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ABSTRACT. In the present paper, we discuss Grothendieck's conjecture of anabelian geometry for *hyperbolic polycurves*, i.e., successive extensions of families of hyperbolic curves. One of consequences obtained in the present paper is that the isomorphism class of a hyperbolic polycurve of dimension less than or equal to four over a sub-p-adic field is *completely determined* by its étale fundamental group. We also verify the *finiteness* of a set determined by certain isomorphisms between the étale fundamental groups of hyperbolic polycurves of *arbitrary dimension*.

CONTENTS

INTRODUCTION

Let k be a field of characteristic zero, $\overline{k}$ an algebraic closure of k, and $G_k \stackrel{\text{def}}{=} \text{Gal}(\overline{k}/k)$ the absolute Galois group of k determined by the given algebraic closure $\overline{k}$ of k. Let X be a variety over k [i.e., a scheme that is of finite type, separated, and geometrically connected over k — cf. Definition 1.4]. Then let us write Π_X for the *étale fundamental group* of X [for some choice of basepoint]. The group Π_X is a profinite group which is *uniquely determined* [up to inner automorphisms] by the property that the category of

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discrete finite sets equipped with a continuous Π_X-action is equivalent to the category of finite étale coverings of X. Now since X is a variety over k, the structure morphism $X \to \operatorname{Spec} k$ induces a *surjection*

$$\Pi_X \longrightarrow G_k .$$

In particular, the assignment

$$\Pi\colon \quad (X \to \operatorname{Spec} k) \quad \mapsto \quad (\Pi_X \twoheadrightarrow G_k)$$

defines a *functor* from the category $\mathcal{V}_k$ of varieties over k [whose morphisms are morphisms of schemes over k] to the category $\mathcal{G}_k$ of profinite groups equipped with a surjection onto G_k [whose morphisms are outer homomorphisms of topological groups over G_k]. The following philosophy, i.e., *Grothendieck's conjecture of anabelian geometry* [or, simply, the "Grothendieck conjecture"], was proposed by Grothendieck [cf., e.g., [8], [9]].

> For certain types of k, if one replaces $\mathcal{V}_k$ by "the" subcategory $\mathcal{A}_k$ of $\mathcal{V}_k$ of "*anabelian varieties*" over k, then the restriction of the above functor Π to $\mathcal{A}_k$ should be *fully faithful*.

Although we do not have any general definition of the notion of an "anabelian variety", the following varieties have been regarded as typical examples of anabelian varieties:

- A hyperbolic curve [cf. Definition 2.1, (i)].
- A successive extension of families of anabelian varieties.

In particular, a successive extension of families of hyperbolic curves, i.e., a *hyperbolic polycurve* [cf. Definition 2.1, (ii)], is one of typical examples of anabelian varieties. In the present paper, we discuss the *Grothendieck conjecture for hyperbolic polycurves*.

The following is one of the main results of the present paper [cf. Theorems 3.4; 3.15; Corollaries 3.16; 3.17].

Theorem A. *Let p be a prime number, k a* **sub-p-adic field** [*cf. Definition 3.1*], *$\overline{k}$ an algebraic closure of k, n a positive integer, X a* **hyperbolic polycurve** [*cf. Definition 2.1,* (ii)] *of dimension n over k, and Y a* **normal variety** [*cf. Definition 1.4*] *over k. Write $G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; Π_X, Π_Y for the étale fundamental groups of X, Y, respectively. Let $\phi\colon \Pi_Y \to \Pi_X$ be an* **open** *homomorphism over G_k. Suppose that one of the following conditions* (1), (2), (3), (4) *is satisfied:*

(1) $n = 1$.
(2) *The following conditions are satisfied:*
  (2-i) $n = 2$.
  (2-ii) *The kernel of ϕ is* **topologically finitely generated**.
(3) *The following conditions are satisfied:*

(3-i) $n = 3$.
(3-ii) *The kernel of* ϕ *is* **finite**.
(3-iii) Y *is of* **LFG-type** [*cf. Definition 2.5*].
(3-iv) $3 \leq \dim(Y)$.

(4) *The following conditions are satisfied:*
(4-i) $n = 4$.
(4-ii) ϕ *is* **injective**.
(4-iii) Y *is a* **hyperbolic polycurve** *over* k.
(4-iv) $4 \leq \dim(Y)$.

Then ϕ *arises from a* **uniquely determined dominant** *morphism* $Y \to X$ *over* k.

Remark A.1.

(i) Theorem A in the case where condition (1) is satisfied, k is *finitely generated over the field of rational numbers*, both X and Y are *affine hyperbolic curves* over k, and ϕ is an *isomorphism* was proved in [25] [cf. [25], Theorem (0.3)].
(ii) Theorem A in the case where condition (1) is satisfied was essentially proved in [16] [cf. [16], Theorem A].
(iii) Theorem A in the case where condition (2) is satisfied, Y is a *hyperbolic polycurve of dimension* 2 over k, and ϕ is an *isomorphism* was proved in [16] [cf. [16], Theorem D].

One of the main ingredients of the proof of Theorem A is Theorem A in the case where condition (1) is satisfied [that was essentially proved by Mochizuki — cf. Remark A.1, (ii)]. Another main ingredient of the proof of Theorem A is the *elasticity* [cf. [19], Definition 1.1, (ii)] of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic zero. That is to say, if C is a hyperbolic curve over an algebraically closed field F of characteristic zero, then, for a closed subgroup $H \subseteq \Pi_C$ of the étale fundamental group Π_C of C, it holds that H is *open* in Π_C if and only if H is *topologically finitely generated*, *nontrivial*, and *normal* in an open subgroup of Π_C. An immediate consequence of this *elasticity* is as follows:

> Let V be a variety over F and $\phi\colon \Pi_V \to \Pi_C$ a homomorphism. Suppose that the image of ϕ is *normal* in an open subgroup of Π_C. Then ϕ is *nontrivial* if and only if ϕ is *open*.

Let us observe that this equivalence may be regarded as a *group-theoretic analogue* of the following easily verified *scheme-theoretic fact*:

> Let V be a variety over F and $f\colon V \to C$ a morphism over F. Then the image of f is *not a point* if and only if f is *dominant*.

The following result follows immediately from Theorem A [cf. Corollary 3.19 in the case where both X and Y are *hyperbolic polycurves*]. That is to say, roughly speaking, the isomorphism class of a hyperbolic polycurve of dimension less than or equal to four over a sub-p-adic field is *completely determined* by its étale fundamental group.

Theorem B. *Let p be a prime number; k a* **sub-p-adic field** [*cf. Definition 3.1*]*; $\overline{k}$ an algebraic closure of k; X, Y* **hyperbolic polycurves** [*cf. Definition 2.1,* (ii)] *over k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; Π_X, Π_Y for the étale fundamental groups of X, Y, respectively;*

$$\mathrm{Isom}_k(X, Y)$$

for the set of isomorphisms of X with Y over k;

$$\mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)$$

for the set of isomorphisms of Π_X with Π_Y over G_k; $\Delta_{Y/k}$ for the kernel of the natural surjection $\Pi_Y \twoheadrightarrow G_k$. Suppose that either X or Y is of **dimension** ≤ 4. *Then the natural map*

$$\mathrm{Isom}_k(X, Y) \longrightarrow \mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_{Y/k})$$

is **bijective**.

Next, let us observe that if X and Y are *hyperbolic polycurves* over a sub-p-adic field k, then the *finiteness* of the set of isomorphisms over k

$$\mathrm{Isom}_k(X, Y)$$

may be easily verified [cf., e.g., Proposition 4.5]. Thus, if the natural map discussed in Theorem B is *bijective* for *arbitrary hyperbolic polycurves over sub-p-adic fields* [i.e., Theorem B without the assumption that "either X or Y is of dimension ≤ 4" holds], then it follows that the set

$$\mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_{Y/k})$$

is *finite*. Unfortunately, it is not clear to the author at the time of writing whether or not such a generalization of Theorem B holds. Nevertheless, the following result asserts that the above set is, in fact, *finite* [cf. Theorem 4.4].

Theorem C. *Let p be a prime number; k a* **sub-p-adic field** [*cf. Definition 3.1*]*; $\overline{k}$ an algebraic closure of k; X, Y* **hyperbolic polycurves** [*cf. Definition 2.1,* (ii)] *over k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; Π_X, Π_Y for the étale fundamental groups of X, Y, respectively; $\mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)$ for the set of isomorphisms of Π_X with Π_Y over G_k; $\Delta_{Y/k}$ for the kernel of the natural surjection $\Pi_Y \twoheadrightarrow G_k$. Then the quotient set*

$$\mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_{Y/k})$$

is **finite**.

In the notation of Theorem C, if k is *finite over the field of rational numbers*, then we also prove the *finiteness* of the set of outer isomorphisms of Π_X with Π_Y [cf. Corollary 4.6].

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1. Exactness of certain homotopy sequences

In the present §1, we consider the *exactness* of certain homotopy sequences [cf. Proposition 1.10, (i)] and prove that the *topological finite generation* of the kernel of the outer homomorphism between étale fundamental groups induced by a certain morphism of schemes [cf. Corollary 1.11]. In the present §1, let k be a field of *characteristic zero*, $\overline{k}$ an algebraic closure of k, and $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

Definition 1.1. Let X be a *connected noetherian* scheme.

(i) We shall write
$$\Pi_X$$
for the étale fundamental group of X [for some choice of basepoint].

(ii) Let Y be a *connected noetherian* scheme and $f\colon X \to Y$ a morphism. Then we shall write
$$\Delta_f = \Delta_{X/Y} \subseteq \Pi_X$$
for the kernel of the outer homomorphism $\Pi_X \to \Pi_Y$ induced by f.

Lemma 1.2. *Let X be a connected noetherian* **normal** *scheme. Write $\eta \to X$ for the generic point of X. Then the outer homomorphism $\Pi_\eta \to \Pi_X$ induced by the morphism $\eta \to X$ is* **surjective***. In particular, if $U \subseteq X$ is an open subscheme, then the outer homomorphism $\Pi_U \to \Pi_X$ induced by the open immersion $U \hookrightarrow X$ is* **surjective***.*

Proof. This follows from [26], Exposé V, Proposition 8.2. □

Lemma 1.3. *Let X, Y be connected noetherian schemes and $f\colon X \to Y$ a morphism. Suppose that Y is* **normal***, and that f is* **dominant** *and of* **finite type***. Then the outer homomorphism $\Pi_X \to \Pi_Y$ induced by f is* **open***.*

Proof. Since f is *dominant* and of *finite type*, it follows that there exists a *finite* extension K of the function field of Y such that the natural morphism $\mathrm{Spec}\, K \to Y$ factors through f. Thus, it follows immediately from Lemma 1.2 that $\Pi_X \to \Pi_Y$ is *open*. This completes the proof of Lemma 1.3. □

Definition 1.4. Let X be a scheme over k. Then we shall say that X is a *variety* over k if X is of finite type, separated, and geometrically connected over k.

Lemma 1.5. *Let X be a variety over k. Then the sequence of schemes $X \otimes_k \overline{k} \stackrel{\mathrm{pr}_1}{\to} X \to \mathrm{Spec}\, k$ determines an* **exact** *sequence of profinite groups*
$$1 \longrightarrow \Pi_{X\otimes_k \overline{k}} \longrightarrow \Pi_X \longrightarrow G_k \longrightarrow 1\,.$$

In particular, we obtain an **isomorphism** $\Pi_{X\otimes_k\overline{k}} \xrightarrow{\sim} \Delta_{X/k}$ [*which is well-defined up to* Π_X*-conjugation*].

Proof. This follows from [26], Exposé IX, Théorème 6.1. □

Lemma 1.6. *Let* X, Y *be connected noetherian schemes and* $f\colon X \to Y$ *a morphism. Suppose that* f *is* **of finite type**, **separated**, **dominant** *and* **generically geometrically connected**. *Suppose, moreover, that* Y *is* **normal**. *Then the outer homomorphism* $\Pi_X \twoheadrightarrow \Pi_Y$ *induced by* f *is* **surjective**.

Proof. Write $\eta \to Y$ for the generic point of Y. Then since $X \to Y$ is *dominant* and *generically geometrically connected*, we obtain a commutative diagram of *connected* schemes

$$\begin{array}{ccc} X \times_Y \eta & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \eta & \longrightarrow & Y\,. \end{array}$$

Now since Y is *normal*, and [one verifies easily that] $X \times_Y \eta$ is a variety over η [i.e., over the function field of Y], it follows immediately from Lemmas 1.2; 1.5 that the outer homomorphism $\Pi_X \to \Pi_Y$ is *surjective*. This completes the proof of Lemma 1.6. □

Lemma 1.7. *Let* X *be a variety over* k. *Suppose that* G_k *is* **topologically finitely generated** [*e.g.,* $k = \overline{k}$]. *Then the profinite group* Π_X *is* **topologically finitely generated**.

Proof. Since [we have assumed that] k is of *characteristic zero*, this follows from [27], Exposé II, Théorème 2.3.1, together with Lemma 1.5. □

Definition 1.8. Let X, Y be *integral* noetherian schemes and $f\colon X \to Y$ a *dominant* morphism of *finite type*. Then we shall write

$$\mathrm{Nor}(f) = \mathrm{Nor}(X/Y) \longrightarrow Y$$

for the *normalization* of Y in [the necessarily finite extension of the function field of Y obtained by forming its algebraic closure in the function field of] X. Note that it follows immediately from the various definitions involved that $\mathrm{Nor}(f) = \mathrm{Nor}(X/Y)$ is *irreducible* and *normal*, and the morphism $\mathrm{Nor}(f) = \mathrm{Nor}(X/Y) \to Y$ is *dominant* and *affine*.

Lemma 1.9. *Let* X, Y *be integral noetherian schemes and* $f\colon X \to Y$ *a* **dominant** *morphism of* **finite type**. *Suppose that* X *is* **normal**. *Then* f *factors through the natural morphism* $\mathrm{Nor}(f) \to Y$, *and the resulting morphism* $X \to \mathrm{Nor}(f)$ *is* **dominant** *and* **generically geometrically irreducible** [*i.e., there exists an open subscheme* $U \subseteq \mathrm{Nor}(f)$ *of* $\mathrm{Nor}(f)$ *such that the geometric fiber of* $X \times_{\mathrm{Nor}(f)}$

$U \stackrel{\mathrm{pr}_2}{\to} U$ *at any geometric point of* U *is* **irreducible** — *cf.* [6], *Proposition* (*9.7.8*)]. *If, moreover,* X *and* Y *are* **varieties** *over* k*, then the natural morphism* $\mathrm{Nor}(f) \to Y$ *is* **finite** *and* **surjective***, and* $\mathrm{Nor}(f)$ *is a* **normal variety** *over* k*.*

Proof. The assertion that f factors through the natural morphism $\mathrm{Nor}(f) \to Y$ and the assertion that the resulting morphism $X \to \mathrm{Nor}(f)$ is *dominant* follow immediately from the various definitions involved. The assertion that the resulting morphism $X \to \mathrm{Nor}(f)$ is *generically geometrically irreducible* follows immediately from [5], Proposition (4.5.9). Finally, we verify that if, moreover, X and Y are *varieties* over k, then the natural morphism $\mathrm{Nor}(f) \to Y$ is *finite* and *surjective*, and $\mathrm{Nor}(f)$ is a *normal variety* over k. Now since Y is a *variety* over k, it follows immediately from the discussion following [13], §33, Lemma 2, that $\mathrm{Nor}(f) \to Y$ is *finite*. Thus, since $\mathrm{Nor}(f) \to Y$ is *dominant* [cf. Definition 1.8], we conclude that $\mathrm{Nor}(f) \to Y$ is *surjective*. On the other hand, since $\mathrm{Nor}(f) \to Y$ is *separated* and of *finite type* [cf. the *finiteness* of $\mathrm{Nor}(f) \to Y$], to verify that $\mathrm{Nor}(f)$ is a *normal variety* over k, it suffices to verify that $\mathrm{Nor}(f)$ is *geometrically irreducible* over k. On the other hand, since $\mathrm{Nor}(f) \to Y$ is *dominant*, this follows immediately from [5], Proposition (4.5.9), together with our assumption that X is *geometrically irreducible* over k [cf. the fact that X is a *normal variety* over k]. This completes the proof of Lemma 1.9. □

Proposition 1.10. *Let* S*,* X*, and* Y *be connected noetherian* **normal** *schemes and* $Y \to X \to S$ *morphisms of schemes. Suppose that the following conditions are satisfied:*

(1) $Y \to X$ *is* **dominant** *and induces an outer* **surjection** $\Pi_Y \twoheadrightarrow \Pi_X$.
(2) $X \to S$ *is* **surjective***,* **of finite type***,* **separated***, and* **generically geometrically integral**.
(3) $Y \to S$ *is* **of finite type***,* **separated***,* **faithfully flat***,* **geometrically normal***, and* **generically geometrically connected**.

Then the following hold:

(i) *Let* $\overline{s} \to S$ *be a geometric point of* S *that satisfies the following condition*
  (4) *For any connected finite étale covering* $X' \to X$ *and any geometric point* $\overline{s}' \to \mathrm{Nor}(X'/S)$ *of* $\mathrm{Nor}(X'/S)$ *that lifts the geometric point* $\overline{s}$ *of* S*, the geometric fiber* $X' \times_{\mathrm{Nor}(X'/S)} \overline{s}'$ *of* $X' \to \mathrm{Nor}(X'/S)$ [*cf. Lemma 1.9*] *at* $\overline{s}' \to \mathrm{Nor}(X'/S)$ *is* **connected**. [*Note that it follows from Lemma 1.9 that a geometric point of* S *whose image is the generic point of* S *satisfies condition* (4)].

Then the sequence of connected schemes $X \times_S \overline{s} \stackrel{\mathrm{pr}_1}{\to} X \to S$ [*note that* $X \times_S \overline{s}$ *is connected by conditions* (2), (4) *— cf. also* [5], *Corollaire* (*4.6.3*)] *determines an* **exact** *sequence of profinite groups*

$$\Pi_{X\times_S\overline{s}} \longrightarrow \Pi_X \longrightarrow \Pi_S \longrightarrow 1\,.$$

(ii) *If, moreover, the function field of* S *is of* **characteristic zero**, *then* $\Delta_{X/S}$ *is* **topologically finitely generated**.

Proof. Let us first observe that it follows from Lemma 1.7, together with the fact that a geometric point of S whose image is the generic point of S satisfies condition (4) [cf. condition (4)], that assertion (ii) follows from assertion (i). Thus, to verify Proposition 1.10, it suffices to verify assertion (i). Next, let us observe that since the composite $X \times_S \overline{s} \to X \to S$ factors through $\overline{s} \to S$, it follows that the composite $\Pi_{X\times_S\overline{s}} \to \Pi_X \to \Pi_S$ is *trivial*. On the other hand, it follows immediately from Lemma 1.6 that the outer homomorphism $\Pi_X \to \Pi_S$ is *surjective*. Thus, it follows immediately from the various definitions involved that, to verify Proposition 1.10, it suffices to verify that the following assertion holds:

> Claim 1.10.A: Let $X' \to X$ be a connected finite étale covering of X such that the natural morphism $X' \times_S \overline{s} \to X \times_S \overline{s}$ has a section. Then there exists a finite étale covering of S whose pullback by $X \to S$ is isomorphic to X' over X.

To verify Claim 1.10.A, write $T \stackrel{\mathrm{def}}{=} \mathrm{Nor}(X'/S) \to S$. Now let us observe that since X is *connected*, and $X' \to X$ is *finite* and *étale* [hence *closed* and *open*], it follows that $X' \to X$, hence also $Y' \stackrel{\mathrm{def}}{=} Y \times_X X' \stackrel{\mathrm{pr}_1}{\to} Y$, is *surjective*.

Next, to verify Claim 1.10.A, I claim that the following assertion holds:

> Claim 1.10.A.1: Y, $Y_T \stackrel{\mathrm{def}}{=} Y \times_S T$, and Y' are *irreducible* and *normal*.

Indeed, we have assumed that Y is *normal*. Thus, since $X' \to X$, hence also $Y' \to Y$, is *étale*, it follows that Y' is *normal*. On the other hand, since T is *normal*, and $Y \to S$, hence also $Y_T \to T$, is *geometrically normal*, it follows from [6], Proposition (11.3.13), (ii), that Y_T is *normal*.

Since Y_T and Y' are *normal*, to verify Claim 1.10.A.1, it suffices to verify that Y_T and Y' are *connected*. Now let us observe that the assertion that Y' is *connected* follows from our assumption that the natural outer homomorphism $\Pi_Y \to \Pi_X$ is *surjective*. Next, to verify that Y_T is *connected*, let $U \subseteq Y_T$ be a *nonempty* connected component of Y_T. Then since $Y \to S$, hence also $Y_T \to T$, is *flat* and of *finite type*, hence *open*, the images of U and $Y_T \setminus U$ in T are *open* in T. Thus, since $Y \to S$, hence also $Y_T \to T$, is *generically geometrically connected*, it follows that the image of $Y_T \setminus U$ in T, hence also $Y_T \setminus U$, is *empty*. This completes the proof of the assertion that Y_T is *connected*, hence also of Claim 1.10.A.1.

Next, to verify Claim 1.10.A, I claim that the following assertion holds:

> Claim 1.10.A.2: The natural morphism $T \to S$, hence also $Y_T \to Y$, is *finite*.

Indeed, since $Y \to S$ is *geometrically normal*, one verifies easily that $Y' \to S$ is *geometrically reduced*. Thus, it follows from [5], Corollaire (4.6.3), that the [necessarily finite] extension of the function field of S obtained by forming its algebraic closure in the function field of Y' [cf. Claim 1.10.A.1], hence also X', is *separable*. In particular, since S is *normal*, the natural morphism $T \to S$ is *finite* [cf., e.g., [13], §33, Lemma 1]. This completes the proof of Claim 1.10.A.2.

Next, to verify Claim 1.10.A, I claim that the following assertion holds:

> Claim 1.10.A.3: The natural morphism $Y' \to Y_T$ is *finite* and *étale* [hence *closed* and *open*; thus, $Y' \to Y_T$ is *surjective* — cf. Claim 1.10.A.1].

Indeed, since Y' and Y_T are *finite* over Y [cf. Claim 1.10.A.2], and $Y' \to Y_T$ is a *morphism over* Y, one verifies easily that $Y' \to Y_T$ is *finite* [cf. [4], Proposition (4.4.2)]. In particular, in light of the *surjectivity* of $Y_T \to Y$ [that follows from the *surjectivity* of $Y' \to Y$ — cf. the discussion preceding Claim 1.10.A.1], by considering the fibers of $Y' \to Y_T \to Y$ at the generic point of Y, together with Claim 1.10.A.1, we conclude that $Y' \to Y_T$ is *dominant*, hence *surjective*. On the other hand, since $Y' \to Y$ is *unramified*, it follows from [7], Proposition (17.3.3), (v), that $Y' \to Y_T$ is *unramified*. Thus, since Y_T is *normal* [cf. Claim 1.10.A.1], it follows from [26], Exposé I, Corollaire 9.11, that $Y' \to Y_T$ is *étale*. This completes the proof of Claim 1.10.A.3.

