}_{\underline{v}} \stackrel{\mathrm{def}}{=} q_{\underline{v}}^{1/2l} \in \mathcal{O}^{\triangleright}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$$

— where the notation "$q_{\underline{v}}^{1/2l}$" [hence also $\underline{\underline{q}}_{\underline{v}}$] is *completely determined up to a* $\boldsymbol{\mu}_{2l}(\mathbb{T}_{\underline{\underline{X}}_{\underline{v}}})$*-multiple*. Write $\Phi_{\mathcal{C}_{\underline{v}}}$ for the *divisor monoid* [cf. [FrdI], Definition 1.1, (iv)] of the $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}_{\underline{v}}$. Then the image of $\underline{\underline{q}}_{\underline{v}}$ determines a *constant section* [i.e., a sub-monoid on $\mathcal{D}_{\underline{v}}$ isomorphic to $\mathbb{N}$] "$\log_{\Phi}(\underline{\underline{q}}_{\underline{v}})$" of $\Phi_{\mathcal{C}_{\underline{v}}}$. Moreover, the resulting submonoid [cf. Remark 3.2.2 below]

$$\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathbb{N} \cdot \log_{\Phi}(\underline{\underline{q}}_{\underline{v}})|_{\mathcal{D}^{\vdash}_{\underline{v}}} \subseteq \Phi_{\mathcal{C}_{\underline{v}}}|_{\mathcal{D}^{\vdash}_{\underline{v}}}$$

determines a $p_{\underline{v}}$*-adic Frobenioid* with *base category* given by $\mathcal{D}^{\vdash}_{\underline{v}}$ [cf. [FrdII], Example 1.1, (ii)]

$$\mathcal{C}^{\vdash}_{\underline{v}} \quad (\subseteq \ \mathcal{C}_{\underline{v}} \ \subseteq \ \underline{\underline{\mathcal{F}}}_{\underline{v}} \ \to \ \underline{\underline{\mathcal{F}}}^{\div}_{\underline{v}})$$

— which may be thought of as a subcategory of $\mathcal{C}_{\underline{v}}$. Also, we observe that [since the q-parameter $\underline{\underline{q}}_{\underline{v}} \in K_{\underline{v}}$, it follows that] $\underline{\underline{q}}_{\underline{v}}$ determines a $\boldsymbol{\mu}_{2l}(-)$*-orbit* of *characteristic splittings* [cf. [FrdI], Definition 2.3]

$$\tau^{\vdash}_{\underline{v}}$$

on $\mathcal{C}^{\vdash}_{\underline{v}}$.

(v) Next, let us recall that the *base field* of $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ is equal to $K_{\underline{v}}$ [cf. the discussion of Definition 3.1, (e)]. Write

$$\mathcal{D}^{\Theta}_{\underline{v}} \subseteq (\mathcal{D}_{\underline{v}})_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}$$

for the *full subcategory* of the category of the category $(\mathcal{D}_{\underline{v}})_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}$ [cf. the notational conventions concerning categories discussed in §0] determined by the *products* in $\mathcal{D}_{\underline{v}}$ of $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ with objects of $\mathcal{D}^{\vdash}_{\underline{v}}$. Thus, one verifies immediately that "forming the product with $\underline{\underline{\ddot{Y}}}_{\underline{v}}$" determines a *natural equivalence of categories* $\mathcal{D}^{\vdash}_{\underline{v}} \stackrel{\sim}{\to} \mathcal{D}^{\Theta}_{\underline{v}}$. Moreover, for $A^{\Theta} \in \mathrm{Ob}(\mathcal{D}^{\Theta}_{\underline{v}})$, the assignment

$$A^{\Theta} \mapsto \mathcal{O}^{\times}(\mathbb{T}_{A^{\Theta}}) \cdot (\underline{\underline{\Theta}}^{\mathbb{N}}_{\underline{v}}|_{\mathbb{T}_{A^{\Theta}}}) \subseteq \mathcal{O}^{\times}(\mathbb{T}^{\div}_{A^{\Theta}})$$

determines a *monoid* $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ *on* $\mathcal{D}^{\Theta}_{\underline{v}}$ [in the sense of [FrdI], Definition 1.1, (ii)]; write $\mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) \subseteq \mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ for the submonoid determined by the invertible elements. Next, let us observe that, relative to the natural equivalence of categories $\mathcal{D}^{\vdash}_{\underline{v}} \stackrel{\sim}{\to} \mathcal{D}^{\Theta}_{\underline{v}}$ — which we think of as mapping $\mathrm{Ob}(\mathcal{D}^{\vdash}_{\underline{v}}) \ni A \mapsto A^{\Theta} \stackrel{\mathrm{def}}{=} \underline{\underline{\ddot{Y}}}_{\underline{v}} \times A \in \mathrm{Ob}(\mathcal{D}^{\Theta}_{\underline{v}})$ — we have *natural isomorphisms*

$$\mathcal{O}^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \ \stackrel{\sim}{\to} \ \mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-); \quad \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \ \stackrel{\sim}{\to} \ \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$$

[where $\mathcal{O}^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-)$, $\mathcal{O}^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-)$ are the monoids associated to the Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}}$ as in [FrdI], Proposition 2.2] which are *compatible* with the assignment

$$\underline{\underline{q}}_{\underline{v}}|_{\mathbb{T}_A} \mapsto \underline{\underline{\Theta}}_{\underline{v}}|_{\mathbb{T}_{A^{\Theta}}}$$

and the natural isomorphism [i.e., induced by the natural projection $A^{\Theta} = \ddot{\underline{\underline{Y}}}_{v} \times A \to A$] $\mathcal{O}^{\times}(\mathbb{T}_A) \xrightarrow{\sim} \mathcal{O}^{\times}(\mathbb{T}_{A^{\Theta}})$. In particular, we conclude that the monoid $\mathcal{O}^{\triangleright}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ determines — in a fashion consistent with the notation of [FrdI], Proposition 2.2! — a $p_{\underline{v}}$*-adic Frobenioid* with *base category* given by $\mathcal{D}^{\Theta}_{\underline{v}}$ [cf. [FrdII], Example 1.1, (ii)]

$$\mathcal{C}^{\Theta}_{\underline{v}} \quad (\subseteq \underline{\underline{\mathcal{F}}}^{\div}_{v})$$

— which may be thought of as a subcategory of $\underline{\underline{\mathcal{F}}}^{\div}_{v}$, and which is equipped with a $\boldsymbol{\mu}_{2l}(-)$*-orbit* of *characteristic splittings* [cf. [FrdI], Definition 2.3]

$$\tau^{\Theta}_{\underline{v}}$$

determined by $\underline{\underline{\Theta}}_{v}$. Moreover, we have a *natural equivalence of categories*

$$\mathcal{C}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{C}^{\Theta}_{\underline{v}}$$

that maps $\tau^{\vdash}_{\underline{v}}$ to $\tau^{\Theta}_{\underline{v}}$. This fact may be stated more succinctly by writing

$$\mathcal{F}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{F}^{\Theta}_{\underline{v}}$$

— where we write $\mathcal{F}^{\vdash}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$; $\mathcal{F}^{\Theta}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$. In the following, we shall refer to a pair such as $\mathcal{F}^{\vdash}_{\underline{v}}$ or $\mathcal{F}^{\Theta}_{\underline{v}}$ consisting of a Frobenioid equipped with a collection of characteristic splittings as a *split Frobenioid*.

(vi) Here, it is useful to recall [cf. Remark 3.2.1 below] that:

(a) the subcategory $\mathcal{D}^{\vdash}_{\underline{v}} \subseteq \mathcal{D}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\mathcal{D}_{\underline{v}}$ [cf. [AbsAnab], Lemma 1.3.8];

(b) the category $\mathcal{D}^{\Theta}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\mathcal{D}_{\underline{v}}$ [cf. (a); the discussion at the beginning of [EtTh], §5];

(c) the category $\mathcal{D}^{\vdash}_{\underline{v}}$ (respectively, $\mathcal{D}^{\Theta}_{\underline{v}}$) may be *reconstructed category-theoretically from* $\mathcal{C}^{\vdash}_{\underline{v}}$ (respectively, $\mathcal{C}^{\Theta}_{\underline{v}}$) [cf. [FrdI], Theorem 3.4, (v); [FrdII], Theorem 1.2, (i); [FrdII], Example 1.3, (i); [AbsAnab], Theorem 1.1.1, (ii)];

(d) the category $\mathcal{D}_{\underline{v}}$ may be *reconstructed category-theoretically either from* $\underline{\underline{\mathcal{F}}}_{v}$ [cf. [EtTh], Theorem 4.4; [EtTh], Proposition 5.1] *or from* $\mathcal{C}_{\underline{v}}$ [cf. [FrdI], Theorem 3.4, (v); [FrdII], Theorem 1.2, (i); [FrdII], Example 1.3, (i); [SemiAnbd], Example 3.10; [SemiAnbd], Remark 3.4.1]

Next, let us observe that by (b), (d), together with the discussion of (ii) concerning the *category-theoreticity* of $\underline{\underline{\Theta}}_{v}$, it follows [cf. Remark 3.2.1 below] that

(e) one may *reconstruct the split Frobenioid* $\mathcal{F}^{\Theta}_{\underline{v}}$ [up to the $l \cdot \underline{\mathbb{Z}}$ indeterminacy in $\underline{\underline{\Theta}}_{v}$ discussed in (ii); cf. also Remark 3.2.3 below] *category-theoretically from* $\underline{\underline{\mathcal{F}}}_{v}$ [cf. [FrdI], Theorem 3.4, (i), (v); [EtTh], Proposition 5.1].

Next, let us recall that the *values of* $\underline{\underline{\Theta}}_{\underline{v}}$ may be computed by restricting the corresponding *Kummer class*, i.e., the *"étale theta function"* [cf. [EtTh], Proposition 1.4, (iii); the proof of [EtTh], Theorem 5.7], which may be reconstructed *category-theoretically* from $\mathcal{D}_{\underline{v}}$ [cf. [EtTh], Corollary 2.8, (i)]. Thus, by applying the *isomorphisms of cyclotomes* of [AbsTopIII], Corollary 1.10, (c); [AbsTopIII], Remark 3.2.1 [cf. also [AbsTopIII], Remark 3.1.1] to these Kummer classes, one concludes from (a), (d) that

(f) one may *reconstruct the split Frobenioid* $\mathcal{F}^{\vdash}_{\underline{v}}$ *category-theoretically from* $\mathcal{C}_{\underline{v}}$, hence also [cf. (iii)] from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. Remark 3.2.1 below].

Remark 3.2.1.

(i) In [FrdI], [FrdII], and [EtTh] [cf. [EtTh], Remark 5.1.1], the phrase *"reconstructed category-theoretically"* is interpreted as meaning *"preserved by equivalences of categories"*. From the point of view of the theory of [AbsTopIII] — i.e., the discussion of *"mono-anabelian"* versus *"bi-anabelian"* geometry [cf., [AbsTopIII], §I2, (Q2)] — this sort of definition is "bi-anabelian" in nature. In fact, it is not difficult to verify that the techniques of [FrdI], [FrdII], and [EtTh] all result in *explicit reconstruction algorithms*, whose *input data* consists solely of the category structure of the given category, of a *"mono-anabelian"* nature that *do not require the use of some* **fixed reference model** *that arises from scheme theory* [cf. the discussion of [AbsTopIII], §I4]. For more on the *foundational aspects* of such "mono-anabelian reconstruction algorithms", we refer to the discussion of [IUTchIV], Example 3.5.

(ii) One reason that we do not develop in detail here a "mono-anabelian approach to the geometry of categories" along the lines of [AbsTopIII] is that, unlike the case with the mono-anabelian theory of [AbsTopIII], which plays a *quite essential role* in the theory of the present series of papers, much of the category-theoretic reconstruction theory of [FrdI], [FrdII], and [EtTh] is *not of essential importance* in the development of theory of the present series of papers. That is to say, for instance, instead of quoting results to the effect that the base categories or divisor monoids of various Frobenioids may be reconstructed category-theoretically, one could instead simply work with the data consisting of "the category constituted by the Frobenioid equipped with its pre-Frobenioid structure" [cf. [FrdI], Definition 1.1, (iv)]. Nevertheless, we chose to apply the theory of [FrdI], [FrdII], and [EtTh] partly because it *simplifies the exposition* [i.e., reduces the number of auxiliary structures that one must carry around], but more importantly because it renders explicit precisely which structures arising from scheme-theory are *"categorically intrinsic"* and which merely amount to *"arbitrary, non-intrinsic choices"* which, when formulated intrinsically, correspond to various *"indeterminacies"*. This explicitness is of particular importance with respect to phenomena related to the *unit-linear Frobenius functor* [cf. [FrdI], Proposition 2.5] and the *Frobenioid-theoretic indeterminacies* studied in [EtTh], §5.

Remark 3.2.2. Although the submonoid $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ is not "absolutely primitive" in the sense of [FrdII], Example 1.1, (ii), it is "very close to being absolutely primitive", in the sense that [as is easily verified] there exists a positive integer N such that $N \cdot \Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ is *absolutely primitive*. This proximity to absolute primitiveness may also be seen in the existence of the *characteristic splittings* $\tau^{\vdash}_{\underline{v}}$.

Remark 3.2.3.

(i) Let $\alpha \in \mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{v})$. Then observe that α determines, in a natural way, an *automorphism* $\alpha_{\mathcal{D}}$ *of the functor* $\mathcal{D}^{\vdash}_{\underline{v}} \to \mathcal{D}_{\underline{v}}$ obtained by composing the equivalence of categories $\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}^{\Theta}_{\underline{v}}$ [i.e., which maps $\mathrm{Ob}(\mathcal{D}^{\vdash}_{\underline{v}}) \ni A \mapsto A^{\Theta} \in \mathrm{Ob}(\mathcal{D}^{\Theta}_{\underline{v}})$] discussed in Example 3.2, (v), with the natural functor $\mathcal{D}^{\Theta}_{\underline{v}} \subseteq (\mathcal{D}_{\underline{v}})_{\underline{\underline{\ddot{Y}}}_{v}} \to \mathcal{D}_{\underline{v}}$. Moreover, $\alpha_{\mathcal{D}}$ induces, in a natural way, an *isomorphism* $\alpha_{\mathcal{O}^{\rhd}}$ of the *monoid* $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ *on* $\mathcal{D}^{\Theta}_{\underline{v}}$ *associated to* $\underline{\underline{\Theta}}_{v}$ in Example 3.2, (v), onto the *corresponding monoid on* $\mathcal{D}^{\Theta}_{\underline{v}}$ *associated to the* α*-conjugate* $\underline{\underline{\Theta}}^{\alpha}_{v}$ of $\underline{\underline{\Theta}}_{v}$. Thus, it follows immediately from the discussion of Example 3.2, (v), that

> $\alpha_{\mathcal{O}^{\rhd}}$ — hence also α — induces an isomorphism of the *split Frobenioid* $\mathcal{F}^{\Theta}_{\underline{v}}$ *associated to* $\underline{\underline{\Theta}}_{v}$ onto the *split Frobenioid* $\mathcal{F}^{\Theta^{\alpha}}_{\underline{v}}$ *associated to* $\underline{\underline{\Theta}}^{\alpha}_{v}$ which lies over the *identity functor on* $\mathcal{D}^{\Theta}_{\underline{v}}$.

In particular, the expression *"$\mathcal{F}^{\Theta}_{\underline{v}}$, regarded up to the $l \cdot \underline{\mathbb{Z}}$ indeterminacy in $\underline{\underline{\Theta}}_{v}$ discussed in Example 3.2, (ii)"* may be understood as referring to the various split Frobenioids "$\mathcal{F}^{\Theta^{\alpha}}_{\underline{v}}$", as α ranges over the elements of $\mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{v})$, relative to the identifications given by these isomorphisms of split Frobenioids induced by the various elements of $\mathrm{Aut}_{\mathcal{D}_{\underline{v}}}(\underline{\underline{\ddot{Y}}}_{v})$.

(ii) Suppose that $A \in \mathrm{Ob}(\mathcal{D}_{\underline{v}})$ lies in the image of the natural functor $\mathcal{D}^{\Theta}_{\underline{v}} \subseteq (\mathcal{D}_{\underline{v}})_{\underline{\underline{\ddot{Y}}}_{v}} \to \mathcal{D}_{\underline{v}}$, and that $\psi : B \to \mathbb{T}_A$ is a linear morphism in the Frobenioid $\underline{\underline{\mathcal{F}}}_{v}$. Then ψ induces an injective homomorphism

$$\mathcal{O}^{\times}(\mathbb{T}^{\div}_{A}) \hookrightarrow \mathcal{O}^{\times}(B^{\div})$$

[cf. [FrdI], Proposition 1.11, (iv)]. In particular, one may pull-back sections of the *monoid* $\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-)$ *on* $\mathcal{D}^{\Theta}_{\underline{v}}$ of Example 3.2, (v), to B. Such pull-backs are useful, for instance, when one considers the *roots of* $\underline{\underline{\Theta}}_{v}$, as in the theory of [EtTh], §5.

Remark 3.2.4. Before proceeding, we pause to discuss a certain minor oversight on the part of the author in the discussion of the theory of *tempered Frobenioids* in [EtTh], §3, §4. Let $\mathfrak{Z}^{\log}_{\infty}$ be as in the discussion at the beginning of [EtTh], §3. Here, we recall that $\mathfrak{Z}^{\log}_{\infty}$ is obtained as the *"universal combinatorial covering"* of the formal log scheme associated to a stable log curve with split special fiber over the ring of integers of a finite extension of an MLF [cf. *loc. cit.* for more details]; we write $Z^{\log}$ for the generic fiber of the stable log curve under consideration.

(i) First, let us consider the following conditions on a nonzero meromorphic function f on $\mathfrak{Z}^{\log}_{\infty}$:

(a) For every $N \in \mathbb{N}_{\geq 1}$, it holds that f admits an *N-th root* over some tempered covering of $Z^{\log}$.

(b) For every $N \in \mathbb{N}_{\geq 1}$ which is *prime to p*, it holds that f admits an *N-th root* over some tempered covering of $Z^{\log}$.

(c) The divisor of zeroes and poles of f is a *log-divisor*.

It is immediate that (a) implies (b). Moreover, one verifies immediately, by considering the ramification divisors of the tempered coverings that arise from extracting roots of f, that (b) implies (c). When N is *prime to p*, if f satisfies (c), then it follows immediately from the theory of *admissible coverings* [cf., e.g., [PrfGC], §2, §8] that there exists a *finite* log étale covering $Y^{\log} \to Z^{\log}$ whose pull-back $Y^{\log}_{\infty} \to Z^{\log}_{\infty}$ to the generic fiber $Z^{\log}_{\infty}$ of $\mathfrak{Z}^{\log}_{\infty}$ is sufficient

(R1) to annihilate all ramification over the cusps or special fiber of $\mathfrak{Z}^{\log}_{\infty}$ that might arise from extracting an N-th root of f, as well as

(R2) to split all extensions of the function fields of irreducible components of the special fiber of $\mathfrak{Z}^{\log}_{\infty}$ that might arise from extracting an N-th root of f.

That is to say, in this situation, it follows that f *admits an N-th root over the tempered covering of $Z^{\log}$ given by the "universal combinatorial covering" of* $Y^{\log}$. In particular, it follows that (c) implies (b). Thus, in summary, we have:

$$(a) \implies (b) \iff (c).$$

On the other hand, unfortunately, it is not clear to the author at the time of writing *whether or not (c) [or (b)] implies (a).*

(ii) Observe that it follows from the theory of [EtTh], §1 [cf., especially, [EtTh], Proposition 1.3] that the *theta function* that forms the main topic of interest of [EtTh] *satisfies condition (a)* of (i).

(iii) In [EtTh], Definition 3.1, (ii), a meromorphic function f as in (i) is defined to be *"log-meromorphic"* if it satisfies condition (c) of (i). On the other hand, in the proof of [EtTh], Proposition 4.2, (iii), it is necessary to use property (a) of (ii) — i.e., despite the fact that, as remarked in (i), it is not clear whether or not property (c) implies property (a). The author apologizes for any confusion caused by this oversight on his part.

(iv) The problem pointed out in (iii) may be remedied — at least *from the point of view of the theory of [EtTh]* — via either of following two approaches:

(A) One may modify [EtTh], Definition 3.1, (ii), by taking the definition of a *"log-meromorphic"* function to be a function that satisfies condition (a) [i.e., as opposed to condition (c)] of (i). [In light of the content of this modified definition, perhaps a better term for this class of meromorphic functions would be *"tempered-meromorphic"*.] Then the remainder of the text of [EtTh] goes through without change.

(B) One may modify [EtTh], Definition 4.1, (i), by assuming that the meromorphic function "$f \in \mathcal{O}^\times(A^{\mathrm{birat}})$" of [EtTh], Definition 4.1, (i), satisfies the following "Frobenioid-theoretic version" of condition (a):

(d) For every $N \in \mathbb{N}_{\geq 1}$, there exists a linear morphism $A' \to A$ in $\mathcal{C}$ such that the pull-back of f to A' admits an N-th root.

[Here, we recall that, as discussed in (ii), the Frobenioid-theoretic theta functions that appear in [EtTh] satisfy (d).] Note that since the rational function monoid of the Frobenioid $\mathcal{C}$, as well as the linear morphisms of $\mathcal{C}$, are *category-theoretic* [cf. [FrdI], Theorem 3.4, (iii), (v); [FrdI], Corollary 4.10], this condition (d) is *category-theoretic*. Thus, if one modifies [EtTh], Definition 4.1, (i), in this way, then the remainder of the text of [EtTh] goes through without change, except that one must replace the reference to the definition of "log-meromorphic" [i.e., [EtTh], Definition 3.1, (ii)] that occurs in the proof of [EtTh], Proposition 4.2, (iii), by a reference to condition (d) [i.e., in the modified version of [EtTh], Definition 4.1, (i)].

(v) In the discussion of (iv), we note that the approach of (A) results in a slightly different definition of the notion of a *"tempered Frobenioid"* from the original definition given in [EtTh]. Put another way, the approach of (B) has the advantage that it does *not* result in any modification of the definition of the notion of a *"tempered Frobenioid"*; that is to say, the approach of (B) only results in a slight reduction in the range of applicability of the theory of [EtTh], §4, which is essentially *irrelevant* from the point of view of the present series of papers, since [cf. (ii)] *theta functions* lie within this reduced range of applicability. On the other hand, the approach of (A) has the advantage that one may consider the *Kummer theory of arbitrary rational functions* of the tempered Frobenioid *without imposing any further hypotheses*. Thus, for the sake of simplicity, in the present series of papers, *we shall interpret the notion of a "tempered Frobenioid" via the approach of (A).*

Example 3.3. Frobenioids at Good Nonarchimedean Primes. Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$. Then:

(i) Write

$$\mathcal{D}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{B}(\underline{X}_{\to \underline{v}})^0; \quad \mathcal{D}^{\vdash}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{B}(K_{\underline{v}})^0$$

[cf. §0]. Thus, $\mathcal{D}^{\vdash}_{\underline{v}}$ may be naturally regarded [by pulling back finite étale coverings via the structure morphism $\underline{X}_{\to \underline{v}} \to \mathrm{Spec}(K_{\underline{v}})$] as a *full subcategory*

$$\mathcal{D}^{\vdash}_{\underline{v}} \subseteq \mathcal{D}_{\underline{v}}$$

of $\mathcal{D}_{\underline{v}}$, and we have a *natural functor* $\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\vdash}_{\underline{v}}$, which is *left-adjoint* to the natural inclusion functor $\mathcal{D}^{\vdash}_{\underline{v}} \hookrightarrow \mathcal{D}_{\underline{v}}$ [cf. [FrdII], Example 1.3, (ii)]. For $\mathrm{Spec}(L) \in \mathrm{Ob}(\mathcal{D}^{\vdash}_{\underline{v}})$ [i.e., L is a finite separable extension of $K_{\underline{v}}$], write $\mathrm{ord}(\mathcal{O}^{\triangleright}_L) \stackrel{\mathrm{def}}{=} \mathcal{O}^{\triangleright}_L / \mathcal{O}^{\times}_L$ as in [FrdII], Example 1.1, (i). Thus, the assignment [cf. §0]

$$\Phi_{\mathcal{C}_{\underline{v}}} : \mathrm{Spec}(L) \mapsto \mathrm{ord}(\mathcal{O}^{\triangleright}_L)^{\mathrm{pf}}$$

determines a *monoid* $\Phi_{\mathcal{C}_{\underline{v}}}$ on $[\mathcal{D}^{\vdash}_{\underline{v}}$, hence, by pull-back via the natural functor $\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\vdash}_{\underline{v}}$, on$]$ $\mathcal{D}_{\underline{v}}$; the assignment

$$\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} : \mathrm{Spec}(L) \mapsto \mathrm{ord}(\mathbb{Z}^{\rhd}_{p_{\underline{v}}}) \ (\subseteq \mathrm{ord}(\mathcal{O}^{\rhd}_{L})^{\mathrm{pf}})$$

determines an *absolutely primitive* [cf. [FrdII], Example 1.1, (ii)] *submonoid* $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}} \subseteq \Phi_{\mathcal{C}_{\underline{v}}}|_{\mathcal{D}^{\vdash}_{\underline{v}}}$ on $\mathcal{D}^{\vdash}_{\underline{v}}$; these monoids $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$, $\Phi_{\mathcal{C}_{\underline{v}}}$ determine *$p_{\underline{v}}$-adic Frobenioids*

$$\mathcal{C}^{\vdash}_{\underline{v}} \subseteq \mathcal{C}_{\underline{v}}$$

[cf. [FrdII], Example 1.1, (ii), where we take "Λ" to be $\mathbb{Z}$], whose *base categories* are given by $\mathcal{D}^{\vdash}_{\underline{v}}$, $\mathcal{D}_{\underline{v}}$ [in a fashion compatible with the natural inclusion $\mathcal{D}^{\vdash}_{\underline{v}} \subseteq \mathcal{D}_{\underline{v}}$], respectively. Also, we shall write

$$\underline{\underline{\mathcal{F}}}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{C}_{\underline{v}}$$

[cf. the notation of Example 3.2, (i)]. Finally, let us observe that the element $p_{\underline{v}} \in \mathbb{Z}_{p_{\underline{v}}} \subseteq \mathcal{O}^{\rhd}_{K_{\underline{v}}}$ determines a *characteristic splitting*

$$\tau^{\vdash}_{\underline{v}}$$

on $\mathcal{C}^{\vdash}_{\underline{v}}$ [cf. [FrdII], Theorem 1.2, (v)]. Write $\mathcal{F}^{\vdash}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$ for the resulting *split Frobenioid*.

(ii) Next, let us write $\log(p_{\underline{v}})$ for the element $p_{\underline{v}}$ of (i) *considered additively* and consider the monoid on $\mathcal{D}^{\vdash}_{\underline{v}}$

$$\mathcal{O}^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) = \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-) \times \ (\mathbb{N} \cdot \log(p_{\underline{v}}))$$

associated to $\mathcal{C}^{\vdash}_{\underline{v}}$ [cf. [FrdI], Proposition 2.2]. By replacing "$\log(p_{\underline{v}})$" by the *formal symbol* "$\log(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}}) = \log(p_{\underline{v}}^{\log(\underline{\underline{\Theta}})})$", we obtain a monoid

$$\mathcal{O}^{\rhd}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) \stackrel{\mathrm{def}}{=} \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) \times \ (\mathbb{N} \cdot \log(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}}))$$

[i.e., where $\mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(-) \stackrel{\mathrm{def}}{=} \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}(-)$], which is *naturally isomorphic* to $\mathcal{O}^{\rhd}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ and which arises as the monoid "$\mathcal{O}^{\rhd}(-)$" of [FrdI], Proposition 2.2, associated to some *$p_{\underline{v}}$-adic Frobenioid* $\mathcal{C}^{\Theta}_{\underline{v}}$ with base category $\mathcal{D}^{\Theta}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{D}^{\vdash}_{\underline{v}}$ equipped with a *characteristic splitting* $\tau^{\Theta}_{\underline{v}}$ determined by $\log(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}})$. In particular, we have a *natural equivalence*

$$\mathcal{F}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{F}^{\Theta}_{\underline{v}}$$

— where $\mathcal{F}^{\Theta}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$ — of *split Frobenioids*.

(iii) Here, it is useful to recall that

(a) the subcategory $\mathcal{D}^{\vdash}_{\underline{v}} \subseteq \mathcal{D}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\mathcal{D}_{\underline{v}}$ [cf. [AbsAnab], Lemma 1.3.8];

(b) the category $\mathcal{D}^{\vdash}_{\underline{v}}$ (respectively, $\mathcal{D}^{\Theta}_{\underline{v}}$) may be *reconstructed category-theoretically from* $\mathcal{C}^{\vdash}_{\underline{v}}$ (respectively, $\mathcal{C}^{\Theta}_{\underline{v}}$) [cf. [FrdI], Theorem 3.4, (v); [FrdII], Theorem 1.2, (i); [FrdII], Example 1.3, (i); [AbsAnab], Theorem 1.1.1, (ii)];

(c) the category $\mathcal{D}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}} = \mathcal{C}_{\underline{v}}$ [cf. [FrdI], Theorem 3.4, (v); [FrdII], Theorem 1.2, (i); [FrdII], Example 1.3, (i); [AbsAnab], Lemma 1.3.1].

Note that it follows immediately from the *category-theoreticity of the divisor monoid* $\Phi_{\mathcal{C}_{\underline{v}}}$ [cf. [FrdI], Corollary 4.11, (iii); [FrdII], Theorem 1.2, (i)], together with (a), (c), and the definition of $\mathcal{C}^{\vdash}_{\underline{v}}$, that

(d) $\mathcal{C}^{\vdash}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.

Finally, by applying the *algorithmically constructed field structure* on the image of the *Kummer map* of [AbsTopIII], Proposition 3.2, (iii) [cf. Remark 3.1.2; Remark 3.3.2 below], it follows that one may construct the element "$p_{\underline{v}}$" of $\mathcal{O}^{\rhd}_{K_{\underline{v}}}$ category-theoretically from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, hence that the characteristic splitting $\tau^{\vdash}_{\underline{v}}$ may be *reconstructed category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. [Here, we recall that the curve X_F is *"of strictly Belyi type"* — cf. [AbsTopIII], Remark 2.8.3.] In particular,

(e) one may *reconstruct the split Frobenioids* $\mathcal{F}^{\vdash}_{\underline{v}}$, $\mathcal{F}^{\Theta}_{\underline{v}}$ *category-theoretically from* $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.

Remark 3.3.1. A similar remark to Remark 3.2.1 [i.e., concerning the phrase *"reconstructed category-theoretically"*] applies to the Frobenioids $\mathcal{C}_{\underline{v}}$, $\mathcal{C}^{\vdash}_{\underline{v}}$ constructed in Example 3.3.

Remark 3.3.2. Note that the $p_{\underline{v}}$-adic Frobenioids $\mathcal{C}_{\underline{v}}$ (respectively, $\mathcal{C}^{\vdash}_{\underline{v}}$) of Examples 3.2, (iii), (iv); 3.3, (i) consist of *essentially the same data* as an *"MLF-Galois* $\mathbb{TM}$*-pair of strictly Belyi type"* (respectively, *"MLF-Galois* $\mathbb{TM}$*-pair of mono-analytic type"*), in the sense of [AbsTopIII], Definition 3.1, (ii) [cf. [AbsTopIII], Remark 3.1.1]. A similar remark applies to the $p_{\underline{v}}$-adic Frobenioid $\mathcal{C}_{\underline{v}}$ (respectively, $\mathcal{C}^{\vdash}_{\underline{v}}$) of Example 3.2 [cf. [AbsTopIII], Remark 3.1.3].

Example 3.4. Frobenioids at Archimedean Primes. Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Then:

(i) Write

$$\mathbb{X}_{\underline{v}},\ \mathbb{C}_{\underline{v}},\ \underline{\mathbb{X}}_{\underline{v}},\ \underline{\mathbb{C}}_{\underline{v}},\ \underline{\mathbb{X}}_{\to_{\underline{v}}},\ \underline{\mathbb{C}}_{\to_{\underline{v}}}$$

for the Aut-*holomorphic orbispaces* [cf. [AbsTopIII], Remark 2.1.1] determined, respectively, by the hyperbolic orbicurves X_K, C_K, $\underline{X}_K$, $\underline{C}_K$, $\underline{X}_{\to_K}$, $\underline{C}_{\to_K}$ at $\underline{v}$. Thus, for $\square \in \{\mathbb{X}_{\underline{v}}, \mathbb{C}_{\underline{v}}, \underline{\mathbb{X}}_{\underline{v}}, \underline{\mathbb{C}}_{\underline{v}}, \underline{\mathbb{X}}_{\rightarrow \underline{v}}, \underline{\mathbb{C}}_{\rightarrow \underline{v}}\}$, we have a *complex archimedean topological field* [i.e., a "CAF" — cf. §0]

$$\overline{\mathcal{A}}_{\square}$$

[cf. [AbsTopIII], Definition 4.1, (i)] which may be *algorithmically constructed from* $\square$; write $\mathcal{A}_{\square} \stackrel{\text{def}}{=} \overline{\mathcal{A}}_{\square} \setminus \{0\}$. Next, let us write

$$\mathcal{D}_{\underline{v}} \stackrel{\text{def}}{=} \underline{\mathbb{X}}_{\rightarrow \underline{v}}$$

and

$$\mathcal{C}_{\underline{v}}$$

for the *archimedean Frobenioid* as in [FrdII], Example 3.3, (ii) [i.e., "$\mathcal{C}$" of *loc. cit.*], where we take the *base category* [i.e., "$\mathcal{D}$" of *loc. cit.*] to be the one-morphism category determined by $\operatorname{Spec}(K_{\underline{v}})$. Thus, the *linear morphisms* among the pseudo-terminal objects of $\mathcal{C}$ determine *unique isomorphisms* [cf. [FrdI], Definition 1.3, (iii), (c)] among the respective *topological monoids* "$\mathcal{O}^{\triangleright}(-)$" — where we recall [cf. [FrdI], Theorem 3.4, (iii); [FrdII], Theorem 3.6, (i), (vii)] that these topological monoids may be *reconstructed category-theoretically* from $\mathcal{C}$. In particular, it makes sense to write "$\mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}})$", "$\mathcal{O}^{\times}(\mathcal{C}_{\underline{v}}) \subseteq \mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}})$". Moreover, we observe that, by construction, there is a *natural isomorphism*

$$\mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}}) \xrightarrow{\sim} \mathcal{O}^{\triangleright}_{K_{\underline{v}}}$$

of *topological monoids*. Thus, one may also think of $\mathcal{C}_{\underline{v}}$ as a *"Frobenioid-theoretic representation"* of the topological monoid $\mathcal{O}^{\triangleright}_{K_{\underline{v}}}$ [cf. [AbsTopIII], Remark 4.1.1]. Observe that there is a *natural topological isomorphism* $K_{\underline{v}} \xrightarrow{\sim} \overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$, which may be restricted to $\mathcal{O}^{\triangleright}_{K_{\underline{v}}}$ to obtain an *inclusion of topological monoids*

$$\kappa_{\underline{v}} : \mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}_{\mathcal{D}_{\underline{v}}}$$

— which we shall refer to as the *Kummer structure* on $\mathcal{C}_{\underline{v}}$ [cf. Remark 3.4.2 below]. Write

$$\underline{\underline{\mathcal{F}}}_{\underline{v}} \stackrel{\text{def}}{=} (\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$$

[cf. Example 3.2, (i); Example 3.3, (i)].

(ii) Next, recall the category $\mathbb{TM}^{\vdash}$ of *"split topological monoids"* of [AbsTopIII], Definition 5.6, (i) — i.e., the category whose *objects* $(C, \overrightarrow{C})$ consist of a topological monoid C isomorphic to $\mathcal{O}^{\triangleright}_{\mathbb{C}}$ and a topological submonoid $\overrightarrow{C} \subseteq C$ [necessarily isomorphic to $\mathbb{R}_{\geq 0}$] such that the natural inclusions $C^{\times} \hookrightarrow C$ [where $C^{\times}$, which is necessarily isomorphic to $\mathbb{S}^1$, denotes the topological submonoid of *invertible elements*], $\overrightarrow{C} \hookrightarrow C$ determine an isomorphism $C^{\times} \times \overrightarrow{C} \xrightarrow{\sim} C$ of topological monoids, and whose *morphisms* $(C_1, \overrightarrow{C}_1) \to (C_2, \overrightarrow{C}_2)$ are isomorphisms of topological monoids $C_1 \xrightarrow{\sim} C_2$ that induce isomorphisms $\overrightarrow{C}_1 \xrightarrow{\sim} \overrightarrow{C}_2$. Note that the CAF's $K_{\underline{v}}$, $\overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$ determine, in a natural way, objects of $\mathbb{TM}^{\vdash}$. Write

$$\tau^{\vdash}_{\underline{v}}$$

for the resulting *characteristic splitting* of the Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{C}_{\underline{v}}$, i.e., so that we may think of the pair $(\mathcal{O}^{\rhd}(\mathcal{C}^{\vdash}_{\underline{v}}), \tau^{\vdash}_{\underline{v}})$ as the object of $\mathbb{TM}^{\vdash}$ determined by $K_{\underline{v}}$;

$$\mathcal{D}^{\vdash}_{\underline{v}}$$

for the object of $\mathbb{TM}^{\vdash}$ determined by $\overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$;

$$\mathcal{F}^{\vdash}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\vdash}_{\underline{v}}, \mathcal{D}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$$

for the [ordered] *triple* consisting of $\mathcal{C}^{\vdash}_{\underline{v}}$, $\mathcal{D}^{\vdash}_{\underline{v}}$, and $\tau^{\vdash}_{\underline{v}}$. Thus, the object $(\mathcal{O}^{\rhd}(\mathcal{C}^{\vdash}_{\underline{v}}), \tau^{\vdash}_{\underline{v}})$ of $\mathbb{TM}^{\vdash}$ is *isomorphic* to $\mathcal{D}^{\vdash}_{\underline{v}}$. Moreover, $\mathcal{C}^{\vdash}_{\underline{v}}$ (respectively, $\mathcal{D}^{\vdash}_{\underline{v}}$; $\mathcal{F}^{\vdash}_{\underline{v}}$) may be *algorithmically reconstructed* from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ (respectively, $\mathcal{D}_{\underline{v}}$; $\underline{\underline{\mathcal{F}}}_{\underline{v}}$).

(iii) Next, let us observe that $p_{\underline{v}} \in K_{\underline{v}}$ [cf. §0] may be thought of as a(n) [non-identity] element of the *noncompact factor* $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ [i.e., the factor denoted by a "$\rightarrow$" in the definition of $\mathbb{TM}^{\vdash}$] of the object $(\mathcal{O}^{\rhd}(\mathcal{C}^{\vdash}_{\underline{v}}), \tau^{\vdash}_{\underline{v}})$ of $\mathbb{TM}^{\vdash}$. This noncompact factor $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ is isomorphic, as a topological monoid, to $\mathbb{R}_{\geq 0}$; let us write $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ *additively* and denote by $\log(p_{\underline{v}})$ the element of $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ determined by $p_{\underline{v}}$. Thus, relative to the natural action [by multiplication!] of $\mathbb{R}_{\geq 0}$ on $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$, it follows that $\log(p_{\underline{v}})$ is a *generator* of $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$. In particular, we may form a *new topological monoid*

$$\Phi_{\mathcal{C}^{\Theta}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathbb{R}_{\geq 0} \cdot \log(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}})$$

isomorphic to $\mathbb{R}_{\geq 0}$ that is generated by a *formal symbol* "$\log(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}}) = \log(p_{\underline{v}}^{\log(\underline{\underline{\Theta}})})$". Moreover, if we denote by $\mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ the *compact factor* of the object $(\mathcal{O}^{\rhd}(\mathcal{C}^{\vdash}_{\underline{v}}), \tau^{\vdash}_{\underline{v}})$ of $\mathbb{TM}^{\vdash}$, and set $\mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}} \stackrel{\mathrm{def}}{=} \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$, then we obtain a *new split Frobenioid* $(\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$, isomorphic to $(\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}})$, such that

$$\mathcal{O}^{\rhd}(\mathcal{C}^{\Theta}_{\underline{v}}) = \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}} \times \Phi_{\mathcal{C}^{\Theta}_{\underline{v}}}$$

— where we note that this equality gives rise to a *natural isomorphism of split Frobenioids* $(\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}}) \xrightarrow{\sim} (\mathcal{C}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$, obtained by "forgetting the formal symbol $\log(\underline{\underline{\Theta}})$". In particular, we thus obtain a *natural isomorphism*

$$\mathcal{F}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{F}^{\Theta}_{\underline{v}}$$

— where we write $\mathcal{F}^{\Theta}_{\underline{v}} \stackrel{\mathrm{def}}{=} (\mathcal{C}^{\Theta}_{\underline{v}}, \mathcal{D}^{\Theta}_{\underline{v}}, \tau^{\Theta}_{\underline{v}})$ for the [ordered] *triple* consisting of $\mathcal{C}^{\Theta}_{\underline{v}}$, $\mathcal{D}^{\Theta}_{\underline{v}} \stackrel{\mathrm{def}}{=} \mathcal{D}^{\vdash}_{\underline{v}}$, $\tau^{\Theta}_{\underline{v}}$. Finally, we observe that $\mathcal{F}^{\Theta}_{\underline{v}}$ may be *algorithmically reconstructed* from $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.

Remark 3.4.1. A similar remark to Remark 3.2.1 [i.e., concerning the phrase *"reconstructed category-theoretically"*] applies to the phrase *"algorithmically reconstructed"* that was applied in the discussion of Example 3.4.

Remark 3.4.2. One way to think of the *Kummer structure*

$$\kappa_{\underline{v}} : \mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}_{\mathcal{D}_{\underline{v}}}$$

discussed in Example 3.4, (i), is as follows. In the terminology of [AbsTopIII], Definition 2.1, (i), (iv), the structure of CAF on $\overline{\mathcal{A}}_{\mathcal{D}_{\underline{v}}}$ determines, via pull-back by $\kappa_{\underline{v}}$, an Aut-*holomorphic structure* on the *groupification* $\mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}})^{\mathrm{gp}}$ of $\mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}})$, together with a [tautological!] *co-holomorphicization* $\mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}})^{\mathrm{gp}} \to \mathcal{A}_{\mathcal{D}_{\underline{v}}}$. Conversely, if one starts with this Aut-holomorphic structure on [the groupification of] the topological monoid $\mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}})$, together with the co-holomorphicization $\mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}})^{\mathrm{gp}} \to \mathcal{A}_{\mathcal{D}_{\underline{v}}}$, then one verifies immediately that one may recover the *inclusion of topological monoids* $\kappa_{\underline{v}}$. [Indeed, this follows immediately from the elementary fact that every holomorphic automorphism of the complex Lie group $\mathbb{C}^{\times}$ that preserves the submonoid of elements of norm ≤ 1 is equal to the *identity*.] That is to say, in summary,

> the **Kummer structure** $\kappa_{\underline{v}}$ is completely **equivalent** to the collection of data consisting of the Aut-**holomorphic structure** [induced by $\kappa_{\underline{v}}$] on the *groupification* $\mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}})^{\mathrm{gp}}$ of $\mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}})$, together with the **co-holomorphicization** [induced by $\kappa_{\underline{v}}$] $\mathcal{O}^{\triangleright}(\mathcal{C}_{\underline{v}})^{\mathrm{gp}} \to \mathcal{A}_{\mathcal{D}_{\underline{v}}}$.

The significance of thinking of Kummer structures in this way lies in the *observation* that [unlike inclusions of topological monoids!]

> the **co-holomorphicization** induced by $\kappa_{\underline{v}}$ is **compatible** with the **logarithm** operation discussed in [AbsTopIII], Corollary 4.5.

Indeed, this observation may be thought of as a rough summary of a substantial portion of the content of [AbsTopIII], Corollary 4.5. Put another way, thinking of Kummer structures in terms of co-holomorphicizations allows one to *separate* out the portion of the structures involved that is *not compatible* with this logarithm operation — i.e., the *monoid* structures! — from the portion of the structures involved that *is compatible* with this logarithm operation — i.e., the tautological *co-holomorphicization*.

Example 3.5. Global Realified Frobenioids.

(i) Write

$$\mathcal{C}^{\Vdash}_{\mathrm{mod}}$$

for the *realification* [cf. [FrdI], Theorem 6.4, (ii)] of the *Frobenioid* of [FrdI], Example 6.3 [cf. also Remark 3.1.5 of the present paper], associated to the *number field* F_{mod} and the *trivial Galois extension of* F_{mod} [so the base category of $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ is, in the terminology of [FrdI], equivalent to a *one-morphism category*]. Thus, the *divisor monoid* $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}}$ of $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ may be thought of a single abstract monoid, whose *set of primes*, which we denote $\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}})$ [cf. [FrdI], §0], is in *natural bijective correspondence* with $\mathbb{V}_{\mathrm{mod}}$ [cf. [FrdI], Theorem 6.4, (iii)]. Moreover, the submonoid $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v}$ of $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}}$ corresponding to $v \in \mathbb{V}_{\mathrm{mod}}$ is *naturally isomorphic* to $\mathrm{ord}(\mathcal{O}^{\triangleright}_{(F_{\mathrm{mod}})_v}) \otimes \mathbb{R}_{\geq 0}$ $(\cong \mathbb{R}_{\geq 0})$. In particular, p_v determines an element

$\log^{\vdash}_{\mathrm{mod}}(p_v) \in \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v}$. Write $\underline{v} \in \underline{\mathbb{V}}$ for the element of $\underline{\mathbb{V}}$ that corresponds to v. Then observe that regardless of whether $\underline{v}$ belongs to $\underline{\mathbb{V}}^{\mathrm{bad}}$, $\underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$, or $\underline{\mathbb{V}}^{\mathrm{arc}}$, the *realification* $\Phi^{\mathrm{rlf}}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ of the divisor monoid $\Phi_{\mathcal{C}^{\vdash}_{\underline{v}}}$ of $\mathcal{C}^{\vdash}_{\underline{v}}$ [which, as is easily verified, is a *constant monoid* over the corresponding base category] may be regarded as a *single abstract monoid* isomorphic to $\mathbb{R}_{\geq 0}$. Write $\log_{\Phi}(p_{\underline{v}}) \in \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ for the element defined by $p_{\underline{v}}$ and

$$\mathcal{C}_{\rho_{\underline{v}}} : \mathcal{C}^{\Vdash}_{\mathrm{mod}} \to (\mathcal{C}^{\vdash}_{\underline{v}})^{\mathrm{rlf}}$$

for the *natural restriction functor* [cf. the theory of poly-Frobenioids developed in [FrdII], §5] to the *realification* of the Frobenioid $\mathcal{C}^{\vdash}_{\underline{v}}$ [cf. [FrdI], Proposition 5.3]. Thus, one verifies immediately that $\mathcal{C}_{\rho_{\underline{v}}}$ is determined, up to isomorphism, by the *isomorphism of topological monoids* [which are isomorphic to $\mathbb{R}_{\geq 0}$]

$$\rho_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\vdash}_{\underline{v}}}$$

induced by $\mathcal{C}_{\rho_{\underline{v}}}$ — which, by considering the natural *"volume interpretations"* of the arithmetic divisors involved, is easily computed to be given by the assignment $\log^{\vdash}_{\mathrm{mod}}(p_v) \mapsto \frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]} \log_{\Phi}(p_{\underline{v}})$.

(ii) In a similar vein, one may construct a *"Θ-version"* [i.e., as in Examples 3.2, (v); 3.3, (ii); 3.4, (iii)] of the various data constructed in (i). That is to say, we set

$$\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}}} \stackrel{\mathrm{def}}{=} \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}} \cdot \log(\underline{\underline{\Theta}})$$

— i.e., an isomorphic copy of $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}}$ generated by a *formal symbol* $\log(\underline{\underline{\Theta}})$. This monoid $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}}}$ thus determines a *Frobenioid* $\mathcal{C}^{\Vdash}_{\mathrm{tht}}$, equipped with a *natural equivalence of categories* $\mathcal{C}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathrm{tht}}$ and a *natural bijection* $\mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{tht}}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$. For $v \in \mathbb{V}_{\mathrm{mod}}$, the element $\log^{\vdash}_{\mathrm{mod}}(p_v)$ of the submonoid $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}},v} \subseteq \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod}}}$ thus determines an element $\log^{\vdash}_{\mathrm{mod}}(p_v) \cdot \log(\underline{\underline{\Theta}})$ of a submonoid $\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}},v} \subseteq \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}}}$. Write $\underline{v} \in \underline{\mathbb{V}}$ for the element of $\underline{\mathbb{V}}$ that corresponds to v. Then the *realification* $\Phi^{\mathrm{rlf}}_{\mathcal{C}^{\Theta}_{\underline{v}}}$ of the divisor monoid $\Phi_{\mathcal{C}^{\Theta}_{\underline{v}}}$ of $\mathcal{C}^{\Theta}_{\underline{v}}$ [which, as is easily verified, is a *constant monoid* over the corresponding base category] may be regarded as a *single abstract monoid* isomorphic to $\mathbb{R}_{\geq 0}$. Write

$$\mathcal{C}_{\rho^{\Theta}_{\underline{v}}} : \mathcal{C}^{\Vdash}_{\mathrm{tht}} \to (\mathcal{C}^{\Theta}_{\underline{v}})^{\mathrm{rlf}}$$

for the *natural restriction functor* [cf. (i) above; the theory of poly-Frobenioids developed in [FrdII], §5] to the *realification* of the Frobenioid $\mathcal{C}^{\Theta}_{\underline{v}}$ [cf. [FrdI], Proposition 5.3]. Thus, one verifies immediately that $\mathcal{C}_{\rho^{\Theta}_{\underline{v}}}$ is determined, up to isomorphism, by the *isomorphism of topological monoids* [which are isomorphic to $\mathbb{R}_{\geq 0}$]

$$\rho^{\Theta}_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{tht}},v} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\Theta}_{\underline{v}}}$$

induced by $\mathcal{C}_{\rho^{\Theta}_{\underline{v}}}$. If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, then write $\log_{\Phi}(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}}) \in \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\Theta}_{\underline{v}}}$ for the element determined by $\log_{\Phi}(p_{\underline{v}})$; thus, [cf. (i)] $\rho^{\Theta}_{\underline{v}}$ is given by the assignment $\log^{\vdash}_{\mathrm{mod}}(p_v) \cdot$

$\log(\underline{\underline{\Theta}}) \mapsto \frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]}\log_{\Phi}(p_{\underline{v}}) \cdot \log(\underline{\underline{\Theta}})$. On the other hand, if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, then let us write

$$\log_{\Phi}(\underline{\underline{\Theta}}_{\underline{v}}) \in \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\Theta}_{\underline{v}}}$$

for the element determined by $\underline{\underline{\Theta}}_{\underline{v}}$ [cf. Example 3.2, (v)] and $\log_{\Phi}(p_{\underline{v}})$ for the constant section of $\Phi_{\mathcal{C}_{\underline{v}}}$ determined by $p_{\underline{v}}$ [cf. the notation "$\log_{\Phi}(\underline{\underline{q}}_{\underline{v}})$" of Example 3.2, (iv)]; in particular, it makes sense to write $\log_{\Phi}(p_{\underline{v}})/\log_{\Phi}(\underline{\underline{q}}_{\underline{v}}) \in \mathbb{Q}_{>0}$; thus, [cf. (i)] $\rho^{\Theta}_{\underline{v}}$ is given by the assignment

$$\log^{\vdash}_{\mathrm{mod}}(p_v) \cdot \log(\underline{\underline{\Theta}}) \mapsto \frac{\log_{\Phi}(p_{\underline{v}})}{[K_{\underline{v}} : (F_{\mathrm{mod}})_v]} \cdot \frac{\log(\underline{\underline{\Theta}}_{\underline{v}})}{\log_{\Phi}(\underline{\underline{q}}_{\underline{v}})}$$

— cf. Remark 3.5.1, (i), below. Note that, for arbitrary $\underline{v} \in \underline{\mathbb{V}}$, the various $\rho_{\underline{v}}$, $\rho^{\Theta}_{\underline{v}}$ are *compatible* with the *natural isomorphisms* $\mathcal{C}^{\Vdash}_{\mathrm{mod}} \overset{\sim}{\to} \mathcal{C}^{\Vdash}_{\mathrm{tht}}$, $\mathcal{C}^{\vdash}_{\underline{v}} \overset{\sim}{\to} \mathcal{C}^{\Theta}_{\underline{v}}$ [cf. §0]. This fact may be expressed as a *natural isomorphism between collections of data* [consisting of a category, a bijection of sets, a collection of data indexed by $\underline{\mathbb{V}}$, and a collection of isomorphisms indexed by $\underline{\mathbb{V}}$]

$$\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \quad \overset{\sim}{\to} \quad \mathfrak{F}^{\Vdash}_{\mathrm{tht}}$$

— where we write

$$\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \overset{\mathrm{def}}{=} (\mathcal{C}^{\Vdash}_{\mathrm{mod}},\ \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \overset{\sim}{\to} \underline{\mathbb{V}},\ \{\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$
$$\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \overset{\mathrm{def}}{=} (\mathcal{C}^{\Vdash}_{\mathrm{tht}},\ \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{tht}}) \overset{\sim}{\to} \underline{\mathbb{V}},\ \{\mathcal{F}^{\Theta}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{\rho^{\Theta}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

[and we apply the natural bijection $\underline{\mathbb{V}} \overset{\sim}{\to} \mathbb{V}_{\mathrm{mod}}$]; cf. Remark 3.5.2 below.

