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The state sum invariant of 3-manifolds constructed from the E_6 linear skein

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THE STATE SUM INVARIANT OF 3-MANIFOLDS CONSTRUCTED FROM THE E_6 LINEAR SKEIN

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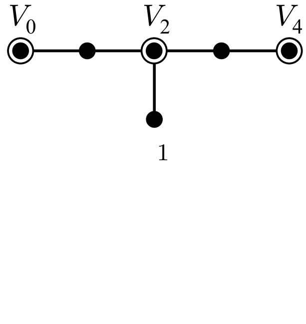
Abstract. The E_6 state sum invariant is a topological invariant of closed 3-manifolds constructed by using the $6j$-symbols of the E_6 subfactor. In this paper, we introduce the E_6 linear skein as a certain vector space motivated by E_6 subfactor planar algebra, and develop its linear skein theory by showing many relations in it. By using this linear skein, we give an elementary self-contained construction of the E_6 state sum invariant.

1. Introduction

In [TV], Turaev and Viro constructed a state sum invariant of 3-manifolds based on their triangulations, by using the $6j$-symbols of representations of the quantum group $U_q(\mathfrak{sl}_2)$. Further, Ocneanu [O] (see also [EK, KS]) generalized the construction to the case of other types of $6j$-symbols, say, the $6j$-symbols of subfactors. In the construction, we consider colorings (called states, historically) of edges and faces of a triangulation of a 3-manifold, and associate colored tetrahedra to values of the $6j$-symbols. A state sum invariant is defined by a sum of the product of such values of tetrahedra, where the sum runs over all admissible colorings. When the $6j$-symbols can be obtained from representations of a quantum group, it is known (see [T]) that the state sum invariant is equal to the square of the absolute value of the Reshetikhin-Turaev invariant, and the calculation of the state sum invariant is reduced to the calculation of the Reshetikhin-Turaev invariant. However, in the case of the $6j$-symbols of the E_6 subfactor, such Reshetikhin-Turaev invariant can not be defined, and it is necessary to calculate the state sum invariant directly. For some calculations of the E_6 state sum invariant, see [SuW, W], where they construct the E_6 state sum invariant directly from concrete values of the $6j$-symbols of the E_6 subfactor given in [I].

We briefly recall the E_6 subfactor; see, for example, [EK] for details. A subfactor $\mathcal{N}$ is a certain subalgebra of a certain C*-algebra $\mathcal{M}$. A principal graph of a subfactor is (roughly speaking) a graph whose vertices are irreducible $\mathcal{N}$-$\mathcal{N}$ bimodules and irreducible $\mathcal{N}$-$\mathcal{M}$ bimodules, and an irreducible $\mathcal{N}$-$\mathcal{N}$ bimodule X is connected to an irreducible $\mathcal{N}$-$\mathcal{M}$ bimodule Y by an edge when Y appears in the irreducible decomposition of $X \underset{\mathcal{N}}{\otimes} \mathcal{M}$. The E_6 subfactor is the subfactor whose principal graph is of the following form,

(1.1)

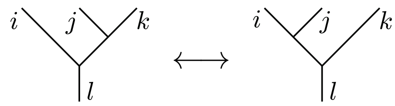


where circled vertices are $\mathcal{N}$-$\mathcal{N}$ bimodules and the other vertices are $\mathcal{N}$-$\mathcal{M}$ bimodules. The $6j$-symbols are coefficients of a transformation between bases of

$$\mathrm{Hom}\big(V_l,\,(V_i\otimes V_j)\otimes V_k\big)\quad\text{and}\quad \mathrm{Hom}\big(V_l,\,V_i\otimes(V_j\otimes V_k)\big),\tag{1.2}$$

though it is difficult in general to calculate their concrete values directly from the subfactor, since these bimodules are infinite dimensional.

We briefly recall the $\mathfrak{sl}_2$ linear skein; see [KL, L] for details. It is known that the Jones polynomial of links can be defined by using the Kauffman bracket, which is defined by a recursive relation among link diagrams. Lickorish introduced the $\mathfrak{sl}_2$ linear skein, which is the vector space spanned by link diagrams subject to the recursive relation of the Kauffman bracket. It is a key point that we can calculate the value of any link diagram by graphical calculation using the defining relations of the linear skein recursively. Further, he introduced the Jones-Wenzl idempotents as elements of the $\mathfrak{sl}_2$ linear skein (white boxes defined in (2.5)), corresponding to irreducible representations of the quantum group $U_q(\mathfrak{sl}_2)$. By using these Jones-Wenzl idempotents, he gave an elementary self-contained construction of the Reshetikhin-Turaev invariant; in fact, this construction is quite useful when calculating the Reshetikhin-Turaev invariant of concrete 3-manifolds; see [KL]. Moreover, it is known [KL, L] that we can describe the $6j$-symbols of representations of $U_q(\mathfrak{sl}_2)$ in terms of the $\mathfrak{sl}_2$ linear skein, as coefficients of a transformation between the following two graphs,

(1.3) 

,

which describes the transformation of (1.2) graphically.

As a graphical approach to subfactors, Jones [J] introduced planar algebras, which are a kind of algebras given graphically in the plane. As the Kuperberg program says (see [MPS]), it is a problem to

(i) give a presentation by generators and relations for each planar algebra, and
(ii) show basic properties of the planar algebra based on such a presentation.

For the D_{2n} planar algebra, (i) and (ii) have been done in [MPS]. For the E_6 and E_8 planar algebras, Bigelow [B] has done (i), and has partially done (ii) by using the existence of the subfactor planar algebra, though idempotents corresponding to $\mathcal{N}$-$\mathcal{N}$ bimodules are not given in the E_6 planar algebra in [B].

In this paper, we introduce the E_6 linear skein, motivated by Bigelow's generators and relations of the E_6 planar algebra. We define the E_6 linear skein $\mathcal{S}(\mathbb{R}^2)$ of $\mathbb{R}^2$ to be the vector space spanned by certain 6-valent graphs (which we call planar diagrams) subject to certain relations (Definition 2.1). Our relations are a modification of Bigelow's relations; we show that they are equivalent in Section 6.1. We show that $\mathcal{S}(\mathbb{R}^2)$ is 1-dimensional (Proposition 2.2), which means the key point that we can calculate the value of any planar diagram by graphical calculation using the defining relations of the linear skein recursively. That is, in order to prove Proposition 2.2, we show that

(1) any planar diagram is equal to a scalar multiple of the empty diagram in $\mathcal{S}(\mathbb{R}^2)$, and

(2) such a scalar is uniquely determined for any planar diagram.
We give a self-contained combinatorial proof of them. To show them, it is important to give an efficient algorithm to reduce any planar diagram to the empty diagram. Such a reduction is done by decreasing the number of 6-valent vertices of a planar diagram. To do this, we use the relation (2.4) (one of our relations), which can reduce two vertices connected by two parallel edges, while the corresponding relation (6.1) (one of Bigelow's relations) reduces two vertices connected by three parallel edges. In fact, to reduce planar diagrams, our relations are more efficient than Bigelow's relations, and this is a reason why we define the E_6 linear skein by our relations, instead of Bigelow's relations. We show (1) by decreasing the number of vertices of any planar diagram by using (2.4). To show (2), we show that the resulting value does not depend on the choice of a process of decreasing the number of vertices; we consider all such processes and show the independence on them concretely.

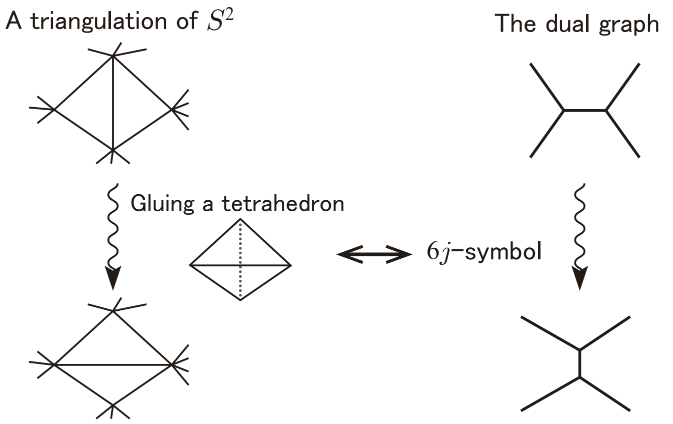
Further, we introduce idempotents (gray boxes in Section 3) in our linear skein, corresponding to the irreducible $\mathcal{N}$-$\mathcal{N}$ bimodules V_0, V_2, V_4 in (1.1). It is known, see [I], that the fusion rule algebra of the E_6 subfactor is given by the product shown in the following table, and the quantum dimensions of V_0, V_2, V_4 are equal to 1, $1+\sqrt{3}$, 1 respectively.

	V_0	V_2	V_4
V_0	V_0	V_2	V_4
V_2	V_2	$V_0+2V_2+V_4$	V_2
V_4	V_4	V_2	V_0

In particular, V_0 is the $\mathcal{N}$-$\mathcal{N}$ bimodule $\mathcal{N}$, which gives the unit of the fusion rule algebra. Corresponding to V_0, we define the gray box over 0 strand to be the empty diagram. Further, V_2 is an irreducible $\mathcal{N}$-$\mathcal{N}$ bimodule in the irreducible decomposition of $\mathcal{M} \underset{\mathcal{M}}{\otimes} \mathcal{M}$. Corresponding to V_2, we define the gray box over 2 strands to be the Jones-Wenzl idempotent over 2 strands. Furthermore, V_4 is an irreducible $\mathcal{N}$-$\mathcal{N}$ bimodule in the irreducible decomposition of $\mathcal{M} \underset{\mathcal{M}}{\otimes} \mathcal{M} \underset{\mathcal{N}}{\otimes} \mathcal{M} \underset{\mathcal{M}}{\otimes} \mathcal{M}$. Corresponding to V_4, we define the gray box over 4 strands to be a certain idempotent over 4 strands. We show that the values of the closures of these gray boxes are equal to the quantum dimensions of V_0, V_2, V_4 (Lemma 3.2). By using these gray boxes, we introduce colored planar trivalent graphs, whose edges are colored by these gray boxes, where we define admissible trivalent vertices in (3.5) corresponding to the above mentioned fusion rule algebra. In particular, we note that we consider two kinds of trivalent vertices when the adjacent three edges are colored by 2, 2, 2, since the summand V_2 in $V_2 \otimes V_2$ has multiplicity 2. Moreover, we consider the linear skein $H(i_1, i_2, \cdots, i_n)$ spanned by planar diagrams on a disk bounded by the gray boxes over i_1 strands, i_2 strands, $\cdots$, i_n strands, corresponding to the intertwiner space $\mathrm{Hom}(V_0,\, V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n})$. We show that a basis of this space is given by colored trivalent trees (Proposition 4.9). In particular, when $n = 4$, we can describe the $6j$-symbols in terms of colored planar trivalent graphs as coefficients of a transformation between bases of the forms in (1.3).

By using these $6j$-symbols, we give a construction of our state sum invariant in terms of colored planar trivalent graphs (Definition 5.1). It is known as a general procedure that the topological invariance of such a state sum invariant is shown from the defining relation of the $6j$-symbols. We review this procedure in terms of our E_6 linear skein (the proof of Theorem 5.2). In particular, in the proof, we show a pentagon relation of the $6j$-symbols by using a basis of $H(i_1, i_2, \cdots, i_5)$ given in Proposition 4.9. We show that our state sum invariant is equal to the E_6 state sum invariant (Proposition 6.2), since our $6j$-symbols can be transformed into the $6j$-symbols of the E_6 subfactor given in [I].

We comment on a topological aspect of our construction; see [KL] for this aspect. A triangulation of a 3-manifold is locally described by a triangulation of a 3-ball. When a 3-ball has a triangulation, it induces a triangulation of the boundary 2-sphere, and we consider its dual planar trivalent graph. In this correspondence, "gluing a tetrahedron on a 3-ball" corresponds to a fusion of the dual trivalent graph, which is described by $6j$-symbols.



From this viewpoint, we calculate our state sum invariant for some concrete 3-manifolds in Section 7. It is expected that, when we study topological aspects of the invariant, it is useful to construct a state sum invariant in terms of the linear skein.

The paper is organized as follows. In Section 2, we introduce the E_6 linear skein $\mathcal{S}(\mathbb{R}^2)$ of $\mathbb{R}^2$, and show that $\mathcal{S}(\mathbb{R}^2)$ is 1-dimensional. Further, we show that $\mathcal{S}(\mathbb{R}^2)$ is spherical, that is, we can regard planar diagrams in $\mathbb{R}^2$ as in $S^2 = \mathbb{R}^2 \cup \{\infty\}$. In Section 3, we introduce gray boxes and colored planar trivalent graphs. In Section 4, we introduce the space $H(i_1, i_2, \cdots, i_n)$, and give a basis of this space. By using this basis, we define the $6j$-symbols. In Section 5, we construct our state sum invariant by using these $6j$-symbols. In Section 6, we show that the defining relations of our E_6 linear skein are equivalent to Bigelow's relations. Further, we show that our state sum invariant is equal to the E_6 state sum invariant. In Section 7, we calculate our state sum invariant for the lens spaces $L(4,1)$, $L(5,2)$ and $L(5,1)$ in terms of the E_6 linear skein. In Appendix A, we present the concrete values of the weights. In Appendix B, we show that our $6j$-symbols can be transformed into the $6j$-symbols of the E_6 subfactor.

The author would like to thank Tomotada Ohtsuki for his encouragement and proving Lemma B.1. The author is also grateful to Stephen Bigelow, Kazuo Habiro, Masaki Izumi, Scott Morrison and Michihisa Wakui for their helpful comments.

Notation. Throughout the paper, the scalar field for every vector space is the complex field $\mathbb{C}$. We put $q = \exp(\pi\sqrt{-1}/12)$, $[n] = (q^n - q^{-n})/(q - q^{-1})$, and $\omega = \exp(4\pi\sqrt{-1}/3)$. Further, $d_0 = 1$, $d_2 = [3]$, $d_4 = 1$ (by Lemma 3.2), and we put $w = d_0^2 + d_2^2 + d_4^2 = 2 + [3]^2 = 6 + 2\sqrt{3}$. We note that

$$
\begin{aligned}
&[2] = \frac{\sqrt{2}+\sqrt{6}}{2}, \quad [3] = \sqrt{2}\,[2] = 1+\sqrt{3}, \quad [4] = \sqrt{3}\,[2] = \frac{3\sqrt{2}+\sqrt{6}}{2},\\
&[5] = [2]^2 = 2+\sqrt{3}, \quad [6] = 2\,[2] = \sqrt{2}+\sqrt{6}, \quad [12-n] = [n].
\end{aligned} \tag{1.4}
$$

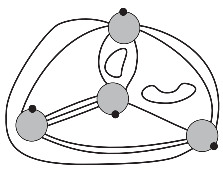
2. The E_6 linear skein

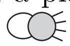
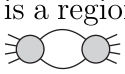
In this section, we introduce the E_6 linear skein of $\mathbb{R}^2$ and show that it is 1-dimensional in Section 2.1. Further, we introduce the E_6 linear skein of a disk and show some properties in the E_6 linear skein in Section 2.2.

2.1. The E_6 linear skein of $\mathbb{R}^2$.

In this section, we introduce the E_6 linear skein $\mathcal{S}(\mathbb{R}^2)$ of $\mathbb{R}^2$ as a vector space spanned by certain planar graphs in Definition 2.1, and show that $\mathcal{S}(\mathbb{R}^2)$ is a 1-dimensional vector space spanned by the empty diagram in Proposition 2.2.

We define a *planar diagram* to be a 6-valent graph (possibly containing closed curves) embedded in $\mathbb{R}^2$ such that each vertex is depicted by a disk whose boundary has a base point, as shown in the following picture.

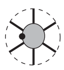


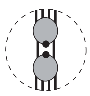
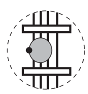
We regard isotopic planar diagrams as equivalent planar diagrams. A planar diagram is said to be *connected* if it is connected as a graph. A *cap* of a planar diagram is an edge bounding a region of the shape of a disk as shown in  . A *digon* of a planar graph is a region of the shape of a disk bounded by two edges and two vertices as shown in  .

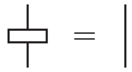
Definition 2.1. We define the E_6 *linear skein* of $\mathbb{R}^2$, denoted by $\mathcal{S}(\mathbb{R}^2)$, to be the vector space spanned by planar diagrams subject to the following relations,

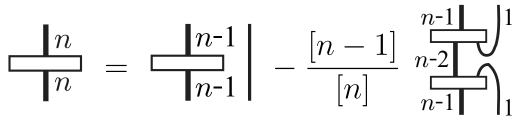
(2.1) $D \cup (\text{a closed curve}) \;=\; [2]\,D$ for any planar diagram D,

(2.2) (A planar diagram containing a cap) $= 0$,

(2.3)  $= \omega$  ,

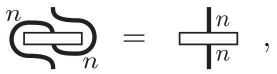
(2.4)  $= [4]$  $+ [3][4]$  .

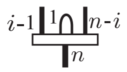
Here, in each of (2.3) and (2.4), pictures in the formula mean planar diagrams, which are identical except for a disk, where they differ as shown in the pictures. The white boxes, called the *Jones-Wenzl idempotents*, are inductively defined by , and

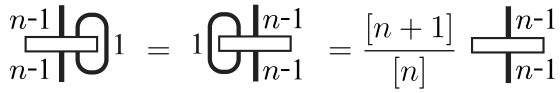
(2.5)  for $2 \leq n \leq 11$,

where a thick strand attached with an integer n means n parallel strands.

It is known, see for example [L], that the Jones-Wenzl idempotents satisfy the following properties in the linear skein,

(2.6)  ,

(2.7)  $= 0 \quad (i = 1, \ldots, n-1)$,

(2.8) 

for $1 \leq n \leq 11$.

The aim of this section is to show the following proposition, which implies that $\mathcal{S}(\mathbb{R}^2)$ is 1-dimensional.

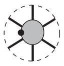
Proposition 2.2. *There exists an isomorphism* $\langle\,\rangle : \mathcal{S}(\mathbb{R}^2) \to \mathbb{C}$ *which takes the empty diagram* $\emptyset$ *to* 1.