Next, to verify Claim 1.10.A, I claim that the following assertion holds:

> Claim 1.10.A.4: The morphism $Y_T \to Y$ is *finite* and *étale*.

Indeed, the *finiteness* of $Y_T \to Y$ was already verified in Claim 1.10.A.2. Thus, since Y and Y_T are *irreducible* and *normal* [cf. Claim 1.10.A.1], and $Y_T \to Y$ is *surjective* [cf. the proof of Claim 1.10.A.3], it follows from [26], Exposé I, Corollaire 9.11, that, to verify Claim 1.10.A.4, it suffices to verify that $Y_T \to Y$ is *unramified*. To this end, let Ω be a separably closed field and $\overline{y} \stackrel{\text{def}}{=}$

$\mathrm{Spec}\,\Omega \to Y$ a morphism of schemes. Then since $Y' \to Y$ is *unramified*, $Y' \times_Y \overline{y}$ is isomorphic to the disjoint union of finitely many copies of $\mathrm{Spec}\,\Omega$. Thus, since $Y' \to Y_T$ is *surjective* and *étale* [cf. Claim 1.10.A.3], we conclude that $Y_T \times_Y \overline{y}$ is isomorphic to the disjoint union of finitely many copies of $\mathrm{Spec}\,\Omega$, i.e., $Y_T \to Y$ is *unramified*. This completes the proof of Claim 1.10.A.4.

Next, to verify Claim 1.10.A, I claim that the following assertion holds:

> Claim 1.10.A.5: The morphism $T \to S$, hence also $X_T \stackrel{\mathrm{def}}{=} X \times_S T \stackrel{\mathrm{pr}_1}{\to} X$, is *finite* and *étale*. Moreover, X_T is *connected*, and the natural morphism $X' \to X_T$ is *finite* and *étale* [hence *closed* and *open*; thus, $X' \to X_T$ is *surjective*].

Indeed, since [we have assumed that] the composite $Y \to X \to S$ is *faithfully flat* and *quasi-compact*, it follows from Claim 1.10.A.4, together with [5], Proposition (2.7.1); [7], Corollaire (17.7.3), (ii), that $T \to S$, hence also $X_T \to X$, is *finite* and *étale*. Thus, the *connectedness* of X_T follows immediately from the *surjectivity* of the natural outer homomorphism $\Pi_X \to \Pi_S$ [cf. the discussion preceding Claim 1.10.A]. Finally, we verify that $X' \to X_T$ is *finite* and *étale*. The *finiteness* and *unramifiedness* of $X' \to X_T$ follow immediately from a similar argument to the argument used in the proof of the assertion that $Y' \to Y_T$ is *finite* and *unramified* [cf. the proof of 1.10.A.3]. On the other hand, since X' and X_T are *flat* over X, the *flatness* of $X' \to X_T$ follows immediately from [26], Exposé I, Corollaire 5.9, together with the *unramifiedness* of $X_T \to X$, which implies that the fiber of $X_T \to X$ at any point of X is isomorphic to the disjoint union of finitely many spectrums of fields. This completes the proof of Claim 1.10.A.5.

Since $T \to S$ is a *finite étale covering* [cf. Claim 1.10.A.5], it is immediate that, to verify Proposition 1.10, i.e., to verify Claim 1.10.A, it suffices to verify that the *finite étale covering* $X' \to X_T$ [cf. Claim 1.10.A.5] is an *isomorphism*. On the other hand, let us observe that, since X' and X_T are *connected* [cf. Claim 1.10.A.5], to verify Claim 1.10.A, it suffices to verify that the finite étale covering $X' \to X_T$ is of *degree one*. Write d for the degree of the finite étale covering $T \to S$. Then since [we have assumed that] $X \times_S \overline{s}$ is connected, it follows immediately that the number of the connected components of $X_T \times_S \overline{s}$ is d. Moreover, it follows immediately from our choice of $\overline{s} \to S$ [cf. condition (4)] that the number of the connected components of $X' \times_S \overline{s}$ is d. Thus, since $X' \to X_T$ is *surjective* [cf. Claims 1.10.A.5], the morphism $X' \to X_T$ determines a *bijection* between the set of the connected components of $X' \times_S \overline{s}$ and the set of the connected components of $X_T \times_S \overline{s}$. On the other hand, let us recall that we have assumed that the natural morphism $X' \times_S \overline{s} \to X \times_S \overline{s}$ has a section. Thus, by considering the connected component of $X' \times_S \overline{s}$ obtained by forming the image of a section of $X' \times_S \overline{s} \to X \times_S \overline{s}$, one verifies easily that the finite étale covering $X' \to X_T$ is of *degree one*. This completes the proof of Claim 1.10.A, hence also of Proposition 1.10. □

Corollary 1.11. *Let S, X be connected noetherian* **normal** *schemes and $X \to S$ a morphism of schemes that is* **surjective***,* **of finite type***,* **separated***, and* **generically geometrically irreducible***. Suppose that the function field of S is of* **characteristic zero***. Suppose, moreover, that one of the following conditions is satisfied:*

(1) *There exists an open subscheme $U \subseteq X$ of X such that the composite $U \hookrightarrow X \to S$ is* **surjective** *and* **smooth***.*
(2) *There exist a connected* **normal** *scheme Y and a* **modification** *$Y \to X$ [i.e., $Y \to X$ is* **proper***,* **surjective***, and* **induces** *an isomorphism between their function fields] such that the composite $Y \to X \to S$ is* **smooth***.*

Proof. Suppose that condition (1) (respectively, (2)) is satisfied. Then, to verify Corollary 1.11, it follows from Proposition 1.10, (ii), that it suffices to verify that the scheme U (respectively, Y) over X in condition (1) (respectively, (2)) satisfies the condition for "Y" in the statement of Proposition 1.10. On the other hand, this follows immediately from Lemma 1.2. This completes the proof of Corollary 1.11. □

2. ÉTALE FUNDAMENTAL GROUPS OF HYPERBOLIC POLYCURVES

In the present §2, we discuss the generalities on the étale fundamental groups of *hyperbolic polycurves*. In the present §2, let k be a field of *characteristic zero*, $\overline{k}$ an algebraic closure of k, and $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

Definition 2.1. Let S be a scheme and X a scheme over S.

(i) We shall say that X is a *hyperbolic curve* [*of type* (g, r)] over S if there exist
- a pair of nonnegative integers (g, r);
- a scheme X^{cpt} which is smooth, proper, geometrically connected, and of relative dimension one over S;
- a [possibly empty] closed subscheme $D \subseteq X^{\mathrm{cpt}}$ of X^{cpt} which is finite and étale over S

such that
- $2g - 2 + r > 0$;
- any geometric fiber of $X^{\mathrm{cpt}} \to S$ is [a necessarily smooth proper curve] of genus g;
- the finite étale covering $D \hookrightarrow X^{\mathrm{cpt}} \to S$ is of degree r;
- X is isomorphic to $X^{\mathrm{cpt}} \setminus D$ over S.

(ii) We shall say that X is a *hyperbolic polycurve* [*of relative dimension* n] over S if there exist a positive integer n and a [not necessarily unique] factorization of the structure morphism $X \to S$

$$X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow S = X_0$$

such that, for each $i \in \{1, \cdots, n\}$, $X_i \to X_{i-1}$ is a hyperbolic curve [cf. (i)]. We shall refer to the above morphism $X \to X_{n-1}$ as a *parametrizing morphism* of X and refer to the above factorization of $X \to S$ as a *sequence of parametrizing morphisms*.

Remark 2.1.1. In the notation of Definition 2.1, (ii), suppose that S is a *normal* (respectively, *regular*) variety of dimension m over k. Then one verifies easily that any *hyperbolic polycurve of relative dimension* n over S is a *normal* (respectively, *regular*) *variety of dimension* $n + m$ over k.

Definition 2.2. In the notation of Definition 2.1, (i), suppose that S is *normal*. Then it follows from the argument given in the discussion entitled "Curves" in [17], §0, that the pair "(X^{cpt}, D)" of Definition 2.1, (i), is uniquely determined up to canonical isomorphism over S. We shall refer to X^{cpt} as the *smooth compactification* of X over S and refer to D as the *divisor of cusps* of X over S.

Proposition 2.3. *Let n be a positive integer, S a connected noetherian separated* **normal** *scheme over k, X a* **hyperbolic polycurve** *of relative dimension n over S,*

$$X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow S = X_0$$

a sequence of parametrizing morphisms, and $Y \to X$ a connected finite étale covering of X. For each $i \in \{0, \cdots, n\}$, write $Y_i \stackrel{\text{def}}{=} \mathrm{Nor}(Y/X_i)$. Then the following hold:

(i) *For each $i \in \{1, \cdots, n\}$, Y_i is a* **hyperbolic curve** *over Y_{i-1}. Moreover, if we write Y_i^{cpt} for the smooth compactification of the hyperbolic curve Y_i over Y_{i-1} [cf. Definition 2.2], then the composite $Y_i^{\text{cpt}} \to Y_{i-1} \to X_{i-1}$ is* **proper** *and* **smooth**. *Furthermore, if we write $Y_i^{\text{cpt}} \to Z_{i-1} \to X_{i-1}$ for the Stein factorization of the proper morphism $Y_i^{\text{cpt}} \to X_{i-1}$, then Z_{i-1} is isomorphic to Y_{i-1} over X_{i-1}.*

(ii) *For each $i \in \{0, \cdots, n\}$, the natural morphism $Y_i \to X_i$ is a connected* **finite étale covering**.

In particular, Y is a **hyperbolic polycurve** *of relative dimension n over $\mathrm{Nor}(Y/S)$, and the factorization*

$$Y = Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow \mathrm{Nor}(Y/S) = Y_0$$

is a sequence of parametrizing morphisms.

Proof. First, I claim that the following assertion holds:

Claim 2.3.A: If $n = 1$, then Proposition 2.3 holds.

Indeed, write X^{cpt} for the smooth compactification of X over S [cf. Definition 2.2]; $D \subseteq X^{\text{cpt}}$ for the divisor of cusps of X over S [cf. Definition 2.2]; $Y^{\text{cpt}} \stackrel{\text{def}}{=} \mathrm{Nor}(Y/X^{\text{cpt}})$; E for the *reduced* closed subscheme of Y^{cpt} whose support is the complement $Y^{\text{cpt}} \setminus Y$ [cf. [4], Corollaire (4.4.9)]; $T \stackrel{\text{def}}{=} \mathrm{Nor}(Y/S)$. Let us observe that since S and X^{cpt} are *normal schemes over k*, and k is of *characteristic zero*, the natural morphisms $T \to S$ and $Y^{\text{cpt}} \to X^{\text{cpt}}$ are *finite* [cf., e.g., [13], §33, Lemma 1], and, moreover, the basechange by a geometric generic point of S of the natural morphism $Y^{\text{cpt}} \to X^{\text{cpt}}$ is a *tamely ramified covering* along [the basechange by the geometric generic point of S of] $D \subseteq X^{\text{cpt}}$. [Note that it follows immediately from the definition of the term "hyperbolic curve" that D is a *divisor with normal crossings of X^{cpt} relative to S* — cf. [26], Exposé XIII, §2.1.] In particular, it follows immediately from Abhyankar's lemma [cf. [26], Exposé XIII, Proposition 5.5] that Y^{cpt} is *smooth* over S, and, moreover, E is *étale* over S. Write $Y^{\text{cpt}} \to Z \to S$ for the Stein factorization of $Y^{\text{cpt}} \to S$. [Note that since Y^{cpt} is *finite* over X^{cpt}, and X^{cpt} is *proper* over S, Y^{cpt} is *proper* over S.] Then since [one verifies easily that] Z and T are *irreducible* and *normal*, and the resulting morphism $Z \to T$ is *finite* and induces an isomorphism between their function fields, it follows from [4], Corollaire (4.4.9) that Z is isomorphic to T over S. On the other hand, since Y^{cpt} is *proper* and *smooth* over S, it follows from [26], Exposé X, Proposition 1.2, that Z, hence also T, is a *finite étale covering* of S. In particular, it follows from [7], Proposition (17.3.4), together with the fact that Y^{cpt} (respectively, E) is *smooth* (respectively, *étale*) over S, we conclude that Y^{cpt} (respectively, E) is *smooth* (respectively, *étale*) over T. Now one verifies easily that the pair $(Y^{\mathrm{cpt}}, E \subseteq Y^{\mathrm{cpt}})$ satisfies the condition in Definition 2.1, (i), for "$(X^{\mathrm{cpt}}, D \subseteq X^{\mathrm{cpt}})$". This completes the proof of Claim 2.3.A.

Next, I claim that the following assertion holds:

> Claim 2.3.B: For a fixed $i_0 \in \{1, \cdots, n\}$, if assertion (i) in the case where we take "i" to be i_0 holds, then assertion (ii) in the case where we take "i" to be $i_0 - 1$ holds.

Indeed, it follows from assertion (i) in the case where we take "i" to be i_0 that, to verify assertion (ii) in the case where we take "i" to be $i_0 - 1$, it suffices to verify that $Z_{i_0-1} \to X_{i_0-1}$ is a *finite étale covering*. On the other hand, since the composite $Y_{i_0}^{\mathrm{cpt}} \to Y_{i_0-1} \to X_{i_0-1}$ is *proper* and *smooth* [cf. assertion (i) in the case where we take "i" to be i_0], this follows from [26], Exposé X, Proposition 1.2. This completes the proof of Claim 2.3.B.

Next, I claim that the following assertion holds:

> Claim 2.3.C: For a fixed $i_0 \in \{1, \cdots, n\}$, if assertion (ii) in the case where we take "i" to be i_0 holds, then assertion (i) in the case where we take "i" to be i_0 holds.

Indeed, by applying Claim 2.3.A to the connected *finite étale covering* $Y_{i_0} \to X_{i_0}$ [cf. assertion (ii) in the case where we take "i" to be i_0] of the hyperbolic curve X_{i_0} over X_{i_0-1}, we conclude that assertion (i) in the case where we take "i" to be i_0 holds. This completes the proof of Claim 2.3.C.

Now let us observe that assertion (ii) in the case where we take "i" to be n is immediate. Thus, Proposition 2.3 follows immediately from Claims 2.3.B and 2.3.C. This completes the proof of Proposition 2.3. $\square$

Proposition 2.4. *Let $0 \leq m < n$ be integers, S a connected noetherian separated* **normal** *scheme over k, X a* **hyperbolic polycurve** *of relative dimension n over S, and*

$$X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow S = X_0$$

a sequence of parametrizing morphisms. Then the following hold:

(i) *For any geometric point $\overline{x}_m \to X_m$ of X_m, the sequence of connected schemes $X \times_{X_m} \overline{x}_m \overset{\mathrm{pr}_1}{\to} X \to X_m$ determines an* **exact** *sequence of profinite groups*

$$1 \longrightarrow \Pi_{X \times_{X_m} \overline{x}_m} \longrightarrow \Pi_X \longrightarrow \Pi_{X_m} \longrightarrow 1 .$$

In particular, we obtain an **isomorphism** $\Pi_{X \times_{X_m} \overline{x}_m} \overset{\sim}{\to} \Delta_{X/X_m}$ [*which is well-defined up to* Π_X*-conjugation*].

(ii) *Let T be a connected noetherian separated* **normal** *scheme over S and $T \to X_m$ a morphism over S. Then the natural morphisms $X \times_{X_m} T \overset{\mathrm{pr}_1}{\to} X$ and $X \times_{X_m} T \overset{\mathrm{pr}_2}{\to} T$ determine an outer* **isomorphism**

$$\Pi_{X \times_{X_m} T} \overset{\sim}{\longrightarrow} \Pi_X \times_{\Pi_{X_m}} \Pi_T$$

and an **isomorphism**

$$\Delta_{X \times_{X_m} T/T} \overset{\sim}{\longrightarrow} \Delta_{X/X_m}$$

[*which is well-defined up to* Π_X*-conjugation*].

(iii) Δ_{X/X_m} *is* **nontrivial**, **topologically finitely generated**, *and* **torsion-free**. *In particular,* Δ_{X/X_m} *is* **infinite**.

(iv) Δ_{X_{m+1}/X_m} *is* **elastic** [*cf.* [19], *Definition 1.1,* (ii)], *i.e., the following holds: Let $N \subseteq \Delta_{X_{m+1}/X_m}$ be a* **topologically finitely generated** *closed subgroup of Δ_{X_{m+1}/X_m} that is* **normal** *in an open subgroup of Δ_{X_{m+1}/X_m}. Then N is* **nontrivial** *if and only if N is* **open** *in Δ_{X_{m+1}/X_m}.*

(v) *Suppose that the hyperbolic curve X_{m+1} over X_m is of type (g, r)* [*cf. Definition 2.1,* (i)]. *Then the abelianization of Δ_{X_{m+1}/X_m} is a* **free $\widehat{\mathbb{Z}}$-module of rank $2g + \max\{r-1, 0\}$**.

(vi) *For any positive integer N, there exists an open subgroup $H \subseteq \Delta_{X_{m+1}/X_m}$ of Δ_{X_{m+1}/X_m} such that the abelianization of H is* [*a free $\widehat{\mathbb{Z}}$-module*] *of* **rank $\geq N$**.

Proof. First, we verify assertion (i). Let us observe that it follows immediately from Lemma 1.6; Proposition 2.3, (i), together with the various definitions involved, that $(X_m, X, X, \overline{x}_m \to X_m)$ satisfies the four conditions (1), (2), (3), and (4) [for "$(S, X, Y, \overline{s} \to S)$"] in the statement of Proposition 1.10. Thus, It follows immediately from Proposition 1.10, (i), that the sequence of profinite groups

$$\Pi_{X \times_{X_m} \overline{x}_m} \longrightarrow \Pi_X \longrightarrow \Pi_{X_m} \longrightarrow 1$$

is *exact*. Thus, to verify assertion (i), it suffices to verify that $\Pi_{X \times_{X_m} \overline{x}_m} \to \Pi_X$ is *injective*. Now I claim that the following assertion holds:

> Claim 2.4.A: If $n = 1$ [thus, $m = 0$], i.e., X is a *hyperbolic curve* over S, and the finite étale covering of S obtained by forming the divisor of cusps of

the hyperbolic curve X over S [cf. Definition 2.2] is *trivial*, then $\Pi_{X\times_{X_m}\overline{x}_m} \to \Pi_X$ is *injective*.

Indeed, write (g,r) for the *type* of the hyperbolic curve X over S; $\mathcal{M}_{g,r}$, $\mathcal{M}_{g,r+1}$ for the moduli stacks over k of ordered r-, $(r+1)$-pointed smooth proper curves of genus g, respectively; $\Pi_{\mathcal{M}_{g,r}}$, $\Pi_{\mathcal{M}_{g,r+1}}$ for the étale fundamental groups of $\mathcal{M}_{g,r}$, $\mathcal{M}_{g,r+1}$, respectively. Then since [we have assumed that] the finite étale covering of S obtained by forming the divisor of cusps of the hyperbolic curve X over S is *trivial*, it follows immediately from the various definitions involved that there exists a morphism of stacks $s_X \colon S \to \mathcal{M}_{g,r}$ over k such that the fiber product of s_X and the morphism of stacks $\mathcal{M}_{g,r+1} \to \mathcal{M}_{g,r}$ over k obtained by forgetting the last marked point is isomorphic to X over S. Thus, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} & & \Pi_{X\times_S\overline{x}_0} & \longrightarrow & \Pi_X & \longrightarrow & \Pi_S & \longrightarrow & 1 \\ & & \wr\downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi_{\mathcal{M}_{g,r+1}\times_{\mathcal{M}_{g,r}}\overline{x}_0} & \longrightarrow & \Pi_{\mathcal{M}_{g,r+1}} & \longrightarrow & \Pi_{\mathcal{M}_{g,r}} & \longrightarrow & 1 \end{array}$$

— where the right-hand vertical arrow is the outer homomorphism induced by s_X, the left-hand vertical arrow is an *isomorphism*, and the horizontal sequences are *exact* [cf., e.g., [12], Lemma 2.1; the discussion preceding Claim 2.4.A]. In particular, it follows that $\Pi_X \times_S \overline{x}_0 \to \Pi_X$ is *injective*. This completes the proof of Claim 2.4.A.

Next, I claim that the following assertion holds:

Claim 2.4.B: If $n = 1$ [thus, $m = 0$], then $\Pi_{X\times_{X_m}\overline{x}_m} \to \Pi_X$ is *injective*.

Indeed, since the divisor of cusps of X over S is a *finite étale covering* of S, there exists a connected finite étale covering $S' \to S$ of S such that the finite étale covering of S' obtained by forming the divisor of cusps of the hyperbolic curve $X \times_S S'$ over S' is *trivial*. Thus, we have a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{X\times_S\overline{x}_0} & \longrightarrow & \Pi_{X\times_S S'} & \longrightarrow & \Pi_{S'} & \longrightarrow & 1 \\ & & \Vert & & \downarrow & & \downarrow & & \\ & & \Pi_{X\times_S\overline{x}_0} & \longrightarrow & \Pi_X & \longrightarrow & \Pi_S & \longrightarrow & 1 \end{array}$$

— where the vertical arrows are outer *open injections*, and the horizontal sequences are *exact* [cf. Claim 2.4.A; the discussion preceding Claim 2.4.A]. In particular, it follows that $\Pi_{X\times_S\overline{x}_0} \to \Pi_X$ is *injective*. This completes the proof of Claim 2.4.B.

Now, we verify the *injectivity* of $\Pi_{X\times_{X_m}\overline{x}_m} \to \Pi_X$ by induction on $n-m$. If $n-m=1$, then the *injectivity* of $\Pi_{X\times_{X_m}\overline{x}_m} \to \Pi_X$ follows immediately from Claim 2.4.B. Suppose that $n-m \geq 2$, and that the *induction hypothesis* is in force. Let $\overline{x}_{n-1} \to X_{n-1}$ be a geometric point of X_{n-1} that lifts the geometric point $\overline{x}_m \to X_m$. Then it follows immediately from various definitions involved that we have a commutative diagram of profinite groups

$$\begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & \downarrow \\ 1 & \longrightarrow & \Pi_{X \times_{X_{n-1}} \overline{x}_{n-1}} & \longrightarrow & \Pi_{X \times_{X_m} \overline{x}_m} & \longrightarrow & \Pi_{X_{n-1} \times_{X_m} \overline{x}_m} \\ & & \| & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \Pi_{X \times_{X_{n-1}} \overline{x}_{n-1}} & \longrightarrow & \Pi_X & \longrightarrow & \Pi_{X_{n-1}} \,; \end{array}$$

moreover, since X, $X \times_{X_m} \overline{x}_m$, X_{n-1} are *hyperbolic polycurves over* X_{n-1}*,* $X_{n-1} \times_{X_m} \overline{x}_m$*,* X_m *of relative dimension* 1*,* 1*,* $n-m-1$, respectively, it follows immediately from the *induction hypothesis* that the two horizontal sequences and the right-hand vertical sequence of the above diagram are *exact*. Thus, one verifies easily that $\Pi_{X \times_{X_m} \overline{x}_m} \to \Pi_X$ is *injective*. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i), together with the "Five lemma". Next, we verify assertion (iii). Let us observe that it follows from assertion (i) that, to verify assertion (iii), we may assume without loss of generality that $m = n-1$. On the other hand, if $m = n-1$, i.e., X is a hyperbolic curve over X_m, assertion (iii) is well-known [cf., e.g., [25], Proposition 1.1, (i); [25], Proposition 1.6]. This completes the proof of assertion (iii). Assertion (iv) follows from [20], Theorem 1.5. Assertion (v) is well-known [cf., e.g., [25], Corollary 1.2]. Assertion (vi) follows immediately from Hurwitz's formula [cf., e.g., [10], Chapter IV, Corollary 2.4], together with assertions (iii), (v). This completes the proof of Proposition 2.4. □

Definition 2.5 (cf. [11], §4.5)**.** Let X be a variety over k. Then we shall say that X is of *LFG-type* [where the "LFG" stands for "large fundamental group"] if, for any normal variety Y over $\overline{k}$ and any morphism $Y \to X \otimes_k \overline{k}$ over $\overline{k}$ that is not constant, the image of the outer homomorphism $\Pi_Y \to \Pi_{X \otimes_k \overline{k}}$ is infinite. Note that one verifies easily that the issue of whether or not X satisfies this condition does *not depend* on the choice of "$\overline{k}$" [cf. also Lemma 1.5].

