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GALOIS ACTION ON MAPPING CLASS GROUPS

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Abstract. Let l be a prime number. In this present paper, we study the outer Galois action on the profinite and the relative pro-l completions of mapping class groups of pointed orientable topological surfaces. In the profinite case, we prove that the outer Galois action is faithful. In the pro-l case, we prove that the kernel of the outer Galois action has certain stability properties with respect to the genus and the number of punctures.

1. Introduction

Let k be a (commutative) field of characteristic zero, X a smooth geometrically connected curve over k, and (g, n) a pair of nonnegative integers such that $2g - 2 + n > 0$ (hyperbolicity). We call X a (g, n)-curve if there exists a proper smooth genus g curve C over k and a closed subscheme $D \subseteq C$ such that $X = C \setminus D$ and the composite $D \hookrightarrow C \to \operatorname{Spec} k$ is a finite étale covering over $\operatorname{Spec} k$ of degree n. Let $\overline{k}$ be an algebraic closure of k. For a (g, n)-curve X, by SGA1 [1], we have a short exact sequence

$$1 \longrightarrow \pi_1(X \otimes_k \overline{k}) \longrightarrow \pi_1(X) \longrightarrow G_k \longrightarrow 1$$

where π_1 denotes the algebraic fundamental groups and $G_k := \operatorname{Gal}(\overline{k}/k)$ is the absolute Galois group of k. Let $\mathit{\Pi}_{g,n}$ denote the profinite completion of the fundamental group $\pi_1(g, n)$ of a compact Riemann surface of genus g with n points punctured. By the comparison theorem, $\pi_1(X \otimes_k \overline{k})$ is isomorphic to $\mathit{\Pi}_{g,n}$. Since $\pi_1(X)$ acts on $\pi_1(X \otimes_k \overline{k})$ by conjugation in the above short exact sequence, $\pi_1(X)$ also acts on $\mathit{\Pi}_{g,n}$. This gives the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathit{\Pi}_{g,n} & \longrightarrow & \pi_1(X) & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \operatorname{Inn}(\mathit{\Pi}_{g,n}) & \longrightarrow & \operatorname{Aut}(\mathit{\Pi}_{g,n}) & \longrightarrow & \operatorname{Out}(\mathit{\Pi}_{g,n}) & \longrightarrow & 1,
\end{array}$$

where Aut (respectively Inn) denotes the continuous automorphism group (respectively the inner automorphism group) of $\mathit{\Pi}_{g,n}$, and Out denotes the quotient, so that the horizontal sequences are both exact. The right vertical map gives the outer Galois representation

$$\rho_X : G_k \longrightarrow \operatorname{Out}(\mathit{\Pi}_{g,n}).$$

Belyĭ proved that ρ_X is injective when $X = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ and k is a number field (Corollary to Theorem 4, [6]). Voevodskiĭ proved the injectivity of ρ_X when the genus of X is one and k is a number field, and suggested a conjecture that the ρ_X is

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injective when X is an affine hyperbolic curve and k is a number field ([34]). This conjecture was solved by Matsumoto ([20]). Moreover, the proper case was proved by Hoshi and Mochizuki ([14]). Therefore, we have the following theorem:

Theorem 1.1. *The outer Galois representation ρ_X is injective when X is a hyperbolic curve and k is a number field.*

Grothendieck considered that any hyperbolic curve over a number field is anabelian, i.e., the geometry of any hyperbolic curve X over a number field is determined by ρ_X (the Grothendieck conjecture for algebraic curves, [12]). This conjecture was proved by Mochizuki ([22, 23]). The above theorem can be regarded as an evidence that ρ_X has high complexity when k is a number field.

On the other hand, Grothendieck considered that the moduli space of hyperbolic curves is also anabelian ([12]). Therefore, it is a natural problem that we consider Voevodskiĭ's conjecture in the case when X is the moduli space of hyperbolic curves. Let $\mathcal{M}_{g,n}$ be the moduli stack over k of smooth geometrically connected proper curves of genus g with n (ordered) marked points ([9, 18]). It is known that $\pi_1(\mathcal{M}_{g,n} \otimes \overline{k})$ is isomorphic to the profinite completion $\mathit{\Gamma}_{g,n}$ of the mapping class group $\mathrm{MCG}_{g,n}$ of an n-pointed genus g topological surface ([30]). As above, we have the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathit{\Gamma}_{g,n} & \longrightarrow & \pi_1(\mathcal{M}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\mathit{\Gamma}_{g,n}) & \longrightarrow & \mathrm{Aut}(\mathit{\Gamma}_{g,n}) & \longrightarrow & \mathrm{Out}(\mathit{\Gamma}_{g,n}) & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are both exact. The right vertical map gives the outer Galois representation

$$\rho_{g,n} : G_k \longrightarrow \mathrm{Out}(\mathit{\Gamma}_{g,n}).$$

For the injectivity of $\rho_{g,n}$, our result in the present paper is summarized in the following (cf. Theorem 2.3):

Theorem 1.2. *Let k be a number field and (g, n) a pair of nonnegative integers such that $2g - 2 + n > 0$. Then the homomorphism $\rho_{g,n+1}$ is injective.*

Remark 1.3. As $\mathcal{M}_{0,4} = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$, the injectivity of $\rho_{0,4}$ follows from the above theorem of Belyĭ (Corollary to Theorem 4, [6]).

The proof of Theorem 1.2 yields a variant, where we consider an arbitrary family of hyperbolic curves instead of the universal family $\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}$. As above, for any geometrically connected locally noetherian scheme X over k, we can consider the outer Galois representation $\rho_X : G_k \to \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))$ determined by the exact sequence

$$1 \longrightarrow \pi_1(X \otimes_k \overline{k}) \longrightarrow \pi_1(X) \longrightarrow G_k \longrightarrow 1.$$

Grothendieck considered that hyperbolic polycurves (i.e., successive families of hyperbolic curves) are also anabelian ([12]). The injectivity of ρ_X is implicit in [14] when X is a hyperbolic polycurve and k is a number field. We can prove the injectivity of ρ_X when X is an arbitrary family of hyperbolic curves (cf. Theorem 4.3):

Theorem 1.4. *Let k be a number field and (g,n) a pair of nonnegative integers such that $2g-2+n>0$, S a geometrically connected regular scheme of finite type over k and $X \to S$ a family of (g,n)-curves over S. Then the homomorphism ρ_X is injective.*

Hoshi and Tamagawa informed the author of a different proof of Theorem 1.2. In fact, their proof gave a result stronger than Theorem 1.2, as follows. By Oda's theory ([30]) and using the Birman exact sequence (Chapter 4, [10])

$$1 \longrightarrow \pi_1(g,n) \longrightarrow \mathrm{MCG}_{g,n+1} \longrightarrow \mathrm{MCG}_{g,n} \longrightarrow 1,$$

we have the following exact sequence:

$$1 \longrightarrow \mathit{\Pi}_{g,n} \longrightarrow \pi_1(\mathcal{M}_{g,n+1}) \longrightarrow \pi_1(\mathcal{M}_{g,n}) \longrightarrow 1.$$

This exact sequence gives the universal monodromy representation

$$\rho_{g,n}^{univ} : \pi_1(\mathcal{M}_{g,n}) \longrightarrow \mathrm{Out}(\mathit{\Pi}_{g,n}).$$

It is known that the homomorphism $\rho_{g,n}^{univ}$ is injective if and only if $\rho_{g,n}^{univ}|_{\mathit{\Gamma}_{g,n}}$ is injective (Corollary 6.5, [14]).