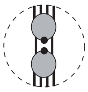
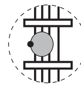
Proof. We show that $\mathcal{S}(\mathbb{R}^2)$ is spanned by the empty diagram $\emptyset$, *i.e.*, at most 1-dimensional, as follows. Let D be a planar diagram. We show that D is equal to a scalar multiple of $\emptyset$ in $\mathcal{S}(\mathbb{R}^2)$. By considering an innermost connected component of D, we can reduce the proof to the case where D is connected. If D has no vertices, then D is the empty diagram or a closed curve. Thus, by (2.1), D is equal to $\emptyset$ or $[2]\emptyset$. If D has just one vertex, then D must have a cap, and thus $D = 0$ by (2.2). Hence, we can assume that D is a connected planar diagram with at least two vertices and no caps. Then, by Lemma 2.3 below, D has a digon. By using (2.3), we move the base points of the vertices of this digon as shown in the left-hand side of (2.4). Further, by applying the left-hand side of (2.4) to this digon, D is presented by a linear sum of planar diagrams with fewer vertices. By repeating this argument, D can be presented by a scalar multiple of $\emptyset$ in $\mathcal{S}(\mathbb{R}^2)$. Hence, $\mathcal{S}(\mathbb{R}^2)$ is spanned by the empty diagram $\emptyset$.

We show the proposition by improving the above argument. Let $\tilde{\mathcal{S}}_k(\mathbb{R}^2)$ be the vector space freely spanned by planar diagrams with at most k vertices. We will inductively define the linear map $\langle\,\rangle_k : \tilde{\mathcal{S}}_k(\mathbb{R}^2) \to \mathbb{C}$ for $k = 0, 1, 2, \cdots$, extending $\langle\,\rangle_{k-1}$, satisfying that $\langle \emptyset \rangle_k = 1$ and

$$\big\langle D \cup (\text{a closed curve}) \big\rangle_k = [2]\, \langle D \rangle_k \quad \text{for any planar diagram } D, \tag{2.9}$$

$$\big\langle (\text{A planar diagram containing a cap}) \big\rangle_k = 0, \tag{2.10}$$

(2.11) $\Big\langle$  $\Big\rangle_k = \omega \Big\langle$  $\Big\rangle_k$,

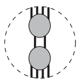
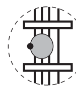
(2.12) $\Big\langle$  $\Big\rangle_k = [4]\Big\langle$  $\Big\rangle_k + [3][4] \Big\langle$  $\Big\rangle_k$.

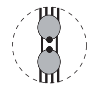
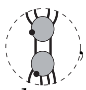
If such linear maps exist, we obtain a non-trivial linear map $\langle\,\rangle : \mathcal{S}(\mathbb{R}^2) \to \mathbb{C}$ as the inductive limit of them, and such a linear map $\langle\,\rangle$ must be isomorphic, since $\mathcal{S}(\mathbb{R}^2)$ is at most 1-dimensional as shown above. In the following of this proof, we define $\langle\,\rangle_k$ for $k = 0, 1, \cdots$ by induction on k showing (2.9)–(2.12).

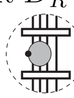
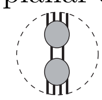
When $k = 0$, we define $\langle\,\rangle_0$, as follows. Let D be a planar diagram with no vertices. Then, D is a union of closed curves. We define $\langle D\rangle_0 = [2]^m$, where m is the number of closed curves of D. We can verify (2.9) for $k = 0$ by definition, and the conditions (2.10)–(2.12) are trivial in this case.

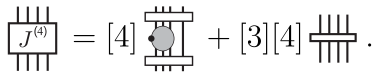
When $k = 1$, we define $\langle\,\rangle_1$, as follows. For a planar diagram D with no vertices, we put $\langle D\rangle_1 = \langle D\rangle_0$. For a planar diagram D with just one vertex, we put $\langle D\rangle_1 = 0$, noting that D must have a cap. We can verify (2.9)–(2.11) for $k = 1$ by definition, and the condition (2.12) is trivial in this case.

When $k \geq 2$, assuming that there exists a linear map $\langle\,\rangle_{k-1} : \tilde{\mathcal{S}}_{k-1}(\mathbb{R}^2) \to \mathbb{C}$ satisfying (2.9)–(2.12) for $k-1$, we define a map $\langle\,\rangle_k$, as follows. For a planar diagram D with at most $k-1$ vertices, we put $\langle D\rangle_k = \langle D\rangle_{k-1}$. For a planar diagram D with just k vertices, we define $\langle D\rangle_k$, as follows. When D is disconnected, we put $\langle D\rangle_k$ to be the product of $\langle$connected component of $D\rangle_k$. If D contains a cap, we put $\langle D\rangle_k = 0$. Hence, it is sufficient to define $\langle D\rangle_k$ for a connected planar diagram D with no caps. By Lemma 2.3 below, such a planar diagram has a digon. By applying the left-hand side of the following formula to this digon, we define $\langle D\rangle_k$ by

(2.13) $\Big\langle$  $\Big\rangle_k = \omega^\eta \Big([4]\Big\langle$  $\Big\rangle_{k-1} + [3][4] \Big\langle$  $\Big\rangle_{k-1} \Big)$,

where the integer η is defined by the position of the base points of the diagram in the left-hand side, as follows. We move each of the base points around the vertices clockwisely until the diagram become , and η is the number of times the base points pass the edges. For example, if the diagram in the left-hand side is , then $\eta = 5 + 2 = 7$. We note that the planar diagram in the left-hand side has k vertices, and the planar diagrams in the right-hand side have $k-1$ and $k-2$ vertices. We also note that the definition (2.13) is well defined independently of the π-rotation of this substitution, since the right-hand side of (2.13) is invariant under the π-rotation by the $k-1$ case of Lemma 2.5 below. Further, in order to complete the proof, we must show that $\langle D\rangle_k$ does not depend on the choice of a digon, and that this $\langle\,\rangle_k$ satisfies (2.9)–(2.12).

We show that $\langle D\rangle_k$ does not depend on the choice of a digon, as follows. For a planar diagram D with a digon R, we put D_R to be the linear sum of planar diagrams obtained from D by substituting $\omega^\eta \Big([4]$  $+ [3][4]$  $\Big)$ into  of this digon. We define $J^{(4)}$ by



Let D be a planar diagram with two digons R_1 and R_2. Then, we have the following three cases of the mutual positions of R_1 and R_2; see Figure 1.
(a) The vertices of R_1 and R_2 are distinct.
(b) R_1 and R_2 have one common vertex.
(c) The vertices of R_1 and R_2 are equal.
We assume that the base points of vertices of R_1 and R_2 are as shown in Figure 1, since the other cases are reduced to this case from the definition of η. It is sufficient to show that $\langle D_{R_1}\rangle_{k-1} = \langle D_{R_2}\rangle_{k-1}$ in each of Cases (a)–(c).

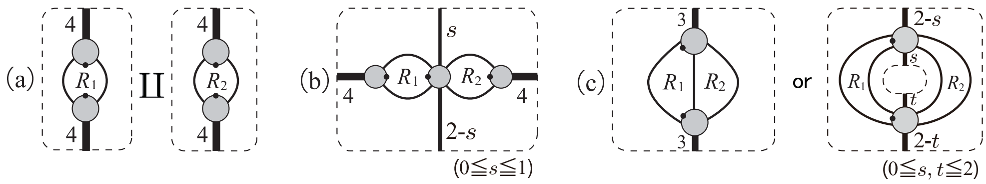
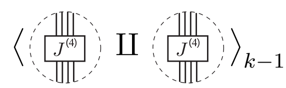
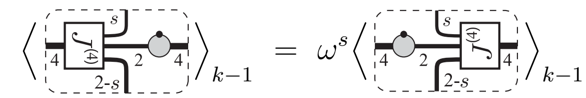


FIGURE 1. Possible positions of two digons R_1 and R_2.

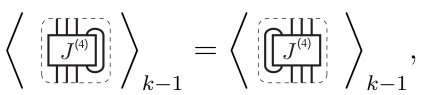
Case (a). $\langle D_{R_1}\rangle_{k-1} = \langle D_{R_2}\rangle_{k-1}$, since they are equal to  by (2.12) for $k-1$, completing this case.

Case (b). The equation $\langle D_{R_1}\rangle_{k-1} = \langle D_{R_2}\rangle_{k-1}$ is rewritten as

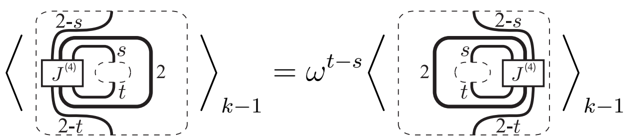
(2.14)  $(s = 0, 1)$,

and we show this formula in Lemma 2.10 below, completing this case.

Case (c). When R_1 and R_2 have one common edge, it is enough to show that



and this follows from (2.8) and Lemma 2.4 below. When the edges of R_1 and R_2 are distinct, it is enough to show that

(2.15)  $(0 \leq s, t \leq 2)$

with $s+t$ being even, and we show this formula in Lemma 2.8 below, completing this case.

Therefore, we showed that $\langle D\rangle_k$ does not depend on the choice of a digon, and hence, we obtain a well-defined linear map $\langle\,\rangle_k : \tilde{\mathcal{S}}_k(\mathbb{R}^2) \to \mathbb{C}$.

Finally, we show that $\langle\,\rangle_k$ satisfies (2.9)–(2.12), as follows. We recall that $\langle\,\rangle_k$ is defined by

$$\langle D\rangle_k = \begin{cases} \prod\langle\text{connected component of } D\rangle_k & \text{if } D \text{ is disconnected,}\\ 0 & \text{if } D \text{ is a connected planar diagram with a cap,}\\ \langle D_R\rangle_{k-1} & \text{if } D \text{ is a connected planar diagram with no cap.}\end{cases}$$

For any planar diagram D with k vertices, we have that

$$\big\langle D\cup(\text{a closed curve})\big\rangle_k = \langle D\rangle_k\big\langle(\text{a closed curve})\big\rangle_k = [2]\langle D\rangle_k,$$

from the definition of $\langle\,\rangle_k$ for disconnected planar diagrams, and hence, we obtain (2.9). From the definition of $\langle\,\rangle_k$, we obtain (2.10). From the definition of $\langle\,\rangle_k$ and (2.11) for $k-1$, we obtain (2.11). The remaining case is to show (2.12). Let D be the planar diagram in the left-hand side of (2.12). It is sufficient to show (2.12) when D is connected. If D does not have a cap, (2.12) is obtained from (2.13). We assume that D has a cap. If the cap is on a vertex outside the picture of the left-hand side of (2.12), both sides of (2.12) are 0 by definition. Otherwise, the cap is on a vertex in the picture of the left-hand side of (2.12). In this case, the left-hand side of (2.12) is 0 by definition, and the right-hand side of (2.12) is also 0 by (2.7). Hence, we obtain (2.12). Therefore, we showed that $\langle\,\rangle_k$ satisfies (2.9)–(2.12), completing the proof. $\square$

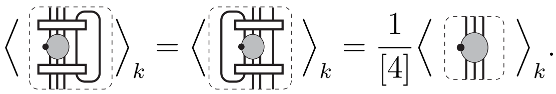
In the proof of Proposition 2.2, we used Lemmas 2.3, 2.4, 2.5, 2.8 and 2.10 below. We show them in the following of this section.

Lemma 2.3. *A connected planar diagram with at least two vertices and no caps has a digon.*

Proof. Let D be a planar diagram with no caps. In this proof, we regard D as on $\mathbb{R}^2\cup\{\infty\}=S^2$. It is sufficient to show that D has at least two digons in S^2.

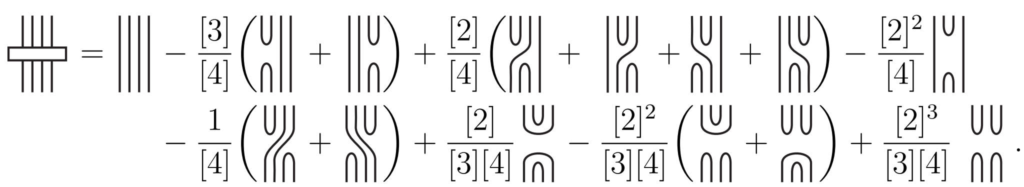
Let v, e, and f be the numbers of vertices, edges, and faces of D respectively. Let C_n be the number of n-gons of D. By definition, $f=\sum_{k\geq 2}C_k$. Further, $6v=2e=\sum_{k\geq 2}kC_k$, since D is 6-valent. From these equations and Euler's formula $v-e+f=2$, we obtain $6=C_2-\sum_{k\geq 4}(k-3)C_k\leq C_2$. Hence, D has at least two digons in S^2, as required. $\square$

Lemma 2.4. *For an integer $k\geq 2$, let $\langle\,\rangle_k$ be a linear map $\tilde{\mathcal{S}}_k(\mathbb{R}^2)\to\mathbb{C}$ satisfying* (2.9)–(2.11)*. Then,*

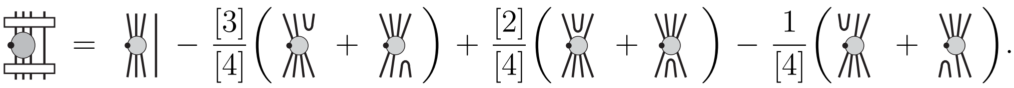


Proof. By calculating the Jones-Wenzl idempotent concretely by definition, we have that

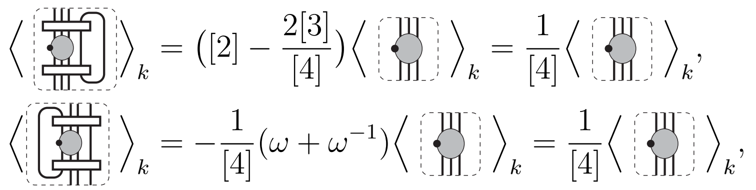
(2.16)



By using the above formula, we have that

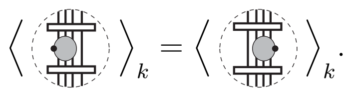
(2.17) 

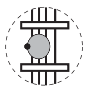
Hence,



which imply the required formula of the lemma. □

Lemma 2.5. *For an integer $k \geq 2$, let $\langle\,\rangle_k$ be a linear map $\tilde{\mathcal{S}}_k(\mathbb{R}^2) \to \mathbb{C}$ satisfying* (2.9)–(2.12)*. Then,*



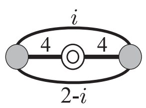
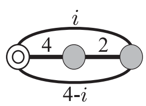
Proof. We put $D =$  and $D' =$ , *i.e.*, D and D' are planar diagrams which are identical except for a disk, where they differ as shown in these pictures. Let Γ be a planar graph obtained from D by replacing the disk with an 8-valent vertex. If Γ has a cap on a 6-valent vertex, then both $\langle D\rangle_k$ and $\langle D'\rangle_k$ are equal to 0 by (2.10). Hence, we assume that there are no caps on 6-valent vertices of Γ.

If Γ has at least three 6-valent vertices, then, by Lemma 2.6 below, Γ has a digon whose vertices are 6-valent. By applying (2.12) to this digon, we can decrease the number of vertices of D and D', keeping the required formula unchanged. Hence, repeating this argument, we can reduce the proof of the lemma to the case where Γ has at most two vertices.

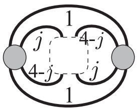
If Γ has no 6-valent vertex, then both $\langle D\rangle_k$ and $\langle D'\rangle_k$ are equal to 0 by (2.10), since any planar diagram with just one vertex must have a cap. Hence, the lemma holds in this case.

If Γ has one 6-valent vertex, then Γ must have a cap on the 8-valent vertex. Hence, by (2.7) and Lemma 2.4, we have that $\langle D\rangle_k = \langle D'\rangle_k$. Therefore, the lemma holds in this case.

If Γ has two 6-valent vertices, then we show the lemma, as follows. If Γ has a cap on the 8-valent vertex, we obtain the lemma as shown above. Hence, we assume that Γ has no cap on the 8-valent vertex. Then, Γ must be either

$\Gamma_{1,i} =$  $(0 \leq i \leq 2)$ or $\Gamma_{2,i} =$  $(0 \leq i \leq 4)$,

where we depict the 8-valent vertex by $\circledcirc$. Among them, $\Gamma_{1,i}$ for $i \neq 1$ and $\Gamma_{2,i}$ for any i have a digon, and we can show the lemma as shown above in this case. The remaining case is $\Gamma_{1,1}$. In this case, the outer region of D and D' is depicted by



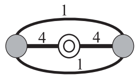
$(0 \leq j \leq 4)$, and hence, D and D' are isotopic. Therefore, the lemma holds in this case. $\square$

Lemma 2.6. *Let Γ be a connected planar graph with no caps, whose vertices are one 8-valent vertex and at least three 6-valent vertices. Then, Γ has a digon whose vertices are 6-valent.*

Proof. In this proof, we regard Γ as on $\mathbb{R}^2 \cup \{\infty\} = S^2$. We put v, e, f and C_n $(n = 2, 3, \dots)$ of Γ in the same way as in the proof of Lemma 2.3. Let C_2' be the number of digons of Γ whose vertices are 6-valent, and let C_2'' be the number of digons of Γ which have the 8-valent vertex. By definition, $C_2 = C_2' + C_2''$. It is sufficient to show that $C_2' \geq 2$.

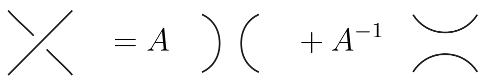
Let m be the number of the vertices adjacent to the 8-valent vertex. We can verify that $m \geq 2$ and $C_2'' \leq 8 - m$. In a similar way as in the proof of Lemma 2.3, we have that $6(v-1) + 8 = 2e = \sum_{k \geq 2} kC_k$ and $f = \sum_{k \geq 2} C_k$. From these equations and Euler's formula $v - e + f = 2$, we have that

$$C_2' - \sum_{k \geq 4} (k-3) C_k \;=\; 7 - C_2'' \;\geq\; m - 1.$$

If $m \geq 3$ or Γ has a k-gon with $k \geq 4$, we have that $C_2' \geq 2$, as required. Hence, we assume that $m = 2$, and that each face of Γ is a digon or a 3-gon. Then, the neighborhood of the 8-valent vertex must be 

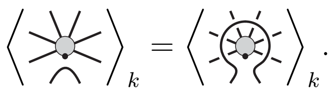
. Since Γ has at least three 6-valent vertices, this contradicts to the connectivity of Γ. Hence, we obtain the lemma. $\square$

In order to show Lemma 2.8 and 2.10, we show Lemma 2.7 below, which says that an edge can "pass-over" a vertex. It is known, see for example [L], that a tangle diagram is regarded as in the linear skein by putting

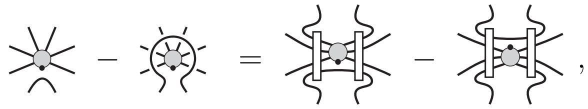


with $A = \sqrt{-1}\, q^{1/2} = \sqrt{-1} \exp(\pi\sqrt{-1}/24)$, noting that $[2] = -A^2 - A^{-2}$. Further, it is known, see [L], that the value of a tangle diagram in the linear skein is invariant under Reidemeister moves II and III.