(iii) One may also construct a *"$\mathcal{D}$-version"* — which, from the point of view of the theory of [AbsTopIII], one may also think of as a *"log-shell version"* — of the various data constructed in (i), (ii). To this end, we write

$$\mathcal{D}^{\Vdash}_{\mathrm{mod}}$$

for a [i.e., another] copy of $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$. Thus, one may associate to $\mathcal{D}^{\Vdash}_{\mathrm{mod}}$ various objects $\Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}}}$, $\mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{mod}}) \overset{\sim}{\to} \mathbb{V}_{\mathrm{mod}}$, $\log^{\mathcal{D}}_{\mathrm{mod}}(p_v) \in \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}},v} \subseteq \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}}}$ [for $v \in \mathbb{V}_{\mathrm{mod}}$] that map to the corresponding objects associated to $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ under the *tautological equivalence of categories* $\mathcal{C}^{\Vdash}_{\mathrm{mod}} \overset{\sim}{\to} \mathcal{D}^{\Vdash}_{\mathrm{mod}}$. Write $\underline{v} \in \underline{\mathbb{V}}$ for the element of $\underline{\mathbb{V}}$ that corresponds to v. Next, suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$; then let us recall from [AbsTopIII], Proposition 5.8, (iii), that [since the profinite group associated to $\mathcal{D}^{\vdash}_{\underline{v}}$ is the absolute Galois group of an MLF] one may *construct algorithmically* from $\mathcal{D}^{\vdash}_{\underline{v}}$ a *topological monoid* isomorphic to $\mathbb{R}_{\geq 0}$

$$(\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$$

[i.e., the topological monoid determined by the nonnegative elements of the ordered topological group "$\mathbb{R}_{\mathrm{non}}(G)$" of *loc. cit.*] equipped with a *distinguished "Frobenius element"* $\in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$; if $e_{\underline{v}}$ is the *absolute ramification index* of the MLF $K_{\underline{v}}$, then we shall write $\log^{\mathcal{D}}_{\Phi}(p_{\underline{v}}) \in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$ for the result of multiplying this Frobenius element by [the positive real number] $e_{\underline{v}}$. Next, suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$; then let us recall from [AbsTopIII], Proposition 5.8, (vi), that [since, by definition, $\mathcal{D}^{\vdash}_{\underline{v}} \in \mathrm{Ob}(\mathbb{TM}^{\vdash})$] one may *construct algorithmically* from $\mathcal{D}^{\vdash}_{\underline{v}}$ a *topological monoid* isomorphic to $\mathbb{R}_{\geq 0}$

$$(\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$$

[i.e., the topological monoid determined by the nonnegative elements of the ordered topological group "$\mathbb{R}_{\mathrm{arc}}(G)$" of *loc. cit.*] equipped with a *distinguished "Frobenius element"* $\in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$; we shall write $\log^{\mathcal{D}}_{\Phi}(p_{\underline{v}}) \in (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$ for the result of dividing this Frobenius element by [the positive real number] 2π. In particular, for every $\underline{v} \in \underline{\mathbb{V}}$, we obtain a uniquely determined *isomorphism of topological monoids* [which are isomorphic to $\mathbb{R}_{\geq 0}$]

$$\rho^{\mathcal{D}}_{\underline{v}} : \Phi_{\mathcal{D}^{\Vdash}_{\mathrm{mod}}, v} \xrightarrow{\sim} (\mathbb{R}^{\vdash}_{\geq 0})_{\underline{v}}$$

by assigning $\log^{\mathcal{D}}_{\mathrm{mod}}(p_v) \mapsto \frac{1}{[K_{\underline{v}}:(F_{\mathrm{mod}})_v]} \log^{\mathcal{D}}_{\Phi}(p_{\underline{v}})$. Thus, we obtain *data* [consisting of a Frobenioid, a bijection of sets, a collection of data indexed by $\underline{\mathbb{V}}$, and a collection of isomorphisms indexed by $\underline{\mathbb{V}}$]

$$\mathfrak{F}^{\Vdash}_{\mathcal{D}} \stackrel{\mathrm{def}}{=} (\mathcal{D}^{\Vdash}_{\mathrm{mod}}, \ \mathrm{Prime}(\mathcal{D}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \ \{\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}, \ \{\rho^{\mathcal{D}}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}})$$

[where we apply the natural bijection $\underline{\mathbb{V}} \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$], which, by [AbsTopIII], Proposition 5.8, (iii), (vi), may be *reconstructed algorithmically* from the *data* $\{\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$.

Remark 3.5.1.

(i) The formal symbol "$\log(\underline{\underline{\Theta}})$" may be thought of as the result of *identifying* the various formal quotients "$\log(\underline{\underline{\Theta}}_{\underline{v}})/\log_{\Phi}(\underline{\underline{q}}_{\underline{v}})$", as $\underline{v}$ varies over the elements of $\underline{\mathbb{V}}^{\mathrm{bad}}$.

(ii) The global Frobenioids $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$, $\mathcal{C}^{\Vdash}_{\mathrm{tht}}$ of Example 3.5 may be thought of as *"devices for currency exchange"* between the various *"local currencies"* constituted by the divisor monoids at the various $\underline{v} \in \underline{\mathbb{V}}$.

(iii) One may also formulate the data contained in $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$, $\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$ via the language of *poly-Frobenioids* as developed in [FrdII], §5, but we shall not pursue this topic in the present series of papers.

Remark 3.5.2. In Example 3.5, as well as in the following discussion, we shall often speak of *"isomorphisms of collections of data"*, relative to the following conventions.

(i) Such isomorphisms are always assumed to satisfy various *evident compatibility conditions*, relative to the various relationships stipulated between the various constituent data, whose explicit mention we shall omit for the sake of simplicity.

(ii) In situations where the collections of data consist partially of various *categories*, the portion of the "isomorphism of collections of data" involving corresponding categories is to be understood as an *isomorphism class of equivalences of categories* [cf. §0].

Definition 3.6. Fix a collection of *initial* Θ-*data* $(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \underline{\epsilon})$ as in Definition 3.1. In the following, we shall use the various notations introduced in Definition 3.1 for various objects associated to this initial Θ-data. Then we define a Θ-*Hodge theater* [relative to the given initial Θ-data] to be a collection of data

$$^{\dagger}\mathcal{HT}^{\Theta} = (\{^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ ^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$$

that satisfies the following conditions:

(a) If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is a *category* which admits an equivalence of categories $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} \xrightarrow{\sim} \underline{\underline{\mathcal{F}}}_{\underline{v}}$ [where $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is as in Examples 3.2, (i); 3.3, (i)]. In particular, $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ admits a natural *Frobenioid structure* [cf. [FrdI], Corollary 4.11, (iv)], which may be *constructed solely from the category-theoretic structure of* $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$. Write $^{\dagger}\mathcal{D}_{\underline{v}}$, $^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$, $^{\dagger}\mathcal{D}^{\Theta}_{\underline{v}}$, $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$, $^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}$ for the objects *constructed category-theoretically from* $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ that correspond to the objects without a "$\dagger$" discussed in Examples 3.2, 3.3 [cf., especially, Examples 3.2, (vi); 3.3, (iii)].

(b) If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is a *collection of data* $(^{\dagger}\mathcal{C}_{\underline{v}}, {}^{\dagger}\mathcal{D}_{\underline{v}}, {}^{\dagger}\kappa_{\underline{v}})$ — where $^{\dagger}\mathcal{C}_{\underline{v}}$ is a *category* equivalent to the category $\mathcal{C}_{\underline{v}}$ of Example 3.4, (i); $^{\dagger}\mathcal{D}_{\underline{v}}$ is an Aut-*holomorphic orbispace*; and $^{\dagger}\kappa_{\underline{v}} : \mathcal{O}^{\triangleright}(^{\dagger}\mathcal{C}_{\underline{v}}) \hookrightarrow \mathcal{A}_{^{\dagger}\mathcal{D}_{\underline{v}}}$ is an inclusion of topological monoids, which we shall refer to as the *Kummer structure* on $^{\dagger}\mathcal{C}_{\underline{v}}$ — such that there exists an isomorphism of collections of data $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} \xrightarrow{\sim} \underline{\underline{\mathcal{F}}}_{\underline{v}}$ [where $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is as in Example 3.4, (i)]. Write $^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$, $^{\dagger}\mathcal{D}^{\Theta}_{\underline{v}}$, $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$, $^{\dagger}\mathcal{F}^{\Theta}_{\underline{v}}$ for the objects *constructed algorithmically from* $^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}$ that correspond to the objects without a "$\dagger$" discussed in Example 3.4, (ii), (iii).

(c) $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ is a *collection of data*

$$(^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}},\ \mathrm{Prime}(^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}},\ \{^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ \{^{\dagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$$

— where $^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ is a *category* which admits an equivalence of categories $^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathrm{mod}}$ [which implies that $^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ admits a natural category-theoretically constructible *Frobenioid structure* — cf. [FrdI], Corollary 4.11, (iv); [FrdI], Theorem 6.4, (i)]; $\mathrm{Prime}(^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \xrightarrow{\sim} \underline{\mathbb{V}}$ is a bijection of sets, where we write $\mathrm{Prime}(^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}})$ for the set of primes constructed from the category $^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ [cf. [FrdI], Theorem 6.4, (iii)]; $^{\dagger}\mathcal{F}^{\vdash}_{\underline{v}}$ is as discussed in (a), (b) above; $^{\dagger}\rho_{\underline{v}} : \Phi_{^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{mod}},\underline{v}} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}}$ [where we use notation as in the discussion of Example 3.5, (i)] is an *isomorphism of topological monoids*. Moreover, we require that there exist an *isomorphism of collections of data* $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ [where $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ is as in Example 3.5, (ii)]. Write $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}}$, $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathcal{D}}$ for the objects *constructed algorithmically from* $^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ that correspond to the objects without a "$\dagger$" discussed in Example 3.5, (ii), (iii).

Remark 3.6.1. When we discuss various collections of Θ-Hodge theaters, labeled by some symbol "$\Box$" in place of a "$\dagger$", we shall use apply the notation of Definition

3.6 with "†" replaced by "□" to denote the various objects associated to the Θ-Hodge theater labeled by "□".

Remark 3.6.2. If ${}^{\dagger}\mathcal{HT}^{\Theta}$ and ${}^{\ddagger}\mathcal{HT}^{\Theta}$ are Θ-*Hodge theaters*, then there is an evident notion of *isomorphism of* Θ-*Hodge theaters* ${}^{\dagger}\mathcal{HT}^{\Theta} \xrightarrow{\sim} {}^{\ddagger}\mathcal{HT}^{\Theta}$ [cf. Remark 3.5.2]. We leave the routine details to the interested reader.

Corollary 3.7. **(Θ-Links Between Θ-Hodge Theaters)** *Fix a collection of* **initial Θ-data** *(* $\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \underline{\epsilon}$ *) as in Definition 3.1. Let*

$$ {}^{\dagger}\mathcal{HT}^{\Theta} = (\{{}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}); \quad {}^{\ddagger}\mathcal{HT}^{\Theta} = (\{{}^{\ddagger}\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}) $$

be **Θ-Hodge theaters** *[relative to the given initial Θ-data]. Then:*

(i) **(Θ-Link)** *The* **full poly-isomorphism** *[cf. §0] between collections of data [cf. Remark 3.5.2]*

$$ {}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}} $$

is **nonempty** *[cf. Remark 3.7.1 below]. We shall refer to this full poly-isomorphism as the* **Θ-link**

$$ {}^{\dagger}\mathcal{HT}^{\Theta} \quad \xrightarrow{\Theta} \quad {}^{\ddagger}\mathcal{HT}^{\Theta} $$

from ${}^{\dagger}\mathcal{HT}^{\Theta}$ *to* ${}^{\ddagger}\mathcal{HT}^{\Theta}$*.*

(ii) **(Preservation of "$\mathcal{D}^{\vdash}$")** *Let* $\underline{v} \in \underline{\mathbb{V}}$*. Recall the* **tautological isomorphisms** ${}^{\square}\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} {}^{\square}\mathcal{D}^{\Theta}_{\underline{v}}$ *for* $\square = \dagger, \ddagger$ *— i.e., which arise from the definitions when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$*, and which arise from a natural product functor [cf. Example 3.2, (v)] when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*. Then we obtain isomorphisms*

$$ {}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\Theta}_{\underline{v}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}^{\vdash}_{\underline{v}} $$

by composing the tautological isomorphism just mentioned with any isomorphism induced by a **Θ-link isomorphism** *as in (i).*

(iii) **(Preservation of "$\mathcal{O}^{\times}$")** *Let* $\underline{v} \in \underline{\mathbb{V}}$*. Recall the* **tautological isomorphisms** $\mathcal{O}^{\times}_{{}^{\square}\mathcal{C}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^{\times}_{{}^{\square}\mathcal{C}^{\Theta}_{\underline{v}}}$ *[where we omit the notation "$(-)$"] for* $\square = \dagger, \ddagger$ *— i.e., which arise from the definitions when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ *[cf. Examples 3.3, (ii); 3.4, (iii)], and which are induced by the natural product functor [cf. Example 3.2, (v)] when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$*. Then, relative to the corresponding composite isomorphism of (ii), we obtain a composite isomorphism*

$$ \mathcal{O}^{\times}_{{}^{\dagger}\mathcal{C}^{\vdash}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^{\times}_{{}^{\dagger}\mathcal{C}^{\Theta}_{\underline{v}}} \xrightarrow{\sim} \mathcal{O}^{\times}_{{}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}} $$

by composing the tautological isomorphism just mentioned with any isomorphism induced by a **Θ-link isomorphism** *as in (i).*

Proof. The various assertions of Corollary 3.7 follow immediately from the definitions and the discussion of Examples 3.2, 3.3, 3.4, and 3.5. ◯

Remark 3.7.1. One verifies immediately that there exist *many distinct isomorphisms* ${}^{\dagger}\mathfrak{F}^{\Vdash}_{\mathrm{tht}} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ as in Corollary 3.7, (i), *none* of which is conferred a *"distinguished"* status, i.e., in the fashion of the *"natural isomorphism* $\mathfrak{F}^{\Vdash}_{\mathrm{mod}} \xrightarrow{\sim} \mathfrak{F}^{\Vdash}_{\mathrm{tht}}$*"* discussed in Example 3.5, (ii).

The following result follows formally from Corollary 3.7.

Corollary 3.8. **(Frobenius-pictures of Θ-Hodge Theaters)** *Fix a collection of initial Θ-data as in Corollary 3.7. Let $\{{}^{n}\mathcal{HT}^{\Theta}\}_{n\in\mathbb{Z}}$ be a* **collection of distinct Θ-Hodge theaters** *indexed by the integers. Then by applying Corollary 3.7, (i), with ${}^{\dagger}\mathcal{HT}^{\Theta} \stackrel{\mathrm{def}}{=} {}^{n}\mathcal{HT}^{\Theta}$, ${}^{\ddagger}\mathcal{HT}^{\Theta} \stackrel{\mathrm{def}}{=} {}^{(n+1)}\mathcal{HT}^{\Theta}$, we obtain an* **infinite chain**

$$\dots \quad \xrightarrow{\Theta} \quad {}^{(n-1)}\mathcal{HT}^{\Theta} \quad \xrightarrow{\Theta} \quad {}^{n}\mathcal{HT}^{\Theta} \quad \xrightarrow{\Theta} \quad {}^{(n+1)}\mathcal{HT}^{\Theta} \quad \xrightarrow{\Theta} \quad \dots$$

of **Θ-linked Θ-Hodge theaters**. *This infinite chain may be represented symbolically as an* **oriented graph $\vec{\Gamma}$** *[cf. [AbsTopIII], §0]*

$$\dots \quad \rightarrow \quad \bullet \quad \rightarrow \quad \bullet \quad \rightarrow \quad \bullet \quad \rightarrow \quad \dots$$

— i.e., where the arrows correspond to the "$\xrightarrow{\Theta}$'s", and the "$\bullet$'s" correspond to the "${}^{n}\mathcal{HT}^{\Theta}$". This oriented graph $\vec{\Gamma}$ admits a natural action by $\mathbb{Z}$ — i.e., a **translation symmetry** *— but it does* **not admit arbitrary permutation symmetries**. *For instance, $\vec{\Gamma}$ does not admit an automorphism that switches two adjacent vertices, but leaves the remaining vertices fixed. Put another way, from the point of view of the discussion of [FrdI], Introduction, the mathematical structure constituted by this infinite chain is* **"Frobenius-like"**, *or* **"order-conscious"**. *It is for this reason that we shall refer to this infinite chain in the following discussion as the* **Frobenius-picture**.

Remark 3.8.1.

(i) Perhaps the central defining aspect of the Frobenius-picture is the fact that the Θ-link maps

$${}^{n}\underline{\underline{\Theta}}_{\underline{v}} \quad \mapsto \quad {}^{(n+1)}\underline{\underline{q}}_{\underline{v}}$$

[cf. the discussion of Example 3.2, (v)].

$$\dots \quad \text{- - - -} \quad \boxed{{}^{n}\underline{\underline{q}}_{\underline{v}} \quad \rightsquigarrow \quad {}^{n}\underline{\underline{\Theta}}_{\underline{v}}} \quad \text{- - - -} \quad \boxed{{}^{(n+1)}\underline{\underline{q}}_{\underline{v}} \quad \rightsquigarrow \quad {}^{(n+1)}\underline{\underline{\Theta}}_{\underline{v}}} \quad \text{- - - -} \quad \dots$$

Fig. 3.1: Frobenius-picture of Θ-Hodge theaters

From this point of view, the Frobenius-picture may be depicted as in Fig. 3.1 — i.e., each box is a Θ-Hodge theater; the "$\rightsquigarrow$" may be thought of as denoting the scheme theory that lies between "$\underline{\underline{q}}_{\underline{v}}$" and "$\underline{\underline{\Theta}}_{\underline{v}}$"; the "- - - -" denotes the Θ-link.

(ii) It is perhaps not surprising [cf. the theory of [FrdI]] that the Frobenius-picture involves, in an essential way, the *divisor monoid* portion [i.e., "$\underline{\underline{q}}_{\underline{v}}$" and "$\underline{\underline{\Theta}}_{\underline{v}}$"] of the various Frobenioids that appear in a Θ-Hodge theater. Put another way,

> it is as if the *"Frobenius-like nature"* of the divisor monoid portion of the Frobenioids involved *induces the "Frobenius-like nature" of the Frobenius-picture.*

By contrast, *observe* that for $\underline{v} \in \underline{\mathbb{V}}$, the isomorphisms

$$\ldots \overset{\sim}{\rightarrow} {}^n\mathcal{D}^{\vdash}_{\underline{v}} \overset{\sim}{\rightarrow} {}^{(n+1)}\mathcal{D}^{\vdash}_{\underline{v}} \overset{\sim}{\rightarrow} \ldots$$

of Corollary 3.7, (ii), imply that if one thinks of the various ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ as being only known *up to isomorphism*, then

> *one may regard ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ as a sort of* **constant invariant** *of the various Θ-Hodge theaters that constitute the Frobenius-picture*

— cf. Remark 3.9.1 below. This *observation* is the starting point of the theory of the *étale-picture* [cf. Corollary 3.9, (i), below]. Note that by Corollary 3.7, (iii), we also obtain isomorphisms

$$\ldots \overset{\sim}{\rightarrow} \mathcal{O}^{\times}_{{}^n\mathcal{C}^{\vdash}_{\underline{v}}} \overset{\sim}{\rightarrow} \mathcal{O}^{\times}_{{}^{(n+1)}\mathcal{C}^{\vdash}_{\underline{v}}} \overset{\sim}{\rightarrow} \ldots$$

lying over the isomorphisms involving the "${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$" discussed above.

(iii) In the situation of (ii), suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. Then ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ is simply the category of connected objects of the Galois category associated to the profinite group $G_{\underline{v}}$. That is to say, one may think of ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ as representing *"$G_{\underline{v}}$ up to isomorphism"*. Then each ${}^n\mathcal{D}_{\underline{v}}$ represents an *"isomorph of the topological group $\Pi_{\underline{v}}$,* **labeled** *by n, which is regarded as an extension of some isomorph of $G_{\underline{v}}$ that is* **independent** *of n"*. In particular, the quotients corresponding to $G_{\underline{v}}$ of the copies of $\Pi_{\underline{v}}$ that arise from ${}^n\mathcal{HT}^{\Theta}$ for different n are only related to one another via some *indeterminate* isomorphism. Thus, from the point of view of the theory of [AbsTopIII] [cf. [AbsTopIII], §I3; [AbsTopIII], Remark 5.10.2, (ii)], each $\Pi_{\underline{v}}$ gives rise to a *well-defined ring structure* — i.e., a *"holomorphic structure"* — which is *obliterated* by the indeterminate isomorphism between the quotient isomorphs of $G_{\underline{v}}$ arising from ${}^n\mathcal{HT}^{\Theta}$ for distinct n.

(iv) In the situation of (ii), suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Then ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ is an object of $\mathbb{TM}^{\vdash}$; each ${}^n\mathcal{D}_{\underline{v}}$ represents an *"isomorph of the* Aut-*holomorphic orbispace $\underline{\mathbb{X}}_{\rightarrow \underline{v}}$,* **labeled** *by n, whose associated [complex archimedean] topological field $\overline{\mathcal{A}}_{\underline{\mathbb{X}}_{\rightarrow \underline{v}}}$ gives rise to an isomorph of $\mathcal{D}^{\vdash}_{\underline{v}}$ that is* **independent** *of n"*. In particular, the various isomorphs of $\mathcal{D}^{\vdash}_{\underline{v}}$ associated to the copies of $\underline{\mathbb{X}}_{\rightarrow \underline{v}}$ that arise from ${}^n\mathcal{HT}^{\Theta}$ for different n are only related to one another via some *indeterminate* isomorphism. Thus, from the point of view of the theory of [AbsTopIII] [cf. [AbsTopIII], §I3; [AbsTopIII], Remark 5.10.2, (ii)], each $\underrightarrow{\mathbb{X}}_{\underline{v}}$ gives rise to a *well-defined ring structure* — i.e., a *"holomorphic structure"* — which is *obliterated* by the indeterminate isomorphism between the isomorphs of $\mathcal{D}^{\vdash}_{\underline{v}}$ arising from ${}^{n}\mathcal{HT}^{\Theta}$ for distinct n.

The discussion of Remark 3.8.1, (iii), (iv), may be summarized as follows.

Corollary 3.9. **(Étale-pictures of Θ-Hodge Theaters)** *In the situation of Corollary 3.8, let* $\underline{v} \in \underline{\mathbb{V}}$. *Then:*

$$\begin{tikzcd}
\cdots & \boxed{{}^{n}\mathcal{D}_{\underline{v}}} \arrow[d, dash, dashed] & \cdots \\
\boxed{{}^{(n-1)}\mathcal{D}_{\underline{v}}} \arrow[r, dash, dashed] & \boxed{\mathcal{D}^{\vdash}_{\underline{v}}} \arrow[r, dash, dashed] \arrow[d, dash, dashed] & \boxed{{}^{(n+1)}\mathcal{D}_{\underline{v}}} \\
\cdots & \boxed{{}^{(n+2)}\mathcal{D}_{\underline{v}}} & \cdots
\end{tikzcd}$$

Fig. 3.2: Étale-picture of Θ-Hodge theaters

(i) We have a **diagram** *as in Fig. 3.2, which we refer to as the* **étale-picture**. *Here, each horizontal and vertical "— —" denotes the relationship between* ${}^{(-)}\mathcal{D}_{\underline{v}}$ *and* $\mathcal{D}^{\vdash}_{\underline{v}}$ *— i.e., an extension of topological groups when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, or the underlying object of* $\mathbb{TM}^{\vdash}$ *arising from the associated topological field when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ *— discussed in Remark 3.8.1, (iii), (iv). [Unlike the Frobenius-picture!] the étale-picture admits* **arbitrary permutation symmetries** *among the labels* $n \in \mathbb{Z}$ *corresponding to the various Θ-Hodge theaters. Put another way, the étale-picture may be thought of as a sort of* **canonical splitting** *of the Frobenius-picture.*

(ii) In a similar vein, we have a **diagram** *as in Fig. 3.3 below, obtained by replacing the "*$\mathcal{D}^{\vdash}_{\underline{v}}$*" in the middle of Fig. 3.2 by "*$\mathcal{D}^{\vdash}_{\underline{v}} \curvearrowright \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$*". Here, each horizontal and vertical "— —" denotes the relationship between* ${}^{(-)}\mathcal{D}_{\underline{v}}$ *and* $\mathcal{D}^{\vdash}_{\underline{v}}$ *discussed in (i); when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$*, the notation "*$\mathcal{D}^{\vdash}_{\underline{v}} \curvearrowright \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$*" denotes an isomorph of the pair consisting of the category* $\mathcal{D}^{\vdash}_{\underline{v}}$ *together with the group-like monoid* $\mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ *on* $\mathcal{D}^{\vdash}_{\underline{v}}$*; when* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$*, the notation "*$\mathcal{D}^{\vdash}_{\underline{v}} \curvearrowright \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$*" denotes an isomorph of the pair consisting of the object* $\mathcal{D}^{\vdash}_{\underline{v}} \in \mathrm{Ob}(\mathbb{TM}^{\vdash})$ *and the topological group* $\mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$ *[which*

is isomorphic — but not canonically! — to the compact factor of $\mathcal{D}^{\vdash}_{\underline{v}}$*]. Just as in the case of (i), this diagram admits* **arbitrary permutation symmetries** *among the labels* $n \in \mathbb{Z}$ *corresponding to the various* Θ*-Hodge theaters.*

$$
\begin{array}{ccccc}
 & & \boxed{{}^{n}\mathcal{D}_{\underline{v}}} & & \\
\dots & & | & & \dots \\
 & & | & & \\
\boxed{{}^{(n-1)}\mathcal{D}_{\underline{v}}} & - \ - & \boxed{\mathcal{D}^{\vdash}_{\underline{v}} \ \curvearrowright \ \mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}} & - \ - & \boxed{{}^{(n+1)}\mathcal{D}_{\underline{v}}} \\
 & & | & & \\
\dots & & | & & \dots \\
 & & \boxed{{}^{(n+2)}\mathcal{D}_{\underline{v}}} & &
\end{array}
$$

Fig. 3.3: Étale-picture plus units

Remark 3.9.1. If one formulates things relative to the language of [AbsTopIII], Definition 3.5, then ${}^{(-)}\mathcal{D}^{\vdash}_{\underline{v}}$ constitutes a **core**. Relative to the theory of [AbsTopIII], §5, this core is essentially the **mono-analytic core** discussed in [AbsTopIII], §I3; [AbsTopIII], Remark 5.10.2, (ii). Indeed, the symbol "$\vdash$" is intended — both in [AbsTopIII] and in the present series of papers! — as an abbreviation for the term *"mono-analytic"*.

Remark 3.9.2. Whereas the *étale-picture* of Corollary 3.9, (i), will remain valid throughout the development of the remainder of the theory of the present series of papers, the local units "$\mathcal{O}^{\times}_{\mathcal{C}^{\vdash}_{\underline{v}}}$" that appear in Corollary 3.9, (ii), will ultimately *cease to be a constant invariant* of various enhanced versions of the Frobenius-picture that will arise in the theory of [IUTchIII]. In a word, these enhancements revolve around the incorporation into each Hodge theater of the *"rotation of addition (i.e., '$\boxplus$') and multiplication (i.e., '$\boxtimes$')"* in the style of the theory of [AbsTopIII].

Remark 3.9.3.

(i) As discussed in [AbsTopIII], §I3; [AbsTopIII], Remark 5.10.2, (ii), the *"mono-analytic core"* $\{\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ may be thought of as a sort of **fixed underlying real-analytic surface** associated to a number field on which various *holomorphic structures* are imposed. Then the Frobenius-picture in its entirety may be thought of as a sort of **global arithmetic analogue** of the notion of a **Teichmüller geodesic** *in classical complex Teichmüller theory* or, alternatively, as a global arithmetic analogue of the **canonical liftings** *of p-adic Teichmüller theory* [cf. the discussion of [AbsTopIII], §I5].

(ii) Recall that in classical complex Teichmüller theory, **one** of the **two** real dimensions of the surface is **dilated** as one moves along the Teichmüller geodesic, while the **other** of the two real dimensions is **held fixed**. In the case of the Frobenius-picture of Corollary 3.8, the **local units "$\mathcal{O}^\times$"** correspond to the dimension that is **held fixed**, while the **local value groups** are subject to **"Θ-dilations"** as one moves along the diagram constituted by the Frobenius-picture. Note that in order to construct such a mathematical structure in which the local units and local value groups are treated **independently**, it is of crucial importance to avail oneself of the various **characteristic splittings** that appear in the split Frobenioids of Examples 3.2, 3.3. Here, we note in passing that, in the case of Example 3.2, this splitting corresponds to the **"constant multiple rigidity"** *of the étale theta function*, which forms a central theme of the theory of [EtTh].

(iii) In classical complex Teichmüller theory, the two real dimensions of the surface that are treated independently of one another correspond to the **real** and **imaginary** parts of the coordinate obtained by locally integrating the square root of a given square differential. In particular, it is of crucial importance in classical complex Teichmüller theory that these real and imaginary parts *not be "subject to confusion with one another"*. In the case of the square root of a square differential, the only indeterminacy that arises is indeterminacy with respect to *multiplication by* -1, an operation that satisfies the *crucial property of preserving the real and imaginary parts of a complex number*. By contrast, it is interesting to note that

> if, for $n \geq 3$, one attempts to construct Teichmüller deformations in the fashion of classical complex Teichmüller theory by means of coordinates obtained by *locally integrating the n-th root of a given section of the n-th tensor power of the sheaf of differentials*, then one must contend with an indeterminacy with respect to *multiplication by an n-th root of unity*, an operation that results in an *essential confusion between the real and imaginary parts of a complex number*.

(iv) Whereas linear movement along the oriented graph $\vec{\Gamma}$ of Corollary 3.8 corresponds to the *linear flow* along a Teichmüller geodesic, the *"rotation of addition (i.e., '$\boxplus$') and multiplication (i.e., '$\boxtimes$')"* in the style of the theory of [AbsTopIII] — which will be incorporated into the theory of the present series of papers in [IUTchIII] [cf. Remark 3.9.2] — corresponds to *rotations around a fixed point* in the complex geometry arising from Teichmüller theory [cf., e.g., the discussion of [AbsTopIII], §I3; the hyperbolic geometry of the upper half-plane, regarded as the "Teichmüller space" of compact Riemann surfaces of genus 1]. Alternatively, in the analogy with p-adic Teichmüller theory, this "rotation of $\boxplus$ and $\boxtimes$" corresponds to the *Frobenius morphism in positive characteristic* — cf. the discussion of [AbsTopIII], §I5.

Remark 3.9.4. At first glance, the assignment "${}^n\underline{\underline{\Theta}}_{\underline{v}} \mapsto {}^{(n+1)}\underline{\underline{q}}_{\underline{v}}$" [cf. Remark 3.8.1, (i)] may strike the reader as being nothing more than a *"conventional evaluation map"* [i.e., of the theta function at a torsion point — cf. the discussion of Example 3.2, (iv)]. Although we shall ultimately be interested, in the theory of the present series of papers, in such *"Hodge-Arakelov-style evaluation maps"* [within a fixed Hodge theater!] *of the theta function at torsion points"* [cf. the theory of [IUTchII]], the Θ-link considered here *differs quite fundamentally* from such conventional evaluation maps in the following respect:

> *the* **value** ${}^{(n+1)}\underline{\underline{q}}_{\underline{v}}$ *belongs to a* **distinct scheme theory** *— i.e., the scheme theory represented by the distinct Θ-Hodge theater* ${}^{(n+1)}\mathcal{HT}^{\Theta}$ *— from the* **base** ${}^{n}\underline{\underline{q}}_{\underline{v}}$ *[which belongs to the scheme theory represented by the Θ-Hodge theater* ${}^{n}\mathcal{HT}^{\Theta}$*] over which the theta function* ${}^{n}\underline{\underline{\Theta}}_{\underline{v}}$ *is constructed.*

The distinctness of the ring/scheme theories of distinct Θ-Hodge theaters may be seen, for instance, in the *indeterminacy* of the isomorphism between the associated isomorphs of $\mathcal{D}^{\vdash}_{\underline{v}}$, an indeterminacy which has the effect of *obliterating* the ring structure — i.e., the *"arithmetic holomorphic structure"* — associated to ${}^{n}\mathcal{D}_{\underline{v}}$ for distinct n [cf. the discussion of Remark 3.8.1, (iii), (iv)].

Section 4: Multiplicative Combinatorial Teichmüller Theory

In the present §4, we begin to prepare for the construction of the various "enhancements" to the Θ-Hodge theaters of §3 that will be made in §5. More precisely, in the present §4, we discuss the *combinatorial aspects* of the *"$\mathcal{D}$"* — i.e., in the terminology of the theory of Frobenioids, the *"base category" — portion* of the notions to be introduced in §5 below. In a word, these combinatorial aspects revolve around the **"functorial dynamics"** *imposed upon the various number fields and local fields involved by the* **"labels"**

$$\in \quad \mathbb{F}_l^{\circledast} \quad \stackrel{\mathrm{def}}{=} \quad \mathbb{F}_l^{\times}/\{\pm 1\}$$

— where we note that the set $\mathbb{F}_l^{\circledast}$ is of *cardinality* $l^{\circledast} \stackrel{\mathrm{def}}{=} (l-1)/2$ — of the l-torsion points at which we intend to conduct, in [IUTchII], the *"Hodge-Arakelov-theoretic evaluation"* of the étale theta function studied in [EtTh] [cf. Remarks 4.3.1; 4.3.2; 4.5.1, (v); 4.9.1, (i)].

In the following, we fix a collection of *initial* Θ*-data*

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \underline{\epsilon})$$

as in Definition 3.1; also, we shall use the various notations introduced in Definition 3.1 for various objects associated to this initial Θ-data.

Definition 4.1.

(i) We define a *holomorphic base-prime-strip*, or *$\mathcal{D}$-prime-strip*, [relative to the given initial Θ-data] to be a collection of data

$$^{\dagger}\mathfrak{D} = \{^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then $^{\dagger}\mathcal{D}_{\underline{v}}$ is a *category* which admits an equivalence of categories $^{\dagger}\mathcal{D}_{\underline{v}} \stackrel{\sim}{\to} \mathcal{D}_{\underline{v}}$ [where $\mathcal{D}_{\underline{v}}$ is as in Examples 3.2, (i); 3.3, (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then $^{\dagger}\mathcal{D}_{\underline{v}}$ is an Aut-*holomorphic orbispace* such that there exists an isomorphism of Aut-holomorphic orbispaces $^{\dagger}\mathcal{D}_{\underline{v}} \stackrel{\sim}{\to} \mathcal{D}_{\underline{v}}$ [where $\mathcal{D}_{\underline{v}}$ is as in Example 3.4, (i)]. Observe that if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then $\pi_1(^{\dagger}\mathcal{D}_{\underline{v}})$ determines, in a functorial fashion, a *profinite group corresponding to "$\underline{C}_{\underline{v}}$"* [cf. Corollary 1.2 if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$; [EtTh], Proposition 2.4, if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$], which contains $\pi_1(^{\dagger}\mathcal{D}_{\underline{v}})$ as an open subgroup; thus, if we write $^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$ for $\mathcal{B}(-)^0$ of this profinite group, then we obtain a *natural morphism* $^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$ [cf. §0]. In a similar vein, if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then since $\underrightarrow{X}_{\underline{v}}$ admits a $K_{\underline{v}}$-core, a routine translation into the "language of Aut-holomorphic orbispaces" of the argument given in the proof of Corollary 1.2 [cf. also [AbsTopIII], Corollary 2.4] reveals that $^{\dagger}\mathcal{D}_{\underline{v}}$ determines, in a functorial fashion, an Aut-holomorphic orbispace $^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$ corresponding to "$\underline{C}_{\underline{v}}$", together with a *natural morphism* $^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$ *of* Aut-*holomorphic orbispaces.* Thus, in summary, one obtains a collection of data

$$^{\dagger}\underline{\mathfrak{D}} = \{^{\dagger}\underline{\mathcal{D}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

completely determined by ${}^{\dagger}\mathfrak{D}$.

(ii) Suppose that we are in the situation of (i). Then observe that by applying the *group-theoretic algorithm* of [AbsTopI], Lemma 4.5, to the topological group $\pi_1({}^{\dagger}\mathcal{D}_{\underline{v}})$ when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, or by considering $\pi_0(-)$ of a cofinal collection of "neighborhoods of infinity" [i.e., complements of compact subsets] of the underlying topological space of ${}^{\dagger}\mathcal{D}_{\underline{v}}$ when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, it makes sense to speak of the *set of cusps* of ${}^{\dagger}\mathcal{D}_{\underline{v}}$; a similar observation applies to ${}^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}$. If $\underline{v} \in \underline{\mathbb{V}}$, then we define a *label class of cusps* of ${}^{\dagger}\mathcal{D}_{\underline{v}}$ to be the set of cusps of ${}^{\dagger}\mathcal{D}_{\underline{v}}$ that lie over a single *"nonzero cusp"* [i.e., a cusp that arises from a *nonzero element* of the quotient "Q" that appears in the definition of a "hyperbolic orbicurve of type $(1, l\text{-tors})_{\pm}$" given in [EtTh], Definition 2.1] of ${}^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$; write

$$\mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}})$$

for the *set of label classes of cusps* of ${}^{\dagger}\mathcal{D}_{\underline{v}}$. Thus, [for any $\underline{v} \in \underline{\mathbb{V}}$!] $\mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}})$ admits a natural $\mathbb{F}_l^{\circledast}$*-torsor structure* [cf. [EtTh], Definition 2.1]. Moreover, for each $\underline{v} \in \underline{\mathbb{V}}$, one may construct, solely from ${}^{\dagger}\mathcal{D}_{\underline{v}}$, a *canonical element*

$${}^{\dagger}\underline{\eta}_{\underline{v}} \in \mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}})$$

determined by "$\underline{\epsilon}_{\underline{v}}$" [cf. the notation of Definition 3.1, (f)]. [Indeed, this follows from [EtTh], Corollary 2.9, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, from Corollary 1.2 for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$, and from the evident translation into the "language of Aut-holomorphic orbispaces" of Corollary 1.2 for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$.]

(iii) We define a *mono-analytic base-prime-strip*, or $\mathcal{D}^{\vdash}$*-prime-strip*, [relative to the given initial Θ-data] to be a collection of data

$${}^{\dagger}\mathfrak{D}^{\vdash} = \{{}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$ is a *category* which admits an equivalence of categories ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}^{\vdash}_{\underline{v}}$ [where $\mathcal{D}^{\vdash}_{\underline{v}}$ is as in Examples 3.2, (i); 3.3, (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}}$ is an object of the category $\mathbb{TM}^{\vdash}$ [so, if $\mathcal{D}^{\vdash}_{\underline{v}}$ is as in Example 3.4, (ii), then there exists an isomorphism ${}^{\dagger}\mathcal{D}^{\vdash}_{\underline{v}} \xrightarrow{\sim} \mathcal{D}^{\vdash}_{\underline{v}}$ in $\mathbb{TM}^{\vdash}$].

(iv) A *morphism of* $\mathcal{D}$*- (respectively,* $\mathcal{D}^{\vdash}$*-) prime-strips* is defined to be a collection of morphisms, indexed by $\underline{\mathbb{V}}$, between the various constituent objects of the prime-strips. Following the conventions of §0, one thus has a notion of *capsules of* $\mathcal{D}$*- (respectively,* $\mathcal{D}^{\vdash}$*-)* and *morphisms of capsules of* $\mathcal{D}$*- (respectively,* $\mathcal{D}^{\vdash}$*-) prime-strips*. Note that to any $\mathcal{D}$*-prime-strip* ${}^{\dagger}\mathfrak{D}$, one may associate, in a natural way, a $\mathcal{D}^{\vdash}$*-prime strip* ${}^{\dagger}\mathfrak{D}^{\vdash}$ — which we shall refer to as the *mono-analyticization* of ${}^{\dagger}\mathfrak{D}$ — by considering appropriate subcategories at the *nonarchimedean primes* [cf. Examples 3.2, (i), (vi); 3.3, (i), (iii)], or by applying the construction of Example 3.4, (ii), at the *archimedean primes*.

(v) Write

$$\mathcal{D}^{\circledcirc} \stackrel{\mathrm{def}}{=} \mathcal{B}(\underline{C}_K)^0$$

[cf. §0]. Then recall from [AbsTopIII], Theorem 1.9 [cf. Remark 3.1.2] that there exists a *group-theoretic algorithm* for reconstructing, from $\pi_1(\mathcal{D}^{\circledcirc})$ [cf. §0], the algebraic closure "$\overline{F}$" of the base field "K", hence also the *set of valuations* "$\mathbb{V}(\overline{F})$" [e.g., as a collection of topologies on $\overline{F}$ — cf., e.g., [AbsTopIII], Corollary 2.8]. Moreover, for $\underline{w} \in \mathbb{V}(K)^{\text{arc}}$, let us recall [cf. Remark 3.1.2; [AbsTopIII], Corollaries 2.8, 2.9] that one may *reconstruct group-theoretically*, from $\pi_1(\mathcal{D}^{\circledcirc})$, the Aut-*holomorphic orbispace* $\underline{\mathbb{C}}_{\underline{w}}$ associated to $\underline{C}_{\underline{w}}$. Let $^{\dagger}\mathcal{D}^{\circledcirc}$ be a category equivalent to $\mathcal{D}^{\circledcirc}$. Then let us write

$$\overline{\mathbb{V}}(^{\dagger}\mathcal{D}^{\circledcirc})$$

for the set of valuations [i.e., "$\mathbb{V}(\overline{F})$"], equipped with its natural $\pi_1(^{\dagger}\mathcal{D}^{\circledcirc})$-action,

$$\mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc}) \quad \stackrel{\text{def}}{=} \quad \overline{\mathbb{V}}(^{\dagger}\mathcal{D}^{\circledcirc})/\pi_1(^{\dagger}\mathcal{D}^{\circledcirc})$$

for the quotient of $\overline{\mathbb{V}}(^{\dagger}\mathcal{D}^{\circledcirc})$ by $\pi_1(^{\dagger}\mathcal{D}^{\circledcirc})$ [i.e., "$\mathbb{V}(K)$"], and, for $\underline{w} \in \mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc})^{\text{arc}}$,

$$\underline{\mathbb{C}}(^{\dagger}\mathcal{D}^{\circledcirc}, \underline{w})$$

[i.e., "$\underline{\mathbb{C}}_{\underline{w}}$" — cf. the discussion of [AbsTopIII], Definition 5.1, (ii)] for the Aut-holomorphic orbispace obtained by applying these group-theoretic reconstruction algorithms to $\pi_1(^{\dagger}\mathcal{D}^{\circledcirc})$. Now if $\mathbb{U}$ is an arbitrary Aut-*holomorphic orbispace*, then let us define a *morphism*

$$\mathbb{U} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$$

to be a morphism of Aut-holomorphic orbispaces [cf. [AbsTopIII], Definition 2.1, (ii)] $\mathbb{U} \to \underline{\mathbb{C}}(^{\dagger}\mathcal{D}^{\circledcirc}, \underline{w})$ for some $\underline{w} \in \mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc})^{\text{arc}}$. Thus, it makes sense to speak of the pre-composite (respectively, post-composite) of such a morphism $\mathbb{U} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$ with a morphism of Aut-holomorphic orbispaces (respectively, with an isomorphism [cf. §0] $^{\dagger}\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}^{\circledcirc}$ [i.e., where $^{\ddagger}\mathcal{D}^{\circledcirc}$ is a category equivalent to $\mathcal{D}^{\circledcirc}$]). Finally, just as in the discussion of (ii) in the case of "$\underline{v} \in \underline{\mathbb{V}}^{\text{good}} \bigcap \underline{\mathbb{V}}^{\text{non}}$", it makes sense [cf. [AbsTopI], Lemma 4.5] to speak of the *set of cusps* of $^{\dagger}\mathcal{D}^{\circledcirc}$, as well as the *set of label classes of cusps*

$$\text{LabCusp}(^{\dagger}\mathcal{D}^{\circledcirc})$$

of $^{\dagger}\mathcal{D}^{\circledcirc}$, which admits a natural $\mathbb{F}_l^{\circledast}$*-torsor structure*.

(vi) Let $^{\dagger}\mathcal{D}^{\circledcirc}$ be a category equivalent to $\mathcal{D}^{\circledcirc}$, $^{\dagger}\mathfrak{D} = \{^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ a $\mathcal{D}$*-prime-strip*. If $\underline{v} \in \underline{\mathbb{V}}$, then we define a *poly-morphism* $^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$ be a collection of morphisms $^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$ [cf. §0 when $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$; (v) when $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$]. We define a *poly-morphism*

$$^{\dagger}\mathfrak{D} \to {}^{\dagger}\mathcal{D}^{\circledcirc}$$

be a collection of poly-morphisms $\{^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\mathcal{D}^{\circledcirc}\}_{\underline{v} \in \underline{\mathbb{V}}}$. Finally, if $\{^{e}\mathfrak{D}\}_{e \in E}$ is a *capsule of* $\mathcal{D}$*-prime-strips*, then we define a *poly-morphism*

$$\{^{e}\mathfrak{D}\}_{e \in E} \to {}^{\dagger}\mathcal{D}^{\circledcirc} \text{ (respectively, } \{^{e}\mathfrak{D}\}_{e \in E} \to {}^{\dagger}\mathfrak{D})$$

to be a collection of poly-morphisms $\{^{e}\mathfrak{D} \to {}^{\dagger}\mathcal{D}^{\circledcirc}\}_{e \in E}$ (respectively, $\{^{e}\mathfrak{D} \to {}^{\dagger}\mathfrak{D}\}_{e \in E}$).

The following result follows immediately from the discussion of Definition 4.1, (ii).

Proposition 4.2. (The Set of Label Classes of Cusps of a Base-Prime-Strip) *Let* ${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ *be a* $\mathcal{D}$**-prime-strip**. *Then for any* $\underline{v}, \underline{w} \in \underline{\mathbb{V}}$*, there exist* **bijections**

$$\mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{w}})$$

that are **uniquely determined** *by the condition that they be* **compatible** *with the assignments* ${}^{\dagger}\underline{\eta}_{\underline{v}} \mapsto {}^{\dagger}\underline{\eta}_{\underline{w}}$ *[cf. Definition 4.1, (ii)], as well as with the* $\mathbb{F}_l^{\circledast}$**-torsor structures** *on either side. In particular, these bijections are* **preserved** *by arbitrary* **isomorphisms** *of* $\mathcal{D}$*-prime-strips. Thus, by identifying the various "*$\mathrm{LabCusp}({}^{\dagger}\mathcal{D}_{\underline{v}})$*" via these bijections, it* **makes sense to write** $\mathrm{LabCusp}({}^{\dagger}\mathfrak{D})$*. Finally,* $\mathrm{LabCusp}({}^{\dagger}\mathfrak{D})$ *is equipped with a* **canonical element***, arising from the* ${}^{\dagger}\underline{\eta}_{\underline{v}}$ *[for* $\underline{v} \in \underline{\mathbb{V}}$*], as well as a natural* $\mathbb{F}_l^{\circledast}$**-torsor structure***; in particular, this canonical element and* $\mathbb{F}_l^{\circledast}$*-torsor structure determine a* **natural bijection**

$$\mathrm{LabCusp}({}^{\dagger}\mathfrak{D}) \quad \xrightarrow{\sim} \quad \mathbb{F}_l^{\circledast}$$

that is preserved by isomorphisms of $\mathcal{D}$*-prime-strips.*

Remark 4.2.1. Note that if, in Examples 3.3, 3.4 — i.e., at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ — *one defines "*$\mathcal{D}_{\underline{v}}$*" by means of "*$\underline{C}_{\underline{v}}$*" instead of "*$\underline{X}_{\rightarrow\underline{v}}$*"*, then there does *not* exist a system of bijections as in Proposition 4.2. Indeed, by the *Tchebotarev density theorem* [cf., e.g., [Lang], Chapter VIII, §4, Theorem 10], it follows immediately that there exist $\underline{v} \in \underline{\mathbb{V}}$ such that the decomposition subgroup in $\mathrm{Gal}(K/F) \cong GL_2(\mathbb{F}_l)$ determined [up to conjugation] by $\underline{v}$ is equal to the *subgroup of scalar matrices*. Thus, if ${}^{\dagger}\mathfrak{D} = \{{}^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$, ${}^{\dagger}\underline{\mathfrak{D}} = \{{}^{\dagger}\underline{\mathcal{D}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ are as in Definition 4.1, (i), then for such a $\underline{v}$, the automorphism group of ${}^{\dagger}\mathcal{D}_{\underline{v}}$ acts *transitively* on the set of nonzero cusps of ${}^{\dagger}\underline{\mathcal{D}}_{\underline{v}}$, while the automorphism group of ${}^{\dagger}\mathcal{D}_{\underline{w}}$ acts *trivially* [by [EtTh], Corollary 2.9] on the set of cusps of ${}^{\dagger}\underline{\mathcal{D}}_{\underline{w}}$ for any $\underline{w} \in \underline{\mathbb{V}}^{\mathrm{bad}}$.