Remark 1.5. The problem of the injectivity of $\rho_{g,n}^{univ}|_{\mathit{\Gamma}_{g,n}}$ is called the congruence subgroup problem for $\mathrm{MCG}_{g,n}$. The congruence subgroup problem was proved for $g \leq 1$ by Asada ([5]) and for $g=2$, $n>0$ by Boggi ([7]). Boggi called the image of $\rho_{g,n}^{univ}|_{\mathit{\Gamma}_{g,n}}$ the geometric profinite completion of $\mathrm{MCG}_{g,n}$ in [7].

We denote by

$$\rho_{g,n}^{geom} : G_k \longrightarrow G_k^{g,n} \longrightarrow \mathrm{Out}(\rho_{g,n}^{univ}(\mathit{\Gamma}_{g,n}))$$

the natural homomorphism determined by the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathit{\Gamma}_{g,n} & \longrightarrow & \pi_1(\mathcal{M}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \rho_{g,n}^{univ}(\mathit{\Gamma}_{g,n}) & \longrightarrow & \rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n})) & \longrightarrow & G_k^{g,n} & \longrightarrow & 1,
\end{array}$$

where $G_k^{g,n} := \rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n}))/\rho_{g,n}^{univ}(\mathit{\Gamma}_{g,n})$, and the horizontal sequences are exact.

Theorem 1.6 (Hoshi-Tamagawa)**.** *Let k be a number field and (g,n) a pair of nonnegative integers such that $3g-3+n>0$. Then the homomorphism $\rho_{g,n}^{geom}$ is injective. In particular, $\rho_{g,n}$ is injective.*

We remark that Boggi also announced a similar result (Corollary 7.6, [8]).

Next, we consider a pro-l version of Theorem 1.6, which l is a prime number. Let $\mathit{\Pi}_{g,n}^{l}$ denote the pro-l completion of the fundamental group of a Riemann surface of genus g with n points punctured. For a (g,n)-curve X over k, by the functoriality of pro-l completion, we obtain

$$\rho_X^l : G_k \longrightarrow \mathrm{Out}(\mathit{\Pi}_{g,n}^{l}).$$

As above, we have the pro-l universal monodromy representation

$$\rho_{g,n}^{univ,l} : \pi_1(\mathcal{M}_{g,n}) \longrightarrow \mathrm{Out}(\mathit{\Pi}_{g,n}^{l}).$$

Therefore, we also have the natural homomorphism

$$\rho_{g,n}^{geom,l} : G_k \longrightarrow G_k^{l,g,n} \longrightarrow \mathrm{Out}(\rho_{g,n}^{univ,l}(\mathit{\Gamma}_{g,n}))$$

determined by the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathit{\Gamma}_{g,n} & \longrightarrow & \pi_1(\mathcal{M}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \rho_{g,n}^{univ,l}(\mathit{\Gamma}_{g,n}) & \longrightarrow & \rho_{g,n}^{univ,l}(\pi_1(\mathcal{M}_{g,n})) & \longrightarrow & G_k^{l,g,n} & \longrightarrow & 1,
\end{array}$$

where $G_k^{l,g,n} := \rho_{g,n}^{univ,l}(\pi_1(\mathcal{M}_{g,n}))/\rho_{g,n}^{univ,l}(\mathit{\Gamma}_{g,n})$, and the horizontal sequences are exact. The field determined by $\mathrm{im}(\ker(\rho_{g,n}^{univ,l}) \to G_k)(= \ker(G_k \to G_k^{l,g,n}))$ can be regarded as the field of definition of the Teichmüller modular function field with l-power level structures. Oda conjectured that this field is independent of (g, n) ([29]). This conjecture was proved by using the weight filtration and the universal deformation of a maximally degenerate stable curve ([28, 27, 20, 17, 33]). We prove the second main result in the present paper by using Oda's conjecture (cf. Theorem 3.4):

Theorem 1.7. *Let (g, n) be a pair of nonnegative integers such that $3g-3+n > 0$ and either $(g, n) \neq (1, 1)$ or $l = 2$. Then the kernel of the homomorphism $\rho_{g,n}^{geom,l}$ coincides with the kernel of the homomorphism*

$$\rho^l_{\mathbb{P}^1_k\setminus\{0,1,\infty\}} : G_k \longrightarrow \mathrm{Out}(\mathit{\Pi}^l_{0,3}).$$

We apply Theorem 1.7 to the relative pro-l representation (Corollary 3.8).

The present paper is organized as follows: In section 2, we study the profinite case. Firstly, we prove a technical lemma (Lemma 2.2) in group theory and we derive Theorem 1.2 from this lemma. Secondly, we explain a proof of Theorem 1.6 due to Hoshi and Tamagawa by using a geometric version of the Grothendieck conjecture. In section 3, we prove Theorem 1.7 by using a geometric version of the Grothendieck conjecture and Oda's conjecture. Finally, we study the kernel of the relative pro-l representation. In section 4, we prove a variant of Theorem 1.2 (including Theorem 1.4) which does not follow from the method of Hoshi and Tamagawa.

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Notations and Conventions

Numbers: The notation $\mathbb{Z}$ will be used to denote the set, group, or ring of rational integers and the notation $\mathbb{Q}$ will be used to denote the set, group, or field of rational numbers. We shall refer to a finite extension of $\mathbb{Q}$ as a number field. For a prime number l, the notation $\mathbb{Z}_l$ will be used to denote the set, group, or ring of l-adic integers and the notation $\mathbb{Q}_l$ will be used to denote the set, group, or field of l-adic numbers. We shall refer to a finite extension of $\mathbb{Q}_l$ as an l-adic local field. The notation $\mathbb{C}$ will be used to denote the set, group, or field of complex numbers.

Profinite groups: If G is a profinite group, $H \subseteq G$ is a closed subgroup of G, and g is an element of G, then we shall write $Z_G(H)$ for the centralizer of H in G, i.e.,

$$Z_G(H) := \{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\} \subseteq G,$$

and we shall write $N_G(H)$ for the normalizer of H in G, i.e.,

$$N_G(H) := \{g \in G \mid gHg^{-1} = H\} \subseteq G.$$

If G is a profinite group, then we shall denote by $\mathrm{Aut}(G)$ the group of automorphisms of G, by $\mathrm{Inn}(G)$ the group of inner automorphisms of G, by $\mathrm{Out}(G)$ the quotient of $\mathrm{Aut}(G)$ by the normal subgroup $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$.