Lemma 2.7. *For an integer $k \geq 2$, let $\langle\,\rangle_k$ be a linear map $\tilde{\mathcal{S}}_k(\mathbb{R}^2) \to \mathbb{C}$ satisfying (2.9)–(2.12). Then,*



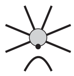
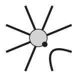
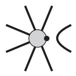
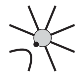
Proof. Since $\langle \cdot \rangle_k$ of the right-hand side of the following formula is equal to 0 by Lemma 2.5,

(2.18) 

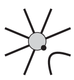
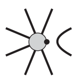
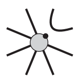
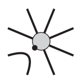
it is sufficient to show the above formula. In the following of this proof, we show that each side of (2.18) is equal to

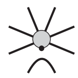
$$\sum_{j=0}^{7} \zeta^j D_j\,,$$

where we put $\zeta = \exp(-\pi\sqrt{-1}/4)$ and

$D_0 =$  $,\ D_1 =$  $,\ D_2 =$  $,\ \dots,\ D_7 =$  .

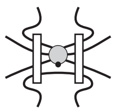
We show that the left-hand side of (2.18) is equal to $\sum_{j=0}^{7}\zeta^j D_j$, as follows. By expanding all crossings of the left-hand side of the following formula and by moving the base point, we have that

 $= A^{-6}$  $+\,\omega^{-1}A^{-4}$  $+\,\omega^{-2}A^{-2}$  $+\cdots+A^{6}$ 

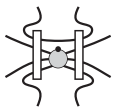
$$= \sum_{j=1}^{7} \omega^{-j+1}A^{2j-8} D_j \;=\; -\sum_{j=1}^{7}\zeta^j D_j \;=$$  $$-\sum_{j=0}^{7}\zeta^j D_j\,.$$

Hence, the left-hand side of (2.18) is equal to $\sum_{j=0}^{7}\zeta^j D_j$.

We show that the right-hand side of (2.18) is equal to $\sum_{j=0}^{7}\zeta^j D_j$, as follows. By (2.17), we have that

 $$=\; D_0 - \frac{[3]}{[4]}(\omega^{-1}D_1+\omega D_7) + \frac{[2]}{[4]}(\omega D_2 + \omega^{-1}D_6) - \frac{1}{[4]}(D_3+D_5).$$

By considering its mirror image, we have that

 $$=\; D_4 - \frac{[3]}{[4]}(\omega D_3+\omega^{-1} D_5) + \frac{[2]}{[4]}(\omega^{-1} D_2 + \omega D_6) - \frac{1}{[4]}(D_1+D_7).$$

Hence, the right-hand side of (2.18) is equal to

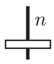
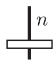
$$D_0 - D_4 + \frac{1-[3]\omega^{-1}}{[4]}(D_1-D_5) + \frac{[2](\omega-\omega^{-1})}{[4]}(D_2-D_6) + \frac{[3]\omega-1}{[4]}(D_3-D_7).$$

Further, we can verify that

$$\frac{1-[3]\omega^{-1}}{[4]} = \zeta, \qquad \frac{[2](\omega-\omega^{-1})}{[4]} = \zeta^2, \qquad \frac{[3]\omega-1}{[4]} = \zeta^3$$

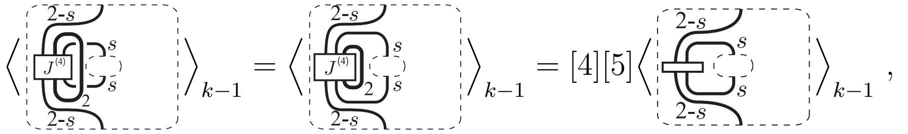
by direct calculation. Therefore, the right-hand side of (2.18) is equal to $\sum_{j=0}^{7}\zeta^j D_j$, as required. $\square$

It is known, see for example [L], that

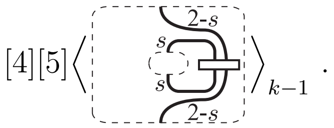
(2.19)  $= (-1)^n A^{n(n+2)}$  ,  $= (-1)^n A^{-n(n+2)}$  $\quad (1 \le n \le 11)$.

Lemma 2.8. *The formula* (2.15) *holds for* $s,t \in \{0,1,2\}$ *with* $s+t$ *being even.*

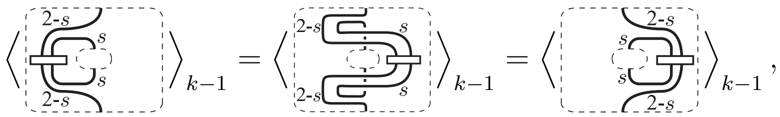
Proof. When $s \neq t$, that is, $(s,t) = (2,0)$ or $(0,2)$, the both sides of (2.15) are equal to 0 from Lemma 2.9 below. Hence, we may assume $s = t$. By Lemma 2.7, the left-hand side of (2.15) is equal to



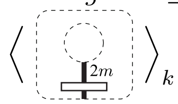
where the first equality is obtained by expanding the crossings, the second one is obtained by Lemma 2.4 and (2.8). In the same way, we can verify that the right-hand side of (2.15) is equal to



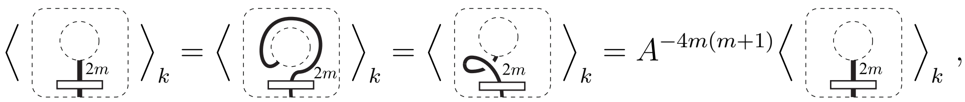
Further, we have that



where the first equality is obtained by Lemma 2.7, the second one is obtained by (2.19) and by expanding the crossings. Therefore, we obtain (2.15), as required. $\square$

Lemma 2.9. *For an integer $k \geq 2$, let $\langle\rangle_k$ be a linear map $\tilde{\mathcal{S}}(\mathbb{R}^2) \longrightarrow \mathbb{C}$ satisfying (2.9)–(2.12). Then,*  *$= 0$ for $m \in \{1,2,4,5\}$.*

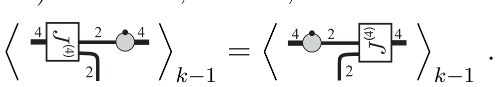
Proof. We have that



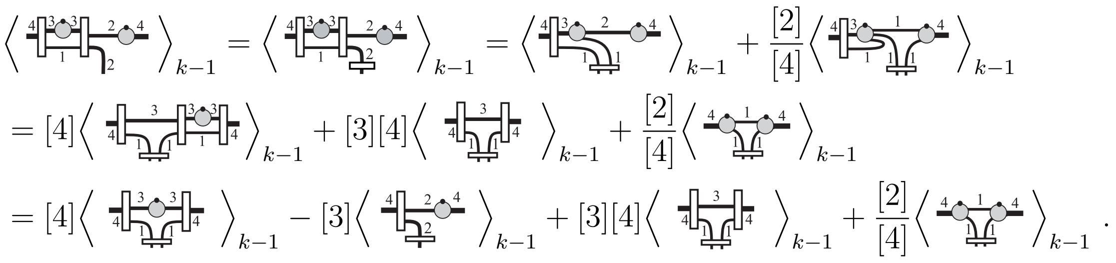
where the second equality is obtained by Lemma 2.7 and the third one is obtained by (2.19). Since $A^{-4m(m+1)} = \exp(-2\pi\sqrt{-1}\cdot\frac{m(m+1)}{12}) \neq 1$, we obtain the required formula. $\square$

Lemma 2.10. *The formula* (2.14) *holds for $s = 0, 1$.*

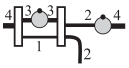
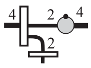
Proof. We first show (2.14) for $s = 0$, that is,

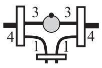
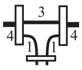
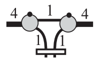
(2.20) 

By using (2.16), we have that



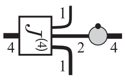
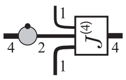
Therefore, the left-hand side of (2.20) is equal to

$[4]\big\langle$  $\big\rangle_{k-1} + [3][4]\big\langle$  $\big\rangle_{k-1}$

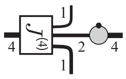
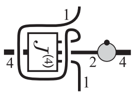
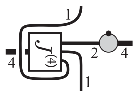
$= [4]^2\big\langle$  $\big\rangle_{k-1} + [3][4]^2\big\langle$  $\big\rangle_{k-1} + [2]\big\langle$  $\big\rangle_{k-1}.$

By using Lemma 2.5, we can verify that the right-hand side of (2.20) is also equal to the above formula. Hence, we obtain (2.20).

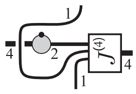
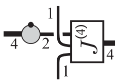
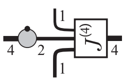
We next show (2.14) for $s = 1$, that is,

(2.21) $\big\langle$  $\big\rangle_{k-1} = \omega\big\langle$  $\big\rangle_{k-1}.$

We have

$\big\langle$  $\big\rangle_{k-1} = \big\langle$  $\big\rangle_{k-1} = -A^6\big\langle$  $\big\rangle_{k-1},$

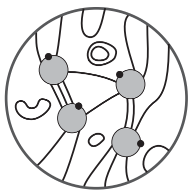
where the first equality is obtained by Lemma 2.7 and the second one is obtained by expanding the crossings. By applying (2.20), this is equal to

$-A^6\big\langle$  $\big\rangle_{k-1} = -A^6\big\langle$  $\big\rangle_{k-1} = -A^8\big\langle$  $\big\rangle_{k-1}.$

Since $-A^8 = \omega$, we obtain (2.21). $\square$

2.2. **Some properties in the E_6 linear skein.** In this section, we introduce the E_6 linear skein of a disk and show some properties in the E_6 linear skein.

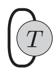
For an integer $m \geq 0$, let $(D^2, 2m)$ denote a disk D^2 with fixed distinct $2m$ points on its boundary. We define a *planar diagram in* $(D^2, 2m)$ to be a graph (possibly containing closed curves) embedded in D^2 whose vertices are $2m$ univalent vertices on the fixed points of the boundary of D^2 and 6-valent vertices, such that each 6-valent vertex is depicted by a disk whose boundary has a base point, as shown in the following picture.



$(m = 7)$

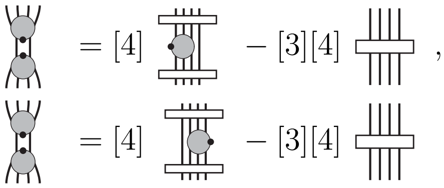
We regard isotopic planar diagrams as equivalent planar diagrams. In the following of this paper, we omit to draw the disk D^2 of a planar diagram. For an integer $m \geq 0$, we define the E_6 *linear skein* of $(D^2, 2m)$, denoted by $\mathcal{S}(D^2, 2m)$, to be the vector space spanned by planar diagrams in $(D^2, 2m)$ subject to the relations (2.1)–(2.4).

Lemma 2.11. *For any planar diagram T in $(D^2, 2)$,*

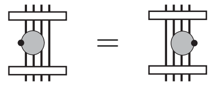
 $=$ 

in $\mathcal{S}(\mathbb{R}^2)$.

Proof. From (2.4), we have that

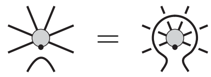


in $\mathcal{S}(D^2, 8)$. Comparing these equations, we have that

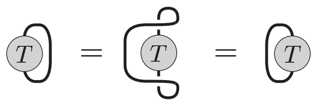


in $\mathcal{S}(D^2, 8)$. Hence, in the same way as in the proof of Lemma 2.7, we obtain

(2.22)



in $\mathcal{S}(D^2, 8)$. Therefore, we have that

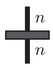


where the first equality is obtained by (2.22) and the second one is obtained by (2.19). Hence, we obtain the required formula. □

By this lemma, we can regard planar diagrams in $\mathbb{R}^2$ as in $S^2 = \mathbb{R}^2 \cup \{\infty\}$.

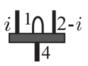
3. Colored planar trivalent graphs

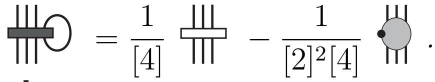
In this section, we introduce gray boxes as certain idempotents in the E_6 linear skein, and introduce colored planar trivalent graphs as planar trivalent graphs whose edges are colored by such gray boxes. Further, we calculate the values of some simple colored planar trivalent graphs.

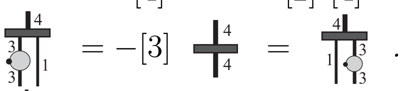
We define gray boxes  $\in \mathcal{S}(D^2, 2n)$ for $n = 0, 2, 4$ by

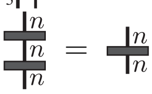
We note that, by definition, these gray boxes are symmetric with respect to π rotation, like the white boxes of the Jones-Wenzl idempotents. We show some basic properties of the gray boxes in the following lemma.

Lemma 3.1.

(1)  $= 0$ *for* $i = 0, 1, 2$.

(2) 

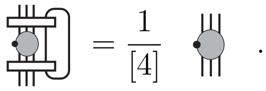
(3) 

(4)  *for* $n = 0, 2, 4$.

Proof. We obtain (1) from the definition of gray boxes and (2.2) and (2.7).

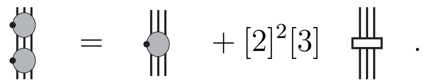
We show (2), as follows. In the same way as in the proof of Lemma 2.4, we have that

(3.1)

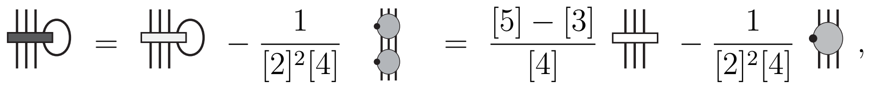


Applying this formula and (2.8) to (2.4), we have that

(3.2)



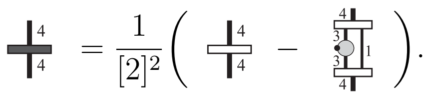
Hence,



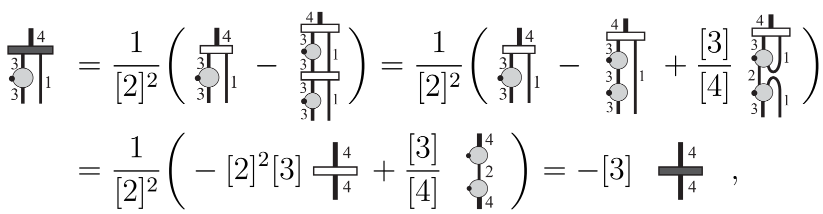
where we obtain the second equality by (2.8) and (3.2). Since $[5]-[3]=1$, we obtain (2).

We show (3), as follows. From (2.4) and the definition of the gray box, we have that

(3.3)

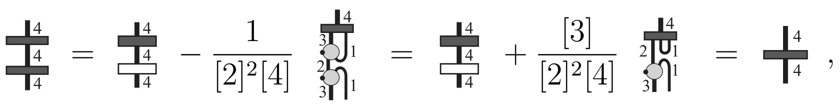


Hence, we have that



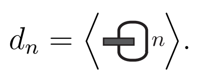
where the first equality is obtained by (3.3), the second one is obtained by (2.16) and the third one is obtained by (2.2), (2.7), (2.16) and (3.2). Hence, we obtain the first equality of (3). We can obtain the second equality of (3) in a similar way.

We show (4), as follows. When $n = 0$ or 2, the required formula is obtained by definition. When $n = 4$, we have that



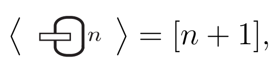
where the second equality is obtained by (3) and the third one is obtained by (2.16) and (1). Hence, we obtain (4), completing the proof. □

We define d_n for $n \in \{0, 2, 4\}$ by



We recall (see, for example, [L]) that

(3.4)



which can be obtained by using (2.8) repeatedly.

Lemma 3.2. *The values of d_n are given by*

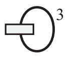
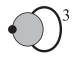
$$d_n = \begin{cases} 1 & \text{if } n = 0, 4, \\ [3] = 1 + \sqrt{3} & \text{if } n = 2. \end{cases}$$

In particular, these values are positive real numbers.

Proof. When $n = 0$, the required formula is trivial.

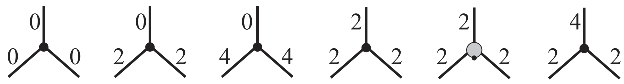
When $n = 2$, we obtain the required formula from the definition of the gray box and (3.4).

When $n = 4$, we show the required formula, as follows. By Lemma 3.1 (2), we have that

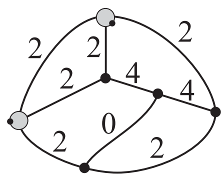
$d_4 = \frac{1}{[4]} \langle$  $\rangle - \frac{1}{[2]^2[4]} \langle$  $\rangle = 1,$

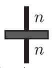
where we obtain the second equality by (3.4) and (2.2). Hence, we obtain the required formula. $\square$

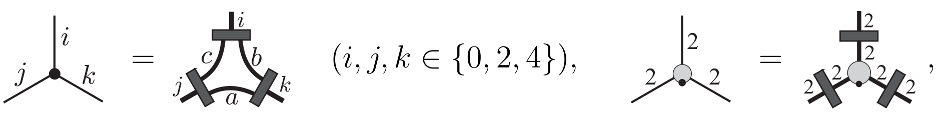
A *planar trivalent graph* is a trivalent graph embedded in $\mathbb{R}^2$. We consider two kinds of vertices; one is depicted by $\bullet$, and the other is depicted by a disk whose boundary has a base point. A *coloring* of a planar trivalent graph Γ is a map from the set of edges of Γ to $\{0, 2, 4\}$ and a map from the set of vertices of Γ to $\{\bullet, \ \}$. A coloring of a planar trivalent graph Γ is said to be *admissible* if the neighborhood of each vertex of Γ is colored as shown in either of the following pictures.