Lemma 2.6. *Let* X*,* Y *be varieties over* k*. Suppose that* X *is of* **LFG-type**. *Then the following hold:*

(i) *Suppose that* Y *is* **quasi-finite** *over* X*. Then* Y *is of* **LFG-type**.
(ii) *Let* $f\colon X \to Y$ *be a morphism over* k*. Suppose that the kernel* Δ_f *is* **finite**. *Then* f *is* **quasi-finite**. *If, moreover,* f *is* **surjective***, then* Y *is of* **LFG-type**.

Proof. First, let us observe that, it follows from Lemma 1.5, together with the various definitions involved, that, by replacing k by $\overline{k}$, to verify Lemma 2.6, we may assume without loss of generality that $k = \overline{k}$. Now we verify assertion (i). Let Z be a normal variety over k and $Z \to Y$ a *nonconstant* morphism over k. Then since Y is *quasi-finite* over X, it follows that the composite $Z \to Y \to X$ is *nonconstant*. In particular, since X is of *LFG-type*, the image of the composite $\Pi_Z \to \Pi_Y \to \Pi_X$, hence also $\Pi_Z \to \Pi_Y$, is *infinite*. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $\overline{y} \to Y$ be a k-valued geometric point of Y and F a connected component [which is necessarily a *normal variety* over k] of the normalization of the geometric fiber of $X \to Y$ at $\overline{y}$. Then one verifies easily that the outer homomorphism $\Pi_F \to \Pi_X$ induced by the natural morphism $F \to X$ over k factors through $\Delta_f \subseteq \Pi_X$; in particular, since Δ_f is *finite*, the image of $\Pi_F \to \Pi_X$ is *finite*. Thus, since X is of *LFG-type*, one verifies easily that F is *finite* over k. This completes the proof of the fact that f is *quasi-finite*.

Finally, to verify that if, moreover, f is *surjective*, then Y is of *LFG-type*, let Z be a normal variety over k and $Z \to Y$ a *nonconstant* morphism over k. Then since f is a *quasi-finite surjection*, and $Z \to Y$ is *nonconstant*, one verifies easily that there exists a connected component C [which is necessarily a *normal variety* over k] of the normalization of $Z \times_Y X$ such that the natural morphism $C \to X$ over k is *nonconstant*. Thus, since X is of *LFG-type*, the image of $\Pi_C \to \Pi_X$, hence also $\Pi_C \to \Pi_X \to \Pi_Y$ [cf. the *finiteness* of Δ_f], is *infinite*. In particular, since the composite $C \to X \to Y$ factors through $Z \to Y$, we conclude that the image of $\Pi_Z \to \Pi_Y$ is *infinite*. This completes the proof of assertion (ii). □

Proposition 2.7. *Let S be a* **normal** *variety over k which is either* $\operatorname{Spec} k$ *or of* **LFG-type**. *Then every* **hyperbolic polycurve** *over S is of* **LFG-type**.

Proof. First, let us observe that it follows from Lemma 1.5, together with the various definitions involved, that, by replacing k by $\overline{k}$, to verify Proposition 2.7, we may assume without loss of generality that $k = \overline{k}$. Let X be a hyperbolic polycurve of relative dimension n over S. Then it follows immediately from *induction on* n that, to verify Proposition 2.7, we may assume without loss of generality that $n = 1$. Let Y be a normal variety over k and $Y \to X$ a *nonconstant* morphism over k.

Now suppose that the composite $Y \to X \to S$ is *nonconstant*, which thus implies that $S \neq \operatorname{Spec} k$. Then it follows from our assumption that S is of *LFG-type* that the image of the composite

$\Pi_Y \to \Pi_X \twoheadrightarrow \Pi_S$, hence also $\Pi_Y \to \Pi_X$, is *infinite*. This completes the proof of the *infiniteness* of the image of $\Pi_Y \to \Pi_X$ in the case where the composite $Y \to X \to S$ is *nonconstant*.

Next, suppose that the composite $Y \to X \to S$ is *constant*. Write $\overline{s} \to S$ for the k-valued geometric point of S through which the *constant* morphism $Y \to X \to S$ factors [cf. the fact that Y is a *normal variety* over k]. Then it is immediate that the composite $Y \to X \to S$ determines a *nonconstant*, hence *dominant*, morphism $Y \to X \times_S \overline{s}$ over k. Thus, since $\Pi_{X \times_S \overline{s}} \xrightarrow{\sim} \Delta_{X/S}$ [cf. Proposition 2.4, (i)] is *infinite* [cf. Proposition 2.4, (iii)], it follows immediately from Lemma 1.3 that the image of $\Pi_Y \to \Pi_{X \times_S \overline{s}} \xrightarrow{\sim} \Delta_{X/S} \hookrightarrow \Pi_X$ is *infinite*. This completes the proof of the *infiniteness* of the image of $\Pi_Y \to \Pi_X$ in the case where the composite $Y \to X \to S$ is *constant*, hence also of Proposition 2.7. $\square$

Lemma 2.8. *Let S be a connected noetherian separated* **normal** *scheme over k, X a* **hyperbolic curve** *over S, R a strictly henselian discrete valuation ring over S, K the field of fractions of R, and* $\mathrm{Spec}\, K \to X$ *a morphism over S. Then it holds that the morphism* $\mathrm{Spec}\, K \to X$ *factors through the open immersion* $\mathrm{Spec}\, K \hookrightarrow \mathrm{Spec}\, R$ *if and only if the image of the outer homomorphism* $\Pi_{\mathrm{Spec}\, K} \to \Pi_X$ *induced by the morphism* $\mathrm{Spec}\, K \to X$ *is* **trivial**.

Proof. First, let us recall [cf., e.g., [7], Théorème (18.5.11)] that $\Pi_{\mathrm{Spec}\, R} = \{1\}$. Thus, *necessity* is immediate; moreover, it follows immediately from Proposition 2.4, (ii), that, by replacing S by $\mathrm{Spec}\, R$, to verify *sufficiency*, we may assume without loss of generality that $S = \mathrm{Spec}\, R$. Next, let us observe that, by considering the exact sequence (1-5) of [25] with respect to a suitable connected finite étale covering of X, one verifies easily that, for each cusp of the hyperbolic curve X over R, the natural outer homomorphism from the étale fundamental group of the formal neighborhood of the cusp to $\Delta_{X/\mathrm{Spec}\, R}$ is *injective*. Thus, *sufficiency* follows immediately from the well-known explicit description of the universal profinite étale covering of the formal neighborhood of a cusp of a hyperbolic curve [given by, for instance, Abhyankar's Lemma — cf. [26], Exposé XIII, Proposition 5.5], together with the easily verified fact that every nonzero element of the maximal ideal of R is *not divisible* in $K^\times$. This completes the proof of Lemma 2.8. $\square$

Lemma 2.9. *Let S, Y, Z be* **normal** *varieties over k; $Z \to Y \to S$ morphisms over k; X a* **hyperbolic polycurve** *over S; $f \colon Z \to X$ a morphism over S.*

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S. \end{array}$$

Suppose that the following conditions are satisfied:

(1) $Z \to Y$ *is* **dominant** *and* **generically geometrically irreducible**. [*Thus, it follows from Lemma 1.6 that the natural outer homomorphism* $\Pi_Z \to \Pi_Y$ *is* **surjective**.]

(2) $\Delta_{Z/Y} \subseteq \Delta_{Z/X}$. [*Thus, it follows from the* **surjectivity** *of* $\Pi_Z \to \Pi_Y$ *— cf.* (1) *— that the natural outer homomorphism* $\Pi_Z \to \Pi_X$ *induced by* f *determines an outer homomorphism* $\Pi_Y \to \Pi_X$.]

Then the morphism $f\colon Z \to X$ *factors through* $Z \to Y$.

Proof. First, let us observe that, by *induction on the relative dimension of* X *over* S, to verify Lemma 2.9, we may assume without loss of generality that X is a *hyperbolic curve* over S. Write $\Gamma_0 \subseteq X \times_S Y$ for the scheme-theoretic image of the natural morphism $Z \to X \times_S Y$ over S and $\Gamma \stackrel{\mathrm{def}}{=} \mathrm{Nor}(Z/\Gamma_0)$. [Note that one verifies easily that Γ_0 is an *integral variety* over k, and the natural morphism $Z \to \Gamma_0$ is *dominant*.] Now let us observe that it follows from Lemma 1.9 that Γ is a *normal variety* over k, the resulting morphism $Z \to \Gamma$ is *dominant* and *generically geometrically irreducible*, and the natural morphism $\Gamma \to \Gamma_0$ is *finite* and *surjective*.

Here, to verify Lemma 2.9, I claim that the following assertion holds:

> Claim 2.9.A: Let $\overline{y} \to Y$ be a geometric point of Y. Then the image of the morphism $Z \times_Y \overline{y} \to X \times_k \overline{y}$ determined by f consists of *finitely many closed points* of $X \times_k \overline{y}$.

Indeed, let $F \to Z \times_Y \overline{y}$ be a connected component [which is necessarily a *normal variety* over $\overline{y}$] of the normalization of $Z \times_Y \overline{y}$. Then it follows immediately from condition (2) that the image of the outer homomorphism $\Pi_F \to \Pi_X$ induced by the composite of natural morphisms $F \to Z \times_Y \overline{y} \stackrel{\mathrm{pr}_1}{\to} Z \to X$ is *trivial*. On the other hand, it is immediate that the composite of natural morphisms $F \to Z \times_Y \overline{y} \stackrel{\mathrm{pr}_1}{\to} Z \to X$ *factors* through the projection $X \times_S \overline{y} \stackrel{\mathrm{pr}_1}{\to} X$. Thus, since the outer homomorphism $\Pi_{X \times_S \overline{y}} \to \Pi_X$ induced by the projection $X \times_S \overline{y} \stackrel{\mathrm{pr}_1}{\to} X$ is *injective* [cf. Proposition 2.4, (i)], it follows that the image of the outer homomorphism $\Pi_F \to \Pi_{X \times_S \overline{y}}$ induced by the morphism $F \to X \times_S \overline{y}$ is *trivial*. In particular, since $X \times_S \overline{y}$ is a *hyperbolic curve* over $\overline{y}$, hence of *LFG-type* [cf. Proposition 2.7], and the morphism $F \to X \times_S \overline{y}$ is a *morphism between varieties over* $\overline{y}$, one verifies easily that the image of the morphism $F \to X \times_S \overline{y}$ consists of a *closed point* of $X \times_S \overline{y}$. Thus, since the morphism $Z \times_Y \overline{y} \to X \times_k \overline{y}$ in question *factors* through

$Z \times_S \overline{y} \to X \times_S \overline{y}$, we conclude that Claim 2.9.A holds. This completes the proof of Claim 2.9.A.

Next, to verify Lemma 2.9, I claim that the following assertion holds:

> Claim 2.9.B: The composite $\Gamma \to \Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$, hence also the composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$, is *dominant* and induces an *isomorphism* between their function fields.

Indeed, since $Z \to Y$ is *dominant* and *generically geometrically irreducible* [cf. condition (1)] and factors through the composite in question $\Gamma \to Y$, one verifies easily from [5], Proposition (4.5.9), that the composite in question $\Gamma \to Y$ is *dominant* and *generically geometrically irreducible*. Thus, one verifies easily that, to verify Claim 2.9.B, it suffices to verify that $\Gamma \to Y$ is *generically quasi-finite*. To verify that $\Gamma \to Y$ is *generically quasi-finite*, let $\overline{\eta}_Y \to Y$ be a geometric point of Y whose image is the generic point of Y. Then since [one verifies easily that] the operation of taking scheme-theoretic image commutes with basechange by a flat morphism, $\Gamma_0 \times_Y \overline{\eta}_Y$ is naturally isomorphic to the scheme-theoretic image of the natural morphism $Z \times_Y \overline{\eta}_Y \to X \times_S \overline{\eta}_Y$. On the other hand, since the natural morphism $X \times_S \overline{\eta}_Y \to X \times_k \overline{\eta}_Y$ is a *closed immersion*, it follows immediately from Claim 2.9.A that the image of the natural morphism $Z \times_Y \overline{\eta}_Y \to X \times_S \overline{\eta}_Y$ consists of *finitely many closed points* of $X \times_S \overline{\eta}_Y$. Thus, we conclude that the composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$, hence [by the finiteness of $\Gamma \to \Gamma_0$ — cf. the discussion preceding Claim 2.9.A] also the composite $\Gamma \to \Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$, is *generically quasi-finite*. This completes the proof of Claim 2.9.B.

Next, to verify Lemma 2.9, I claim that the following assertion holds:

> Claim 2.9.C: $\Delta_{\Gamma/Y} \subseteq \Delta_{\Gamma/X}$.

Indeed, let us observe that it follows immediately from Claim 2.9.B; condition (1), together with Lemma 1.6, that the natural outer homomorphism $\Pi_Z \to \Pi_\Gamma$ is *surjective*. Thus, one verify easily from condition (2) that Claim 2.9.C holds. This completes the proof of Claim 2.9.C.

Next, to verify Lemma 2.9, I claim that the following assertion holds:

> Claim 2.9.D: Let $\overline{y} \to Y$ be a geometric point of Y. Then the image of the morphism $\Gamma \times_Y \overline{y} \to X \times_k \overline{y}$ determined by the composite $\Gamma \to \Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_1}{\to} X$ consists of *finitely many closed points* of $X \times_k \overline{y}$.

Indeed, this follows immediately from Claim 2.9.C, together with a similar argument to the argument used in the proof of Claim 2.9.A. This completes the proof of Claim 2.9.D.

Next, to verify Lemma 2.9, I claim that the following assertion holds:

> Claim 2.9.E: The composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$ is an *open immersion*.

Indeed, let $\overline{y} \to Y$ be a geometric point of Y. Then let us first observe that it follows immediately from Claim 2.9.D that the image of the composite $\Gamma \times_Y \overline{y} \to \Gamma_0 \times_Y \overline{y} \hookrightarrow X \times_S \overline{y}$ consists of *finitely many closed points* of $X \times_S \overline{y}$. Thus, since $\Gamma \to \Gamma_0$ is *surjective* [cf. the discussion preceding Claim 2.9.A], and the morphism $\Gamma_0 \times_Y \overline{y} \hookrightarrow X \times_S \overline{y}$ is a *closed immersion*, we conclude that $\Gamma_0 \times_Y \overline{y}$ is *quasi-finite* over $\overline{y}$. In particular, the composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$ is *quasi-finite*. Thus, it follows immediately from Claim 2.9.B, together with [4], Corollaire (4.4.9), that the composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$ is an *open immersion*. This completes the proof of Claim 2.9.E.

Next, to verify Lemma 2.9, I claim that the following assertion holds:

> Claim 2.9.F: If X is *proper* over S, then $f\colon Z \to X$ factors through $Z \to Y$.

Indeed, if X is *proper* over S, then one verifies easily that the composite $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$ is *proper*. Thus, it follows immediately from Claim 2.9.E that $\Gamma_0 \hookrightarrow X \times_S Y \overset{\mathrm{pr}_2}{\to} Y$ is an *isomorphism*. In particular, we conclude that $f\colon Z \to X$ factors through $Z \to Y$. This completes the proof of Claim 2.9.F.

Next, to verify Lemma 2.9, I claim that the following assertion holds:

> Claim 2.9.G: If the genus [i.e., "g" in Definition 2.1, (i)] of the hyperbolic curve X over S is ≥ 2, then f factors through $Z \to Y$.

Indeed, write X^{cpt} for the smooth compactification of the hyperbolic curve X over S [cf. Definition 2.2]. Then since [one verifies easily that] X^{cpt} is a *proper hyperbolic curve* over S, by applying Claim 2.9.F [where we take "(S, Y, Z, X)" to be $(S, Y, Z, X^{\mathrm{cpt}})$], we conclude that the natural morphism $Z \to X^{\mathrm{cpt}}$ over S factors as the composite $Z \to Y \to X^{\mathrm{cpt}}$. Thus, to verify Claim 2.9.G, it suffices to verify that this morphism $Y \to X^{\mathrm{cpt}}$ factors through $X \subseteq X^{\mathrm{cpt}}$. In particular, by considering a suitable discrete valuation of the function field of Y [cf. [3], Proposition (7.1.7)], one verifies easily that, to verify this, it suffices to verify that, for any strictly henselian discrete valuation ring R and any morphism $\mathrm{Spec}\, R \to Y$ such that the image of the generic point η_R of $\mathrm{Spec}\, R$ is the generic point of Y, it holds that the composite $\mathrm{Spec}\, R \to Y \to X^{\mathrm{cpt}}$ factors through $X \subseteq X^{\mathrm{cpt}}$. On the other hand, since the composite $\eta_R \to \mathrm{Spec}\, R \to Y$ for such a $\mathrm{Spec}\, R \to Y$ factors as the composite $\eta_R \to \Gamma \to Y$ [cf. Claim 2.9.B], this follows immediately from Claim 2.9.C, together with Lemma 2.8. This completes the proof of Claim 2.9.G.

Finally, to verify Lemma 2.9, I claim that the following assertion holds:

Claim 2.9.H: f factors through $Z \to Y$.

Indeed, it follows immediately that there exists a connected finite étale Galois covering $X' \to X$ of X such that the genus [i.e., "g" in Definition 2.1, (i)] of the hyperbolic curve X' over $S' \stackrel{\mathrm{def}}{=} \mathrm{Nor}(X'/S)$ [cf. Proposition 2.3] is ≥ 2. Write $Y' \to Y$ for the connected finite étale Galois covering of Y corresponding to $X' \to X$ [by the outer homomorphism $\Pi_Y \to \Pi_X$ — cf. condition (2)]; $Z' \stackrel{\mathrm{def}}{=} Z \times_Y Y' \to Z$ for the connected [cf. condition (1)] finite étale Galois covering of Z corresponding to $Y' \to Y$. Then, by applying Claim 2.9.G [where we take "(S, Y, Z, X)" to be (S', Y', Z', X')], we conclude that the natural morphism $Z' \to X'$ over S' factors as the composite $Z' \to Y' \to X'$; in particular, the natural morphism $Z' \to X$ over S factors as the composite $Z' \to Y' \to X$. Now since [one verifies easily that] the operation taking scheme-theoretic image commutes with basechange by a flat morphism, this implies that the composite of natural morphisms $\Gamma_0 \times_Y Y' \hookrightarrow X \times_S Y' \stackrel{\mathrm{pr}_2}{\to} Y'$ is an *isomorphism*. Thus, since $Y' \to Y$ is *faithfully flat* and *quasi-compact*, it follows from [5], Proposition (2.7.1), that the composite $\Gamma_0 \hookrightarrow X \times_S Y \stackrel{\mathrm{pr}_2}{\to} Y$ is an *isomorphism*; in particular, we conclude that f factors through $Z \to Y$. This completes the proof of Claim 2.9.H, hence also of Lemma 2.9. □

Lemma 2.10. *Let S, Y be* **normal** *varieties over k; $Y \to S$ a morphism; X a* **hyperbolic polycurve** *over S; $\phi \colon \Pi_Y \to \Pi_X$ a homomorphism. Write $\eta \to Y$ for the generic point of Y. Then the following conditions are equivalent:*

(1) *The homomorphism ϕ arises from a morphism $Y \to X$ over S.*
(2) *There exists a morphism $\eta \to X$ over S such that the outer homomorphism $\Pi_\eta \to \Pi_X$ induced by this morphism $\eta \to X$ coincides with the composite of the outer surjection* [*cf. Lemma 1.2*] *$\Pi_\eta \twoheadrightarrow \Pi_Y$ induced by $\eta \to Y$ and the outer homomorphism determined by ϕ.*

Proof. The implication (1) $\Rightarrow$ (2) is immediate; thus, it remains to verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied. Then it follows immediately that there exists an open subscheme $U \subseteq Y$ of Y such that the morphism $\eta \to X$ in condition (2) *extends* to a morphism $U \to X$ over S. Moreover, it follows immediately from Lemma 1.2 that the outer homomorphism $\Pi_U \to \Pi_X$ induced by this morphism $U \to X$ coincides with the composite of the outer surjection [cf. Lemma 1.2] $\Pi_U \twoheadrightarrow \Pi_Y$ induced by the natural open immersion $U \hookrightarrow Y$ and the outer homomorphism determined by ϕ. Thus, in light of Lemma 1.2, by applying Lemma 2.9 [where we take "(S, Y, Z, X)" in the statement of Lemma 2.9 to be (S, Y, U, X)], we conclude that condition (1) is satisfied. This completes the proof of Lemma 2.10. □

Lemma 2.11. *Let X be a* **hyperbolic curve** *over k, Y a* **normal** *variety over k, and $f\colon Y \to X$ a morphism over k. Write $\phi_f\colon \Pi_Y \to \Pi_X$ for the outer homomorphism induced by f. Consider the following conditions:*

(1) *f is* **surjective***,* **smooth***, and* **generically geometrically connected**.
(2) *ϕ_f is* **surjective***, and the kernel Δ_f of ϕ_f is* **topologically finitely generated**.
(3) *f is* **surjective** *and* **generically geometrically connected**.
(4) *Let C be a* **hyperbolic curve** *over k and $C \to X$ a morphism over k. Then if f factors through $C \to X$, then $C \to X$ is an* **isomorphism**.