Surface groups and mapping class groups: For a pair (g, n) of nonnegative integers and a prime number l, the notation $\Pi_{g,n}$ will be used to denote the profinite completion of the fundamental group $\pi_1(g, n)$ of a compact Riemann surface of genus g with n points punctured, the notation $\Pi^l_{g,n}$ will be used to denote the pro-l completion of the fundamental group $\pi_1(g, n)$ of a Riemann surface of genus g with n points punctured, the notation $\mathrm{MCG}_{g,n}$ will be used to denote the mapping class group of (g, n)-type, namely the discrete group of isotopy classes of orientation preserving self-diffeomorphisms of an orientable surface of genus g with n points punctured which fix the n points pointwise, the notation $\mathrm{MCG}_{g,[n]}$ will be used to denote the discrete group of isotopy classes of orientation preserving self-diffeomorphisms of an orientable surface of genus g with n points punctured which preserve the set of punctures, and the notation $\Gamma_{g,n}$ will be used to denote the profinite completion of $\mathrm{MCG}_{g,n}$. We shall denote by $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,n})$ the subgroup of $\mathrm{Out}(\Pi_{g,n})$ consisting of elements which preserve the set of cuspidal inertia subgroups of $\Pi_{g,n}$, and by $\mathrm{Out}^{\mathrm{C}}(\Pi^l_{g,n})$ the subgroup of $\mathrm{Out}(\Pi^l_{g,n})$ consisting of elements which preserve the set of cuspidal inertia subgroups of $\Pi^l_{g,n}$.

Curves: Let $f : X \to S$ be a morphism of schemes. Then for a pair (g, n) of nonnegative integers such that $2g-2+n > 0$, we shall say that f is a family of (g, n)-curves over S if there exist a proper smooth geometrically connected morphism $f^{\mathrm{cpt}} : X^{\mathrm{cpt}} \to S$ whose geometric fibers are of dimension one and of genus g, and a relative divisor $D \subseteq X^{\mathrm{cpt}}$ which is finite étale over S of degree n such that X and $X^{\mathrm{cpt}} \setminus D$ are isomorphic over S. We shall say that $f^{\mathrm{cpt}} : X^{\mathrm{cpt}} \to S$ is a compactification of $f : X \to S$ and $D \subseteq X^{\mathrm{cpt}}$ is a divisor at infinity of $f : X \to S$. We shall say that a family of (g, n)-curves $X \to S$ is split if a finite étale covering $D \to S$ obtained by a divisor at infinity of $X \to S$ is trivial, i.e., D is isomorphic to the disjoint union of n copies of S over S. Note that the pair (X^{cpt}, D) is unique up to canonical isomorphism if S is normal (e.g., Section 0, [24]). In particular, we shall refer to a family of (g, n)-curves over the spectrum of a field k as a (g, n)-curve over k.

Fundamental groups: Let l be a prime number, k a field, and $\overline{k}$ an algebraic closure of k. For a scheme X which is a geometrically connected and of finite type over k, we shall write $\pi_1(X \otimes_k \overline{k})^l$ for the maximal pro-l quotient of $\pi_1(X \otimes_k \overline{k})$, and $\pi_1(X)^{\underline{l}}$ for the quotient of $\pi_1(X)$ by the kernel of the natural surjection $\pi_1(X \otimes_k \overline{k}) \to \pi_1(X \otimes_k \overline{k})^l$.

2. Profinite mapping class groups

In the present section, we prove the main result of the present paper in the profinite case. Let k be a field of characteristic zero, (g, n) a pair of nonnegative integers such that $2g - 2 + n > 0$, $\mathcal{M}_{g,n}$ the moduli stack over k of the smooth geometrically connected proper curves of genus g with n (ordered) marked points, $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ determined by a fixed algebraic closure $\overline{k}$ of k, and $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The following theorem plays an essential role in our proof.

Theorem 2.1 (Corollary 6.4, [14])**.** *Let X be a (g, n)-curve over k. Then the subgroup*

$$\rho_X^{-1}(\rho_{g,n}^{univ}(\mathit{\Gamma}_{g,n})) \subseteq G_k$$

of G_k is contained in the kernel of the homomorphism

$$G_k \longrightarrow G_{\mathbb{Q}}$$

determined by the natural inclusion $\mathbb{Q} \hookrightarrow k$.

Theorem 2.1 was proved by Matsumoto and Tamagawa (Theorem 1.1, [21]) in the affine case, and more recently by Hoshi and Mochizuki (Corollary 6.4, [14]) in the proper case.

Lemma 2.2. *Consider the commutative diagram of groups where the vertical and horizontal sequences are exact:*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & K & \longrightarrow & \mathit{\Gamma}' & \longrightarrow & \mathit{\Gamma} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & G & \longrightarrow & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1. & & & &
\end{array}$$

Let $\rho_G : H \to \mathrm{Out}(K), \rho_{G'} : H \to \mathrm{Out}(\mathit{\Gamma}'), \rho_{\mathit{\Gamma}'} : \mathit{\Gamma} \to \mathrm{Out}(K)$ denote the natural homomorphisms determined by the above commutative diagram. Then the subgroup

$$\rho_G(\ker(\rho_{G'})) \subseteq \mathrm{Out}(K)$$

of $\mathrm{Out}(K)$ is contained in the image of $\rho_{\mathit{\Gamma}'}$.

Proof. Let h be an element of the kernel of $\rho_{G'}$. Since G surjects onto H, we can take $h' \in G$ mapped to $h \in H$. By the injectivity of the homomorphism $G \to G'$, we may regard h as an element of H'. Then there exists an element γ of $\mathit{\Gamma}'$ such that $\mathrm{Inn}(h')$ acts on $\mathit{\Gamma}'$ by $\mathrm{Inn}(\gamma)$. In particular, $\mathrm{Inn}(h')$ acts on K by $\mathrm{Inn}(\gamma)$. This means $\rho_G(h) \in \mathrm{im}(\rho_{\mathit{\Gamma}'})$. $\square$

Theorem 2.3. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$. Then the kernel of the homomorphism $\rho_{g,n+1}$ is contained in the kernel of the homomorphism*

$$G_k \longrightarrow G_{\mathbb{Q}}$$

determined by the natural inclusion $\mathbb{Q} \hookrightarrow k$.

In particular, if k is a number field or an l-adic local field, then the homomorphism $\rho_{g,n+1}$ is injective.

Proof. By the commutative diagram

$$\begin{array}{ccccc}
1 & & 1 & & \\
\downarrow & & \downarrow & & \\
\Gamma_{g,n+1} & \longrightarrow & \Gamma_{g,n} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
\pi_1(\mathcal{M}_{g,n+1}) & \longrightarrow & \pi(\mathcal{M}_{g,n}) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
G_k & = & G_k & & \\
\downarrow & & \downarrow & & \\
1 & & 1, & &
\end{array}$$

where the vertical and horizontal sequences are exact, we may assume that n is small, so that there exists a (g,n)-curve X over k such that a divisor at infinity of $X \to \operatorname{Spec} k$ is split by considering a hyperelliptic curve. Since $\mathcal{M}_{g,n+1}$ is the universal curve over $\mathcal{M}_{g,n}$ (see [18]), we obtain a cartesian square

$$\begin{array}{ccc}
X & \longrightarrow & \operatorname{Spec} k \\
\downarrow & \square & \downarrow \\
\mathcal{M}_{g,n+1} & \longrightarrow & \mathcal{M}_{g,n}.
\end{array}$$

This induces a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \Gamma_{g,n+1} & \longrightarrow & \Gamma_{g,n} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(\mathcal{M}_{g,n+1}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_k & = & G_k & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1, & & & &
\end{array}$$

where the vertical and horizontal sequences are exact. Then Lemma 2.2 implies that
$$\rho_X(\ker(\rho_{g,n+1})) \subseteq \mathrm{im}(\mathit{\Gamma}_{g,n} \longrightarrow \mathrm{Out}(\mathit{\Pi}_{g,n})).$$
By using Theorem 2.1, the result follows. □

Next, we explain a different proof of Theorem 2.3 due to Hoshi and Tamagawa, using a geometric version of the Grothendieck conjecture. In fact, their proof gives a result stronger than Theorem 2.3. The following theorem plays an essential role in their proof.