Example 4.3. Model Base-NF-Bridges. In the following, we construct the *"models"* for the notion of a "base-NF-bridge" [cf. Definition 4.6, (i), below].

(i) Write

$$\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \quad \subseteq \quad \mathrm{Aut}(\underline{C}_K) \quad \cong \quad \mathrm{Out}(\Pi_{\underline{C}_K}) \quad \cong \quad \mathrm{Aut}(\mathcal{D}^{\circledcirc})$$

— where the first "$\cong$" follows, for instance, from [AbsTopIII], Theorem 1.9 — for the subgroup of elements which *fix the cusp* $\underline{\epsilon}$. Now let us recall that the profinite group Δ_X may be *reconstructed group-theoretically* from $\Pi_{\underline{C}_K}$ [cf. [AbsTopII], Corollary 3.3, (i), (ii); [AbsTopII], Remark 3.3.2; [AbsTopI], Example 4.8]. Since inner automorphisms of $\Pi_{\underline{C}_K}$ clearly act *by multiplication by* ± 1 on the l-torsion points of $E_{\overline{F}}$ [i.e., on $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l$], we obtain a natural homomorphism $\mathrm{Out}(\Pi_{\underline{C}_K}) \to \mathrm{Aut}(\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l)/\{\pm 1\}$. Thus, relative to a suitable isomorphism $\mathrm{Aut}(\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l)/\{\pm 1\} \cong GL_2(\mathbb{F}_l)/\{\pm 1\}$, the images of the groups $\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$, $\mathrm{Aut}(\underline{C}_K)$ may be identified with the subgroups

$$\left\{ \begin{pmatrix} * & * \\ 0 & \pm 1 \end{pmatrix} \right\} \quad \subseteq \quad \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \quad \subseteq \quad GL_2(\mathbb{F}_l)/\{\pm 1\}$$

— i.e., *"semi-unipotent, up to ±1"* and *Borel* subgroups — of $GL_2(\mathbb{F}_l)/\{\pm 1\}$. Write

$$\underline{\mathbb{V}}^{\pm \mathrm{un}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \cdot \underline{\mathbb{V}} \quad \subseteq \quad \underline{\mathbb{V}}^{\mathrm{Bor}} \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\underline{C}_K) \cdot \underline{\mathbb{V}} \quad \subseteq \quad \mathbb{V}(K)$$

for the resulting subsets of $\mathbb{V}(K)$. Thus, one verifies immediately that the subgroup $\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \subseteq \mathrm{Aut}(\underline{C}_K)$ is *normal*, and that we have a *natural isomorphism*

$$\mathrm{Aut}(\underline{C}_K)/\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \stackrel{\sim}{\rightarrow} \mathbb{F}_l^{\circledast}$$

— so we may think of $\underline{\mathbb{V}}^{\mathrm{Bor}}$ as the $\mathbb{F}_l^{\circledast}$*-orbit of* $\underline{\mathbb{V}}^{\pm \mathrm{un}}$. Also, we observe that [in light of the above discussion] it follows immediately that there exists a *group-theoretic algorithm* for reconstructing, from $\pi_1(\mathcal{D}^{\circledcirc})$ [i.e., an isomorph of $\Pi_{\underline{C}_K}$] the subgroup

$$\mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \subseteq \mathrm{Aut}(\mathcal{D}^{\circledcirc})$$

determined by $\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$.

(ii) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$. Then the natural restriction functor on finite étale coverings arising from the natural composite morphism $\underline{X}_{\rightarrow \underline{v}} \rightarrow \underline{C}_{\underline{v}} \rightarrow \underline{C}_K$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ (respectively, $\underline{\underline{X}}_{\underline{v}} \rightarrow \underline{C}_{\underline{v}} \rightarrow \underline{C}_K$ if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$) determines [cf. Examples 3.2, (i); 3.3, (i)] a *natural morphism* $\phi^{\mathrm{NF}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \rightarrow \mathcal{D}^{\circledcirc}$ [cf. §0 for the definition of the term "morphism"]. Write

$$\phi^{\mathrm{NF}}_{\underline{v}} : \mathcal{D}_{\underline{v}} \rightarrow \mathcal{D}^{\circledcirc}$$

for the *poly-morphism* given by the collection of morphisms $\mathcal{D}_{\underline{v}} \rightarrow \mathcal{D}^{\circledcirc}$ of the form

$$\beta \circ \phi^{\mathrm{NF}}_{\bullet,\underline{v}} \circ \alpha$$

— where $\alpha \in \mathrm{Aut}(\mathcal{D}_{\underline{v}}) \cong \mathrm{Aut}(\underline{X}_{\rightarrow \underline{v}})$ (respectively, $\alpha \in \mathrm{Aut}(\mathcal{D}_{\underline{v}}) \cong \mathrm{Aut}(\underline{\underline{X}}_{\underline{v}})$); $\beta \in \mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \cong \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$ [cf., e.g., [AbsTopIII], Theorem 1.9].

(iii) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. Thus, [cf. Example 3.4, (i)] we have a *tautological morphism* $\mathcal{D}_{\underline{v}} = \underline{\mathbb{X}}_{\rightarrow \underline{v}} \rightarrow \underline{\mathbb{C}}_{\underline{v}} \stackrel{\sim}{\rightarrow} \underline{\mathbb{C}}(\mathcal{D}^{\circledcirc}, \underline{v})$, hence a morphism $\phi^{\mathrm{NF}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \rightarrow \mathcal{D}^{\circledcirc}$ [cf. Definition 4.1, (v)]. Write

$$\phi^{\mathrm{NF}}_{\underline{v}} : \mathcal{D}_{\underline{v}} \rightarrow \mathcal{D}^{\circledcirc}$$

for the *poly-morphism* given by the collection of morphisms $\mathcal{D}_{\underline{v}} \rightarrow \mathcal{D}^{\circledcirc}$ of the form

$$\beta \circ \phi^{\mathrm{NF}}_{\bullet,\underline{v}} \circ \alpha$$

— where $\alpha \in \mathrm{Aut}(\mathcal{D}_{\underline{v}}) \cong \mathrm{Aut}(\underline{\mathbb{X}}_{\rightarrow \underline{v}})$ [cf. [AbsTopIII], Corollary 2.3, (i)]; $\beta \in \mathrm{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \cong \mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$.

(iv) For each $j \in \mathbb{F}_l^{\circledast}$, let

$$\mathfrak{D}_j = \{\mathcal{D}_{\underline{v}_j}\}_{\underline{v} \in \underline{\mathbb{V}}}$$

— where we use the notation $\underline{v}_j$ to denote the pair $(j, \underline{v})$ — be a *copy* of the "tautological $\mathcal{D}$-prime-strip" $\{\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$. Let us denote by

$$\phi_1^{\mathrm{NF}} : \mathfrak{D}_1 \to \mathcal{D}^{\circledcirc}$$

[where, by abuse of notation, we write "1" for the element of $\mathbb{F}_l^{\circledast}$ determined by 1] the *poly-morphism* determined by the collection $\{\phi_{\underline{v}_1}^{\mathrm{NF}} : \mathcal{D}_{\underline{v}_1} \to \mathcal{D}^{\circledcirc}\}_{\underline{v} \in \underline{\mathbb{V}}}$ of copies of the poly-morphisms $\phi_{\underline{v}}^{\mathrm{NF}}$ constructed in (ii), (iii). Note that ϕ_1^{NF} is *stabilized by the action of* $\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$ *on* $\mathcal{D}^{\circledcirc}$. Thus, it makes sense to consider, for arbitrary $j \in \mathbb{F}_l^{\circledast}$, the *poly-morphism*

$$\phi_j^{\mathrm{NF}} : \mathfrak{D}_j \to \mathcal{D}^{\circledcirc}$$

obtained [via any isomorphism $\mathfrak{D}_1 \cong \mathfrak{D}_j$] by *post-composing with the "poly-action"* [i.e., action via poly-automorphisms — cf. (i)] *of* $j \in \mathbb{F}_l^{\circledast}$ *on* $\mathcal{D}^{\circledcirc}$. Let us write

$$\mathfrak{D}_{\circledast} \stackrel{\mathrm{def}}{=} \{\mathfrak{D}_j\}_{j \in \mathbb{F}_l^{\circledast}}$$

for the *capsule of $\mathcal{D}$-prime-strips* indexed by $j \in \mathbb{F}_l^{\circledast}$ [cf. Definition 4.1, (iv)] and denote by

$$\phi_{\circledast}^{\mathrm{NF}} : \mathfrak{D}_{\circledast} \to \mathcal{D}^{\circledcirc}$$

the *poly-morphism* given by the collection of poly-morphisms $\{\phi_j^{\mathrm{NF}}\}_{j \in \mathbb{F}_l^{\circledast}}$. Thus, $\phi_{\circledast}^{\mathrm{NF}}$ is *equivariant* with respect to the *natural poly-action of* $\mathbb{F}_l^{\circledast}$ *on* $\mathcal{D}^{\circledcirc}$ and the *natural permutation poly-action* of $\mathbb{F}_l^{\circledast}$, via capsule-full [cf. §0] poly-automorphisms, on the constituents of the capsule $\mathfrak{D}_{\circledast}$. In particular, we obtain a *natural poly-action* of $\mathbb{F}_l^{\circledast}$ on the collection of data $(\mathfrak{D}_{\circledast}, \mathcal{D}^{\circledcirc}, \phi_{\circledast}^{\mathrm{NF}})$.

Remark 4.3.1.

(i) Suppose, for simplicity, in the following discussion that $F = F_{\mathrm{mod}}$. Note that the morphism of schemes $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$ [or, equivalently, the homomorphism of rings $F \hookrightarrow K$] *does not admit a section.* This nonexistence of a section is closely related to the *nonexistence of a "global multiplicative subspace"* of the sort discussed in [HASurII], Remark 3.7. In the context of *loc. cit.*, this nonexistence of a "global multiplicative subspace" may be thought of as a concrete way of representing the *principal obstruction* to applying the scheme-theoretic Hodge-Arakelov theory of [HASurI], [HASurII] to diophantine geometry. From this point of view, if one thinks of the ring structure of F, K as a sort of *"arithmetic holomorphic structure"* [cf. [AbsTopIII], Remark 5.10.2, (ii)], then one may think of the *[$\mathcal{D}$-]prime-strips* that appear in the discussion of Example 4.3 as defining, via the arrows ϕ_j^{NF} of Example 4.3, (iv),

"arithmetic collections of local analytic sections" of $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$

— cf. Fig. 4.1, where each "$\cdot - \cdot - \ldots - \cdot - \cdot$" represents a [$\mathcal{D}$-]prime-strip. In fact, if, for the sake of brevity, we abbreviate the phrase "collection of local analytic" by the term *"local-analytic"*, then each of these sections may be thought of as yielding not only an **"arithmetic local-analytic global multiplicative subspace"**, but also an **"arithmetic local-analytic global canonical generator"** [i.e., up to multiplication by ± 1, of the quotient of the module of l-torsion points of the elliptic curve in question by the "arithmetic local-analytic global multiplicative subspace"]. We refer to Remark 4.9.1, (i), below, for more on this point of view.

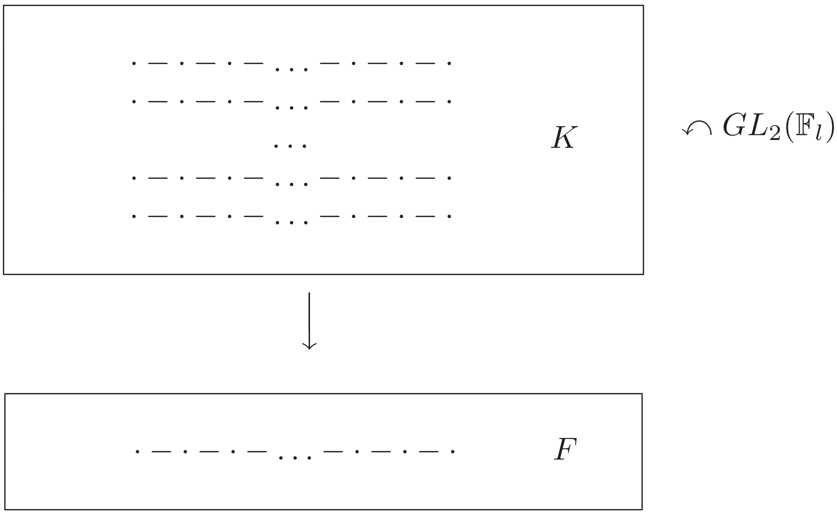

Fig. 4.1: Prime-strips as "sections" of $\mathrm{Spec}(K) \to \mathrm{Spec}(F)$

(ii) The way in which these *"arithmetic local-analytic sections"* constituted by the [$\mathcal{D}$-]prime-strips *fail to be [globally] "arithmetically holomorphic"* may be understood from several closely related points of view. The first point of view was already noted above in (i) — namely:

(a) these sections *fail* to extend to *ring homomorphisms* $K \to F$.

The second point of view involves the classical phenomonenon of *decomposition of primes* in extensions of number fields. The decomposition of primes in extensions of number fields may be represented by a *tree*, as in Fig. 4.2, below. If one thinks of the tree in large parentheses of Fig. 4.2 as representing the decomposition of primes over a prime v of F in extensions of F [such as K!], then the "arithmetic local-analytic sections" constituted by the $\mathcal{D}$-prime-strips may be thought of as

$$\begin{pmatrix} \cdots & & \\ \backslash|/ & \cdots & \cdots \\ v' & v'' & v''' \\ \backslash & | & / \\ & v & \end{pmatrix} \quad \supseteq \quad \begin{pmatrix} \cdots \\ \backslash|/ \\ v' \end{pmatrix}$$

Fig. 4.2: Prime decomposition trees

(b) an isomorphism, or identification, between v [i.e., a prime of F] and v' [i.e., a prime of K] which [manifestly — cf., e.g., [NSW], Theorem 12.2.5] fails to extend to an *isomorphism between the respective prime decomposition trees* over v and v'.

If one thinks of the relation "$\in$" between sets in axiomatic set theory as determining a *"tree"*, then

> the point of view of (b) is *reminiscent* of the point of view of [IUTchIV], §3, where one is concerned with *constructing some sort of artificial solution to the "membership equation* $a \in a$*"* [cf. the discussion of [IUTchIV], Remark 3.3.1, (i)].

The third point of view consists of the observation that although the the "arithmetic local-analytic sections" constituted by the $\mathcal{D}$-prime-strips involve isomorphisms of the various *local absolute Galois groups*,

(c) these isomorphisms of local absolute Galois groups fail to extend to a *section of global absolute Galois groups* $G_F \twoheadrightarrow G_K$ [i.e., a section of the natural inclusion $G_K \hookrightarrow G_F$].

Here, we note that in fact, by the *Neukirch-Uchida theorem* [cf. [NSW], Chapter XII, §2], one may think of (a) and (c) as *essentially equivalent*. Moreover, (b) is *closely related* to this equivalence, in the sense that the proof [cf., e.g., [NSW], Chapter XII, §2] of the Neukirch-Uchida theorem *depends in an essential fashion* on a careful analysis of the *prime decomposition trees* of the number fields involved.

(iii) In some sense, understanding more precisely the content of the failure of these "arithmetic local-analytic sections" constituted by the $\mathcal{D}$-prime-strips to be "arithmetically holomorphic" is a *central theme* of the theory of the present series of papers — a theme which is very much in line with the *spirit of classical complex Teichmüller theory*.

Remark 4.3.2. The *incompatibility* of the "arithmetic local-analytic sections" of Remark 4.3.1, (i), with *global prime distributions* and *global absolute Galois groups* [cf. the discussion of Remark 4.3.1, (ii)] is precisely the technical obstacle that will necessitate the application — in [IUTchIII] — of the *absolute p-adic mono-anabelian geometry* developed in [AbsTopIII], in the form of *"panalocalization along the various prime-strips"* [cf. [IUTchIII] for more details]. Indeed,

> the mono-anabelian theory developed in [AbsTopIII] represents the *culmination* of earlier research of the author during the years 2000 to 2007 concerning **absolute *p*-adic anabelian geometry** — research that was motivated precisely by the goal of *developing a geometry* that would allow one to work with the "arithmetic local-analytic sections" constituted by the prime-strips, so as to overcome the principal technical obstruction to applying the Hodge-Arakelov theory of [HASurI], [HASurII] [cf. Remark 4.3.1, (i)].

Note that the "desired geometry" in question will also be subject to other requirements. For instance, in [IUTchIII] [cf. also [IUTchII], §4], we shall make essential use of the *global arithmetic — i.e., the ring structure and absolute Galois groups — of number fields*. As observed above in Remark 4.3.1, (ii), these global arithmetic structures are *not compatible* with the "arithmetic local-analytic sections" constituted by the prime-strips. In particular, this state of affairs imposes the further requirement that the "geometry" in question be *compatible with globalization*, i.e., that it give rise to the global arithmetic of the number fields in question in a fashion that is *independent of the various local geometries* that appear in the "arithmetic local-analytic sections" constituted by the prime-strips, but nevertheless admits *localization operations* to these various local geometries [cf. Fig. 4.3; the discussion of [IUTchII], Remark 4.11.2, (iii); [AbsTopIII], Remark 3.7.6, (iii), (v)].

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{local geometry}\\ \text{at } v\end{array}} & \dots & \boxed{\begin{array}{c}\text{local geometry}\\ \text{at } v'\end{array}} & \dots & \boxed{\begin{array}{c}\text{local geometry}\\ \text{at } v''\end{array}} \\
 & & \boxed{\text{global geometry}} \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
$$

Fig. 4.3: Globalizability

Finally, in order for the "desired geometry" to be applicable to the theory developed in the present series of papers, it is necessary for it to be based on *"étale-like structures"*, so as to give rise to *canonical splittings*, as in the *étale-picture* discussed in Corollary 3.9, (i). Thus, in summary, the requirements that we wish to impose on the "desired geometry" are the following:

(a) **local independence** *of global structures*,
(b) **globalizability**, in a fashion that is **independent** *of local structures*,
(c) the property of being based on **étale-like structures**.

Note, in particular, that properties (a), (b) at first glance *almost appear to contradict one another*. In particular, the simultaneous realization of (a), (b) is *highly nontrivial*. For instance, in the case of a function field of dimension one over a base field, the simultaneous realization of properties (a), (b) appears to require that one restrict oneself essentially to working with structures that *descend to the base field*! It is thus a *highly nontrivial consequence* of the theory of [AbsTopIII] that the *mono-anabelian geometry* of [AbsTopIII] does indeed satisfy all of these requirements (a), (b), (c) [cf. the discussion of [AbsTopIII], §I1].

Remark 4.3.3.

(i) One important theme of [AbsTopIII] is the analogy between the **mono-anabelian theory** of [AbsTopIII] and the theory of *Frobenius-invariant indigenous bundles* of the sort that appear in p-adic Teichmüller theory [cf. [AbsTopIII], §I5]. In fact, [although this point of view is not mentioned in [AbsTopIII]] one may *"compose"* this analogy with the analogy between the p-adic and complex theories discussed in [pOrd], Introduction; [pTeich], Introduction, §0, and consider the analogy between the mono-anabelian theory of [AbsTopIII] and the **classical geometry of the upper half-plane $\mathfrak{H}$**. In addition to being *more elementary* than the p-adic theory, this analogy with the classical geometry of the upper half-plane $\mathfrak{H}$ also has the virtue that

> since it revolves around the **canonical Kähler metric** — i.e., the **Poincaré metric** — on the upper half-plane, it renders more transparent the relationship between the theory of the present series of papers and *classical Arakelov theory* [which also revolves, to a substantial extent, around Kähler metrics at the archimedean primes].

(ii) The essential content of the mono-anabelian theory of [AbsTopIII] may be summarized by the diagram

$$\Pi \curvearrowright \overline{k}^{\times} \quad \xrightarrow{\mathfrak{log}} \quad \overline{k} \curvearrowleft \Pi \tag{$*$}$$

— where k is a finite extension of $\mathbb{Q}_p$; $\overline{k}$ is an algebraic closure of k; Π is the arithmetic fundamental group of a hyperbolic orbicurve over k; $\mathfrak{log}$ is the p-adic logarithm [cf. [AbsTopIII], §I1]. On the other hand, if $(\mathcal{E}, \nabla_{\mathcal{E}})$ denotes the *"tautological indigenous bundle"* on $\mathfrak{H}$ [i.e., the first de Rham cohomology of the tautological elliptic curve over $\mathfrak{H}$], then one has a natural *Hodge filtration* $0 \to \omega \to \mathcal{E} \to \tau \to 0$ [where ω, $\tau \stackrel{\mathrm{def}}{=} \omega^{-1}$ are holomorphic line bundles on $\mathfrak{H}$], together with a natural *complex conjugation operation* $\iota_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$. The composite

$$\omega \quad \hookrightarrow \quad \mathcal{E} \quad \xrightarrow{\iota_{\mathcal{E}}} \quad \mathcal{E} \quad \twoheadrightarrow \quad \tau$$

then determines an *Hermitian metric* $|-|_{\omega}$ on ω. For any trivializing section f of ω, the $(1,1)$-form

$$\kappa_{\mathfrak{H}} \stackrel{\mathrm{def}}{=} \frac{1}{2\pi i} \partial\overline{\partial} \log(|f|_{\omega})$$

is the **canonical Kähler metric** [i.e., Poincaré metric] on $\mathfrak{H}$. Then one can already readily identify various *formal similarities* between $\kappa_{\mathfrak{H}}$ and the diagram $(*)$ reviewed above: Indeed, at a somewhat superficial level, the "log" that appears in the definition of $\kappa_{\mathfrak{H}}$ is reminiscent of the "log-Frobenius operation" $\mathfrak{log}$. At a less superficial level, the "Galois group" Π is reminiscent — cf. the point of view that *"Galois groups are arithmetic tangent bundles"*, a point of view that underlies the theory of the *arithmetic Kodaira-Spencer morphism* discussed in [HASurI]! — of ∂. If one thinks of *complex conjugation* as a sort of *"archimedean Frobenius"* [cf. [pTeich], Introduction, §0], then $\overline{\partial}$ is reminiscent of the "Galois group" Π operating on the *opposite side* [cf. $\iota_{\mathcal{E}}$] of the log-Frobenius operation $\mathfrak{log}$. The Hodge filtration of $\mathcal{E}$ corresponds to the *ring structures* of the copies of $\overline{k}$ on either side of $\mathfrak{log}$ [cf. the discussion of [AbsTopIII], Remark 3.7.2]. Finally, perhaps most importantly from the point of view of the theory of the present series of papers:

> the fact that *log-shells* play the role in the theory of [AbsTopIII] of *"canonical rigid integral structures"* [cf. [AbsTopIII], §I1] — i.e., *"canonical standard units of volume"* — is reminiscent of the fact that the Kähler metric $\kappa_{\mathfrak{H}}$ also plays the role of *determining a canonical notion of volume* on $\mathfrak{H}$.

(iii) From the point of view of the analogy discussed in (ii), property (a) of Remark 4.3.2 may be thought of as corresponding to the **local representability** *via the [positive]* $(1,1)$*-form* $\kappa_{\mathfrak{H}}$ — on, say, a compact quotient S of $\mathfrak{H}$ — of the [positive] **global degree** of [the result of descending to S] the line bundle ω; property (b) of Remark 4.3.2 may be thought of as corresponding to the fact that this $(1,1)$-form $\kappa_{\mathfrak{H}}$ that gives rise to a local representation on S of the notion of a positive global degree not only exists locally on S, but also admits a **canonical global extension** to the entire Riemann surface S which may be related to the **algebraic theory** [i.e., of algebraic rational functions on S].

(iv) The analogy discussed in (ii) may be summarized as follows:

mono-anabelian theory	*geometry of the upper-half plane* $\mathfrak{H}$
the Galois group Π	the differential operator ∂
the Galois group Π on the opposite side of $\mathfrak{log}$	the differential operator $\overline{\partial}$
the ring structures of the copies of $\overline{k}$ on either side of $\mathfrak{log}$	the Hodge filtration of $\mathcal{E}$, $\iota_{\mathcal{E}}$, $\lvert - \rvert_{\mathcal{E}}$
log-shells as *canonical units of volume*	the *canonical Kähler volume* $\kappa_{\mathfrak{H}}$

Example 4.4. Model Base-Θ-Bridges. In the following, we construct the *"models"* for the notion of a "base-Θ-bridge" [cf. Definition 4.6, (ii), below]. We continue to use the notation of Example 4.3.

(i) Let $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$. Recall that there is a *natural bijection* between the set

$$|\mathbb{F}_l| \quad \stackrel{\text{def}}{=} \quad \mathbb{F}_l/\{\pm 1\} = 0 \bigcup \mathbb{F}_l^{*}$$

[i.e., the set of $\{\pm 1\}$-orbits of $\mathbb{F}_l$] and the set of *cusps* of the hyperbolic orbicurve $\underline{C}_{\underline{v}}$ [cf. [EtTh], Corollary 2.9]. Thus, [by considering fibers over $\underline{C}_{\underline{v}}$] we obtain *labels* $\in |\mathbb{F}_l|$ of various collections of cusps of $\underline{X}_{\underline{v}}$, $\underline{\underline{X}}_{\underline{v}}$. Write

$$\mu_- \in \underline{X}_{\underline{v}}(K_{\underline{v}})$$

for the *unique torsion point of order* 2 whose closure in any stable model of $\underline{X}_{\underline{v}}$ over $\mathcal{O}_{K_{\underline{v}}}$ intersects the same irreducible component of the special fiber of the stable model as the [unique] cusp labeled $0 \in |\mathbb{F}_l|$. Now observe that it makes sense to speak of the points $\in \underline{X}_{\underline{v}}(K_{\underline{v}})$ obtained as *μ_--translates of the cusps*, relative to the group scheme structure of the elliptic curve determined by $\underline{X}_{\underline{v}}$ [i.e., whose origin is given by the cusp labeled $0 \in |\mathbb{F}_l|$]. We shall refer to these μ_--translates of the cusps with labels $\in |\mathbb{F}_l|$ as the **evaluation points** of $\underline{X}_{\underline{v}}$. Note that the **value** of the **theta function** "$\underline{\underline{\Theta}}_{\underline{v}}$" of Example 3.2, (ii), at a point lying over an evaluation point arising from a cusp with label $j \in |\mathbb{F}_l|$ is contained in the $\boldsymbol{\mu}_{2l}$-orbit of

$$\{ \, \underline{\underline{q}}_{\underline{v}}^{\underline{j}^2} \, \}_{\underline{j} \equiv j}$$

[cf. Example 3.2, (iv); [EtTh], Proposition 1.4, (ii)] — where $\underline{\underline{j}}$ ranges over the elements of $\mathbb{Z}$ that map to $j \in |\mathbb{F}_l|$. In particular, it follows immediately from the *definition* of the covering $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}}$ [i.e., by considering l-th roots of the theta function! — cf. [EtTh], Definition 2.5, (i)] that the points of $\underline{\underline{X}}_{\underline{v}}$ that lie over evaluation points of $\underline{X}_{\underline{v}}$ are *all defined over* $K_{\underline{v}}$. We shall refer to the points $\in \underline{\underline{X}}_{\underline{v}}(K_{\underline{v}})$ that lie over the evaluation points of $\underline{X}_{\underline{v}}$ as the *evaluation points* of $\underline{\underline{X}}_{\underline{v}}$ and to the various sections

$$G_{\underline{v}} \to \Pi_{\underline{v}} = \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}$$

of the natural surjection $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$ that arise from the evaluation points as the **evaluation sections** of $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$. Thus, each evaluation section has an associated **label** $\in |\mathbb{F}_l|$. Note that there is a *group-theoretic algorithm* for constructing the evaluation sections from [isomorphs of] the *topological group* $\Pi_{\underline{v}}$. Indeed, this follows immediately from [the proofs of] [EtTh], Corollary 2.9 [concerning the *group-theoreticity of the labels*]; [EtTh], Proposition 2.4 [concerning the *group-theoreticity of* $\Pi_{\underline{C}_{\underline{v}}}$, $\Pi_{\underline{X}_{\underline{v}}}$]; [SemiAnbd], Corollary 3.11 [concerning the dual semi-graphs of the special fibers of stable models], applied to $\Delta^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}} \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}} = \Pi_{\underline{v}}$; [SemiAnbd], Theorem 6.8, (iii) [concerning the *group-theoreticity of the decomposition groups of* μ_-*-translates of the cusps*].

(ii) We continue to suppose that $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Let

$$\mathfrak{D}_{>} = \{\mathcal{D}_{>,\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$$

be a *copy* of the *"tautological* $\mathcal{D}$*-prime-strip"* $\{\mathcal{D}_{\underline{w}}\}_{\underline{w}\in\underline{\mathbb{V}}}$. For each $j \in \mathbb{F}_l^{*}$, write

$$\phi^{\Theta}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} \to \mathcal{D}_{>,\underline{v}}$$

for the *poly-morphism* given by the collection of morphisms [cf. §0] obtained by composing with arbitrary *isomorphisms* $\mathcal{D}_{\underline{v}_j} \xrightarrow{\sim} \mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0$, $\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0 \xrightarrow{\sim} \mathcal{D}_{>,\underline{v}}$ the various morphisms $\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0 \to \mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0$ that arise [i.e., via composition with the natural surjection $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$] from the *evaluation sections labeled* j. Now if $\mathcal{C}$ is any isomorph of $\mathcal{B}^{\mathrm{temp}}(\Pi_{\underline{v}})^0$, then let us write

$$\pi_1^{\mathrm{geo}}(\mathcal{C}) \subseteq \pi_1(\mathcal{C})$$

for the subgroup corresponding to $\Delta^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}} \subseteq \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}} = \Pi_{\underline{v}}$, a subgroup which we recall may be *reconstructed group-theoretically* [cf., e.g., [AbsTopI], Theorem 2.6, (v); [AbsTopI], Proposition 4.10, (i)]. Then we observe that for each constituent morphism $\mathcal{D}_{\underline{v}_j} \to \mathcal{D}_{>,\underline{v}}$ of the poly-morphism $\phi^{\Theta}_{\underline{v}_j}$, the induced homomorphism $\pi_1(\mathcal{D}_{\underline{v}_j}) \to \pi_1(\mathcal{D}_{>,\underline{v}})$ [well-defined, up to composition with an inner automorphism] is *compatible with the respective outer actions* [of the domain and codomain of this homomorphism] on $\pi_1^{\mathrm{geo}}(\mathcal{D}_{\underline{v}_j})$, $\pi_1^{\mathrm{geo}}(\mathcal{D}_{>,\underline{v}})$ for some [not necessarily unique, but determined up to *finite ambiguity* — cf. [SemiAnbd], Theorem 6.4!] outer isomorphism $\pi_1^{\mathrm{geo}}(\mathcal{D}_{\underline{v}_j}) \xrightarrow{\sim} \pi_1^{\mathrm{geo}}(\mathcal{D}_{>,\underline{v}})$. We shall refer to this fact by saying that "$\phi^{\Theta}_{\underline{v}_j}$ *is* *compatible with the outer actions on the respective geometric [tempered] fundamental groups".*

(iii) Let $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$. For each $j \in \mathbb{F}_l^{\circledast}$, write

$$\phi^{\Theta}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} \xrightarrow{\sim} \mathcal{D}_{>,\underline{v}}$$

for the *full poly-isomorphism* [cf. §0].

(iv) For each $j \in \mathbb{F}_l^{\circledast}$, write

$$\phi^{\Theta}_j : \mathfrak{D}_j \to \mathfrak{D}_>$$

for the *poly-morphism* determined by the collection $\{\phi^{\Theta}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} \to \mathcal{D}_{>,\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ and

$$\phi^{\Theta}_{\circledast} : \mathfrak{D}_{\circledast} \to \mathfrak{D}_>$$

for the *poly-morphism* $\{\phi^{\Theta}_j\}_{j \in \mathbb{F}_l^{\circledast}}$. Thus, whereas the capsule $\mathfrak{D}_{\circledast}$ admits a *natural permutation poly-action by* $\mathbb{F}_l^{\circledast}$, the *"labels"* — i.e., in effect, elements of $\text{LabCusp}(\mathfrak{D}_>)$ [cf. Proposition 4.2] — determined by the various collections of *evaluation sections* corresponding to a given $j \in \mathbb{F}_l^{\circledast}$ are *held fixed by arbitrary automorphisms of* $\mathfrak{D}_>$ [cf. Proposition 4.2].

Example 4.5. Transport of Label Classes of Cusps via Model Base-Bridges. We continue to use the notation of Examples 4.3, 4.4.

(i) Let $j \in \mathbb{F}_l^{\circledast}$, $\underline{v} \in \underline{\mathbb{V}}$. Recall from Example 4.3, (iv), that the data of the arrow $\phi^{\text{NF}}_j : \mathfrak{D}_j \to \mathcal{D}^{\circledcirc}$ at $\underline{v}$ consists of an arrow $\phi^{\text{NF}}_{\underline{v}_j} : \mathcal{D}_{\underline{v}_j} \to \mathcal{D}^{\circledcirc}$. If $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, then $\phi^{\text{NF}}_{\underline{v}_j}$ induces various outer homomorphisms $\pi_1(\mathcal{D}_{\underline{v}_j}) \to \pi_1(\mathcal{D}^{\circledcirc})$; thus,

> by considering *cuspidal inertia groups of* $\pi_1(\mathcal{D}^{\circledcirc})$ whose unique index l subgroup is *contained in the image* of this homomorphism [cf. Corollary 2.5 when $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$; the discussion of Remark 4.5.1 below],

we conclude that these homomorphisms induce a *natural isomorphism of* $\mathbb{F}_l^{\circledast}$*-torsors* $\text{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \text{LabCusp}(\mathcal{D}_{\underline{v}_j})$. In a similar vein, if $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, then it follows from Definition 4.1, (v), that $\phi^{\text{NF}}_{\underline{v}_j}$ consists of certain morphisms of Aut-holomorphic orbispaces which induce various outer homomorphisms $\pi_1(\mathcal{D}_{\underline{v}_j}) \to \pi_1(\mathcal{D}^{\circledcirc})$ from the [discrete] topological fundamental group $\pi_1(\mathcal{D}_{\underline{v}_j})$ to the profinite group $\pi_1(\mathcal{D}^{\circledcirc})$; thus,

> by considering the *closures in* $\pi_1(\mathcal{D}^{\circledcirc})$ *of the images of cuspidal inertia groups of* $\pi_1(\mathcal{D}_{\underline{v}_j})$ [cf. the discussion of Remark 4.5.1 below],

we conclude that these homomorphisms induce a *natural isomorphism of* $\mathbb{F}_l^{\circledast}$*-torsors* $\text{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \text{LabCusp}(\mathcal{D}_{\underline{v}_j})$. Now let us observe that it follows immediately from the definitions that, as one allows $\underline{v}$ to *vary*, these isomorphisms of $\mathbb{F}_l^{\circledast}$-torsors $\mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathcal{D}_{\underline{v}_j})$ are compatible with the natural bijections in the first display of Proposition 4.2, hence determine an isomorphism of $\mathbb{F}_l^{\circledast}$-torsors $\mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathfrak{D}_j)$. Next, let us note that the data of the arrow $\phi_j^{\Theta} : \mathfrak{D}_j \to \mathfrak{D}_>$ at the various $\underline{v} \in \underline{\mathbb{V}}$ determines an isomorphism of $\mathbb{F}_l^{\circledast}$-torsors $\mathrm{LabCusp}(\mathfrak{D}_j) \xrightarrow{\sim} \mathrm{LabCusp}(\mathfrak{D}_>)$ [which may be *composed* with the previous isomorphism of $\mathbb{F}_l^{\circledast}$-torsors $\mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathfrak{D}_j)$]. Indeed, this is immediate from the definitions when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$; when $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, it follows immediately from the discussion of Example 4.4, (ii).

(ii) The discussion of (i) may be summarized as follows:

for each $j \in \mathbb{F}_l^{\circledast}$, *restriction* at the various $\underline{v} \in \underline{\mathbb{V}}$ via ϕ_j^{NF}, ϕ_j^{Θ} determines an *isomorphism of* $\mathbb{F}_l^{\circledast}$*-torsors*

$$\phi_j^{\mathrm{LC}} : \mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(\mathfrak{D}_>)$$

such that ϕ_j^{LC} is obtained from ϕ_1^{LC} by composing with the action by $j \in \mathbb{F}_l^{\circledast}$.

Write $[\underline{\epsilon}] \in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$ for the element determined by $\underline{\epsilon}$. Then we observe that

$$\phi_j^{\mathrm{LC}}([\underline{\epsilon}]) \mapsto j; \quad \phi_1^{\mathrm{LC}}(j \cdot [\underline{\epsilon}]) \mapsto j$$

via the *natural bijection* $\mathrm{LabCusp}(\mathfrak{D}_>) \xrightarrow{\sim} \mathbb{F}_l^{\circledast}$ of Proposition 4.2. In particular, the element $[\underline{\epsilon}] \in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$ may be *characterized* as the *unique element* $\in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$ such that *evaluation* at the element yields the assignment $\phi_j^{\mathrm{LC}} \mapsto j$.

Remark 4.5.1.

(i) Let G be a group. If $H \subseteq G$ is a subgroup, $g \in G$, then we shall write $H^g \stackrel{\mathrm{def}}{=} g \cdot H \cdot g^{-1}$. Let $J \subseteq H \subseteq G$ be subgroups. Suppose further that each of the subgroups J, H of G is *only known up to conjugacy in* G. Put another way, we suppose that we are in a situation in which there are **independent G-conjugacy indeterminacies** in the specification of the subgroups J and H. Thus, for instance, there is *no natural natural way* to distinguish the given inclusion $\iota : J \hookrightarrow H$ from its γ-conjugate $\iota^{\gamma} : J^{\gamma} \hookrightarrow H^{\gamma}$, for $\gamma \in G$. Moreover, it may happen to be the case that for some $g \in G$, not only J, but also $J^g \subseteq H$ [or, equivalently $J \subseteq H^{g^{-1}}$]. Here, the subgroups J, J^g of H are *not necessarily conjugate in* H; indeed, the abstract pairs of a group and a subgroup given by (H, J) and (H, J^g) *need not be isomorphic* [i.e., it is not even necessarily the case that there exists an automorphism of H that maps J onto J^g]. In particular, the existence of the independent G-conjugacy indeterminacies in the specification of J and H means that *one cannot specify the inclusion* $\iota : J \hookrightarrow H$ *independently of the inclusion* $\zeta : J \hookrightarrow H^{g^{-1}}$ [i.e., arising from $J^g \subseteq H$]. One way to express this state of affairs is as follows. Write "$\stackrel{\mathrm{out}}{\hookrightarrow}$" for the outer homomorphism determined by an injective homomorphism between groups. Then the collection of **factorizations** $J \stackrel{\mathrm{out}}{\hookrightarrow} H \stackrel{\mathrm{out}}{\hookrightarrow} G$ of the natural "outer" inclusion $J \overset{\text{out}}{\hookrightarrow} G$ through some G-conjugate of H — i.e., put another way,

the collection of **outer homomorphisms**

$$J \overset{\text{out}}{\hookrightarrow} H$$

that are **compatible** with the **"structure morphisms"** $J \overset{\text{out}}{\hookrightarrow} G$, $H \overset{\text{out}}{\hookrightarrow} G$ determined by the natural inclusions

— is **well-defined**, in a fashion that is *compatible with independent G-conjugacy indeterminacies in the specification of J and H.* That is to say, this collection of outer homomorphisms amounts to the collection of inclusions $J^{g_1} \hookrightarrow H^{g_2}$, for $g_1, g_2 \in G$. By contrast, to specify the inclusion $\iota : J \hookrightarrow H$ [together with, say, its G-conjugates $\{\iota^\gamma\}_{\gamma \in G}$] *independently* of the inclusion $\zeta : J \hookrightarrow H^{g^{-1}}$ [and its G-conjugates $\{\zeta^\gamma\}_{\gamma \in G}$] amounts to the imposition of a **partial synchronization** — *i.e., a* **partial deactivation** — *of the [a priori!] independent G-conjugacy indeterminacies in the specification of J and H.* Moreover, such a "partial deactivation" can only be effected at the cost of introducing certain **arbitrary choices** into the construction under consideration.

(ii) Relative to the *factorizations* considered in (i), we make the following observation. Given a G-conjugate H^* of H and a subgroup $I \subseteq H^*$, the condition on I that

$$(*^{\subseteq}) \ \textit{I be a G-conjugate of J}$$

is a condition that is *independent* of the datum H^*, while the condition on I that

$$(*^{\cong}) \ \textit{I be a G-conjugate of J such that } (H^*, I) \cong (H, J)$$

[where the "$\cong$" denotes an isomorphism of pairs consisting of a group and a subgroup — cf. the discussion of (i)] is a condition that *depends*, in an essential fashion, on the datum H^*. Here, $(*^{\subseteq})$ is precisely the condition that one must impose when one considers *arbitrary factorizations* as in (i), while $(*^{\cong})$ is the condition that one must impose when one wishes to restrict one's attention to factorizations whose first arrow gives rise to a pair isomorphic to the pair determined by ι. That is to say, the *dependence* of $(*^{\cong})$ on the datum H^* may be regarded as an explicit formulation of the necessity for the *"imposition of a partial synchronization"* as discussed in (i), while the corresponding *independence*, exhibited by $(*^{\subseteq})$, of the datum H^* may be regarded as an explicit formulation of the *lack* of such a necessity when one considers arbitrary factorizations as in (i). Finally, we note that by reversing the direction of the inclusion "$\subseteq$", one may consider a subgroup of $I \subseteq G$ that *contains* a given G-conjugate J^* of J, i.e., $I \supseteq J^*$; then analogous observations may be made concerning the condition $(*^{\supseteq})$ on I that I *be a G-conjugate of H.*

(iii) The abstract situation described in (i) occurs in the discussion of Example 4.5, (i), at $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$. That is to say, the group "G" (respectively, "H"; "J") of (i) corresponds to the group $\pi_1(\mathcal{D}^{\circledcirc})$ (respectively, the image of $\pi_1(\mathcal{D}_{\underline{v}_j})$ in $\pi_1(\mathcal{D}^{\circledcirc})$; the unique index l open subgroup of a cuspidal inertia group of $\pi_1(\mathcal{D}^{\circledcirc})$)

of Example 4.5, (i). Here, we recall that the homomorphism $\pi_1(\mathcal{D}_{\underline{v}_j}) \to \pi_1(\mathcal{D}^{\circledcirc})$ is only known up to composition with an inner automorphism — i.e., up to $\pi_1(\mathcal{D}^{\circledcirc})$-*conjugacy*; a cuspidal inertia group of $\pi_1(\mathcal{D}^{\circledcirc})$ is also only determined by an element $\in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$ up to $\pi_1(\mathcal{D}^{\circledcirc})$-*conjugacy*. Moreover, it is immediate from the construction of the *"model $\mathcal{D}$-NF-bridges"* of Example 4.3 [cf. also Definition 4.6, (i), below] that *there is* **no natural** *way to* **synchronize these indeterminacies**. Indeed, from the point of view of the discussion of Remark 4.3.1, (ii), by considering the actions of the absolute Galois groups of the local and global base fields involved on the cuspidal inertia groups that appear, one sees that such a synchronization would amount, roughly speaking, to a *Galois-equivariant splitting* [i.e., relative to the global absolute Galois groups that that appear] *of the "prime decomposition trees"* of Remark 4.3.1, (ii) — which is absurd [cf. [IUTchII], Remark 2.5.2, (iii), for a more detailed discussion of this sort of phenomenon]. This phenomenon of the "non-synchronizability" of indeterminacies arising from local and global absolute Galois groups is reminiscent of the discussion of [EtTh], Remark 2.16.2. On the other hand, by Corollary 2.5, one concludes in the present situation the *highly nontrivial* fact that

> a **factorization** "$J \hookrightarrow H \hookrightarrow G$" is **uniquely determined** by the composite $J \hookrightarrow G$, i.e., by the G-conjugate of J that one starts with, **without** resorting to any *a priori* **"synchronization of indeterminacies"**.

(iv) A similar situation to the situation of (iii) occurs in the discussion of Example 4.5, (i), at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$. That is to say, in this case, the group "G" (respectively, "H"; "J") of (i) corresponds to the group $\pi_1(\mathcal{D}^{\circledcirc})$ (respectively, the image of $\pi_1(\mathcal{D}_{\underline{v}_j})$ in $\pi_1(\mathcal{D}^{\circledcirc})$; a cuspidal inertia group of $\pi_1(\mathcal{D}_{\underline{v}_j})$) of Example 4.5, (i). In this case, although it does not hold that a *factorization* "$J \hookrightarrow H \hookrightarrow G$" is *uniquely determined* by the composite $J \hookrightarrow G$, i.e., by the G-conjugate of J that one starts with [cf. Remark 2.6.1], it does nevertheless hold, by Corollary 2.8, that the H-conjugacy class of the image of J via the arrow $J \hookrightarrow H$ that occurs in such a factorization is *uniquely determined*.

(v) The property observed at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$ in (iv) is somewhat *weaker* than the *rather strong* property observed at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ in (iii). In the present series of papers, however, we shall only be concerned with such subtle factorization properties at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, where we wish to develop, in [IUTchII], the theory of *"Hodge-Arakelov-theoretic evaluation"* by restricting certain cohomology classes via an arrow "$J \hookrightarrow H$" appearing in a *factorization* "$J \hookrightarrow H \hookrightarrow G$" of the sort discussed in (i). In fact, in the context of the theory of Hodge-Arakelov-theoretic evaluation that will be developed in [IUTchII], a slightly modified version of the phenomenon discussed in (iii) — which involves the *"additive"* version to be developed in §6 of the "multiplicative" theory developed in the present §4 — will be of central importance.

Definition 4.6.

(i) We define a *base-NF-bridge*, or *$\mathcal{D}$-NF-bridge*, [relative to the given initial Θ-data] to be a poly-morphism

$$^{\dagger}\mathfrak{D}_J \quad \xrightarrow{^{\dagger}\phi^{\mathrm{NF}}_{\circledast}} \quad {}^{\dagger}\mathcal{D}^{\circledcirc}$$

— where ${}^{\dagger}\mathcal{D}^{\circledcirc}$ is a *category equivalent to* $\mathcal{D}^{\circledcirc}$; ${}^{\dagger}\mathfrak{D}_J = \{{}^{\dagger}\mathfrak{D}_j\}_{j \in J}$ is a *capsule of* $\mathcal{D}$-*prime-strips*, indexed by a finite index set J — such that there exist isomorphisms $\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\circledcirc}$, $\mathfrak{D}_{\circledast} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_J$, conjugation by which maps $\phi^{\mathrm{NF}}_{\circledast} \mapsto {}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}$. We define a(n) *[iso]morphism of* $\mathcal{D}$*-NF-bridges*

$$({}^{\dagger}\mathfrak{D}_J \quad \xrightarrow{{}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}} \quad {}^{\dagger}\mathcal{D}^{\circledcirc}) \quad \to \quad ({}^{\ddagger}\mathfrak{D}_{J'} \quad \xrightarrow{{}^{\ddagger}\phi^{\mathrm{NF}}_{\circledast}} \quad {}^{\ddagger}\mathcal{D}^{\circledcirc})$$

to be a pair of poly-morphisms

$${}^{\dagger}\mathfrak{D}_J \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{J'}; \quad {}^{\dagger}\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}^{\circledcirc}$$

— where ${}^{\dagger}\mathfrak{D}_J \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{J'}$ is a *capsule-full poly-isomorphism* [cf. §0]; ${}^{\dagger}\mathcal{D}^{\circledcirc} \to {}^{\ddagger}\mathcal{D}^{\circledcirc}$ is a poly-morphism which is an $\mathrm{Aut}_{\underline{\epsilon}}({}^{\dagger}\mathcal{D}^{\circledcirc})$- [or, equivalently, $\mathrm{Aut}_{\underline{\epsilon}}({}^{\ddagger}\mathcal{D}^{\circledcirc})$-] *orbit* [cf. the discussion of Example 4.3, (i)] of isomorphisms — which are *compatible* with ${}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}$, ${}^{\ddagger}\phi^{\mathrm{NF}}_{\circledast}$. There is an evident notion of composition of morphisms of $\mathcal{D}$-NF-bridges.

(ii) We define a *base*-Θ-*bridge*, or $\mathcal{D}$-Θ-*bridge*, [relative to the given initial Θ-data] to be a poly-morphism

$${}^{\dagger}\mathfrak{D}_J \quad \xrightarrow{{}^{\dagger}\phi^{\Theta}_{\circledast}} \quad {}^{\dagger}\mathfrak{D}_{>}$$

— where ${}^{\dagger}\mathfrak{D}_{>}$ is a $\mathcal{D}$*-prime-strip*; ${}^{\dagger}\mathfrak{D}_J = \{{}^{\dagger}\mathfrak{D}_j\}_{j \in J}$ is a *capsule of* $\mathcal{D}$*-prime-strips*, indexed by a finite index set J — such that there exist isomorphisms $\mathfrak{D}_{>} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_{>}$, $\mathfrak{D}_{\circledast} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_J$, conjugation by which maps $\phi^{\Theta}_{\circledast} \mapsto {}^{\dagger}\phi^{\Theta}_{\circledast}$. We define a(n) *[iso]morphism of* $\mathcal{D}$*-*Θ*-bridges*

$$({}^{\dagger}\mathfrak{D}_J \quad \xrightarrow{{}^{\dagger}\phi^{\Theta}_{\circledast}} \quad {}^{\dagger}\mathfrak{D}_{>}) \quad \to \quad ({}^{\ddagger}\mathfrak{D}_{J'} \quad \xrightarrow{{}^{\ddagger}\phi^{\Theta}_{\circledast}} \quad {}^{\ddagger}\mathfrak{D}_{>})$$

to be a pair of poly-morphisms

$${}^{\dagger}\mathfrak{D}_J \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{J'}; \quad {}^{\dagger}\mathfrak{D}_{>} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{>}$$

— where ${}^{\dagger}\mathfrak{D}_J \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{J'}$ is a *capsule-full poly-isomorphism*; ${}^{\dagger}\mathfrak{D}_{>} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{>}$ is the *full poly-isomorphism* — which are *compatible* with ${}^{\dagger}\phi^{\Theta}_{\circledast}$, ${}^{\ddagger}\phi^{\Theta}_{\circledast}$. There is an evident notion of composition of morphisms of $\mathcal{D}$-Θ-bridges.