Then we have implications and an equivalence:

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longleftrightarrow (4).$$

Proof. The implication (1) $\Rightarrow$ (2) follows immediately from Corollary 1.11. Next, we verify the implication (2) $\Rightarrow$ (4). Suppose that condition (2) is satisfied. First, let us observe that it follows immediately from Lemma 1.5 that, by replacing k by $\overline{k}$, to verify that condition (4) is satisfied, we may assume without loss of generality that $k = \overline{k}$. Let C be a hyperbolic curve over k and $C \to X$ a morphism over k. Suppose that f factors through $C \to X$. Then since ϕ_f is *surjective*, $\Pi_C \to \Pi_X$ is *surjective*. On the other hand, since the kernel of ϕ_f is *topologically finitely generated*, one verifies easily that the kernel of $\Pi_C \to \Pi_X$ is *topologically finitely generated*. Thus, it follows immediately from Proposition 2.4, (iii), (iv), that $\Pi_C \to \Pi_X$ is an *outer isomorphism*. In particular, it follows immediately from Proposition 2.4, (v), together with Hurwitz's formula [cf., e.g., [10], Chapter IV, Corollary 2.4], that $C \to X$ is an *isomorphism*. This completes the proof of the implication (2) $\Rightarrow$ (4).

Next, we verify the implication (3) $\Rightarrow$ (4). Suppose that condition (3) is satisfied. Let C be a hyperbolic curve over k and $C \to X$ a morphism over k. Suppose that f factors through $C \to X$. Then since f is *surjective*, $C \to X$ is *surjective*, hence *quasi-finite*. On the other hand, since f is *generically geometrically connected*, and k is of *characteristic zero*, one verifies easily that $C \to X$ induces an *isomorphism between their function fields*. Thus, since X and C are *irreducible* and *normal*, it follows from [4], Corollaire (4.4.9), that $C \to X$ is an *isomorphism*. This completes the proof of the implication (3) $\Rightarrow$ (4).

Finally, we verify the implication (4) $\Rightarrow$ (3). Suppose that condition (3) is *not satisfied*. If f is *not surjective*, then one verifies easily that f factors through the natural open immersion from a suitable open subscheme of X. [Note that one verifies easily that every open subscheme of a hyperbolic curve over k is a *hyperbolic curve over* k.] Thus, condition (4) is *not satisfied*. On the other hand, if f is *surjective* but *not generically geometrically connected*, then the morphism $C \stackrel{\text{def}}{=} \mathrm{Nor}(Y/X) \to X$ over k is *not an isomorphism*, and, moreover, f factors through this morphism $C \to X$ [cf. Lemma 1.9]. Since [one verifies easily that] C is a *hyperbolic curve over* k, we conclude that condition (4) is *not satisfied*. This completes the proof of the implication (4) $\Rightarrow$ (3), hence also of Lemma 2.11. □

Lemma 2.12. *In the notation of Lemma 2.11, suppose, moreover, that Y is of* **LFG-type**. *Then the following hold:*

(i) *The following conditions are equivalent:*
  (i-1) *f is a* **finite étale covering**.
  (i-2) *ϕ_f is an outer* **open injection**.
  (i-3) *ϕ_f is* **open***, and, moreover, the kernel Δ_f of ϕ_f is* **finite**.

(ii) *The following conditions are equivalent:*
  (ii-1) *f is an* **isomorphism**
  (ii-2) *ϕ_f is an outer* **isomorphism**.
  (ii-3) *ϕ_f is* **surjective***, and, moreover, the kernel Δ_f of ϕ_f is* **finite**.

Proof. First, we verify assertion (ii). The implications (ii-1) $\Rightarrow$ (ii-2) $\Rightarrow$ (ii-3) are immediate; thus, it remains to verify the implication (ii-3) $\Rightarrow$ (ii-1). To verify this implication, suppose that condition (ii-3) is satisfied. Then it follows from the implication (2) $\Rightarrow$ (3) of Lemma 2.11 that f is *surjective* and *generically geometrically connected*. On the other hand, it follows from Lemma 2.6, (ii), that f is *quasi-finite*. Thus, it follows from [4], Corollaire (4.4.9), that f is an *isomorphism*. This completes the proof of the implication (ii-3) $\Rightarrow$ (ii-1), hence also of assertion (ii).

Finally, we verify assertion (i). The implications (i-1) $\Rightarrow$ (i-2) $\Rightarrow$ (i-3) are immediate; thus, it remains to verify the implication (i-3) $\Rightarrow$ (i-1). To verify this implication, suppose that condition (i-3) is satisfied. Then, by replacing X by a connected finite étale covering of X corresponding to the image of [an open homomorphism that lifts] ϕ_f, we may assume without loss of generality that ϕ_f is an outer *isomorphism*. Thus, the implication (i-3) $\Rightarrow$ (i-1) follows from the implication (ii-3) $\Rightarrow$ (ii-1) of assertion (ii). This completes the proof of assertion (i). □

Lemma 2.13. *In the notation of Lemma 2.11, suppose, moreover, that Y is a* **hyperbolic curve** *over k. Then the following hold:*

(i) *The following conditions are equivalent:*
  - (i-1) *f is a* **finite étale covering**.
  - (i-2) *ϕ_f is an outer* **open injection**.
  - (i-3) *ϕ_f is* **open***, and, moreover, the kernel Δ_f of ϕ_f is* **topologically finitely generated**.

(ii) *The following conditions are equivalent:*
  - (ii-1) *f is an* **isomorphism**
  - (ii-2) *ϕ_f is an outer* **isomorphism**.
  - (ii-3) *ϕ_f is* **surjective***, and, moreover, the kernel Δ_f of ϕ_f is* **topologically finitely generated**.

Proof. First, we verify assertion (ii). The implications (ii-1) $\Rightarrow$ (ii-2) $\Rightarrow$ (ii-3) are immediate; thus, it remains to verify the implication (ii-3) $\Rightarrow$ (ii-1). To verify this implication, suppose that condition (ii-3) is satisfied. Now let us observe that it follows immediately from Lemma 1.5 that, by replacing k by $\overline{k}$, to verify that condition (ii-1) is satisfied, we may assume without loss of generality that $k = \overline{k}$. Then it follows from Proposition 2.4, (iii), together with the *surjectivity* of ϕ_f, that the image of ϕ_f is *infinite*, i.e., Δ_f is *not open* in Π_Y. Thus, since Y is a *hyperbolic curve* over k, and Δ_f is *topologically finitely generated*, it follows from Proposition 2.4, (iv), that ϕ_f is *injective*. In particular, it follows from the implication (ii-2) $\Rightarrow$ (ii-1) of Lemma 2.12, together with Proposition 2.7, that f is an *isomorphism*. This completes the proof of the implication (ii-3) $\Rightarrow$ (ii-1), hence also of assertion (ii).

Finally, we verify assertion (i). The implications (i-1) $\Rightarrow$ (i-2) $\Rightarrow$ (i-3) are immediate; thus, it remains to verify the implication (i-3) $\Rightarrow$ (i-1). To verify this implication, suppose that condition (i-3) is satisfied. Then, by replacing X by a connected finite étale covering of X corresponding to the image of [an open homomorphism that lifts] ϕ_f, we may assume without loss of generality that ϕ_f is an outer *isomorphism*. Thus, the implication (i-3) $\Rightarrow$ (i-1) follows from the implication (ii-3) $\Rightarrow$ (ii-1) of assertion (ii). This completes the proof of the implication (i-3) ⇒ (i-1), hence also of assertion (i). □

Lemma 2.14. *Suppose that* $k = \overline{k}$. *Let* n *be a positive integer,* X *a* **hyperbolic polycurve** *over* k, F *a* **normal** *variety over* k *of* **dimension** $\geq n$, *and* $F \to X$ *a* **quasi-finite** *morphism over* k. [*Thus, it holds that* $n \leq \dim(X)$.] *Write* $\Pi_{F\to X} \stackrel{\text{def}}{=} \Pi_F/\Delta_{F/X}$. *Then there exists a sequence of* **normal** *closed subgroups of* $\Pi_{F\to X}$

$$\{1\} = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{n-1} \subseteq H_n = \Pi_{F\to X}$$

such that, for each $i \in \{1, \cdots, n\}$, *the closed subgroup* H_i *is* **topologically finitely generated**, *and, moreover, the quotient* H_i/H_{i-1} *is* **infinite**.

Proof. Write $d \stackrel{\text{def}}{=} \dim(X)$. Let

$$X = X_d \longrightarrow X_{d-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \operatorname{Spec} k = X_0$$

be a sequence of parametrizing morphisms. For each $j \in \{0, \cdots, d\}$, write $F[j] \to X_j$ for the normalization in F of the scheme-theoretic image of the composite $F \to X \to X_j$. Then it follows immediately from the various definitions involved, together with Lemma 1.9, that we obtain a commutative diagram of *normal varieties over* k

$$\begin{array}{ccccccccc} F & \longrightarrow & F[d] & \longrightarrow & \cdots & \longrightarrow & F[1] & \longrightarrow & \operatorname{Spec} k = F[0] \\ \downarrow & & \downarrow & & & & \downarrow & & \| \\ X & \xrightarrow{=} & X_d & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & \operatorname{Spec} k = X_0 \end{array}$$

— where the horizontal arrows are *dominant* and *generically geometrically connected*, and the vertical arrows are *finite*, which implies that $F[i]$ is of *dimension* $\leq i$, and that $0 \leq \dim(F[i+1]) - \dim(F[i]) \leq 1$ [cf. also [5], Proposition (5.5.2)]. Now since F is of *dimension* $\geq n$, one verifies easily that there exists a uniquely determined subset $\{D_0, \cdots, D_{n-1}\} \subseteq \{0, \cdots, d-1\}$ of cardinality n such that, for each $i \in \{0, \cdots, n-1\}$, the normal variety $F[D_i + 1]$ is of *dimension* $i + 1$, but the normal variety $F[D_i]$ is of *dimension* i. Write, moreover, $F[D_n] \stackrel{\text{def}}{=} F$. Next, let us observe that since k is of *characteristic zero*, and the horizontal arrows in the above commutative diagram of normal varieties over k are *dominant* and *generically geometrically connected*, one verifies easily that, for each $i \in \{0, \cdots, n\}$, there exists a *nonempty* open subscheme $U[D_i] \subseteq F[D_i]$ of $F[D_i]$ such that, for each $i \in \{1, \cdots, n\}$, the image of $U[D_i] \subseteq F[D_i]$ by $F[D_i] \to F[D_{i-1}]$ is contained in $U[D_{i-1}] \subseteq F[D_{i-1}]$, and, moreover, the resulting morphism $U[D_i] \to U[D_{i-1}]$ is *surjective*, *smooth*, and *geometrically connected*. Thus, we obtain a commutative diagram of *normal varieties over* k

$$\begin{array}{ccccccccc} U[D_n] & \longrightarrow & U[D_{n-1}] & \longrightarrow & \cdots & \longrightarrow & U[D_1] & \longrightarrow & \operatorname{Spec} k = U[D_0] \\ \downarrow & & \downarrow & & & & \downarrow & & \| \\ F[D_n] & \longrightarrow & F[D_{n-1}] & \longrightarrow & \cdots & \longrightarrow & F[D_1] & \longrightarrow & \operatorname{Spec} k = F[D_0] \end{array}$$

— where the vertical arrows are *open immersions*, and the upper horizontal arrows are *surjective*, *smooth*, and *geometrically connected*.

Now, for each $i \in \{0, \cdots, n\}$, let us write $N_i \subseteq \Pi_F$ for the *normal* closed subgroup obtained by forming the image of the *normal* closed subgroup $\Delta_{U[D_n]/U[D_{n-i}]} \subseteq \Pi_{U[D_n]}$ by the outer *surjection* $\Pi_{U[D_n]} \twoheadrightarrow \Pi_F$ [cf. Lemma 1.2]; $H_i \stackrel{\text{def}}{=} N_i/(N_i \cap \Delta_{F/X}) \subseteq \Pi_{F\to X}$. [Thus, one verifies easily that $N_0 = \{1\}$; $H_0 = \{1\}$; $N_n = \Pi_F$; $H_n = \Pi_{F\to X}$.] The rest of the proof of Lemma 2.14 is devoted to verifying that this sequence of normal closed subgroups of $\Pi_{F\to X}$

$$H_0 = \{1\} \subseteq H_1 \subseteq \cdots \subseteq H_{n-1} \subseteq H_n = \Pi_{F\to X}$$

satisfies the condition in the statement of Lemma 2.14.

First, let us observe that it follows from Corollary 1.11 that $\Delta_{U[D_n]/U[D_{n-i}]}$, hence also H_i, is *topologically finitely generated*. Thus, it remains to verify that, for each $i \in \{0, \cdots, n-1\}$, the quotient H_{i+1}/H_i is *infinite*. To verify this, let $\overline{a} \to U[D_{n-i-1}]$ be a k-valued geometric point of $U[D_{n-i-1}]$. Write $U_{D_{n-i};D_{n-i-1}} \stackrel{\text{def}}{=} U[D_{n-i}] \times_{U[D_{n-i-1}]} \overline{a}$ [which is a *regular variety over* k *of dimension* 1 (respectively, $\dim(F) - n + 1$) if $i \neq 0$ (respectively, $i = 0$) by our choices of $U[D_{n-i}]$ and $U[D_{n-i-1}]$]. Then one verifies easily from Proposition 1.10, (i), that the natural morphism $U_{D_{n-i};D_{n-i-1}} \to X_{D_{n-i}}$ [where we write $X_{D_n} \stackrel{\text{def}}{=} X$] determines a sequence of profinite groups

$$\Pi_{U_{D_{n-i};D_{n-i-1}}} \longrightarrow H_{i+1}/H_i \longrightarrow \Pi_{X_{D_{n-i}}}.$$

On the other hand, since [one verifies easily that] the natural morphism $U_{D_{n-i};D_{n-i-1}} \to X_{D_{n-i}}$ is *quasi-finite*, hence *nonconstant*, and $X_{D_{n-i}}$ is of *LFG-type* [cf. Proposition 2.7], the image of the outer homomorphism $\Pi_{U_{D_{n-i};D_{n-i-1}}} \to \Pi_{X_{D_{n-i}}}$, hence also the image of $H_{i+1}/H_i \to \Pi_{X_{D_{n-i}}}$, is *infinite*. Thus, we conclude that H_{i+1}/H_i is *infinite*. This completes the proof of Lemma 2.14. □

3. Results on the Grothendieck conjecture for hyperbolic polycurves

In the present §3, we prove some results on the Grothendieck conjecture for hyperbolic polycurves [cf. Theorems 3.4; 3.8; 3.9; 3.12; 3.15; Corollaries 3.13; 3.16; 3.17; 3.19; 3.21 3.22]. In the present §3, let k be a field of *characteristic zero*, $\overline{k}$ an algebraic closure of k, and $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

Definition 3.1 (cf. [16], Definition 15.4, (i))**.** Let p be a prime number. Then we shall say that k is *sub-p-adic* if k is isomorphic to a subfield of a finitely generated extension of the p-adic completion of the field of rational numbers.

Proposition 3.2. *Let X be a* **hyperbolic polycurve** *over k and Y an integral variety over k. Then the following hold:*

(i) *Write $\mathrm{Hom}_k^{\mathrm{dom}}(Y,X) \subseteq \mathrm{Hom}_k(Y,X)$ for the subset of* **dominant** *morphisms from Y to X over k and $\mathrm{Hom}_{G_k}^{\mathrm{open}}(\Pi_Y,\Pi_X) \subseteq \mathrm{Hom}_{G_k}(\Pi_Y,\Pi_X)$ for the subset of* **open** *homomorphisms from Π_Y to Π_X over G_k. Then the natural map*

$$\mathrm{Hom}_k^{\mathrm{dom}}(Y,X) \longrightarrow \mathrm{Hom}_{G_k}^{\mathrm{open}}(\Pi_Y,\Pi_X)/\mathrm{Inn}(\Delta_{X/k})$$

[*cf. Lemma 1.3*] *is* **injective**.

(ii) *Suppose that k is* **sub-p-adic** *for some prime number p. Then the natural map*

$$\mathrm{Hom}_k(Y,X) \longrightarrow \mathrm{Hom}_{G_k}(\Pi_Y,\Pi_X)/\mathrm{Inn}(\Delta_{X/k})$$

is **injective**.

Proof. Write $n \stackrel{\mathrm{def}}{=} \dim(X)$. First, we verify assertion (i). Now I claim that the following assertion holds:

Claim 3.2.A: If $n = 1$, then assertion (i) holds.

Indeed, let $F \subseteq Y \otimes_k \overline{k}$ be an irreducible component of $Y \otimes_k \overline{k}$. Write $F_{\mathrm{red}} \subseteq Y \otimes_k \overline{k}$ for the reduced closed subscheme of $Y \otimes_k \overline{k}$ whose support is $F \subseteq Y \otimes_k \overline{k}$. [Thus, F_{red} is an *integral variety* over $\overline{k}$]. Then we have natural Π_X-, Π_Y-conjugacy classes of isomorphisms $\Pi_{X\otimes_k\overline{k}} = \Delta_{X\otimes_k\overline{k}/\overline{k}} \stackrel{\sim}{\to} \Delta_{X/k}$, $\Pi_{Y\otimes_k\overline{k}} = \Delta_{Y\otimes_k\overline{k}/\overline{k}} \stackrel{\sim}{\to} \Delta_{Y/k}$ [cf. Lemma 1.5], respectively, and a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_k^{\mathrm{dom}}(Y,X) & \longrightarrow & \mathrm{Hom}_{G_k}^{\mathrm{open}}(\Pi_Y,\Pi_X)/\mathrm{Inn}(\Delta_{X/k}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\overline{k}}^{\mathrm{dom}}(F_{\mathrm{red}}, X\otimes_k \overline{k}) & \longrightarrow & \mathrm{Hom}(\Pi_{F_{\mathrm{red}}},\Pi_{X\otimes_k\overline{k}})/\mathrm{Inn}(\Pi_{X\otimes_k\overline{k}}) \end{array}$$

— where the left-hand vertical arrow is *injective* [cf. the fact that the natural morphism $F_{\mathrm{red}} \to Y$ is *schematically dense*], and the lower horizontal arrow factors through the subset

$$\mathrm{Hom}^{\mathrm{open}}(\Pi_{F_{\mathrm{red}}},\Pi_{X\otimes_k\overline{k}})/\mathrm{Inn}(\Pi_{X\otimes_k\overline{k}})$$

[cf. Lemma 1.3]. Thus, by replacing k, Y by $\overline{k}$, F_{red}, respectively, to verify Claim 3.2.A, we may assume without loss of generality that $k = \overline{k}$.

Let f, $g \colon Y \to X$ be dominant morphisms over k that induce the *same* outer homomorphism $\Pi_Y \to \Pi_X$. Now one verifies easily that there exists a normal open subgroup $H \subseteq \Pi_X$ of Π_X such that the genus [i.e., "g" in Definition 2.1, (i)] of the hyperbolic curve over k obtained by forming the connected finite étale Galois covering of X corresponding to $H \subseteq \Pi_X$ is ≥ 2. Thus, by replacing Y by the connected finite étale Galois covering of Y corresponding to the inverse image of $H \subseteq \Pi_X$ by the outer homomorphism $\Pi_Y \to \Pi_X$ induced by f [and considering a similar commutative diagram to the above commutative diagram], to verify that $f = g$ [i.e., Claim 3.2.A], we may assume without loss of generality that the genus [i.e., "g" in Definition 2.1, (i)] of X is ≥ 2. In particular, since an open immersion is a monomorphism [cf. [7], Proposition (17.2.6)], by replacing X by the smooth compactification of X over k [cf. Definition 2.2], to verify that $f = g$ [i.e., Claim 3.2.A], we may assume without loss of generality that X is *proper* over k. Next, let us observe that, it follows from [1], Theorem 7.3, that there exist *regular projective* variety Z over k, a divisor with normal crossings $D \subseteq Z$ of Z, and a *surjection* $Z \setminus D \twoheadrightarrow Y$. Thus, by replacing Y by $Z \setminus D$ [and considering a similar commutative diagram to the above commutative diagram], to verify that $f = g$ [i.e., Claim 3.2.A], we may assume without loss of generality that Y admits an *Albanese morphism* $\iota_Y \colon Y \to A_Y$ [cf., e.g., [19], Proposition A.8, (i)]. Write J_X for the Jacobian variety of X and $\iota_X \colon X \hookrightarrow J_X$ for the closed immersion determined by some k-rational point of X. Then the composites $Y \xrightarrow{f} X \xrightarrow{\iota_X} J_X$, $Y \xrightarrow{g} X \xrightarrow{\iota_X} J_X$ determine morphisms α_f, $\alpha_g \colon A_Y \to J_X$ such that $\alpha_f \circ \iota_Y = \iota_X \circ f$, $\alpha_g \circ \iota_Y = \iota_X \circ g$, respectively. Now since ι_X is a *closed immersion*, and a closed immersion is a monomorphism [cf. [7], Proposition (17.2.6)], one verifies easily that, to verify $f = g$ [i.e., Claim 3.2.A], it suffices to verify that $\alpha_f = \alpha_g$. On the other hand, since f and g induce the *same* outer homomorphism $\Pi_Y \to \Pi_X$, it follows immediately from [19], Proposition A.8, (iii), that α_f, α_g induce the *same* outer homomorphism $\Pi_{A_Y} \to \Pi_{J_X}$. Thus, it follows immediately from [21], §19, Theorem 3, together with [21], §4, Corollary 1, that the difference between α_f and α_g is the translation by a k-rational point $j \in J_X(k)$ of J_X. On the other hand, since f and g are *dominant*, one verifies easily from the various definitions involved that the translation by $j \in J_X(k)$ *preserves* the image of ι_X. Thus, it follows from Lemma 3.3 below that $j \in J_X(k)$ is the *identity element*, i.e., that $\alpha_f = \alpha_g$. This completes the proof of Claim 3.2.A.

Next, we verify assertion (i) by *induction on* n. If $n = 1$, then assertion (i) follows from Claim 3.2.A. Now suppose that $n \geq 2$, and that the *induction hypothesis* is in force. Let $X \to X_{n-1}$ be a parametrizing morphism of X; $f, g \colon Y \to X$ dominant morphisms over k that induce the *same* $\Delta_{X/k}$-conjugacy class of homomorphisms $\Pi_Y \to \Pi_X$. Then since the composites $f_{n-1} \colon Y \xrightarrow{f} X \to X_{n-1}$, $g_{n-1} \colon Y \xrightarrow{g} X \to X_{n-1}$ induce the *same* $\Delta_{X_{n-1}/k}$-conjugacy class of homomorphisms $\Pi_Y \to \Pi_{X_{n-1}}$, it follows from the *induction hypothesis* that $f_{n-1} = g_{n-1}$. Let $\overline{\eta} \to X_{n-1}$ be a geometric point of X_{n-1} whose image is the generic point of X_{n-1} and $C \subseteq Y \times_{X_{n-1}} \overline{\eta}$ [where we take the implicit morphism $Y \to X_{n-1}$ to be $f_{n-1} = g_{n-1}$] an irreducible component of $Y \times_{X_{n-1}} \overline{\eta}$. Write $C_{\mathrm{red}} \subseteq Y \times_{X_{n-1}} \overline{\eta}$ for the reduced closed subscheme of $Y \times_{X_{n-1}} \overline{\eta}$ whose support is $C \subseteq Y \times_{X_{n-1}} \overline{\eta}$. [Thus, C_{red} is an *integral variety* over $\overline{\eta}$]. Then, in light of the easily verified fact that the natural morphism $C_{\mathrm{red}} \to Y$ is *schematically dense*, by replacing $(\mathrm{Spec}\, k, X, Y)$ by $(\overline{\eta}, X \times_{X_{n-1}} \overline{\eta}, C_{\mathrm{red}})$ and applying Proposition 2.4, (ii), to verify assertion (i), it suffices to verify assertion (i) in the case where $n = 1$, which follows from Claim 3.2.A. This completes the proof of assertion (i).