Theorem 2.4 (Theorem D, [15]). *Let (g,n) be a pair of nonnegative integers such that $3g-3+n>0$ and l a prime number.*
(i) *The group $Z_{\mathrm{Out}^{\mathrm{C}}(\mathit{\Pi}_{g,n})}(\rho_{g,n}^{univ}(\mathit{\Gamma}_{g,n}))$ is isomorphic to*
$$\begin{cases} \mathbb{Z}/2\times\mathbb{Z}/2 & \text{if } (g,n)=(0,4); \\ \mathbb{Z}/2 & \text{if } (g,n)\in\{(1,1),(1,2),(2,0)\}; \\ \{1\} & \text{if } (g,n)\notin\{(0,4),(1,1),(1,2),(2,0)\}. \end{cases}$$
(ii) *Suppose that*
$$(g,n)\neq(1,1).$$
Then the group $Z_{\mathrm{Out}^{\mathrm{C}}(\mathit{\Pi}^l_{g,n})}(\rho_{g,n}^{univ,l}(\mathit{\Gamma}_{g,n}))$ is isomorphic to
$$\begin{cases} \mathbb{Z}/2\times\mathbb{Z}/2 & \text{if } (g,n)=(0,4); \\ \mathbb{Z}/2 & \text{if } (g,n)\in\{(1,2),(2,0)\}; \\ \{1\} & \text{if } (g,n)\notin\{(0,4),(1,2),(2,0)\}. \end{cases}$$
(iii) *Suppose that $l=2$. Then the group $Z_{\mathrm{Out}^{\mathrm{C}}(\mathit{\Pi}^l_{1,1})}(\rho_{1,1}^{univ,l}(\mathit{\Gamma}_{1,1}))$ is isomorphic to $\mathbb{Z}/2$.*

The proof of Theorem 2.4 is very sophisticated using the theory of profinite Dehn twists developed in [15].

Theorem 2.5 (Hoshi-Tamagawa). *Let (g,n) be a pair of nonnegative integers such that $3g-3+n>0$. Then the kernel of the homomorphism $\rho_{g,n}^{geom}$ is contained in the kernel of the homomorphism*
$$G_k \longrightarrow G_{\mathbb{Q}}$$
determined by the natural inclusion $\mathbb{Q}\hookrightarrow k$.

In particular, if k is a number field or an l-adic local field, then the homomorphisms $\rho_{g,n}^{geom}$ and $\rho_{g,n}$ are injective.

Proof. We may assume that k is $\mathbb{Q}$. Note that $G_{\mathbb{Q}}^{g,n} := \rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n}))/\rho_{g,n}^{univ}(\mathit{\Gamma}_{g,n})$ is isomorphic to $G_{\mathbb{Q}}$ by Theorem 2.1. Also, by Theorem 2.3 and the injectivity of $\rho_{g,n}^{univ}$ when g is zero (Theorem 3A, [5]), we may assume that $g>0$. Then the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \rho_{g,n}^{univ}(\mathit{\Gamma}_{g,n}) & \longrightarrow & \rho_{g,n}^{univ}(\pi_1(\mathcal{M}_{g,n})) & \longrightarrow & G_{\mathbb{Q}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\rho_{g,n}^{univ}(\mathit{\Gamma}_{g.n})) & \longrightarrow & \mathrm{Aut}(\rho_{g,n}^{univ}(\mathit{\Gamma}_{g.n})) & \longrightarrow & \mathrm{Out}(\rho_{g,n}^{univ}(\mathit{\Gamma}_{g,n})) & \longrightarrow & 1 \end{array}$$

induces an isomorphism

$$Z_{\rho^{univ}_{g,n}(\pi_1(\mathcal{M}_{g,n}))}(\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n}))/Z_{\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n})}(\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n}))$$

$$\simeq \ker(G_{\mathbb{Q}} \longrightarrow \mathrm{Out}(\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n}))).$$

Therefore, it is enough to prove

$$Z_{\rho^{univ}_{g,n}(\pi_1(\mathcal{M}_{g,n}))}(\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n}))/Z_{\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n})}(\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n})) = \{1\}.$$

Note that the image of $\rho^{univ}_{g,n}$ is contained in $\mathrm{Out}^{\mathrm{C}}(\mathit{\Pi}_{g,n})$. By the injectivity of $\mathrm{MCG}_{g,[n]} \to \mathrm{Out}(\pi_1(g,n))$ (e.g., Theorem 8.8, in [10]) and $\mathrm{Out}(\pi_1(g,n)) \to \mathrm{Out}(\mathit{\Pi}_{g,n})$ (Lemma 3.2.1 in [3] for $n > 0$ and [11] for $n = 0$), we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{MCG}_{g,n} & \longrightarrow & \mathrm{Out}(\pi_1(g,n)) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{MCG}_{g,[n]} & \hookrightarrow & \mathrm{Out}(\mathit{\Pi}_{g,n}). \end{array}$$

Since an element of $\mathrm{MCG}_{g,[n]}$ induces an action on the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$, an element of $\mathrm{MCG}_{g,[n]}$ induces an action on the set of conjugacy classes of cuspidal inertia subgroups of $\mathit{\Pi}_{g,n}$. Note that there exits a canonical bijection between the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$ and the set of conjugacy classes of cuspidal inertia subgroups of $\mathit{\Pi}_{g,n}$. Hence, the image of $\mathrm{MCG}_{g,[n]} \hookrightarrow \mathrm{Out}(\mathit{\Pi}_{g,n})$ is contained in $\mathrm{Out}^{\mathrm{C}}(\mathit{\Pi}_{g,n})$. In particular, we have the natural inclusion $Z_{\mathrm{MCG}_{g,[n]}}(\mathrm{MCG}_{g,[n]}) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\mathit{\Pi}_{g,n})}(\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n}))$ and this inclusion is isomorphism by Theorem 2.4 (i) and section 4 of Chapter 3 in [10]. If the image σ' of an element σ of $Z_{\mathrm{MCG}_{g,[n]}}(\mathrm{MCG}_{g,[n]})$ is not contained in $\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n})$, σ is not contained in $\mathrm{MCG}_{g,n}$. Since the action of $\mathrm{MCG}_{g,[n]} / \mathrm{MCG}_{g,n}$ on the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$ is faithful, σ induces a nontrivial action on the set of conjugacy classes of cuspidal inertia subgroups of $\pi_1(g,n)$. Therefore, σ' induces a nontrivial action on the set of conjugacy classes of cuspidal inertia subgroups of $\mathit{\Pi}_{g,n}$. Since the action of $\rho^{univ}_{g,n}(\pi_1(\mathcal{M}_{g,n}))$ on the set of conjugacy classes of cuspidal inertia subgroups of $\mathit{\Pi}_{g,n}$ is trivial by the definition of $\pi_1(\mathcal{M}_{g,n})$, σ' is not contained in $\rho^{univ}_{g,n}(\pi_1(\mathcal{M}_{g,n}))$. Hence, we have $Z_{\rho^{univ}_{g,n}(\pi_1(\mathcal{M}_{g,n}))}(\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n}))/Z_{\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n})}(\rho^{univ}_{g,n}(\mathit{\Gamma}_{g,n})) = \{1\}$. □

3. Pro-l mapping class groups

In the present section, we prove the pro-l version of the main result of the present paper. Let l be a prime number and assume that the base field k is a field of characteristic zero.