(3.5) 

We define a *colored planar trivalent graph* to be a planar trivalent graph with an admissible coloring, for example, as shown in the following picture.



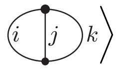
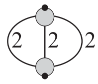
We regard a colored planar trivalent graph as in the E_6 linear skein, by substituting  into each of the edges colored by n, and substituting the following diagrams into vertices,



where we put

$$a = \frac{-i + j + k}{2}, \quad b = \frac{i - j + k}{2}, \quad c = \frac{i + j - k}{2}.$$

We remark that, abusing the notation, the symbol  denotes n parallel edges in a planar diagram, while it denotes an edge colored by n in a colored planar trivalent graph. Further, the symbol denotes a 6-valent vertex in a planar diagram, while it denotes a kind of a trivalent vertex in a colored planar trivalent graph.

We put $\theta(i,j,k,\bullet) = \Big\langle$  $\Big\rangle$ and $\theta(2,2,2,\odot) = \Big\langle$  $\Big\rangle$ for a triple $i,j,k \in \{0,2,4\}$ such that  is one of the pictures in (3.5) up to rotation.

Lemma 3.3. *The values of $\theta(\cdot)$ are given by*

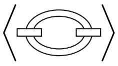
$$\begin{aligned}
\theta(i,j,k,\bullet) &= \theta(j,k,i,\bullet),\\
\theta(0,n,n,\bullet) &= d_n,\\
\theta(2,2,2,\bullet) &= [3][4]/[2]^2 = \sqrt{6},\\
\theta(2,2,2,\odot) &= [2]^2[3][4] = 12\sqrt{2}+7\sqrt{6},\\
\theta(2,2,4,\bullet) &= 1.
\end{aligned}$$

In particular, these values are positive real numbers.

Proof. We obtain the first formula of the lemma by Lemma 2.11.

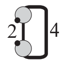
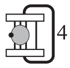
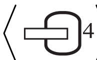
We obtain the second formula of the lemma from the definition of $\theta(i,j,k)$ and Lemma 3.1 (4).

We obtain the third formula of the lemma, since

$\theta(2,2,2,\bullet) = \Big\langle$  $\Big\rangle = \Big\langle$  $\Big\rangle - \dfrac{1}{[2]}\Big\langle$  $\Big\rangle$

$$= \left(\frac{[3]}{[2]}\right)^2 \cdot [2] - \frac{1}{[2]} \cdot [3] = \frac{([3]-1)[3]}{[2]} = \frac{[3][4]}{[2]^2}.$$

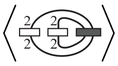
We obtain the fourth formula of the lemma, since

$\theta(2,2,2,\odot) = \Big\langle$  $\Big\rangle = [4]\Big\langle$  $\Big\rangle + [3][4]\Big\langle$  $\Big\rangle$

$$= [3][4][5] = [2]^2[3][4],$$

where the third equality is obtained by (3.4).

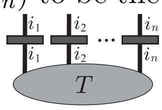
We obtain the last formula of the lemma, since

$\theta(2,2,4,\bullet) = \Big\langle$  $\Big\rangle = \Big\langle$  $\Big\rangle = d_4 = 1,$

where the second equality is obtained by Lemma 3.1 (1). □

4. $6j$-symbols in the E_6 linear skein

In this section, we consider the vector space $H(i_1,\cdots,i_n)$ spanned by planar diagrams whose ends are gray boxes colored by $i_1,\cdots,i_n$, and give a basis of this vector space in Propositions 4.2 and 4.9. Further, we introduce $6j$-symbols of the E_6 linear skein as coefficients of a transformation between certain bases of $H(i,j,k,l)$ as we show in Proposition 4.10.

For $i_1,\ldots,i_n \in \{0,2,4\}$ $(n \geq 2)$, we define the vector space $H(i_1,\ldots,i_n)$ to be the subspace of $\mathcal{S}(D^2, i_1+\cdots+i_n)$ spanned by planar diagrams of the form  with T being a planar diagram in $(D^2, i_1+\cdots+i_n)$.

In order to prove Proposition 4.2, we show the following lemma.

Lemma 4.1. *For an integer $m \leq 4$, let T be a planar diagram in $(D^2, 2m)$. Then, T can be presented by a linear sum of planar diagrams in $(D^2, 2m)$ with at most one vertex.*

Proof. By (2.1), we may assume that T has no closed curves. If T has at most one vertex, the assertion of the lemma is trivial. If T has at least two vertices, we show the lemma, as follows.

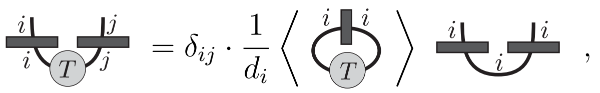
Let Γ be a planar graph on S^2 obtained from T by regarding the outer region of the unit disk of T as a $2m$-valent vertex. We call this $2m$-valent vertex ∞. Similarly as the proof of Lemma 2.5, it is enough to show that Γ has a digon whose vertices does not contain ∞. If Γ is disconnected, then, by considering an innermost connected component which does not contain ∞, we see that T has a digon from Lemma 2.3. Hence, we may assume that Γ is connected.

When $m = 4$, we obtain the lemma in a similar way as the proof of Lemma 2.6.

When $m \leq 3$, we show the lemma, as follows. In a similar way as the proof of Lemma 2.3, we can verify that there exists at least $m + 3$ digons. The number of digons which contain ∞ is at most $2m - 1$, because ∞ is $2m$-valent. Thus, Γ has at least $m + 3 - (2m - 1) = 4 - m$ digons whose vertices does not contain ∞. Since $4 - m \geq 1$, we obtain the assertion of the lemma. $\square$

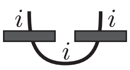
In the following proposition, we give a basis of $H(i, j)$.

Proposition 4.2. *For any $i, j \in \{0, 2, 4\}$ and $T \in \mathcal{S}(D^2, i + j)$,*



where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise. As a consequence, we have that

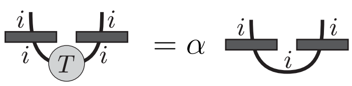
$$H(i,j) = \begin{cases} \mathrm{span}_{\mathbb{C}}\{ \text{[diagram]} \} & i = j, \\ 0 & i \neq j. \end{cases}$$

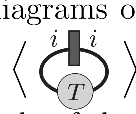


Proof. By Lemma 4.1, we may assume that T has at most one vertex.

If T has just one vertex, then we can verify that there exists a cap, or parallel three edges connecting the vertex and  . Hence, by (2.2) and Lemma 3.1 (3), we can reduce the proof of the proposition to the case where T has no vertices.

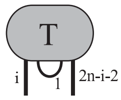
If T has no vertices, we show the proposition, as follows. When $i \neq j$, the left-hand side of the first formula of the proposition is equal to 0, by Lemma 3.1 (1). When $i = j$, the left-hand side is equal to a scalar multiple of the identity diagram by (2.1) and Lemma 3.1 (1). Hence, we can put



for some $\alpha \in \mathbb{C}$. By closing the diagrams of both sides, using Lemma 3.1 (4) and taking the bracket, we obtain $\alpha = \langle$  $\rangle / d_i$, noting that $d_i \neq 0$ by Lemma 3.2. Therefore, we obtain the first formula of the proposition.

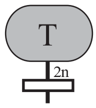
The second formula of the proposition is obtained from the first formula, noting that  is non-zero in $\mathcal{S}(D^2, 2i)$, since the value d_i of its closure is non-zero by Lemma 3.2. $\square$

In order to prove Proposition 4.9, we need Lemmas 4.7 and 4.8 below. In order to show Lemma 4.7, we show the following four lemmas.

Lemma 4.3. *For $n \in \{1, 2, 4, 5\}$ and $T \in \mathcal{S}(D^2, 2n)$, we suppose that* 

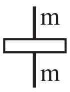
$= 0$ *for any $i = 0, 1, \ldots, 2n-2$. Then, $T = 0$ in $\mathcal{S}(D^2, 2n)$.*

Proof. In the same way as the proof of Lemma 2.9 using (2.22), we obtain that

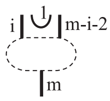


$= 0$

in $\mathcal{S}(D^2, 2n)$. Further, we have that



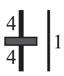
$=$ 

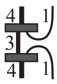
$+$ (a linear sum of planar diagrams of the form 

)

in the linear skein, which can be shown by induction on m from the definition of the Jones-Wenzl idempotents. Hence, by the assumption of the lemma putting $m = 2n$, we obtain that $T = 0$. $\square$

Lemma 4.4. *In $\mathcal{S}(D^2, 10)$,*



$= [2]$ 

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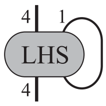
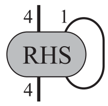
Proof. By Lemma 4.3, it is sufficient to show that

(4.1) 

$=$ 

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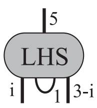
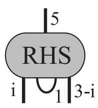
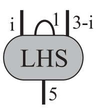
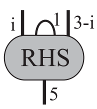
(4.2) 

$=$ 

,

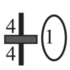
(4.3) 

$=$ 

,

for any $i = 0, 1, \cdots, 3$.

We show (4.1), as follows. (4.1) is rewritten as

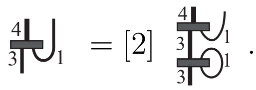


$= [2]$ 

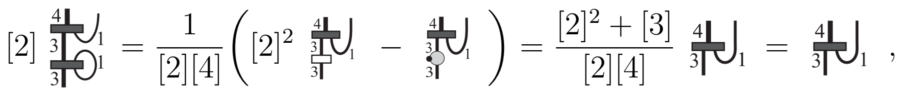
,

and this is shown by Lemma 3.1 (4).

We show (4.2), as follows. When $i = 0, 1, 2$, this is shown by Lemma 3.1 (1). When $i = 3$, (4.2) is rewritten as



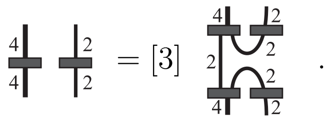
By Lemma 3.1 (2), we have that



where the second equality is obtained by Lemma 3.1 (1) and (3). Therefore, we obtain (4.2).

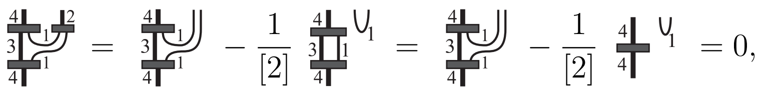
We can verify (4.3) in a similar way as above. $\square$

Lemma 4.5. *In $\mathcal{S}(D^2, 12)$,*

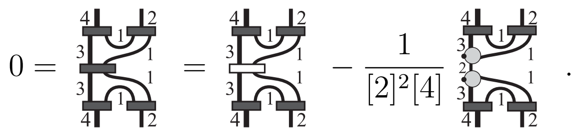


Proof. We have that

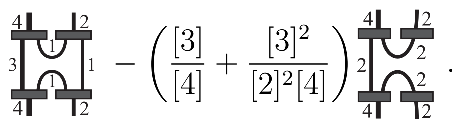
(4.4)



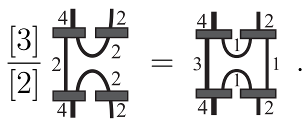
where the second equality is obtained by Lemma 3.1 (4), and the last one is obtained by Lemma 4.4. Hence,



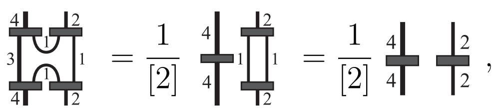
From (2.16), Lemma 3.1 (1) and (3), this is equal to



Since $\frac{[3]}{[4]} + \frac{[3]^2}{[2]^2[4]} = \frac{[3]}{[2]} \cdot \frac{[2]^2+[3]}{[2]^2[4]} = \frac{[3]}{[2]}$, we obtain that

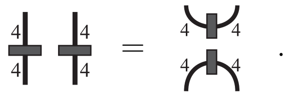


Further, the right-hand side is calculated as

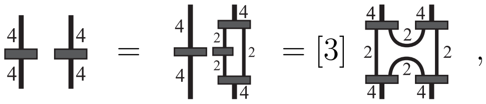


where we obtain the first equality by Lemma 4.4. Therefore, we obtain the required formula of the lemma. $\square$

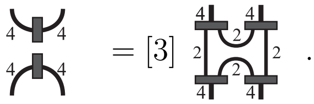
Lemma 4.6. *In $\mathcal{S}(D^2, 16)$,*



Proof. We have that

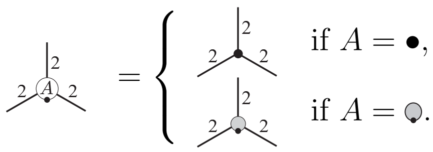


where the first equality is obtained by Lemma 3.1 (1) and (4), and the second one is obtained by Lemma 4.5. As the $\pi/2$ rotation of this formula, we have that

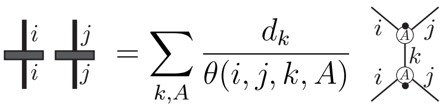


From the above two formulas, we obtain the required formula. ◻

For $A \in \{\bullet, \circ\}$, we denote



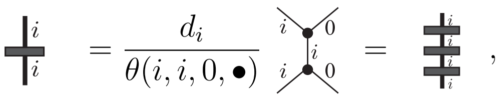
Lemma 4.7. *For any $i, j \in \{0, 2, 4\}$,*



in $\mathcal{S}(D^2, 2(i+j))$, where $k \in \{0, 2, 4\}$ and $A \in \{\bullet, \circ\}$ of the sum run over all admissible colorings of the colored planar trivalent graph in the summand.

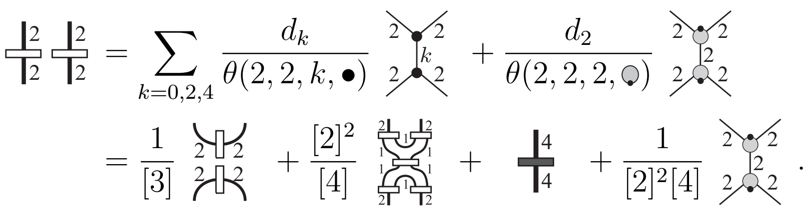
Proof. Since the gray boxes are symmetric with respect to π rotation, we may assume that $i \geq j$ without loss of generality.

When $j = 0$, the required formula is rewritten as

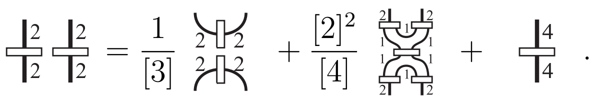


and this is obtained by Lemma 3.1 (4).

When $(i, j) = (2, 2)$, the required formula is rewritten as

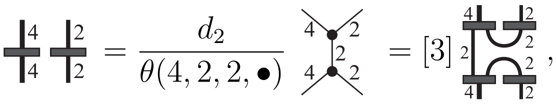


From the definition of the gray box, this is rewritten as



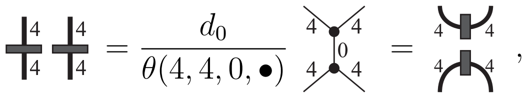
We can verify the above formula by direct calculation, using (2.16).

When $(i,j) = (4,2)$, the required formula is rewritten as



and this is obtained by Lemma 4.5.

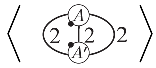
When $(i,j) = (4,4)$, the required formula is rewritten as



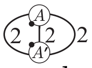
and this is obtained by Lemma 4.6.

Therefore, the proof of the lemma is completed. ☐

Lemma 4.8. *For $A, A' \in \{\bullet, \circ\}$,*

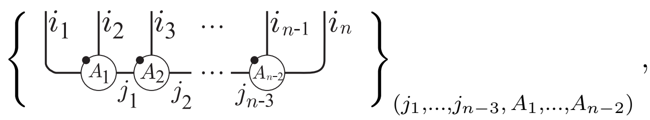
 $= \delta_{AA'}\,\theta(2,2,2,A)\,.$

Proof. When $A = A'$, we obtain the lemma from the definition of $\theta(\cdot)$.

When $A \neq A'$,  is presented by a linear sum of planar diagrams with just one vertex. Since any planar diagram with just one vertex has a cap, the left-hand side of the lemma is equal to 0. Hence, we obtain the required formula. ☐

In the following proposition, we give a basis of $H(i_1, \dots, i_n)$ for $n \geq 3$.

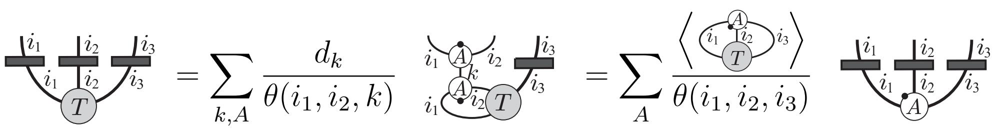
Proposition 4.9. *For $i_1, \dots, i_n \in \{0,2,4\}$ $(n \geq 3)$, the vector space $H(i_1, \dots, i_n)$ has a basis*

(4.5) 

where $j_1, \dots, j_{n-3} \in \{0,2,4\}$ and $A_1, \dots, A_{n-2} \in \{\bullet, \circ\}$ run over all admissible colorings.

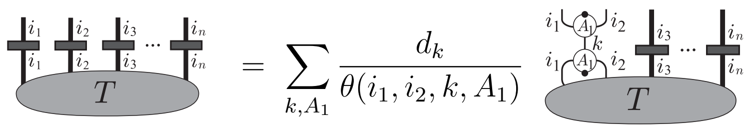
Proof. We first show that $H(i_1, \dots, i_n)$ is spanned by the planar diagrams in (4.5) by induction on n, as follows.

When $n = 3$, we have that



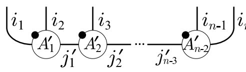
for any $T \in \mathcal{S}(D^2, i_1 + i_2 + i_3)$, where the first equality is obtained by Lemma 4.7, and the second one is obtained by Proposition 4.2. Hence, $H(i_1, i_2, i_3)$ is spanned by the planar diagrams in (4.5).

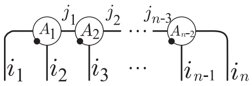
When $n \geq 4$, we show that $H(i_1, \dots, i_n)$ is spanned by the planar diagrams in (4.5), assuming the case of $n-1$, as follows. In a similar way as above, we have that

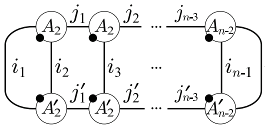


for any $T \in \mathcal{S}(D^2, i_1 + \cdots + i_n)$. By the assumption of the induction, this can be presented by a linear sum of planar diagrams in (4.5). Therefore, $H(i_1, \ldots, i_n)$ is spanned by the planar diagrams in (4.5).