Next, we verify assertion (ii). Write $\eta \to Y$ for the generic point of Y. Fix a homomorphism $\Pi_\eta \to \Pi_Y$ arising from the natural morphism $\eta \to Y$. Then we have a natural Π_X-conjugacy class of isomorphisms $\Delta_{X \times_k \eta/\eta} \xrightarrow{\sim} \Delta_{X/k}$, [cf. Proposition 2.4, (ii)], a natural outer isomorphism $\Pi_{X \times_k \eta} \xrightarrow{\sim} \Pi_X \times_{G_k} \Pi_\eta$ [cf. Proposition 2.4, (ii)], and a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_k(Y, X) & \longrightarrow & \mathrm{Hom}_{G_k}(\Pi_Y, \Pi_X)/\mathrm{Inn}(\Delta_{X/k}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_\eta(\eta, X \times_k \eta) & \longrightarrow & \mathrm{Hom}_{\Pi_\eta}(\Pi_\eta, \Pi_{X \times_k \eta})/\mathrm{Inn}(\Delta_{X \times_k \eta/\eta}) \end{array}$$

— where the left-hand vertical arrow is *injective* [cf. the fact that the natural morphism $\eta \to Y$ is *schematically dense*]. Thus, by replacing k by the function field of Y [i.e., η] and Y by $\mathrm{Spec}\, k$, to verify assertion (ii), we may assume without loss of generality that $Y = \mathrm{Spec}\, k$. Then — in light of Proposition 2.4, (ii) — assertion (ii) follows immediately from [16], Theorem C, together with *induction on* n. This completes the proof of assertion (ii). □

Lemma 3.3. *Let* X *be a* **proper hyperbolic curve** *over* k *such that* $X(k) \neq \emptyset$. *Write* J_X *for the Jacobian variety of* X *and* $\iota_X \colon X \hookrightarrow J_X$ *for the closed immersion determined by a* k*-rational point of* X. *Let* $j \in J_X(k)$ *be a* k*-rational point of* J_X. *Suppose that the translation by* j **preserves** *the image of* ι_X. *Then* $j \in J_X(k)$ *is the* **identity element** *of* J_X.

Proof. Write g for the genus of X and $\Theta \subseteq J_X$ for the divisor of J_X obtained by forming the scheme-theoretic image of the morphism

$$X \times_k \cdots \times_k X \longrightarrow J_X$$

— where the fiber product is of $g-1$ copies of X — given by adding $g-1$ copies of ι_X. Then since the translation by j *preserves* the image of ι_X, one verifies easily that the translation by j *preserves* the divisor Θ. Thus, j is *contained* in the [set of k-rational points of the] kernel of the homomorphism "$\phi_{\mathcal{O}_{J_X}(\Theta)}$" defined in the discussion following [21], §6, Corollary 4, associated to the invertible sheaf $\mathcal{O}_{J_X}(\Theta)$. On the other hand, it is well-known [cf., e.g., [15], Theorem 6.6] that $\phi_{\mathcal{O}_{J_X}(\Theta)}$ is an *isomorphism*. This completes the proof of Lemma 3.3. □

Theorem 3.4. *Let p be a prime number, k a* **sub-p-adic field** [*cf. Definition 3.1*], *$\overline{k}$ an algebraic closure of k, X a* **hyperbolic curve** [*cf. Definition 2.1,* (i)] *over k, and Y a* **normal variety** [*cf. Definition 1.4*] *over k. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; Π_X, Π_Y for the étale fundamental groups of X, Y, respectively. Let $\phi\colon \Pi_Y \to \Pi_X$ be an* **open** *homomorphism over G_k. Then ϕ arises from a* **uniquely determined dominant** *morphism $Y \to X$ over k.*

Proof. Since there exists an open subscheme of Y which is *smooth* over k, this follows immediately from [16], Theorem A, together with Lemma 2.10; Proposition 3.2, (i). □

Lemma 3.5. *Let n be a positive integer; S, Y* **normal** *varieties over k; X a* **hyperbolic polycurve** *over S of relative dimension n; $\phi\colon \Pi_Y \to \Pi_X$ an* **open** *homomorphism over G_k. Suppose that the composite $\Pi_Y \xrightarrow{\phi} \Pi_X \twoheadrightarrow \Pi_S$ arises from a morphism $Y \to S$ over k. Write $S' \subseteq S$ for the scheme-theoretic image of the morphism $Y \to S$, $Z \stackrel{\mathrm{def}}{=} \mathrm{Nor}(Y/S')$, and $\eta \to Z$ for the generic point of Z. Then the following hold:*

(i) *The morphism $Y \to Z$ [cf. Lemma 1.9] over k is* **dominant** *and* **generically geometrically connected**. *In particular, $Y_\eta \stackrel{\mathrm{def}}{=} Y \times_Z \eta$ is a [nonempty]* **normal variety** *over η.*

(ii) *There exist* **nonempty** *open subschemes $U_Y \subseteq Y$, $U_Z \subseteq Z$ of Y, Z, respectively, such that the image of $U_Y \subseteq Y$ by the natural morphism $Y \to Z$ is contained in $U_Z \subseteq Z$, and, moreover, the resulting morphism $U_Y \to U_Z$ is* **surjective**, **smooth**, *and* **geometrically connected**.

(iii) *The image of the composite $\Delta_{Y_\eta/\eta} \hookrightarrow \Pi_{Y_\eta} \twoheadrightarrow \Pi_Y \xrightarrow{\phi} \Pi_X \twoheadrightarrow \Pi_S$ [cf. (i)] is* **trivial**. *In particular, we obtain a natural Π_X-conjugacy class of homomorphisms $\Delta_{Y_\eta/\eta} \to \Delta_{X/S}$.*

(iv) *If, moreover,* $n = 1$, k *is* **sub-p-adic***, and the image of a homomorphism that belongs to the* Π_X*-conjugacy class of homomorphisms* $\Delta_{Y_\eta/\eta} \to \Delta_{X/S}$ *of* (iii) *is* **nontrivial***, then* ϕ *arises from a morphism* $Y \to X$ *over* k.

Proof. Assertion (i) follows from Lemma 1.9. Assertion (ii) follows immediately from the fact that k is of *characteristic zero*, together with assertion (i). Assertion (iii) follows immediately from the definition of $\Delta_{Y_\eta/\eta}$, together with the fact that the composite $Y_\eta \hookrightarrow Y \to S$ factors through the natural morphism $\eta \to S$. Finally, we verify assertion (iv). Let us observe that since ϕ is *open*, $\Pi_{Y_\eta} \twoheadrightarrow \Pi_Y$ is *surjective* [cf. Lemma 1.2], and $\Delta_{Y_\eta/\eta} \subseteq \Pi_{Y_\eta}$ is *normal*, it follows that the image of $\Delta_{Y_\eta/\eta} \to \Delta_{X/S}$ of assertion (iii) is *normal* in an open subgroup of $\Delta_{X/S}$; in particular, it follows from Proposition 2.4, (iv), together with Lemmas 1.5; 1.7, that the image of $\Delta_{Y_\eta/\eta} \to \Delta_{X/S}$ of assertion (iii) is *open*. Write $X_\eta \stackrel{\text{def}}{=} X \times_S \eta$. Let us fix an isomorphism $\Pi_{X_\eta} \stackrel{\sim}{\to} \Pi_X \times_{\Pi_S} \Pi_\eta$ [cf. Proposition 2.4, (ii)] over Π_η arising from morphisms $X_\eta \to X$, $X_\eta \to \eta$ over S; a homomorphism $\Pi_{Y_\eta} \to \Pi_Y \times_{\Pi_Z} \Pi_\eta$ over Π_η arising from morphisms $Y_\eta \to Y$, $Y_\eta \to \eta$ over Z. Then the open homomorphism $\phi\colon \Pi_Y \to \Pi_X$ determines a homomorphism $\phi_\eta\colon \Pi_{Y_\eta} \to \Pi_Y \times_{\Pi_Z} \Pi_\eta \to \Pi_X \times_{\Pi_S} \Pi_\eta \stackrel{\sim}{\leftarrow} \Pi_{X_\eta}$ over Π_η. On the other hand, since [we already verified that] the image of $\Delta_{Y_\eta/\eta} \to \Delta_{X/S}$ of assertion (iii) is *open*, it follows immediately from Proposition 2.4, (ii), together with the various definitions involved, that the homomorphism ϕ_η over Π_η is *open*. Thus, since X_η is a *hyperbolic curve* over η, it follows from Theorem 3.4 that ϕ_η arises from a morphism $Y_\eta \to X_\eta$ over η. In particular, it follows immediately from Lemma 2.10 that ϕ arises from a morphism $Y \to X$ over k. This completes the proof of assertion (iv). □

Lemma 3.6. *In the notation of Lemma 3.5, suppose, moreover, that* $\dim(X)$ $(= \dim(S) + n) \leq \dim(Y)$. *Write* $N \subseteq \Pi_Y$ *for the* **normal** *closed subgroup of* Π_Y *obtained by forming the image of the normal closed subgroup* $\Delta_{U_Y/U_Z} \subseteq \Pi_{U_Y}$ [*cf. Lemma 3.5,* (ii)] *of* Π_{U_Y} *by the outer surjection* [*cf. Lemma 1.2*] $\Pi_{U_Y} \twoheadrightarrow \Pi_Y$ *induced by the natural open immersion* $U_Y \hookrightarrow Y$. *Then the following hold:*

(i) *The image of the composite* $\Delta_{Y_\eta/\eta} \hookrightarrow \Pi_{Y_\eta} \twoheadrightarrow \Pi_Y$*, hence also the composite* $\Pi_{Y_{\overline{\eta}}} \to \Pi_{Y_\eta} \twoheadrightarrow \Pi_Y$ [*cf. Lemma 1.5; Lemma 3.5,* (i)], **coincides** *with* $N \subseteq \Pi_Y$.

(ii) *If, moreover,* Y *is of* **LFG-type***, then* N *is* **infinite**.

(iii) *If, moreover,* Y *is a* **hyperbolic polycurve** *over* k*, then there exists a sequence of* **normal** *closed subgroups of* N

$$\{1\} = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{\dim(Y)-\dim(\mathrm{S})-1} \subseteq H_{\dim(Y)-\dim(\mathrm{S})} = N$$

such that, for each $i \in \{1, \cdots, \dim(Y) - \dim(\mathrm{S})\}$*, the closed subgroup* H_i *is* **topologically finitely generated***, and, moreover, the quotient* H_i/H_{i-1} *is* **infinite**.

Proof. Let $\overline{\eta} \to U_Z$ be a geometric point of U_Z whose image is the generic point η of U_Z. Write $Y_{\overline{\eta}} \stackrel{\text{def}}{=} Y \times_Z \overline{\eta}$ and $(U_Y)_{\overline{\eta}} \stackrel{\text{def}}{=} U_Y \times_{U_Z} \overline{\eta}$. [Note that $Y_{\overline{\eta}}$ (respectively, $(U_Y)_{\overline{\eta}}$) is a *normal* (respectively, *regular*) variety over $\overline{\eta}$ of dimension $\geq \dim(Y) - \dim(S)$ by Lemma 3.5, (i) (respectively, our choice of (U_Y, U_Z)) — cf. also [5], Proposition (5.5.2).] First, we verify assertion (i). It follows from Lemma 1.5 that we have a natural Π_{Y_η}-conjugacy class of *isomorphisms* $\Pi_{Y_{\overline{\eta}}} \xrightarrow{\sim} \Delta_{Y_\eta/\eta}$; moreover, it follows from Proposition 1.10, (i), together with our choice of (U_Y, U_Z), that there exists a natural Π_{U_Y}-conjugacy class of *surjections* $\Pi_{(U_Y)_{\overline{\eta}}} \twoheadrightarrow \Delta_{U_Y/U_Z}$. Thus, one verifies easily from the *surjectivity* of $\Pi_{(U_Y)_{\overline{\eta}}} \twoheadrightarrow \Pi_{Y_{\overline{\eta}}}$ [cf. Lemma 1.2] that the image of the composite $\Delta_{Y_\eta/\eta} \hookrightarrow \Pi_{Y_\eta} \twoheadrightarrow \Pi_Y$ coincides with $N \subseteq \Pi_Y$. This completes the proof of assertion (i). Next, we verify assertion (ii). It follows immediately from our choice of (U_Y, U_Z) that the geometric fiber F of $U_Y \to U_Z$ at a $\overline{k}$-valued geometric point of U_Z is a *regular variety* over $\overline{k}$ of *dimension* $\geq \dim(Y) - \dim(S) > 0$. In particular, one verifies easily that the natural morphism $F \to Y \otimes_k \overline{k}$ over $\overline{k}$ is *nonconstant*. Thus, since [we have assumed that] Y is of *LFG-type*, it follows immediately from Lemma 1.5 that the image of $\Pi_F \to \Pi_Y$ induced by the natural morphism $F \to Y$ is *infinite*. On the other hand, one verifies easily that $\Pi_F \to \Pi_Y$ factors through the composite $\Delta_{U_Y/U_Z} \hookrightarrow \Pi_{U_Y} \twoheadrightarrow \Pi_Y$. Thus, it follows immediately from the definition of N that N is *infinite*. This completes the proof of assertion (ii). Finally, we verify assertion (iii). Now let us observe that the natural morphism $Y_{\overline{\eta}} \to Y$ factors through a natural *closed immersion* $Y_{\overline{\eta}} \hookrightarrow Y \times_k \overline{\eta}$. Thus, since $Y \times_k \overline{\eta}$ is a *hyperbolic polycurve* over $\overline{\eta}$, it follows from Lemma 2.14 that the image of $\Pi_{Y_{\overline{\eta}}} \to \Pi_{Y \times_k \overline{\eta}}$ admits a sequence of closed subgroups as in the statement of assertion (iii). On the other hand, any homomorphism $\Pi_{Y \times_k \overline{\eta}} \to \Pi_Y$ that arises from the morphism $Y \times_k \overline{\eta} \stackrel{\mathrm{pr}_1}{\to} Y$ determines an *isomorphism* $\Pi_{Y \times_k \overline{\eta}} \xrightarrow{\sim} \Delta_{Y/k}$ [cf. Lemma 1.5; Proposition 2.4, (ii)]. Thus, it follows immediately from assertion (i) that assertion (iii) holds. This completes the proof of assertion (iii). □

Definition 3.7. Let X, Y be *normal* varieties over k and $\phi\colon \Pi_Y \to \Pi_X$ a homomorphism over G_k.

(i) We shall say that ϕ is *nondegenerate* if ϕ is open, and, moreover, for any open subscheme $U \subseteq Y$, any normal variety Z over k such that $\dim(Z) < \dim(X)$, and any smooth and geometrically connected surjection $U \to Z$ over k, the composite $\Pi_U \twoheadrightarrow \Pi_Y \to \Pi_X$ of the outer homomorphism $\Pi_U \twoheadrightarrow \Pi_Y$ induced by the open immersion $U \hookrightarrow Y$ and the outer homomorphism $\Pi_Y \to \Pi_X$ determined by ϕ does not factor through the outer homomorphism $\Pi_U \to \Pi_Z$ induced by the morphism $U \to Z$.

(ii) Suppose that X is a *hyperbolic polycurve* of relative dimension n over k. Then we shall say that the homomorphism ϕ is *poly-nondegenerate* if there exists a sequence of parametrizing morphisms

$$X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \operatorname{Spec} k = X_0$$

such that, for each $i \in \{0, \cdots, n\}$, the composite $\Pi_Y \to \Pi_X \twoheadrightarrow \Pi_{X_i}$ of the outer homomorphism $\Pi_Y \to \Pi_X$ determined by ϕ and the natural outer homomorphism $\Pi_X \twoheadrightarrow \Pi_{X_i}$ is nondegenerate [cf. (i)].

Theorem 3.8. *Let p be a prime number, k a* **sub-p-adic field** *[cf. Definition 3.1], $\overline{k}$ an algebraic closure of k, X a* **hyperbolic polycurve** *[cf. Definition 2.1, (ii)] over k, and Y a* **normal variety** *[cf. Definition 1.4] over k. Write $G_k \stackrel{\text{def}}{=} \operatorname{Gal}(\overline{k}/k)$; Π_X, Π_Y for the étale fundamental groups of X, Y, respectively; $\Delta_{X/k} \subseteq \Pi_X$ for the kernel of the natural surjection $\Pi_X \twoheadrightarrow G_k$; $\operatorname{Hom}_k^{\text{dom}}(Y, X)$ for the set of* **dominant** *morphisms from X to Y over k; $\operatorname{Hom}_k^{\text{PND}}(\Pi_Y, \Pi_X)$ for the set of* **poly-nondegenerate** *homomorphisms [cf. Definition 3.7, (ii)] from Π_Y to Π_X over G_k. Then the natural map*

$$\operatorname{Hom}_k^{\text{dom}}(Y, X) \longrightarrow \operatorname{Hom}_{G_k}(\Pi_Y, \Pi_X)/\operatorname{Inn}(\Delta_{X/k})$$

determines a **bijection**

$$\operatorname{Hom}_k^{\text{dom}}(Y, X) \xrightarrow{\sim} \operatorname{Hom}_{G_k}^{\text{PND}}(\Pi_Y, \Pi_X)/\operatorname{Inn}(\Delta_{X/k}) .$$

Proof. First, I claim that the following assertion holds:

Claim 3.8.A: A [necessarily *open* — cf. Lemma 1.3] homomorphism $\phi_f \colon \Pi_Y \to \Pi_X$ over G_k arising from a *dominant* morphism $f \colon Y \to X$ over k is *poly-nondegenerate*.

Indeed, suppose that there exist a sequence of parametrizing morphisms

$$X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \operatorname{Spec} k = X_0 ,$$

an integer $i \in \{0, \cdots, n\}$, an open subscheme $U \subseteq Y$ of Y, a normal variety Z over k, and a smooth and geometrically connected surjection $U \to Z$ over k such that the composite $\Pi_U \twoheadrightarrow \Pi_Y \stackrel{\phi_f}{\to} \Pi_X \twoheadrightarrow \Pi_{X_i}$ factors through $\Pi_U \to \Pi_Z$. Then, by applying Lemma 2.9 [where we take "(S, Y, Z, X, f)" in the statement of Lemma 2.9 to be $(\mathrm{Spec}\, k, Z, U, X_i, U \hookrightarrow Y \xrightarrow{f} X \to X_i)$], we conclude that the composite $U \hookrightarrow Y \xrightarrow{f} X \to X_i$ factors through $U \to Z$. In particular, since f is *dominant*, it holds that $\dim(Z) \geq i$. This completes the proof of Claim 3.8.A.

It follows from Claim 3.8.A, together with Proposition 3.2, (ii), that, to verify Theorem 3.8, it suffices to verify the *surjectivity* of the natural map

$$\mathrm{Hom}_k^{\mathrm{dom}}(Y, X) \longrightarrow \mathrm{Hom}_{G_k}^{\mathrm{PND}}(\Pi_Y, \Pi_X)/\mathrm{Inn}(\Delta_{X/k}) .$$

Let $\phi \colon \Pi_Y \to \Pi_X$ be a *poly-nondegenerate* homomorphism over G_k and

$$X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \mathrm{Spec}\, k = X_0$$

a sequence of parametrizing morphisms as in Definition 3.7, (ii). Now I claim that the following assertion holds:

> Claim 3.8.B: Suppose that there exists a morphism $f \colon Y \to X$ over k from which ϕ arises. Then f is *dominant*.

Indeed, assume that f is *not dominant*. Write $X' \subseteq X$ for the scheme-theoretic image of f and $S \stackrel{\mathrm{def}}{=} \mathrm{Nor}(Y/X')$. Then since the natural morphism $Y \to S$ over k is *dominant* and *generically geometrically irreducible* [cf. Lemma 1.9], and k is of *characteristic zero*, one verifies easily that there exist open subschemes $U_Y \subseteq Y$, $U_S \subseteq S$ of Y, S, respectively, such that the image of $U_Y \subseteq Y$ by $Y \to S$ is *contained* in $U_S \subseteq S$, and, moreover, the resulting morphism $U_Y \to U_S$ is *surjective*, *smooth*, and *geometrically connected*. On the other hand, since f is *not dominant*, one verifies easily that X', hence also U_S, is of *dimension* $< \dim(X)$. Thus, since ϕ is *poly-nondegenerate*, we obtain a contradiction. This completes the proof of Claim 3.8.B.

Next, let us observe that, to verify that ϕ arises from a dominant morphism $Y \to X$ over k, it suffices to verify that the following assertion holds:

> Claim 3.8.C: For each $i \in \{0, \cdots, n-1\}$, if the composite $\Pi_Y \xrightarrow{\phi} \Pi_X \twoheadrightarrow \Pi_{X_i}$ arises from a dominant morphism $Y \to X_i$ over k, then the composite $\Pi_Y \xrightarrow{\phi} \Pi_X \twoheadrightarrow \Pi_{X_{i+1}}$ arises from a dominant morphism $Y \to X_{i+1}$ over k.

The rest of the proof of Theorem 3.8 is devoted to verifying Claim 3.8.C.

Write $Z \stackrel{\mathrm{def}}{=} \mathrm{Nor}(Y/X_i)$; $\eta \to Z$ for the generic point of Z, $Y_\eta \stackrel{\mathrm{def}}{=} Y \times_Z \eta$. Now I claim that the following assertion holds:

> Claim 3.8.C.1: The image of a homomorphism that belongs to the Π_{X_i}-conjugacy class of homomorphisms $\Delta_{Y_\eta/\eta} \to \Delta_{X_{i+1}/X_i}$ of Lemma 3.5, (iii) [where we take "(S, Y, X)" in the statement of Lemma 3.5 to be (X_i, Y, X_{i+1})], is *nontrivial*.

Indeed, assume that $\Delta_{Y_\eta/\eta} \to \Delta_{X_{i+1}/X_i}$ of Lemma 3.5, (iii), is *trivial*. Then it follows immediately from Lemma 3.5, (ii); Lemma 3.6, (i), that there exists a *nonempty* open subschemes $U_Y \subseteq Y$, $U_Z \subseteq Z$ such that the natural morphism $Y \to Z$ induces a morphism $U_Y \to U_Z$ which is *surjective*, *smooth*, and *geometrically connected*, and, moreover, the image of the composite $\Delta_{U_Y/U_Z} \hookrightarrow \Pi_{U_Y} \twoheadrightarrow \Pi_Y \stackrel{\phi}{\to} \Pi_X \twoheadrightarrow \Pi_{X_{i+1}}$ is *trivial*. [Here, we note that it follows immediately from the existence of the *poly-nondegenerate* homomorphism ϕ, together with the definition of *poly-nondegeneracy*, that $\dim(X) \leq \dim(Y)$.] Thus, it follows immediately that the composite $\Pi_{U_Y} \twoheadrightarrow \Pi_Y \stackrel{\phi}{\to} \Pi_X \twoheadrightarrow \Pi_{X_{i+1}}$ factors through $\Pi_{U_Y} \twoheadrightarrow \Pi_{U_Z}$. On the other hand, since $\dim(Z) = i < i + 1 = \dim(X_{i+1})$, and ϕ is *poly-nondegenerate*, we obtain a contradiction. This completes the proof of Claim 3.8.C.1.