Lemma 3.1. *Let (g,n) be a pair of nonnegative integers such that $2g - 2 + n > 0$. Then the natural homomorphism $\pi_1(g,n) \to \mathit{\Pi}^l_{g.n}$ is injective.*

Proof. It follows immediately from the fact that $\pi_1(g,n)$ is conjugacy l-separable (Theorem 3.2, Theorem 4.1 in [31]). □

By above lemma, we can consider $\pi_1(g,n)$ as a subgroup of $\mathit{\Pi}^l_{g,n}$.

Lemma 3.2. *Let (g, n) be a pair of nonnegative integers such that $2g - 2 + n > 0$. Then the group $N_{\Pi^l_{g,n}}(\pi_1(g, n))$ is equal to $\pi_1(g, n)$. In particular, the natural homomorphism* $\mathrm{Out}(\pi_1(g, n)) \to \mathrm{Out}(\Pi^l_{g,n})$ *induced by* $\pi_1(g, n) \hookrightarrow \Pi^l_{g,n}$ *is injective.*

Proof. It is clear that $N_{\Pi^l_{g,n}}(\pi_1(g, n)) \supseteq \pi_1(g, n)$ by the definition of normalizer. Let a be an element of $N_{\Pi^l_{g,n}}(\pi_1(g, n))$. Then, for any element γ of $\pi_1(g, n)$, γ is conjugate to $a\gamma a^{-1}$ in $\pi_1(g, n)$ by the fact that $\pi_1(g, n)$ is conjugacy l-separable (Theorem 3.2, Theorem 4.1 in [31]). Therefore, since $\pi_1(g, n)$ has Property A (Lemma 1, Theorem 3 in [11]), there exists an element h of $\pi_1(g, n)$ such that $a\gamma a^{-1} = h\gamma h^{-1}$ for any element γ of $\pi_1(g, n)$. Since $\Pi^l_{g,n}$ is center-free (Corollary 1.3.4 in [26]) and $\pi_1(g, n)$ is dense in $\Pi^l_{g,n}$, we have $a = h \in \pi_1(g, n)$. □

Remark 3.3. These lemmas may be well-known. At least, Lemma 3.2 was proved for special cases by several people (e.g., Proposition 1, [19], Corollary 2 to Proposition B2, [4]).

Theorem 3.4. *Let (g, n) be a pair of nonnegative integers such that $3g-3+n > 0$ and either $(g, n) \neq (1, 1)$ or $l = 2$. Then the kernel of the homomorphism $\rho^{geom,l}_{g,n}$ coincides with the kernel of the homomorphism*

$$\rho^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}} : G_k \longrightarrow \mathrm{Out}(\Pi^l_{0,3}).$$

Proof. By the Galois Kernel Theorem in [16] (or Theorem C in [14]) and $\rho^{univ,l}_{g,n}(\Gamma_{g,n})$ is isomorphic to $\Gamma^l_{g,n}$ when g is zero (Remark to Theorem 1, [5]), we may assume that $g > 0$. Here, $\Gamma^l_{g,n}$ is the pro-l completion of $\Gamma_{g.n}$. As the proof of Theorem 2.5, we can show that the natural homomorphism

$$G^{l,g,n}_k \longrightarrow \mathrm{Out}(\rho^{univ,l}_{g,n}(\Gamma_{g,n}))$$

is injective. Here, $G^{l,g,n}_k$ is the group

$$\rho^{univ,l}_{g,n}(\pi_1(\mathcal{M}_{g,n}))/\rho^{univ,l}_{g,n}(\Gamma_{g,n}).$$

Indeed, the arguments of the proof of Theorem 2.5 go well as they are, if we replace Theorem 2.4 (i) with Theorem 2.4 (ii), (iii) and the injectivity of $\mathrm{Out}(\pi_1(g, n)) \to \mathrm{Out}(\Pi_{g.n})$ with the injectivity of $\mathrm{Out}(\pi_1(g, n)) \to \mathrm{Out}(\Pi^l_{g,n})$ (Lemma 3.2). Therefore, it is sufficient to prove that

$$\ker(G_k \longrightarrow G^{l,g,n}_k) = \ker(\rho^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}}).$$

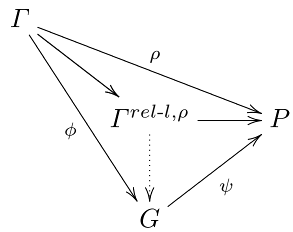
Let $p_{g,n} : \pi_1(\mathcal{M}_{g,n}) \to G_k$ be the natural homomorphism. Then we have

$$\ker(G_k \longrightarrow G^{l,g,n}_k) = p_{g,n}(\ker(\rho^{univ,l}_{g,n})).$$

However, it is known that $p_{g,n}(\ker(\rho^{univ,l}_{g,n}))$ coincides with $\ker(\rho^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}})$ (Oda's conjecture, cf. Theorem 3.3, [33]). This completes the proof. □

Next, we consider the relative pro-l case. Since all mapping class groups in genus g are perfect when $g \geq 3$, their pro-l completions are trivial. However, Hain and Matsumoto developed a theory of relative pro-l completion of groups, and showed that the natural relative pro-l completions of mapping class groups are large and more closely reflect their structure ([13]). We explain below their theory.

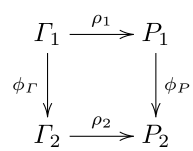
Let Γ be a discrete or profinite group, P a profinite group, and $\rho : \Gamma \to P$ a continuous dense homomorphism. (Here, a dense homomorphism means a homomorphism with dense image.) The relative pro-l completion $\Gamma^{rel\text{-}l,\rho}$ of Γ with respect to ρ is characterized by a universal mapping property: if G is a profinite group, $\psi : G \to P$ a continuous homomorphism with pro-l kernel, and if $\phi : \Gamma \to G$ is a continuous homomorphism whose composition with ψ is ρ, then there is a unique continuous homomorphism $\Gamma^{rel\text{-}l,\rho} \to G$ that extends ϕ:



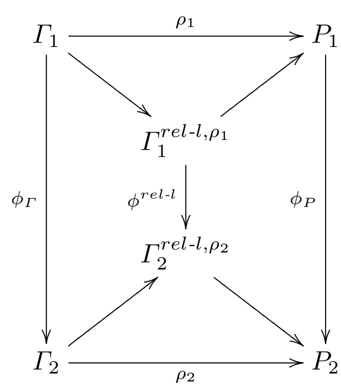
The following properties are direct consequences of the universal mapping property:

Proposition 3.5 (Proposition 2.1, [13]). *A dense homomorphism $\rho : \Gamma \to P$ from a discrete group to a profinite group induces a homomorphism $\overline{\rho} : \hat{\Gamma} \to P$ from the profinite completion of Γ to P. The natural homomorphism $\Gamma \to \hat{\Gamma}$ induces a natural isomorphism $\Gamma^{rel\text{-}l,\rho} \to \hat{\Gamma}^{rel\text{-}l,\overline{\rho}}$.*

Proposition 3.6 (Proposition 2.3, [13]). *Suppose that Γ_1 and Γ_2 are both discrete groups or both profinite groups and that P_1 and P_2 are profinite groups. Suppose that $\rho_j : \Gamma_j \to P_j$ ($j \in \{1, 2\}$) are continuous dense homomorphisms. If*



is a commutative diagram of topological groups, then there is a unique continuous homomorphism $\phi^{rel\text{-}\ l} : \Gamma_1^{rel\text{-}l,\rho_1} \to \Gamma_1^{rel\text{-}l,\rho_2}$ such that the diagram



commutes.