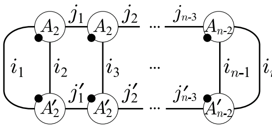
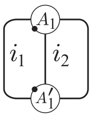
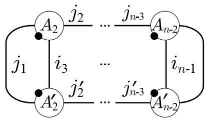
We next show that the planar diagrams in (4.5) are linearly independent, as follows. We denote the index set of (4.5) by J. We assume that

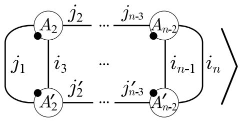
$$\sum_{(j'_1,\ldots,j'_{n-3},\, A'_1,\ldots,A'_{n-2}) \in J} \alpha^{A'_1,\ldots,A'_{n-2}}_{j'_1,\ldots,j'_{n-3}}$$  $$= 0$$

for some scalars $\alpha^{A'_1,\ldots,A'_{n-2}}_{j'_1,\ldots,j'_{n-3}} \in \mathbb{C}$. For any $(j_1, \ldots, j_{n-3},\, A_1, \ldots, A_{n-2}) \in J$, by gluing the planar diagram  to the above formula and taking the bracket, we have that

(4.6) $$\sum_{(j'_1,\ldots,j'_{n-3},\, A'_1,\ldots,A'_{n-2}) \in J} \alpha^{A'_1,\ldots,A'_{n-2}}_{j'_1,\ldots,j'_{n-3}} \Big\langle$$  $$\Big\rangle = 0.$$

By Proposition 4.2, we have that

$$\Big\langle$$  $$\Big\rangle = \frac{\delta_{j_1 j'_1}}{d_{j_1}} \Big\langle$$  $$\Big\rangle\Big\langle$$  $$\Big\rangle$$

$$= \frac{\delta_{j_1 j'_1}\, \delta_{A_1 A'_1}\, \theta(i_1, i_2, j_1, A_1)}{d_{j_1}} \Big\langle$$  $$\Big\rangle$$

$$= \cdots$$

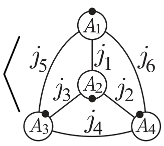
$$= \prod_{k=1}^{n-1} \delta_{j_k j'_k} \prod_{l=1}^{n-2} \delta_{A_l A'_l} \cdot \frac{\theta(i_1, i_2, j_1, A_1)\, \theta(j_1, i_3, j_2, A_2) \cdots \theta(j_{n-1}, i_{n-1}, i_n, A_{n-2})}{d_{j_1} d_{j_2} \cdots d_{j_{n-1}}},$$

where we obtain the second equality by Lemma 4.8, and obtain the last line by repeating this procedure. Hence, from (4.6), we obtain that

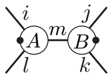
$$\alpha^{A_1,\ldots,A_{n-2}}_{j_1,\ldots,j_{n-3}} \cdot \frac{\theta(i_1, i_2, j_1, A_1)\, \theta(j_1, i_3, j_2, A_2) \cdots \theta(j_{n-1}, i_{n-1}, i_n, A_{n-2})}{d_{j_1} d_{j_2} \cdots d_{j_{n-1}}} = 0.$$

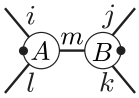
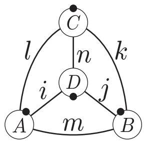
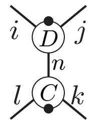
Since all of $\theta(\cdot)$ in the above formula are non-zero by Lemma 3.3, we obtain that $\alpha^{A_1,\ldots,A_{n-2}}_{j_1,\ldots,j_{n-3}} = 0$ for any $(j_1, \ldots, j_{n-3},\, A_1, \ldots, A_{n-2}) \in J$. Hence, the planar diagrams in (4.5) are linearly independent, as required. $\square$

We consider the values

$$\frac{d_{j_4}}{\theta(j_3, j_4, j_5, A_3)\, \theta(j_2, j_4, j_6, A_4)} \Big\langle$$  $$\Big\rangle$$

for $j_1, j_2, \ldots, j_6 \in \{0, 2, 4\}$ and $A_1, A_2, A_3, A_4 \in \{\bullet, \circ\}$ with the colored planar trivalent graph in the above formula being admissible. These values satisfy the formula of the following proposition, which is the defining relation of the $6j$-symbols. In this sense, the above values give the *6j-symbols* of the E_6 linear skein.

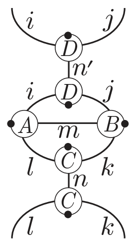
Proposition 4.10 (the defining relation of the $6j$-symbols)**.** *For $i, j, k, l, m \in \{0, 2, 4\}$ and $A, B \in \{\bullet, \circ\}$ with*  *being admissible,*


$$= \sum_{n,C,D} \frac{d_n}{\theta(k, l, n, C)\, \theta(i, j, n, D)} \Big\langle$$

$$\Big\rangle$$

.

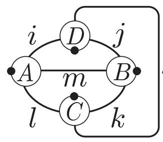
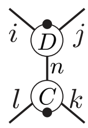
where $n \in \{0, 2, 4\}$ and $C, D \in \{\bullet, \circ\}$ of the sum run over all admissible colorings of the colored planar trivalent graph in $\langle\ \rangle$.

Proof. In the proof, indices of a sum run over all admissible colorings of the colored planar trivalent graphs in the summand.

By using Lemma 4.7 twice, the left-hand side of the required formula is equal to

$$\sum_{n,n',C,D} \frac{d_n d_{n'}}{\theta(k, l, n, C)\, \theta(i, j, n', D)}$$

.

By Proposition 4.2, this is equal to

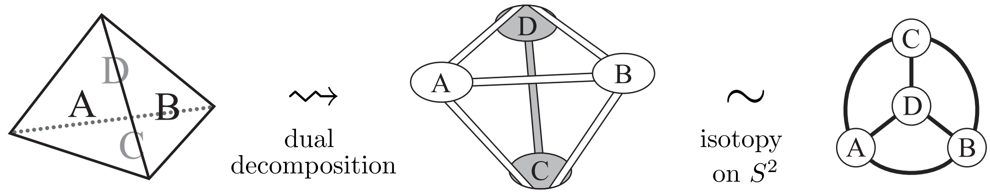
$$\sum_{n,C,D} \frac{d_n}{\theta(k, l, n, C)\, \theta(i, j, n, D)} \Big\langle$$

$$\Big\rangle$$

.

From Lemma 2.11, this is equal to the right-hand side of the required formula. □

5. A state sum invariant of 3 manifolds

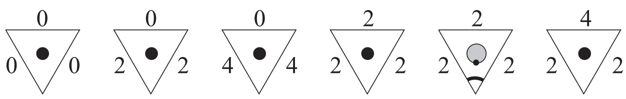
It is known, see [EK, KS], that a state sum invariant of 3-manifolds can be constructed from a set of $6j$-symbols, and the topological invariance of the invariant is shown by using the orthogonal relation and the pentagon relation of the $6j$-symbols, which are naturally obtained from the defining relation of the $6j$-symbols. See also [T, TV] for similar procedures of such constructions, and see [W] for a construction of the E_6 state sum invariant. In this section, along such a general procedure, we construct a state sum invariant of 3-manifolds based on our E_6 linear skein in Definition 5.1, and show its topological invariance in Theorem 5.2. The outline of the proof of the topological invariance is the known procedure, and we check it concretely based on our E_6 linear skein in this section.

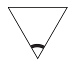
We relate an oriented tetrahedron to a planar trivalent graph, as follows, where, in the left picture below, we regard gray characters as on faces of the hidden side.

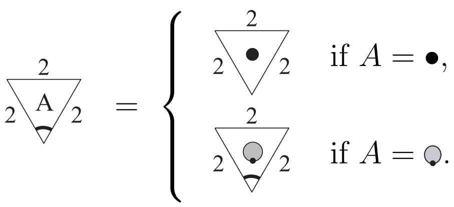


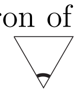
For a given tetrahedron (the left picture), we consider the triangulation of the boundary of the tetrahedron, and consider its dual decomposition on S^2 (the middle picture). Further, by isotopy on S^2, we obtain a planar trivalent graph (the right picture). Here, we fix an orientation of the plane of the trivalent graph, and we make the above correspondence in such a way that the induced orientation of the boundary of the tetrahedron coincides with the orientation of the plane under this correspondence.

We consider colorings of a tetrahedron corresponding to colorings of a planar trivalent graph, as follows. A *coloring* of a tetrahedron T is a map from the set of faces of T to $\{\bullet, \circ\}$ and a map from the set of edges of T to $\{0,2,4\}$. A coloring of a tetrahedron T is said to be *admissible* if each face of T is colored as shown in either of the following pictures, noting that these are dual to the pictures of (3.5).

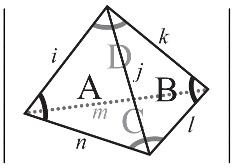


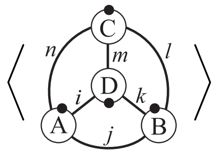
Here, when a face is colored by $\circ$, we consider a marking such as  at a vertex of a triangle of the face corresponding to the base point of $\circ$. For $A \in \{\bullet, \circ\}$, we denote



Let M be a closed oriented 3-manifold, and let $\mathcal{T}$ be a triangulation of M. A *coloring* of $\mathcal{T}$ is a map from the set of faces of $\mathcal{T}$ to $\{\bullet, \circ\}$ and a map from the set of edges of $\mathcal{T}$ to $\{0,2,4\}$. A coloring of $\mathcal{T}$ is said to be *admissible* if the coloring of each tetrahedron of $\mathcal{T}$ is admissible. When a face is colored by $\circ$, we consider a marking such as  at a vertex of a triangle of the face. We define a *weight* $|\cdot|$

of a colored oriented tetrahedron by





$$\left|\ \cdot\ \right| = \frac{\langle\ \cdot\ \rangle}{\sqrt{\theta(i,j,n,A)\,\theta(j,k,l,B)\,\theta(l,m,n,C)\,\theta(i,k,m,D)}},$$

where the colored planar trivalent graph in the right-hand side is obtained from the tetrahedron as mentioned above. We note that the values of $\theta(\cdot)$ are positive real numbers by Lemma 3.3. We also note that, by Lemma 2.11, the colored planar trivalent graph in the right-hand side can be regarded as in S^2, which guarantees that its value is well determined from a colored oriented tetrahedron. When the marking of the tetrahedron is given in other ways, we define its weight similarly. We denote by v the number of vertices of $\mathcal{T}$. We put $w = d_0^2 + d_2^2 + d_4^2 = 2 + [3]^2$.

Definition 5.1. We define the E_6 *state sum* of a closed oriented 3-manifold M with a triangulation $\mathcal{T}$ by

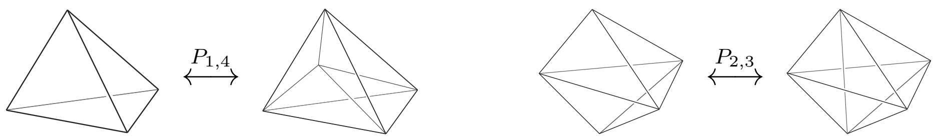
$$Z^{E_6}(M,\mathcal{T}) \ = \ w^{-v} \sum_{\lambda} \prod_{E} d_{\lambda(E)} \prod_{T} \big|(T,\lambda)\big|,$$

where the sum of λ runs over all admissible colorings of $\mathcal{T}$, the product of E runs over all edges of $\mathcal{T}$, and the product of T runs all tetrahedra of $\mathcal{T}$.

We note that $Z^{E_6}(M,\mathcal{T})$ does not depend on a choice of markings of faces, because, when we change a marking of a face, the changes of the weights of the adjacent tetrahedra cancel together. Further, we show that $Z^{E_6}(M,\mathcal{T})$ does not depend on a choice of a triangulation $\mathcal{T}$ in the following theorem.

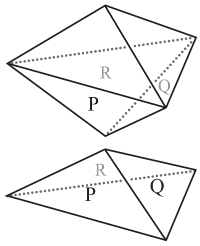
Theorem 5.2. *$Z^{E_6}(M,\mathcal{T})$ is a topological invariant of a closed oriented 3-manifold M, independently of a choice of a triangulation $\mathcal{T}$.*

We will show a proof of the theorem later in this section. In order to show the theorem, we recall the Pachner moves. Let M and M' be closed 3-manifolds with triangulations $\mathcal{T}$ and $\mathcal{T}'$ respectively. It is known [P] that M and M' are homeomorphic if and only if $\mathcal{T}$ and $\mathcal{T}'$ are related by a finite sequence of simplicial isomorphisms and the Pachner moves $P_{1,4}$ and $P_{2,3}$ shown below,



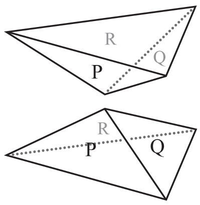
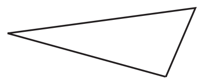
where the pictures consist of one tetrahedron, four tetrahedra, two tetrahedra and three tetrahedra, respectively. Hence, in order to show Theorem 5.2, it is sufficient to show that $Z^{E_6}(M,\mathcal{T})$ is invariant under the Pachner moves $P_{1,4}$ and $P_{2,3}$.

Further, it is known [TV] that the $P_{1,4}$ move can be simplified by using the $P_{2,3}$ move, as follows. We consider singular triangulations, extending the usual triangulations. By applying the $P_{2,3}$ move to the upper three tetrahedra of the right-hand side of the $P_{1,4}$ move, the right-hand side of the $P_{1,4}$ move is rewritten as

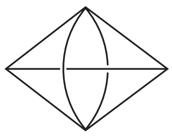
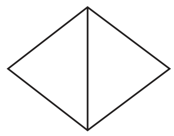
the union obtained by gluing  along P, Q, R.

Hence, by using the $P_{2,3}$ move, the $P_{1,4}$ move can be replaced with the following move.

(5.1)

$\Bigg($ the union obtained by gluing  along P, Q, R $\Bigg)$ $\longleftrightarrow$ 

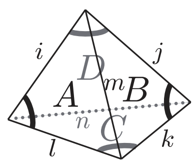
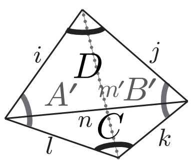
Further, we will show later in the proof of Theorem 5.2 that invariance under this move can be reduced to invariance under the following move [TV],

(5.2)  $\longleftrightarrow$ 

where the left-hand side consists of two (singular) tetrahedra, and the right-hand side consists of two triangles (with no tetrahedra). Hence, in order to show Theorem 5.2, it is sufficient to show that $Z^{E_6}(M, \mathcal{T})$ is invariant under the move (5.2) and the $P_{2,3}$ move.

Furthermore, it is known as general procedures (see [EK, KS, T, TV]) that invariance under the move (5.2) and the $P_{2,3}$ move are obtained from the orthogonal relation and the pentagon relation of the $6j$-symbols, which are naturally obtained from the defining relation of the $6j$-symbols (Proposition 4.10). We show them in the following two lemmas.

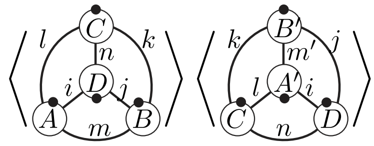
Lemma 5.3 (the orthogonal relation)**.**

$$d_m \sum_{n,C,D} d_n \left| \vphantom{\int} \right.$$  $$\left| \vphantom{\int} \right| \left| \vphantom{\int} \right.$$  $$\left| \vphantom{\int} \right. = \delta_{mm'}\delta_{AA'}\delta_{BB'}.$$

Proof. In the proof, indices of a sum run over all admissible colorings of the colored trivalent graph in the summand, and we denote $\theta(i,j,k,A) = \theta^A_{ijk}$.