(iii) We define a *base*-ΘNF-*Hodge theater*, or $\mathcal{D}$-ΘNF-*Hodge theater*, [relative to the given initial Θ-data] to be a collection of data

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = ({}^{\dagger}\mathcal{D}^{\circledcirc} \quad \xleftarrow{{}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}} \quad {}^{\dagger}\mathfrak{D}_J \quad \xrightarrow{{}^{\dagger}\phi^{\Theta}_{\circledast}} \quad {}^{\dagger}\mathfrak{D}_{>})$$

— where ${}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}$ is a $\mathcal{D}$-NF-bridge; ${}^{\dagger}\phi^{\Theta}_{\circledast}$ is a $\mathcal{D}$-Θ-bridge — such that there exist isomorphisms

$$\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\circledcirc}; \quad \mathfrak{D}_{\circledast} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_J; \quad \mathfrak{D}_{>} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_{>}$$

conjugation by which maps $\phi^{\mathrm{NF}}_{\circledast} \mapsto {}^{\dagger}\phi^{\mathrm{NF}}_{\circledast}$, $\phi^{\Theta}_{\circledast} \mapsto {}^{\dagger}\phi^{\Theta}_{\circledast}$. A(n) *[iso]morphism of* $\mathcal{D}$*-*ΘNF*-Hodge theaters* is defined to be a pair of morphisms between the respective associated $\mathcal{D}$-NF- and $\mathcal{D}$-Θ-bridges that are *compatible* with one another in the sense that they induce the *same bijection* between the index sets of the respective capsules of $\mathcal{D}$-prime-strips. There is an evident notion of composition of morphisms of $\mathcal{D}$-ΘNF-Hodge theaters.

Proposition 4.7. (Transport of Label Classes of Cusps via Base-Bridges) *Let*

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = (^{\dagger}\mathcal{D}^{\circledcirc} \quad \xleftarrow{^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}} \quad ^{\dagger}\mathfrak{D}_J \quad \xrightarrow{^{\dagger}\phi_{\divideontimes}^{\Theta}} \quad ^{\dagger}\mathfrak{D}_{>})$$

be a $\mathcal{D}$-ΘNF-**Hodge theater** *[relative to the given initial* Θ*-data]. Then:*

(i) The structure at the $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *of the* $\mathcal{D}$-Θ-**bridge** $^{\dagger}\phi_{\divideontimes}^{\Theta}$ *determines a* **bijection**

$$^{\dagger}\chi : \pi_0(^{\dagger}\mathfrak{D}_J) = J \xrightarrow{\sim} \mathbb{F}_l^{\divideontimes}$$

— i.e., determines **labels** $\in \mathbb{F}_l^{\divideontimes}$ *for the constituent* $\mathcal{D}$-**prime-strips** *of the capsule* $^{\dagger}\mathfrak{D}_J$*.*

(ii) For each $j \in J$*,* **restriction** *at the various* $\underline{v} \in \underline{\mathbb{V}}$ *[cf. Example 4.5] via the portion of* $^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}$*,* $^{\dagger}\phi_{\divideontimes}^{\Theta}$ *indexed by* j *determines an* **isomorphism of** $\mathbb{F}_l^{\divideontimes}$**-torsors**

$$^{\dagger}\phi_j^{\mathrm{LC}} : \mathrm{LabCusp}(^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathrm{LabCusp}(^{\dagger}\mathfrak{D}_{>})$$

such that $^{\dagger}\phi_j^{\mathrm{LC}}$ *is obtained from* $^{\dagger}\phi_1^{\mathrm{LC}}$ *[where, by abuse of notation, we write "*$1 \in J$*" for the element of* J *that maps via* $^{\dagger}\chi$ *to the image of* 1 *in* $\mathbb{F}_l^{\divideontimes}$*] by composing with the action by* $^{\dagger}\chi(j) \in \mathbb{F}_l^{\divideontimes}$*.*

(iii) There exists a **unique element**

$$[^{\dagger}\underline{\epsilon}] \in \mathrm{LabCusp}(^{\dagger}\mathcal{D}^{\circledcirc})$$

such that for each $j \in J$*, the* **natural bijection** $\mathrm{LabCusp}(^{\dagger}\mathfrak{D}_{>}) \xrightarrow{\sim} \mathbb{F}_l^{\divideontimes}$ *of the second display of Proposition 4.2 maps* $^{\dagger}\phi_j^{\mathrm{LC}}([^{\dagger}\underline{\epsilon}]) = {}^{\dagger}\phi_1^{\mathrm{LC}}(^{\dagger}\chi(j) \cdot [^{\dagger}\underline{\epsilon}]) \mapsto {}^{\dagger}\chi(j)$*. In particular, the element* $[^{\dagger}\underline{\epsilon}]$ *determines an* **isomorphism of** $\mathbb{F}_l^{\divideontimes}$**-torsors**

$$^{\dagger}\zeta_{\divideontimes} : \mathrm{LabCusp}(^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} J \quad (\xrightarrow{\sim} \mathbb{F}_l^{\divideontimes})$$

[where the bijection in parentheses is the bijection $^{\dagger}\chi$ *of (i)] between* **"global cusps"** *[i.e., "*$^{\dagger}\chi(j) \cdot [^{\dagger}\underline{\epsilon}]$*"] and* **capsule indices** *[i.e.,* $j \in J \xrightarrow{\sim} \mathbb{F}_l^{\divideontimes}$*]. Finally, when considered up to composition with multiplication by an element of* $\mathbb{F}_l^{\divideontimes}$*, the bijection* $^{\dagger}\zeta_{\divideontimes}$ *is* **independent** *of the choice of* $^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}$ *within the* $\mathbb{F}_l^{\divideontimes}$**-orbit** *of* $^{\dagger}\phi_{\divideontimes}^{\mathrm{NF}}$ *relative to the natural poly-action of* $\mathbb{F}_l^{\divideontimes}$ *on* $^{\dagger}\mathcal{D}^{\circledcirc}$ *[cf. Example 4.3, (iii); Fig. 4.4 below].*

Proof. Assertion (i) follows immediately from the definitions [cf. Example 4.4, (i), (ii), (iv); Definition 4.6], together with the bijection of the second display of Proposition 4.2. Assertions (ii) and (iii) follow immediately from the *intrinsic nature* of the constructions of Example 4.5. ◯

Remark 4.7.1. The significance of the natural bijection $^{\dagger}\zeta_{\divideontimes}$ of Proposition 4.7, (iii), lies in the following observation: Suppose that one wishes to work with the *global data* $^{\dagger}\mathcal{D}^{\circledcirc}$ in a fashion that is *independent* of the *local data* [i.e., "prime-strip data"] $^{\dagger}\mathfrak{D}_{>}$, $^{\dagger}\mathfrak{D}_J$ [cf. Remark 4.3.2, (b)]. Then

> by replacing the *capsule index set* J by the *set of global label classes of cusps* $\mathrm{LabCusp}(^{\dagger}\mathcal{D}^{\circledcirc})$ via $^{\dagger}\zeta_{*}$, one obtains an object — i.e., $\mathrm{LabCusp}(^{\dagger}\mathcal{D}^{\circledcirc})$ — constructed via [i.e., "native to"] the *global data* that is **immune** to the **"collapsing"** of $J \xrightarrow{\sim} \mathbb{F}_l^{*}$ — i.e., of $\mathbb{F}_l^{*}$-orbits of $\underline{\mathbb{V}}^{\pm \mathrm{un}}$ — *even* at primes $\underline{v} \in \underline{\mathbb{V}}$ of the sort discussed in Remark 4.2.1!

That is to say, this "collapsing" of [i.e., failure of $\mathbb{F}_l^{*}$ to act *freely* on] $\mathbb{F}_l^{*}$-orbits of $\underline{\mathbb{V}}^{\pm \mathrm{un}}$ is a *characteristically global* consequence of the *global prime decomposition trees* discussed in Remark 4.3.1, (ii) [cf. the example discussed in Remark 4.2.1]. We refer to Remark 4.9.3, (ii), below for a discussion of a closely related phenomenon.

Remark 4.7.2.

(i) At the level of *labels* [cf. the content of Proposition 4.7], the structure of a $\mathcal{D}$-ΘNF-*Hodge theater* may be summarized via the diagram of Fig. 4.4 below — i.e., where the expression "$[\,1 \;<\; 2 < \;\ldots\; < \;(l^{*}-1)\; < \;l^{*}\,]$" corresponds to $^{\dagger}\mathfrak{D}_{>}$; the expression "$(\,1 \quad 2 \quad \ldots \quad l^{*}-1 \quad l^{*}\,)$" corresponds to $^{\dagger}\mathfrak{D}_{J}$; the lower right-hand "$\mathbb{F}_l^{*}$-cycle of $*$'s" corresponds to $^{\dagger}\mathcal{D}^{\circledcirc}$; the "$\Uparrow$" corresponds to the associated $\mathcal{D}$-Θ-*bridge*; the "$\Rightarrow$" corresponds to the associated $\mathcal{D}$-*NF-bridge*.

(ii) Note that the labels arising from $^{\dagger}\mathfrak{D}_{>}$ correspond, ultimately, to various **irreducible components** in the special fiber of a certain tempered covering of a **["geometric"!] Tate curve** [a special fiber which consists of a *chain of copies of the projective line* — cf. [EtTh], Corollary 2.9]. In particular, these labels are obtained by *counting* — in an intuitive, *archimedean*, *additive* fashion — the number of irreducible components between a given irreducible component and the "origin". In particular, the portion of the diagram of Fig. 4.4 corresponding to $^{\dagger}\mathfrak{D}_{>}$ may be described by the following terms:

> **geometric**, **additive**, **archimedean**, hence **Frobenius-like** [cf. Corollary 3.8].

By contrast, the various "$*$'s" in the portion of the diagram of Fig. 4.4 corresponding to $^{\dagger}\mathcal{D}^{\circledcirc}$ arise, ultimately, from various **primes** of an **["arithmetic"!] number field**. These primes are permuted by the *multiplicative group* $\mathbb{F}_l^{*} = \mathbb{F}_l^{\times}/\{\pm 1\}$, in a *cyclic* — i.e., *nonarchimedean* — fashion. Thus, the portion of the diagram of Fig. 4.4 corresponding to $^{\dagger}\mathcal{D}^{\circledcirc}$ may be described by the following terms:

> **arithmetic**, **multiplicative**, **nonarchimedean**, hence **étale-like** [cf. the discussion of Remark 4.3.2].

That is to say, the portions of the diagram of Fig. 4.4 corresponding to $^{\dagger}\mathfrak{D}_{>}$, $^{\dagger}\mathcal{D}^{\circledcirc}$ *differ quite fundamentally in structure*. In particular, it is not surprising that the only "common ground" of these two fundamentally structurally different portions consists of an *underlying set of cardinality* l^{*} [i.e., the portion of the diagram of Fig. 4.4 corresponding to $^{\dagger}\mathfrak{D}_{J}$].

(iii) The bijection $^{\dagger}\zeta_{*}$ — or, perhaps more appropriately, its inverse

$$(^{\dagger}\zeta_{*})^{-1} : J \xrightarrow{\sim} \mathrm{LabCusp}(^{\dagger}\mathcal{D}^{\circledcirc})$$

— may be thought of as relating *arithmetic* [i.e., if one thinks of the elements of the capsule index set J as *collections of primes of a number field*] to *geometry* [i.e., if one thinks of the elements of $\mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc})$ as corresponding to the [geometric!] *cusps* of the hyperbolic orbicurve]. From this point of view,

> $({}^{\dagger}\zeta_{\divideontimes})^{-1}$ may be thought of as a sort of **"combinatorial Kodaira-Spencer morphism"** [cf. the point of view of [HASurI], §1.4].

We refer to Remark 4.9.2, (iv), below, for another way to think about ${}^{\dagger}\zeta_{\divideontimes}$.

$$[\,1\ <\ 2 <\ \ldots\ <\ j\ <\ \ldots\ <\ (l^{\divideontimes}-1)\ <\ l^{\divideontimes}\,]$$

$$\Uparrow$$

$$\begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & l^{\divideontimes}-1 & \\ & & & & l^{\divideontimes} \end{pmatrix} \quad \Rightarrow \quad \begin{array}{ccccc} & * & \longrightarrow & * & \\ \nearrow & & & & \searrow \\ * & & & & * \\ \uparrow & & \mathbb{F}_l^{\divideontimes} \curvearrowright & & \downarrow \\ * & & & & * \\ \nwarrow & & & & \swarrow \\ & * & \ldots & * & \end{array}$$

Fig. 4.4: The combinatorial structure of a $\mathcal{D}$-ΘNF-Hodge theater

The following result follows immediately from the definitions.

Proposition 4.8. (First Properties of Base-NF-Bridges, Base-Θ-Bridges, and Base-ΘNF-Hodge Theaters) *Relative to a fixed collection of* **initial Θ-data***:*

(i) The **set of isomorphisms** *between* **two $\mathcal{D}$-NF-bridges** *forms an* **$\mathbb{F}_l^{\divideontimes}$-torsor***.*

(ii) The **set of isomorphisms** *between* **two $\mathcal{D}$-Θ-bridges** *(respectively,* **two $\mathcal{D}$-ΘNF-Hodge theaters***) is* **of cardinality one***.*

(iii) Given a $\mathcal{D}$-NF-bridge and a $\mathcal{D}$-Θ-bridge, the set of capsule-full poly-isomorphisms between the respective capsules of $\mathcal{D}$-prime-strips which allow one to **glue** *the given $\mathcal{D}$-NF- and $\mathcal{D}$-Θ-bridges together to form a $\mathcal{D}$-ΘNF-Hodge theater forms an* **$\mathbb{F}_l^{\divideontimes}$-torsor***.*

(iv) Given a $\mathcal{D}$-NF-bridge, there exists a [relatively simple — cf. the discussion of Example 4.4, (i), (ii), (iii)] **functorial algorithm** *for constructing, up to an*

$\mathbb{F}_l^{\circledast}$**-indeterminacy** *[cf. (i), (iii)], from the given $\mathcal{D}$-NF-bridge a* $\mathcal{D}$-ΘNF**-Hodge theater** *whose underlying $\mathcal{D}$-NF-bridge is the given $\mathcal{D}$-NF-bridge.*

Proposition 4.9. (Symmetries arising from Forgetful Functors) *Relative to a fixed collection of* **initial Θ-data***:*

(i) **(Base-NF-Bridges)** *The operation of associating to a* $\mathcal{D}$-ΘNF*-Hodge theater the underlying $\mathcal{D}$-NF-bridge of the* $\mathcal{D}$-ΘNF*-Hodge theater determines a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-NF-bridges}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-NF-bridges}\end{array}}$$

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\text{NF}} \quad \mapsto \quad ({}^{\dagger}\mathcal{D}^{\circledcirc} \ \overset{{}^{\dagger}\phi_{\circledast}^{\text{NF}}}{\longleftarrow} \ {}^{\dagger}\mathfrak{D}_J)$$

whose output data admits a $\mathbb{F}_l^{\circledast}$**-symmetry** *which acts* **simply transitively** *on the index set [i.e., "J"] of the underlying capsule of $\mathcal{D}$-prime-strips [i.e., "${}^{\dagger}\mathfrak{D}_J$"] of this output data.*

(ii) **(Holomorphic Capsules)** *The operation of associating to a* $\mathcal{D}$-ΘNF*-Hodge theater the underlying capsule of $\mathcal{D}$-prime-strips of the* $\mathcal{D}$-ΘNF*-Hodge theater determines a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of } l^{\circledast}\text{-capsules}\\ \text{of } \mathcal{D}\text{-prime-strips}\\ \text{and capsule-full poly-}\\ \text{isomorphisms of } l^{\circledast}\text{-capsules}\end{array}}$$

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\text{NF}} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}_J$$

whose output data admits an $\mathfrak{S}_{l^{\circledast}}$**-symmetry** *[where we write $\mathfrak{S}_{l^{\circledast}}$ for the symmetric group on $l^{\circledast}$ letters] which acts* **transitively** *on the index set [i.e., "J"] of this output data. Thus, this functor may be thought of as an operation that consists of* **forgetting** *the* **labels** $\in \mathbb{F}_l^{\circledast}$ *of Proposition 4.7, (i). In particular, if one is only given this output data ${}^{\dagger}\mathfrak{D}_J$ up to isomorphism, then there is a total of precisely $l^{\circledast}$* **possibilities** *for the element $\in \mathbb{F}_l^{\circledast}$ to which a given index $j \in J$ corresponds [cf. Proposition 4.7, (i)], prior to the application of this functor.*

(iii) **(Mono-analytic Capsules)** *By composing the functor of (ii) with the* **mono-analyticization** *operation discussed in Definition 4.1, (iv), one obtains a*

natural functor

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta\text{NF-Hodge theaters}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of } l^{\divideontimes}\text{-capsules}\\ \text{of } \mathcal{D}^{\vdash}\text{-prime-strips}\\ \text{and capsule-full poly-}\\ \text{isomorphisms of } l^{\divideontimes}\text{-capsules}\end{array}}$$

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}^{\vdash}_{J}$$

whose output data satisfies the same symmetry properties with respect to labels as the output data of the functor of (ii).

Proof. Assertions (i), (ii), (iii) follow immediately from the definitions [cf. also Proposition 4.8, (i), in the case of assertion (i)]. ○

Remark 4.9.1.

(i) Ultimately, in the theory of the present series of papers [cf., especially, [IUTchII], §2], we shall be interested in

> **evaluating** the **étale theta function** of [EtTh] — i.e., in the spirit of the **Hodge-Arakelov theory** of [HASurI], [HASurII]— at the various $\mathcal{D}$-prime-strips of ${}^{\dagger}\mathfrak{D}_J$, in the fashion stipulated by the **labels** discussed in Proposition 4.7, (i).

These values of the étale theta function will be used to construct various *arithmetic line bundles*. We shall be interested in computing the *arithmetic degrees* — in the form of various *"log-volumes"* — of these arithmetic line bundles. In order to compute these *global* log-volumes, it is necessary to be able to *compare* the log-volumes that arise at $\mathcal{D}$-prime-strips with *different labels*. It is for this reason that the *non-labeled* output data of the functors of Proposition 4.9, (i), (ii), (iii) [cf. also Proposition 4.11, (i), (ii), below], are of *crucial importance* in the theory of the present series of papers. That is to say,

> the **non-labeled** output data of the functors of Proposition 4.9, (i), (ii), (iii) [cf. also Proposition 4.11, (i), (ii), below] — which allow one to consider **isomorphisms** between the $\mathcal{D}$-prime-strips that were originally assigned **different labels** — make possible the **comparison** of objects [e.g., log-volumes] constructed relative to different labels.

In Proposition 4.11, (i), (ii), below, we shall see that by considering *"processions"*, one may perform such comparisons in a fashion that *minimizes the label indeterminacy* that arises.

(ii) Since the $\mathbb{F}_l^{\divideontimes}$-symmetry that appears in Proposition 4.9, (i), is *transitive*, it follows that one may use this action to perform **comparisons** as discussed in (i). This prompts the question:

> What is the difference between this $\mathbb{F}_l^{\circledast}$**-symmetry** and the $\mathfrak{S}_{l^{\circledast}}$**-symmetry** of the output data of the functors of Proposition 4.9, (ii), (iii)?

In a word, restricting to the $\mathbb{F}_l^{\circledast}$-symmetry of Proposition 4.9, (i), amounts to the *imposition of a "cyclic structure"* on the index set J [i.e., a structure of $\mathbb{F}_l^{\circledast}$-torsor on J]. Thus, relative to the issue of *comparability* raised in (i), this $\mathbb{F}_l^{\circledast}$-symmetry allows *comparison* between — i.e., involves *isomorphisms* between the non-labeled $\mathcal{D}$-prime-strips corresponding to — distinct members of this index set J, *without disturbing the cyclic structure* on J. This cyclic structure may be thought of as a sort of *combinatorial manifestation* of the link to the **global object** ${}^{\dagger}\mathcal{D}^{\circledcirc}$ that appears in a $\mathcal{D}$-NF-bridge. On the other hand,

> in order to **compare** these $\mathcal{D}$-prime-strips indexed by J **"in the absolute"** to $\mathcal{D}$-prime-strips that have nothing to do with J, it is necessary the **"forget the cyclic structure on** J**"**.

This is precisely what is achieved by considering the functors of Proposition 4.9, (ii), (iii), i.e., by working with the "full $\mathfrak{S}_{l^{\circledast}}$-symmetry".

Remark 4.9.2.

(i) The various elements of the index set of the capsule of $\mathcal{D}$-prime-strips of a $\mathcal{D}$-NF-bridge are *synchronized* in their correspondence with the labels "$1, 2, \dots, l^{\circledast}$", in the sense that this correspondence is completely determined up to composition with the action of an element of $\mathbb{F}_l^{\circledast}$. In particular, this correspondence is always **bijective**.

> One may regard this phenomenon of **synchronization**, or *cohesion*, as an *important consequence* of the fact that the number field in question is represented in the $\mathcal{D}$-NF-bridge via a **single copy** [i.e., as opposed to a *capsule* whose index set is of cardinality ≥ 2] of $\mathcal{D}^{\circledcirc}$.

Indeed, consider a situation in which *each* $\mathcal{D}$-prime-strip in the capsule ${}^{\dagger}\mathfrak{D}_J$ is equipped with its own *"independent globalization"*, i.e., copy of $\mathcal{D}^{\circledcirc}$, to which it is related by a copy of "ϕ_j^{NF}", which [in order not to invalidate the *comparability of distinct labels* — cf. Remark 4.9.1, (i)] is regarded as being known *only up to composition with the action of an element of* $\mathbb{F}_l^{\circledast}$. Then if one thinks of the [manifestly *mutually disjoint* — cf. Definition 3.1, (f); Example 4.3, (i)] $\mathbb{F}_l^{\circledast}$-translates of $\underline{\mathbb{V}}^{\pm\mathrm{un}} \bigcap \mathbb{V}(K)^{\mathrm{bad}}$ [whose union is equal to $\underline{\mathbb{V}}^{\mathrm{Bor}} \bigcap \mathbb{V}(K)^{\mathrm{bad}}$] as being *labeled* by the elements of $\mathbb{F}_l^{\circledast}$, then *each* $\mathcal{D}$-prime-strip in the capsule ${}^{\dagger}\mathfrak{D}_J$ — i.e., each "$\bullet$" in Fig. 4.5 below — is subject, as depicted in Fig. 4.5, to an *independent indeterminacy* concerning the label $\in \mathbb{F}_l^{\circledast}$ to which it is associated. In particular, the *set of all possibilities for each association* includes correspondences between the index set J of the capsule ${}^{\dagger}\mathfrak{D}_J$ and the set of labels $\mathbb{F}_l^{\circledast}$ which **fail to be bijective**. Moreover, although $\mathbb{F}_l^{\circledast}$ arises essentially as a subquotient of a Galois group of extensions of number fields [cf. the *faithful poly-action* of $\mathbb{F}_l^{\circledast}$ on primes of $\mathbb{V}(K)$], the fact that it also acts *faithfully* on conjugates of the cusp $\underline{\epsilon}$ [cf. Example 4.3, (i)] implies that *"working with elements of* $\mathbb{V}(K)$ *up to* $\mathbb{F}_l^{\circledast}$*-indeterminacy"* may only be done *at the* *expense of "working with conjugates of the cusp $\underline{\epsilon}$ up to $\mathbb{F}_l^{\circledast}$-indeterminacy"*. That is to say, "working with nonsynchronized labels" is *inconsistent* with the construction of the *crucial bijection* $^{\dagger}\zeta_{\circledast}$ in Proposition 4.7, (iii).

$$\begin{array}{ccccccc}
\bullet & \mapsto & 1? & 2? & 3? & \cdots & l^{\circledast}? \\
\bullet & \mapsto & 1? & 2? & 3? & \cdots & l^{\circledast}? \\
\vdots & & & & \vdots & & \\
\bullet & \mapsto & 1? & 2? & 3? & \cdots & l^{\circledast}?
\end{array}$$

Fig. 4.5: Nonsynchronized labels

(ii) In the context of the discussion of (i), we observe that the *"single copy"* of $\mathcal{D}^{\circledcirc}$ may also be thought of as a **"single connected component"**, hence — from the point of view of *Galois categories* — as a **"single basepoint"**.

(iii) In the context of the discussion of (i), it is interesting to note that since the natural action of $\mathbb{F}_l^{\circledast}$ on $\mathbb{F}_l^{\circledast}$ is *transitive*, one obtains the same "set of all possibilities for each association", regardless of whether one considers independent $\mathbb{F}_l^{\circledast}$*-indeterminacies* at each index of J or independent $\mathfrak{S}_{l^{\circledast}}$*-indeterminacies* at each index of J [cf. the discussion of Remark 4.9.1, (ii)].

(iv) The **synchronized indeterminacy** [cf. (i)] exhibited by a $\mathcal{D}$-NF-bridge — i.e., at a more concrete level, the *crucial bijection* $^{\dagger}\zeta_{\circledast}$ of Proposition 4.7, (iii) — may be thought of as a sort of **combinatorial model** of the notion of a **"holomorphic structure"**. By contrast, the **nonsynchronized indeterminacies** discussed in (i) may be thought of as a sort of combinatorial model of the notion of a **"real analytic structure"**. Moreover, we observe that the theme of the above discussion — in which one considers

> *"how a given combinatorial holomorphic structure is* **'embedded'** *within its underlying combinatorial real analytic structure"*

— is very much in line with the *spirit of classical complex Teichmüller theory.*

(v) From the point of view discussed in (iv), the *main results* of the **"multiplicative combinatorial Teichmüller theory"** developed in the present §4 may be summarized as follows:

(a) *globalizability of labels*, in a fashion that is *independent of local structures* [cf. Remark 4.3.2, (b); Proposition 4.7, (iii)];
(b) *comparability of distinct labels* [cf. Proposition 4.9; Remark 4.9.1, (i)];
(c) *absolute comparability* [cf. Proposition 4.9, (ii), (iii); Remark 4.9.1, (ii)];
(d) *minimization of label indeterminacy — without sacrificing the symmetry necessary to perform comparisons! — via processions* [cf. Proposition 4.11, (i), (ii), below].

Remark 4.9.3.

(i) Ultimately, in the theory of the present series of papers [cf. [IUTchIII]], we would like to apply the *mono-anabelian theory* of [AbsTopIII] to the various local and global arithmetic fundamental groups [i.e., isomorphs of $\Pi_{\underline{C}_K}$, $\Pi_{\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$] that appear in a $\mathcal{D}$-ΘNF-Hodge theater [cf. the discussion of Remark 4.3.2]. To do this, it is of essential importance to have available not only the absolute Galois groups of the various local and global base fields involved, but also the *geometric fundamental groups* that lie inside the isomorphs of $\Pi_{\underline{C}_K}$, $\Pi_{\underline{v}}$ involved. Indeed, in the theory of [AbsTopIII], it is precisely the outer Galois action of the absolute Galois group of the base field on the geometric fundamental group that allows one to reconstruct the *ring structures group-theoretically* in a fashion that is compatible with *localization/globalization* operations as shown in Fig. 4.3. Here, we pause to recall that in [AbsTopIII], Remark 5.10.3, (i), one may find a discussion of the analogy between this phenomenon of **"entrusting of arithmetic moduli"** [to the outer Galois action on the geometric fundamental group] and the **Kodaira-Spencer isomorphism of an indigenous bundle** — an analogy that is reminiscent of the discussion of Remark 4.7.2, (iii).

(ii) Next, let us observe that the state of affairs discussed in (i) has important implications concerning the *circumstances that necessitate the use of "$\underline{X}_{\to_{\underline{v}}}$"* [i.e., as opposed to "$\underline{C}_{\underline{v}}$"] in the definition of "$\mathcal{D}_{\underline{v}}$" in Examples 3.3, 3.4 [cf. Remark 4.2.1]. Indeed, *localization/globalization* operations as shown in Fig. 4.3 give rise, when applied to the various geometric fundamental groups involved, to various *bijections* between local and global *sets of label classes of cusps*. Now suppose that *one uses "$\underline{C}_{\underline{v}}$" instead of "$\underline{X}_{\to_{\underline{v}}}$"* in the definition of "$\mathcal{D}_{\underline{v}}$" in Examples 3.3, 3.4. Then the existence of $\underline{v} \in \underline{\mathbb{V}}$ of the sort discussed in Remark 4.2.1, together with the condition of *compatiblity with localization/globalization* operations as shown in Fig. 4.3 — where we take, for instance,

$$(v \text{ of Fig. 4.3}) \stackrel{\mathrm{def}}{=} (\underline{v} \text{ of Remark 4.2.1})$$
$$(v' \text{ of Fig. 4.3}) \stackrel{\mathrm{def}}{=} (\underline{w} \text{ of Remark 4.2.1})$$

— imply that, at a combinatorial level, one is led, in effect, to a situation of the sort discussed in Remark 4.9.2, (i), i.e., a situation involving **nonsynchronized labels** [cf. Fig. 4.5], which, as discussed in Remark 4.9.2, (i), is *incompatible* with the construction of the *crucial bijection* ${}^{\dagger}\zeta$ of Proposition 4.7, (iii), an object which will play an important role in the theory of the present series of papers.

Definition 4.10. Let $\mathcal{C}$ be a *category*, n a positive integer. Then we shall refer to as a *procession of length* n, or n-*procession*, of $\mathcal{C}$ any diagram of the form

$$P_1 \hookrightarrow P_2 \hookrightarrow \ldots \hookrightarrow P_n$$

— where each P_j [for $j = 1, \ldots, n$] is a j-*capsule* [cf. §0] of objects of $\mathcal{C}$; each arrow $P_j \hookrightarrow P_{j+1}$ [for $j = 1, \ldots, n-1$] denotes the collection of *all capsule-full* *poly-morphisms* [cf. §0] from P_j to P_{j+1}. A *morphism* from an n-procession of $\mathcal{C}$ to an m-procession of $\mathcal{C}$

$$(P_1 \hookrightarrow \ldots \hookrightarrow P_n) \quad \rightarrow \quad (Q_1 \hookrightarrow \ldots \hookrightarrow Q_m)$$

consists of an *order-preserving injection* $\iota : \{1, \ldots, n\} \hookrightarrow \{1, \ldots, m\}$ [so $n \leq m$], together with a capsule-full poly-morphism $P_j \hookrightarrow Q_{\iota(j)}$ for each $j = 1, \ldots, n$.

$$/^{\circledast} \quad \hookrightarrow \quad /^{\circledast}/^{\circledast} \quad \hookrightarrow \quad /^{\circledast}/^{\circledast}/^{\circledast} \quad \hookrightarrow \quad \ldots \quad \hookrightarrow \quad (/^{\circledast} \ \ldots \ /^{\circledast})$$

Fig. 4.6: An $l^{\circledast}$-procession of $\mathcal{D}$-prime-strips

Proposition 4.11. **(Processions of Base-Prime-Strips)** *Relative to a fixed collection of* **initial Θ-data***:*

(i) **(Holomorphic Processions)** *Given a $\mathcal{D}$-Θ-bridge ${}^{\dagger}\phi^{\Theta}_{\circledast} : {}^{\dagger}\mathfrak{D}_J \rightarrow {}^{\dagger}\mathfrak{D}_{>}$, with underlying capsule of $\mathcal{D}$-prime-strips ${}^{\dagger}\mathfrak{D}_J$, write* $\mathrm{Prc}({}^{\dagger}\mathfrak{D}_J)$ *for the* $l^{\circledast}$**-procession of $\mathcal{D}$-prime-strips** *[cf. Fig. 4.6, where each "$/^{\circledast}$" denotes a $\mathcal{D}$-prime-strip] determined by considering the ["sub"]capsules of ${}^{\dagger}\mathfrak{D}_J$ corresponding to the subsets $\mathbb{S}^{\circledast}_1 \subseteq \ldots \subseteq \mathbb{S}^{\circledast}_j \stackrel{\mathrm{def}}{=} \{1, 2, \ldots, j\} \subseteq \ldots \subseteq \mathbb{S}^{\circledast}_{l^{\circledast}} \stackrel{\mathrm{def}}{=} \mathbb{F}^{\circledast}_l$ [where, by abuse of notation, we use the notation for positive integers to denote the images of these positive integers in $\mathbb{F}^{\circledast}_l$], relative to the bijection ${}^{\dagger}\chi : J \stackrel{\sim}{\rightarrow} \mathbb{F}^{\circledast}_l$ of Proposition 4.7, (i). Then the assignment ${}^{\dagger}\phi^{\Theta}_{\circledast} \mapsto \mathrm{Prc}({}^{\dagger}\mathfrak{D}_J)$ determines a* **natural functor**

category of $\mathcal{D}$-Θ-bridges and isomorphisms of $\mathcal{D}$-Θ-bridges	$\rightarrow$	category of processions of $\mathcal{D}$-prime-strips and morphisms of processions
${}^{\dagger}\phi^{\Theta}_{\circledast}$	$\mapsto$	$\mathrm{Prc}({}^{\dagger}\mathfrak{D}_J)$

whose output data satisfies the following property: there are precisely $\boldsymbol{n}$ **possibilities** *for the element $\in \mathbb{F}^{\circledast}_l$ to which a given index of the index set of the n-capsule that appears in the procession constituted by this output data corresponds, prior to the application of this functor. That is to say, by taking the product, over elements of $\in \mathbb{F}^{\circledast}_l$, of cardinalities of "sets of possibilies", one concludes that*

> *by considering* **processions** *— i.e., the functor discussed above, possibly pre-composed with the functor ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \mapsto {}^{\dagger}\phi^{\Theta}_{\circledast}$ that associates to a $\mathcal{D}$-ΘNF-Hodge theater its associated $\mathcal{D}$-Θ-bridge — the indeterminacy consisting of $(l^{\circledast})^{(l^{\circledast})}$ possibilities that arises in Proposition 4.9, (ii), is* **reduced** *to an* **indeterminacy** *consisting of a total of* $l^{\circledast}!$ **possibilities**.

(ii) **(Mono-analytic Processions)** *By composing the functor of (i) with the* **mono-analyticization** *operation discussed in Definition 4.1, (iv), one obtains a* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta\text{-bridges}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta\text{-bridges}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of processions}\\ \text{of } \mathcal{D}^{\vdash}\text{-prime-strips}\\ \text{and morphisms of}\\ \text{processions}\end{array}}$$

$$^{\dagger}\phi^{\Theta}_{\divideontimes} \quad \mapsto \quad \mathrm{Prc}(^{\dagger}\mathfrak{D}^{\vdash}_{J})$$

whose output data satisfies the same indeterminacy properties with respect to labels as the output data of the functor of (i).

Proof. Assertions (i), (ii) follow immediately from the definitions. ○

The following result is an immediate consequence of our discussion.

Corollary 4.12. **(Étale-pictures of Base-ΘNF-Hodge Theaters)** *Relative to a fixed collection of* **initial Θ-data***:*

(i) Consider the [composite] **functor**

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \mapsto \quad ^{\dagger}\mathfrak{D}_{>} \quad \mapsto \quad ^{\dagger}\mathfrak{D}^{\vdash}_{>}$$

— from the category of $\mathcal{D}$-ΘNF-Hodge theaters and isomorphisms of $\mathcal{D}$-ΘNF-Hodge theaters to the category of $\mathcal{D}^{\vdash}$-prime-strips and isomorphisms of $\mathcal{D}^{\vdash}$-prime-strips — obtained by assigning to the $\mathcal{D}$-ΘNF-Hodge theater $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ the **mono-analyticization** *[cf. Definition 4.1, (iv)] $^{\dagger}\mathfrak{D}^{\vdash}_{>}$ of the $\mathcal{D}$-prime-strip $^{\dagger}\mathfrak{D}_{>}$ that appears as the codomain of the* **underlying $\mathcal{D}$-Θ-bridge** *[cf. Definition 4.6, (ii)] of $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$. If $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$, $^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ are* **$\mathcal{D}$-ΘNF-Hodge theaters***, then we define the* **base-NF-, or $\mathcal{D}$-NF-, link**

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \xrightarrow{\mathcal{D}} \quad ^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$$

from $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ to $^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ to be the **full poly-isomorphism**

$$^{\dagger}\mathfrak{D}^{\vdash}_{>} \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}^{\vdash}_{>}$$

between the $\mathcal{D}^{\vdash}$-prime-strips obtained by applying the functor discussed above to $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$, $^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$.

(ii) If

$$\ldots \quad \xrightarrow{\mathcal{D}} \quad ^{(n-1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \xrightarrow{\mathcal{D}} \quad ^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \xrightarrow{\mathcal{D}} \quad ^{(n+1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} \quad \xrightarrow{\mathcal{D}} \quad \ldots$$

[where $n \in \mathbb{Z}$] is an **infinite chain** *of* $\mathcal{D}$**-NF-linked** $\mathcal{D}$**-**ΘNF**-Hodge theaters** *[cf. the situation discussed in Corollary 3.8], then we obtain a resulting* **chain of full poly-isomorphisms**

$$\dots \overset{\sim}{\to} {}^{n}\mathfrak{D}^{\vdash}_{>} \overset{\sim}{\to} {}^{(n+1)}\mathfrak{D}^{\vdash}_{>} \overset{\sim}{\to} \dots$$

[cf. the situation discussed in Remark 3.8.1, (ii)] between the $\mathcal{D}^{\vdash}$*-prime-strips obtained by applying the functor of (i). That is to say, the output data of the functor of (i) forms a* **constant invariant** *[cf. the discussion of Remark 3.8.1, (ii)] — i.e., a* **mono-analytic core** *[cf. the situation discussed in Remark 3.9.1] — of the above infinite chain.*

(iii) If we regard each of the $\mathcal{D}$-ΘNF*-Hodge theaters of the chain of (ii) as a* **spoke** *emanating from the mono-analytic core discussed in (ii), then we obtain a* **diagram** *— i.e., an* **étale-picture of** $\mathcal{D}$**-**ΘNF**-Hodge-theaters** *— as in Fig. 4.7 below [cf. the situation discussed in Corollary 3.9, (i)]. In Fig. 4.7, "$>^{\vdash}$" denotes the mono-analytic core; "$/^{\ast} \hookrightarrow /^{\ast}/^{\ast} \hookrightarrow \dots$" denotes the "holomorphic" processions of Proposition 4.11, (i), together with the remaining ["holomorphic"] data of the corresponding* $\mathcal{D}$-ΘNF*-Hodge theater. Finally, [cf. the situation discussed in Corollary 3.9, (i)] this diagram satisfies the important property of admitting* **arbitrary permutation symmetries** *among the spokes [i.e., among the labels $n \in \mathbb{Z}$ of the* $\mathcal{D}$-ΘNF*-Hodge-theaters].*

$$\begin{array}{ccccc} & & \boxed{/^{\ast} \hookrightarrow /^{\ast}/^{\ast} \hookrightarrow \dots} & & \\ \dots & & & & \dots \\ & & | & & \\ \boxed{/^{\ast} \hookrightarrow /^{\ast}/^{\ast} \hookrightarrow \dots} & — & \boxed{>^{\vdash}} & — & \boxed{/^{\ast} \hookrightarrow /^{\ast}/^{\ast} \hookrightarrow \dots} \\ \dots & & | & & \dots \\ & & \boxed{/^{\ast} \hookrightarrow /^{\ast}/^{\ast} \hookrightarrow \dots} & & \end{array}$$

Fig. 4.7: Étale-picture of $\mathcal{D}$-ΘNF-Hodge theaters

Section 5: ΘNF-Hodge Theaters

In the present §5, we continue our discussion of various "enhancements" to the Θ-Hodge theaters of §3. Namely, we define the notion of a ΘNF-**Hodge theater** [cf. Definition 5.5, (iii)] and observe that these ΘNF-Hodge theaters satisfy the **same "functorial dynamics"** [cf. Corollary 5.6; Remark 5.6.1] as the base-ΘNF-Hodge theaters discussed in §4.

Let

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = ({}^{\dagger}\mathcal{D}^{\circledcirc} \quad \overset{{}^{\dagger}\phi_{\circledast}^{\mathrm{NF}}}{\longleftarrow} \quad {}^{\dagger}\mathfrak{D}_J \quad \overset{{}^{\dagger}\phi_{\circledast}^{\Theta}}{\longrightarrow} \quad {}^{\dagger}\mathfrak{D}_{>})$$

be a $\mathcal{D}$-ΘNF-*Hodge theater* [cf. Definition 4.6], relative to fixed *initial* Θ-*data* $(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \underline{\epsilon})$ as in Definition 3.1.

Example 5.1. Global Frobenioids.

(i) By applying the *anabelian result* of [AbsTopIII], Theorem 1.9, via the *"Θ-approach"* discussed in Remark 3.1.2, to $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$, we may construct *group-theoretically* from $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$ an isomorph of "$\overline{F}^{\times}$" — which we shall denote

$$\mathbb{M}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

— equipped with its *natural* $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$-*action.* Here, we recall that this construction includes a reconstruction of the *field structure* on $\overline{\mathbb{M}}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc}) \stackrel{\mathrm{def}}{=} \mathbb{M}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc}) \bigcup \{0\}$. Next, let us observe that the F-*core* C_F [cf. [CanLift], Remark 2.1.1; [EtTh], the discussion at the beginning of §2] admits a *unique model* $C_{F_{\mathrm{mod}}}$ over F_{mod}. In particular, it follows that one may construct *group-theoretically* from $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$, in a functorial fashion, a *profinite group* corresponding to "$C_{F_{\mathrm{mod}}}$" [cf. the algorithms of [AbsTopII], Corollary 3.3, (i), which are applicable in light of [AbsTopI], Example 4.8; the definition of "F_{mod}" in Definition 3.1, (b)], which contains $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$ as an *open subgroup*; write ${}^{\dagger}\mathcal{D}^{\circledast}$ for $\mathcal{B}(-)^0$ of this profinite group, so we obtain a *natural morphism*

$${}^{\dagger}\mathcal{D}^{\circledcirc} \to {}^{\dagger}\mathcal{D}^{\circledast}$$

— i.e., a "category-theoretic version" of the natural morphism of hyperbolic orbicurves $\underline{C}_K \to C_{F_{\mathrm{mod}}}$ — together with a *natural extension* of the action of $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$ on $\mathbb{M}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$ to $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$. In particular, by taking $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$-*invariants*, we obtain a submonoid/subfield

$$\mathbb{M}^{\circledcirc}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \mathbb{M}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc}), \quad \overline{\mathbb{M}}^{\circledcirc}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \overline{\mathbb{M}}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$$

corresponding to $F^{\times}_{\mathrm{mod}} \subseteq \overline{F}^{\times}$, $F_{\mathrm{mod}} \subseteq \overline{F}$.

(ii) Next, let us recall [cf. Definition 4.1, (v)] that the field structure on $\overline{\mathbb{M}}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$ [i.e., "$\overline{F}$"] allows one to reconstruct *group-theoretically* from $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$ the *set of valuations* $\overline{\mathbb{V}}({}^{\dagger}\mathcal{D}^{\circledcirc})$ [i.e., "$\mathbb{V}(\overline{F})$"] on $\overline{\mathbb{M}}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$ equipped with its natural

$\pi_1({}^\dagger\mathcal{D}^\circledast)$-action, hence also the *monoid* on ${}^\dagger\mathcal{D}^\circledast$ [i.e., in the sense of [FrdI], Definition 1.1, (ii)]

$$\Phi^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)(-)$$

that associates to an object $A \in \mathrm{Ob}({}^\dagger\mathcal{D}^\circledast)$ the *monoid* $\Phi^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)(A)$ *of "stack-theoretic"* [cf. Remark 3.1.5] *arithmetic divisors* on the corresponding subfield $\overline{\mathbb{M}}^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)^A \subseteq \overline{\mathbb{M}}^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)$ [i.e., the monoid denoted "$\Phi(-)$" in [FrdI], Example 6.3; cf. also Remark 3.1.5 of the present paper], together with the natural morphism of monoids $\mathbb{M}^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)^A \to \Phi^\circledcirc({}^\dagger\mathcal{D}^\circledcirc)(A)^{\mathrm{gp}}$ [cf. the discussion of [FrdI], Example 6.3; Remark 3.1.5 of the present paper]. As discussed in [FrdI], Example 6.3 [cf. also Remark 3.1.5 of the present paper], this data determines, by applying [FrdI], Theorem 5.2, (ii), a *model Frobenioid*

$$\mathcal{F}^\circledast({}^\dagger\mathcal{D}^\circledcirc)$$

over the base category ${}^\dagger\mathcal{D}^\circledast$.

(iii) Let ${}^\dagger\mathcal{F}^\circledast$ be any *category* equivalent to $\mathcal{F}^\circledast({}^\dagger\mathcal{D}^\circledcirc)$. Thus, ${}^\dagger\mathcal{F}^\circledast$ is equipped with a *natural Frobenioid structure* [cf. [FrdI], Corollary 4.11; [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper]; write $\mathrm{Base}({}^\dagger\mathcal{F}^\circledast)$ for the *base category* of this Frobenioid. Suppose further that we have been given a *morphism*

$${}^\dagger\mathcal{D}^\circledcirc \to \mathrm{Base}({}^\dagger\mathcal{F}^\circledast)$$

which is *abstractly equivalent* [cf. §0] to the natural morphism ${}^\dagger\mathcal{D}^\circledcirc \to {}^\dagger\mathcal{D}^\circledast$ [cf. (i)]. In the following discussion, we shall use the resulting [*uniquely determined*, in light of the F*-coricity* of C_F, together with [AbsTopIII], Theorem 1.9!] isomorphism $\mathrm{Base}({}^\dagger\mathcal{F}^\circledast) \xrightarrow{\sim} {}^\dagger\mathcal{D}^\circledast$ to *identify* $\mathrm{Base}({}^\dagger\mathcal{F}^\circledast)$ with ${}^\dagger\mathcal{D}^\circledast$. Let us denote by

$${}^\dagger\mathcal{F}^\circledcirc \quad \stackrel{\mathrm{def}}{=} \quad {}^\dagger\mathcal{F}^\circledast|_{{}^\dagger\mathcal{D}^\circledcirc} \quad (\subseteq {}^\dagger\mathcal{F}^\circledast)$$

the *restriction* of ${}^\dagger\mathcal{F}^\circledast$ to ${}^\dagger\mathcal{D}^\circledcirc$ via the natural morphism ${}^\dagger\mathcal{D}^\circledcirc \to {}^\dagger\mathcal{D}^\circledast$ and by

$${}^\dagger\mathcal{F}^\circledcirc_{\mathrm{mod}} \quad \stackrel{\mathrm{def}}{=} \quad {}^\dagger\mathcal{F}^\circledast|_{\text{terminal objects}} \quad (\subseteq {}^\dagger\mathcal{F}^\circledast)$$

the *restriction* of ${}^\dagger\mathcal{F}^\circledast$ to the full subcategory of ${}^\dagger\mathcal{D}^\circledast$ determined by the terminal objects [i.e., "$C_{F_{\mathrm{mod}}}$"] of ${}^\dagger\mathcal{D}^\circledast$. Thus, when the data denoted here by the label "†" arises [in the evident sense] from data as discussed in Definition 3.1, the Frobenioid ${}^\dagger\mathcal{F}^\circledcirc_{\mathrm{mod}}$ may be thought of as the Frobenioid of *arithmetic line bundles on the stack* "S_{mod}" of Remark 3.1.5.

(iv) We continue to use the notation of (iii). We shall denote by a superscript "birat" the *birationalizations* [which are *category-theoretic* — cf. [FrdI], Corollary 4.10; [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper] of the Frobenioids ${}^\dagger\mathcal{F}^\circledcirc$, ${}^\dagger\mathcal{F}^\circledast$, ${}^\dagger\mathcal{F}^\circledcirc_{\mathrm{mod}}$; we shall also use this superscript to denote the images of objects and morphisms of these Frobenioids in their birationalizations. Thus, if $A \in \mathrm{Ob}({}^\dagger\mathcal{F}^\circledast)$, then $\mathcal{O}^\times(A^{\mathrm{birat}})$ may be naturally identified with the multiplicative group of nonzero elements of the number field [i.e., finite extension of F_{mod}] corresponding to A. In particular, by allowing A to *vary* among the *Frobenius-trivial* *objects* of ${}^{\dagger}\mathcal{F}^{\circledast}$ that lie over *Galois objects* of ${}^{\dagger}\mathcal{D}^{\circledast}$, we obtain a *pair* [i.e., consisting of a topological group acting continuously on a discrete abelian group]

$$\pi_1({}^{\dagger}\mathcal{D}^{\circledast}) \quad \curvearrowright \quad \widetilde{\mathcal{O}}^{\circledcirc\times}$$

— which we consider up to the action by the *"inner automorphisms of the pair"* arising from conjugation by $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$. Write $\Phi_{{}^{\dagger}\mathcal{F}^{\circledast}}$ for the *divisor monoid* of the Frobenioid ${}^{\dagger}\mathcal{F}^{\circledast}$ [which is *category-theoretic* — cf. [FrdI], Corollary 4.11, (iii); [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper]. Thus, for each $\mathfrak{p} \in \text{Prime}(\Phi_{{}^{\dagger}\mathcal{F}^{\circledast}}(A))$ [where we use the notation "Prime$(-)$" as in [FrdI], §0], the natural homomorphism $\mathcal{O}^{\times}(A^{\text{birat}}) \to \Phi_{{}^{\dagger}\mathcal{F}^{\circledast}}(A)^{\text{gp}}$ [cf. [FrdI], Proposition 4.4, (iii)] determines — i.e., by taking the inverse image via this homomorphism of [the union with $\{0\}$ of] the subset of $\Phi_{{}^{\dagger}\mathcal{F}^{\circledast}}(A)$ constituted by $\mathfrak{p}$ — a *submonoid* $\mathcal{O}^{\rhd}_{\mathfrak{p}} \subseteq \mathcal{O}^{\times}(A^{\text{birat}})$. That is to say, in more intuitive terms, this submonoid is the submonoid of integral elements of $\mathcal{O}^{\times}(A^{\text{birat}})$ with respect to the valuation determined by $\mathfrak{p}$ of the number field corresponding to A. Write $\mathcal{O}^{\times}_{\mathfrak{p}} \subseteq \mathcal{O}^{\rhd}_{\mathfrak{p}}$ for the submonoid of *invertible elements*. Thus, by allowing A to *vary* among the *Frobenius-trivial objects* of ${}^{\dagger}\mathcal{F}^{\circledast}$ that lie over *Galois objects* of ${}^{\dagger}\mathcal{D}^{\circledast}$ and considering the way in which the natural action of $\text{Aut}_{{}^{\dagger}\mathcal{F}^{\circledast}}(A)$ on $\mathcal{O}^{\times}(A^{\text{birat}})$ *permutes* the various submonoids $\mathcal{O}^{\rhd}_{\mathfrak{p}}$, it follows that for each $\mathfrak{p}_0 \in \text{Prime}(\Phi_{{}^{\dagger}\mathcal{F}^{\circledast}}(A_0))$, where $A_0 \in \text{Ob}({}^{\dagger}\mathcal{F}^{\circledast})$ lies over a *terminal object* of ${}^{\dagger}\mathcal{D}^{\circledast}$, we obtain a *closed subgroup* [well-defined up to conjugation]

$$\Pi_{\mathfrak{p}_0} \subseteq \pi_1({}^{\dagger}\mathcal{D}^{\circledast})$$

by considering the elements of $\text{Aut}_{{}^{\dagger}\mathcal{F}^{\circledast}}(A)$ that *fix* the submonoid $\mathcal{O}^{\rhd}_{\mathfrak{p}}$, for some system of $\mathfrak{p}$'s lying over $\mathfrak{p}_0$. That is to say, in more intuitive terms, the subgroup $\Pi_{\mathfrak{p}_0}$ is simply the *decomposition group* associated to some $v \in \mathbb{V}_{\text{mod}}$. In particular, it follows that $\mathfrak{p}_0$ is *nonarchimedean* if and only if the p-cohomological dimension of $\Pi_{\mathfrak{p}_0}$ is equal to $2+1=3$ for infinitely many prime numbers p [cf., e.g., [NSW], Theorem 7.1.8, (i)]. Suppose that $\mathfrak{p}$ is *nonarchimedean* [i.e., lies over a nonarchimedean $\mathfrak{p}_0$]. Write $\mathcal{O}^{\times}_{\widehat{\mathfrak{p}}}$ for the *profinite completion* of the abelian group $\mathcal{O}^{\times}_{\mathfrak{p}}$ and $\mathcal{O}^{\rhd}_{\widehat{\mathfrak{p}}}$ for the result of applying a change of structure group via the natural morphism $\mathcal{O}^{\times}_{\mathfrak{p}} \to \mathcal{O}^{\times}_{\widehat{\mathfrak{p}}}$ to the "disjoint union of $\mathcal{O}^{\times}_{\mathfrak{p}}$-torsors" $\mathcal{O}^{\rhd}_{\mathfrak{p}}$. Then $\mathcal{O}^{\rhd}_{\widehat{\mathfrak{p}}}$ may be identified with the multiplicative monoid of nonzero integral elements of the completion of the number field corresponding to A at the prime of this number field determined by $\mathfrak{p}$. Thus, again by allowing A to *vary* and considering the resulting *system of topological monoids* "$\mathcal{O}^{\rhd}_{\widehat{\mathfrak{p}}}$", we obtain a construction, for *nonarchimedean* $\mathfrak{p}_0$, of the *pair* [i.e., consisting of a topological group acting continuously on a topological monoid]

$$\Pi_{\mathfrak{p}_0} \quad \curvearrowright \quad \widetilde{\mathcal{O}}^{\rhd}_{\widehat{\mathfrak{p}}_0}$$

— which [since $\Pi_{\mathfrak{p}_0}$ is *commensurably terminal* in $\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$ — cf., e.g., [AbsAnab], Theorem 1.1.1, (i)] we consider up to the action by the *"inner automorphisms of the pair"* arising from conjugation by $\Pi_{\mathfrak{p}_0}$. In the language of [AbsTopIII], §3, this pair is an *"MLF-Galois* $\mathbb{TM}$*-pair of strictly Belyi type"* [cf. [AbsTopIII], Definition 3.1, (ii)].