Proposition 3.7 (Proposition 2.4, [13]). *Suppose that P_1, P_2 and P_3 are profinite groups and that $\rho_j : \Gamma_j \to P_j$ ($j \in \{1, 2, 3\}$) are continuous dense homomorphisms of topological groups. Suppose that the Γ_j are all discrete groups or all profinite* *groups. If the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varGamma_1 & \longrightarrow & \varGamma_2 & \longrightarrow & \varGamma_3 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_3} & & \\ 1 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 1 \end{array}$$

of topological groups commutes and has two raws exact, then the sequence

$$\varGamma_1^{rel\text{-}l,\rho_1} \longrightarrow \varGamma_2^{rel\text{-}l,\rho_2} \longrightarrow \varGamma_3^{rel\text{-}l,\rho_3} \longrightarrow 1$$

is exact.

Let $\mathcal{A}_g$ be the moduli stack of principally polarized abelian varieties of dimension g. It is known that the orbifold fundamental groups $\pi_1^{\mathrm{orb}}(\mathcal{M}_{g,n}(\mathbb{C}))$ and $\pi_1^{\mathrm{orb}}(\mathcal{A}_g(\mathbb{C}))$ of $\mathcal{M}_{g,n}(\mathbb{C})$ and $\mathcal{A}_g(\mathbb{C})$ are isomorphic to $\mathrm{MCG}_{g,n}$ and $Sp_g(\mathbb{Z})$ respectively. Here, $Sp_g(A)$ is the group of symplectic $2g \times 2g$ matrices with entries in a commutative ring A. Let

$$\rho^{period} : \mathrm{MCG}_{g,n} \longrightarrow Sp_g(\mathbb{Z})$$

be the surjective homomorphism determined by the period map $\mathcal{M}_{g,n}(\mathbb{C}) \to \mathcal{A}_g(\mathbb{C})$ which takes the moduli point $[C]$ of a compact Riemann surface C (equipped with n marked points) to that of its jacobian $[\mathrm{Jac}(C)]$ (also see Chapter 6, [10]). Then ρ^{period} induces the continuous dense homomorphism

$$\rho^{period,l} : \mathrm{MCG}_{g,n} \longrightarrow Sp_g(\mathbb{Z}_l).$$

Hain and Matsumoto defined the relative pro-l completion of mapping class group by

$$\varGamma_{g,n}^{rel\text{-}l} := \mathrm{MCG}_{g,n}^{rel\text{-}l,\rho^{period,l}}.$$

Let $\overline{\rho}^{period,l} : \varGamma_{g,n} \to Sp_g(\mathbb{Z}/l)$ be the homomorphism determined by ρ^{period}. Then, by using Proposition 3.5 and the universal mapping property, we have the natural isomorphism

$$\varGamma_{g,n}^{rel\text{-}l} \simeq \varGamma_{g,n}^{rel\text{-}l,\overline{\rho}^{period,l}}.$$

This means that $\varGamma_{g,n}^{rel\text{-}l}$ is an almost pro-l group (i.e. there exists a closed subgroup of $\varGamma_{g,n}^{rel\text{-}l}$ with finite index that is a pro-l group). Also, Hain and Matsumoto proved that the natural homomorphism $\mathrm{MCG}_{g,n} \to \varGamma_{g,n}^{rel\text{-}l}$ is injective for $n > 0$ (Proposition 3.1, [13]). (In fact, since the injectivity of $\mathrm{MCG}_{g,n} \to \varGamma_{g,n}^{rel\text{-}l}$ is reduced to the injectivity of $\mathrm{MCG}_{g,n+1} \to \varGamma_{g,n+1}^{rel\text{-}l}$ by using Lemma 3.2, we also have the injectivity of $\mathrm{MCG}_{g,0} \to \varGamma_{g,0}^{rel\text{-}l}$ (for $g > 1$).)

The functoriality of relative pro-l completion implies that there is an outer Galois action

$$\rho_{g,n}^{rel\text{-}l} : G_k \longrightarrow \mathrm{Out}(\varGamma_{g,n}^{rel\text{-}l}).$$

Since the representation $\rho_{g,n}^{rel\text{-}l}$ is unramified outside l when k is a number field (Theorem 3, [13]), $\rho_{g,n}^{rel\text{-}l}$ is not injective. By using Theorem 3.4, we have the following corollary.

Corollary 3.8. *Let (g,n) be a pair of natural numbers such that $3g-3+n > 0$ and either $(g,n) \neq (1,1)$ or $l = 2$. Then the kernel of the homomorphism $\rho_{g,n}^{rel\text{-}l}$ is contained in the kernel of the homomorphism*

$$\rho^l_{\mathbb{P}^1_k \setminus \{0,1,\infty\}} : G_k \longrightarrow \mathrm{Out}(\varPi^l_{0,3}).$$

Proof. The commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \Gamma_{g,n+1} & \longrightarrow & \Gamma_{g,n} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi^{l}_{g,n} & \longrightarrow & \Gamma^{rel\text{-}l}_{g,n+1} & \longrightarrow & \Gamma^{rel\text{-}l}_{g,n} & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact (Proposition 3.1 (2), [13]), induces the following commutative diagram

$$\begin{array}{ccc} \Gamma_{g,n} & \longrightarrow & \Gamma^{rel\text{-}l}_{g,n} \\ & \searrow^{\rho^{univ,l}_{g,n}} & \downarrow \\ & & \mathrm{Out}(\Pi^{l}_{g,n}). \end{array}$$

Therefore, we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma^{rel\text{-}l}_{g,n} & \longrightarrow & \pi_1(\mathcal{M}_{g,n})/\ker(\Gamma_{g,n} \to \Gamma^{rel\text{-}l}_{g,n}) & \longrightarrow & G_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \rho^{univ,l}_{g,n}(\Gamma_{g,n}) & \longrightarrow & \rho^{univ,l}_{g,n}(\pi_1(\mathcal{M}_{g,n})) & \longrightarrow & G^{l,g,n}_{k} & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact and the vertical homomorphisms are surjective. Hence, this induces

$$\ker(\rho^{rel\text{-}l}_{g,n}) \subseteq \ker(\rho^{geom,l}_{g,n}) = \ker(\rho^{l}_{\mathbb{P}^1_k \setminus \{0,1,\infty\}}).$$

□

4. The case of an arbitrary family of hyperbolic curves

In the present section, we prove a variant of Theorem 2.3. Let l be a prime number, k a field of characteristic zero, and $\overline{k}$ an algebraic closure of k. For any geometrically connected regular scheme S of finite type over k and any family $X \to S$ of (g,n)-curves over S, we denote by $\varphi^{l}_{X/S} : \pi_1(S \otimes_k \overline{k}) \to \mathrm{Aut}(\Pi^{\mathrm{ab}}_{g,n} \otimes_{\mathbb{Z}} (\mathbb{Z}/l))$ the natural monodromy action arising from the family of (g,n)-curves $X \to S$. Here, the group $\Pi^{\mathrm{ab}}_{g,n}$ is the abelianization of $\Pi_{g,n}$.