From the definition of the weights, the required formula is rewritten as

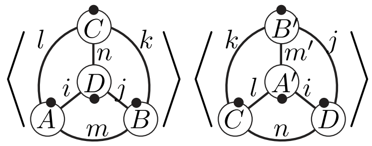
$$\frac{d_m}{\sqrt{\theta^A_{ilm}\theta^B_{kjm}\theta^{A'}_{ilm'}\theta^{B'}_{jkm'}}} \sum_{n,C,D} \frac{d_n}{\theta^C_{kln}\theta^D_{ijn}}$$



$$= \delta_{mm'}\delta_{AA'}\delta_{BB'}.$$

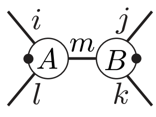
$\sqrt{\cdot}$ in the above formula is equal to $\theta^A_{ilm}\theta^B_{kjm}$ when $m = m'$, $A = A'$ and $B = B'$. Thus, it is sufficient to show that

(5.3) $$\frac{d_m}{\theta^A_{ilm}\theta^B_{kjm}} \sum_{n,C,D} \frac{d_n}{\theta^C_{kln}\theta^D_{ijn}}$$

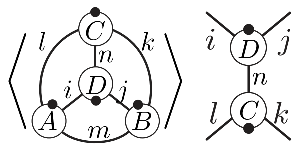


$$= \delta_{mm'}\delta_{AA'}\delta_{BB'}.$$

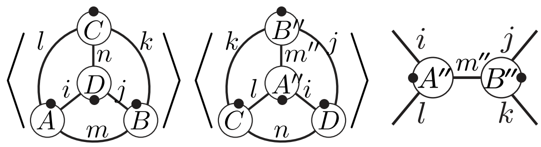
By using Proposition 4.10 twice, we have that



$$= \sum_{n,C,D} \frac{d_n}{\theta^C_{kln}\theta^D_{ijn}}$$



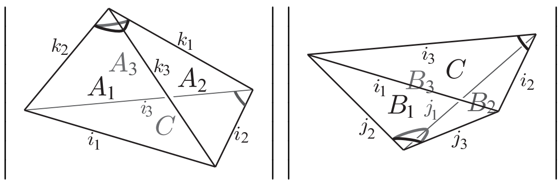
$$= \sum_{\substack{m'',n,\\ A'',B'',C,D}} \frac{d_{m''}d_n}{\theta^{A''}_{ilm''}\theta^{B''}_{jkm''}\theta^C_{kln}\theta^D_{ijn}}$$



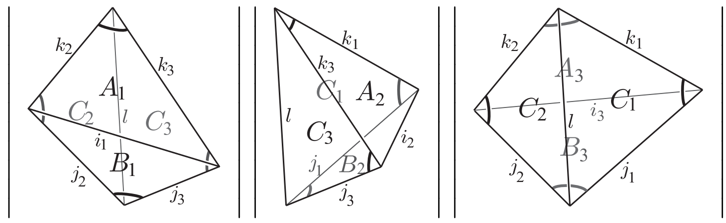
Hence, by Proposition 4.9 for $n = 4$, we obtain (5.3), as required. ☐

Lemma 5.4 (the pentagon relation)**.**

$$\sum_C$$

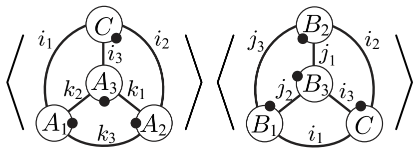


$$= \sum_{l,C_1,C_2,C_3} d_l$$

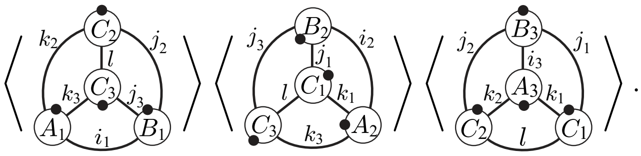


Proof. In the proof, indices of a sum run over all admissible colorings of the colored planar trivalent graphs in the summand, and we denote $\theta^A_{ijk} = \theta(i,j,k,A)$. From the definition of the weights, the required formula is rewritten as

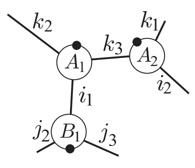
(5.4) $$\sum_C \frac{1}{\theta^C_{i_1i_2i_3}}$$



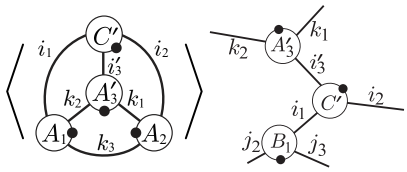
$$= \sum_{l,C_1,C_2,C_3} \frac{d_l}{\theta^{C_1}_{j_1k_1l}\theta^{C_2}_{j_2k_2l}\theta^{C_3}_{j_3k_3l}}$$



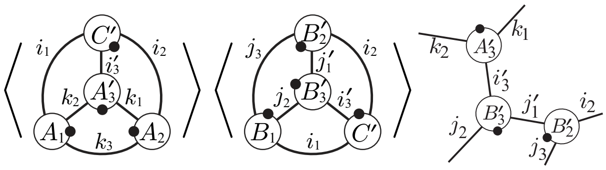
We show the above formula by calculating a certain colored planar trivalent graph in two ways, as follows. By using Proposition 4.10 twice, we have that



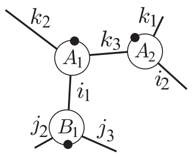
$$= \sum_{i'_3,A'_3,C'} \frac{d_{i'_3}}{\theta^{A'_3}_{i'_3k_1k_2}\theta^{C'}_{i_1i_2i'_3}}$$



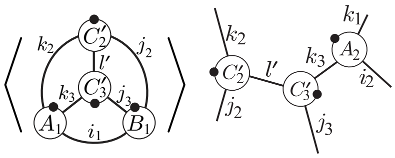
$$= \sum_{\substack{i'_3,j'_1,\\ A'_3,B'_2,B'_3,C'}} \frac{d_{j'_1}d_{i'_3}}{\theta^{A'_3}_{i'_3k_1k_2}\theta^{B'_2}_{i_2j'_1j_3}\theta^{B'_3}_{i'_3j'_1j_2}\theta^{C'}_{i_1i_2i'_3}}$$



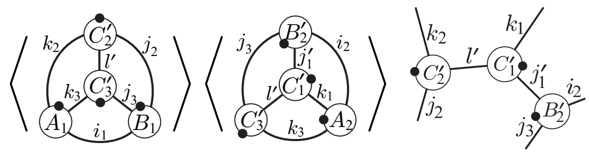
On the other hand, by using Proposition 4.10 three times, we have that



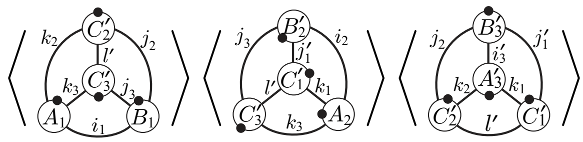
$$= \sum_{l',C'_2,C'_3} \frac{d_{l'}}{\theta^{C'_2}_{j_2k_2l'}\theta^{C'_3}_{j_3k_3l'}}$$



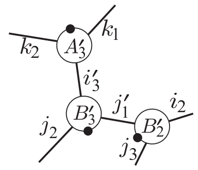
$$= \sum_{\substack{l',j'_1,\\ B'_2,C'_1,C'_2,C'_3}} \frac{d_{j'_1}d_{l'}}{\theta^{B'_2}_{i_2j'_1j_3}\theta^{C'_1}_{j'_1k_1l'}\theta^{C'_2}_{j_2k_2l'}\theta^{C'_3}_{j_3k_3l'}}$$



$d_{j'_1}d_{i'_3}d_{l'}$



$$= \sum_{\substack{l',j'_1,i'_3,\\ A'_3,B'_2,B'_3,C'_1,C'_2,C'_3}} \frac{}{\theta^{A'_3}_{i'_3k_1k_2}\theta^{B'_2}_{i_2j'_1j_3}\theta^{B'_3}_{i'_3j'_1j_2}\theta^{C'_1}_{j'_1k_1l'}\theta^{C'_2}_{j_2k_2l'}\theta^{C'_3}_{j_3k_3l'}}$$



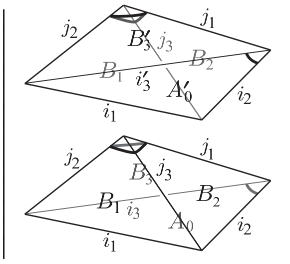
From these formulae and Proposition 4.9 for $n = 5$, we obtain (5.4), as required. $\square$

We now show a proof of Theorem 5.2.

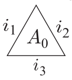
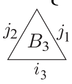
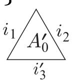
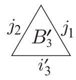
Proof of Theorem 5.2. As mentioned before, it is sufficient to show that $Z^{E_6}(M, \mathcal{T})$ is invariant under the $P_{2,3}$ move and the move (5.1).

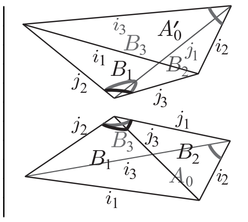
We obtain the invariance under the $P_{2,3}$ move by Lemma 5.4.

We show the invariance under the move (5.1), as follows. In the following of this proof, indices of a sum run over all admissible colorings of tetrahedra in the summand. By Lemma 5.3, we have that

$$d_{i_3} \sum_{j_3, B_1, B_2} d_{j_3}$$ 

$$= \delta_{i_3 i_3'} \delta_{A_0 A_0'} \delta_{B_3 B_3'}$$

for any $i_1, i_2, j_1, j_2 \in \{0, 2, 4\}$ and $A_0, A_0', B_3, B_3' \in \{\bullet, \circ\}$ such that the colored triangles  ,  ,  and  are admissible. By putting $i_3 = i_3'$, $B_3 = B_3'$, by multiplying $d_{i_3}^{-1} d_{j_1} d_{j_2}$ by both sides and by summing over j_1, j_2, B_3, we have that

$$\sum_{\substack{j_1, j_2, j_3, \\ B_1, B_2, B_3}} d_{j_1} d_{j_2} d_{j_3}$$ 

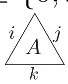
$$= d_{i_3}^{-1} \sum_{j_1, j_2, A_3} d_{j_1} d_{j_2} \cdot \delta_{A_0 A_0'} = w \, \delta_{A_0 A_0'},$$

where the last equality is obtained by Lemma 5.5 below. This means the invariance under the move (5.1), as required. □

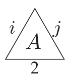
In the proof of Theorem 5.2, we used the following lemma.

Lemma 5.5. *For any $k \in \{0, 2, 4\}$,*

$$d_k^{-1} \sum_{i,j,A} d_i d_j = w,$$

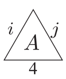
where $i, j \in \{0, 2, 4\}$ and $A \in \{\bullet, \circ\}$ in the sum run over all admissible colorings of a triangle  .

Proof. When $k = 0$, we obtain the required formula from the definition of admissible colorings and the definition of w.

When $k = 2$,  is admissible for $(i, j, A) = (0, 2, \bullet)$, $(2, 0, \bullet)$, $(2, 2, \bullet)$, $(2, 2, \circ)$, $(2, 4, \bullet)$, $(4, 2, \bullet)$. Hence, the left-hand side of the lemma is equal to

$$d_2^{-1} \cdot 2(d_0 d_2 + d_2^2 + d_2 d_4) = 4 + 2[3] = 6 + 2\sqrt{3} = w,$$

as required.

When $k = 4$,  is admissible for $(i, j, A) = (0, 4, \bullet)$, $(4, 0, \bullet)$, $(2, 2, \bullet)$. Hence, the left-hand side of the lemma is equal to

$$d_4^{-1}(2 d_0 d_4 + d_2^2) = 2 + [3]^2 = w,$$

as required. □

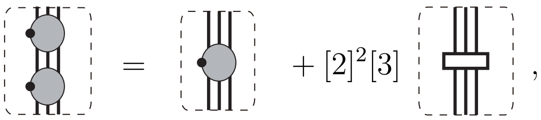
6. Equality to the E_6 state sum invariant

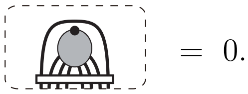
In this section, we show our defining relations of the E_6 linear skein are equivalent to Bigelow's relations of the E_6 subfactor planar algebra in Section 6.1. Further, we show our state sum invariant is equal to the E_6 state sum invariant in Section 6.2.

6.1. **Equivalence to Bigelow's relations of the E_6 subfactor planar algebra.**

Bigelow [B] defined a planar algebra $\{\mathcal{S}'(D^2,2n)\}_{n=0,1,\dots}$ (in his paper this is denoted by $\mathcal{P}$) by giving generators and relations, and proved that its principal graph is the E_6 Dynkin diagram. However, his proof relies on the existence of the E_6 subfactor planar algebra and some of its known properties. In this section, we show that $\mathcal{S}(D^2,2n)$ is isomorphic to $\mathcal{S}'(D^2,2n)$ for any $n \geq 0$. We note that an S-labeled disc of [B] corresponds to a vertex of this paper. As a consequence of this section, $\{\mathcal{S}(D^2,2n)\}_{n=0,1,\dots}$ forms a subfactor planar algebra.

As in [B], for an integer $n \geq 0$, we define $\mathcal{S}'(D^2,2n)$ to be the vector space spanned by planar diagrams in $(D^2,2n)$ subject to the relations (2.1)–(2.3) and (6.1), (6.2) below,

(6.1) 

(6.2) 

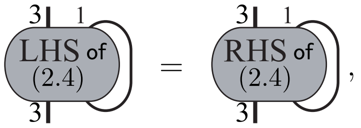
We recall that $\mathcal{S}(D^2,2n)$ is the vector space spanned by planar diagrams in $(D^2,2n)$ subject to the relations (2.1)–(2.4).

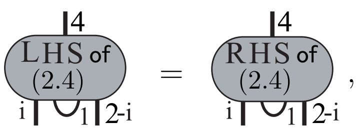
Proposition 6.1. *For any $n \geq 0$, $\mathcal{S}(D^2,2n)$ is isomorphic to $\mathcal{S}'(D^2,2n)$.*

Proof. We assume (2.1)–(2.3) in this proof. It is enough to show that (2.4) is equivalent to (6.1) and (6.2).

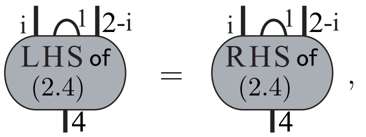
Assuming (2.4), we show (6.1) and (6.2), as follows. We obtain (6.1) from (3.2). Further, we obtain (6.2) in the similar way as the proof of Lemma 2.9 in the case $m = 4$, using (2.22).

Assuming (6.1) and (6.2), we show (2.4), as follows. From the E_6 version of the proof of [B, Lemma 3.1], an edge can pass-over a vertex. Hence, the formula of Lemma 4.3 holds in $\mathcal{S}'(D^2,2m)$ in the similar way as the proof of Lemma 2.9. Hence, it is sufficient to show that

(6.3) 

(6.4) 

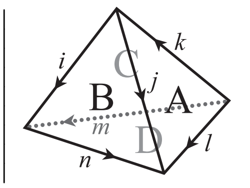
(6.5)



for any $i = 0, 1, 2, 3$. We obtain (6.3) by (2.8), (3.1) and (6.1), and we obtain (6.4) and (6.5) by (2.2) and (2.7), noting that the relations (2.1)–(2.3) implies (2.7), (2.8) and (3.1). Therefore, we obtain (2.4), as required. □

6.2. **Equality of our state sum invariant to the E_6 state sum invariant.** The E_6 state sum invariant of 3-manifolds is the state sum invariant defined from the $6j$-symbols of the E_6 subfactor. The E_6 state sum invariant is concretely formulated and calculated in [SuW, W]. In this section, we show that our state sum invariant defined in Section 5 is equal to the E_6 state sum invariant in Proposition 6.2.

We briefly review the formulation of the E_6 state sum invariant; for details, see [SuW, W]. Similarly as our formulation, edges are colored by $0, 2, 4$ (they are denoted by "id", "ρ", "α" in [SuW, W]). Admissible colorings are defined similarly as our formulation. A face whose edges are colored by i, j, k is colored by • unless $i = j = k = 2$, and is colored by S_3 and S_4 if $i = j = k = 2$, while a face whose edges are colored by $2, 2, 2$ is colored by • and ◌ in our formulation. (To be precise, their faces are colored by S_1, S_2, S_3, S_4, though the color is uniquely determined unless $i = j = k = 2$, and we denote it by • here.) Unlike our formulation, a total order of the vertices is given, and edges are oriented by using this order. The weight of a tetrahedron

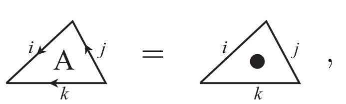


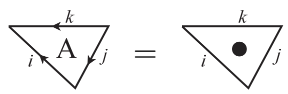
is given by the $6j$-symbols of the E_6 subfactor; we review their concrete values in Appendix B.

Proposition 6.2. *Our state sum invariant of a closed oriented 3-manifold M defined in Section 5 is equal to the E_6 state sum invariant of M.*

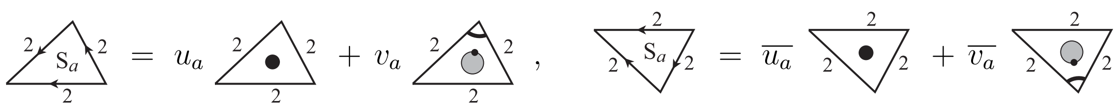
Proof. We note that basic parts of their formulation are similar to our formulation. The differences are that their edges are oriented, and that a face whose edges are colored by $2, 2, 2$ is colored by S_3 and S_4, while such a face is colored by • and ◌ in our formulation.

We transfer the colors of faces by putting

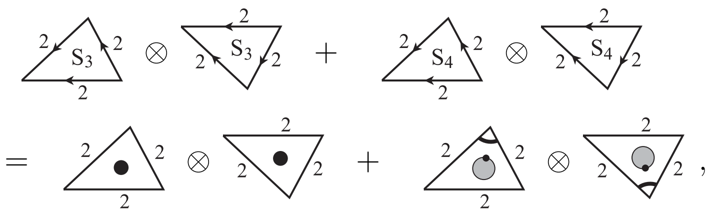




unless $i = j = k = 2$,



with some unitary matrix $\begin{pmatrix} u_3 & v_3 \\ u_4 & v_4 \end{pmatrix}$. By the unitarity of this matrix, we have that



which justifies that the substitution of S_3, S_4 can be transformed into the substitution of $\bullet$, $\circ$ in the definition of the state sum invariant.

It is shown, see Lemma B.1 (due to T. Ohtsuki), that our $6j$-symbols can be transformed into the $6j$-symbols of the E_6 subfactor by such a transformation as above. Hence, our state sum invariant is equal to the E_6 state sum invariant. □

7. Properties and examples of our state sum invariant

The values of the E_6 state-sum invariant have been calculated for the lens space $L(p,q)$ for $q = 1, 2, 3$ in [SuW], and for some other 3-manifolds in [SaW]. In this section, we calculate the values of the E_6 state-sum invariant for the lens spaces $L(4,1)$, $L(5,2)$ and $L(5,1)$ in terms of our E_6 linear skein in Examples 7.5, 7.6 and 7.7. Further, we review some property of the E_6 state sum invariant in Proposition 7.1.

The following proposition is a well-known basic property of the E_6 state sum invariant.

Proposition 7.1. *For any closed oriented 3-manifold M,*

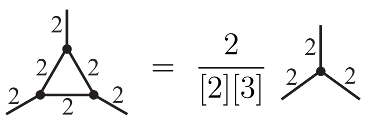
$$Z^{E_6}(\overline{M}) \;=\; \overline{Z^{E_6}(M)},$$

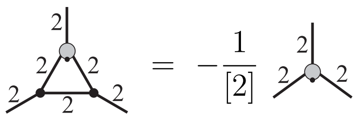
where $\overline{M}$ denotes M with the opposite orientation.

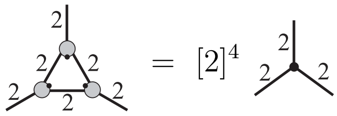
Proof. We review the proof based on our construction of the state sum invariant. When we change the orientation of M, the colored planar trivalent graph corresponding to a tetrahedron becomes its mirror image. Hence, the value of $Z^{E_6}(M)$ becomes its complex conjugate. □

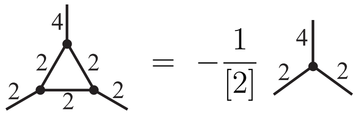
In order to calculate our state sum invariant later, we show some properties of colored planar trivalent graphs in the following three lemmas.

Lemma 7.2.