(v) We continue to use the notation of (iv). Now if "$(-)$" is a [commutative] *topological monoid*, then let us write

$$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}((-)) \stackrel{\text{def}}{=} \text{Hom}(\mathbb{Q}/\mathbb{Z}, (-))$$

[cf. [AbsTopIII], Definition 3.1, (v); [AbsTopIII], Definition 5.1, (v)]. Also, let us write $\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^\dagger\mathcal{D}^{\circledast}))$ for the *cyclotome* "$\boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\Pi_{(-)})$" of [AbsTopIII], Theorem 1.9, which we think of as being applied *"via the* Θ*-approach"* [cf. Remark 3.1.2] to $\pi_1({}^\dagger\mathcal{D}^{\circledast})$; write $\pi_1^{\Theta}({}^\dagger\mathcal{D}^{\circledcirc}) = \pi_1^{\Theta}({}^\dagger\mathcal{D}^{\circledast}) \subseteq \pi_1({}^\dagger\mathcal{D}^{\circledcirc}) \subseteq \pi_1({}^\dagger\mathcal{D}^{\circledast})$ for the closed subgroup — which is *characteristic* as a subgroup of $\pi_1({}^\dagger\mathcal{D}^{\circledcirc})$ — determined by "$\Delta_{\underline{\underline{X}}}$" [cf. [AbsAnab], Lemma 1.1.4, (i); [AbsTopII], Corollary 3.3, (i), (ii); [AbsTopI], Example 4.8; [EtTh], Proposition 2.2, (i)]. Next, let us *observe* that there exists a **unique isomorphism of cyclotomes**

$$\boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^\dagger\mathcal{D}^{\circledast})) \xrightarrow{\sim} \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\widetilde{\mathcal{O}}^{\circledcirc\times})$$

such that the *image* of the *natural injection*

$$\widetilde{\mathcal{O}}^{\circledcirc\times} \hookrightarrow \varinjlim_{H} \ H^1(H, \boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^\dagger\mathcal{D}^{\circledast})))$$

obtained by applying the inverse of the above displayed isomorphism to the [tautological] *Kummer map* $\widetilde{\mathcal{O}}^{\circledcirc\times} \hookrightarrow \varinjlim_H \ H^1(H, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}(\widetilde{\mathcal{O}}^{\circledcirc\times}))$, where H ranges over the open subgroups of $\pi_1({}^\dagger\mathcal{D}^{\circledast})$ containing $\pi_1^{\Theta}({}^\dagger\mathcal{D}^{\circledast})$, *coincides* with the subset "$\overline{k}^{\times}_{\mathrm{NF}}$" constructed in [AbsTopIII], Theorem 1.9, (d), in a fashion that is *compatible* with the *integral submonoids* "$\mathcal{O}^{\triangleright}_{\mathfrak{p}}$" [cf. the discussion of (iv)], relative to the *ring structure* on [the union with $\{0\}$ of] "$\overline{k}^{\times}_{\mathrm{NF}}$" constructed in [AbsTopIII], Theorem 1.9, (e). [Here, we note in passing that the *natural injection* of the above display was constructed in a purely *category-theoretic* fashion from ${}^\dagger\mathcal{F}^{\circledast}$ [cf. the slightly different construction discussed in [AbsTopIII], Corollary 5.2, (iii)], while the *codomain* of this natural injection was constructed in a purely *category-theoretic* fashion from ${}^\dagger\mathcal{D}^{\circledast}$.] Indeed, the *existence* portion of the above *observation* follows immediately by observing that the *isomorphism of cyclotomes* under consideration is simply the *"usual common sense identification of cyclotomes"* that is typically applied without mention in discussions of arithmetic geometry over number fields; on the other hand, the *uniqueness* portion of the above *observation* follows immediately, in light of the condition on the *image* of the natural injection of the above display [cf. also the discussion of (iv)], from the elementary observation that

$$\mathbb{Q}_{>0} \bigcap \widehat{\mathbb{Z}}^{\times} = \{1\}$$

[relative to the natural inclusion $\mathbb{Q} \hookrightarrow \widehat{\mathbb{Z}}$]. Thus, by applying the *anabelian result* of [AbsTopIII], Theorem 1.9, *"via the* Θ*-approach"* [cf. Remark 3.1.2] to $\pi_1({}^\dagger\mathcal{D}^{\circledast})$, we conclude that

> one may **reconstruct category-theoretically** from ${}^\dagger\mathcal{F}^{\circledast}$ the *pair*
>
> $$\pi_1({}^\dagger\mathcal{D}^{\circledast}) \quad \curvearrowright \quad \widetilde{\mathcal{O}}^{\circledcirc\times}$$
>
> [up to "inner automorphism"], as well as the **additive structure** on $\{0\} \bigcup \widetilde{\mathcal{O}}^{\circledcirc\times}$, and the topologies on $\widetilde{\mathcal{O}}^{\circledcirc\times}$ determined by the **set of valuations** of the resulting field $\{0\} \bigcup \widetilde{\mathcal{O}}^{\circledcirc\times}$.

In particular, we obtain a purely *category-theoretic* construction, from ${}^\dagger\mathcal{F}^{\circledast}$, of the *natural bijection*

$$\mathrm{Prime}({}^\dagger\mathcal{F}^{\circledcirc}_{\mathrm{mod}}) \quad \xrightarrow{\sim} \quad \mathbb{V}_{\mathrm{mod}}$$

— where we write $\mathrm{Prime}({}^\dagger\mathcal{F}^{\circledcirc}_{\mathrm{mod}})$ for the *set of primes* [cf. [FrdI], §0] of the divisor monoid of ${}^\dagger\mathcal{F}^{\circledcirc}_{\mathrm{mod}}$; we think of $\mathbb{V}_{\mathrm{mod}}$ as the set of $\pi_1({}^\dagger\mathcal{D}^{\circledast})$*-orbits* of $\overline{\mathbb{V}}({}^\dagger\mathcal{D}^{\circledcirc})$.

(vi) Before proceeding, we observe that the discussion of (iv), (v) concerning ${}^\dagger\mathcal{F}^{\circledast}$, ${}^\dagger\mathcal{D}^{\circledast}$ may also be carried out for ${}^\dagger\mathcal{F}^{\circledcirc}$, ${}^\dagger\mathcal{D}^{\circledcirc}$. We leave the routine details to the reader.

(vii) Next, let us consider the *index set* J of the capsule of $\mathcal{D}$-prime-strips ${}^\dagger\mathfrak{D}_J$. For $j \in J$, write $\underline{\mathbb{V}}_j \stackrel{\mathrm{def}}{=} \{\underline{v}_j\}_{\underline{v}\in\underline{\mathbb{V}}}$. Thus, we have a *natural bijection* $\underline{\mathbb{V}}_j \xrightarrow{\sim} \underline{\mathbb{V}}$, i.e., given by sending $\underline{v}_j \mapsto \underline{v}$. These bijections determine a *"diagonal subset"*

$$\underline{\mathbb{V}}_{\langle J\rangle} \quad \subseteq \quad \underline{\mathbb{V}}_J \quad \stackrel{\mathrm{def}}{=} \quad \prod_{j\in J} \underline{\mathbb{V}}_j$$

— which also admits a *natural bijection* $\underline{\mathbb{V}}_{\langle J\rangle} \xrightarrow{\sim} \underline{\mathbb{V}}$. Thus, we obtain *natural bijections*

$$\underline{\mathbb{V}}_{\langle J\rangle} \xrightarrow{\sim} \underline{\mathbb{V}}_j \xrightarrow{\sim} \mathrm{Prime}({}^\dagger\mathcal{F}^{\circledcirc}_{\mathrm{mod}}) \xrightarrow{\sim} \mathbb{V}_{\mathrm{mod}}$$

for $j \in J$. Write

$${}^\dagger\mathcal{F}^{\circledcirc}_{\langle J\rangle} \stackrel{\mathrm{def}}{=} \{{}^\dagger\mathcal{F}^{\circledcirc}_{\mathrm{mod}}, \underline{\mathbb{V}}_{\langle J\rangle} \xrightarrow{\sim} \mathrm{Prime}({}^\dagger\mathcal{F}^{\circledcirc}_{\mathrm{mod}})\}$$

$${}^\dagger\mathcal{F}^{\circledcirc}_{j} \stackrel{\mathrm{def}}{=} \{{}^\dagger\mathcal{F}^{\circledcirc}_{\mathrm{mod}}, \underline{\mathbb{V}}_{j} \xrightarrow{\sim} \mathrm{Prime}({}^\dagger\mathcal{F}^{\circledcirc}_{\mathrm{mod}})\}$$

for $j \in J$. That is to say, we think of ${}^\dagger\mathcal{F}^{\circledcirc}_j$ as a copy of ${}^\dagger\mathcal{F}^{\circledcirc}_{\mathrm{mod}}$ *"situated on"* the constituent labeled j of the capsule ${}^\dagger\mathfrak{D}_J$; we think of ${}^\dagger\mathcal{F}^{\circledcirc}_{\langle J\rangle}$ as a copy of ${}^\dagger\mathcal{F}^{\circledcirc}_{\mathrm{mod}}$ *"situated in a diagonal fashion on"* all the constituents of the capsule ${}^\dagger\mathfrak{D}_J$. Thus, we have a *natural embedding of categories*

$${}^\dagger\mathcal{F}^{\circledcirc}_{\langle J\rangle} \quad \hookrightarrow \quad {}^\dagger\mathcal{F}^{\circledcirc}_J \quad \stackrel{\mathrm{def}}{=} \quad \prod_{j\in J} {}^\dagger\mathcal{F}^{\circledcirc}_j$$

— where, by abuse of notation, we write ${}^\dagger\mathcal{F}^{\circledcirc}_{\langle J\rangle}$ for the *underlying category* of [i.e., the first member of the pair] ${}^\dagger\mathcal{F}^{\circledcirc}_{\langle J\rangle}$. Here, we observe that the category ${}^\dagger\mathcal{F}^{\circledcirc}_J$ is *not equipped with a Frobenioid structure*. Write

$${}^\dagger\mathcal{F}^{\circledcirc\mathbb{R}}_j; \quad {}^\dagger\mathcal{F}^{\circledcirc\mathbb{R}}_{\langle J\rangle}; \quad {}^\dagger\mathcal{F}^{\circledcirc\mathbb{R}}_J \quad \stackrel{\mathrm{def}}{=} \quad \prod_{j\in J} {}^\dagger\mathcal{F}^{\circledcirc\mathbb{R}}_j$$

for the respective *realifications* [or product of the underlying categories of the realifications] of the corresponding Frobenioids whose notation does not contain a superscript "$\mathbb{R}$". [Here, we recall that the theory of realifications of Frobenioids is discussed in [FrdI], Proposition 5.3.]

Remark 5.1.1. Thus, ${}^\dagger\mathcal{F}^{\circledcirc}_{\langle J\rangle}$ may be thought of as the Frobenioid associated to *divisors on* $\underline{\mathbb{V}}_J$ [i.e., finite formal sums of elements of this set with coefficients in $\mathbb{Z}$ or

$\mathbb{R}$] whose dependence on $j \in J$ is *constant* — that is to say, divisors corresponding to *"constant distributions"* on $\underline{\mathbb{V}}_J$. Such constant distributions are depicted in Fig. 5.1 below. On the other hand, the product of [underlying categories of] Frobenioids ${}^{\dagger}\mathcal{F}_J^{\circledcirc}$ may be thought of as a sort of category of *"arbitrary distributions"* on $\underline{\mathbb{V}}_J$, i.e., divisors on $\underline{\mathbb{V}}_J$ whose dependence on $j \in J$ is *arbitrary.*

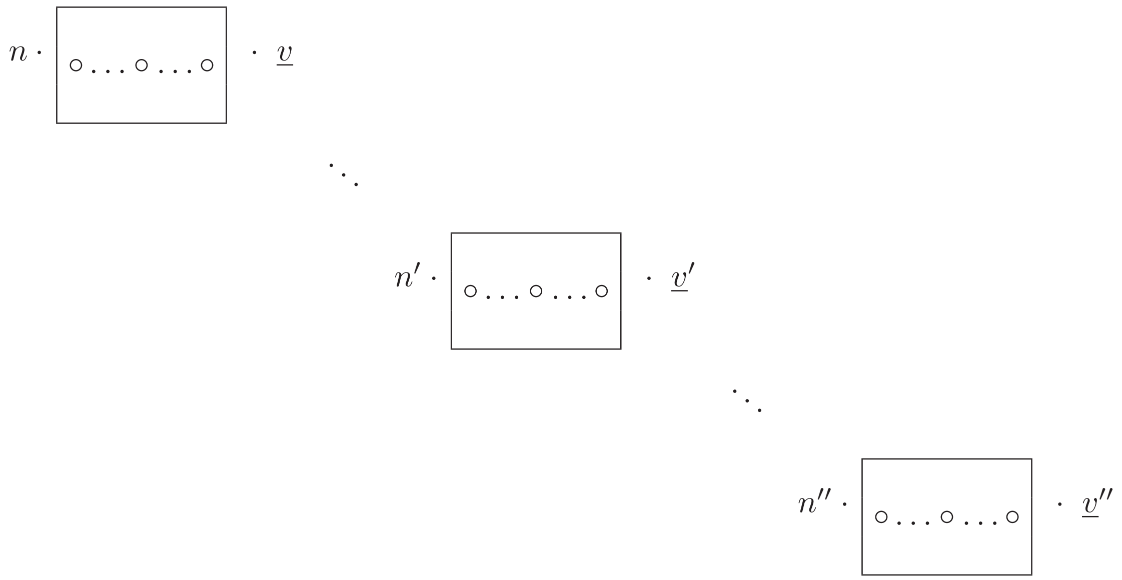

Fig. 5.1: Constant distribution

Remark 5.1.2. The constructions of Example 5.1 manifestly only require the *$\mathcal{D}$-NF-bridge* portion ${}^{\dagger}\phi_{\circledast}^{\mathrm{NF}}$ of the $\mathcal{D}$-ΘNF-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$.

Remark 5.1.3. Note that unlike the case with ${}^{\dagger}\mathcal{F}^{\circledcirc}$, ${}^{\dagger}\mathcal{F}^{\circledast}$, or $\mathbb{M}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$, one cannot perform *Kummer theory* [cf. [FrdII], Definition 2.1, (ii)] with ${}^{\dagger}\mathcal{F}_{\mathrm{mod}}^{\circledcirc}$ or $\mathbb{M}_{\mathrm{mod}}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$. [That is to say, in more concrete terms, unlike the case with $\overline{F}^{\times}$, elements of $F_{\mathrm{mod}}^{\times}$ *do not necessarily admit N-th roots*, for N a nonnegative integer!] The fact that one *can* perform Kummer theory with ${}^{\dagger}\mathcal{F}^{\circledcirc}$, ${}^{\dagger}\mathcal{F}^{\circledast}$, or $\mathbb{M}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$ implies that $\mathbb{M}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$ equipped with its natural $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$-action, as well as the "birational monoid portions" of ${}^{\dagger}\mathcal{F}^{\circledcirc}$ or ${}^{\dagger}\mathcal{F}^{\circledast}$, satisfy various *strong rigidity properties* [cf. Corollary 5.3, (i), below]. For instance, these rigidity properties allow one to recover the *additive structure* on [the union with $\{0\}$ of] $\mathbb{M}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$ equipped with its natural $\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})$-action [cf. the discussion of Example 5.1, (v)]. That is to say,

> the *additive structure* on [the union with $\{0\}$ of] the "birational monoid portion" of ${}^{\dagger}\mathcal{F}_{\mathrm{mod}}^{\circledcirc}$ may only be recovered if one is given the *additional datum consisting of the natural embedding* ${}^{\dagger}\mathcal{F}_{\mathrm{mod}}^{\circledcirc} \hookrightarrow {}^{\dagger}\mathcal{F}^{\circledast}$ [cf. Example 5.1, (iii)].

Put another way, if one only considers the category ${}^{\dagger}\mathcal{F}_{\mathrm{mod}}^{\circledcirc}$ *without* the embedding ${}^{\dagger}\mathcal{F}_{\mathrm{mod}}^{\circledcirc} \hookrightarrow {}^{\dagger}\mathcal{F}^{\circledast}$, then ${}^{\dagger}\mathcal{F}_{\mathrm{mod}}^{\circledcirc}$ is subject to a **"Kummer black-out"** — one consequence of which is that there is no way to recover the additive structure on the "birational monoid portion" of ${}^{\dagger}\mathcal{F}_{\mathrm{mod}}^{\circledcirc}$. In subsequent discussions, we shall refer to these observations concerning ${}^{\dagger}\mathcal{F}^{\circledcirc}$, ${}^{\dagger}\mathcal{F}^{\circledast}$, $\mathbb{M}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$, ${}^{\dagger}\mathcal{F}^{\circledcirc}_{\mathrm{mod}}$, and $\mathbb{M}^{\circledcirc}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})$ by saying that ${}^{\dagger}\mathcal{F}^{\circledcirc}$, ${}^{\dagger}\mathcal{F}^{\circledast}$, and $\mathbb{M}^{\circledcirc}({}^{\dagger}\mathcal{D}^{\circledcirc})$ are **Kummer-ready**, whereas ${}^{\dagger}\mathcal{F}^{\circledcirc}_{\mathrm{mod}}$ and $\mathbb{M}^{\circledcirc}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc})$ are **Kummer-blind**. In particular, the various copies of [and products of copies of] ${}^{\dagger}\mathcal{F}^{\circledcirc}_{\mathrm{mod}}$ — i.e., ${}^{\dagger}\mathcal{F}^{\circledcirc}_{j}$, ${}^{\dagger}\mathcal{F}^{\circledcirc}_{\langle J\rangle}$, ${}^{\dagger}\mathcal{F}^{\circledcirc}_{J}$ — considered in Example 5.1, (vii), are also *Kummer-blind*.

Definition 5.2.

(i) We define a *holomorphic Frobenioid-prime-strip*, or *$\mathcal{F}$-prime-strip*, [relative to the given initial Θ-data] to be a collection of data

$$ {}^{\ddagger}\mathfrak{F} = \{{}^{\ddagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} $$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then ${}^{\ddagger}\mathcal{F}_{\underline{v}}$ is a *category* ${}^{\ddagger}\mathcal{C}_{\underline{v}}$ which admits an equivalence of categories ${}^{\ddagger}\mathcal{C}_{\underline{v}} \overset{\sim}{\to} \mathcal{C}_{\underline{v}}$ [where $\mathcal{C}_{\underline{v}}$ is as in Examples 3.2, (iii); 3.3, (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then ${}^{\ddagger}\mathcal{F}_{\underline{v}} = ({}^{\ddagger}\mathcal{C}_{\underline{v}}, {}^{\ddagger}\mathcal{D}_{\underline{v}}, {}^{\ddagger}\kappa_{\underline{v}})$ is a *collection of data* consisting of a category, an Aut-holomorphic orbispace, and a Kummer structure such that there exists an isomorphism of collections of data ${}^{\ddagger}\mathcal{F}_{\underline{v}} \overset{\sim}{\to} \underline{\underline{\mathcal{F}}}_{\underline{v}} = (\mathcal{C}_{\underline{v}}, \mathcal{D}_{\underline{v}}, \kappa_{\underline{v}})$ [where $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is as in Example 3.4, (i)].

(ii) We define a *mono-analytic Frobenioid-prime-strip*, or *$\mathcal{F}^{\vdash}$-prime-strip*, [relative to the given initial Θ-data] to be a collection of data

$$ {}^{\ddagger}\mathfrak{F}^{\vdash} = \{{}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} $$

that satisfies the following conditions: (a) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, then ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$ is a *split Frobenioid*, whose underlying Frobenioid we denote by ${}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}$, which admits an isomorphism ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}} \overset{\sim}{\to} \mathcal{F}^{\vdash}_{\underline{v}}$ [where $\mathcal{F}^{\vdash}_{\underline{v}}$ is as in Examples 3.2, (v); 3.3, (i)]; (b) if $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}$ is a triple of data, consisting of a Frobenioid ${}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}$, an object of $\mathbb{TM}^{\vdash}$, and a splitting of the Frobenioid, such that there exists an isomorphism of collections of data ${}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}} \overset{\sim}{\to} \mathcal{F}^{\vdash}_{\underline{v}}$ [where $\mathcal{F}^{\vdash}_{\underline{v}}$ is as in Example 3.4, (ii)].

(iii) A *morphism of $\mathcal{F}$- (respectively, $\mathcal{F}^{\vdash}$-) prime-strips* is defined to be a collection of isomorphisms, indexed by $\underline{\mathbb{V}}$, between the various constituent objects of the prime-strips. Following the conventions of §0, one thus has notions of *capsules of $\mathcal{F}$- (respectively, $\mathcal{F}^{\vdash}$-)* and *morphisms of capsules of $\mathcal{F}$- (respectively, $\mathcal{F}^{\vdash}$-) prime-strips.*

(iv) We define a *globally realified mono-analytic Frobenioid-prime-strip*, or *$\mathcal{F}^{\Vdash}$-prime-strip*, [relative to the given initial Θ-data] to be a collection of data

$$ {}^{\ddagger}\mathfrak{F}^{\Vdash} \;=\; ({}^{\ddagger}\mathcal{C}^{\Vdash},\ \mathrm{Prime}({}^{\ddagger}\mathcal{C}^{\Vdash}) \overset{\sim}{\to} \underline{\mathbb{V}},\ {}^{\ddagger}\mathfrak{F}^{\vdash},\ \{{}^{\ddagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$

that satisfies the following conditions: (a) ${}^{\ddagger}\mathcal{C}^{\Vdash}$ is a category [which is, in fact, equipped with a Frobenioid structure] that is isomorphic to the category $\mathcal{C}^{\Vdash}_{\mathrm{mod}}$ of Example 3.5, (i); (b) "Prime(−)" is defined as in the discussion of Example 3.5, (i); (c) $\mathrm{Prime}({}^{\ddagger}\mathcal{C}^{\Vdash}) \overset{\sim}{\to} \underline{\mathbb{V}}$ is a bijection of sets; (d) ${}^{\ddagger}\mathfrak{F}^{\vdash} = \{{}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ is an $\mathcal{F}^{\vdash}$-prime-strip; (e) ${}^{\ddagger}\rho_{\underline{v}} : \Phi_{{}^{\ddagger}\mathcal{C}^{\Vdash},\underline{v}} \overset{\sim}{\to} \Phi^{\mathrm{rlf}}_{{}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}$, where "$\Phi_{{}^{\ddagger}\mathcal{C}^{\Vdash},\underline{v}}$" and "$\Phi^{\mathrm{rlf}}_{{}^{\ddagger}\mathcal{C}^{\vdash}_{\underline{v}}}$" are defined as in the discussion of Example 3.5, (i), is an isomorphism of topological monoids [both of which are, in fact, isomorphic to $\mathbb{R}_{\geq 0}$]; (f) the collection of data in the above display is *isomorphic to the collection of data* $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ of Example 3.5, (ii). A *morphism of* $\mathcal{F}^{\Vdash}$*-prime-strips* is defined to be an isomorphism between collections of data as discussed above. Following the conventions of §0, one thus has notions of *capsules of* $\mathcal{F}^{\Vdash}$*-prime-strips* and *morphisms of capsules of* $\mathcal{F}^{\Vdash}$*-prime-strips*.

Remark 5.2.1.

(i) Note that it follows immediately from Definitions 4.1, (i), (iii); 5.2, (i), (ii); Examples 3.2, (vi), (c), (d); 3.3, (iii), (b), (c), that there exists a *functorial algorithm* for constructing $\mathcal{D}$- (respectively, $\mathcal{D}^{\vdash}$-) *prime-strips* from $\mathcal{F}$- (respectively, $\mathcal{F}^{\vdash}$-) *prime-strips*.

(ii) In a similar vein, it follows immediately from Definition 5.2, (i), (ii); Examples 3.2, (vi), (f); 3.3, (iii), (e); 3.4, (i), (ii), that there exists a *functorial algorithm* for constructing from an $\mathcal{F}$*-prime-strip* ${}^{\ddagger}\mathfrak{F} = \{{}^{\ddagger}\mathcal{F}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ an $\mathcal{F}^{\vdash}$*-prime-strip* ${}^{\ddagger}\mathfrak{F}^{\vdash}$

$$ {}^{\ddagger}\mathfrak{F} \mapsto {}^{\ddagger}\mathfrak{F}^{\vdash} = \{{}^{\ddagger}\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} $$

— which we shall we shall refer to as the *mono-analyticization* of ${}^{\ddagger}\mathfrak{F}$. Moreover, one may also construct from the $\mathcal{F}$*-prime-strip* ${}^{\ddagger}\mathfrak{F}$, via a *functorial algorithm* [cf. the constructions of Example 3.5, (i), (ii)], a collection of data

$$ {}^{\ddagger}\mathfrak{F} \mapsto {}^{\ddagger}\mathfrak{F}^{\Vdash} \stackrel{\mathrm{def}}{=} ({}^{\ddagger}\mathcal{C}^{\Vdash},\ \mathrm{Prime}({}^{\ddagger}\mathcal{C}^{\Vdash}) \stackrel{\sim}{\to} \underline{\mathbb{V}},\ {}^{\ddagger}\mathfrak{F}^{\vdash},\ \{{}^{\ddagger}\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}) $$

— i.e., consisting of a category [which is, in fact, equipped with a Frobenioid structure], a bijection, the $\mathcal{F}^{\vdash}$-prime-strip ${}^{\ddagger}\mathfrak{F}^{\vdash}$, and an isomorphism of topological monoids associated to ${}^{\ddagger}\mathcal{C}^{\Vdash}$ and ${}^{\ddagger}\mathfrak{F}^{\vdash}$, respectively, at each $\underline{v} \in \underline{\mathbb{V}}$ — which is isomorphic to the collection of data $\mathfrak{F}^{\Vdash}_{\mathrm{mod}}$ of Example 3.5, (ii), i.e., which forms an $\mathcal{F}^{\Vdash}$*-prime-strip* [cf. Definition 5.2, (iv)].

Remark 5.2.2. Thus, from the point of view of the discussion of Remark 5.1.3, $\mathcal{F}$*-prime-strips* are *Kummer-ready* [i.e., at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$ — cf. the theory of [FrdII], §2], whereas $\mathcal{F}^{\vdash}$*-prime-strips* are *Kummer-blind*.

Corollary 5.3. (Isomorphisms of Global Frobenioids, Frobenioid-Prime-Strips, and Tempered Frobenioids) *Relative to a fixed collection of* **initial Θ-data***:*

(i) For $i = 1, 2$*, let* ${}^{i}\mathcal{F}^{\circledast}$ *(respectively,* ${}^{i}\mathcal{F}^{\circledcirc}$*) be a* **category** *which is equivalent to the category* ${}^{\dagger}\mathcal{F}^{\circledast}$ *(respectively,* ${}^{\dagger}\mathcal{F}^{\circledcirc}$*) of Example 5.1, (iii). Thus,* ${}^{i}\mathcal{F}^{\circledast}$ *(respectively,* ${}^{i}\mathcal{F}^{\circledcirc}$*) is equipped with a* **natural Frobenioid structure** *[cf. [FrdI], Corollary 4.11; [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper]. Write* $\mathrm{Base}({}^{i}\mathcal{F}^{\circledast})$ *(respectively,* $\mathrm{Base}({}^{i}\mathcal{F}^{\circledcirc})$*) for the base category of this Frobenioid. Then the natural map*

$$ \mathrm{Isom}({}^{1}\mathcal{F}^{\circledast}, {}^{2}\mathcal{F}^{\circledast}) \to \mathrm{Isom}(\mathrm{Base}({}^{1}\mathcal{F}^{\circledast}), \mathrm{Base}({}^{2}\mathcal{F}^{\circledast})) $$

$$ \textit{(respectively, } \mathrm{Isom}({}^{1}\mathcal{F}^{\circledcirc}, {}^{2}\mathcal{F}^{\circledcirc}) \to \mathrm{Isom}(\mathrm{Base}({}^{1}\mathcal{F}^{\circledcirc}), \mathrm{Base}({}^{2}\mathcal{F}^{\circledcirc}))\textit{)} $$

[cf. [FrdI], Corollary 4.11; [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper] is **bijective***.*

*(ii) For $i = 1, 2$, let ${}^i\mathfrak{F}$ be an $\mathcal{F}$-***prime-strip***; ${}^i\mathfrak{D}$ the $\mathcal{D}$-***prime-strip*** associated to ${}^i\mathfrak{F}$ [cf. Remark 5.2.1, (i)]. Then the natural map*

$$\mathrm{Isom}({}^1\mathfrak{F}, {}^2\mathfrak{F}) \to \mathrm{Isom}({}^1\mathfrak{D}, {}^2\mathfrak{D})$$

[cf. Remark 5.2.1, (i)] is **bijective***.*

*(iii) For $i = 1, 2$, let ${}^i\mathfrak{F}^{\vdash}$ be an $\mathcal{F}^{\vdash}$-***prime-strip***; ${}^i\mathfrak{D}^{\vdash}$ the $\mathcal{D}^{\vdash}$-***prime-strip*** associated to ${}^i\mathfrak{F}^{\vdash}$ [cf. Remark 5.2.1, (i)]. Then the natural map*

$$\mathrm{Isom}({}^1\mathfrak{F}^{\vdash}, {}^2\mathfrak{F}^{\vdash}) \to \mathrm{Isom}({}^1\mathfrak{D}^{\vdash}, {}^2\mathfrak{D}^{\vdash})$$

[cf. Remark 5.2.1, (i)] is **surjective***.*

(iv) Let $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Recall the category $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ of Example 3.2, (i). Thus, $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is equipped with a **natural Frobenioid structure** *[cf. [FrdI], Corollary 4.11; [EtTh], Proposition 5.1], with base category $\mathcal{D}_{\underline{v}}$. Then the natural homomorphism $\mathrm{Aut}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \to \mathrm{Aut}(\mathcal{D}_{\underline{v}})$ [cf. Example 3.2, (vi), (d)] is* **bijective***.*

Proof. Assertion (i) follows immediately from the *category-theoreticity* of the *"natural injection $\widetilde{\mathcal{O}}^{\circledcirc\times} \hookrightarrow \varinjlim_H H^1(H, \boldsymbol{\mu}^{\Theta}_{\widehat{\mathbb{Z}}}(\pi_1({}^{\dagger}\mathcal{D}^{\circledast})))$"* of Example 5.1, (v) [cf. also the surrounding discussion; Example 5.1, (vi)]. [Here, we note in passing that this argument is entirely similar to the technique applied to the proof of the equivalence "$\mathfrak{Th}^{\circledcirc}_{\mathbb{T}} \xrightarrow{\sim} \mathbb{EA}^{\circledcirc}$" of [AbsTopIII], Corollary 5.2, (iv).] Assertion (ii) (respectively, (iii)) follows immediately from [AbsTopIII], Proposition 3.2, (iv); [AbsTopIII], Proposition 4.2, (i) [cf. also [AbsTopIII], Remarks 3.1.1, 4.1.1] (respectively, [AbsTopIII], Proposition 5.8, (ii), (v)).

Finally, we consider assertion (iv). First, we recall that since automorphisms of $\mathcal{D}_{\underline{v}} = \mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})^0$ necessarily arise from *automorphisms of the scheme* $\underline{\underline{X}}_{\underline{v}}$ [cf., [AbsTopIII], Theorem 1.9; [AbsTopIII], Remark 1.9.1], *surjectivity* follows immediately from the construction of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. Thus, it remains to verify *injectivity.* To this end, let $\alpha \in \mathrm{Ker}(\mathrm{Aut}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) \to \mathrm{Aut}(\mathcal{D}_{\underline{v}}))$. For simplicity, we suppose [without loss of generality] that α lies over the identity self-equivalence of $\mathcal{D}_{\underline{v}}$. Then I *claim* that to show that α is [isomorphic to — cf. §0] the *identity self-equivalence* of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, it suffices to verify that

> α induces [cf. [FrdI], Corollary 4.11; [EtTh], Proposition 5.1] the *identity* on the *rational function* and *divisor monoids* of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$.

Indeed, recall that since $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ is a *Frobenioid of model type* [cf. [EtTh], Definition 3.6, (ii)], it follows [cf. Remark 5.3.3 below] from [FrdI], Corollary 5.7, (i), (iv), that α preserves *base-Frobenius pairs.* Thus, once one shows that α induces the *identity* on the *rational function* and *divisor monoids* of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, it follows, by arguing as in the construction of the equivalence of categories given in the proof of [FrdI], Theorem 5.2, (iv), that the various *units* obtained in [FrdI], Proposition 5.6, determine [cf. the argument of the first paragraph of the proof of [FrdI], Proposition 5.6] an *isomorphism* between α and the *identity self-equivalence* of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, as desired.

Thus, we proceed to show that α induces the *identity* on the *rational function* and *divisor monoids* of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$, as follows. In light of the *category-theoreticity* [cf. [EtTh], Theorem 5.6] of the *cyclotomic rigidity isomorphism* of [EtTh], Proposition 5.5, the fact that α induces the *identity* on the rational function monoid follows immediately from the *naturality of the Kummer map* [cf. the discussion of Remark 3.2.4; [FrdII], Definition 2.1, (ii)], which is *injective* by [EtTh], Proposition 3.2, (iii) — cf. the argument of [EtTh], Theorem 5.7, applied to verify the category-theoreticity of the Frobenioid-theoretic theta function. Next, we consider the effect of α on the divisor monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. To this end, let us first recall that α preserves *cuspidal* and *non-cuspidal* elements of the monoids that appear in this divisor monoid [cf. [EtTh], Proposition 5.3, (i)]. In particular, by considering the non-cuspidal portion of the divisor of the Frobenioid-theoretic theta function and its conjugates [each of which is preserved by α, since α has already been shown to induce the identity on the rational function monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$], we conclude that α induces the *identity* on the *non-cuspidal* elements of the monoids that appear in the divisor monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ [cf. [EtTh], Proposition 5.3, (v), (vi), for a discussion of closely related facts]. In a similar vein, since any divisor of degree zero on an elliptic curve that is supported on the *torsion points* of the elliptic curve admits a positive multiple which is *principal*, it follows by considering the cuspidal portions of divisors of appropriate rational functions [each of which is preserved by α, since α has already been shown to induce the identity on the rational function monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$] that α also induces the *identity* on the *cuspidal* elements of the monoids that appear in the divisor monoid of $\underline{\underline{\mathcal{F}}}_{\underline{v}}$. This completes the proof of assertion (iv). $\bigcirc$

Remark 5.3.1.

(i) In the situation of Corollary 5.3, (ii), let

$$\phi : {}^1\mathfrak{D} \to {}^2\mathfrak{D}$$

be a *morphism of $\mathcal{D}$-prime-strips* [i.e., which is not necessarily an isomorphism!] that induces an *isomorphism* between the respective collections of data indexed by $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, as well as an *isomorphism* $\phi^{\vdash} : {}^1\mathfrak{D}^{\vdash} \xrightarrow{\sim} {}^2\mathfrak{D}^{\vdash}$ between the associated $\mathcal{D}^{\vdash}$*-prime-strips* [cf. Definition 4.1, (iv)]. Then let us *observe* that by applying Corollary 5.3, (ii), it follows that ϕ *lifts* to a *uniquely determined "arrow"*

$$\psi : {}^1\mathfrak{F} \to {}^2\mathfrak{F}$$

— which we think of as *"lying over"* ϕ — defined as follows: First, let us recall that, in light of our assumptions on ϕ, it follows immediately from the construction [cf. Examples 3.2, (iii); 3.3, (i); 3.4, (i)] of the various *p-adic* and *archimedean Frobenioids* [cf. [FrdII], Example 1.1, (ii); [FrdII], Example 3.3, (ii)] that appear in an $\mathcal{F}$-prime-strip that it makes sense to speak of the *"pull-back"* — i.e., by forming the *"categorical fiber product"* [cf. [FrdI], §0; [FrdI], Proposition 1.6] — of the Frobenioids that appear in the $\mathcal{F}$-prime-strip ${}^2\mathfrak{F}$ via the various morphisms at $\underline{v} \in \underline{\mathbb{V}}$ that constitute ϕ. That is to say, it follows from our assumptions on ϕ [cf. also [AbsTopIII], Proposition 3.2, (iv)] that ϕ determines a *pulled-back $\mathcal{F}$-prime-strip* "$\phi^*({}^2\mathfrak{F})$", whose associated $\mathcal{D}$-prime-strip [cf. Remark 5.2.1, (i)] is *tautologically equal* to ${}^1\mathfrak{D}$. On the other hand, by Corollary 5.3, (ii), it follows that this tautological equality of associated $\mathcal{D}$-prime-strips *uniquely determines* an isomorphism ${}^1\mathfrak{F} \xrightarrow{\sim} \phi^*({}^2\mathfrak{F})$. Then we *define* the arrow $\psi : {}^1\mathfrak{F} \to {}^2\mathfrak{F}$ to be this isomorphism ${}^1\mathfrak{F} \xrightarrow{\sim} \phi^*({}^2\mathfrak{F})$ and refer to ψ as the *"morphism uniquely determined by ϕ"* or the *"uniquely determined morphism that lies over ϕ"*. Also, we shall apply various terms used to *describe a morphism ϕ of $\mathcal{D}$-prime-strips* to the "arrow" of $\mathcal{F}$-prime-strips determined by ϕ.

(ii) The conventions discussed in (i) concerning liftings of morphisms of $\mathcal{D}$-prime-strips may also be applied to *poly-morphisms*. We leave the routine details to the reader.

Remark 5.3.2. Just as in the case of Corollary 5.3, (i), (ii), the *rigidity property* of Corollary 5.3, (iv), may be regarded as being essentially a consequence of *"Kummer-readiness"* [cf. Remarks 5.1.3, 5.2.2] of the tempered Frobenioid $\underline{\underline{\mathcal{F}}}_{\underline{v}}$ — cf. also the arguments applied in the proofs of [AbsTopIII], Proposition 3.2, (iv); [AbsTopIII], Corollary 5.2, (iv).

Remark 5.3.3. We take this opportunity to rectify a minor oversight in [FrdI]. The hypothesis that the Frobenioids under consideration be of *"unit-profinite type"* in [FrdI], Proposition 5.6 — hence also in [FrdI], Corollary 5.7, (iii) — may be *removed*. Indeed, if, in the notation of the proof of [FrdI], Proposition 5.6, one writes $\phi'_p = c_p \cdot \phi_p$, where $c_p \in \mathcal{O}^\times(A)$, for $p \in \mathfrak{Primes}$, then one has

$$\begin{aligned} c_2 \cdot c_p^2 \cdot \phi_2 \cdot \phi_p &= c_2 \cdot \phi_2 \cdot c_p \cdot \phi_p = \phi'_2 \cdot \phi'_p = \phi'_p \cdot \phi'_2 \\ &= c_p \cdot \phi_p \cdot c_2 \cdot \phi_2 = c_p \cdot c_2^p \cdot \phi_p \cdot \phi_2 = c_p \cdot c_2^p \cdot \phi_2 \cdot \phi_p \end{aligned}$$

— so $c_2 \cdot c_p^2 = c_p \cdot c_2^p$, i.e., $c_p = c_2^{p-1}$, for $p \in \mathfrak{Primes}$. Thus, $\phi'_p = c_2^{-1} \cdot \phi_p \cdot c_2$, so by taking $u \stackrel{\mathrm{def}}{=} c_2^{-1}$, one may *eliminate the final two paragraphs* of the proof of [FrdI], Proposition 5.6.

Let

$${}^\dagger\mathcal{HT}^\Theta = (\{{}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}},\ {}^\dagger\mathfrak{F}^{\Vdash}_{\mathrm{mod}})$$

be a Θ-*Hodge theater* [relative to the given initial Θ-data] such that the associated $\mathcal{D}$-*prime-strip* $\{{}^\dagger\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ is [for simplicity] *equal* to the $\mathcal{D}$-prime-strip ${}^\dagger\mathfrak{D}_>$ of the $\mathcal{D}$-ΘNF-*Hodge theater* in the discussion preceding Example 5.1. Write

$${}^\dagger\mathfrak{F}_>$$

for the $\mathcal{F}$-*prime-strip* tautologically associated to this Θ-Hodge theater [cf. the data "$\{{}^\dagger\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$" of Definition 3.6; Definition 5.2, (i)]. Thus, ${}^\dagger\mathfrak{D}_>$ may be identified with the $\mathcal{D}$-prime-strip associated [cf. Remark 5.2.1, (i)] to ${}^\dagger\mathfrak{F}_>$.

Example 5.4. Model Θ- and NF-Bridges.

(i) For $j \in J$, let

$$^{\dagger}\mathfrak{F}_j = \{^{\dagger}\mathcal{F}_{\underline{v}_j}\}_{\underline{v}_j \in \underline{\mathbb{V}}_j}$$

be an *$\mathcal{F}$-prime-strip* whose associated $\mathcal{D}$-prime-strip [cf. Remark 5.2.1, (i)] is equal to $^{\dagger}\mathfrak{D}_j$,

$$^{\dagger}\mathfrak{F}_{\langle J \rangle} = \{^{\dagger}\mathcal{F}_{\underline{v}_{\langle J \rangle}}\}_{\underline{v}_{\langle J \rangle} \in \underline{\mathbb{V}}_{\langle J \rangle}}$$

an *$\mathcal{F}$-prime-strip* whose associated $\mathcal{D}$-prime-strip we denote by $^{\dagger}\mathfrak{D}_{\langle J \rangle}$. Write

$$^{\dagger}\mathfrak{F}_J \stackrel{\text{def}}{=} \prod_{j \in J} {^{\dagger}\mathfrak{F}_j}$$

— where the "formal product $\prod$" is to be understood as denoting the capsule with index set J for which the datum indexed by j is given by $^{\dagger}\mathfrak{F}_j$. Thus, $^{\dagger}\mathfrak{F}_{\langle J \rangle}$ may be related to $^{\dagger}\mathfrak{F}_{>}$, in a *natural fashion*, via the *full poly-isomorphism*

$$^{\dagger}\mathfrak{F}_{\langle J \rangle} \stackrel{\sim}{\rightarrow} {^{\dagger}\mathfrak{F}_{>}}$$

and to $^{\dagger}\mathfrak{F}_J$ via the *"diagonal arrow"*

$$^{\dagger}\mathfrak{F}_{\langle J \rangle} \rightarrow {^{\dagger}\mathfrak{F}_J} = \prod_{j \in J} {^{\dagger}\mathfrak{F}_j}$$

— i.e., the arrow defined as the collection of data indexed by J for which the datum indexed by j is given by the *full poly-isomorphism* $^{\dagger}\mathfrak{F}_{\langle J \rangle} \stackrel{\sim}{\rightarrow} {^{\dagger}\mathfrak{F}_j}$. Thus, we think of $^{\dagger}\mathfrak{F}_j$ as a copy of $^{\dagger}\mathfrak{F}_{>}$ *"situated on"* the constituent labeled j of the capsule $^{\dagger}\mathfrak{D}_J$; we think of $^{\dagger}\mathfrak{F}_{\langle J \rangle}$ as a copy of $^{\dagger}\mathfrak{F}_{>}$ *"situated in a diagonal fashion on"* all the constituents of the capsule $^{\dagger}\mathfrak{D}_J$.

(ii) Note that in addition to thinking of $^{\dagger}\mathfrak{F}_{>}$ as being related to $^{\dagger}\mathfrak{F}_j$ [for $j \in J$] via the *full poly-isomorphism* $^{\dagger}\mathfrak{F}_{>} \stackrel{\sim}{\rightarrow} {^{\dagger}\mathfrak{F}_j}$, we may also regard $^{\dagger}\mathfrak{F}_{>}$ as being related to $^{\dagger}\mathfrak{F}_j$ [for $j \in J$] via the *poly-morphism*

$$^{\dagger}\psi^{\Theta}_j : {^{\dagger}\mathfrak{F}_j} \rightarrow {^{\dagger}\mathfrak{F}_{>}}$$

uniquely determined by $^{\dagger}\phi^{\Theta}_j$ [cf. Remark 5.3.1]. Write

$$^{\dagger}\psi^{\Theta}_{\divideontimes} : {^{\dagger}\mathfrak{F}_J} \rightarrow {^{\dagger}\mathfrak{F}_{>}}$$

for the *collection of arrows* $\{^{\dagger}\psi^{\Theta}_j\}_{j \in J}$ — which we think of as *"lying over"* the collection of arrows $^{\dagger}\phi^{\Theta}_{\divideontimes} = \{^{\dagger}\phi^{\Theta}_j\}_{j \in J}$.

(iii) Next, let $^{\dagger}\mathcal{F}^{\circledast}$, $^{\dagger}\mathcal{F}^{\circledcirc}$ be as in Example 5.1, (iii); $\delta \in \text{LabCusp}(^{\dagger}\mathcal{D}^{\circledcirc})$. Thus, [cf. the discussion of Example 4.3, (i)] there exists a *unique* $\text{Aut}_{\underline{\epsilon}}(^{\dagger}\mathcal{D}^{\circledcirc})$-orbit of isomorphisms $^{\dagger}\mathcal{D}^{\circledcirc} \stackrel{\sim}{\rightarrow} \mathcal{D}^{\circledcirc}$ that maps $\delta \mapsto [\underline{\epsilon}] \in \text{LabCusp}(\mathcal{D}^{\circledcirc})$. We shall refer to as a *δ-valuation* $\in \mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc})$ [cf. Definition 4.1, (v)] any element that maps to an element of $\underline{\mathbb{V}}^{\pm \text{un}}$ [cf. Example 4.3, (i)] via this $\text{Aut}_{\underline{\epsilon}}(^{\dagger}\mathcal{D}^{\circledcirc})$-orbit of isomorphisms. Note that the notion of a δ-valuation may also be defined *intrinsically* by means of the structure of $\mathcal{D}$-NF-bridge $^{\dagger}\phi_{\circledast}^{\mathrm{NF}}$. Indeed, [one verifies immediately that] a *δ-valuation* may be defined as a valuation $\in \mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc})$ that lies in the "image" [in the evident sense] via $^{\dagger}\phi_{\circledast}^{\mathrm{NF}}$ of a $\mathcal{D}$-prime-strip $^{\dagger}\mathfrak{D}_j$ of the capsule $^{\dagger}\mathfrak{D}_J$ such that the map $\mathrm{LabCusp}(^{\dagger}\mathcal{D}^{\circledcirc}) \to \mathrm{LabCusp}(^{\dagger}\mathfrak{D}_j)$ induced by $^{\dagger}\phi_{\circledast}^{\mathrm{NF}}$ maps δ to the element of $\mathrm{LabCusp}(^{\dagger}\mathfrak{D}_j)$ that is *"labeled 1"*, relative to the bijection of the second display of Proposition 4.2.