It follows from Claim 3.8.C.1, together with Lemma 3.5, (iv), that the composite $\Pi_Y \stackrel{\phi}{\to} \Pi_X \twoheadrightarrow \Pi_{X_{i+1}}$ arises from a [necessarily *dominant* — cf. Claim 3.8.B] morphism $Y \to X_{i+1}$ over k. This completes the proof of Claim 3.8.C, hence also of Theorem 3.8. □

Theorem 3.9. *Let p be a prime number; k a* **sub-p-adic field** [*cf. Definition 3.1*]*; $\overline{k}$ an algebraic closure of k; Y, S* **normal varieties** [*cf. Definition 1.4*] *over k; X a* **hyperbolic curve** [*cf. Definition 2.1,* (i)] *over S. Write $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$; Π_X, Π_Y, Π_S for the étale fundamental groups of X, Y, S, respectively. Let $\phi\colon \Pi_Y \to \Pi_X$ be a homomorphism over G_k. Suppose that the following conditions are satisfied:*

1. *The composite $\Pi_Y \stackrel{\phi}{\to} \Pi_X \twoheadrightarrow \Pi_S$ arises from a morphism $Y \to S$ over k.*
2. *ϕ is* **open***, and its kernel is* **finite***.*
3. *Y is of* **LFG-type** [*cf. Definition 2.5*].
4. $\dim(X)$ $(= \dim(S) + 1) \leq \dim(Y)$.

Then ϕ arises from a **quasi-finite dominant** *morphism $Y \to X$ over k. In particular,* $\dim(X) = \dim(Y)$.

Proof. Let us observe that, in light of Lemma 2.6, (ii), by applying Lemmas 3.5, (iv), 3.6, (i) [where we take "(S, Y, X, ϕ)" in the statement of Lemma 3.5 to be (S, Y, X, ϕ)], to verify that ϕ arises from a *quasi-finite* morphism $Y \to X$ over k [which is necessarily *dominant* by condition (4)], it suffices to verify that the image of the closed subgroup $N \subseteq \Pi_Y$ defined in the statement of Lemma 3.6 by $\phi\colon \Pi_Y \to \Pi_X$ is *nontrivial*. On the other hand, since Y is of *LFG-type*, it follows from Lemma 3.6, (ii), that N is *infinite*. Thus, it follows from condition (2) that the image of $N \subseteq \Pi_Y$ by $\phi\colon \Pi_Y \to \Pi_X$ is *nontrivial*. This completes the proof of Theorem 3.9. □

Definition 3.10. Let n be a positive integer. Then we shall say that the *assertion* $(\dagger_n)$ *holds* if, for any hyperbolic polycurve X of dimension n over $\overline{k}$, Π_X does not admit a sequence of closed subgroups of Π_X

$$\{1\} = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n \subseteq H_{n+1} = \Pi_X$$

such that, for each $i \in \{0, \cdots, n\}$, the closed subgroup H_i is *topologically finitely generated*, *normal* in H_{i+1}, and, moreover, the quotient H_{i+1}/H_i is *infinite*.

Lemma 3.11. *The assertion* $(\dagger_1)$ [*cf. Definition 3.10*] *holds.*

Proof. This follows immediately from Proposition 2.4, (iv). □

Theorem 3.12. *Let* n *be a positive integer,* p *a prime number,* k *a* **sub-p-adic field** [*cf. Definition 3.1*], $\overline{k}$ *an algebraic closure of* k, S *a* **normal variety** [*cf. Definition 1.4*] *over* k, X *a* **hyperbolic polycurve** [*cf. Definition 2.1,* (ii)] *of* **relative dimension** n *over* S, *and* Y *a* **hyperbolic polycurve** *over* k. *Write* $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$; Π_X, Π_Y, Π_S *for the étale fundamental groups of* X, Y, S, *respectively. Let* $\phi\colon \Pi_Y \to \Pi_X$ *be a homomorphism over* G_k. *Suppose that the following conditions are satisfied:*

1. *The composite* $\Pi_Y \xrightarrow{\phi} \Pi_X \twoheadrightarrow \Pi_S$ *arises from a morphism* $Y \to S$ *over* k.
2. ϕ *is an* **open injection**.
3. $\dim(X)$ $(= \dim(S) + n) \leq \dim(Y)$.
4. *For each* $i \in \{1, \cdots, n-1\}$, *the assertion* $(\dagger_i)$ [*cf. Definition 3.10*] *holds.*

Then ϕ *arises from a* **quasi-finite dominant** *morphism* $Y \to X$ *over* k. *In particular,* $\dim(X) = \dim(Y)$.

Proof. Let

$$X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow S = X_0$$

be a sequence of parametrizing morphisms. Fix a *surjection* [cf. Proposition 2.4, (i)] $\Pi_X \twoheadrightarrow \Pi_{X_1}$ over G_k that arises from the morphism $X \to X_1$ over k. First, I claim that the following assertion holds:

Claim 3.12.A: If $n \geq 2$, then the composite $\Pi_Y \to \Pi_{X_1}$ of $\phi\colon \Pi_Y \to \Pi_X$ and the fixed surjection $\Pi_X \twoheadrightarrow \Pi_{X_1}$ arises from a morphism $Y \to X_1$ over k.

Indeed, write $S' \subseteq S$ for the scheme-theoretic image of the morphism $Y \to S$ [cf. condition (1)], $Z \stackrel{\text{def}}{=} \mathrm{Nor}(Y/S')$; $\eta \to Z$ for the generic point of Z; $Y_\eta \stackrel{\text{def}}{=} Y \times_Z \eta$. Then, to verify Claim 3.12.A, by applying Lemmas 3.5, (iv); 3.6, (i) [where we take "(S, Y, X, ϕ)" in the statement of Lemma 3.5 to be "$(S, Y, X_1, \Pi_Y \stackrel{\phi}{\to} \Pi_X \twoheadrightarrow \Pi_{X_1})$"], it suffices to verify that the image of $N \subseteq \Pi_Y$ defined in the statement of Lemma 3.6 in Π_{X_1} is *nontrivial*. To verify this, assume that the image of $N \subseteq \Pi_Y$ in Π_{X_1} is *trivial*, i.e., the image of $N \subseteq \Pi_Y$ in Π_X is contained in $\Delta_{X/X_1} \subseteq \Pi_X$. On the other hand, since $N \subseteq \Pi_Y$ is *normal* in Π_Y, and ϕ is an *open injection*, it follows that the image of N in Π_X is *normal* in an open subgroup of Π_X. In particular, again by the fact that ϕ is an *open injection*, we conclude that there exists an open subgroup $U \subseteq \Delta_{X/X_1}$ of Δ_{X/X_1} such that if, for each $i \in \{0, \cdots, \dim(Y) - \dim(S)\}$, we write $H_i^U \subseteq \Delta_{X/X_1}$ for the image in Δ_{X/X_1} of "H_i" in the statement of Lemma 3.6, (iii), for our "N", then

- $H^U_{\dim(Y)-\dim(S)} \subseteq U$ [so we obtain a sequence of closed subgroups of U
$$\{1\} = H_0^U \subseteq H_1^U \subseteq \cdots \subseteq H^U_{\dim(Y)-\dim(S)-1}$$
$$\subseteq H^U_{\dim(Y)-\dim(S)} \subseteq H^U_{\dim(Y)-\dim(S)+1} \stackrel{\text{def}}{=} U\,,$$
- for each $i \in \{1, \cdots, \dim(Y) - \dim(S) + 1\}$, the closed subgroup H_i^U is *topologically finitely generated*,
- for each $i \in \{1, \cdots, \dim(Y) - \dim(S)\}$, the closed subgroup H_i^U is *normal* in $H^U_{\dim(Y)-\dim(S)}$,
- the closed subgroup $H^U_{\dim(Y)-\dim(S)} \subseteq H^U_{\dim(Y)-\dim(S)+1}$ is *normal* in $H^U_{\dim(Y)-\dim(S)+1}$, and,
- for each $i \in \{1, \cdots, \dim(Y)-\dim(S)\}$, the quotient H_i^U/H_{i-1}^U is *infinite*.

Now let us recall that we have assumed that $n \leq \dim(Y) - \dim(S)$, and that the assertion $(\dagger_{n-1})$ holds. Moreover, it follows immediately from Propositions 2.3; 2.4, (ii), that U may be regarded as "Π_X" for a hyperbolic polycurve over $\overline{k}$ of dimension $n - 1$. Thus,

- if $H^U_{\dim(Y)-\dim(S)+1}/H^U_{\dim(Y)-\dim(S)}$ is *finite*, then by replacing U $(= H^U_{\dim(Y)-\dim(S)+1})$ by $H^U_{\dim(Y)-\dim(S)}$ and, for each $i \in \{1, \cdots, n\}$, taking "H_i" in Definition 3.10 [in the case where we take "Π_X" in Definition 3.10 to be U — cf. the above discussion] to be $H^U_{\dim(Y)-\dim(S)-n+i}$, and

- if $H^U_{\dim(Y)-\dim(S)+1}/H^U_{\dim(Y)-\dim(S)}$ is *infinite*, then, for each $i \in \{1, \cdots, n\}$, by taking "H_i" in Definition 3.10 [in the case where we take "Π_X" in Definition 3.10 to be U — cf. the above discussion] to be $H^U_{\dim(Y)-\dim(S)-n+1+i}$,

we obtain a contradiction. This completes the proof of Claim 3.12.A.

By applying Claim 3.12.A inductively and replacing S by X_{n-1}, to verify Theorem 3.12, we may assume without loss of generality that X is a *hyperbolic curve over* S. Then it follows from Theorem 3.9, together with Proposition 2.7, that ϕ arises from a *quasi-finite dominant* morphism $Y \to X$ over k. This completes the proof of Theorem 3.12. □

Corollary 3.13. *Let p be a prime number, k a* **sub-p-adic field** [*cf. Definition 3.1*], *$\overline{k}$ an algebraic closure of k, S a* **normal variety** [*cf. Definition 1.4*] *over k, X a* **hyperbolic polycurve** [*cf. Definition 2.1,* (ii)] *of* **dimension 2** *over S, and Y a* **hyperbolic polycurve** *over k. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$; Π_X, Π_Y, Π_S for the étale fundamental groups of X, Y, S, respectively. Let $\phi\colon \Pi_Y \to \Pi_X$ be a homomorphism over G_k. Suppose that the following conditions are satisfied:*

1. *The composite $\Pi_Y \xrightarrow{\phi} \Pi_X \twoheadrightarrow \Pi_S$ arises from a morphism $Y \to S$ over k.*
2. *ϕ is an* **open injection**.
3. $\dim(X)$ $(= \dim(S) + 2) \leq \dim(Y)$.

Then ϕ arises from a **quasi-finite dominant** *morphism $Y \to X$ over k. In particular,* $\dim(X) = \dim(Y)$.

Proof. This follows from Theorem 3.12, together with Lemma 3.11. □

Lemma 3.14. *Let G_1, G_2 be profinite groups; $H_1 \subseteq G_1$, $H_2 \subseteq G_2$ closed subgroups; $\phi\colon G_1 \to G_2$ a homomorphism. Suppose that $\phi(H_1) \subseteq H_2$. Then the homomorphism $H_1 \to H_2$ induced by ϕ is* **surjective** *if and only if the following condition is satisfied: For any open subgroup $U \subseteq G_2$ of G_2 and normal open subgroup $N \subseteq U$ of U, if the composite $H_2 \cap U \hookrightarrow U \twoheadrightarrow U/N$ is* **surjective**, *then the composite $H_1 \cap \phi^{-1}(U) \hookrightarrow \phi^{-1}(U) \xrightarrow{\phi} U \twoheadrightarrow U/N$ is* **surjective**.

Proof. This follows immediately from the various definitions involved. □

Theorem 3.15. *Let p be a prime number, k a* **sub-p-adic field** [*cf. Definition 3.1*], *$\overline{k}$ an algebraic closure of k, X a* **hyperbolic polycurve** [*cf. Definition 2.1,* (ii)] *of* **dimension 2** *over k, and Y a* **normal variety** [*cf. Definition 1.4*] *over k. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$;*

Π_X, Π_Y for the étale fundamental groups of X, Y, respectively. Let $\phi\colon \Pi_Y \to \Pi_X$ be an **open** *homomorphism over G_k. Suppose that the kernel of ϕ is* **topologically finitely generated**. *Then ϕ arises from a* **uniquely determined dominant** *morphism $Y \to X$ over k. In particular, Y is of* **dimension** ≥ 2.

Proof. First, let us observe that it follows from Proposition 2.3 that, by replacing X by the connected finite étale covering of X corresponding to the image of ϕ, to verify Theorem 3.15, we may assume without loss of generality that ϕ is *surjective*. Let $X \to X_1$ be a parametrizing morphism of X. Then since the kernel Δ_{X/X_1} of the outer *surjection* [cf. Proposition 2.4, (i)] $\Pi_X \twoheadrightarrow \Pi_{X_1}$ is *topologically finitely generated* [cf. Proposition 2.4, (iii)], it follows from Theorem 3.4, together with the implication (2) $\Rightarrow$ (3) of Lemma 2.11, that the composite $\Pi_Y \stackrel{\phi}{\to} \Pi_X \twoheadrightarrow \Pi_{X_1}$ arises from a morphism $Y \to X_1$ over k which is *surjective* and *generically geometrically connected*. Write $\eta \to X_1$ for the generic point of X_1; $Y_\eta \stackrel{\mathrm{def}}{=} Y \times_{X_1} \eta$; $X_\eta \stackrel{\mathrm{def}}{=} X \times_{X_1} \eta$. [Thus, Y_η is a *normal variety* over η.]

Now I claim that the following assertion holds:

> Claim 3.15.A: A homomorphism that belongs to the Π_X-conjugacy class of homomorphisms $\Delta_{Y_\eta/\eta} \to \Delta_{X/X_1}$ of Lemma 3.5, (iii) [where we take "(S, Y, X)" in the statement of Lemma 3.5 to be "(X_1, Y, X)"], is *surjective*.

Indeed, it follows immediately from Lemma 3.14 that, to verify Claim 3.15.A, it suffices to verify that the following assertion holds:

> Claim 3.15.A.1: Let $X' \to X$ be a connected finite étale covering of X and $X'' \to X'$ a connected finite étale *Galois* covering of X'. Write $Y' \to Y$ for the connected finite étale covering of Y corresponding to $X' \to X$ by ϕ; $Y'' \to Y'$ for the connected finite étale *Galois* covering of Y' corresponding to $X'' \to X'$ by ϕ. Write, moreover, $Y'_\eta \stackrel{\mathrm{def}}{=} Y' \times_{X_1} \eta$ $(= Y' \times_Y Y_\eta)$; $Y''_\eta \stackrel{\mathrm{def}}{=} Y'' \times_{X_1} \eta$ $(= Y'' \times_Y Y_\eta)$. [Here, let us observe that since the natural morphism $Y_\eta \to Y$ induces an outer *surjection* $\Pi_{Y_\eta} \twoheadrightarrow \Pi_Y$ — cf. Lemma 1.2 — it holds that Y'_η and Y''_η are *connected*.] Suppose that the composite $\Delta_{X/X_1} \cap \Pi_{X'} \hookrightarrow \Pi_{X'} \twoheadrightarrow \Pi_{X'}/\Pi_{X''}$ is *surjective*. Then the composite $\Delta_{Y_\eta/\eta} \cap \Pi_{Y'_\eta} \hookrightarrow \Pi_{Y'_\eta} \twoheadrightarrow \Pi_{Y'_\eta}/\Pi_{Y''_\eta}$ is *surjective*.

Now, to verify Claim 3.15.A.1, let us observe that, in the notation of Claim 3.15.A.1, it follows immediately from Proposition 2.3 that the sequence of schemes $X' \to X'_1 \stackrel{\text{def}}{=} \mathrm{Nor}(X'/X_1) \to X'_0 \stackrel{\text{def}}{=} \mathrm{Nor}(X'/\mathrm{Spec}\, k)$ determines a structure of *hyperbolic polycurve of dimension* 2 on X', and, moreover, the natural morphisms $X'_1 \to X_1$, $\eta' \to \eta$ — where we write $\eta' \to X'_1$ for the generic point of X'_1 — are connected *finite étale coverings*. In particular, one verifies easily that the natural inclusions $\Pi_{X'} \hookrightarrow \Pi_X$, $\Pi_{Y'_\eta} \hookrightarrow \Pi_{Y_\eta}$ determine equalities

$$\Delta_{X/X_1} \cap \Pi_{X'} = \Delta_{X'/X'_1}, \quad \Delta_{Y_\eta/\eta} \cap \Pi_{Y'_\eta} = \Delta_{Y'_\eta/\eta'}\,.$$

Thus, to verify Claim 3.15.A, i.e., Claim 3.15.A.1, by replacing X by X', it suffices to verify that the following assertion holds [cf. also Lemma 1.5; Proposition 2.4, (ii)]:

> Claim 3.15.A.2: In the notation of Claim 3.15.A.1, let $\overline{\eta} \to X_1$ be a geometric point of X_1 whose image is the generic point η. Suppose that $X'' \to X$ is *Galois*, and that $X'' \times_{X_1} \overline{\eta}$ is *connected*. Then $Y''_\eta \times_\eta \overline{\eta}$ $(= Y'' \times_{X_1} \overline{\eta})$ is *connected*.

Now, to verify Claim 3.15.A.2, let us observe that since $X'' \times_{X_1} \overline{\eta}$ is *connected*, i.e., $X'' \to X_1$ is *generically geometrically connected*, and [one verifies easily that] the composite $X'' \to X \to X_1$ is *smooth* and *surjective*, it follows from the implication (1) $\Rightarrow$ (2) of Lemma 2.11 that the composite $\Pi_{X''} \hookrightarrow \Pi_X \twoheadrightarrow \Pi_{X_1}$ is *surjective*, and its kernel is *topologically fintiely generated*. Thus, since [we have assumed that] the kernel of ϕ is *topologically fintiely generated*, it follows immediately that the composite $\Pi_{Y''} \twoheadrightarrow \Pi_{X''} \hookrightarrow \Pi_X \twoheadrightarrow \Pi_{X_1}$ [where the first arrow is the *surjection* induced by ϕ] is *surjective*, and its kernel is *topologically fintiely generated*. Therefore, by the implication (2) $\Rightarrow$ (3) of Lemma 2.11, we conclude that the natural morphism $Y'' \to X_1$ is *surjective* and *generically geometrically connected*; in particular, $Y'' \times_{X_1} \overline{\eta}$ is *connected*. This completes the proof of Claim 3.15.A.2, hence also of Claim 3.15.A.

It follows from Claim 3.15.A, together with Lemma 3.5, (iv), that ϕ arises from a morphism $Y \to X$ over k. On the other hand, since the composite $Y \to X \to X_1$ is *dominant*, one verifies easily from Claim 3.15.A, together with Proposition 2.4, (iii), that this morphism $Y \to X$ is *dominant*. This completes the proof of Theorem 3.15. □

Remark 3.15.1. The argument given in the proof of Theorem 3.15 is essentially the same as the argument applied in [16] to prove [16], Theorem D.

Corollary 3.16. *Let p be a prime number, k a* **sub-p-adic field** *[cf. Definition 3.1], $\overline{k}$ an algebraic closure of k, Y a* **normal variety**

[*cf. Definition 1.4*] *over* k*, and* X *a* **hyperbolic polycurve** [*cf. Definition 2.1,* (ii)] *of* **dimension** 3 *over* k*. Write* $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$*;* Π_X*,* Π_Y *for the étale fundamental groups of* X*,* Y*, respectively. Let* $\phi\colon \Pi_Y \to \Pi_X$ *be a homomorphism over* G_k*. Suppose that the following conditions are satisfied:*

(1) ϕ *is* **open***, and its kernel is* **finite**.

(2) Y *is of* **LFG-type** [*cf. Definition 2.5*].

(3) $3 \leq \dim(Y)$.

Then ϕ *arises from a* **uniquely determined quasi-finite dominant** *morphism* $Y \to X$ *over* k*. In particular,* Y *is of* **dimension** 3.

Proof. Let $X \to X_2$ be a parametrizing morphism of X. Then it follows immediately from condition (1), together with Proposition 2.4, (iii), that the kernel of the composite $\Pi_Y \stackrel{\phi}{\to} \Pi_X \twoheadrightarrow \Pi_{X_2}$ is *topologically finitely generated*. Thus, it follows from Theorem 3.15 that the composite $\Pi_Y \stackrel{\phi}{\to} \Pi_X \twoheadrightarrow \Pi_{X_2}$ arises from a uniquely determined dominant morphism $Y \to X_2$ over k. In particular, it follows from Proposition 3.2, (ii); Theorem 3.9, together with Lemma 2.6, (ii), that ϕ arises from a *uniquely determined quasi-finite dominant* morphism $Y \to X$ over k. This completes the proof of Corollary 3.16. □

Corollary 3.17. *Let* p *be a prime number,* k *a* **sub-p-adic field** [*cf. Definition 3.1*], $\overline{k}$ *an algebraic closure of* k*,* X *a* **hyperbolic polycurve** [*cf. Definition 2.1,* (ii)] *of* **dimension** 4 *over* k*, and* Y *a* **hyperbolic polycurve** *over* k*. Write* $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$*;* Π_X*,* Π_Y *for the étale fundamental groups of* X*,* Y*, respectively. Let* $\phi\colon \Pi_Y \to \Pi_X$ *be a homomorphism over* G_k*. Suppose that the following conditions are satisfied:*

(1) ϕ *is an* **open injection** (*respectively,* **isomorphism**).

(2) $4 \leq \dim(Y)$.

Then ϕ *arises from a* **uniquely determined finite étale covering** (*respectively,* **uniquely determined isomorphism**) $Y \to X$ *over* k*. In particular,* $\dim(Y) = 4$.