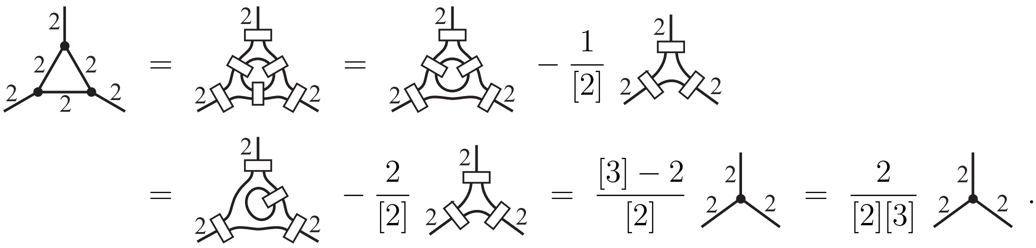
(1) 

(2) 

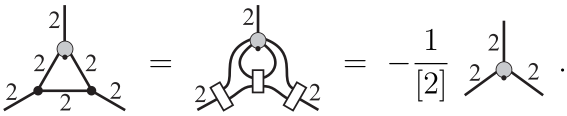
(3) 

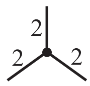
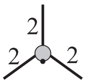
(4) 

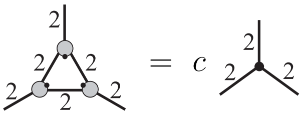
Proof. We obtain (1), since



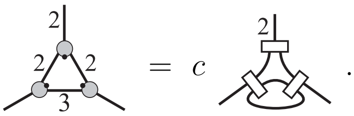
We obtain (2), since



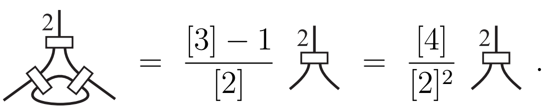
We show (3), as follows. By Proposition 4.9, $H(2,2,2)$ is spanned by  and  . Further, since the left-hand side of the required formula is symmetric with respect to $\frac{2}{3}\pi$ rotation, we can put

(7.1) 

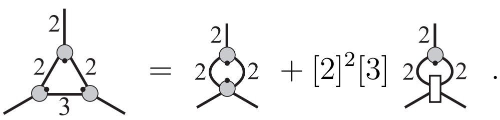
with some scalar c. By closing one strand at the bottom, we have that

(7.2) 

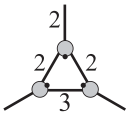
By expanding white boxes, we calculate the diagram of the right-hand side as



Further, by (3.2), the left-hand side of (7.2) is calculated as

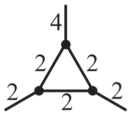
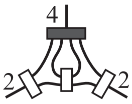
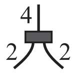


We can verify that the second summand of the right-hand side is equal to 0 by expanding the white box. Hence, by using (3.2) again, we can show that

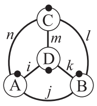
 $= [2]^2[4]$  .

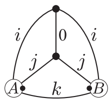
Therefore, from (7.2), $c = [2]^4$. Hence, from (7.1), we obtain (3) of the lemma.

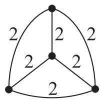
We obtain (4), since

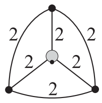
 $=$  $= -\dfrac{1}{[2]}$  $= -\dfrac{1}{[2]}$  .

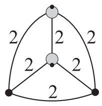
□

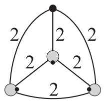
Lemma 7.3. *The values of* $\Big\langle$  $\Big\rangle$ *are given, as follows.*

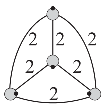
(1) $\Big\langle$  $\Big\rangle = \delta_{AB}\,\theta(i,j,k,A)$ *for* $i,j,k \in \{0,2,4\}$ *and* $A, B \in \{\bullet, \circ\}$.

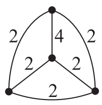
(2) $\Big\langle$  $\Big\rangle = \dfrac{2}{[2][3]}\,\theta(2,2,2,\bullet) = \dfrac{2\,[4]}{[2]^3}$.

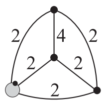
(3) $\Big\langle$  $\Big\rangle = 0$.

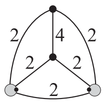
(4) $\Big\langle$  $\Big\rangle = -\dfrac{1}{[2]}\,\theta(2,2,2,\circ) = -[2][3][4]$.

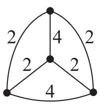
(5) $\Big\langle$  $\Big\rangle = [2]^2[3][4]$.

(6) $\Big\langle$  $\Big\rangle = 0$.

(7) $\Big\langle$  $\Big\rangle = -\dfrac{1}{[2]}$.

(8) $\Big\langle$  $\Big\rangle = -[3]$.

(9) $\Big\langle$  $\Big\rangle = [2]^3$.

(10) $\Big\langle$  $\Big\rangle = -\dfrac{1}{[3]}$.

Proof. We obtain (1) by Lemma 3.1 (4) and Lemma 4.8.

We obtain (2) by applying Lemma 7.2 (1) to a triangle of a diagram of the required formula.

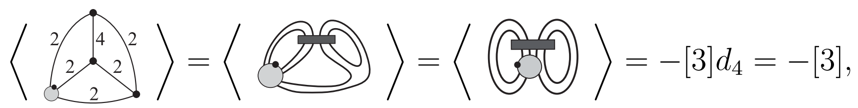
We obtain (3), since any planar diagram with just one vertex must have a cap and such a diagram is equal to 0 in the linear skein.

We obtain (4) by applying Lemma 7.2 (2) to a triangle of a diagram of the required formula.

We obtain (5) and (6) by applying Lemma 7.2 (3) to a triangle of a diagram of the required formulas and by using Lemma 4.8.

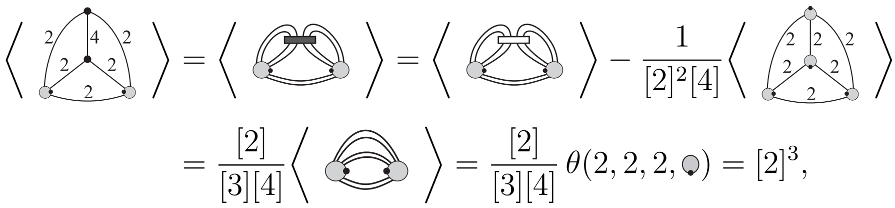
We obtain (7) by applying Lemma 7.2 (4) to a triangle of a diagram of the required formula and by using Lemma 4.8.

We obtain (8), since



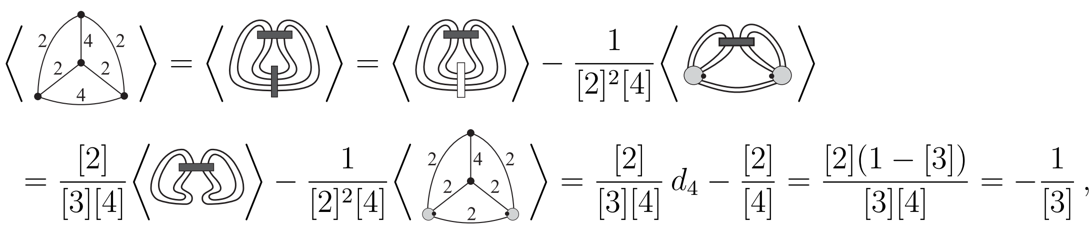
where the third equality is obtained by Lemma 3.1 (3).

We obtain (9), since



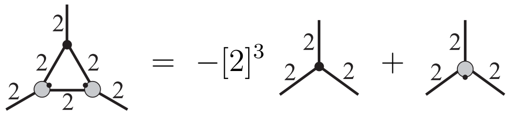
where the third equality is obtained by (2.16) and (6) of the lemma.

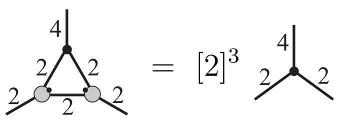
We obtain (10), since

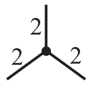
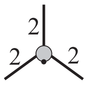


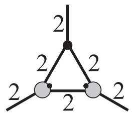
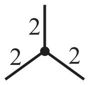
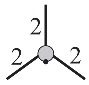
where the fourth equality is obtained by (9) of the lemma. ☐

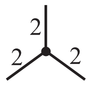
Lemma 7.4.

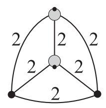
(1) 

(2) 

Proof. We show (1), as follows. Since the space $H(2,2,2)$ is spanned by  and  by Proposition 4.9, we can put

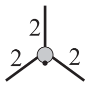
(7.3)  $= c_1$  $+ c_2$ 

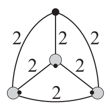
with some scalars c_1 and c_2. By closing the diagrams of (7.3) with  , we have that

$\Big\langle$  $\Big\rangle = c_1 \cdot \theta(2,2,2,\bullet)$

by Lemma 4.8. Further, by Lemmas 3.3 and 7.3,

$$-[2][3][4] = c_1 \cdot \frac{[3][4]}{[2]^2}\,.$$

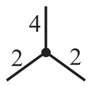
Hence, $c_1 = -[2]^3$. By closing the diagrams of (7.3) with  , we have that

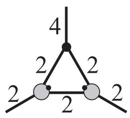
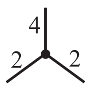
$\Big\langle$  $\Big\rangle = c_2 \cdot \theta(2,2,2,$$)$

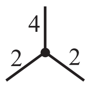
by Lemma 4.8. Further, by Lemmas 3.3 and 7.3,

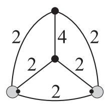
$$[2]^2[3][4] = c_2 \cdot [2]^2[3][4]\,.$$

Hence, $c_2 = 1$. Therefore, from (7.3), we obtain (1) of the lemma.

We show (2), as follows. Since the space $H(4,2,2)$ is spanned by  by Proposition 4.9, we can put

(7.4)  $= c$ 

with some scalars c. By closing the diagrams of (7.4) with  , we have that

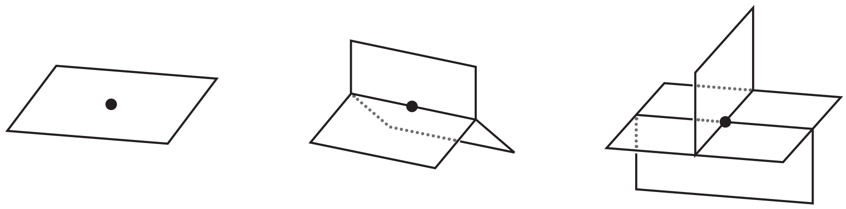
$\Big\langle$  $\Big\rangle = c \cdot \theta(2,2,4,\bullet)$.

Further, by Lemmas 3.3 and 7.3,

$$[2]^3 = c\,.$$

Hence, from (7.4), we obtain (2) of the lemma. $\square$

We briefly review the construction of the state sum invariant based on spines of 3-manifolds; see [T, TV] for details. A *spine* of a closed oriented 3-manifold is a 2-polyhedron obtained from $M - (\text{3-balls})$ by collapsing all 3-cells in such a way that each point of the resulting 2-polyhedron has a neighborhood of either of the following forms.



Thus, a spine consists of vertices (the right picture), edges (the middle picture) and faces (the left picture). A typical spine of a 3-manifold M is the 2-skeleton of the dual decomposition of a triangulation of M. A *coloring* of a spine X is a map from the set of edges of X to $\{\bullet, \circ\}$ and a map from the set of edges of X to $\{0, 2, 4\}$. We can define an *admissible* coloring of a spine in an appropriate way, corresponding to an admissible coloring of a triangulation. We can define a *weight* of a colored vertex of a spine corresponding to the weight of a colored tetrahedron, by using a colored planar trivalent graph obtained as the intersection of the spine and the boundary of a neighborhood of the vertex. It is known, see [T, TV], that the state sum invariant of a 3-manifold M with a spine X is presented by

$$Z^{E_6}(M) \;=\; w^{-v} \sum_{\lambda} \prod_{F} d_{\lambda(F)} \prod_{V} \big(\text{the weight of } V \text{ colored by } \lambda\big),$$

where the sum of λ runs over all admissible colorings of X, the product of F runs over all faces of X, the product of V runs all vertices of X, and v denotes the number of 3-balls when we make X from M.

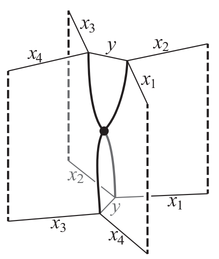
Example 7.5 ([SuW])**.** The value of the E_6 state sum invariant of the lens space $L(4,1)$ is given by

$$Z^{E_6}\big(L(4,1)\big) \;=\; \frac{3+\sqrt{-3}}{6}\,.$$

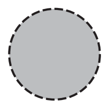
Proof. In this proof, we calculate the required value based on our construction of the state sum invariant.

A spine of $L(4,1)$ is given as follows,

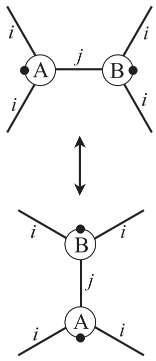
(7.5) $\Bigg($ 2-polyhedron obtained by gluing

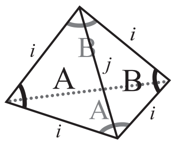


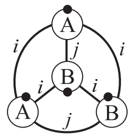
along $x_1, \cdots, x_4, y$ $\Bigg) \displaystyle\bigcup_{\substack{\text{dashed}\\ \text{line}}}$



where the resulting 2-polyhedron obtained from the left picture by gluing the edges labeled by $x_1, \cdots, x_4, y$ has a boundary of a dashed line, and we consider the union of this resulting 2-polyhedron and a disk along this dashed line. This spine has one vertex. This vertex corresponds to the fusion of the left picture below.

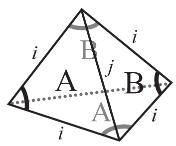


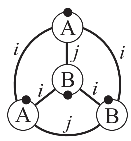




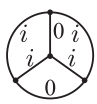
The fusion of the left picture corresponds to the tetrahedron of the middle picture; the upper graph of the fusion is dual to the faces of the front side of the tetrahedron, and the lower graph is dual to the faces of the hidden side of the tetrahedron. The tetrahedron of the middle picture corresponds to the planar graph of the right picture; we note that this planar graph is the union of the upper graph of the fusion and the mirror image of the lower graph of the fusion, which is equal to the boundary of the left picture of (7.5).

Hence, we calculate the value of $Z^{E_6}(L(4,1))$, as follows,

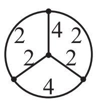
$$Z^{E_6}\big(L(4,1)\big) \;=\; w^{-1} \sum_{\substack{i,j\in\{0,2,4\}\\ A,B\in\{\bullet,\circ\}}} d_i d_j \left|\; \right.$$

$$\left.\;\right|$$

(7.6)
$$\;=\; w^{-1} \sum_{\substack{i,j\in\{0,2,4\}\\ A,B\in\{\bullet,\circ\}}} \frac{d_i d_j}{\theta(i,i,j,A)\,\theta(i,i,j,B)} \Big\langle \;$$

$$\;\Big\rangle\,.$$

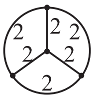
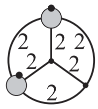
When $j=0$, the sum of (7.6) is equal to

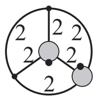
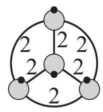
$$\sum_{i\in\{0,2,4\}} \frac{d_i d_0}{\theta(i,i,0,\bullet)^2} \Big\langle \;$$

$$\;\Big\rangle \;=\; \sum_{i\in\{0,2,4\}} \frac{d_i^2}{d_i^2} \;=\; 3\,,$$

by Lemmas 3.3 and 7.3. When $j=4$, the sum of (7.6) is equal to

$$\frac{d_2 d_4}{\theta(2,2,4,\bullet)^2} \Big\langle \;$$

$$\;\Big\rangle \;=\; \frac{[3]}{1^2}\cdot\Big(-\frac{1}{[3]}\Big) \;=\; -1\,,$$

by Lemmas 3.3 and 7.3. When $j = 2$, the sum of (7.6) is equal to

$$\frac{d_2^2}{\theta(2,2,2,\bullet)^2}\Big\langle$$  $$\Big\rangle + \frac{d_2^2}{\theta(2,2,2,\bullet)\,\theta(2,2,2,\circ)}\Big\langle$$  $$\Big\rangle$$

$$+ \frac{d_2^2}{\theta(2,2,2,\bullet)\,\theta(2,2,2,\circ)}\Big\langle$$  $$\Big\rangle + \frac{d_2^2}{\theta(2,2,2,\circ)^2}\Big\langle$$  $$\Big\rangle$$

$$= \frac{[3]^2}{\big([3][4]/[2]^2\big)^2}\cdot\frac{2\,[4]}{[2]^3} + 2\cdot\frac{[3]^2}{\big([3][4]/[2]^2\big)\big([2]^2[3][4]\big)}\big(-[2][3][4]\,\omega^{-2}\big)$$
$$= \frac{2\,[2]}{[4]} - \frac{2\,[2][3]\,\omega^{-2}}{[4]},$$

by Lemmas 3.3 and 7.3. Hence,

$$Z^{E_6}\big(L(4,1)\big) = w^{-1}\Big(3 - 1 + \frac{2\,[2]}{[4]} - \frac{2\,[2][3]\,\omega^{-2}}{[4]}\Big) = \frac{3+\sqrt{-3}}{6},$$

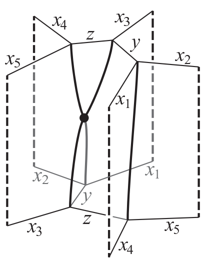
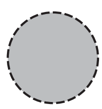
by (1.4), as required. □

Example 7.6 ([SuW])**.** The value of the E_6 state sum invariant of the lens space $L(5,2)$ is given by

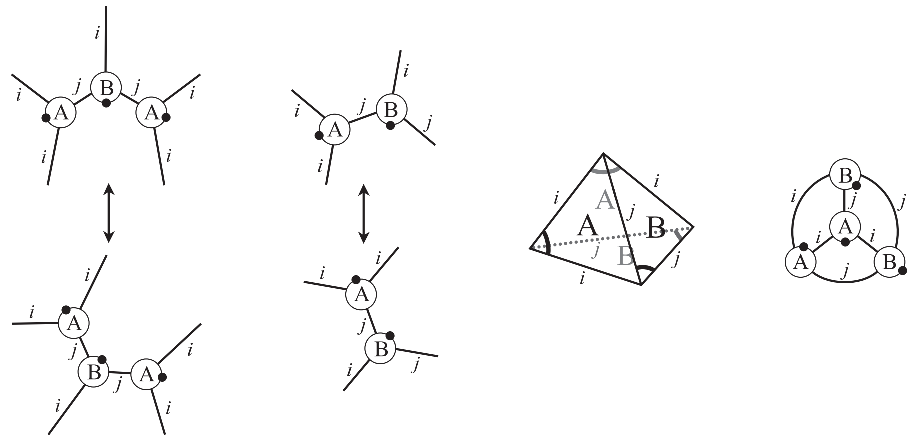
$$Z^{E_6}\big(L(5,2)\big) = \frac{3+\sqrt{3}}{12}.$$

Proof. In this proof, we calculate the required value based on our construction of the state sum invariant.