(iv) We continue to use the notation of (iii). Then let us observe that by *localizing* at the various δ-valuations $\in \mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc})$, one may construct, in a natural way, an *$\mathcal{F}$-prime-strip*

$$^{\dagger}\mathcal{F}^{\circledcirc}|_{\delta}$$

— which is well-defined *up to isomorphism* — from $^{\dagger}\mathcal{F}^{\circledcirc}$ [i.e., in the notation of Example 5.1, (iv), from $\widetilde{\mathcal{O}}^{\circledcirc\times}$, equipped with its natural $\pi_1(^{\dagger}\mathcal{D}^{\circledcirc})$-action]. Indeed, at a *nonarchimedean* δ-valuation $\underline{v}$, this follows by considering the *$p_{\underline{v}}$-adic Frobenioids* [cf. Remark 3.3.2] associated to the restrictions to $\Pi_{\mathfrak{p}_0} \bigcap \pi_1(^{\dagger}\mathcal{D}^{\circledcirc})$ $(\subseteq \pi_1(^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \pi_1(^{\dagger}\mathcal{D}^{\circledast}))$ of the *pairs*

$$\text{“}\Pi_{\mathfrak{p}_0} \quad \curvearrowright \quad \widetilde{\mathcal{O}}^{\triangleright}_{\widehat{\mathfrak{p}}_0}\text{”}$$

of Example 5.1, (iv) [cf. also Example 5.1, (vi)]. On the other hand, at an *archimedean* δ-valuation $\underline{v}$, this follows by applying the *functorial algorithm* for reconstructing the Aut-*holomorphic orbispace at* $\underline{v}$ given in [AbsTopIII], Corollaries 2.8, 2.9, together with the discussion concerning the *"natural injection $\widetilde{\mathcal{O}}^{\circledcirc\times} \hookrightarrow \varinjlim_H \; H^1(H, \boldsymbol{\mu}_{\widehat{\mathbb{Z}}}^{\Theta}(\pi_1(^{\dagger}\mathcal{D}^{\circledast})))$"* in Example 5.1, (v) [cf. also Example 5.1, (vi)]. Here, we observe that since the natural projection map $\underline{\mathbb{V}}^{\pm\mathrm{un}} \to \mathbb{V}_{\mathrm{mod}}$ *fails to be injective*, in order to relate the restrictions obtained at different elements in a fiber of this map in a well-defined fashion, it is necessary to regard $^{\dagger}\mathcal{F}^{\circledcirc}|_{\delta}$ as being well-defined only up to isomorphism. Nevertheless, despite this indeterminacy inherent in the definition of $^{\dagger}\mathcal{F}^{\circledcirc}|_{\delta}$, it still makes sense to define, for an $\mathcal{F}$-prime-strip $^{\ddagger}\mathfrak{F}$ with underlying $\mathcal{D}$-prime-strip $^{\ddagger}\mathfrak{D}$ [cf. Remark 5.2.1, (i)], a *poly-morphism*

$$^{\ddagger}\mathfrak{F} \to {^{\dagger}\mathcal{F}^{\circledcirc}}$$

to be a *full poly-isomorphism* $^{\ddagger}\mathfrak{F} \xrightarrow{\sim} {^{\dagger}\mathcal{F}^{\circledcirc}}|_{\delta}$ for some $\delta \in \mathrm{LabCusp}(^{\dagger}\mathcal{D}^{\circledcirc})$ [cf. Definition 4.1, (vi)]. Moreover, it makes sense to *pre-compose* such poly-morphisms with isomorphisms of $\mathcal{F}$-prime-strips and to *post-compose* such poly-morphisms with isomorphisms between isomorphs of $^{\dagger}\mathcal{F}^{\circledcirc}$. Here, we note that such a poly-morphism $^{\ddagger}\mathfrak{F} \to {^{\dagger}\mathcal{F}^{\circledcirc}}$ may be thought of as "lying over" an induced poly-morphism $^{\ddagger}\mathfrak{D} \to {^{\dagger}\mathcal{D}^{\circledcirc}}$ [cf. Definition 4.1, (vi)]. Also, we observe that any poly-morphism $^{\ddagger}\mathfrak{F} \to {^{\dagger}\mathcal{F}^{\circledcirc}}$ is *fixed* by pre-composition with automorphisms of $^{\ddagger}\mathfrak{F}$, as well as by post-compostion with automorphisms $\in \mathrm{Aut}_{\underline{\epsilon}}(^{\dagger}\mathcal{F}^{\circledcirc})$. In a similar vein, if $\{^{e}\mathfrak{F}\}_{e\in E}$ is a *capsule of $\mathcal{F}$-prime-strips* whose associated capsule of $\mathcal{D}$-prime-strips [cf. Remark 5.2.1, (i)] we denote by $\{^{e}\mathfrak{D}\}_{e\in E}$, then we define a *poly-morphism*

$$\{^{e}\mathfrak{F}\}_{e\in E} \to {^{\dagger}\mathcal{F}^{\circledcirc}} \text{ (respectively, } \{^{e}\mathfrak{F}\}_{e\in E} \to {^{\dagger}\mathfrak{F}})$$

to be a collection of poly-morphisms $\{^{e}\mathfrak{F} \to {^{\dagger}\mathcal{F}^{\circledcirc}}\}_{e\in E}$ (respectively, $\{^{e}\mathfrak{F} \to {^{\dagger}\mathfrak{F}}\}_{e\in E}$) [cf. Definition 4.1, (vi)]. Thus, a poly-morphism $\{^{e}\mathfrak{F}\}_{e\in E} \to {^{\dagger}\mathcal{F}^{\circledcirc}}$ (respectively, $\{^{e}\mathfrak{F}\}_{e\in E} \to {^{\dagger}\mathfrak{F}}$) may be thought of as "lying over" an induced poly-morphism $\{^{e}\mathfrak{D}\}_{e\in E} \to {^{\dagger}\mathcal{D}^{\circledcirc}}$ (respectively, $\{^{e}\mathfrak{D}\}_{e\in E} \to {^{\dagger}\mathfrak{D}}$) [cf. Definition 4.1, (vi)].

(v) We continue to use the notation of (iv). Now observe that by Corollary 5.3, (i), (ii), there exists a *unique* poly-morphism

$$^{\dagger}\psi^{\mathrm{NF}}_{\circledast} : {}^{\dagger}\mathfrak{F}_J \to {}^{\dagger}\mathcal{F}^{\circledcirc}$$

that lies over $^{\dagger}\phi^{\mathrm{NF}}_{\circledast}$.

(vi) We continue to use the notation of (v). Now observe that it follows from the definition of $^{\dagger}\mathcal{F}^{\circledcirc}_{\mathrm{mod}}$ in terms of *terminal objects* of $^{\dagger}\mathcal{D}^{\circledast}$ [cf. Example 5.1, (iii)] that any poly-morphism $^{\dagger}\mathfrak{F}_{\langle J\rangle} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$ [cf. the notation of (i)] induces, via "restriction" [in the evident sense], an *isomorphism class of functors* [cf. the notation of Example 5.1, (vii)]

$$({}^{\dagger}\mathcal{F}^{\circledcirc} \subseteq {}^{\dagger}\mathcal{F}^{\circledast} \supseteq) \quad {}^{\dagger}\mathcal{F}^{\circledcirc}_{\mathrm{mod}} \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledcirc}_{\langle J\rangle} \;\to\; {}^{\dagger}\mathcal{F}_{\underline{v}_{\langle J\rangle}}$$

for each $\underline{v}_{\langle J\rangle} \in \underline{\mathbb{V}}_{\langle J\rangle}$ which is *independent* of the choice of the poly-morphism $^{\dagger}\mathfrak{F}_{\langle J\rangle} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$ [i.e., among its $\mathbb{F}^{\circledast}_l$*-conjugates*]. That is to say, the fact that $^{\dagger}\mathcal{F}^{\circledcirc}_{\mathrm{mod}}$ is defined in terms of *terminal objects* of $^{\dagger}\mathcal{D}^{\circledast}$ [cf. also the definition of F_{mod} given in Definition 3.1, (b)!] implies that this particular isomorphism class of functors is *immune to* [i.e., fixed by] the various *indeterminacies* that appear in the definition of the poly-morphism $^{\dagger}\mathfrak{F}_{\langle J\rangle} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$, as well as to the choice of $^{\dagger}\mathfrak{F}_{\langle J\rangle} \to {}^{\dagger}\mathcal{F}^{\circledcirc}$. Let us write

$$({}^{\dagger}\mathcal{F}^{\circledcirc} \subseteq {}^{\dagger}\mathcal{F}^{\circledast} \supseteq) \quad {}^{\dagger}\mathcal{F}^{\circledcirc}_{\mathrm{mod}} \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledcirc}_{\langle J\rangle} \;\to\; {}^{\dagger}\mathfrak{F}_{\langle J\rangle}$$

for the collection of isomorphism classes of restriction functors just defined, as $\underline{v}_{\langle J\rangle}$ ranges over the elements of $\underline{\mathbb{V}}_{\langle J\rangle}$. In a similar vein, we also obtain collections of *natural isomorphism classes of restriction functors*

$$^{\dagger}\mathcal{F}^{\circledcirc}_{J} \;\to\; {}^{\dagger}\mathfrak{F}_J; \quad {}^{\dagger}\mathcal{F}^{\circledcirc}_{j} \;\to\; {}^{\dagger}\mathfrak{F}_j$$

for $j \in J$. Finally, just as in Example 5.1, (vii), we obtain natural *realifications*

$$^{\dagger}\mathcal{F}^{\circledcirc\mathbb{R}}_{\langle J\rangle} \;\to\; {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{\langle J\rangle}; \quad {}^{\dagger}\mathcal{F}^{\circledcirc\mathbb{R}}_{J} \;\to\; {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{J}; \quad {}^{\dagger}\mathcal{F}^{\circledcirc\mathbb{R}}_{j} \;\to\; {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{j}$$

of the various $\mathcal{F}$-prime-strips and isomorphism classes of restriction functors discussed so far.

(vii) We shall refer to as *"pivotal distributions"* the objects constructed in (vi)

$$^{\dagger}\mathcal{F}^{\circledcirc}_{\mathrm{pvt}} \;\to\; {}^{\dagger}\mathfrak{F}_{\mathrm{pvt}}; \quad {}^{\dagger}\mathcal{F}^{\circledcirc\mathbb{R}}_{\mathrm{pvt}} \;\to\; {}^{\dagger}\mathfrak{F}^{\mathbb{R}}_{\mathrm{pvt}}$$

in the case $j = 1$ — cf. Fig. 5.2.

Remark 5.4.1. The constructions of Example 5.4, (i), (ii) (respectively, Example 5.4, (iii), (iv), (v), (vi)) manifestly only require the $\mathcal{D}$-Θ-*bridge* portion $^{\dagger}\phi^{\Theta}_{\circledast}$ (respectively, $\mathcal{D}$-*NF-bridge* portion $^{\dagger}\phi^{\mathrm{NF}}_{\circledast}$) of the $\mathcal{D}$-ΘNF-Hodge theater $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ [cf. Remark 5.1.2].

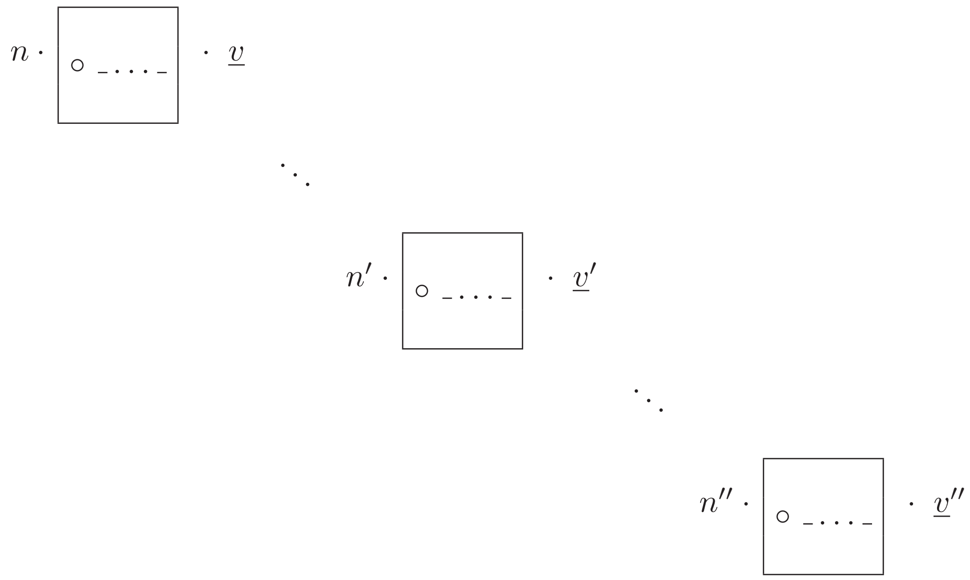

Fig. 5.2: Pivotal distribution

Remark 5.4.2.

(i) At this point, it is useful to consider the various copies of ${}^{\dagger}\mathcal{F}^{\circledcirc}_{\mathrm{mod}}$ and its realifications introduced so far from the point of view of **"log-volumes"**, i.e., **arithmetic degrees** [cf., e.g., the discussion of [FrdI], Example 6.3; [FrdI], Theorem 6.4; Remark 3.1.5 of the present paper]. That is to say, since ${}^{\dagger}\mathcal{F}^{\circledcirc}_{j}$ may be thought of as a sort of *"section of* ${}^{\dagger}\mathcal{F}^{\circledcirc}_{J}$ *over* ${}^{\dagger}\mathcal{F}^{\circledcirc}_{\mathrm{mod}}$*"* — i.e., a sort of *"section of* K *over* F_{mod}*"* [cf. the discussion of *prime-strips* in Remark 4.3.1] — one way to think of log-volumes of ${}^{\dagger}\mathcal{F}^{\circledcirc}_{\langle J \rangle}$ is as quantities that *differ by a factor of* l^{*} — i.e., corresponding, to the cardinality of $J \xrightarrow{\sim} \mathbb{F}^{*}_{l}$ — from log-volumes of ${}^{\dagger}\mathcal{F}^{\circledcirc}_{j}$. Put another way, this amounts to thinking of arithmetic degrees that appear in the various factors of ${}^{\dagger}\mathcal{F}^{\circledcirc}_{J}$ as being

> **averaged** *over the elements of* J and hence of arithmetic degrees that appear in ${}^{\dagger}\mathcal{F}^{\circledcirc}_{\langle J \rangle}$ as the **"resulting averages"**.

This sort of averaging may be thought of as a sort of abstract, Frobenioid-theoretic analogue of the *normalization of arithmetic degrees* that is often used in the theory of heights [cf., e.g., [GenEll], Definition 1.2, (i)] that allows one to work with heights in such a way that the height of a point remains *invariant* with respect to change of the base field.

(ii) On the other hand, to work with the various *isomorphisms of Frobenioids* — such as ${}^{\dagger}\mathcal{F}^{\circledcirc}_{j} \xrightarrow{\sim} {}^{\dagger}\mathcal{F}^{\circledcirc}_{\langle J \rangle}$ — involved amounts [since the *arithmetic degree* is an *intrinsic invariant* of the Frobenioids involved — cf. [FrdI], Theorem 6.4, (iv); Remark 3.1.5 of the present paper] to thinking of arithmetic degrees that appear in the various factors of ${}^{\dagger}\mathcal{F}^{\circledcirc}_{J}$ as being

> **summed** *[i.e., without dividing by a factor of* l^{*}*] over the elements of* J and hence of arithmetic degrees that appear in ${}^{\dagger}\mathcal{F}^{\circledcirc}_{\langle J \rangle}$ as the **"resulting sums"**.

This point of view may be thought of as a sort of abstract, Frobenioid-theoretic analogue of the *normalization of arithmetic degrees or heights* in which the height of a point is *multiplied by the degree of the field extension* when one executes a change of the base field.

The notions defined in the following *"Frobenioid-theoretic lifting"* of Definition 4.6 will play a *central role* in the theory of the present series of papers.

Definition 5.5.

(i) We define an *NF-bridge* [relative to the given initial Θ-data] to be a collection of data

$$({}^{\ddagger}\mathfrak{F}_J \xrightarrow{{}^{\ddagger}\psi_{\circledast}^{\mathrm{NF}}} {}^{\ddagger}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{\ddagger}\mathcal{F}^{\circledast})$$

as follows:

(a) ${}^{\ddagger}\mathfrak{F}_J = \{{}^{\ddagger}\mathfrak{F}_j\}_{j\in J}$ is a *capsule of $\mathcal{F}$-prime-strips*, indexed by a finite index set J. Write ${}^{\ddagger}\mathfrak{D}_J = \{{}^{\ddagger}\mathfrak{D}_j\}_{j\in J}$ for the associated *capsule of $\mathcal{D}$-prime-strips* [cf. Remark 5.2.1, (i)].

(b) ${}^{\ddagger}\mathcal{F}^{\circledcirc}$, ${}^{\ddagger}\mathcal{F}^{\circledast}$ are *categories* equivalent to the categories ${}^{\dagger}\mathcal{F}^{\circledcirc}$, ${}^{\dagger}\mathcal{F}^{\circledast}$, respectively, of Example 5.1, (iii). Thus, each of ${}^{\ddagger}\mathcal{F}^{\circledcirc}$, ${}^{\ddagger}\mathcal{F}^{\circledast}$ is equipped with a *natural Frobenioid structure* [cf. [FrdI], Corollary 4.11; [FrdI], Theorem 6.4, (i); Remark 3.1.5 of the present paper]; write ${}^{\ddagger}\mathcal{D}^{\circledcirc}$, ${}^{\ddagger}\mathcal{D}^{\circledast}$ for the respective *base categories* of these Frobenioids.

(c) The arrow "$\dashrightarrow$" consists of the datum of a *morphism* ${}^{\ddagger}\mathcal{D}^{\circledcirc} \to {}^{\ddagger}\mathcal{D}^{\circledast}$ which is *abstractly equivalent* [cf. §0] to the natural morphism ${}^{\dagger}\mathcal{D}^{\circledcirc} \to {}^{\dagger}\mathcal{D}^{\circledast}$ of Example 5.1, (i), together with the datum of an *isomorphism* ${}^{\ddagger}\mathcal{F}^{\circledcirc} \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}^{\circledast}|_{{}^{\ddagger}\mathcal{D}^{\circledcirc}}$ [cf. the discussion of Example 5.1, (iii)].

(d) ${}^{\ddagger}\psi_{\circledast}^{\mathrm{NF}}$ is a *poly-morphism* that lifts [*uniquely*! — cf. Corollary 5.3, (i), (ii)] a poly-morphism ${}^{\ddagger}\phi_{\circledast}^{\mathrm{NF}} : {}^{\ddagger}\mathfrak{D}_J \to {}^{\ddagger}\mathcal{D}^{\circledcirc}$ such that ${}^{\ddagger}\phi_{\circledast}^{\mathrm{NF}}$ forms a *$\mathcal{D}$-NF-bridge* [cf. Example 5.4, (v); Remark 5.4.1].

Thus, one verifies immediately that any NF-bridge as above determines an *associated $\mathcal{D}$-NF-bridge* $({}^{\ddagger}\phi_{\circledast}^{\mathrm{NF}} : {}^{\ddagger}\mathfrak{D}_J \to {}^{\ddagger}\mathcal{D}^{\circledcirc})$. We define a(n) *[iso]morphism of NF-bridges*

$$({}^{1}\mathfrak{F}_{J_1} \xrightarrow{{}^{1}\psi_{\circledast}^{\mathrm{NF}}} {}^{1}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{1}\mathcal{F}^{\circledast}) \to ({}^{2}\mathfrak{F}_{J_2} \xrightarrow{{}^{2}\psi_{\circledast}^{\mathrm{NF}}} {}^{2}\mathcal{F}^{\circledcirc} \dashrightarrow {}^{2}\mathcal{F}^{\circledast})$$

to be a collection of arrows

$${}^{1}\mathfrak{F}_{J_1} \xrightarrow{\sim} {}^{2}\mathfrak{F}_{J_2}; \quad {}^{1}\mathcal{F}^{\circledcirc} \xrightarrow{\sim} {}^{2}\mathcal{F}^{\circledcirc}; \quad {}^{1}\mathcal{F}^{\circledast} \xrightarrow{\sim} {}^{2}\mathcal{F}^{\circledast}$$

— where ${}^{1}\mathfrak{F}_{J_1} \xrightarrow{\sim} {}^{2}\mathfrak{F}_{J_2}$ is a *capsule-full poly-isomorphism* [cf. §0], hence induces a poly-isomorphism ${}^{1}\mathfrak{D}_{J_1} \xrightarrow{\sim} {}^{2}\mathfrak{D}_{J_2}$; ${}^{1}\mathcal{F}^{\circledcirc} \xrightarrow{\sim} {}^{2}\mathcal{F}^{\circledcirc}$ is a *poly-isomorphism* which lifts a poly-isomorphism ${}^{1}\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{2}\mathcal{D}^{\circledcirc}$ such that the pair of arrows ${}^{1}\mathfrak{D}_{J_1} \xrightarrow{\sim} {}^{2}\mathfrak{D}_{J_2}$, ${}^{1}\mathcal{D}^{\circledcirc} \xrightarrow{\sim} {}^{2}\mathcal{D}^{\circledcirc}$ forms a morphism between the associated $\mathcal{D}$-NF-bridges; ${}^{1}\mathcal{F}^{\circledast} \xrightarrow{\sim} {}^{2}\mathcal{F}^{\circledast}$ is an *isomorphism* — which are *compatible* [in the evident sense] with the ${}^{i}\psi_{\circledast}^{\mathrm{NF}}$ [for

$i = 1, 2$], as well as with the respective "$\dashrightarrow$'s". It is immediate that any morphism of NF-bridges induces a morphism between the associated $\mathcal{D}$-NF-bridges. There is an evident notion of composition of morphisms of NF-bridges.

(ii) We define a Θ-*bridge* [relative to the given initial Θ-data] to be a collection of data

$$({}^{\ddagger}\mathfrak{F}_J \quad \xrightarrow{{}^{\ddagger}\psi^{\Theta}_{\divideontimes}} \quad {}^{\ddagger}\mathfrak{F}_{\succ} \quad \dashrightarrow \quad {}^{\ddagger}\mathcal{HT}^{\Theta})$$

as follows:

(a) ${}^{\ddagger}\mathfrak{F}_J = \{{}^{\ddagger}\mathfrak{F}_j\}_{j \in J}$ is a *capsule of $\mathcal{F}$-prime-strips*, indexed by a finite index set J. Write ${}^{\ddagger}\mathfrak{D}_J = \{{}^{\ddagger}\mathfrak{D}_j\}_{j \in J}$ for the associated *capsule of $\mathcal{D}$-prime-strips* [cf. Remark 5.2.1, (i)].

(b) ${}^{\ddagger}\mathcal{HT}^{\Theta}$ is a Θ-*Hodge theater.*

(c) ${}^{\ddagger}\mathfrak{F}_{\succ}$ is the *$\mathcal{F}$-prime-strip* tautologically associated to ${}^{\ddagger}\mathcal{HT}^{\Theta}$ [cf. the discussion preceding Example 5.4]; we use the notation "$\dashrightarrow$" to denote this relationship between ${}^{\ddagger}\mathfrak{F}_{\succ}$ and ${}^{\ddagger}\mathcal{HT}^{\Theta}$. Write ${}^{\ddagger}\mathfrak{D}_{\succ}$ for the *$\mathcal{D}$-prime-strip* associated to ${}^{\ddagger}\mathfrak{F}_{\succ}$ [cf. Remark 5.2.1, (i)].

(d) ${}^{\ddagger}\psi^{\Theta}_{\divideontimes} = \{{}^{\ddagger}\psi^{\Theta}_j\}_{j \in J}$ is the *collection of poly-morphisms* ${}^{\ddagger}\psi^{\Theta}_j : {}^{\ddagger}\mathfrak{F}_j \to {}^{\ddagger}\mathfrak{F}_{\succ}$ determined [cf. Remark 5.3.1] by a $\mathcal{D}$-Θ-bridge ${}^{\ddagger}\phi^{\Theta}_{\divideontimes} = \{{}^{\ddagger}\phi^{\Theta}_j : {}^{\ddagger}\mathfrak{D}_j \to {}^{\ddagger}\mathfrak{D}_{\succ}\}_{j \in J}$.

Thus, one verifies immediately that any Θ-bridge as above determines an *associated $\mathcal{D}$-Θ-bridge* $({}^{\ddagger}\phi^{\Theta}_{\divideontimes} : {}^{\ddagger}\mathfrak{D}_J \to {}^{\ddagger}\mathfrak{D}_{\succ})$. We define a(n) *[iso]morphism of* Θ-*bridges*

$$({}^{1}\mathfrak{F}_{J_1} \quad \xrightarrow{{}^{1}\psi^{\Theta}_{\divideontimes}} \quad {}^{1}\mathfrak{F}_{\succ} \quad \dashrightarrow \quad {}^{1}\mathcal{HT}^{\Theta}) \quad \to \quad ({}^{2}\mathfrak{F}_{J_2} \quad \xrightarrow{{}^{2}\psi^{\Theta}_{\divideontimes}} \quad {}^{2}\mathfrak{F}_{\succ} \quad \dashrightarrow \quad {}^{2}\mathcal{HT}^{\Theta})$$

to be a collection of arrows

$${}^{1}\mathfrak{F}_{J_1} \xrightarrow{\sim} {}^{2}\mathfrak{F}_{J_2}; \quad {}^{1}\mathfrak{F}_{\succ} \xrightarrow{\sim} {}^{2}\mathfrak{F}_{\succ}; \quad {}^{1}\mathcal{HT}^{\Theta} \xrightarrow{\sim} {}^{2}\mathcal{HT}^{\Theta}$$

— where ${}^{1}\mathfrak{F}_{J_1} \xrightarrow{\sim} {}^{2}\mathfrak{F}_{J_2}$ is a *capsule-full poly-isomorphism* [cf. §0]; ${}^{1}\mathfrak{F}_{\succ} \xrightarrow{\sim} {}^{2}\mathfrak{F}_{\succ}$ is a *full poly-isomorphism*; ${}^{1}\mathcal{HT}^{\Theta} \xrightarrow{\sim} {}^{2}\mathcal{HT}^{\Theta}$ is an *isomorphism of* Θ-*Hodge theaters* [cf. Remark 3.6.2] — which are *compatible* [in the evident sense] with the ${}^{i}\psi^{\Theta}_{\divideontimes}$ [for $i = 1, 2$], as well as with the respective "$\dashrightarrow$'s" [cf. Corollary 5.6, (i), below]. It is immediate that any morphism of Θ-bridges induces a morphism between the associated $\mathcal{D}$-Θ-bridges. There is an evident notion of composition of morphisms of Θ-bridges.

(iii) We define a ΘNF-*Hodge theater* [relative to the given initial Θ-data] to be a collection of data

$${}^{\ddagger}\mathcal{HT}^{\Theta \mathrm{NF}} = ({}^{\ddagger}\mathcal{F}^{\circledast} \quad \dashleftarrow \quad {}^{\ddagger}\mathcal{F}^{\circledcirc} \quad \xleftarrow{{}^{\ddagger}\psi^{\mathrm{NF}}_{\divideontimes}} \quad {}^{\ddagger}\mathfrak{F}_J \quad \xrightarrow{{}^{\ddagger}\psi^{\Theta}_{\divideontimes}} \quad {}^{\ddagger}\mathfrak{F}_{\succ} \quad \dashrightarrow \quad {}^{\ddagger}\mathcal{HT}^{\Theta})$$

— where the data $({}^{\ddagger}\mathcal{F}^{\circledast} \quad \dashleftarrow \quad {}^{\ddagger}\mathcal{F}^{\circledcirc} \quad \longleftarrow \quad {}^{\ddagger}\mathfrak{F}_J)$ forms an *NF-bridge*; the data $({}^{\ddagger}\mathfrak{F}_J \quad \longrightarrow \quad {}^{\ddagger}\mathfrak{F}_{\succ} \quad \dashrightarrow \quad {}^{\ddagger}\mathcal{HT}^{\Theta})$ forms a Θ-*bridge* — such that the *associated data* $\{{}^{\ddagger}\phi^{\mathrm{NF}}_{\divideontimes}, {}^{\ddagger}\phi^{\Theta}_{\divideontimes}\}$ [cf. (i), (ii)] forms a $\mathcal{D}$-ΘNF-Hodge theater. A(n) *[iso]morphism of* ΘNF-*Hodge theaters* is defined to be a pair of morphisms between the respective associated NF- and Θ-bridges that are *compatible* with one another in the sense that they induce the *same bijection* between the index sets of the respective capsules of $\mathcal{F}$-prime-strips. There is an evident notion of composition of morphisms of ΘNF-Hodge theaters.

Corollary 5.6. (Isomorphisms of Θ-Hodge Theaters, NF-Bridges, Θ-Bridges, and ΘNF-Hodge Theaters) *Relative to a fixed collection of* **initial Θ-data***:*

(i) The natural functorially induced map from the **set of isomorphisms** *between* **two Θ-Hodge theaters** *to the set of isomorphisms between the respective* **associated $\mathcal{D}$-prime-strips** *[cf. the discussion preceding Example 5.4; Remark 5.2.1, (i)] is* **bijective***.*

(ii) The natural functorially induced map from the **set of isomorphisms** *between* **two NF-bridges** *(respectively,* **two Θ-bridges***;* **two ΘNF-Hodge theaters***) to the set of isomorphisms between the respective* **associated $\mathcal{D}$-NF-bridges** *(respectively,* **associated $\mathcal{D}$-Θ-bridges***;* **associated $\mathcal{D}$-ΘNF-Hodge theaters***) is* **bijective***.*

(iii) Given an NF-bridge and a Θ-bridge, the set of capsule-full poly-isomorphisms between the respective capsules of $\mathcal{F}$-prime-strips which allow one to **glue** *the given NF- and Θ-bridges together to form a ΘNF-Hodge theater forms an* $\mathbb{F}_l^{\circledast}$**-torsor***.*

Proof. First, we consider assertion (i). Sorting through the data listed in the definition of a Θ-Hodge theater ${}^{\dagger}\mathcal{HT}^{\Theta}$ [cf. Definition 3.6], one verifies immediately that the only data that is not contained in the *associated $\mathcal{F}$-prime-strip* ${}^{\dagger}\mathfrak{F}_{>}$ [cf. the discussion preceding Example 5.4] is the *global data* of Definition 3.6, (c), and the *tempered Frobenioids* isomorphic to "$\underline{\underline{\mathcal{F}}}_{\underline{v}}$" [cf. Example 3.2, (i)] at the $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. That is to say, for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$, one verifies immediately that

$$ {}^{\dagger}\mathcal{F}_{>,\underline{v}} \quad = \quad {}^{\dagger}\underline{\underline{\mathcal{F}}}_{\underline{v}} $$

[cf. Example 3.3, (i); Example 3.4, (i); Definition 3.6; Definition 5.2, (i)]. On the other hand, one verifies immediately that this global data is *"rigid"*, i.e., admits no nontrivial automorphisms. Thus, assertion (i) follows by applying Corollary 5.3, (ii), to the associated $\mathcal{F}$-prime-strips and Corollary 5.3, (iv), to the various tempered Frobenioids at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. This completes the proof of assertion (i). In light of assertion (i), assertions (ii), (iii) follow immediately from the definitions and Corollary 5.3, (i), (ii). ◯

Remark 5.6.1. Observe that the various **"functorial dynamics"** studied in §4 — i.e., more precisely, analogues of Propositions 4.8, (i), (ii); 4.9; 4.11 — apply to the *NF-bridges*, *Θ-bridges*, and ΘNF-*Hodge theaters* studied in the present §5. Indeed, such analogues follow immediately from Corollaries 5.3, (ii), (iii); 5.6, (ii).

Section 6: Additive Combinatorial Teichmüller Theory

In the present §6, we discuss the **additive** analogue — i.e., which revolves around the **"functorial dynamics"** that arise from **labels**

$$\in \mathbb{F}_l$$

— of the *"multiplicative combinatorial Teichmüller theory"* developed in §4 for labels $\in \mathbb{F}_l^{\circledast}$. These considerations lead naturally to certain enhancements of the various *Hodge theaters* considered in §5. On the other hand, despite the resemblance of the theory of the present §6 to the theory of §4, §5, the theory of the present §6 is, in certain respects — especially those respects that form the analogue of the theory of §5 — substantially *technically simpler*.

In the following, we fix a collection of *initial* Θ*-data*

$$(\overline{F}/F,\ X_F,\ l,\ \underline{C}_K,\ \underline{\mathbb{V}},\ \underline{\epsilon})$$

as in Definition 3.1; also, we shall use the various notations introduced in Definition 3.1 for various objects associated to this initial Θ-data.

Definition 6.1.

(i) We shall write

$$\mathbb{F}_l^{\rtimes\pm} \stackrel{\mathrm{def}}{=} \mathbb{F}_l \rtimes \{\pm 1\}$$

for the group determined by forming the semi-direct product with respect to the natural inclusion $\{\pm 1\} \hookrightarrow \mathbb{F}_l^{\times}$ and refer to an element of $\mathbb{F}_l^{\rtimes\pm}$ that maps to $+1$ (respectively, -1) via the natural surjection $\mathbb{F}_l^{\rtimes\pm} \twoheadrightarrow \{\pm 1\}$ as *positive* (respectively, *negative*). We shall refer to as an $\mathbb{F}_l^{\pm}$*-group* any set E equipped with a $\{\pm 1\}$-orbit of bijections $E \xrightarrow{\sim} \mathbb{F}_l$. Thus, any $\mathbb{F}_l^{\pm}$-group E is equipped with a natural $\mathbb{F}_l$*-module structure*. We shall refer to as an $\mathbb{F}_l^{\pm}$*-torsor* any set T equipped with a $\mathbb{F}_l^{\rtimes\pm}$-orbit of bijections $T \xrightarrow{\sim} \mathbb{F}_l$ [relative to the action of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathbb{F}_l$ by automorphisms of the form $\mathbb{F}_l \ni z \mapsto \pm z + \lambda \in \mathbb{F}_l$, for $\lambda \in \mathbb{F}_l$]. Thus, if T is an $\mathbb{F}_l^{\pm}$-torsor, then the abelian group of automorphisms of the underlying set of $\mathbb{F}_l$ given by the translations $\mathbb{F}_l \ni z \mapsto z + \lambda \in \mathbb{F}_l$, for $\lambda \in \mathbb{F}_l$, determines an abelian group

$$\mathrm{Aut}_+(T)$$

of *"positive automorphisms"* of the underlying set of T. Moreover, $\mathrm{Aut}_+(T)$ is equipped with a natural structure of $\mathbb{F}_l^{\pm}$-group [such that the abelian group structure of $\mathrm{Aut}_+(T)$ coincides with the $\mathbb{F}_l$-module structure of $\mathrm{Aut}_+(T)$ induced by this $\mathbb{F}_l^{\pm}$-group structure]. Finally, if T is an $\mathbb{F}_l^{\pm}$-torsor, then we shall write

$$\mathrm{Aut}_\pm(T)$$

for the group of automorphisms of the underlying set of T determined [relative to the $\mathbb{F}_l^{\pm}$-torsor structure on T] by the group of automorphisms of the underlying set of $\mathbb{F}_l$ given by $\mathbb{F}_l^{\rtimes\pm}$ [so $\mathrm{Aut}_+(T) \subseteq \mathrm{Aut}_\pm(T)$ is the unique subgroup of index 2].

(ii) Let

$$^{\dagger}\mathfrak{D} = \{^{\dagger}\mathcal{D}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

be a *$\mathcal{D}$-prime-strip* [relative to the given initial Θ-data]. Observe [cf. the discussion of Definition 4.1, (i)] that if $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, then $\pi_1(^{\dagger}\mathcal{D}_{\underline{v}})$ determines, in a functorial fashion, a profinite group corresponding to "$\underline{X}_{\underline{v}}$" [cf. Corollary 1.2 if $\underline{v} \in \underline{\mathbb{V}}^{\text{good}}$; [EtTh], Proposition 2.4, if $\underline{v} \in \underline{\mathbb{V}}^{\text{bad}}$], which contains $\pi_1(^{\dagger}\mathcal{D}_{\underline{v}})$ as an open subgroup; thus, if we write $^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$ for $\mathcal{B}(-)^0$ of this profinite group, then we obtain a *natural morphism* $^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$ [cf. §0]. In a similar vein, if $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, then since $\underline{X}_{\to_{\underline{v}}}$ admits a $K_{\underline{v}}$-core, a routine translation into the "language of Aut-holomorphic orbispaces" of the argument given in the proof of Corollary 1.2 [cf. also [AbsTopIII], Corollary 2.4] reveals that $^{\dagger}\mathcal{D}_{\underline{v}}$ determines, in a functorial fashion, an Aut-holomorphic orbispace $^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$ corresponding to "$\underline{X}_{\underline{v}}$", together with a *natural morphism* $^{\dagger}\mathcal{D}_{\underline{v}} \to {}^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$ *of* Aut-*holomorphic orbispaces*. Thus, in summary, one obtains a collection of data

$$^{\dagger}\underline{\mathfrak{D}}^{\pm} = \{^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

completely determined by $^{\dagger}\mathfrak{D}$.

(iii) Suppose that we are in the situation of (ii). Then observe [cf. the discussion of Definition 4.1, (ii)] that by applying the *group-theoretic algorithm* of [AbsTopI], Lemma 4.5, to the topological group $\pi_1(^{\dagger}\mathcal{D}_{\underline{v}})$ when $\underline{v} \in \underline{\mathbb{V}}^{\text{non}}$, or by considering $\pi_0(-)$ of a cofinal collection of "neighborhoods of infinity" [i.e., complements of compact subsets] of the underlying topological space of $^{\dagger}\mathcal{D}_{\underline{v}}$ when $\underline{v} \in \underline{\mathbb{V}}^{\text{arc}}$, it makes sense to speak of the *set of cusps* of $^{\dagger}\mathcal{D}_{\underline{v}}$; a similar observation applies to $^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$, for $\underline{v} \in \underline{\mathbb{V}}$. If $\underline{v} \in \underline{\mathbb{V}}$, then we define a *$\pm$-label class of cusps* of $^{\dagger}\mathcal{D}_{\underline{v}}$ to be the set of cusps of $^{\dagger}\mathcal{D}_{\underline{v}}$ that lie over a single *cusp* [i.e., corresponding to an *arbitrary element* of the quotient "Q" that appears in the definition of a "hyperbolic orbicurve of type $(1, l$-tors)" given in [EtTh], Definition 2.1] of $^{\dagger}\underline{\mathcal{D}}^{\pm}_{\underline{v}}$; write

$$\text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}})$$

for the *set of $\pm$-label classes of cusps* of $^{\dagger}\mathcal{D}_{\underline{v}}$. Thus, [for any $\underline{v} \in \underline{\mathbb{V}}$!] $\text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}})$ admits a *natural action by* $\mathbb{F}_l^{\times}$ [cf. [EtTh], Definition 2.1], as well as a *zero element*

$$^{\dagger}\underline{\eta}^{0}_{\underline{v}} \in \text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}})$$

and a *$\pm$-canonical element*

$$^{\dagger}\underline{\eta}^{\pm}_{\underline{v}} \in \text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}})$$

— which is *well-defined up to multiplication by* ± 1, and which may be *constructed solely from* $^{\dagger}\mathcal{D}_{\underline{v}}$ [cf. Definition 4.1, (ii)] — such that, relative to the *natural bijection*

$$\left\{\text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}}) \setminus \{^{\dagger}\underline{\eta}^{0}_{\underline{v}}\}\right\}/\{\pm 1\} \xrightarrow{\sim} \text{LabCusp}(^{\dagger}\mathcal{D}_{\underline{v}})$$

[cf. the notation of Definition 4.1, (ii)], we have $^{\dagger}\underline{\eta}^{\pm}_{\underline{v}} \mapsto {}^{\dagger}\underline{\eta}_{\underline{v}}$. In particular, we obtain a *natural bijection*

$$\text{LabCusp}^{\pm}(^{\dagger}\mathcal{D}_{\underline{v}}) \quad \xrightarrow{\sim} \quad \mathbb{F}_l$$

— which is *well-defined up to multiplication by* ± 1 and *compatible*, relative to the natural bijection to "LabCusp($-$)" of the preceding display, with the natural bijection of the second display of Proposition 4.2. That is to say, in the terminology of (i), $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})$ is equipped with a *natural* $\mathbb{F}_l^{\pm}$*-group structure*. This $\mathbb{F}_l^{\pm}$-group structure determines a *natural surjection*

$$\mathrm{Aut}({}^{\dagger}\mathcal{D}_{\underline{v}}) \twoheadrightarrow \{\pm 1\}$$

— i.e., by considering the induced automorphism of $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}})$. Write

$$\mathrm{Aut}_+({}^{\dagger}\mathcal{D}_{\underline{v}}) \subseteq \mathrm{Aut}({}^{\dagger}\mathcal{D}_{\underline{v}})$$

for the index two subgroup of *"positive automorphisms"* [i.e., the kernel of the above surjection] and $\mathrm{Aut}_-({}^{\dagger}\mathcal{D}_{\underline{v}}) \stackrel{\mathrm{def}}{=} \mathrm{Aut}({}^{\dagger}\mathcal{D}_{\underline{v}}) \setminus \mathrm{Aut}_+({}^{\dagger}\mathcal{D}_{\underline{v}})$ [i.e., where "$\setminus$" denotes the *set-theoretic complement*] for the subset of *"negative automorphisms"*. In a similar vein, we shall write

$$\mathrm{Aut}_+({}^{\dagger}\mathfrak{D}) \subseteq \mathrm{Aut}({}^{\dagger}\mathfrak{D})$$

for the subgroup of *"positive automorphisms"* [i.e., automorphisms each of whose components, for $\underline{v} \in \underline{\mathbb{V}}$, is *positive*], and, if $\alpha \in \{\pm 1\}^{\underline{\mathbb{V}}}$ [i.e., where we write $\{\pm 1\}^{\underline{\mathbb{V}}}$ for the set of set-theoretic maps from $\underline{\mathbb{V}}$ to $\{\pm 1\}$],

$$\mathrm{Aut}_\alpha({}^{\dagger}\mathfrak{D}) \subseteq \mathrm{Aut}({}^{\dagger}\mathfrak{D})$$

for the subset of *"α-signed automorphisms"* [i.e., automorphisms each of whose components, for $\underline{v} \in \underline{\mathbb{V}}$, is *positive* if $\alpha(\underline{v}) = +1$ and *negative* if $\alpha(\underline{v}) = -1$].

(iv) Suppose that we are in the situation of (ii). Let

$${}^{\ddagger}\mathfrak{D} = \{{}^{\ddagger}\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$$

be another $\mathcal{D}$*-prime-strip* [relative to the given initial Θ-data]. Then for any $\underline{v} \in \underline{\mathbb{V}}$, we shall refer to as a $+$*-full poly-isomorphism* ${}^{\dagger}\mathcal{D}_{\underline{v}} \stackrel{\sim}{\to} {}^{\ddagger}\mathcal{D}_{\underline{v}}$ any poly-isomorphism obtained as the $\mathrm{Aut}_+({}^{\dagger}\mathcal{D}_{\underline{v}})$- [or, equivalently, $\mathrm{Aut}_+({}^{\ddagger}\mathcal{D}_{\underline{v}})$-] *orbit* of an isomorphism ${}^{\dagger}\mathcal{D}_{\underline{v}} \stackrel{\sim}{\to} {}^{\ddagger}\mathcal{D}_{\underline{v}}$. In particular, if ${}^{\dagger}\mathfrak{D} = {}^{\ddagger}\mathfrak{D}$, then there are precisely *two* $+$-full poly-isomorphisms ${}^{\dagger}\mathcal{D}_{\underline{v}} \stackrel{\sim}{\to} {}^{\ddagger}\mathcal{D}_{\underline{v}}$, namely, the $+$-full poly-isomorphism determined by the identity isomorphism, which we shall refer to as *positive*, and the unique non-positive $+$-full poly-isomorphism, which we shall refer to as *negative*. In a similar vein, we shall refer to as a $+$*-full poly-isomorphism* ${}^{\dagger}\mathfrak{D} \stackrel{\sim}{\to} {}^{\ddagger}\mathfrak{D}$ any poly-isomorphism obtained as the $\mathrm{Aut}_+({}^{\dagger}\mathfrak{D})$- [or, equivalently, $\mathrm{Aut}_+({}^{\ddagger}\mathfrak{D})$-] *orbit* of an isomorphism ${}^{\dagger}\mathfrak{D} \stackrel{\sim}{\to} {}^{\ddagger}\mathfrak{D}$. In particular, if ${}^{\dagger}\mathfrak{D} = {}^{\ddagger}\mathfrak{D}$, then the set of $+$-full poly-isomorphisms ${}^{\dagger}\mathfrak{D} \stackrel{\sim}{\to} {}^{\ddagger}\mathfrak{D}$ is in *natural bijective correspondence* [cf. the discussion of (iii) above] with the set $\{\pm 1\}^{\underline{\mathbb{V}}}$; we shall refer to the $+$-full poly-isomorphism ${}^{\dagger}\mathfrak{D} \stackrel{\sim}{\to} {}^{\ddagger}\mathfrak{D}$ that corresponds to $\alpha \in \{\pm 1\}^{\underline{\mathbb{V}}}$ as the *α-signed $+$-full poly-isomorphism*. Finally, a *capsule-$+$-full poly-morphism* between capsules of $\mathcal{D}$-prime-strips

$$\{{}^{\dagger}\mathfrak{D}_t\}_{t \in T} \stackrel{\sim}{\to} \{{}^{\ddagger}\mathfrak{D}_{t'}\}_{t' \in T'}$$

is defined to be a poly-morphism between two capsules of $\mathcal{D}$-prime-strips determined by $+$*-full poly-isomorphisms* ${}^{\dagger}\mathfrak{D}_t \stackrel{\sim}{\to} {}^{\ddagger}\mathfrak{D}_{\iota(t)}$ [where $t \in T$] between the constituent objects indexed by corresponding indices, relative to some *injection* $\iota : T \hookrightarrow T'$.

(v) Write

$$\mathcal{D}^{\circledcirc\pm} \stackrel{\text{def}}{=} \mathcal{B}(\underline{X}_K)^0$$

[cf. §0; Definition 4.1, (v)]. Thus, we have a *finite étale double covering* $\mathcal{D}^{\circledcirc\pm} \to \mathcal{D}^{\circledcirc} = \mathcal{B}(\underline{C}_K)^0$. Just as in the case of $\mathcal{D}^{\circledcirc}$ [cf. Example 4.3, (i)], one may construct, in a *category-theoretic* fashion from $\mathcal{D}^{\circledcirc\pm}$, the *outer homomorphism*

$$\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \to GL_2(\mathbb{F}_l)/\{\pm 1\}$$

arising from the *l-torsion points* of the elliptic curve $E_{\overline{F}}$ [i.e., on $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l$]. Moreover, it follows from the construction of $\underline{X}_K$ that, relative to the *natural isomorphism* $\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathrm{Aut}(\underline{X}_K)$ [cf., e.g., [AbsTopIII], Theorem 1.9], the *image* of the above outer homomorphism is equal to a *Borel subgroup* of $GL_2(\mathbb{F}_l)/\{\pm 1\}$ [cf. the discussion of Example 4.3, (i)] — i.e., the Borel subgroup corresponding to the *rank one quotient* of $\Delta_X^{\mathrm{ab}} \otimes \mathbb{F}_l$ that gives rise to the covering $\underline{X}_K \to X_K$. In particular, this rank one quotient determines a *natural surjective homomorphism* [which may be *reconstructed category-theoretically* from $\mathcal{D}^{\circledcirc\pm}$!]

$$\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \twoheadrightarrow \mathbb{F}_l^{\circledast}$$

— whose kernel we denote by $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm})$. One verifies immediately that the subgroup $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathrm{Aut}(\underline{X}_K)$ contains the subgroup $\mathrm{Aut}_K(\underline{X}_K) \subseteq \mathrm{Aut}(\underline{X}_K)$ of *K-linear automorphisms* and acts *transitively* on the cusps of $\underline{X}_K$. Next, let us write $\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})$ for the subgroup [which may be *reconstructed category-theoretically* from $\mathcal{D}^{\circledcirc\pm}$! — cf. [AbsTopI], Lemma 4.5] of automorphisms that *fix the cusps* of $\underline{X}_K$. Then one obtains *natural outer isomorphisms*

$$\mathrm{Aut}_K(\underline{X}_K) \quad \xrightarrow{\sim} \quad \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \quad \xrightarrow{\sim} \quad \mathbb{F}_l^{\rtimes\pm}$$

[cf. the discussion preceding [EtTh], Definition 2.1] — where the second outer isomorphism depends, in an essential way, on the choice of the *cusp* $\underline{\epsilon}$ of $\underline{C}_K$ [cf. Definition 3.1, (f)]. Put another way, if we write $\mathrm{Aut}_{+}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})$ for the unique index two subgroup containing $\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})$, then the cusp $\underline{\epsilon}$ determines a *natural* $\mathbb{F}_l^{\pm}$*-group structure* on the subgroup

$$\mathrm{Aut}_{+}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \quad \subseteq \quad \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})$$

[which corresponds to the subgroup $\mathrm{Gal}(\underline{X}_K/X_K) \subseteq \mathrm{Aut}_K(\underline{X}_K)$ via the *natural outer isomorphisms* of the preceding display] and, in the notation of (vi) below, a *natural* $\mathbb{F}_l^{\pm}$*-torsor structure* on the set $\mathrm{LabCusp}^{\pm}(\mathcal{D}^{\circledcirc\pm})$. Write

$$\underline{\mathbb{V}}^{\pm} \quad \stackrel{\text{def}}{=} \quad \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \cdot \underline{\mathbb{V}} \quad = \quad \mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \cdot \underline{\mathbb{V}} \quad \subseteq \quad \mathbb{V}(K)$$

[cf. the discussion of Example 4.3, (i); Remark 6.1.1 below] — where the "=" follows immediately from the *natural outer isomorphisms* discussed above. Then [by considering what happens at the elements of $\underline{\mathbb{V}}^{\pm} \bigcap \underline{\mathbb{V}}^{\mathrm{bad}}$] one verifies immediately that the subgroup $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \cong \mathrm{Aut}(\underline{X}_K)$ may be identified with the subgroup of $\mathrm{Aut}(\underline{X}_K)$ that *stabilizes* $\underline{\mathbb{V}}^{\pm}$.

(vi) Let

$$^{\dagger}\mathcal{D}^{\circledcirc\pm}$$

be any category isomorphic to $\mathcal{D}^{\circledcirc\pm}$. Then just as in the discussion of (iii) in the case of "$\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$", it makes sense [cf. [AbsTopI], Lemma 4.5] to speak of the *set of cusps* of $^{\dagger}\mathcal{D}^{\circledcirc\pm}$, as well as the *set of* $\pm$*-label classes of cusps*

$$\mathrm{LabCusp}^{\pm}(^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

— which, in this case, may be identified with the set of cusps of $^{\dagger}\mathcal{D}^{\circledcirc\pm}$.