Proof. First, let us observe that, to verify Corollary 3.17, by replacing Π_X by the image of ϕ [cf. condition (1)], we may assume without loss of generality that ϕ is an *isomorphism* over G_k. Let $X \to X_3$ be a parametrizing morphism of X and $X_3 \to X_2$ a parametrizing morphism of X_3. Then it follows immediately from our assumption that ϕ is an *isomorphism*, together with Proposition 2.4, (iii), that the kernel of the composite $\Pi_Y \stackrel{\phi}{\to} \Pi_X \twoheadrightarrow \Pi_{X_2}$ is *topologically finitely generated*. Thus, it follows from Theorem 3.15 that the composite $\Pi_Y \stackrel{\phi}{\to} \Pi_X \twoheadrightarrow \Pi_{X_2}$ arises from a uniquely determined dominant morphism $Y \to X_2$ over k. In particular, it follows from Corollary 3.13 that ϕ arises from a *quasi-finite dominant* morphism $Y \to X$ over k; thus, it holds that $4 = \dim(X) = \dim(Y)$. Therefore, in light of Proposition 3.2, (ii), by applying a similar argument to the above argument to ϕ^{-1}, we conclude that the morphism $Y \to X$ is an *isomorphism*. This completes the proof of Corollary 3.17. □

Definition 3.18. Let n be a positive integer and X an algebraic stack over k. Then we shall say that X is a *hyperbolic orbi-polycurve* of dimension n over k if X admits a dense open substack that is a scheme, is geometrically connected over k, and, moreover, admits a finite étale Galois covering that is a hyperbolic polycurve of dimension n over some finite extension of k.

Corollary 3.19. *Let p be a prime number; n_X, n_Y positive integers; k a* **sub-p-adic field** [*cf. Definition 3.1*]*; $\overline{k}$ an algebraic closure of k; X, Y* **hyperbolic orbi-polycurves** *of dimension n_X, n_Y over k, respectively* [*cf. Definition 3.18*]. *Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$; Π_X, Π_Y for the étale fundamental groups of X, Y, respectively; $\mathrm{Isom}_k(X,Y)$ for the set of isomorphisms of X with Y over k; $\mathrm{Isom}_{G_k}(\Pi_X,\Pi_Y)$ for the set of isomorphisms of Π_X with Π_Y over G_k; $\Delta_{Y/k}$ for the kernel of the natural surjection $\Pi_Y \twoheadrightarrow G_k$. Suppose that either $n_X \leq 4$ or $n_Y \leq 4$. Then the natural map*

$$\mathrm{Isom}_k(X,Y) \longrightarrow \mathrm{Isom}_{G_k}(\Pi_X,\Pi_Y)/\mathrm{Inn}(\Delta_{Y/k})$$

is **bijective**.

Proof. First, let us observe that the *injectivity* in question follows immediately from Propositions 2.3; 3.2, (ii), together with the definition of a *hyperbolic orbi-polycurve*. Thus, it remains to verify the *surjectivity* in question. Let $\phi\colon \Pi_X \xrightarrow{\sim} \Pi_Y$ be an isomorphism over G_k. Now I claim that the following assertion holds:

> Claim 3.19.A: If X and Y are *hyperbolic polycurves* over k, then ϕ arises from an isomorphism $X \xrightarrow{\sim} Y$ over k.

Indeed, let us first observe that, to verify that ϕ arises from an isomorphism $X \xrightarrow{\sim} Y$ over k, by replacing (X,Y,ϕ) by (Y,X,ϕ^{-1}) if necessary, we may assume without loss of generality that $n_X \geq n_Y$; in particular, since [we have assumed that] either $n_X \leq 4$ or $n_Y \leq 4$, it holds that $n_Y \leq 4$. Thus, it follows from Theorems 3.4; 3.15; Corollaries 3.16; 3.17, together with Proposition 2.7, that ϕ arises from a *uniquely determined quasi-finite dominant* morphism $X \to Y$ over k. In particular, we obtain that

$n_X = n_Y \leq 4$. Thus, again by applying Theorems 3.4; 3.15; Corollaries 3.16; 3.17, together with Proposition 2.7, to ϕ^{-1}, we conclude from Proposition 3.2, (ii), that the morphism $X \to Y$ is an *isomorphism*. This completes the proof of Claim 3.19.A.

Next, I claim that the following assertion holds:

> Claim 3.19.B: ϕ arises from an isomorphism $X \overset{\sim}{\to} Y$ over k.

Indeed, it follows from the definition of a *hyperbolic orbi-polycurve*, together with Proposition 2.3, that there exist a finite extension k_Z ($\subseteq \overline{k}$) of k and an normal open subgroup $H_X \subseteq \Pi_X$ of Π_X such that the connected finite étale Galois coverings $Z_X \to X$, $Z_Y \to Y$ corresponding to $H_X \subseteq \Pi_X$, $H_Y \stackrel{\mathrm{def}}{=} \phi(H_X) \subseteq \Pi_Y$ are *hyperbolic polycurves* over k_Z. Then it follows from Claim 3.19.A that the isomorphism $H_X \overset{\sim}{\to} H_Y$ induced by ϕ arises from an isomorphism $Z_X \overset{\sim}{\to} Z_Y$ over k_Z. On the other hand, since [we already verified that] the natural map in question is *injective*, and the isomorphism ϕ is *compatible* with the natural outer actions of $\Pi_X/H_X = \mathrm{Gal}(Z_X/X)$, $\Pi_Y/H_Y = \mathrm{Gal}(Z_Y/Y)$ on H_X, H_Y, respectively — relative to the isomorphism $\Pi_X/H_X \overset{\sim}{\to} \Pi_Y/H_Y$ induced by ϕ — we conclude that the isomorphism $Z_X \overset{\sim}{\to} Z_Y$ is *compatible* with the natural actions of $\Pi_X/H_X = \mathrm{Gal}(Z_X/X)$, $\Pi_Y/H_Y = \mathrm{Gal}(Z_Y/Y)$ on Z_X, Z_Y, respectively — relative to the isomorphism $\Pi_X/H_X \overset{\sim}{\to} \Pi_Y/H_Y$ induced by ϕ. Thus, by descending the isomorphism $Z_X \overset{\sim}{\to} Z_Y$, we obtain an isomorphism $X \overset{\sim}{\to} Y$ over k, which, by the various definitions involved, *belongs* to the $\Delta_{Y/k}$-conjugacy class of isomorphisms $\Pi_X \overset{\sim}{\to} \Pi_Y$ determined by ϕ. This completes the proof of Claim 3.19.B, hence also of Corollary 3.19. □

Remark 3.19.1. It seems to the author that the assertion $(\dagger_n)$ [cf. Definition 3.10] holds for every positive integer n. However, it is not clear to the author at the time of writing whether or not there exists an integer $n > 1$ for which the assertion $(\dagger_n)$ holds. Here, let us observe that if one proves that the assertion $(\dagger_n)$ holds for every positive integer n, then it follows immediately from a similar argument to the argument applied in the proof of Corollary 3.19, together with Theorem 3.12, that the conclusion of Corollary 3.19 holds without the assumption that "either $n_X \leq 4$ or $n_Y \leq 4$" in the statement of Corollary 3.19.

Proposition 3.20. *Let k_X, k_Y be a* **finitely generated extension fields over the field of rational numbers***; $\overline{k}_X$, $\overline{k}_Y$ algebraic closures of k_X, k_Y, respectively. Write $G_{k_X} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_X/k_X)$ and $G_{k_Y} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_Y/k_Y)$. Then the following hold:*

(i) *Let* $H \subseteq G_{k_X}$ *be a closed subgroup of* G_{k_X}. *Suppose that* H *is* **topologically finitely generated** *and* **normal** *in an open subgroup of* G_{k_X}. *Then* H *is* **trivial**.

(ii) *Write* $\mathrm{Isom}(\overline{k}_X/k_X, \overline{k}_Y/k_Y)$ *for the set of isomorphisms* $\overline{k}_X \xrightarrow{\sim} \overline{k}_Y$ *that determine isomorphisms* $k_X \xrightarrow{\sim} k_Y$. *Then the natural map*

$$\mathrm{Isom}(\overline{k}_X/k_X, \overline{k}_Y/k_Y) \longrightarrow \mathrm{Isom}(G_{k_Y}, G_{k_X})$$

is **bijective**.

Proof. Assertion (i) follows from [2], Theorem 13.4.2; [2], Proposition 16.11.6. Assertion (ii) follows from the main result of [22] [cf. also [24] for a survey on [22]]. □

Corollary 3.21. *Let* k_X, k_Y *fields of characteristic zero;* $\overline{k}_X$, $\overline{k}_Y$ *algebraic closures of* k_X, k_Y, *respectively;* n *a positive integer;* X *a* **hyperbolic polycurve** [*cf. Definition 2.1,* (ii)] *of dimension* n *over* k_X*;* Y *a* **normal variety** [*cf. Definition 1.4*] *over* k_Y. *Write* $G_{k_X} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_X/k_X)$*;* $G_{k_Y} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}_Y/k_Y)$*;* Π_X, Π_Y *for the étale fundamental groups of* X, Y, *respectively. Let* $\phi\colon \Pi_Y \to \Pi_X$ *be an* **open** *homomorphism. Suppose that one of the following conditions* (1), (2), (3), (4) *is satisfied:*

(1) $n = 1$.

(2) *The following conditions are satisfied:*
  (2-i) $n = 2$.
  (2-ii) *The kernel of* ϕ *is* **topologically finitely generated**.

(3) *The following conditions are satisfied:*
  (3-i) $n = 3$.
  (3-ii) *The kernel of* ϕ *is* **finite**.
  (3-iii) Y *is of* **LFG-type** [*cf. Definition 2.5*].
  (3-iv) $3 \leq \dim(Y)$.

(4) *The following conditions are satisfied:*
  (4-i) $n = 4$.
  (4-ii) ϕ *is* **injective**.
  (4-iii) Y *is a* **hyperbolic polycurve** *over* k_Y.
  (4-iv) $4 \leq \dim(Y)$.

Then the following hold:

(i) *Suppose that both* k_X, k_Y *are* **finitely generated over the field of rational numbers**. *Then the open homomorphism* ϕ *lies over an open homomorphism* $G_{k_Y} \to G_{k_X}$.

(ii) *In the situation of* (i), *suppose that the homomorphism* $G_{k_Y} \to G_{k_X}$ *obtained by* (i) *is* **injective**. *Then* ϕ *arises from a* **dominant** *morphism* $Y \to X$.

(iii) *Suppose that both* k_X, k_Y *are* **finite extensions of the** p**-adic completion of the field of rational numbers** *for* *some prime number p. Suppose, moreover, that one of the following three conditions is satisfied:*

(iii-a) *The open homomorphism ϕ lies over an open homomorphism $G_{k_Y} \to G_{k_X}$ that* **arises** *from a homomorphism $k_X \hookrightarrow k_Y$ of fields.*

(iii-b) *There exist hyperbolic curves* [*cf. Definition 2.1,* (i)] *$Z_X \to \mathrm{Spec}\, k_X$, $Z_Y \to \mathrm{Spec}\, k_Y$ of* **quasi-Belyi type** [*cf.* [18]*, Definition 2.3,* (iii)] *and morphisms $X \to Z_X$, $Y \to Z_Y$ over k_X, k_Y, respectively, such that if we write Π_{Z_X}, Π_{Z_Y} for the étale fundamental groups of Z_X, Z_Y, respectively, then the homomorphism ϕ lies over an* **isomorphism** $\Pi_{Z_Y} \xrightarrow{\sim} \Pi_{Z_X}$.

(iii-c) *The open homomorphism ϕ lies over an open homomorphism $G_{k_Y} \to G_{k_X}$, and, moreover, there exist a hyperbolic curve Z over k_X and a* **dominant** *morphism $X \to Z$ over k_X such that if we write Π_Z for the étale fundamental group of Z, then the extension Π_Z of G_{k_X} is of* **A-qLT-type** [*cf.* [19]*, Definition 3.1,* (v)].

Then ϕ arises from a **dominant** *morphism $Y \to X$.*

Proof. Assertion (i) follows immediately, by considering the composite $\Delta_{Y/k_Y} \hookrightarrow \Pi_Y \xrightarrow{\phi} \Pi_X \twoheadrightarrow G_{k_X}$, from Lemmas 1.5; 1.7; Proposition 3.20, (i). Next, we verify assertion (ii). Let us first observe that, in light of Proposition 2.3, by replacing Π_X by the image of ϕ, to verify assertion (ii), we may assume without loss of generality that ϕ, hence also the *injection* $G_{k_Y} \hookrightarrow G_{k_X}$ obtained by assertion (i), is *surjective*. Then it follows from Proposition 3.20, (ii), that the *isomorphism* $G_{k_Y} \xrightarrow{\sim} G_{k_X}$ arises from an isomorphism $\overline{k}_X \xrightarrow{\sim} \overline{k}_Y$ that determines an isomorphism $k_X \xrightarrow{\sim} k_Y$. In particular, to verify assertion (ii), by replacing $(X, k_X, \overline{k}_X)$ by $(X \otimes_{k_X} k_Y, k_Y, \overline{k}_Y)$ and applying Proposition 2.4, (ii), we may assume without loss of generality that $(k_X, \overline{k}_X) = (k_Y, \overline{k}_Y)$. On the other hand, since $(k_X, \overline{k}_X) = (k_Y, \overline{k}_Y)$, assertion (ii) follows from Theorems 3.4; 3.15; Corollaries 3.16; 3.17. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Now I claim that the following assertion holds:

> Claim 3.21.A: If either condition (iii-b) or condition (iii-c) holds, then condition (iii-a) holds.

Indeed, suppose that condition (iii-b) is satisfied. Then let us observe that it follows from [18], Corollary 2.3, that the isomorphism $\Pi_{Z_Y} \xrightarrow{\sim} \Pi_{Z_X}$ *arises* from an isomorphism $Z_Y \xrightarrow{\sim} Z_X$ of schemes, which thus implies [cf., e.g., the discussion concerning *isogenuous* given in "*Curves*" of [18], §0] that condition (iii-a) is satisfied.

Next, suppose that condition (iii-c) is satisfied. Let $\psi\colon \Pi_X \to \Pi_Z$ be a homomorphism over G_{k_X} that arises from the dominant morphism $X \to Z$ over k_X. Then one verifies easily that the composite $\psi \circ \phi\colon \Pi_Y \to \Pi_Z$ is *open* [cf. Lemma 1.3] and, moreover, lies over an open homomorphism $G_{k_Y} \to G_{k_X}$ [cf. condition (iii-c)]. Thus, it follows immediately from [19], Theorem 3.5, (iii) [cf. also the proof of [19], Theorem 3.5, (iii)], that condition (iii-a) is satisfied. This completes the proof of Claim 3.21.A. In particular, to verify assertion (iii), it suffices to verify assertion (iii) in the case where condition (iii-a) is satisfied.

Suppose that condition (iii-a) is satisfied. Then, in light of Proposition 2.3, by replacing Π_X by the image of ϕ, to verify assertion (iii) in the case where condition (iii-a) is satisfied, we may assume without loss of generality that ϕ, hence also the homomorphism $G_{k_Y} \to G_{k_X}$ of condition (iii-a), is *surjective*. In particular, we conclude that the homomorphism $G_{k_Y} \to G_{k_X}$ of condition (iii-a) *arises* from an isomorphism $k_X \xrightarrow{\sim} k_Y$. Thus, by replacing $(X, k_X, \overline{k}_X)$ by $(X \otimes_{k_X} k_Y, k_Y, \overline{k}_Y)$ and applying Proposition 2.4, (ii), we may assume without loss of generality that $(k_X, \overline{k}_X) = (k_Y, \overline{k}_Y)$. On the other hand, since $(k_X, \overline{k}_X) = (k_Y, \overline{k}_Y)$, assertion (iii) follows from Theorems 3.4; 3.15; Corollaries 3.16; 3.17. This completes the proof of assertion (iii). □

Corollary 3.22. *Let* p *be a prime number and* n *a positive integer. Write* $\mathbb{S}$ *for the set consisting of the set of all prime numbers,* $\mathbb{F}$ *for the set of isomorphism classes of* **sub-p-adic** *fields* [*cf. Definition 3.1*], $\mathbb{V}$ *for the set of isomorphism classes of* **hyperbolic orbi-polycurves** *of dimension* n *over sub-*p*-adic fields* [*cf. Definition 3.18*], *and* $\mathbb{D} \stackrel{\text{def}}{=} \mathbb{S} \times \mathbb{F} \times \mathbb{S}$. *Suppose that* $n \leq 4$. *Then the hypotheses of* [19], *Theorem 4.7,* (i), (ii), *are* **satisfied** *relative to this* $\mathbb{D}$.

Proof. First, let us recall from [16], Lemma 15.8, that the *absolute Galois group of a sub-*p*-adic field* is *slim* [i.e., every open subgroup of the absolute Galois group of a sub-p-adic field is *center-free*]. The fact that $\mathbb{D}$ is *chain-full* [cf. [19], Definition 4.6, (i)] is immediate. The *rel-isom* $\mathbb{D}$*GC* [cf. [19], Definition 4.6, (ii)], as well as the *slimness* of the "Δ_i" in the statement of [19], Theorem 4.7, follows immediately from Corollary 3.19 [cf. also the proof of Corollary 3.21]. □

4. Finiteness of the set of outer isomorphisms between étale fundamental groups of hyperbolic polycurves

In the present §4, we discuss the *finiteness* of a set determined by certain isomorphisms between the étale fundamental groups of hyperbolic polycurves of arbitrary dimension [cf. Theorem 4.4 below]. In the case where the basefield is *finite over the field of rational numbers*, we also prove the *finiteness* of the set of outer isomorphisms between the étale fundamental groups of hyperbolic polycurves [cf. Corollary 4.6 below]. In the present §4, let k be a field of *characteristic zero*, $\overline{k}$ an algebraic closure of k, and $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$.

Lemma 4.1. *Let G be a profinite group, $H \subseteq G$ an open subgroup of G, A a group, and $A \to \mathrm{Aut}(G)$ a homomorphism to the group of automorphisms $\mathrm{Aut}(G)$ of G. Write $A_H \subseteq A$ for the subgroup of A consisting of $a \in A$ such that the automorphism of G obtained by forming the image of a in $\mathrm{Aut}(G)$* **preserves** *$H \subseteq G$. Suppose that G is* **topologically finitely generated***. Then A_H is of* **finite index** *in A.*

Proof. Write $d \stackrel{\text{def}}{=} [G : H]$. Then since G is *topologically finitely generated*, the set S of open subgroups of G of index d is *finite*. On the other hand, the homomorphism $A \to \mathrm{Aut}(G)$ naturally determines an action of A on S, and $A_H \subseteq A$ coincides with the stabilizer of $H \in S$. Thus, A_H is of *finite index* in A. This completes the proof of Lemma 4.1. □

Lemma 4.2. *Let n be a positive integer, X a* **hyperbolic polycurve** *of dimension n over k, and*

$$X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \mathrm{Spec}\, k = X_0$$

a sequence of parametrizing morphisms. Then the following hold:

(i) *There exists an open subgroup $H \subseteq \Delta_{X/k}$ such that, for each $i \in \{0, \cdots, n\}$, if we write $H_i \stackrel{\text{def}}{=} H \cap \Delta_{X/X_i}$ [thus, we have a sequence of* **normal** *closed subgroups of H*

$$H_n = \{1\} \subseteq H_{n-1} \subseteq \cdots \subseteq H_2 \subseteq H_1 \subseteq H_0 = H$$

and a natural **injection** *$H_i/H_{i+1} \hookrightarrow \Delta_{X_{i+1}/X_i}$ for each $i \in \{0, \cdots, n-1\}$ — cf. Proposition 2.4,* (i)*], then, for each $i \in \{1, \cdots, n-1\}$, it holds that*

$$\mathrm{rank}_{\widehat{\mathbb{Z}}}\Big((H_i/H_{i+1})^{\mathrm{ab}}\Big) < \mathrm{rank}_{\widehat{\mathbb{Z}}}\Big((H_{i-1}/H_i)^{\mathrm{ab}}\Big)\,.$$

(ii) *Let ϕ be an automorphism of $\Delta_{X/k}$. Suppose that ϕ preserves the open subgroup $H \subseteq \Delta_{X/k}$ of* (i)*. Then, for each $i \in \{0, \cdots, n\}$, it holds that $\phi(\Delta_{X/X_i}) = \Delta_{X/X_i}$.*

(iii) *Let ψ be an automorphism of Π_X over G_k. Suppose that ψ preserves the open subgroup $H \subseteq \Delta_{X/k}$ of* (i), *and that k is* **sub-p-adic** [*cf. Definition 3.1*] *for some prime number p. Then ψ arises from an* **automorphism** *of X over k.*

Proof. First, we verify assertion (i) by *induction on n*. If $n = 1$, then assertion (i) is immediate. Now suppose that $n \geq 2$, and that the *induction hypothesis* is in force. Then since one may regard Δ_{X/X_1} as the "$\Delta_{X/k}$" of a hyperbolic polycurve over k of dimension $n-1$ [cf. Proposition 2.4, (ii)], by the *induction hypothesis*, there exists an open subgroup $H^* \subseteq \Delta_{X/X_1}$ of Δ_{X/X_1} such that, for each $i \in \{1, \cdots, n\}$, if we write $H_i^* \stackrel{\mathrm{def}}{=} H^* \cap \Delta_{X/X_i}$, then, for each $i \in \{2, \cdots, n-1\}$, it holds that

$$\mathrm{rank}_{\widehat{\mathbb{Z}}}\Big((H_i^*/H_{i+1}^*)^{\mathrm{ab}}\Big) < \mathrm{rank}_{\widehat{\mathbb{Z}}}\Big((H_{i-1}^*/H_i^*)^{\mathrm{ab}}\Big) .$$

Now since the profinite group Δ_{X/X_1} is *normal* in $\Delta_{X/k}$ and *topologically finitely generated* [cf. Proposition 2.4, (iii)], it follows from Lemma 4.1 [where we take "(G, H, A)" in the statement of Lemma 4.1 to be $(\Delta_{X/X_1}, H^*, \Delta_{X/k})$ and the action of "A" on "G" in the statement of Lemma 4.1 to be the action by conjugation] that $N_{\Delta_{X/k}}(H^*)$ is *open* in $\Delta_{X/k}$.

Since $H^* \subseteq N_{\Delta_{X/X_1}}(H^*)$ $(= N_{\Delta_{X/k}}(H^*) \cap \Delta_{X/X_1})$, and H^* is *open* in Δ_{X/X_1}, we have a natural *surjection*

$$N_{\Delta_{X/k}}(H^*)/H^* \longrightarrow N_{\Delta_{X/k}}(H^*)/N_{\Delta_{X/X_1}}(H^*)$$

whose kernel is *finite*. Thus, there exists an *open* subgroup $Q \subseteq N_{\Delta_{X/k}}(H^*)/H^*$ of $N_{\Delta_{X/k}}(H^*)/H^*$ such that the composite

$$Q \xrightarrow{\subseteq} N_{\Delta_{X/k}}(H^*)/H^* \longrightarrow N_{\Delta_{X/k}}(H^*)/N_{\Delta_{X/X_1}}(H^*)$$

is *injective*. In particular, Q may be regarded as an *open* subgroup of

$$N_{\Delta_{X/k}}(H^*)/N_{\Delta_{X/X_1}}(H^*) \xrightarrow{\subseteq} \Delta_{X/k}/\Delta_{X/X_1} \xrightarrow{\sim} \Delta_{X_1/k}$$

[cf. Proposition 2.4, (i)]. Thus, it follows immediately from Proposition 2.4, (vi), that there exists an *open* subgroup $Q_H \subseteq Q$ such that

$$\mathrm{rank}_{\widehat{\mathbb{Z}}}\Big((H_1^*/H_2^*)^{\mathrm{ab}}\Big) < \mathrm{rank}_{\widehat{\mathbb{Z}}}(Q_H^{\mathrm{ab}}) .$$

Let us write $H \subseteq \Delta_{X/k}$ for the open subgroup of $\Delta_{X/k}$ obtained by forming the inverse image of $Q_H \subseteq N_{\Delta_{X/k}}(H^*)/H^*$ by the natural surjection $N_{\Delta_{X/k}}(H^*) \twoheadrightarrow N_{\Delta_{X/k}}(H^*)/H^*$; thus, H fits into an *exact* sequence of profinite groups

$$1 \longrightarrow H^* \longrightarrow H \longrightarrow Q_H \longrightarrow 1 .$$

Now I claim that the following assertion holds:

Claim 4.2.A: This open subgroup $H \subseteq \Delta_{X/k}$ satisfies the condition appearing in the statement of assertion (i).