Proposition 4.1. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n > 0$, S a geometrically connected regular scheme of finite type over k, and $X \to S$ a family of (g,n)-curves over S. Then the natural sequence*

$$1 \longrightarrow \Pi_{g,n} \longrightarrow \pi_1(X) \longrightarrow \pi_1(S) \longrightarrow 1$$

is exact. Moreover, if the image of $\varphi^{l}_{X/S}$ is an l-group, then the natural sequence

$$1 \longrightarrow \Pi^{l}_{g,n} \longrightarrow \pi_1(X)^{\underline{l}} \longrightarrow \pi_1(S)^{\underline{l}} \longrightarrow 1$$

is exact.

Proof. It is enough to prove the case for $k = \overline{k}$. First, we prove the profinite case. Then we have the following exact sequence

$$\Pi_{g,n} \longrightarrow \pi_1(X) \longrightarrow \pi_1(S) \longrightarrow 1$$

by [1]. Let $X^{\mathrm{cpt}} \to S$ be the compactification of $X \to S$ and $D \subseteq X^{\mathrm{cpt}}$ the divisor at infinity of $X \to S$. Then we can take a finite étale (connected) Galois covering $S' \to S$ such that the finite étale covering $D \times_S S' \to S'$ is split. We put $X' := X \times_S S'$, $X'^{\mathrm{cpt}} := X^{\mathrm{cpt}} \times_S S'$, $D' := D \times_S S'$. Then the natural projection $X' \to S'$ is a family of (g,n)-curves and X'^{cpt} (respectively D') is the compactification (respectively the divisor at infinity) of $X' \to S'$. Since $D' \to S'$ is split, by Proposition 2.2 in [25], the natural sequence

$$1 \longrightarrow \varPi_{g,n} \longrightarrow \pi_1(X') \longrightarrow \pi_1(S') \longrightarrow 1$$

is exact. Moreover, by the definition of $X' \to S'$, we have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varPi_{g,n} & \longrightarrow & \pi_1(X') & \longrightarrow & \pi_1(S') & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ & & \varPi_{g,n} & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(S) & \longrightarrow & 1. \end{array}$$

Now, since the natural projection $X' \to X$ is a finite étale covering, $\pi_1(X') \to \pi_1(X)$ is injective. This completes the proof for the profinite case.

Next, we consider the pro-l case. Since the image of $\varphi^l_{X/S}$ is an l-group, by using Lemma 4.5.5 in [32] and Théorème 2.3.1 in [2], the natural homomorphism $\pi_1(S) \to \mathrm{Out}(\varPi_{g,n}) \to \mathrm{Out}(\varPi^l_{g,n})$ factors through the maximal pro-l quotient $\pi_1(S)^l$ of $\pi_1(S)$. Therefore, the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varPi_{g,n} & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(S) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\varPi^l_{g,n}) & \longrightarrow & \mathrm{Aut}(\varPi^l_{g,n}) & \longrightarrow & \mathrm{Out}(\varPi^l_{g,n}) & \longrightarrow & 1 \end{array}$$

induces the following commutative diagram

$$\begin{array}{ccccccccc} & & \varPi^l_{g,n} & \longrightarrow & \pi_1(X)^l & \longrightarrow & \pi_1(S)^l & \longrightarrow & 1 \\ & & \wr\downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\varPi^l_{g,n}) & \longrightarrow & \mathrm{Aut}(\varPi^l_{g,n}) & \longrightarrow & \mathrm{Out}(\varPi^l_{g,n}) & \longrightarrow & 1, \end{array}$$

where the horizontal sequences are exact and the left vertical homomorphism is isomorphism by Corollary 1.3.4 in [26]. This completes the proof for the pro-l case. □

In the notation of the above proposition, we have the natural homomorphisms $\varphi_S : \pi_1(S) \to \mathrm{Out}(\varPi_{g,n})$, $\varphi^l_S : \pi_1(S) \to \mathrm{Out}(\varPi^l_{g,n})$ determined by the following exact sequence

$$1 \longrightarrow \varPi_{g,n} \longrightarrow \pi_1(X) \longrightarrow \pi_1(S) \longrightarrow 1.$$

Note that $\varGamma_{0,4}$ (respectively $\varGamma^{rel\text{-}l}_{0,4}$) is canonically isomorphic to $\varPi_{0,3}$ (respectively $\varPi^l_{0,3}$). By a similar argument used in the proof of Theorem 2.1 (Theorem 1.1, [21] or Corollary 6.4, [14]), we can prove the following proposition.

Proposition 4.2. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$, S a geometrically connected regular scheme of finite type over k with a k-rational point s, $X \to S$ a family of (g,n)-curves over S, X_s the fiber of $X \to S$ at s, and ρ_{X_s} (respectively $\rho^l_{X_s}$) the homomorphism $G_k \to \mathrm{Out}(\Pi_{g,n})$ (respectively $G_k \to \mathrm{Out}(\Pi^l_{g,n})$) associated to the (g,n)-curve X_s over k. Then the subgroup*

$$\rho_{X_s}^{-1}(\varphi_S(\pi_1(S \otimes_k \overline{k}))) \subseteq G_k \ (\textit{respectively } (\rho^l_{X_s})^{-1}(\varphi^l_S(\pi_1(S \otimes_k \overline{k}))) \subseteq G_k)$$

of G_k is contained in the kernel of the homomorphism

$$\rho_{0,4} : G_k \longrightarrow \mathrm{Out}(\Pi_{0,3}) \ (\textit{respectively } \rho_{0,4}^{rel\text{-}l} : G_k \longrightarrow \mathrm{Out}(\Pi^l_{0,3})).$$