(vii) Recall from [AbsTopIII], Theorem 1.9 [cf. Remark 3.1.2] that [just as in the case of $\mathcal{D}^{\circledcirc}$ — cf. the discussion of Definition 4.1, (v)] there exists a *group-theoretic algorithm* for reconstructing, from $\pi_1(\mathcal{D}^{\circledcirc\pm})$ [cf. §0], the algebraic closure "$\overline{F}$" of the base field "K", hence also the *set of valuations* "$\mathbb{V}(\overline{F})$" from $\mathcal{D}^{\circledcirc\pm}$ [e.g., as a collection of topologies on $\overline{F}$ — cf., e.g., [AbsTopIII], Corollary 2.8]. Moreover, for $\underline{w} \in \mathbb{V}(K)^{\mathrm{arc}}$, let us recall [cf. Remark 3.1.2; [AbsTopIII], Corollaries 2.8, 2.9] that one may *reconstruct group-theoretically*, from $\pi_1(\mathcal{D}^{\circledcirc\pm})$, the Aut-*holomorphic orbispace* $\underline{\mathbb{X}}_{\underline{w}}$ associated to $\underline{X}_{\underline{w}}$. Let $^{\dagger}\mathcal{D}^{\circledcirc\pm}$ be as in (vi). Then let us write

$$\overline{\mathbb{V}}(^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

for the set of valuations [i.e., "$\mathbb{V}(\overline{F})$"], equipped with its natural $\pi_1(^{\dagger}\mathcal{D}^{\circledcirc\pm})$-action,

$$\mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc\pm}) \quad \stackrel{\mathrm{def}}{=} \quad \overline{\mathbb{V}}(^{\dagger}\mathcal{D}^{\circledcirc\pm})/\pi_1(^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

for the quotient of $\overline{\mathbb{V}}(^{\dagger}\mathcal{D}^{\circledcirc\pm})$ by $\pi_1(^{\dagger}\mathcal{D}^{\circledcirc\pm})$ [i.e., "$\mathbb{V}(K)$"], and, for $\underline{w} \in \mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc\pm})^{\mathrm{arc}}$,

$$\underline{\mathbb{X}}(^{\dagger}\mathcal{D}^{\circledcirc\pm}, \underline{w})$$

[i.e., "$\underline{\mathbb{X}}_{\underline{w}}$" — cf. the discussion of [AbsTopIII], Definition 5.1, (ii)] for the Aut-holomorphic orbispace obtained by applying these group-theoretic reconstruction algorithms to $\pi_1(^{\dagger}\mathcal{D}^{\circledcirc\pm})$. Now if $\mathbb{U}$ is an arbitrary Aut-*holomorphic orbispace*, then let us define a *morphism*

$$\mathbb{U} \to {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$$

to be a morphism of Aut-holomorphic orbispaces [cf. [AbsTopIII], Definition 2.1, (ii)] $\mathbb{U} \to \underline{\mathbb{X}}(^{\dagger}\mathcal{D}^{\circledcirc\pm}, \underline{w})$ for some $\underline{w} \in \mathbb{V}(^{\dagger}\mathcal{D}^{\circledcirc\pm})^{\mathrm{arc}}$. Thus, it makes sense to speak of the pre-composite (respectively, post-composite) of such a morphism $\mathbb{U} \to {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$ with a morphism of Aut-holomorphic orbispaces (respectively, with an isomorphism [cf. §0] $^{\dagger}\mathcal{D}^{\circledcirc\pm} \stackrel{\sim}{\to} {}^{\ddagger}\mathcal{D}^{\circledcirc\pm}$ [i.e., where $^{\ddagger}\mathcal{D}^{\circledcirc\pm}$ is a category equivalent to $\mathcal{D}^{\circledcirc\pm}$]).

Remark 6.1.1. In fact, in the notation of Example 4.3, (i); Definition 6.1, (v), it is not difficult to verify that $\underline{\mathbb{V}}^{\pm} = \underline{\mathbb{V}}^{\pm\mathrm{un}}$ $(\subseteq \mathbb{V}(K))$.

Example 6.2. Model Base-$\Theta^{\pm}$-Bridges.

(i) In the following, let us think of $\mathbb{F}_l$ as a $\mathbb{F}_l^{\pm}$-*group* [relative to the evident $\mathbb{F}_l^{\pm}$-group structure]. Let

$$\mathfrak{D}_{\succ} = \{\mathcal{D}_{\succ,\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}; \quad \mathfrak{D}_t = \{\mathcal{D}_{\underline{v}_t}\}_{\underline{v}\in\underline{\mathbb{V}}}$$

— where $t \in \mathbb{F}_l$, and we use the notation $\underline{v}_t$ to denote the pair $(t, \underline{v})$ [cf. Example 4.3, (iv)] — be copies of the "tautological $\mathcal{D}$-prime-strip" $\{\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ [cf. Examples 4.3, (iv); 4.4, (ii)]. For each $t \in \mathbb{F}_l$, write

$$\phi^{\Theta^{\pm}}_{\underline{v}_t} : \mathcal{D}_{\underline{v}_t} \to \mathcal{D}_{\succ,\underline{v}}; \quad \phi^{\Theta^{\pm}}_{t} : \mathfrak{D}_t \to \mathfrak{D}_{\succ}$$

for the respective *positive* $+$*-full poly-isomorphisms*, i.e., relative to the respective identifications with the "tautological $\mathcal{D}$-prime-strip" $\{\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$. Write $\mathfrak{D}_{\pm}$ for the *capsule* $\{\mathfrak{D}_t\}_{t \in \mathbb{F}_l}$ [cf. the constructions of Example 4.4, (iv)] and

$$\phi^{\Theta^{\pm}}_{\pm} : \mathfrak{D}_{\pm} \to \mathfrak{D}_{\succ}$$

for the *collection of poly-morphisms* $\{\phi^{\Theta^{\pm}}_{t}\}_{t \in \mathbb{F}_l}$.

(ii) The collection of data

$$(\mathfrak{D}_{\pm}, \mathfrak{D}_{\succ}, \phi^{\Theta^{\pm}}_{\pm})$$

admits a *natural poly-automorphism of order two* $-1_{\mathbb{F}_l}$ defined as follows: the poly-automorphism $-1_{\mathbb{F}_l}$ acts on $\mathbb{F}_l$ as *multiplication by* -1 and induces the poly-isomorphisms $\mathfrak{D}_t \xrightarrow{\sim} \mathfrak{D}_{-t}$ [for $t \in \mathbb{F}_l$] and $\mathfrak{D}_{\succ} \xrightarrow{\sim} \mathfrak{D}_{\succ}$ determined [i.e., relative to the respective identifications with the "tautological $\mathcal{D}$-prime-strip" $\{\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$] by the $+$*-full poly-automorphism* whose *sign* at every $\underline{v} \in \underline{\mathbb{V}}$ is *negative*. One verifies immediately that $-1_{\mathbb{F}_l}$, defined in this way, is *compatible* [in the evident sense] with $\phi^{\Theta^{\pm}}_{\pm}$.

(iii) Let $\alpha \in \{\pm 1\}^{\underline{\mathbb{V}}}$. Then α determines a *natural poly-automorphism of order two* $\alpha^{\Theta^{\pm}}$ of the collection of data

$$(\mathfrak{D}_{\pm}, \mathfrak{D}_{\succ}, \phi^{\Theta^{\pm}}_{\pm})$$

as follows: the poly-automorphism $\alpha^{\Theta^{\pm}}$ acts on $\mathbb{F}_l$ as the *identity* and on $\mathfrak{D}_t$, for $t \in \mathbb{F}_l$, and $\mathfrak{D}_{\succ}$ as the α*-signed* $+$*-full poly-automorphism.* One verifies immediately that $\alpha^{\Theta^{\pm}}$, defined in this way, is *compatible* [in the evident sense] with $\phi^{\Theta^{\pm}}_{\pm}$.

Example 6.3. Model Base-Θ^{ell}-Bridges.

(i) In the following, let us think of $\mathbb{F}_l$ as a $\mathbb{F}_l^{\pm}$*-torsor* [relative to the evident $\mathbb{F}_l^{\pm}$-torsor structure]. Let

$$\mathfrak{D}_t = \{\mathcal{D}_{\underline{v}_t}\}_{\underline{v} \in \underline{\mathbb{V}}}$$

[for $t \in \mathbb{F}_l$] and $\mathfrak{D}_{\pm}$ be as in Example 6.2, (i); $\mathcal{D}^{\circledcirc\pm}$ as in Definition 6.1, (v). In the following, let us *fix* an *isomorphism of* $\mathbb{F}_l^{\pm}$*-torsors*

$$\mathrm{LabCusp}^{\pm}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l$$

[cf. the discussion of Definition 6.1, (v)], which we shall use to *identify* $\mathrm{LabCusp}^{\pm}(\mathcal{D}^{\circledcirc\pm})$ with $\mathbb{F}_l$. Note that this identification induces an *isomorphism of groups*

$$\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$$

[cf. the discussion of Definition 6.1, (v)], which we shall use to *identify* the group $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})$ with the group $\mathbb{F}_l^{\rtimes\pm}$. If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}} \bigcap \underline{\mathbb{V}}^{\mathrm{non}}$ (respectively, $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$), then the natural restriction functor on finite étale coverings arising from the natural composite morphism $\underline{X}_{\to_{\underline{v}}} \to \underline{X}_{\underline{v}} \to \underline{X}_K$ (respectively, $\underline{\underline{X}}_{\underline{v}} \to \underline{X}_{\underline{v}} \to \underline{X}_K$) determines [cf. Examples 3.2, (i); 3.3, (i)] a *natural morphism* $\phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc\pm}$ [cf. the discussion of Example 4.3, (ii)]. If $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$, then [cf. Example 3.4, (i)] we have a *tautological morphism* $\mathcal{D}_{\underline{v}} = \underline{\mathbb{X}}_{\to_{\underline{v}}} \to \underline{\mathbb{X}}_{\underline{v}} \xrightarrow{\sim} \underline{\mathbb{X}}(\mathcal{D}^{\circledcirc\pm}, \underline{v})$, hence a morphism $\phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc\pm}$ [cf. the discussion of Example 4.3, (iii)]. For arbitrary $\underline{v} \in \underline{\mathbb{V}}$, write

$$\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0} : \mathcal{D}_{\underline{v}_0} \to \mathcal{D}^{\circledcirc\pm}$$

for the *poly-morphism* given by the collection of morphisms $\mathcal{D}_{\underline{v}_0} \to \mathcal{D}^{\circledcirc\pm}$ of the form

$$\beta \circ \phi^{\Theta^{\mathrm{ell}}}_{\bullet,\underline{v}} \circ \alpha$$

— where $\alpha \in \mathrm{Aut}_{+}(\mathcal{D}_{\underline{v}_0})$; $\beta \in \mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})$; we apply the tautological identification of $\mathcal{D}_{\underline{v}}$ with $\mathcal{D}_{\underline{v}_0}$ [cf. the discussion of Example 4.3, (ii), (iii), (iv)]. Write

$$\phi^{\Theta^{\mathrm{ell}}}_{0} : \mathfrak{D}_0 \to \mathcal{D}^{\circledcirc\pm}$$

for the *poly-morphism* determined by the collection $\{\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0} : \mathcal{D}_{\underline{v}_0} \to \mathcal{D}^{\circledcirc\pm}\}_{\underline{v}\in\underline{\mathbb{V}}}$ [cf. the discussion of Example 4.3, (iv)]. Note that the existence of "β" in the definition of $\phi^{\Theta^{\mathrm{ell}}}_{\underline{v}_0}$ implies that it makes sense to *post-compose* $\phi^{\Theta^{\mathrm{ell}}}_{0}$ with an element of $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$. Thus, for any $t \in \mathbb{F}_l \subseteq \mathbb{F}_l^{\rtimes\pm}$, let us write

$$\phi^{\Theta^{\mathrm{ell}}}_{t} : \mathfrak{D}_t \to \mathcal{D}^{\circledcirc\pm}$$

for the result of *post-composing* $\phi^{\Theta^{\mathrm{ell}}}_{0}$ with the *"poly-action"* [i.e., action via poly-automorphisms] of t on $\mathcal{D}^{\circledcirc\pm}$ [and *pre-composing* with the tautological identification of $\mathfrak{D}_0$ with $\mathfrak{D}_t$] and

$$\phi^{\Theta^{\mathrm{ell}}}_{\pm} : \mathfrak{D}_{\pm} \to \mathcal{D}^{\circledcirc\pm}$$

for the collection of arrows $\{\phi^{\Theta^{\mathrm{ell}}}_{t}\}_{t\in\mathbb{F}_l}$.

(ii) Let $\gamma \in \mathbb{F}_l^{\rtimes\pm}$. Then γ determines a *natural poly-automorphism* $\gamma_{\pm}$ of $\mathfrak{D}_{\pm}$ as follows: the automorphism $\gamma_{\pm}$ acts on $\mathbb{F}_l$ via the usual action of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathbb{F}_l$ and, for $t \in \mathbb{F}_l$, induces the $+$*-full poly-isomorphism* $\mathfrak{D}_t \xrightarrow{\sim} \mathfrak{D}_{\gamma(t)}$ whose *sign* at every $\underline{v} \in \underline{\mathbb{V}}$ is equal to the sign of γ [cf. the construction of Example 6.2, (ii)]. Thus, we obtain a natural *poly-action* of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathfrak{D}_{\pm}$. On the other hand, the isomorphism $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l^{\rtimes\pm}$ of (i) determines a natural *poly-action* of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathcal{D}^{\circledcirc\pm}$. Moreover, one verifies immediately that $\phi^{\Theta^{\mathrm{ell}}}_{\pm}$ is *equivariant* with respect to these poly-actions of $\mathbb{F}_l^{\rtimes\pm}$ on $\mathfrak{D}_{\pm}$ and $\mathcal{D}^{\circledcirc\pm}$; in particular, we obtain a *natural poly-action*

$$\mathbb{F}_l^{\rtimes\pm} \quad \curvearrowright \quad (\mathfrak{D}_{\pm}, \mathcal{D}^{\circledcirc\pm}, \phi^{\Theta^{\mathrm{ell}}}_{\pm})$$

of $\mathbb{F}_l^{\rtimes\pm}$ on the collection of data $(\mathfrak{D}_{\pm}, \mathcal{D}^{\circledcirc\pm}, \phi^{\Theta^{\mathrm{ell}}}_{\pm})$ [cf. the discussion of Example 4.3, (iv)].

Definition 6.4. In the following, we shall write $l^{\pm} \stackrel{\mathrm{def}}{=} l^{*} + 1 = (l+1)/2$.

(i) We define a *base-*$\Theta^{\pm}$*-bridge*, or $\mathcal{D}$-$\Theta^{\pm}$*-bridge*, [relative to the given initial Θ-data] to be a poly-morphism

$$^{\dagger}\mathfrak{D}_T \quad \stackrel{^{\dagger}\phi_{\pm}^{\Theta^{\pm}}}{\longrightarrow} \quad ^{\dagger}\mathfrak{D}_{\succ}$$

— where $^{\dagger}\mathfrak{D}_{\succ}$ is a *$\mathcal{D}$-prime-strip*; T is an *$\mathbb{F}_l^{\pm}$-group*; $^{\dagger}\mathfrak{D}_T = \{^{\dagger}\mathfrak{D}_t\}_{t\in T}$ is a *capsule of $\mathcal{D}$-prime-strips*, indexed by [the underlying set of] T — such that there exist isomorphisms

$$\mathfrak{D}_{\succ} \stackrel{\sim}{\to} {^{\dagger}\mathfrak{D}_{\succ}}, \quad \mathfrak{D}_{\pm} \stackrel{\sim}{\to} {^{\dagger}\mathfrak{D}_T}$$

— where we require that the bijection of index sets $\mathbb{F}_l \stackrel{\sim}{\to} T$ induced by the second isomorphism determine an *isomorphism of $\mathbb{F}_l^{\pm}$-groups* — conjugation by which maps $\phi_{\pm}^{\Theta^{\pm}} \mapsto {^{\dagger}\phi_{\pm}^{\Theta^{\pm}}}$. In this situation, we shall write

$$^{\dagger}\mathfrak{D}_{|T|}$$

for the $l^{\pm}$-capsule obtained from the l-capsule $^{\dagger}\mathfrak{D}_T$ by forming the quotient $|T|$ of the index set T of this underlying capsule by the action of $\{\pm 1\}$ and identifying the components of the capsule $^{\dagger}\mathfrak{D}_T$ indexed by the elements in the fibers of the quotient $T \twoheadrightarrow |T|$ via the constituent poly-morphisms of $^{\dagger}\phi_{\pm}^{\Theta^{\pm}} = \{^{\dagger}\phi_t^{\Theta^{\pm}}\}_{t\in\mathbb{F}_l}$ [so each consitutent $\mathcal{D}$-prime-strip of $^{\dagger}\mathfrak{D}_{|T|}$ is only well-defined up to a *positive automorphism*, but this *indeterminacy* will not affect applications of this construction — cf. Propositions 6.7; 6.8, (ii); 6.9, (i), below]. Also, we shall write

$$^{\dagger}\mathfrak{D}_{T^{*}}$$

for the l^{*}-capsule determined by the subset $T^{*} \stackrel{\mathrm{def}}{=} |T| \setminus \{0\}$ of nonzero elements of $|T|$. We define a(n) *[iso]morphism of $\mathcal{D}$-$\Theta^{\pm}$-bridges*

$$(^{\dagger}\mathfrak{D}_T \quad \stackrel{^{\dagger}\phi_{\pm}^{\Theta^{\pm}}}{\longrightarrow} \quad ^{\dagger}\mathfrak{D}_{\succ}) \quad \to \quad (^{\ddagger}\mathfrak{D}_{T'} \quad \stackrel{^{\ddagger}\phi_{\pm}^{\Theta^{\pm}}}{\longrightarrow} \quad ^{\ddagger}\mathfrak{D}_{\succ})$$

to be a pair of poly-morphisms

$$^{\dagger}\mathfrak{D}_T \stackrel{\sim}{\to} {^{\ddagger}\mathfrak{D}_{T'}}; \quad ^{\dagger}\mathfrak{D}_{\succ} \stackrel{\sim}{\to} {^{\ddagger}\mathfrak{D}_{\succ}}$$

— where $^{\dagger}\mathfrak{D}_T \stackrel{\sim}{\to} {^{\ddagger}\mathfrak{D}_{T'}}$ is a *capsule-+-full poly-isomorphism* whose induced morphism on index sets $T \stackrel{\sim}{\to} T'$ is an *isomorphism of $\mathbb{F}_l^{\pm}$-groups*; $^{\dagger}\mathfrak{D}_{\succ} \stackrel{\sim}{\to} {^{\ddagger}\mathfrak{D}_{\succ}}$ is a *+-full poly-isomorphism* — which are *compatible* with $^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$, $^{\ddagger}\phi_{\pm}^{\Theta^{\pm}}$. There is an evident notion of composition of morphisms of $\mathcal{D}$-$\Theta^{\pm}$-bridges.

(ii) We define a *base-*Θ^{ell}*-bridge* [i.e., a *"base-Θ-elliptic-bridge"*], or $\mathcal{D}$-Θ^{ell}-*bridge*, [relative to the given initial Θ-data] to be a poly-morphism

$$^{\dagger}\mathfrak{D}_T \quad \stackrel{^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}}{\longrightarrow} \quad ^{\dagger}\mathcal{D}^{\circledcirc\pm}$$

— where $^{\dagger}\mathcal{D}^{\circledcirc\pm}$ is a *category equivalent to* $\mathcal{D}^{\circledcirc\pm}$; T is an *$\mathbb{F}_l^{\pm}$-torsor*; $^{\dagger}\mathfrak{D}_T = \{^{\dagger}\mathfrak{D}_t\}_{t\in T}$ is a *capsule of $\mathcal{D}$-prime-strips*, indexed by [the underlying set of] T — such that there exist isomorphisms

$$\mathcal{D}^{\circledcirc\pm} \stackrel{\sim}{\to} {^{\dagger}\mathcal{D}^{\circledcirc\pm}}, \quad \mathfrak{D}_{\pm} \stackrel{\sim}{\to} {^{\dagger}\mathfrak{D}_T}$$

— where we require that the bijection of index sets $\mathbb{F}_l \xrightarrow{\sim} T$ induced by the second isomorphism determine an *isomorphism of* $\mathbb{F}_l^{\pm}$*-torsors* — conjugation by which maps $\phi_{\pm}^{\Theta^{\mathrm{ell}}} \mapsto {}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$. We define a(n) *[iso]morphism of* $\mathcal{D}$-Θ^{ell}*-bridges*

$$({}^{\dagger}\mathfrak{D}_T \xrightarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}) \quad \to \quad ({}^{\ddagger}\mathfrak{D}_{T'} \xrightarrow{{}^{\ddagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}} {}^{\ddagger}\mathcal{D}^{\circledcirc\pm})$$

to be a pair of poly-morphisms

$${}^{\dagger}\mathfrak{D}_T \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{T'}; \quad {}^{\dagger}\mathcal{D}^{\circledcirc\pm} \xrightarrow{\sim} {}^{\ddagger}\mathcal{D}^{\circledcirc\pm}$$

— where ${}^{\dagger}\mathfrak{D}_T \xrightarrow{\sim} {}^{\ddagger}\mathfrak{D}_{T'}$ is a *capsule-+-full poly-isomorphism* whose induced morphism on index sets $T \xrightarrow{\sim} T'$ is an *isomorphism of* $\mathbb{F}_l^{\pm}$*-torsors*; ${}^{\dagger}\mathcal{D}^{\circledcirc\pm} \to {}^{\ddagger}\mathcal{D}^{\circledcirc\pm}$ is a poly-morphism which is an $\mathrm{Aut}_{\mathrm{csp}}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})$- [or, equivalently, $\mathrm{Aut}_{\mathrm{csp}}({}^{\ddagger}\mathcal{D}^{\circledcirc\pm})$-] *orbit* of isomorphisms — which are *compatible* with ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$, ${}^{\ddagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$. There is an evident notion of composition of morphisms of $\mathcal{D}$-Θ^{ell}-bridges.

(iii) We define a *base-*$\Theta^{\pm\mathrm{ell}}$*-Hodge theater*, or $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*-Hodge theater*, [relative to the given initial Θ-data] to be a collection of data

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} = ({}^{\dagger}\mathfrak{D}_{\succ} \xleftarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}} {}^{\dagger}\mathfrak{D}_T \xrightarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

— where T is a $\mathbb{F}_l^{\pm}$*-group*; ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$ is a $\mathcal{D}$-$\Theta^{\pm}$-bridge; ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$ is a $\mathcal{D}$-Θ^{ell}-bridge [relative to the $\mathbb{F}_l^{\pm}$-torsor structure determined by the $\mathbb{F}_l^{\pm}$-group structure on T] — such that there exist isomorphisms

$$\mathfrak{D}_{\succ} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_{\succ}; \quad \mathfrak{D}_{\pm} \xrightarrow{\sim} {}^{\dagger}\mathfrak{D}_T; \quad \mathcal{D}^{\circledcirc\pm} \xrightarrow{\sim} {}^{\dagger}\mathcal{D}^{\circledcirc\pm}$$

conjugation by which maps $\phi_{\pm}^{\Theta^{\pm}} \mapsto {}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$, $\phi_{\pm}^{\Theta^{\mathrm{ell}}} \mapsto {}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$. A(n) *[iso]morphism of* $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$*-Hodge theaters* is defined to be a pair of morphisms between the respective associated $\mathcal{D}$-$\Theta^{\pm}$- and $\mathcal{D}$-Θ^{ell}-bridges that are *compatible* with one another in the sense that they induce the *same poly-isomorphism* between the respective capsules of $\mathcal{D}$-prime-strips. There is an evident notion of composition of morphisms of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters.

The following *additive* analogue of Proposition 4.7 follows immediately from the various definitions involved. Put another way, the content of Proposition 6.5 below may be thought of as a sort of *"intrinsic version"* of the constructions carried out in Examples 6.2, 6.3.

Proposition 6.5. (Transport of $\pm$-Label Classes of Cusps via Base-Bridges) *Let*

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} = ({}^{\dagger}\mathfrak{D}_{\succ} \xleftarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}} {}^{\dagger}\mathfrak{D}_T \xrightarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

be a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$**-Hodge theater** *[relative to the given initial* Θ*-data]. Then:*

(i) For each $\underline{v} \in \underline{\mathbb{V}}$*,* $t \in T$*, the* $\mathcal{D}$-Θ^{ell}**-bridge** ${}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}$ *induces a [single, well-defined!]* **bijection** *of sets of* $\pm$*-label classes of cusps*

$${}^{\dagger}\zeta_{\underline{v}_t}^{\Theta^{\mathrm{ell}}} : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

that is **compatible** *with the respective* $\mathbb{F}_l^{\pm}$**-torsor** *structures. Moreover, for* $\underline{w} \in \underline{\mathbb{V}}$, *the bijection*

$$^{\dagger}\xi^{\Theta^{\rm ell}}_{\underline{v}_t,\underline{w}_t} \stackrel{\rm def}{=} ({}^{\dagger}\zeta^{\Theta^{\rm ell}}_{\underline{w}_t})^{-1} \circ ({}^{\dagger}\zeta^{\Theta^{\rm ell}}_{\underline{v}_t}) : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\underline{w}_t})$$

is **compatible** *with the respective* $\mathbb{F}_l^{\pm}$**-group** *structures. Write*

$$\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_t)$$

for the $\mathbb{F}_l^{\pm}$*-group obtained by identifying the various* $\mathbb{F}_l^{\pm}$*-groups* $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\underline{v}_t})$, *as* $\underline{v}$ *ranges over the elements of* $\underline{\mathbb{V}}$, *via the various* $^{\dagger}\xi^{\Theta^{\rm ell}}_{\underline{v}_t,\underline{w}_t}$. *Finally, the various* $^{\dagger}\zeta^{\Theta^{\rm ell}}_{\underline{v}_t}$ *determine a [single, well-defined!]* **bijection**

$$^{\dagger}\zeta^{\Theta^{\rm ell}}_{t} : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_t) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

— which is compatible with the respective $\mathbb{F}_l^{\pm}$*-torsor structures.*

(ii) For each $\underline{v} \in \underline{\mathbb{V}}$, $t \in T$, *the* $\mathcal{D}$**-**$\Theta^{\pm}$**-bridge** $^{\dagger}\phi^{\Theta^{\rm ell}}_{\pm}$ *induces a [single, well-defined!]* **bijection** *of sets of* $\pm$*-label classes of cusps*

$$^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{v}_t} : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ,\underline{v}})$$

that is **compatible** *with the respective* $\mathbb{F}_l^{\pm}$**-group** *structures. Moreover, for* $\underline{w} \in \underline{\mathbb{V}}$, *the bijections*

$$^{\dagger}\xi^{\Theta^{\pm}}_{\succ,\underline{v},\underline{w}} \stackrel{\rm def}{=} ({}^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{w}_0}) \circ {}^{\dagger}\xi^{\Theta^{\rm ell}}_{\underline{v}_0,\underline{w}_0} \circ ({}^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{v}_0})^{-1} : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ,\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ,\underline{w}});$$
$$^{\dagger}\xi^{\Theta^{\pm}}_{\underline{v}_t,\underline{w}_t} \stackrel{\rm def}{=} ({}^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{w}_t})^{-1} \circ {}^{\dagger}\xi^{\Theta^{\pm}}_{\succ,\underline{v},\underline{w}} \circ ({}^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{v}_t}) : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\underline{v}_t}) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\underline{w}_t})$$

— where, by abuse of notation, we write "0" for the zero element of the $\mathbb{F}_l^{\pm}$*-group* $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_t)$ *— are* **compatible** *with the respective* $\mathbb{F}_l^{\pm}$**-group** *structures, and we have* $^{\dagger}\xi^{\Theta^{\pm}}_{\underline{v}_t,\underline{w}_t} = {}^{\dagger}\xi^{\Theta^{\rm ell}}_{\underline{v}_t,\underline{w}_t}$. *Write*

$$\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ})$$

for the $\mathbb{F}_l^{\pm}$*-group obtained by identifying the various* $\mathbb{F}_l^{\pm}$*-groups* $\mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ,\underline{v}})$, *as* $\underline{v}$ *ranges over the elements of* $\underline{\mathbb{V}}$, *via the various* $^{\dagger}\xi^{\Theta^{\pm}}_{\succ,\underline{v},\underline{w}}$. *Finally, for any* $t \in T$, *the various* $^{\dagger}\zeta^{\Theta^{\pm}}_{\underline{v}_t}$, $^{\dagger}\zeta^{\Theta^{\rm ell}}_{\underline{v}_t}$ *determine, respectively, a [single, well-defined!]* **bijection**

$$^{\dagger}\zeta^{\Theta^{\pm}}_{t} : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_t) \xrightarrow{\sim} \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathfrak{D}_{\succ});$$

— which is compatible with the respective $\mathbb{F}_l^{\pm}$*-group structures.*

(iii) The assignment

$$T \ni t \mapsto {}^{\dagger}\zeta^{\Theta^{\rm ell}}_{t}(0) \in \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

determines a [single, well-defined!] **bijection**

$$({}^\dagger\zeta_\pm)^{-1} : T \xrightarrow{\sim} \mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}^{\circledcirc\pm})$$

[i.e., whose inverse we denote by ${}^\dagger\zeta_\pm$] — which is **compatible** *with the respective $\mathbb{F}_l^\pm$-torsor structures. Moreover, for any $t \in T$, the composite bijection*

$$({}^\dagger\zeta_0^{\Theta^{\mathrm{ell}}})^{-1} \circ ({}^\dagger\zeta_t^{\Theta^{\mathrm{ell}}}) \circ ({}^\dagger\zeta_t^{\Theta^\pm})^{-1} \circ ({}^\dagger\zeta_0^{\Theta^\pm}) : \mathrm{LabCusp}^\pm({}^\dagger\mathfrak{D}_0) \xrightarrow{\sim} \mathrm{LabCusp}^\pm({}^\dagger\mathfrak{D}_0)$$

coincides with the automorphism of the set $\mathrm{LabCusp}^\pm({}^\dagger\mathfrak{D}_0)$ *determined, relative to the $\mathbb{F}_l^\pm$-structure on this set, by the action of $({}^\dagger\zeta_0^{\Theta^{\mathrm{ell}}})^{-1}(({}^\dagger\zeta_\pm)^{-1}(t))$.*

(iv) Let $\alpha \in \mathrm{Aut}_\pm(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})$. Then if one replaces ${}^\dagger\phi_\pm^{\Theta^{\mathrm{ell}}}$ by $\alpha \circ {}^\dagger\phi_\pm^{\Theta^{\mathrm{ell}}}$ [cf. Proposition 6.6, (iv), below], then the resulting "${}^\dagger\zeta_t^{\Theta^{\mathrm{ell}}}$" is related to the "${}^\dagger\zeta_t^{\Theta^{\mathrm{ell}}}$" determined by the original ${}^\dagger\phi_\pm^{\Theta^{\mathrm{ell}}}$ by post-composition with the image of α via the **natural bijection**

$$\mathrm{Aut}_\pm(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathrm{Aut}_\pm(\mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}^{\circledcirc\pm})) \quad (\cong \mathbb{F}_l^{\rtimes\pm})$$

determined by the tautological action of $\mathrm{Aut}_\pm(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm})$ *on the set of $\pm$-label classes of cusps* $\mathrm{LabCusp}^\pm({}^\dagger\mathcal{D}^{\circledcirc\pm})$*).*

Next, let us observe that it follows immediately from the various definitions involved [cf. the discussion of Definition 6.1; Examples 6.2, 6.3], together with the explicit description of the various *poly-automorphisms* discussed in Examples 6.2, (ii), (iii); 6.3, (ii) [cf. also the various properties discussed in Proposition 6.5], that we have the following *additive* analogue of Proposition 4.8.

Proposition 6.6. (First Properties of Base-$\Theta^\pm$-Bridges, Base-Θ^{ell}-Bridges, and Base-$\Theta^{\pm\mathrm{ell}}$-Hodge Theaters) *Relative to a fixed collection of* **initial Θ-data***:*

(i) The **set of isomorphisms** *between* **two $\mathcal{D}$-$\Theta^\pm$-bridges** *forms a* **torsor** *over the group*

$$\{\pm 1\} \times \left(\{\pm 1\}^{\underline{\mathbb{V}}}\right)$$

— where the first (respectively, second) factor corresponds to poly-automorphisms of the sort described in Example 6.2, (ii) (respectively, Example 6.2, (iii)). Moreover, the first factor may be thought of as corresponding to the induced isomorphisms of **$\mathbb{F}_l^\pm$-groups** *between the index sets of the capsules involved.*

(ii) The **set of isomorphisms** *between* **two $\mathcal{D}$-Θ^{ell}-bridges** *forms an* **$\mathbb{F}_l^{\rtimes\pm}$-torsor** *— i.e., more precisely, a torsor over a finite group that is equipped with a* **natural outer isomorphism** *to $\mathbb{F}_l^{\rtimes\pm}$. Moreover, this set of isomorphisms maps* **bijectively***, by considering the induced bijections, to the set of isomorphisms of* **$\mathbb{F}_l^\pm$-torsors** *between the index sets of the capsules involved.*

(iii) The **set of isomorphisms** *between* **two $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters** *forms a* **$\{\pm 1\}$-torsor***. Moreover, this set of isomorphisms maps* **bijectively***, by considering the induced bijections, to the set of isomorphisms of* **$\mathbb{F}_l^\pm$-groups** *between the index sets of the capsules involved.*

(iv) Given a $\mathcal{D}$-$\Theta^{\pm}$-bridge and a $\mathcal{D}$-Θ^{ell}-bridge, the set of capsule-+-full poly-isomorphisms between the respective capsules of $\mathcal{D}$-prime-strips which allow one to **glue** *the given $\mathcal{D}$-$\Theta^{\pm}$- and $\mathcal{D}$-Θ^{ell}-bridges together to form a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater forms a* **torsor** *over the group*

$$\mathbb{F}_l^{\rtimes\pm} \times \left(\{\pm 1\}^{\underline{\mathbb{V}}}\right)$$

— where the first factor corresponds to the $\mathbb{F}_l^{\rtimes\pm}$ of (ii); the subgroup $\{\pm 1\} \times \left(\{\pm 1\}^{\underline{\mathbb{V}}}\right)$ corresponds to the group of (i). Moreover, the first factor may be thought of as corresponding to the induced isomorphisms of $\mathbb{F}_l^{\pm}$**-torsors** *between the index sets of the capsules involved.*

(v) Given a $\mathcal{D}$-Θ^{ell}-bridge, there exists a [relatively simple — cf. the discussion of Example 6.2, (i)] **functorial algorithm** *for constructing, up to an* $\mathbb{F}_l^{\rtimes\pm}$**-indeterminacy** *[cf. (ii), (iv)], from the given $\mathcal{D}$-Θ^{ell}-bridge a* $\mathcal{D}$**-**$\Theta^{\pm\mathrm{ell}}$**-Hodge theater** *whose underlying $\mathcal{D}$-Θ^{ell}-bridge is the given $\mathcal{D}$-Θ^{ell}-bridge.*

Remark 6.6.1. The underlying combinatorial structure of a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater — or, essentially equivalently [cf. Definition 6.11, Corollary 6.12 below], of a $\Theta^{\pm\mathrm{ell}}$-Hodge theater — is illustrated in Fig. 6.1 below. Thus, Fig. 6.1 may be thought of as a sort of *additive* analogue of the *multiplicative* situation illustrated in Fig. 4.4. In Fig. 6.1, the "$\Uparrow$" corresponds to the associated [$\mathcal{D}$-]$\Theta^{\pm}$-*bridge*, while the "$\Downarrow$" corresponds to the associated [$\mathcal{D}$-]Θ^{ell}-*bridge*.

$$[\,-l^* < \ldots < -2 < -1 < 0 < 1 < 2 < \ldots < l^*\,]$$

$$\Uparrow$$

$$\{\pm 1\} \quad \curvearrowright \quad (\,-l^* < \ldots < -2 < -1 < 0 < 1 < 2 < \ldots < l^*\,)$$

$$\Downarrow$$

$$\begin{array}{ccccc} & \pm & \longrightarrow & \pm & \\ \nearrow & & & & \searrow \\ \pm & & & & \pm \\ \uparrow & & \mathbb{F}_l^{\rtimes\pm} \curvearrowright & & \downarrow \\ \pm & & & & \pm \\ \nwarrow & & & & \swarrow \\ & \pm & \ldots & \pm & \end{array}$$

Fig. 6.1: The combinatorial structure of a $\Theta^{\pm\mathrm{ell}}$-Hodge theater

Proposition 6.7. **(Base-Θ-Bridges Associated to Base-$\Theta^{\pm}$-Bridges)** *Relative to a fixed collection of* **initial Θ-data**, *let*

$${}^{\dagger}\mathfrak{D}_T \xrightarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}} {}^{\dagger}\mathfrak{D}_{\succ}$$

be a $\mathcal{D}$**-**$\Theta^{\pm}$**-bridge**, *as in Definition 6.4, (i). Then by replacing* ${}^{\dagger}\mathfrak{D}_T$ *by* ${}^{\dagger}\mathfrak{D}_{T^*}$ *[cf. Definition 6.4, (i)], identifying the* $\mathcal{D}$*-prime-strip* ${}^{\dagger}\mathfrak{D}_{\succ}$ *with the* $\mathcal{D}$*-prime-strip* ${}^{\dagger}\mathfrak{D}_0$ *via* ${}^{\dagger}\phi_0^{\Theta^{\pm}}$ *[cf. the discussion of Definition 6.4, (i)] to form a* $\mathcal{D}$*-prime-strip* ${}^{\dagger}\mathfrak{D}_{>}$, *replacing the various* $+$*-full poly-morphisms that occur in* ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$ *at the* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{good}}$ *by the corresponding full poly-morphisms, and replacing the various* $+$*-full poly-morphisms that occur in* ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$ *at the* $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ *by the poly-morphisms described [via* **group-theoretic algorithms**!*] in Example 4.4, (i), (ii), we obtain a* **functorial algorithm** *for constructing a [well-defined, up to a unique isomorphism!]* $\mathcal{D}$**-**Θ**-bridge**

$${}^{\dagger}\mathfrak{D}_{T^*} \xrightarrow{{}^{\dagger}\phi_*^{\Theta}} {}^{\dagger}\mathfrak{D}_{>}$$

as in Definition 4.6, (ii). Thus, the newly constructed $\mathcal{D}$*-*Θ*-bridge is related to the given* $\mathcal{D}$*-*$\Theta^{\pm}$*-bridge relative via the following correspondences:*

$${}^{\dagger}\mathfrak{D}_T|_{(T\setminus\{0\})} \mapsto {}^{\dagger}\mathfrak{D}_{T^*}; \qquad {}^{\dagger}\mathfrak{D}_0, {}^{\dagger}\mathfrak{D}_{\succ} \mapsto {}^{\dagger}\mathfrak{D}_{>}$$

— *each of which maps* **precisely two** $\mathcal{D}$*-prime-strips to a* **single** $\mathcal{D}$*-prime-strip.*

Proof. The various assertions of Proposition 6.7 follow immediately from the various definitions involved. ◯

Next, we consider *additive* analogues of Propositions 4.9, 4.11; Corollary 4.12.

Proposition 6.8. (Symmetries arising from Forgetful Functors) *Relative to a fixed collection of* **initial** Θ**-data***:*

(i) **(Base-**Θ^{ell}**-Bridges)** *The operation of associating to a* $\mathcal{D}$*-*$\Theta^{\pm\mathrm{ell}}$*-Hodge theater the underlying* $\mathcal{D}$*-*Θ^{ell}*-bridge of the* $\mathcal{D}$*-*$\Theta^{\pm\mathrm{ell}}$*-Hodge theater determines a* **natural functor**

category of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters and isomorphisms of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters	$\to$	category of $\mathcal{D}$-Θ^{ell}-bridges and isomorphisms of $\mathcal{D}$-Θ^{ell}-bridges

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \mapsto \quad ({}^{\dagger}\mathfrak{D}_T \xrightarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}} {}^{\dagger}\mathcal{D}^{\circledcirc\pm})$$

whose output data admits a $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** — *i.e., more precisely, a symmetry given by the action of a finite group that is equipped with a* **natural outer isomorphism** *to* $\mathbb{F}_l^{\rtimes\pm}$ — *which acts* **doubly transitively** *[i.e., transitively with stabilizers of order two] on the index set [i.e., "T"] of the underlying capsule of* $\mathcal{D}$*-prime-strips [i.e., "${}^{\dagger}\mathfrak{D}_T$"] of this output data.*

(ii) **(Holomorphic Capsules)** *The operation of associating to a* $\mathcal{D}$*-*$\Theta^{\pm\mathrm{ell}}$*-Hodge theater* ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ *the* $l^{\pm}$*-capsule*

$${}^{\dagger}\mathfrak{D}_{|T|}$$

associated to the underlying $\mathcal{D}$-$\Theta^{\pm}$-bridge of ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ [cf. Definition 6.4, (i)] determines a [well-defined, up to a unique isomorphism!] **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of } l^{\pm}\text{-capsules}\\ \text{of } \mathcal{D}\text{-prime-strips}\\ \text{and capsule-full poly-}\\ \text{isomorphisms of } l^{\pm}\text{-capsules}\end{array}}$$

$$ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}_{|T|}$$

whose output data admits an $\mathfrak{S}_{l^{\pm}}$**-symmetry** *[where we write $\mathfrak{S}_{l^{\pm}}$ for the symmetric group on $l^{\pm}$ letters] which acts* **transitively** *on the index set [i.e., "$|T|$"] of this output data. Thus, this functor may be thought of as an operation that consists of* **forgetting** *the* **labels** $\in |\mathbb{F}_l| = \mathbb{F}_l/\{\pm 1\}$ *determined by the $\mathbb{F}_l^{\pm}$-group structure of T [cf. Definition 6.4, (i)]. In particular, if one is only given this output data ${}^{\dagger}\mathfrak{D}_{|T|}$ up to isomorphism, then there is a total of precisely $l^{\pm}$* **possibilities** *for the element $\in |\mathbb{F}_l|$ to which a given index $|t| \in |T|$ corresponds, prior to the application of this functor.*

(iii) **(Mono-analytic Capsules)** *By composing the functor of (ii) with the* **mono-analyticization** *operation discussed in Definition 4.1, (iv), one obtains a [well-defined, up to a unique isomorphism!]* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}\text{-Hodge theaters}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of } l^{\pm}\text{-capsules}\\ \text{of } \mathcal{D}^{\vdash}\text{-prime-strips}\\ \text{and capsule-full poly-}\\ \text{isomorphisms of } l^{\pm}\text{-capsules}\end{array}}$$

$$ {}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}^{\vdash}_{|T|}$$

whose output data satisfies the same symmetry properties with respect to labels as the output data of the functor of (ii).

Proof. Assertions (i), (ii), (iii) follow immediately from the definitions [cf. also Proposition 6.6, (ii), in the case of assertion (i)]. ○

$$/^{\pm} \quad \hookrightarrow \quad /^{\pm}/^{\pm} \quad \hookrightarrow \quad /^{\pm}/^{\pm}/^{\pm} \quad \hookrightarrow \quad \dots \quad \hookrightarrow \quad /^{\pm}/^{\pm}/^{\pm} \ \dots \ /^{\pm}$$

Fig. 6.2: An $l^{\pm}$-procession of $\mathcal{D}$-prime-strips

Proposition 6.9. (Processions of Base-Prime-Strips) *Relative to a fixed collection of* **initial Θ-data***:*

(i) **(Holomorphic Processions)** *Given a $\mathcal{D}$-$\Theta^{\pm}$-bridge ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}} : {}^{\dagger}\mathfrak{D}_T \to {}^{\dagger}\mathfrak{D}_{\succ}$, with underlying capsule of $\mathcal{D}$-prime-strips ${}^{\dagger}\mathfrak{D}_T$, write* $\mathrm{Prc}({}^{\dagger}\mathfrak{D}_T)$ *for the* $l^{\pm}$**-procession of $\mathcal{D}$-prime-strips** *[cf. Fig. 6.2, where each "$/^{\pm}$" denotes a $\mathcal{D}$-prime-strip] determined by considering the ["sub"]capsules of the capsule ${}^{\dagger}\mathfrak{D}_{|T|}$ of Definition 6.4, (i), corresponding to the subsets $\mathbb{S}_1^{\pm} \subseteq \ldots \subseteq \mathbb{S}_t^{\pm} \stackrel{\mathrm{def}}{=} \{0, 1, 2, \ldots, t-1\} \subseteq \ldots \subseteq \mathbb{S}_{l^{\pm}}^{\pm} = |\mathbb{F}_l|$ [where, by abuse of notation, we use the notation for nonnegative integers to denote the images of these nonnegative integers in $|\mathbb{F}_l|$], relative to the bijection $|T| \stackrel{\sim}{\to} |\mathbb{F}_l|$ determined by the $\mathbb{F}_l^{\pm}$-group structure of T [cf. Definition 6.4, (i)]. Then the assignment ${}^{\dagger}\phi_{\pm}^{\Theta^{\pm}} \mapsto \mathrm{Prc}({}^{\dagger}\mathfrak{D}_T)$ determines a [well-defined, up to a unique isomorphism!]* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta^{\pm}\text{-bridges}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta^{\pm}\text{-bridges}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of processions}\\ \text{of } \mathcal{D}\text{-prime-strips}\\ \text{and morphisms of}\\ \text{processions}\end{array}}$$

$$ {}^{\dagger}\phi_{\pm}^{\Theta^{\pm}} \qquad \mapsto \qquad \mathrm{Prc}({}^{\dagger}\mathfrak{D}_T)$$

whose output data satisfies the following property: there are precisely $\boldsymbol{n}$ **possibilities** *for the element $\in |\mathbb{F}_l|$ to which a given index of the index set of the n-capsule that appears in the procession constituted by this output data corresponds, prior to the application of this functor. That is to say, by taking the product, over elements of $\in |\mathbb{F}_l|$, of cardinalities of "sets of possibilies", one concludes that*

> *by considering* **processions** *— i.e., the functor discussed above, possibly pre-composed with the functor ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \mapsto {}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}$ that associates to a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater its associated $\mathcal{D}$-$\Theta^{\pm}$-bridge — the indeterminacy consisting of $(l^{\pm})^{(l^{\pm})}$ possibilities that arises in Proposition 6.8, (ii), is* **reduced** *to an* **indeterminacy** *consisting of a total of* $l^{\pm}!$ **possibilities**.

(ii) **(Mono-analytic Processions)** *By composing the functor of (i) with the* **mono-analyticization** *operation discussed in Definition 4.1, (iv), one obtains a [well-defined, up to a unique isomorphism!]* **natural functor**

$$\boxed{\begin{array}{c}\text{category of}\\ \mathcal{D}\text{-}\Theta^{\pm}\text{-bridges}\\ \text{and isomorphisms of}\\ \mathcal{D}\text{-}\Theta^{\pm}\text{-bridges}\end{array}} \quad \to \quad \boxed{\begin{array}{c}\text{category of processions}\\ \text{of } \mathcal{D}^{\vdash}\text{-prime-strips}\\ \text{and morphisms of}\\ \text{processions}\end{array}}$$

$$ {}^{\dagger}\phi_{\pm}^{\Theta^{\pm}} \qquad \mapsto \qquad \mathrm{Prc}({}^{\dagger}\mathfrak{D}_T^{\vdash})$$

whose output data satisfies the same indeterminacy properties with respect to labels as the output data of the functor of (i).

(iii) The functors of (i), (ii) are **compatible**, *respectively, with the functors of Proposition 4.11, (i), (ii), relative to the functor [i.e., determined by the functorial algorithm] of Proposition 6.7, in the sense that the natural inclusions*

$$\mathbb{S}^{*}_{j} = \{1, \ldots, j\} \hookrightarrow \mathbb{S}^{\pm}_{t} = \{0, 1, \ldots, t-1\}$$

[cf. the notation of Proposition 4.11] — where $j = 1, \ldots, l^{*}$, *and* $t \stackrel{\mathrm{def}}{=} j + 1$ *— determine* **natural transformations**

$$\begin{array}{ccc} {}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} & \mapsto & \Big(\mathrm{Prc}({}^{\dagger}\mathfrak{D}_{T^{*}}) \hookrightarrow \mathrm{Prc}({}^{\dagger}\mathfrak{D}_{T})\Big) \\ {}^{\dagger}\phi^{\Theta^{\pm}}_{\pm} & \mapsto & \Big(\mathrm{Prc}({}^{\dagger}\mathfrak{D}^{\vdash}_{T^{*}}) \hookrightarrow \mathrm{Prc}({}^{\dagger}\mathfrak{D}^{\vdash}_{T})\Big) \end{array}$$

from the respective composites of the functors of Proposition 4.11, (i), (ii), with the functor [determined by the functorial algorithm] of Proposition 6.7 to the functors of (i), (ii).

Proof. Assertions (i), (ii), (iii) follow immediately from the definitions. ○

The following result is an immediate consequence of our discussion.

Corollary 6.10. **(Étale-pictures of Base-$\Theta^{\pm\mathrm{ell}}$-Hodge Theaters)** *Relative to a fixed collection of* **initial Θ-data***:*

(i) Consider the [composite] **functor**

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}_{>} \quad \mapsto \quad {}^{\dagger}\mathfrak{D}^{\vdash}_{>}$$

— from the category of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters and isomorphisms of $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters to the category of $\mathcal{D}^{\vdash}$-prime-strips and isomorphisms of $\mathcal{D}^{\vdash}$-prime-strips — obtained by assigning to the $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theater ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ the **mono-analyticization** *[cf. Definition 4.1, (iv)] ${}^{\dagger}\mathfrak{D}^{\vdash}_{>}$ of the $\mathcal{D}$-prime-strip ${}^{\dagger}\mathfrak{D}_{>}$ associated, via the functorial algorithm of Proposition 6.7, to the* **underlying $\mathcal{D}$-$\Theta^{\pm}$-bridge** *of ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$. If ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$, ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ are* **$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-Hodge theaters***, then we define the* **base-$\Theta^{\pm\mathrm{ell}}$-, or $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$-, link**

$${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \stackrel{\mathcal{D}}{\longrightarrow} \quad {}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$$

from ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ to ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$ to be the **full poly-isomorphism**

$${}^{\dagger}\mathfrak{D}^{\vdash}_{>} \stackrel{\sim}{\rightarrow} {}^{\ddagger}\mathfrak{D}^{\vdash}_{>}$$

between the $\mathcal{D}^{\vdash}$-prime-strips obtained by applying the functor discussed above to ${}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$, ${}^{\ddagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}}$.