Indeed, let us first observe that, by our choice of (H^*, Q_H), one verifies easily that, to verify Claim 4.2.A, it suffices to verify that $H \cap \Delta_{X/X_1} = H^*$. To this end, let us observe that since H^* is *open* in Δ_{X/X_1}, and $H^* \subseteq H \cap \Delta_{X/X_1}$, we have a natural surjection $H/H^* \twoheadrightarrow H/(H \cap \Delta_{X/X_1})$ whose kernel is *finite*. On the other hand, since $H/H^* \xrightarrow{\sim} Q_H$ may be regarded as an open subgroup of $\Delta_{X_1/k}$ [cf. the discussion preceding Claim 4.2.A], it follows from Proposition 2.4, (iii), that H/H^* is *torsion-free*. Thus, we conclude that $H \cap \Delta_{X/X_1} = H^*$. This completes the proof of Claim 4.2.A, hence also of assertion (i).

Next, we verify assertion (ii). Now since [one verifies easily that] the image of the composite $H \hookrightarrow \Delta_{X/k} \twoheadrightarrow \Delta_{X_{n-1}/k}$ satisfies the condition appearing in the statement of assertion (i) for "H", by *induction on* n, to verify assertion (ii), it suffices to verify that the following assertion holds:

Claim 4.2.B: $\phi(\Delta_{X/X_{n-1}}) = \Delta_{X/X_{n-1}}$.

Now, to verify Claim 4.2.B, I claim that the following assertion holds:

Claim 4.2.B.1: $\phi(H_{n-1}) = H_{n-1}$.

Indeed, it is immediate that there exists a *unique* integer $0 \leq m \leq n-1$ such that the image of the composite $H_{n-1} \hookrightarrow H \xrightarrow{\phi} H \twoheadrightarrow H/H_{m+1}$ is *nontrivial*, but the image of the composite $H_{n-1} \hookrightarrow H \xrightarrow{\phi} H \twoheadrightarrow H/H_m$ is *trivial*; thus, $H_{n-1} \hookrightarrow H \xrightarrow{\phi} H \twoheadrightarrow H/H_{m+1}$ determines a *nontrivial* homomorphism $H_{n-1} \to H_m/H_{m+1}$. Now since the composite $H \xrightarrow{\phi} H \twoheadrightarrow H/H_{m+1}$ is *surjective*, and $H_{n-1} \subseteq H$ is *normal* in H, one verifies easily that the image of the *nontrivial* homomorphism $H_{n-1} \to H_m/H_{m+1}$ is *normal*; thus, since H_{n-1} is *topologically finitely generated* [cf. Propositions 2.3; 2.4, (iii)], it follows from Proposition 2.4, (iv), that the image of the *nontrivial* homomorphism $H_{n-1} \to H_m/H_{m+1}$ is *open*, which implies that

$$\mathrm{rank}_{\widehat{\mathbb{Z}}}\Big((H_m/H_{m+1})^{\mathrm{ab}}\Big) \leq \mathrm{rank}_{\widehat{\mathbb{Z}}}(H_{n-1}^{\mathrm{ab}})\,.$$

Thus, it follows from the condition appearing in the statement of assertion (i) that $m = n-1$, i.e., $\phi(H_{n-1}) \subseteq H_{n-1}$. Moreover, by applying a similar argument to the above argument to ϕ^{-1}, we conclude that $\phi(H_{n-1}) = H_{n-1}$. This completes the proof of Claim 4.2.B.1.

Finally, we verify Claim 4.2.B. To verify Claim 4.2.B, write N for the intersection of all $\Delta_{X/k}$-conjugates of H_{n-1}. Then it is immediate that N is *normal* in $\Delta_{X/k}$. Moreover, since $\Delta_{X/X_{n-1}}$ is *topologically finitely generated* [cf. Proposition 2.4, (iii)] and *normal* in $\Delta_{X/k}$, and $H_{n-1} \subseteq \Delta_{X/X_{n-1}}$ is *open* in $\Delta_{X/X_{n-1}}$, one verifies easily that N is *open* in $\Delta_{X/X_{n-1}}$. Thus, $\Delta_{X/X_{n-1}}/N \subseteq \Delta_{X/k}/N$ is a *finite* subgroup of $\Delta_{X/k}/N$; in particular, since $\Delta_{X_{n-1}/k}$ is *torsion-free* [cf. Proposition 2.4, (iii)], $\Delta_{X/X_{n-1}}/N \subseteq \Delta_{X/k}/N$ is the *maximal torsion subgroup* of $\Delta_{X/k}/N$. On the other hand, it follows from Claim 4.2.B.1 that ϕ determines an automorphism of $\Delta_{X/k}/N$. Thus, we conclude that the automorphism of $\Delta_{X/k}/N$ determined by ϕ preserves $\Delta_{X/X_{n-1}}/N$, hence that ϕ preserves $\Delta_{X/X_{n-1}}$. This completes the proof of Claim 4.2.B, hence also of assertion (ii).

Finally, we verify assertion (iii). It follows immediately from assertion (ii), together with Proposition 2.4, (i), that, for each $i \in \{0, \cdots, n\}$, ψ induces an automorphism ψ_i of Π_{X_i} over G_k. [Thus, $\psi_0 = \mathrm{id}_{G_k}$, and $\psi_n = \psi$.] Now it is immediate that, by *induction on* i, to verify assertion (iii), it suffices to verify that the following assertion holds:

> Claim 4.2.C: For each $i \in \{0, \cdots, n-1\}$, if the automorphism ψ_i arises from an automorphism f_i of X_i over k, then ψ_{i+1} arises from an automorphism of X_{i+1} over k.

To verify Claim 4.2.C, write $\eta \to X_i$ for the generic point of X_i, $(X_{i+1})_\eta \stackrel{\mathrm{def}}{=} X_{i+1} \times_{X_i} \eta$, and $(X_{i+1})'_\eta$ for the basechange of the natural morphism $X_{i+1} \to X_i$ by the composite $\eta \to X_i \stackrel{f_i}{\to} X_i$. Then it follows immediately from assertion (ii), together with Proposition 2.4, (ii), that ψ_{i+1} induces an isomorphism $\Pi_{(X_{i+1})_\eta} \stackrel{\sim}{\to} \Pi_{(X_{i+1})'_\eta}$ over Π_η. Thus, it follows from Theorem 3.4, together with the equivalence (ii-1) $\Leftrightarrow$ (ii-2) of Lemma 2.13, that the isomorphism $\Pi_{(X_{i+1})_\eta} \stackrel{\sim}{\to} \Pi_{(X_{i+1})'_\eta}$ arises from an isomorphism $(X_{i+1})_\eta \stackrel{\sim}{\to} (X_{i+1})'_\eta$ over η. In particular, it follows from Lemma 2.10 that ψ_{i+1} arises from an endomorphism of X_{i+1} over k. Therefore, by applying a similar argument to the above argument to ψ_{i+1}^{-1}, we conclude from Proposition 3.2, (ii), that ψ_{i+1} arises from an automorphism of X_{i+1} over k. This completes the proof of Claim 4.2.C, hence also of assertion (iii). □

Theorem 4.3. *Let n be a positive integer, p a prime number, k a* **sub-p-adic field** [*cf. Definition 3.1*], *$\overline{k}$ an algebraic closure of k, X a* **hyperbolic polycurve** [*cf. Definition 2.1,* (ii)] *of dimension n over k, and*

$$X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \operatorname{Spec} k = X_0$$

a sequence of parametrizing morphisms. Write $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$*,* Π_X *for the étale fundamental group of* X*, and* $\Delta_{X/k}$ *for the kernel of the natural surjection* $\Pi_X \twoheadrightarrow G_k$*. For each* $i \in \{1, \cdots, n\}$*, write, moreover,* (g_i, r_i) *for the type of the hyperbolic curve* X_i *over* X_{i-1} [*cf. Definition 2.1,* (i)]. *Suppose that, for each* $i \in \{1, \cdots, n-1\}$*,*

$$2g_{i+1} + \max\{r_{i+1} - 1, 0\} < 2g_i + \max\{r_i - 1, 0\}\,.$$

Then the natural map

$$\mathrm{Aut}_k(X) \longrightarrow \mathrm{Aut}_{G_k}(\Pi_X)/\mathrm{Inn}(\Delta_{X/k})$$

is **bijective***, i.e., every automorphism of* Π_X *over* G_k *arises from a* **uniquely determined automorphism** *of* X *over* k*.*

Proof. The *injectivity* of the map in question follows from Proposition 3.2, (ii). The *surjectivity* of the map in question follows from Lemma 4.2, (iii), together with Proposition 2.4, (v). This completes the proof of Theorem 4.3. □

Theorem 4.4. *Let* p *be a prime number;* k *a* **sub-p-adic field** [*cf. Definition 3.1*]*;* $\overline{k}$ *an algebraic closure of* k*;* X*,* Y **hyperbolic polycurves** [*cf. Definition 2.1,* (ii)] *over* k*. Write* $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$*;* Π_X*,* Π_Y *for the étale fundamental groups of* X*,* Y*, respectively;* $\mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)$ *for the set of isomorphisms of* Π_X *with* Π_Y *over* G_k*;* $\Delta_{Y/k}$ *for the kernel of the natural surjection* $\Pi_Y \twoheadrightarrow G_k$*. Then the set*

$$\mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_{Y/k})$$

is **finite***.*

Proof. If $\mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y) = \emptyset$, then Theorem 4.4 is immediate. Thus, to verify Theorem 4.4, we may assume without loss of generality that $\mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)$ is *nonempty*. Then let us observe that every element of $\mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)$ determines a bijection between $\mathrm{Isom}_{G_k}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Delta_{Y/k})$ and $\mathrm{Aut}_{G_k}(\Pi_X)/\mathrm{Inn}(\Delta_{X/k})$. Thus, to verify Theorem 4.4, by replacing Y by X, we may assume without loss of generality that $X = Y$. Let $H \subseteq \Delta_{X/k}$ be an open subgroup of $\Delta_{X/k}$ which satisfies the condition appearing in the statement of Lemma 4.2, (i), with respect to a sequence of parametrizing morphisms

$$X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow \mathrm{Spec}\, k = X_0\,.$$

Then, by applying Lemma 4.1 [where we take "(G, H, A)" in the statement of Lemma 4.1 to be $(\Delta_{X/k}, H, \mathrm{Aut}(\Delta_{X/k}))$], we conclude that there exists a subgroup $A \subseteq \mathrm{Aut}(\Delta_{X/k})$ of $\mathrm{Aut}(\Delta_{X/k})$ of *finite index* such that, for each $\phi \in A$, it holds that $\phi(H) = H$. Write $B \subseteq \mathrm{Aut}_{G_k}(\Pi_X)$ for the inverse image of $A \subseteq \mathrm{Aut}(\Delta_{X/k})$ by the natural homomorphism $\mathrm{Aut}_{G_k}(\Pi_X) \to \mathrm{Aut}(\Delta_{X/k})$. [Thus, $B \subseteq \mathrm{Aut}_{G_k}(\Pi_X)$ is of *finite index* in $\mathrm{Aut}_{G_k}(\Pi_X)$.] Then it follows immediately from Lemma 4.2, (iii), that every element of B arises from an automorphism of X over k, i.e., the image of the composite $B \hookrightarrow \mathrm{Aut}_{G_k}(\Pi_X) \twoheadrightarrow \mathrm{Aut}_{G_k}(\Pi_X)/\mathrm{Inn}(\Delta_{X/k})$ is contained in the image of the natural injection $\mathrm{Aut}_k(X) \hookrightarrow \mathrm{Aut}_{G_k}(\Pi_X)/\mathrm{Inn}(\Delta_{X/k})$ [cf. Proposition 3.2, (ii)]. In particular, it follows from Proposition 4.5 below that the image of the composite $B \hookrightarrow \mathrm{Aut}_{G_k}(\Pi_X) \twoheadrightarrow \mathrm{Aut}_{G_k}(\Pi_X)/\mathrm{Inn}(\Delta_{X/k})$ is *finite*. On the other hand, since B is of *finite index* in $\mathrm{Aut}_{G_k}(\Pi_X)$, we conclude that $\mathrm{Aut}_{G_k}(\Pi_X)/\mathrm{Inn}(\Delta_{X/k})$ is *finite*. This completes the proof of Theorem 4.4. □

Proposition 4.5. *Let S, Y be integral varieties over k; $Y \to S$ a dominant morphism over k; X a* **hyperbolic polycurve** *over S. Then the set $\mathrm{Hom}_S^{\mathrm{dom}}(Y, X)$ of* **dominant** *morphisms from Y to X over S is* **finite**.

Proof. Write n for the relative dimension of X over S. First, I claim that the following assertion holds:

Claim 4.5.A: If $n = 1$, then Proposition 4.5 holds.

Indeed, let $\overline{\eta} \to S$ be a geometric point of S whose image is the generic point of S and $F \subseteq Y \times_S \overline{\eta}$ an irreducible component of $Y \times_S \overline{\eta}$. Write $F_{\mathrm{red}} \subseteq Y \times_S \overline{\eta}$ for the reduced closed subscheme of $Y \times_S \overline{\eta}$ whose support is $F \subseteq Y \times_S \overline{\eta}$. [Thus, F_{red} is an *integral variety* over $\overline{\eta}$]. Then since Y is *integral*, and $Y \to S$ is *dominant*, one verifies easily that the composite of natural maps $\mathrm{Hom}_S^{\mathrm{dom}}(Y, X) \to \mathrm{Hom}_{\overline{\eta}}(Y \times_S \overline{\eta}, X \times_S \overline{\eta}) \to \mathrm{Hom}_{\overline{\eta}}(F_{\mathrm{red}}, X \times_S \overline{\eta})$ is *injective* [cf. the fact that the composite $F_{\mathrm{red}} \hookrightarrow Y \times_S \overline{\eta} \to Y$ is *schematically dense*] and *factors* through the subset $\mathrm{Hom}_{\overline{\eta}}^{\mathrm{dom}}(F_{\mathrm{red}}, X \times_S \overline{\eta})$. Thus, by replacing S, Y by $\overline{\eta}$, F_{red}, respectively, to verify Claim 4.5.A, we may assume without loss of generality that $k = \overline{k}$ and $S = \mathrm{Spec}\, k$.

Next, to verify Claim 4.5.A, I claim that the following assertion holds:

Claim 4.5.A.1: If Y is of *dimension one* [and $n = 1$], then Proposition 4.5 holds.

Indeed, let us first observe that one verifies easily that there exist a nonnegative integer N and a connected finite étale Galois covering $X' \to X$ of X over k of degree N such that the genus [i.e., "g" in Definition 2.1, (i)] of the hyperbolic curve X' over k is ≥ 2. Then it is immediate that, for each *dominant* morphism $Y \to X$ over k, there exist a connected finite étale Galois covering $Y' \to Y$ of Y over k of degree $\leq N$ and a *dominant* morphism $Y' \to X'$ which lies over the given *dominant* morphism $Y \to X$. Thus, in light of the fact that $Y' \to Y$ is *schematically dense*, since the set of isomorphism classes of connected finite étale Galois coverings of Y over k of degree $\leq N$ is *finite* [cf. Lemma 1.7], and the group of automorphisms of such a Y' over k is *finite* [cf., e.g., [10], Chapter IV, Exercise 2.5], by replacing (X, Y) by (X', Y'), to verify Claim 4.5.A.1, we may assume without loss of generality that X is of genus ≥ 2. Then Claim 4.5.A.1 follows immediately from de Franchis' theorem [cf., e.g., [14], p. 227]. This completes the proof of Claim 4.5.A.1.

It follows from Claim 4.5.A.1 that, to verify Claim 4.5.A, we may assume without loss of generality that Y is of *dimension* ≥ 2. Next, let us observe that, by replacing Y by a suitable affine open subscheme of Y, to verify Claim 4.5.A, we may assume without loss of generality that Y is *regular*, and that Y may be embedded into a projective space P over k [of suitable dimension]. Thus, by applying Bertini's theorem [cf., e.g., the easily verified quasi-projective version of [10], Theorem 8.18] and [23], §V, Corollaire 7.3, inductively [i.e., by considering suitable hyperplane sections], we conclude that there exist a *regular variety* C *of dimension one* over k and a morphism $C \to Y$ over k such that the induced outer homomorphism $\Pi_C \to \Pi_Y$ is *surjective*. Now let us consider the natural commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_k^{\mathrm{dom}}(Y, X) & \longrightarrow & \mathrm{Hom}^{\mathrm{open}}(\Pi_Y, \Pi_X)/\mathrm{Inn}(\Pi_X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_k(C, X) & \longrightarrow & \mathrm{Hom}(\Pi_C, \Pi_X)/\mathrm{Inn}(\Pi_X) \end{array}$$

[cf. Lemma 1.3]. Since the upper horizontal arrow is *injective* [cf. Proposition 3.2, (i)], and the right-hand vertical arrow is *injective* [cf. the *surjectivity* of $\Pi_C \to \Pi_Y$], it holds that the left-hand vertical arrow is *injective*. On the other hand, again by the *surjectivity* of $\Pi_C \to \Pi_Y$, it follows immediately that the left-hand vertical arrow factors through the subset $\mathrm{Hom}_k^{\mathrm{dom}}(C, X) \subseteq \mathrm{Hom}_k(C, X)$ [cf. also Proposition 2.4, (iii)]. Thus, to verify Claim 4.5.A, it suffices to verify the *finiteness* of $\mathrm{Hom}_k^{\mathrm{dom}}(C, X)$, which follows from Claim 4.5.A.1. This completes the proof of Claim 4.5.A.

Finally, we verify Proposition 4.5 by *induction on* n. If $n = 1$, then Proposition 4.5 follows from Claim 4.5.A. Now suppose that $n \geq 2$, and that the *induction hypothesis* is in force. Let $X \to X_{n-1}$ be a parametrizing morphism of X. Then since the *finiteness* of $\mathrm{Hom}_S^{\mathrm{dom}}(Y, X_{n-1})$ follows from the *induction hypothesis*, to verify the *finiteness* of $\mathrm{Hom}_S^{\mathrm{dom}}(Y, X)$, it suffices to verify that, for any $f_{n-1} \in \mathrm{Hom}_S^{\mathrm{dom}}(Y, X_{n-1})$, the inverse image of $\{f_{n-1}\} \subseteq \mathrm{Hom}_S^{\mathrm{dom}}(Y, X_{n-1})$ by the natural map $\mathrm{Hom}_S^{\mathrm{dom}}(Y, X) \to \mathrm{Hom}_S^{\mathrm{dom}}(Y, X_{n-1})$ [induced by the morphism $X \to X_{n-1}$] is *finite*. In other words, to verify the *finiteness* of $\mathrm{Hom}_S^{\mathrm{dom}}(Y, X)$, it suffices to verify that, for any $f_{n-1} \in \mathrm{Hom}_S^{\mathrm{dom}}(Y, X_{n-1})$, the set $\mathrm{Hom}_{X_{n-1}}^{\mathrm{dom}}(Y, X)$ — where we take the structure morphism $Y \to X_{n-1}$ to be f_{n-1} — is *finite*. On the other hand, since $X \to X_{n-1}$ is a *hyperbolic* *curve*, this *finiteness* in question follows from Claim 4.5.A. This completes the proof of Proposition 4.5. □

Corollary 4.6. *Let* k_X*,* k_Y *be* **finite extensions of the field of rational numbers***;* X*,* Y **hyperbolic polycurves** [*cf. Definition 2.1,* (ii)] *over* k_X*,* k_Y*, respectively. Write* Π_X*,* Π_Y *for the étale fundamental groups of* X*,* Y*, respectively;* $\mathrm{Isom}(\Pi_X, \Pi_Y)$ *for the set of isomorphisms of* Π_X *with* Π_Y*. Then the set*

$$\mathrm{Isom}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Pi_Y)$$

is **finite***.*

Proof. If $\mathrm{Isom}(\Pi_X, \Pi_Y) = \emptyset$, then Corollary 4.6 is immediate. Suppose that $\mathrm{Isom}(\Pi_X, \Pi_Y) \neq \emptyset$. Then since every element of $\mathrm{Isom}(\Pi_X, \Pi_Y)$ determines a bijection between $\mathrm{Isom}(\Pi_X, \Pi_Y)/\mathrm{Inn}(\Pi_Y)$ and $\mathrm{Out}(\Pi_X)$, to verify Corollary 4.6, by replacing Y by X, we may assume without loss of generality that $X = Y$.

Now let us observe that, for each $\phi \in \mathrm{Aut}(\Pi_X)$, by considering the composites $\Delta_{X/k_X} \hookrightarrow \Pi_X \overset{\phi}{\overset{\sim}{\to}} \Pi_X \twoheadrightarrow G_{k_X}$, $\Delta_{X/k_X} \hookrightarrow \Pi_X \overset{\phi^{-1}}{\overset{\sim}{\to}} \Pi_X \twoheadrightarrow G_{k_X}$ and applying Propositions 2.4, (iii); 3.20, (i), we conclude that ϕ lies over a(n) [uniquely determined] automorphism of G_{k_X}. Thus, we have a natural *exact* sequence

$$1 \longrightarrow \mathrm{Aut}_{G_{k_X}}(\Pi_X) \longrightarrow \mathrm{Aut}(\Pi_X) \longrightarrow \mathrm{Aut}(G_{k_X}).$$

Write $N \subseteq \mathrm{Out}(\Pi_X)$ for the [necessarily *normal*] subgroup of $\mathrm{Out}(\Pi_X)$ obtained by forming the image of $\mathrm{Aut}_{G_{k_X}}(\Pi_X) \subseteq \mathrm{Aut}(\Pi_X)$ in $\mathrm{Out}(\Pi_X)$. Then since $\Pi_X \to G_{k_X}$ is *surjective*, one verifies easily that the sequence

$$1 \longrightarrow N \longrightarrow \mathrm{Out}(\Pi_X) \longrightarrow \mathrm{Out}(G_{k_X})$$

induced by the above exact sequence is *exact*. Thus, since N is *finite* [cf. Theorem 4.4], and $\mathrm{Out}(G_{k_X})$ is *finite* [cf. Proposition 3.20, (ii)], we conclude that $\mathrm{Out}(\Pi_X)$ is *finite*. This completes the proof of Corollary 4.6. □

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