Proof. Since the pro-l case can be proved by exactly the same argument, we prove only the profinite case. Let i_s be the section $G_k \to \pi_1(S)$ induced by the k-rational point s, $k(S)$ the function field of S, $\overline{k(S)}$ an algebraic closure of $k(S)$, $X_{k(S)} := X \times_S \operatorname{Spec} k(S)$, $\rho_{X_{k(S)}}$ the homomorphism $G_{k(S)} := \mathrm{Gal}(\overline{k(S)}/k(S)) \to \mathrm{Out}(\Pi_{g,n})$ associated to the (g,n)-curve $X_{k(S)}$ over $k(S)$. Then we have $\varphi_S \circ i_s = \rho_{X_s}$, and the natural (outer) homomorphisms $G_{k(S)} \to \pi_1(S)$ is surjective by the geometrically-connectedness of S. Assume that there exist $\gamma \in \pi_1(S \otimes_k \overline{k})$ and $\sigma \in G_k$ such that $\varphi_S(\gamma)$ is equal to $\rho_{X_s}(\sigma)$. By the surjectivity of the above (outer) homomorphism, we can take $\tilde{\gamma}, \tilde{\sigma} \in G_{k(S)}$ mapped to $\gamma, i_s(\sigma) \in \pi_1(S)$, respectively. Since the following diagram

$$\begin{array}{ccc} G_{k(S)} & \twoheadrightarrow & \pi_1(S) \\ & \searrow^{\rho_{X_{k(S)}}} & \downarrow{\scriptstyle \varphi_S} \\ & & \mathrm{Out}(\Pi_{g,n}) \end{array}$$

is commutative, $\tilde{\gamma}\tilde{\sigma}^{-1}$ is contained in the kernel of $\rho_{X_{k(S)}}$. Hence, by Corollary 6.2, in [14], $\tilde{\gamma}\tilde{\sigma}^{-1}$ is contained in the kernel of the natural homomorphism $G_{k(S)} \to \mathrm{Out}(\Pi_{0,3})$. Now, since the following diagram

$$\begin{array}{ccc} G_{k(S)} & \longrightarrow & G_k \\ & \searrow & \downarrow{\scriptstyle \rho_{0,4}} \\ & & \mathrm{Out}(\Pi_{0,3}) \end{array}$$

is commutative and γ is contained in the kernel of $\pi_1(S) \to G_k$, σ is contained in the kernel of $\rho_{0,4}$. $\square$

For a scheme X which is a geometrically connected and of finite type over k, we denote by $\rho^l_X : G_k \to \mathrm{Out}(\pi_1(X \otimes_k \overline{k})^l)$ the composite of $\rho_X : G_k \to \mathrm{Out}(\pi_1(X \otimes_k \overline{k}))$ and the natural homomorphism $\mathrm{Out}(\pi_1(X \otimes_k \overline{k})) \to \mathrm{Out}(\pi_1(X \otimes_k \overline{k})^l)$. The following theorem is a variant of Theorem 2.3.

Theorem 4.3. *Let (g,n) be a pair of nonnegative integers such that $2g-2+n>0$, S a geometrically connected regular scheme of finite type over k, $X \to S$ a family of (g,n)-curves over S. Then the kernel of the homomorphism ρ_X is contained in the kernel of the homomorphism*

$$\rho_{0,4} : G_k \longrightarrow \mathrm{Out}(\Pi_{0,3}).$$

Moreover, if the image of $\varphi^l_{X/S}$ *is an* l*-group, then the kernel of the homomorphism* ρ^l_X *is contained in the kernel of the homomorphism*

$$\rho^{rel\text{-}l}_{0,4} : G_k \longrightarrow \mathrm{Out}(\Pi^l_{0,3}).$$

In particular, if k *is a number field or an* l*-adic local field, then the homomorphism* ρ_X *is injective.*

Proof. First, we prove the profinite case. Let $k(S)$ be the function field of S, $\overline{k(S)}$ an algebraic closure of $k(S)$, $X_{k(S)} := X \times_S \mathrm{Spec}\, k(S)$, $X_{\overline{k(S)}} := X \times_S \mathrm{Spec}\, \overline{k(S)}$, $S_{k(S)} := S \otimes_k k(S)$. Then the diagonal map $S \to S \times_{\mathrm{Spec}\, k} S$ induces a section $\mathrm{Spec}\, k(S) \to S_{k(S)}$ of the natural projection $S_{k(S)} \to \mathrm{Spec}\, k(S)$. Note that we have the following diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_{\overline{k(S)}}) & \longrightarrow & \pi_1(X_{k(S)}) & \longrightarrow & G_{k(S)} & \longrightarrow & 1 \\
 & & \wr\downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(X \otimes_k \overline{k}) & \longrightarrow & \pi(X) & \longrightarrow & G_k & \longrightarrow & 1.
\end{array}$$

This diagram induces the following commutative diagram

$$\begin{array}{ccc}
G_{k(S)} & \longrightarrow & G_k \\
 & \searrow^{\rho_{X_{k(S)}}} & \downarrow \rho_X \\
 & & \mathrm{Out}(\pi_1(X \otimes_k \overline{k})).
\end{array}$$

Also, since S is geometrically connected over k, the natural (outer) homomorphism $G_{k(S)} = \mathrm{Gal}(\overline{k(S)}/k(S)) \to G_k$ is surjective. In particular, $\ker(\rho_{X_{k(S)}})$ surjects onto $\ker(\rho_X)$. Therefore, if $\ker(\rho_{X_{k(S)}})$ is included in $\ker(G_{k(S)} \to \mathrm{Out}(\Pi_{0,3}))$, $\ker(\rho_X)$ is included in $\ker(G_k \to \mathrm{Out}(\Pi_{0,3}))$ by the following commutative diagram

$$\begin{array}{ccc}
G_{k(S)} & \twoheadrightarrow & G_k \\
 & \searrow & \downarrow \\
 & & \mathrm{Out}(\Pi_{0,3}).
\end{array}$$

Hence, replacing $X \to S \to \mathrm{Spec}\, k$ by $X_{k(S)} \to S_{k(S)} \to \mathrm{Spec}\, k(S)$ if necessary, we may assume that S has a k-rational point. Let s be a k-rational point of S, $\overline{s}$ a $\overline{k}$-rational point over s, X_s the fiber of $X \to S$ at s, $X_{\overline{s}}$ the fiber of $X \to S$ at $\overline{s}$. The above k-rational point s of S induces a cartesian square

$$\begin{array}{ccc}
X_s & \longrightarrow & \mathrm{Spec}\, k \\
\downarrow & \square & \downarrow \\
X & \longrightarrow & S.
\end{array}$$

This induces a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \Pi_{g,n} & \longrightarrow & \pi_1(X \otimes_k \overline{k}) & \longrightarrow & \pi_1(S \otimes_k \overline{k}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(X_s) & \longrightarrow & \pi_1(X) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_k & = & G_k & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1, & & & &
\end{array}$$

where the vertical and horizontal sequences are exact. Then Lemma 2.2 implies that

$$\rho_{X_s}(\ker(\rho_X)) \subseteq \mathrm{im}(\varphi_S : \pi_1(S \otimes_k \overline{k}) \longrightarrow \mathrm{Out}(\Pi_{g,n})).$$

Here, ρ_{X_s} is the homomorphism $G_k \to \mathrm{Out}(\Pi_{g,n})$ associated to the hyperbolic curve X_s over k. Hence, by using Proposition 4.2, the result follows for the profinite case.

For the pro-l case, since we have the following commutative diagram

$$\begin{array}{ccc}
\pi_1(S) & \twoheadrightarrow & \pi_1(S)^{\underline{l}} \\
 & \searrow^{\varphi_S^l} & \downarrow \\
 & & \mathrm{Out}(\Pi_{g,n}^l),
\end{array}$$

we can prove by exactly the same argument. □

Remark 4.4. It is trivial that Theorem 2.5 implies Theorem 2.3. However, it seems that Theorem 2.5 (or its proof) does not imply Theorem 4.3.

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