(ii) If

$$\ldots \quad \stackrel{\mathcal{D}}{\longrightarrow} \quad {}^{(n-1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \stackrel{\mathcal{D}}{\longrightarrow} \quad {}^{n}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \stackrel{\mathcal{D}}{\longrightarrow} \quad {}^{(n+1)}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm\mathrm{ell}}} \quad \stackrel{\mathcal{D}}{\longrightarrow} \quad \ldots$$

[where $n \in \mathbb{Z}$] is an **infinite chain** *of* $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$**-linked** $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$**-Hodge theaters** *[cf. the situation discussed in Corollary 3.8], then we obtain a resulting* **chain of full poly-isomorphisms**

$$\dots \ \stackrel{\sim}{\to} \ {}^{n}\mathfrak{D}^{\vdash}_{>} \ \stackrel{\sim}{\to} \ {}^{(n+1)}\mathfrak{D}^{\vdash}_{>} \ \stackrel{\sim}{\to} \ \dots$$

[cf. the situation discussed in Remark 3.8.1, (ii)] between the $\mathcal{D}^{\vdash}$-prime-strips obtained by applying the functor of (i). That is to say, the output data of the functor of (i) forms a **constant invariant** *[cf. the discussion of Remark 3.8.1, (ii)] — i.e., a* **mono-analytic core** *[cf. the situation discussed in Remark 3.9.1] — of the above infinite chain.*

$$\begin{array}{ccccc}
 & & \boxed{/^{\pm} \hookrightarrow /^{\pm}/^{\pm} \hookrightarrow \dots} & & \\
\dots & & | & & \dots \\
\boxed{/^{\pm} \hookrightarrow /^{\pm}/^{\pm} \hookrightarrow \dots} & — & \boxed{>^{\vdash} = \{0, \succ\}^{\vdash}} & — & \boxed{/^{\pm} \hookrightarrow /^{\pm}/^{\pm} \hookrightarrow \dots} \\
\dots & & | & & \dots \\
 & & \boxed{/^{\pm} \hookrightarrow /^{\pm}/^{\pm} \hookrightarrow \dots} & &
\end{array}$$

Fig. 6.3: Étale-picture of $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge theaters

(iii) If we regard each of the $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge theaters of the chain of (ii) as a **spoke** *emanating from the mono-analytic core discussed in (ii), then we obtain a* **diagram** *— i.e., an* **étale-picture of** $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$**-Hodge-theaters** *— as in Fig. 6.3 [cf. the situation discussed in Corollary 3.9, (i)]. In Fig. 6.3, "$>^{\vdash}$" denotes the mono-analytic core, obtained [cf. (i); Proposition 6.7] by* **identifying** *the mono-analyticized $\mathcal{D}$-prime-strips of the $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge theater labeled "0" and "$\succ$"; "$/^{\pm} \hookrightarrow /^{\pm}/^{\pm} \hookrightarrow \dots$" denotes the "holomorphic" processions of Proposition 6.9, (i), together with the remaining ["holomorphic"] data of the corresponding $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge theater. In particular, the mono-analyticizations of the* **zero-labeled** *$\mathcal{D}$-prime-strips — i.e., the $\mathcal{D}$-prime-strips corresponding to the first "$/^{\pm}$" in the processions just discussed — in the various spokes are* **identified with one another**. *Put another way, the coric $\mathcal{D}^{\vdash}$-prime-strip "$>^{\vdash}$" may be thought of as being equipped with* **various distinct "holomorphic structures"** *— i.e., $\mathcal{D}$-prime-strip structures that give rise to the $\mathcal{D}^{\vdash}$-prime-strip structure — corresponding to the various spokes. Finally, [cf. the situation discussed in Corollary 3.9, (i)] this diagram satisfies the important property of admitting* **arbitrary permutation symmetries** *among the spokes [i.e., among the labels $n \in \mathbb{Z}$ of the $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-Hodge-theaters].*

(iv) The constructions of (i), (ii), (iii) are **compatible***, respectively, with the constructions of Corollary 4.12, (i), (ii), (iii), relative to the functor [i.e., determined by the functorial algorithm] of Proposition 6.7, in the evident sense [cf. the compatibility discussed in Proposition 6.9, (iii)].*

Finally, we conclude with *additive* analogues of Definition 5.5, Corollary 5.6.

Definition 6.11.

(i) We define a $\Theta^{\pm}$-*bridge* [relative to the given initial Θ-data] to be a poly-morphism

$$^{\dagger}\mathfrak{F}_T \quad \stackrel{^{\dagger}\psi^{\Theta^{\pm}}_{\pm}}{\longrightarrow} \quad ^{\dagger}\mathfrak{F}_{\succ}$$

— where $^{\dagger}\mathfrak{F}_{\succ}$ is a $\mathcal{F}$*-prime-strip*; T is an $\mathbb{F}_l^{\pm}$*-group*; $^{\dagger}\mathfrak{F}_T = \{^{\dagger}\mathfrak{F}_t\}_{t\in T}$ is a *capsule of* $\mathcal{F}$*-prime-strips*, indexed by [the underlying set of] T — that lifts a $\mathcal{D}$-$\Theta^{\pm}$-bridge $^{\dagger}\phi^{\Theta^{\pm}}_{\pm} : {^{\dagger}\mathfrak{D}_T} \to {^{\dagger}\mathfrak{D}_{\succ}}$ [cf. Corollary 5.3, (ii)]. In this situation, we shall write

$$^{\dagger}\mathfrak{F}_{|T|}$$

for the $l^{\pm}$-capsule obtained from the l-capsule $^{\dagger}\mathfrak{F}_T$ by forming the quotient $|T|$ of the index set T of this underlying capsule by the action of $\{\pm 1\}$ and identifying the components of the capsule $^{\dagger}\mathfrak{F}_T$ indexed by the elements in the fibers of the quotient $T \twoheadrightarrow |T|$ via the constituent poly-morphisms of $^{\dagger}\psi^{\Theta^{\pm}}_{\pm} = \{^{\dagger}\psi^{\Theta^{\pm}}_{t}\}_{t\in\mathbb{F}_l}$ [so each consitutent $\mathcal{F}$-prime-strip of $^{\dagger}\mathfrak{F}_{|T|}$ is only well-defined up to a *positive automorphism*, but this *indeterminacy* will not affect applications of this construction — cf. the discussion of Definition 6.4, (i)]. Also, we shall write

$$^{\dagger}\mathfrak{F}_{T^{*}}$$

for the l^{*}-capsule determined by the subset $T^{*} \stackrel{\mathrm{def}}{=} |T| \setminus \{0\}$ of nonzero elements of $|T|$. We define a(n) *[iso]morphism of* $\mathcal{F}$-$\Theta^{\pm}$*-bridges*

$$(^{\dagger}\mathfrak{F}_T \quad \stackrel{^{\dagger}\psi^{\Theta^{\pm}}_{\pm}}{\longrightarrow} \quad ^{\dagger}\mathfrak{F}_{\succ}) \quad \to \quad (^{\ddagger}\mathfrak{F}_{T'} \quad \stackrel{^{\ddagger}\psi^{\Theta^{\pm}}_{\pm}}{\longrightarrow} \quad ^{\ddagger}\mathfrak{F}_{\succ})$$

to be a pair of poly-isomorphisms

$$^{\dagger}\mathfrak{F}_T \stackrel{\sim}{\to} {^{\ddagger}\mathfrak{F}_{T'}}; \quad ^{\dagger}\mathfrak{F}_{\succ} \stackrel{\sim}{\to} {^{\ddagger}\mathfrak{F}_{\succ}}$$

that lifts a morphism between the associated $\mathcal{D}$-$\Theta^{\pm}$-bridges $^{\dagger}\phi^{\Theta^{\pm}}_{\pm}$, $^{\ddagger}\phi^{\Theta^{\pm}}_{\pm}$. There is an evident notion of composition of morphisms of $\mathcal{F}$-$\Theta^{\pm}$-bridges.

(ii) We define a Θ^{ell}-*bridge* [relative to the given initial Θ-data]

$$^{\dagger}\mathfrak{F}_T \quad \stackrel{^{\dagger}\psi^{\Theta^{\mathrm{ell}}}_{\pm}}{\longrightarrow} \quad ^{\dagger}\mathcal{D}^{\circledcirc\pm}$$

— where $^{\dagger}\mathcal{D}^{\circledcirc\pm}$ is a *category equivalent to* $\mathcal{D}^{\circledcirc\pm}$; T is an $\mathbb{F}_l^{\pm}$*-torsor*; $^{\dagger}\mathfrak{F}_T = \{^{\dagger}\mathfrak{F}_t\}_{t\in T}$ is a *capsule of* $\mathcal{F}$*-prime-strips*, indexed by [the underlying set of] T — to be a

$\mathcal{D}$-Θ^{ell}-bridge ${}^\dagger\phi_{\pm}^{\Theta^{\text{ell}}} : {}^\dagger\mathfrak{D}_T \to {}^\dagger\mathcal{D}^{\circledcirc\pm}$ — where we write ${}^\dagger\mathfrak{D}_T$ for the capsule of $\mathcal{D}$-prime-strips associated to ${}^\dagger\mathfrak{F}_T$ [cf. Remark 5.2.1, (i)]. We define a(n) *[iso]morphism of* Θ^{ell}*-bridges*

$$({}^\dagger\mathfrak{F}_T \quad \stackrel{{}^\dagger\psi_{\pm}^{\Theta^{\text{ell}}}}{\longrightarrow} \quad {}^\dagger\mathcal{D}^{\circledcirc\pm}) \quad \to \quad ({}^\ddagger\mathfrak{F}_{T'} \quad \stackrel{{}^\ddagger\psi_{\pm}^{\Theta^{\text{ell}}}}{\longrightarrow} \quad {}^\ddagger\mathcal{D}^{\circledcirc\pm})$$

to be a pair of poly-isomorphisms

$${}^\dagger\mathfrak{F}_T \stackrel{\sim}{\to} {}^\ddagger\mathfrak{F}_{T'}; \quad {}^\dagger\mathcal{D}^{\circledcirc\pm} \stackrel{\sim}{\to} {}^\ddagger\mathcal{D}^{\circledcirc\pm}$$

that determines a morphism between the associated $\mathcal{D}$-Θ^{ell}-bridges ${}^\dagger\phi_{\pm}^{\Theta^{\text{ell}}}$, ${}^\ddagger\phi_{\pm}^{\Theta^{\text{ell}}}$. There is an evident notion of composition of morphisms of $\mathcal{D}$-Θ^{ell}-bridges.

(iii) We define a $\Theta^{\pm\text{ell}}$*-Hodge theater* [relative to the given initial Θ-data] to be a collection of data

$${}^\dagger\mathcal{HT}^{\Theta^{\pm\text{ell}}} = ({}^\dagger\mathfrak{F}_{\succ} \quad \stackrel{{}^\dagger\psi_{\pm}^{\Theta^{\pm}}}{\longleftarrow} \quad {}^\dagger\mathfrak{F}_T \quad \stackrel{{}^\dagger\psi_{\pm}^{\Theta^{\text{ell}}}}{\longrightarrow} \quad {}^\dagger\mathcal{D}^{\circledcirc\pm})$$

— where the data ${}^\dagger\psi_{\pm}^{\Theta^{\pm}} : {}^\dagger\mathfrak{F}_T \to {}^\dagger\mathfrak{F}_{\succ}$ forms a $\Theta^{\pm}$*-bridge*; the data ${}^\dagger\psi^{\Theta^{\text{ell}}} : {}^\dagger\mathfrak{F}_T \to {}^\dagger\mathcal{D}^{\circledcirc\pm}$ forms a Θ^{ell}*-bridge* — such that the *associated data* $\{{}^\dagger\phi_{\pm}^{\Theta^{\pm}}, {}^\dagger\phi_{\pm}^{\Theta^{\text{ell}}}\}$ [cf. (i), (ii)] forms a $\mathcal{D}$-$\Theta^{\pm\text{ell}}$-Hodge theater. A(n) *[iso]morphism of* $\Theta^{\pm\text{ell}}$*-Hodge theaters* is defined to be a pair of morphisms between the respective associated $\Theta^{\pm}$- and Θ^{ell}-bridges that are *compatible* with one another in the sense that they induce the *same poly-isomorphism* between the respective capsules of $\mathcal{F}$-prime-strips. There is an evident notion of composition of morphisms of $\Theta^{\pm\text{ell}}$-Hodge theaters.

Corollary 6.12. (Isomorphisms of $\Theta^{\pm}$-Bridges, Θ^{ell}-Bridges, and $\Theta^{\pm\text{ell}}$-Hodge Theaters) *Relative to a fixed collection of* **initial Θ-data***:*

(i) The natural functorially induced map from the **set of isomorphisms** *between* **two $\Theta^{\pm}$-bridges** *(respectively,* **two Θ^{ell}-bridges***;* **two $\Theta^{\pm\text{ell}}$-Hodge theaters***) to the set of isomorphisms between the respective* **associated $\mathcal{D}$-$\Theta^{\pm}$-bridges** *(respectively,* **associated $\mathcal{D}$-Θ^{ell}-bridges***;* **associated $\mathcal{D}$-$\Theta^{\pm\text{ell}}$-Hodge theaters***) is* **bijective***.*

(ii) Given a $\Theta^{\pm}$-bridge and a Θ^{ell}-bridge, the set of capsule-+-full poly-isomorphisms between the respective capsules of $\mathcal{F}$-prime-strips which allow one to **glue** *the given $\Theta^{\pm}$- and Θ^{ell}-bridges together to form a $\Theta^{\pm\text{ell}}$-Hodge theater forms a* **torsor** *over the group*

$$\mathbb{F}_l^{\rtimes\pm} \times \left(\{\pm 1\}^{\underline{\mathbb{V}}}\right)$$

[cf. Proposition 6.6, (iv)]. Moreover, the first factor may be thought of as corresponding to the induced isomorphisms of $\mathbb{F}_l^{\pm}$**-torsors** *between the index sets of the capsules involved.*

Proof. Assertions (i), (ii) follow immediately from Definition 6.11; Corollary 5.3, (ii) [cf. also Proposition 6.6, (iv), in the case of assertion (ii)]. ◯

Remark 6.12.1. By applying Corollary 6.12, a similar remark to Remark 5.6.1 may be made concerning the $\Theta^{\pm}$-*bridges*, Θ^{ell}-*bridges*, and $\Theta^{\pm\mathrm{ell}}$-*Hodge theaters* studied in the present §6. We leave the routine details to the reader.

Remark 6.12.2. Relative to a fixed collection of *initial* Θ-*data*:

(i) Suppose that $({}^{\dagger}\mathfrak{F}_T \ \rightarrow \ {}^{\dagger}\mathfrak{F}_{\succ})$ is a $\Theta^{\pm}$-*bridge*; write $({}^{\dagger}\mathfrak{D}_T \ \rightarrow \ {}^{\dagger}\mathfrak{D}_{\succ})$ for the *associated* $\mathcal{D}$-$\Theta^{\pm}$-*bridge* [cf. Definition 6.11, (i)]. Then Proposition 6.7 gives a *functorial algorithm* for constructing a $\mathcal{D}$-Θ-*bridge* $({}^{\dagger}\mathfrak{D}_{T^*} \ \rightarrow \ {}^{\dagger}\mathfrak{D}_{>})$ from this $\mathcal{D}$-$\Theta^{\pm}$-*bridge* $({}^{\dagger}\mathfrak{D}_T \ \rightarrow \ {}^{\dagger}\mathfrak{D}_{\succ})$. Suppose that this $\mathcal{D}$-Θ-bridge $({}^{\dagger}\mathfrak{D}_{T^*} \ \rightarrow \ {}^{\dagger}\mathfrak{D}_{>})$ arises as the $\mathcal{D}$-Θ-bridge associated to a Θ-*bridge* $({}^{\ddagger}\mathfrak{F}_J \ \rightarrow \ {}^{\ddagger}\mathfrak{F}_{>} \ \dashrightarrow \ {}^{\ddagger}\mathcal{HT}^{\Theta})$ [so $J = T^*$ — cf. Definition 5.5, (ii)]. Then since the portion "${}^{\ddagger}\mathfrak{F}_J \rightarrow {}^{\ddagger}\mathfrak{F}_{>}$" of this Θ-bridge is *completely determined* [cf. Definition 5.5, (ii), (d)] by the associated $\mathcal{D}$-Θ-bridge, one verifies immediately that

> one may regard this portion "${}^{\ddagger}\mathfrak{F}_J \rightarrow {}^{\ddagger}\mathfrak{F}_{>}$" of the Θ-bridge as having been constructed via a *functorial algorithm* similar to the functorial algorithm of Proposition 6.7 [cf. also Definition 5.5, (ii), (d); Remark 5.3.1] from the $\Theta^{\pm}$-*bridge* $({}^{\dagger}\mathfrak{F}_T \ \rightarrow \ {}^{\dagger}\mathfrak{F}_{\succ})$.

Since, moreover, *isomorphisms between* Θ-*bridges* are in natural bijective correspondence with isomorphisms between the *associated* $\mathcal{D}$-Θ-*bridges* [cf. Corollary 5.6, (ii)], it thus follows immediately [cf. Corollary 5.3, (ii)] that *isomorphisms between* Θ-*bridges* are in natural bijective correspondence with isomorphisms between the *portions of* Θ-*bridges* [i.e., "${}^{\ddagger}\mathfrak{F}_J \rightarrow {}^{\ddagger}\mathfrak{F}_{>}$"] considered above. Thus, in summary, if $({}^{\ddagger}\mathfrak{F}_J \ \rightarrow \ {}^{\ddagger}\mathfrak{F}_{>} \ \dashrightarrow \ {}^{\ddagger}\mathcal{HT}^{\Theta})$ is a Θ-*bridge* for which the portion "${}^{\ddagger}\mathfrak{F}_J \rightarrow {}^{\ddagger}\mathfrak{F}_{>}$" is obtained via the functorial algorithm discussed above from the $\Theta^{\pm}$-*bridge* $({}^{\dagger}\mathfrak{F}_T \ \rightarrow \ {}^{\dagger}\mathfrak{F}_{\succ})$, then, for simplicity, we shall describe this state of affairs by saying that

> the **Θ-bridge** $({}^{\ddagger}\mathfrak{F}_J \ \rightarrow \ {}^{\ddagger}\mathfrak{F}_{>} \ \dashrightarrow \ {}^{\ddagger}\mathcal{HT}^{\Theta})$ is **glued to the $\Theta^{\pm}$-bridge** $({}^{\dagger}\mathfrak{F}_T \ \rightarrow \ {}^{\dagger}\mathfrak{F}_{\succ})$ via the **functorial algorithm** of Proposition 6.7.

A similar [but easier!] construction may be given for $\mathcal{D}$-Θ-*bridges* and $\mathcal{D}$-$\Theta^{\pm}$-*bridges*. We leave the routine details of giving a more explicit description [say, in the style of the statement of Proposition 6.7] of such functorial algorithms to the reader.

(ii) Now observe that

> by **gluing** a **$\Theta^{\pm\mathrm{ell}}$-Hodge theater** [cf. Definition 6.11, (iii)] to a **ΘNF-Hodge theater** [cf. Definition 5.5, (iii)] along the respective associated $\Theta^{\pm}$- *and* Θ-*bridges* via the functorial algorithm of Proposition 6.7 [cf. (i)], one obtains the notion of a
>
> **"$\Theta^{\pm\mathrm{ell}}$NF-Hodge-theater"**

— cf. Definition 6.13, (i), below. Here, we note that by Proposition 4.8, (ii); Corollary 5.6, (ii), the *gluing isomorphism* that occurs in such a gluing operation is *unique*. Then by applying Propositions 4.8, 6.6, and Corollaries 5.6, 6.12, one may verify *analogues* of these results for such $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters. In a similar vein, one may glue a $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-*Hodge theater* to a $\mathcal{D}$-ΘNF-*Hodge theater* to obtain a *"$\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$NF-Hodge theater"* [cf. Definition 6.13, (ii), below]. We leave the routine details to the reader.

Remark 6.12.3.

(i) One way to think of the notion of a ΘNF-*Hodge theater* studied in §4 is as a sort of

total space *of a local system of* $\mathbb{F}_l^{\circledast}$*-torsors*

over a *"base space"* that represents a sort of *"homotopy"* between a *number field* and a *Tate curve* [i.e., the elliptic curve under consideration at the $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$]. From this point of view, the notion of a $\Theta^{\pm \mathrm{ell}}$-*Hodge theater* studied in the present §6 may be thought of as a sort of

total space *of a local system of* $\mathbb{F}_l^{\rtimes \pm}$*-torsors*

over a similar "base space". Here, it is interesting to note that these $\mathbb{F}_l^{\circledast}$- and $\mathbb{F}_l^{\rtimes \pm}$-torsors arise, on the one hand, from the *l-torsion points* of the elliptic curve under consideration, hence may be thought of as

discrete approximations *of*
[the geometric portion of] this **elliptic curve** *over a number field*

[cf. the point of view of scheme-theoretic Hodge-Arakelov theory discussed in [HA-SurI], §1.3.4]. On the other hand, if one thinks in terms of the *tempered fundamental groups* of the Tate curves that occur at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, then these $\mathbb{F}_l^{\circledast}$- and $\mathbb{F}_l^{\rtimes \pm}$-torsors may be thought of as

finite approximations *of the copy of "$\mathbb{Z}$"*

that occurs as the *Galois group* of a well-known tempered covering of the Tate curve [cf. the discussion of [EtTh], Remark 2.16.2]. Note, moreover, that if one works with $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters* [cf. Remark 6.12.2, (ii)], then one is, in effect, working with both the **additive** and the **multiplicative** structures of this copy of $\mathbb{Z}$ — although, unlike the situation that occurs when one works with **rings**, i.e., in which the additive and multiplicative structures are **"entangled"** with one another in some sort of complicated fashion [cf. the discussion of [AbsTopIII], Remark 5.6.1], if one works with $\Theta^{\pm \mathrm{ell}}$NF-Hodge theaters, then each of the additive and multiplicative structures occurs in an *independent* fashion [i.e., in the form of $\Theta^{\pm \mathrm{ell}}$- and ΘNF-Hodge theaters], i.e., **"extracted"** *from this entanglement*.

(ii) At this point, it is useful to recall that the idea of a *distinct* [i.e., from the copy of $\mathbb{Z}$ implicit in the "base space"] *"local system-theoretic" copy of* $\mathbb{Z}$ occurring over a "base space" that represents a number field is reminiscent not only of the discussion of [EtTh], Remark 2.16.2, but also of the *Teichmüller-theoretic point of view* discussed in [AbsTopIII], §I5. That is to say, relative to the analogy with *p-adic Teichmüller theory*, the "base space" that represents a number field corresponds to a hyperbolic curve in positive characteristic, while the "local system-theoretic" copy of $\mathbb{Z}$ — which, as discussed in (i), also serves as a discrete approximation of the [geometric portion of the] elliptic curve under consideration — corresponds to a *nilpotent ordinary indigenous bundle* over the positive characteristic hyperbolic curve.

(iii) Relative to the analogy discussed in (ii) between the "local system-theoretic" copy of $\mathbb{Z}$ of (i) and the indigenous bundles that occur in p-adic Teichmüller theory, it is interesting to note that the *two combinatorial dimensions* [cf. [AbsTopIII], Remark 5.6.1] corresponding to the **additive** and **multiplicative** [i.e., "$\mathbb{F}_l^{\rtimes\pm}$-" and "$\mathbb{F}_l^{\circledast}$-"] **symmetries** of ΘNF-, $\Theta^{\pm\mathrm{ell}}$-Hodge theaters may be thought of as corresponding, respectively, to the **two real dimensions**

$$\cdot\ z \mapsto z + a, \qquad\qquad z \mapsto -\overline{z} + a;$$

$$\cdot\ z \mapsto \frac{z \cdot \cos(t) - \sin(t)}{z \cdot \sin(t) + \cos(t)}, \qquad\qquad z \mapsto \frac{\overline{z} \cdot \cos(t) + \sin(t)}{\overline{z} \cdot \sin(t) - \cos(t)}$$

— where $a, t \in \mathbb{R}$; z denotes the standard coordinate on $\mathfrak{H}$ — of transformations of the **upper half-plane** $\mathfrak{H}$, i.e., an object that is very closely related to the *canonical indigenous bundles* that occur in the classical complex uniformization theory of hyperbolic Riemann surfaces [cf. the discussion of Remark 4.3.3]. Here, it is also of interest to observe that the above **additive symmetry** of the upper half-plane is closely related to the coordinate on the upper half-plane determined by the **"classical q-parameter"**

$$q \quad \stackrel{\mathrm{def}}{=} \quad e^{2\pi i z}$$

— a situation that is reminiscent of the close relationship, in the theory of the present series of papers, between the $\mathbb{F}_l^{\rtimes\pm}$**-symmetry** and the **Kummer theory** surrounding the *Hodge-Arakelov-theoretic evaluation of the* **theta function** *on the* **l-torsion points** at bad primes [cf. Remark 6.12.6, (ii); the theory of [IUTchII]]. Moreover, the *fixed* basepoint "$\underline{\mathbb{V}}^{\pm}$" [cf. Definition 6.1, (v)] with respect to which one considers l-torsion points in the context of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry is reminiscent of the fact that the above additive symmetries of the upper half-plane *fix* the *cusp at infinity.* Indeed, taken as a whole, the geometry and coordinate naturally associated to this additive symmetry of the upper half-plane may be thought of, at the level of **"combinatorial prototypes"**, as the geometric apparatus associated to a **cusp** [i.e., as opposed to a node — cf. the discussion of [NodNon], Introduction]. By contrast, the **"toral" multiplicative symmetry** of the upper half-plane recalled above is closely related to the coordinate on the upper half-plane that determines a biholomorphic isomorphism with the **unit disc**

$$w \quad \stackrel{\mathrm{def}}{=} \quad \tfrac{z-i}{z+i}$$

— a situation that is reminiscent of the close relationship, in the theory of the present series of papers, between the $\mathbb{F}_l^{\circledast}$**-symmetry** and the **Kummer theory** surrounding the **number field** F_{mod} [cf. Remark 6.12.6, (iii); the theory of §5 of the present paper]. Moreover, the *action* of $\mathbb{F}_l^{\circledast}$ on the "collection of basepoints for the l-torsion points" $\underline{\mathbb{V}}^{\mathrm{Bor}} = \mathbb{F}_l^{\circledast} \cdot \underline{\mathbb{V}}^{\pm\mathrm{un}}$ [cf. Example 4.3, (i)] in the context of the $\mathbb{F}_l^{\circledast}$-symmetry is reminiscent of the fact that the multiplicative symmetries of the upper half-plane recalled above *act transitively* on the *entire boundary of the upper half-plane*. That is to say, taken as a whole, the geometry and coordinate naturally associated to this multiplicative symmetry of the upper half-plane may be thought of, at the level of **"combinatorial prototypes"**, as the geometric apparatus associated to a **node**, i.e., of the sort that occurs in the reduction modulo p of a *Hecke correspondence* [cf. the discussion of [IUTchII], Remark 4.11.4, (iii), (c); [NodNon], Introduction]. Finally, we note that, just as in the case of the $\mathbb{F}_l^{\rtimes\pm}$-, $\mathbb{F}_l^{\circledast}$-symmetries discussed in the present paper, the only "coric" symmetries, i.e., symmetries common to *both* the additive and multiplicative symmetries of the upper half-plane recalled above, are the symmetries "$\{\pm 1\}$" [i.e., the symmetries $z \mapsto z, -\overline{z}$ in the case of the upper half-plane]. The observations of the above discussion are summarized in Fig. 6.4 below.

Remark 6.12.4.

(i) Just as in the case of the $\mathbb{F}_l^{\circledast}$-*symmetry* of Proposition 4.9, (i), the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* of Proposition 6.8, (i), will eventually be applied, in the theory of the present series of papers [cf. theory of [IUTchII], [IUTchIII]], to establish an

explicit **network of comparison isomorphisms**

relating various objects — such as **log-volumes** — associated to the non-labeled prime-strips that are permuted by this symmetry [cf. the discussion of Remark 4.9.1, (i)]. Moreover, just as in the case of the $\mathbb{F}_l^{\circledast}$-symmetry studied in §4 [cf. the discussion of Remark 4.9.2], one important property of this "network of comparison isomorphisms" is that it operates *without* **"label crushing"** [cf. Remark 4.9.2, (i)] — i.e., without disturbing the **bijective** *relationship* between the set of indices of the symmetrized collection of prime-strips and the set of labels $\in T \xrightarrow{\sim} \mathbb{F}_l$ under consideration. Finally, just as in the situation studied in §4,

> *this crucial* **synchronization of labels** *is essentially a consequence of the* **single connected component**

— or, at a more abstract level, the **single basepoint** — of the global object [i.e., "${}^{\dagger}\mathcal{D}^{\circledcirc\pm}$" in the present §6; "${}^{\dagger}\mathcal{D}^{\circledcirc}$" in §4] that appears in the [$\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$- or $\mathcal{D}$-ΘNF-] Hodge theater under consideration [cf. Remark 4.9.2, (ii)].

(ii) At a more concrete level, the "synchronization of labels" discussed in (i) is realized by means of the *crucial bijections*

$$ {}^{\dagger}\zeta_{\circledast} : \mathrm{LabCusp}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} J; \qquad {}^{\dagger}\zeta_{\pm} : \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} T $$

of Propositions 4.7, (iii); 6.5, (iii). Here, we pause to observe that it is precisely the existence of these

> bijections relating **index sets** *of capsules of* $\mathcal{D}$*-prime-strips* to *sets of* **global [±-]label classes** *of cusps*

	Classical upper half-plane	$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theaters in inter-universal Teichmüller theory*
Additive symmetry	$z \mapsto z + a$, $z \mapsto -\overline{z} + a \quad (a \in \mathbb{R})$	$\mathbb{F}_l^{\rtimes \pm}$- **symmetry**
"Functions" assoc'd to *add. symm.*	$q \stackrel{\mathrm{def}}{=} e^{2\pi i z}$	**theta fn.** evaluated at **l-tors.** [cf. I, 6.12.6, (ii)]
Basepoint assoc'd to *add. symm.*	*single* cusp at infinity	$\underline{\mathbb{V}}^{\pm}$ [cf. I, 6.1, (v)]
Combinatorial prototype assoc'd to *add. symm.*	**cusp**	**cusp**
Multiplicative symmetry	$z \mapsto \frac{z \cdot \cos(t) - \sin(t)}{z \cdot \sin(t) + \cos(t)}$, $z \mapsto \frac{\overline{z} \cdot \cos(t) + \sin(t)}{\overline{z} \cdot \sin(t) - \cos(t)} \quad (t \in \mathbb{R})$	$\mathbb{F}_l^{\circledast}$- **symmetry**
"Functions" assoc'd to *mult. symm.*	$w \stackrel{\mathrm{def}}{=} \frac{z-i}{z+i}$	elements of the **number field** F_{mod} [cf. I, 6.12.6, (iii)]
Basepoints assoc'd to *mult. symm.*	$\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$, $\begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix}$ $\curvearrowright$ {*entire* boundary of $\mathfrak{H}$ }	$\mathbb{F}_l^{\circledast} \curvearrowright \underline{\mathbb{V}}^{\mathrm{Bor}} = \mathbb{F}_l^{\circledast} \cdot \underline{\mathbb{V}}^{\pm \mathrm{un}}$ [cf. I, 4.3, (i)]
Combinatorial prototype assoc'd to *mult. symm.*	**nodes** of mod p Hecke correspondence [cf. II, 4.11.4, (iii), (c)]	**nodes** of mod p Hecke correspondence [cf. II, 4.11.4, (iii), (c)]
Coric symmetries	$z \mapsto z, -\overline{z}$	$\{\pm 1\}$

Fig. 6.4: Comparison of $\mathbb{F}_l^{\rtimes \pm}$-, $\mathbb{F}_l^{\circledast}$-symmetries with the geometry of the upper half-plane

that distinguishes the finer **"combinatorially holomorphic"** [cf. Remarks 4.9.1, (ii); 4.9.2, (iv)] $\mathbb{F}_l^{\circledast}$- and $\mathbb{F}_l^{\rtimes\pm}$-**symmetries** of Propositions 4.9.1, (i); 6.8, (i), from the coarser **"combinatorially real analytic"** [cf. Remarks 4.9.1, (ii); 4.9.2, (iv)] $\mathfrak{S}_{l^{\circledast}}$- and $\mathfrak{S}_{l^{\pm}}$-**symmetries** of Propositions 4.9, (ii), (iii); 6.8, (ii), (iii) — i.e., which do *not* admit a *compatible* bijection between the index sets of the capsules involved and some sort of *set of [±-]label classes of cusps* [cf. the discussion of Remark 4.9.2, (i)]. This relationship with a set of [±-]label classes of cusps will play a *crucial role* in the theory of the *Hodge-Arakelov-theoretic evaluation of the étale theta function* that will be developed in [IUTchII].

(iii) On the other hand, one significant feature of the additive theory of the present §6 which does not appear in the multiplicative theory of §4 is the phenomenon of **"global ±-synchronization"** — i.e., at a more concrete level, the various isomorphisms "${}^{\dagger}\xi$" that appear in Proposition 6.5, (i), (ii) — between the ±-indeterminacies that occur at the various $\underline{v} \in \underline{\mathbb{V}}$. Note that this global ±-synchronization is a **necessary "pre-condition"** for the **additive portion** [i.e., corresponding to $\mathbb{F}_l \subseteq \mathbb{F}_l^{\rtimes\pm}$] of the $\mathbb{F}_l^{\rtimes\pm}$-*symmetry* of Proposition 6.8, (i). This "additive portion" of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry plays the *crucial* role of allowing one to relate the *zero* and *nonzero* elements of $\mathbb{F}_l$ [cf. the discussion of Remark 6.12.5 below].

(iv) One important property of both the "${}^{\dagger}\zeta$'s" discussed in (ii) and the "${}^{\dagger}\xi$'s" discussed in (iii) is that they are constructed by means of **functorial algorithms** from the *intrinsic structure* of a $\mathcal{D}$-$\Theta^{\pm\mathrm{ell}}$- or $\mathcal{D}$-ΘNF-Hodge theater [cf. Propositions 4.7, (iii); 6.5, (i), (ii), (iii)] — i.e., not by means of comparison with some **fixed reference model** [cf. the discussion of [AbsTopIII], §I4], such as the objects constructed in Examples 4.3, 4.4, 4.5, 6.2, 6.3. This property will be of *crucial importance* when, in the theory of [IUTchIII], we combine the theory developed in the present series of papers with the theory of *log-shells* developed in [AbsTopIII].

Remark 6.12.5.

(i) One fundamental difference between the $\mathbb{F}_l^{\circledast}$-symmetry of §4 and the $\mathbb{F}_l^{\rtimes\pm}$-symmetry of the present §6 lies in the *inclusion of the zero element* $\in \mathbb{F}_l$ in the symmetry under consideration. This inclusion of the zero element $\in \mathbb{F}_l$ means, in particular, that the resulting *network of comparison isomorphisms* [cf. Remark 6.12.4, (i)]

> allows one to relate the **"zero-labeled"** *prime-strip* to the various **"nonzero-labeled"** *prime-strips*, i.e., the prime-strips labeled by nonzero elements $\in \mathbb{F}_l$ [or, essentially equivalently, $\in \mathbb{F}_l^{\circledast}$].

Moreover, as reviewed in Remark 6.12.4, (ii), the $\mathbb{F}_l^{\rtimes\pm}$-symmetry allows one to relate the zero-labeled and non-zero-labeled prime-strips to one another in a *"combinatorially holomorphic"* fashion, i.e., in a fashion that is compatible with the various natural bijections [i.e., "${}^{\dagger}\zeta$"] with various *sets of global ±-label classes of cusps*. Here, it is useful to recall that *evaluation at* [torsion points closely related to] *the zero-labeled cusps* [cf. the discussion of "evaluation points" in Example 4.4, (i)] plays an important role in the theory of *normalization of the étale theta function*

— cf. the theory of étale theta functions *"of standard type"*, as discussed in [EtTh], Theorem 1.10; the theory to be developed in [IUTchII].

(ii) Whereas the $\mathbb{F}_l^{\rtimes\pm}$-symmetry of the theory of the present §6 has the *advantage* that it allows one to relate zero-labeled and non-zero-labeled prime-strips, it has the [tautological!] *disadvantage* that it does not allow one to **"insulate"** *the non-zero-labeled prime-strips from* **confusion** *with the zero-labeled prime-strip.* This issue will be of substantial importance in the theory of **Gaussian Frobenioids** [to be developed in [IUTchII]], i.e., Frobenioids that, roughly speaking, arise from the *theta values*

$$\{ \underline{\underline{q}}_{\underline{v}}^{\underline{\underline{j}}^2} \}_{\underline{\underline{j}}}$$

[cf. the discussion of Example 4.4, (i)] *at the non-zero-labeled evaluation points.* Moreover, ultimately, in [IUTchII], [IUTchIII], we shall relate these Gaussian Frobenioids to various *global arithmetic line bundles* on the number field F. This will require the use of both the *additive* and the *multiplicative* structures on the number field; in particular, it will require the use of the theory developed in §5.

(iii) By contrast, since, in the theory of the present series of papers, we shall not be interested in analogues of the Gaussian Frobenioids that involve the zero-labeled evaluation points, we shall not require an "additive analogue" of the portion [cf. Example 5.1] of the theory developed in §5 concerning global Frobenioids.

Remark 6.12.6.

(i) Another fundamental difference between the $\mathbb{F}_l^{\circledast}$-symmetry of §4 and the $\mathbb{F}_l^{\rtimes\pm}$-symmetry of the present §6 lies in the **geometric** *nature of the "single basepoint"* [cf. the discussion of Remark 6.12.4] that underlies the $\mathbb{F}_l^{\rtimes\pm}$-symmetry. That is to say, the various *labels* $\in T \xrightarrow{\sim} \mathbb{F}_l$ that appear in a [$\mathcal{D}$-]$\Theta^{\pm\mathrm{ell}}$-Hodge theater correspond — *throughout* the various portions [e.g., bridges] of the [$\mathcal{D}$-]$\Theta^{\pm\mathrm{ell}}$-Hodge theater — to collections of cusps in a **single copy** [i.e., connected component] of "$\mathcal{D}_{\underline{v}}$" at each $\underline{v} \in \underline{\mathbb{V}}$; these collections of cusps are *permuted* by the $\mathbb{F}_l^{\rtimes\pm}$-symmetry of the [$\mathcal{D}$-]$\Theta^{\mathrm{ell}}$-bridge [cf. Proposition 6.8, (i)] *without permuting* [cf. the discussion of Definition 6.1, (v)] the $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})$-orbit of the poly-morphism $\phi_{\underline{v}_0}^{\Theta^{\mathrm{ell}}} : \mathcal{D}_{\underline{v}_0} \to \mathcal{D}^{\circledcirc\pm}$ [cf. the notation of Example 6.3, (i)]. This contrasts sharply with the **arithmetic** *nature of the "single basepoint"* [cf. the discussion of Remark 6.12.4] that underlies the $\mathbb{F}_l^{\circledast}$-symmetry of §4 — in which the $\mathbb{F}_l^{\circledast}$-symmetry [cf. Proposition 4.9, (i)] *permutes* the various $\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K)$-orbits of the poly-morphism $\phi_{\underline{v}_1}^{\mathrm{NF}} : \mathcal{D}_{\underline{v}_1} \to \mathcal{D}^{\circledcirc}$ [cf. the notation of Example 4.3, (iv)].

(ii) The **geometric** *nature of the "single basepoint"* of the $\mathbb{F}_l^{\rtimes\pm}$-symmetry of a [$\mathcal{D}$-]$\Theta^{\pm\mathrm{ell}}$-Hodge theater [cf. (i)] is more suited to the theory of the

Hodge-Arakelov-theoretic evaluation *of the* **étale theta function**

to be developed in [IUTchII], in which the *existence of a "single basepoint"* corresponding to a single connected component of "$\mathcal{D}_{\underline{v}}$" for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ plays a *central role.*

(iii) By contrast, the **arithmetic** *nature of the "single basepoint"* of the $\mathbb{F}_l^{\circledast}$-symmetry of a [$\mathcal{D}$-]$\Theta$NF-Hodge theater [cf. (i)] is more suited to the

$$\textbf{explicit construction } \textit{of the } \textbf{number field } F_{\mathrm{mod}} \textit{ [cf. Example 5.1]}$$

— i.e., to the construction of an object which is invariant with respect to the $\mathrm{Aut}(\underline{C}_K)/\mathrm{Aut}_{\underline{\epsilon}}(\underline{C}_K) \xrightarrow{\sim} \mathbb{F}_l^{\circledast}$-symmetries that appear in the discussion of Example 4.3, (iv). That is to say, if one attempts to carry out a similar construction to the construction of Example 5.1 with respect to the copy of $\mathcal{D}^{\circledcirc\pm}$ that appears in a [$\mathcal{D}$-]Θ^{ell}-bridge, then one must *sacrifice the crucial* **ridigity** *with respect to* $\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm})/\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathbb{F}_l^{\circledast}$ that arises from the structure [i.e., definition] of a [$\mathcal{D}$-]Θ^{ell}-bridge. Moreover, if one sacrifices this $\mathbb{F}_l^{\times}$-rigidity, then one no longer has a situation in which the symmetry under consideration is defined relative to a *single copy of "$\mathcal{D}_{\underline{v}}$"* at each $\underline{v} \in \underline{\mathbb{V}}$, i.e., defined with respect to a *"single geometric basepoint"*. In particular, once one sacrifices this $\mathbb{F}_l^{\times}$-rigidity, the resulting symmetries are *no longer compatible* with the theory of the *Hodge-Arakelov-theoretic evaluation of the étale theta function* to be developed in [IUTchII] [cf. (ii)].

(iv) One way to understand the difference discussed in (iii) between the *global portions* [i.e., the portions involving copies of $\mathcal{D}^{\circledcirc}$, $\mathcal{D}^{\circledcirc\pm}$] of a [$\mathcal{D}$-]$\Theta$NF-Hodge theater and a [$\mathcal{D}$-]$\Theta^{\pm\mathrm{ell}}$-Hodge theater is as a reflection of the fact that whereas the *Borel subgroup*

$$\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \quad \subseteq \quad GL_2(\mathbb{F}_l)$$

is *normally terminal* in $GL_2(\mathbb{F}_l)$ [cf. the discussion of Example 4.3], the *"semi-unipotent" subgroup*

$$\left\{\begin{pmatrix} * & * \\ 0 & \pm 1 \end{pmatrix}\right\} \quad \subseteq \quad GL_2(\mathbb{F}_l)$$

[which corresponds to the subgroup $\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) \subseteq \mathrm{Aut}(\mathcal{D}^{\circledcirc\pm})$ — cf. the discussion of Definition 6.1, (v)] *fails to be normally terminal* in $GL_2(\mathbb{F}_l)$.

(v) In summary, taken as a whole, a [$\mathcal{D}$-]$\Theta^{\pm\mathrm{ell}}$NF-Hodge theater [cf. Remark 6.12.2, (ii)] may be thought of as a sort of

"intricate relay between geometric and arithmetic basepoints"

that allows one to carry out, in a consistent fashion, both

(a) the theory of the *Hodge-Arakelov-theoretic evaluation of the étale theta function* to be developed in [IUTchII] [cf. (ii)], and

(b) the *explicit construction of the number field* F_{mod} in Example 5.1 [cf. (iii)].

Moreover, if one thinks of $\mathbb{F}_l$ as a *finite approximation of* $\mathbb{Z}$ [cf. Remark 6.12.3], then this intricate relay between geometric and arithmetic — or, alternatively, $\mathbb{F}_l^{\rtimes\pm}$ [i.e., *additive!*]- and $\mathbb{F}_l^{\circledast}$ [i.e., *multiplicative!*]- basepoints — may be thought of as a sort of

global combinatorial resolution of the **two combinatorial dimensions** — i.e., **additive** and **multiplicative** [cf. [AbsTopIII], Remark 5.6.1] — of the **ring** $\mathbb{Z}$.

Finally, we observe in passing that — from a *computational point of view* [cf. the theory of [IUTchIV]!] — it is especially natural to regard $\mathbb{F}_l$ as a "good approximation" of $\mathbb{Z}$ when l is **"sufficiently large"**, as is indeed the case in the situations discussed in [GenEll], §4 [cf. also Remark 3.1.2, (iv)].

Definition 6.13.

(i) We define a $\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* [relative to the given initial Θ-data]

$$^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

to be a *triple*, consisting of the following data: (a) a $\Theta^{\pm \mathrm{ell}}$-*Hodge theater* $^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}}$ [cf. Definition 6.11, (iii)]; (b) a ΘNF-*Hodge theater* $^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$ [cf. Definition 5.5, (iii)]; (c) the [necessarily unique!] *gluing isomorphism* between $^{\dagger}\mathcal{HT}^{\Theta^{\pm \mathrm{ell}}}$ and $^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}}$ [cf. the discussion of Remark 6.12.2, (i), (ii)].

(ii) We define a $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$*NF-Hodge theater* [relative to the given initial Θ-data]

$$^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}\mathrm{NF}}$$

to be a *triple*, consisting of the following data: (a) a $\mathcal{D}$-$\Theta^{\pm \mathrm{ell}}$-*Hodge theater* $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}}$ [cf. Definition 6.4, (iii)]; (b) a $\mathcal{D}$-ΘNF-*Hodge theater* $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ [cf. Definition 4.6, (iii)]; (c) the [necessarily unique!] *gluing isomorphism* between $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm \mathrm{ell}}}$ and $^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}}$ [cf. the discussion of Remark 6.12.2, (i), (ii)].

Bibliography

[André] Y. André, On a Geometric Description of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a p-adic Avatar of $\widehat{GT}$, *Duke Math. J.* **119** (2003), pp. 1-39.

[Asada] M. Asada, The faithfulness of the monodromy representations associated with certain families of algebraic curves, *J. Pure Appl. Algebra* **159** (2001), pp. 123-147.

[Falt] G. Faltings, Endlichkeitssätze für Abelschen Varietäten über Zahlkörpern, *Invent. Math.* **73** (1983), pp. 349-366.

[FRS] B. Fine, G. Rosenberger, and M. Stille, Conjugacy pinched and cyclically pinched one-relator groups, *Rev. Mat. Univ. Complut. Madrid* **10** (1997), pp. 207-227.

[Groth] A. Grothendieck, *letter to G. Faltings* (June 1983) in Lochak, L. Schneps, *Geometric Galois Actions; 1. Around Grothendieck's Esquisse d'un Programme, London Math. Soc. Lect. Note Ser.* **242**, Cambridge Univ. Press (1997).

[NodNon] Y. Hoshi, S. Mochizuki, On the Combinatorial Anabelian Geometry of Nodally Nondegenerate Outer Representations, *Hiroshima Math. J.* **41** (2011), pp. 275-342.

[JP] K. Joshi and C. Pauly, *Hitchin-Mochizuki morphism, opers, and Frobenius-destabilized vector bundles over curves*, preprint.

[Kim] M. Kim, The motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel, *Invent. Math.* **161** (2005), pp. 629-656.

[Lang] S. Lang, *Algebraic number theory*, Addison-Wesley Publishing Co. (1970).

[Lehto] O. Lehto, *Univalent Functions and Teichmüller Spaces, Graduate Texts in Mathematics* **109**, Springer, 1987.

[PrfGC] S. Mochizuki, The Profinite Grothendieck Conjecture for Closed Hyperbolic Curves over Number Fields, *J. Math. Sci. Univ. Tokyo* **3** (1996), pp. 571-627.

[pOrd] S. Mochizuki, A Theory of Ordinary p-adic Curves, *Publ. Res. Inst. Math. Sci.* **32** (1996), pp. 957-1151.

[pTeich] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, AMS/IP Studies in Advanced Mathematics **11**, American Mathematical Society/International Press (1999).

[pGC] S. Mochizuki, The Local Pro-p Anabelian Geometry of Curves, *Invent. Math.* **138** (1999), pp. 319-423.

[HASurI] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves I, *Arithmetic Fundamental Groups and Noncommutative Algebra, Proceedings of Symposia in Pure Mathematics* **70**, American Mathematical Society (2002), pp. 533-569.

[HASurII] S. Mochizuki, A Survey of the Hodge-Arakelov Theory of Elliptic Curves II, *Algebraic Geometry 2000, Azumino, Adv. Stud. Pure Math.* **36**, Math. Soc. Japan (2002), pp. 81-114.

[AbsAnab] S. Mochizuki, The Absolute Anabelian Geometry of Hyperbolic Curves, *Galois Theory and Modular Forms*, Kluwer Academic Publishers (2004), pp. 77-122.

[CanLift] S. Mochizuki, The Absolute Anabelian Geometry of Canonical Curves, *Kazuya Kato's fiftieth birthday, Doc. Math. 2003, Extra Vol.*, pp. 609-640.

[SemiAnbd] S. Mochizuki, Semi-graphs of Anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), pp. 221-322.

[CombGC] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math. J.* **59** (2007), pp. 455-479.

[Cusp] S. Mochizuki, Absolute anabelian cuspidalizations of proper hyperbolic curves, *J. Math. Kyoto Univ.* **47** (2007), pp. 451-539.

[FrdI] S. Mochizuki, The Geometry of Frobenioids I: The General Theory, *Kyushu J. Math.* **62** (2008), pp. 293-400.

[FrdII] S. Mochizuki, The Geometry of Frobenioids II: Poly-Frobenioids, *Kyushu J. Math.* **62** (2008), pp. 401-460.

[EtTh] S. Mochizuki, The Étale Theta Function and its Frobenioid-theoretic Manifestations, *Publ. Res. Inst. Math. Sci.* **45** (2009), pp. 227-349.

[AbsTopI] S. Mochizuki, *Topics in Absolute Anabelian Geometry I: Generalities*, RIMS Preprint **1624** (March 2008).

[AbsTopII] S. Mochizuki, *Topics in Absolute Anabelian Geometry II: Decomposition Groups*, RIMS Preprint **1625** (March 2008).

[AbsTopIII] S. Mochizuki, *Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms*, RIMS Preprint **1626** (March 2008).

[GenEll] S. Mochizuki, Arithmetic Elliptic Curves in General Position, *Math. J. Okayama Univ.* **52** (2010), pp. 1-28.

[CombCusp] S. Mochizuki, On the Combinatorial Cuspidalization of Hyperbolic Curves, *Osaka J. Math.* **47** (2010), pp. 651-715.

[IUTchII] S. Mochizuki, *Inter-universal Teichmüller Theory II: Hodge-Arakelov-theoretic Evaluation*, preprint.

[IUTchIII] S. Mochizuki, *Inter-universal Teichmüller Theory III: Canonical Splittings of the Log-theta-lattice*, preprint.

[IUTchIV] S. Mochizuki, *Inter-universal Teichmüller Theory IV: Log-volume Computations and Set-theoretic Foundations*, preprint.

[MNT] S. Mochizuki, H. Nakamura, A. Tamagawa, The Grothendieck conjecture on the fundamental groups of algebraic curves, *Sugaku Expositions* **14** (2001), pp. 31-53.

[Config] S. Mochizuki, A. Tamagawa, The algebraic and anabelian geometry of configuration spaces, *Hokkaido Math. J.* **37** (2008), pp. 75-131.

[NSW] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, Grundlehren *der Mathematischen Wissenschaften* **323**, Springer-Verlag (2000).

[RZ] Ribes and Zaleskii, *Profinite Groups*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **3**, Springer-Verlag (2000).

[Stb1] P. F. Stebe, A residual property of certain groups, *Proc. Amer. Math. Soc.* **26** (1970), pp. 37-42.

[Stb2] P. F. Stebe, Conjugacy separability of certain Fuchsian groups, *Trans. Amer. Math. Soc.* **163** (1972), pp. 173-188.

[Stl] J. Stillwell, *Classical topology and combinatorial group theory. Second edition*, *Graduate Texts in Mathematics* **72**, Springer-Verlag (1993).

[Tama1] A. Tamagawa, The Grothendieck Conjecture for Affine Curves, *Compositio Math.* **109** (1997), pp. 135-194.

[Tama2] A. Tamagawa, Resolution of nonsingularities of families of curves, *Publ. Res. Inst. Math. Sci.* **40** (2004), pp. 1291-1336.

[Wiles] A. Wiles, Modular elliptic curves and Fermat's last theorem, *Ann. of Math.* **141** (1995), pp. 443-551.