A spine of $L(5,2)$ is given as follows.

$\Bigg($ 2-polyhedron obtained by gluing  along $x_1, \cdots, x_5, y, z\Bigg) \bigcup_{\text{dashed line}}$ 

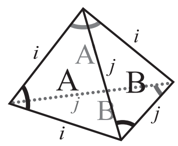
This spine has one vertex. This vertex corresponds to the fusion of the first picture below.



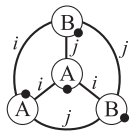
The second picture shows a part of the first picture, removing the identical part. The fusion of the second picture corresponds to the third and fourth pictures, similarly as in the case of $L(4,1)$.

Hence, we calculate the value of $Z^{E_6}(L(5,2))$, as follows,

$$Z^{E_6}\big(L(5,2)\big) \;=\; w^{-1} \sum_{\substack{i,j\in\{0,2,4\}\\ A,B\in\{\bullet,\circ\}}} d_i d_j \left|\; \right|$$

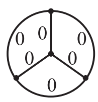


$$(7.7) \qquad = \; w^{-1} \sum_{\substack{i,j\in\{0,2,4\}\\ A,B\in\{\bullet,\circ\}}} \frac{d_i d_j}{\theta(i,i,j,A)\,\theta(i,j,j,B)} \Big\langle \quad \Big\rangle .$$



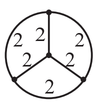
This colored planar trivalent graph has admissible colorings only if $(i,j) = (0,0), (2,2)$. When $(i,j) = (0,0)$, the sum of (7.7) is equal to

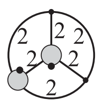
$$\frac{d_0^2}{\theta(0,0,0,\bullet)^2} \Big\langle \quad \Big\rangle \;=\; 1\,.$$



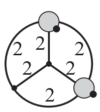
When $(i,j) = (2,2)$, the sum of (7.7) is equal to

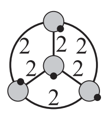
$$\frac{d_2^2}{\theta(2,2,2,\bullet)^2} \Big\langle \quad \Big\rangle + \frac{d_2^2}{\theta(2,2,2,\bullet)\,\theta(2,2,2,\circ)} \Big\langle \quad \Big\rangle$$





$$+ \frac{d_2^2}{\theta(2,2,2,\bullet)\,\theta(2,2,2,\circ)} \Big\langle \quad \Big\rangle + \frac{d_2^2}{\theta(2,2,2,\circ)^2} \Big\langle \quad \Big\rangle$$





$$= \; \frac{[3]^2}{\big([3][4]/[2]^2\big)^2} \cdot \frac{2\,[4]}{[2]^3} \;+\; \frac{[3]^2}{\big([3][4]/[2]^2\big)\big([2]^2[3][4]\big)} \big(-[2][3][4]\big)(\omega^2+\omega^{-2})$$

$$= \frac{2\,[2]}{[4]} - \frac{[2][3](\omega^2+\omega^{-2})}{[4]},$$

by Lemmas 3.3 and 7.3. Hence,

$$Z^{E_6}\big(L(5,2)\big) = w^{-1}\Big(1+\frac{2\,[2]}{[4]} - \frac{[2][3](\omega^2+\omega^{-2})}{[4]}\Big) = \frac{3+\sqrt{3}}{12},$$

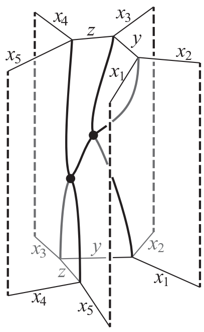
by (1.4), as required. $\square$

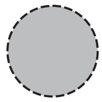
Example 7.7 ([SuW])**.** The value of the E_6 state sum invariant of the lens space $L(5,1)$ is given by

$$Z^{E_6}\big(L(5,1)\big) = \frac{3+\sqrt{3}}{12}.$$

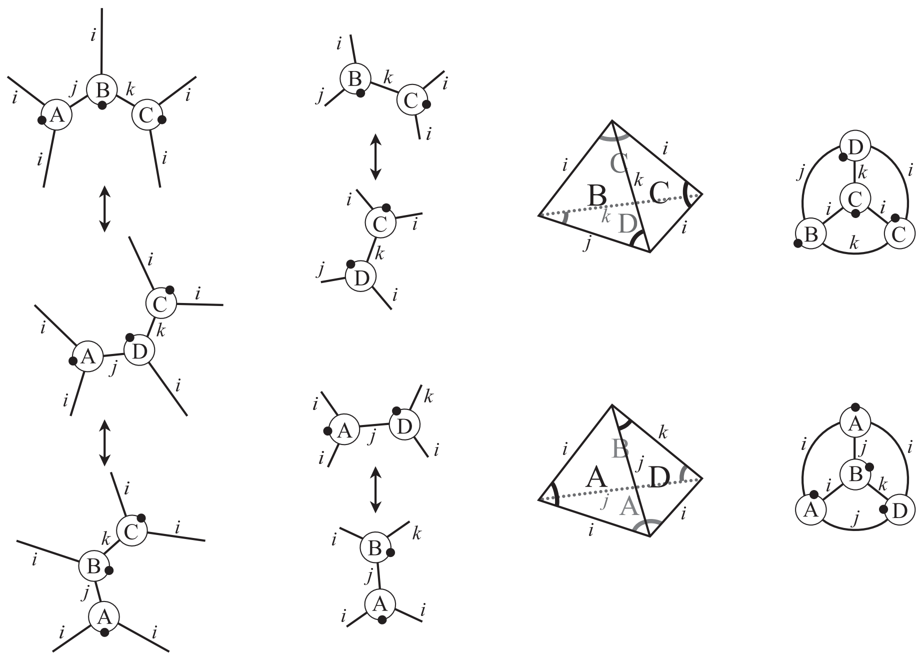
Proof. In this proof, we calculate the required value based on our construction of the state sum invariant.

A spine of $L(5,1)$ is given as follows.

(7.8) $\Bigg($ 2-polyhedron obtained by gluing 

along $x_1, \cdots, x_5, y, z$ $\Bigg) \bigcup_{\text{dashed line}}$ 

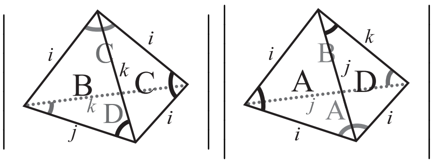
This spine has two vertices. These vertices correspond to the fusions of the first picture below.

(7.9) 

The second column shows parts of the fusions, removing the identical parts. The fusions of the second column correspond to the pictures of the third and fourth columns, similarly as in the cases of $L(4,1)$ and $L(5,2)$.

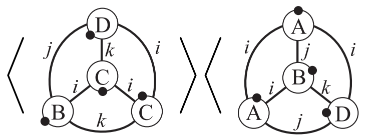
Hence, we calculate the value of $Z^{E_6}(L(5,1))$, as follows,

$$Z^{E_6}\big(L(5,1)\big) \;=\; w^{-1} \sum_{\substack{i,j,k\in\{0,2,4\}\\ A,B,C,D\in\{\bullet,\circ\}}} d_i d_j d_k$$



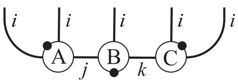
(7.10)

$$= \; w^{-1} \sum_{\substack{i,j,k\in\{0,2,4\}\\ A,B,C,D\in\{\bullet,\circ\}}} \frac{d_i d_j d_k}{\theta(i,i,j,A)\,\theta(i,j,k,B)\,\theta(i,i,k,C)\,\theta(i,j,k,D)}$$

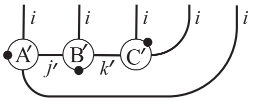


In order to calculate this sum, we consider a relation between the upper left graph and the lower left graph of (7.9). Since the space $H(i,i,i,i,i)$ has a basis given in Proposition 4.9, we can put

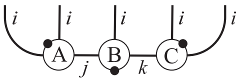
(7.11)



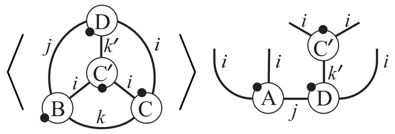
$$= \sum_{\substack{j',k'\in\{0,2,4\}\\ A',B',C'\in\{\bullet,\circ\}}} \Phi^{j'k'}_{A'B'C'}$$

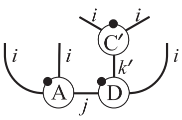


with some scalars $\Phi^{j'k'}_{A'B'C'}$. Further, by Proposition 4.10, we have that

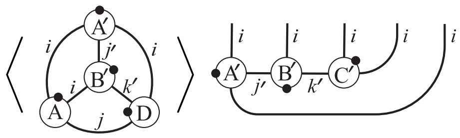


$$= \sum_{\substack{k'\in\{0,2,4\}\\ C',D\in\{\bullet,\circ\}}} \frac{d_{k'}}{\theta(i,i,k',C')\,\theta(i,j,k',D)}$$



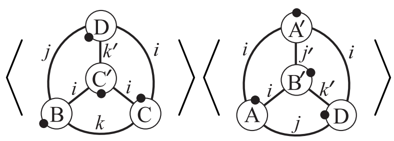


$$= \sum_{\substack{j'\in\{0,2,4\}\\ A',B'\in\{\bullet,\circ\}}} \frac{d_{j'}}{\theta(i,i,j',A')\,\theta(i,j',k',B')}$$

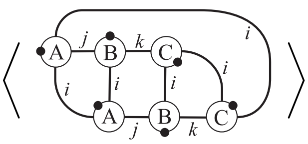


Hence, by comparing the above two formulas to (7.11), we have that

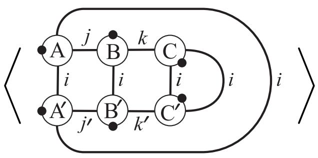
$$\Phi^{j'k'}_{A'B'C'} \;=\; \sum_{D\in\{\bullet,\circ\}} \frac{d_{j'} d_{k'}}{\theta(i,i,j',A')\,\theta(i,j',k',B')\,\theta(i,i,k',C')\,\theta(i,j,k',D)}$$



Further, by closing the diagrams of (7.11) with a certain diagram, we have that

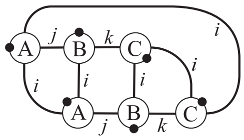
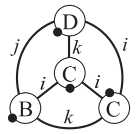
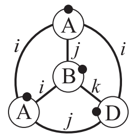


$$= \sum_{\substack{j',k'\in\{0,2,4\}\\ A',B',C'\in\{\bullet,\circ\}}} \Phi^{j'k'}_{A'B'C'}$$

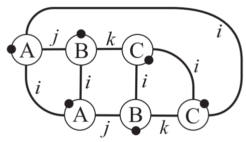


$$= \frac{\theta(i,i,j,A)\,\theta(i,j,k,B)\,\theta(i,i,k,C)}{d_j d_k}\,\Phi^{jk}_{ABC}\,,$$

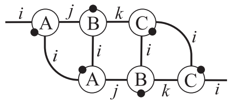
where we obtain the second equality in a similar way as in the proof of Proposition 4.9. Therefore, by presenting Φ^{jk}_{ABC} from the above two formulas in two ways, we have that

$$\Big\langle$$  $$\Big\rangle = \sum_{D\in\{\bullet,\circ\}} \frac{1}{\theta(i,j,k',D)} \Big\langle$$  $$\Big\rangle\Big\langle$$  $$\Big\rangle.$$

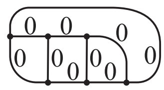
By substituting this formula to (7.10), we have that
(7.12)

$$Z^{E_6}\big(L(5,1)\big) = w^{-1} \sum_{\substack{i,j,k\in\{0,2,4\}\\ A,B,C\in\{\bullet,\circ\}}} \frac{d_i d_j d_k}{\theta(i,i,j,A)\,\theta(i,j,k,B)\,\theta(i,i,k,C)} \Big\langle$$  $$\Big\rangle.$$

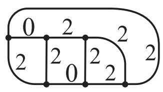
We note that the diagram of this formula is the union the upper left graph of (7.9) and the mirror image of the lower left graph of (7.9), which is equal to the boundary of the left picture of (7.8).

The coloring of the colored planar trivalent graph in (7.12) is admissible only if $(i,j,k) = (0,0,0), (2,0,2), (2,2,0), (2,2,2), (2,2,4), (2,4,2)$. We note that this graph is symmetric with respect to π rotation of  . Hence, it is sufficient to calculate the cases where $(i,j,k) = (0,0,0), (2,0,2), (2,2,2), (2,2,4)$.

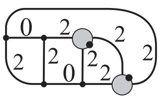
When $(i,j,k) = (0,0,0)$, the sum of (7.12) is equal to

$$\frac{d_0^3}{\theta(0,0,0,\bullet)^3} \Big\langle$$  $$\Big\rangle = 1\,.$$

When $(i,j,k) = (2,0,2)$, the sum of (7.12) is equal to the sum of the following two formulas,

$$\frac{d_0 d_2^2}{\theta(0,2,2,\bullet)^2\,\theta(2,2,2,\bullet)} \Big\langle$$  $$\Big\rangle$$

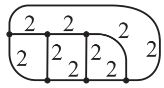
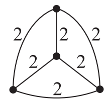
$$= \frac{d_0 d_2^2}{\theta(0,2,2,\bullet)^2\,\theta(2,2,2,\bullet)} \cdot \theta(2,2,2,\bullet) = \frac{d_2^2}{\theta(0,2,2,\bullet)^2} = 1\,,$$

$$\frac{d_0 d_2^2}{\theta(0,2,2,\bullet)^2\,\theta(2,2,2,\circ)^2} \Big\langle$$  $$\Big\rangle$$

$$= \frac{d_0 d_2^2}{\theta(0,2,2,\bullet)^2\,\theta(2,2,2,\circ)} \cdot \omega^{-2}\theta(2,2,2,\circ) = \frac{d_2^2}{\theta(0,2,2,\bullet)^2} \cdot \omega^{-2} = \omega\,,$$

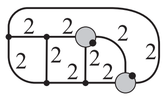
by using Lemma 3.3.

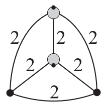
When $(i,j,k) = (2,2,2)$, by the above mentioned symmetry, it is sufficient to calculate the cases where $(A,B,C) = (\bullet,\bullet,\bullet), (\bullet,\bullet,\circ), (\bullet,\circ,\bullet), (\bullet,\circ,\circ), (\circ,\bullet,\circ), (\circ,\circ,\circ)$. When $(A,B,C) = (\bullet,\bullet,\bullet)$, the sum of (7.12) is equal to

$$\frac{d_2^3}{\theta(2,2,2,\bullet)^3}\Big\langle$$  $$\Big\rangle = \Big(\frac{[3]}{[3][4]/[2]^2}\Big)^3 \cdot \frac{2}{[2][3]}\Big\langle$$  $$\Big\rangle$$

$$= \Big(\frac{[2]^2}{[4]}\Big)^3 \cdot \frac{2}{[2][3]} \cdot \frac{2\,[4]}{[2]^3} = \frac{4\,[2]^2}{[3][4]^2},$$

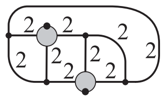
by Lemmas 7.2, 3.3 and 7.3. When $(A,B,C) = (\bullet,\bullet,\circ)$, the sum of (7.12) is equal to

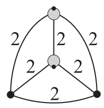
$$\frac{d_2^3}{\theta(2,2,2,\bullet)^2\,\theta(2,2,2,\circ)}\Big\langle$$  $$\Big\rangle$$

$$= \frac{[3]^3}{([3][4]/[2]^2)^2 \cdot [2]^2[3][4]} \cdot \frac{2\,\omega^4}{[2][3]}\Big\langle$$  $$\Big\rangle$$

$$= \frac{[2]^2}{[4]^3} \cdot \frac{2\,\omega^4}{[2][3]}\,(-[2][3][4]) = -\frac{2\,\omega\,[2]^2}{[4]^2},$$

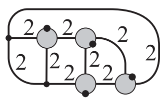
by Lemmas 7.2, 3.3 and 7.3. When $(A,B,C) = (\bullet,\circ,\bullet)$, the sum of (7.12) is equal to

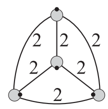
$$\frac{d_2^3}{\theta(2,2,2,\bullet)^2\,\theta(2,2,2,\circ)}\Big\langle$$  $$\Big\rangle$$

$$= \frac{[3]^3}{([3][4]/[2]^2)^2 \cdot [2]^2[3][4]}\Big(-\frac{\omega^{-2}}{[2]}\Big)\Big\langle$$  $$\Big\rangle$$

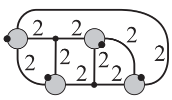
$$= \frac{[2]^2}{[4]^3} \cdot \Big(-\frac{\omega^{-2}}{[2]}\Big)(-[2][3][4]) = \frac{\omega\,[2]^2[3]}{[4]^2},$$

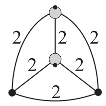
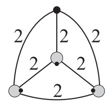
by Lemmas 7.2, 3.3 and 7.3. When $(A,B,C) = (\bullet,\circ,\circ)$, the sum of (7.12) is equal to

$$\frac{d_2^3}{\theta(2,2,2,\bullet)\,\theta(2,2,2,\circ)^2}\Big\langle$$  $$\Big\rangle$$

$$= \frac{[3]^3}{([3][4]/[2]^2) \cdot ([2]^2[3][4])^2}\Big(-\frac{1}{[2]}\Big)\Big\langle$$  $$\Big\rangle = 0,$$

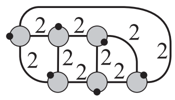
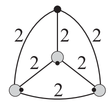
by Lemmas 7.2, 3.3 and 7.3. When $(A,B,C) = (\circ,\bullet,\circ)$, the sum of (7.12) is equal to

$$\frac{d_2^3}{\theta(2,2,2,\bullet)\,\theta(2,2,2,\circ)^2}\Big\langle$$  $$\Big\rangle$$

$$= \frac{[3]^3}{([3][4]/[2]^2)\cdot([2]^2[3][4])^2}\cdot\Big(-[2]^3\,\omega^{-4}\Big\langle$$  $$\Big\rangle + \omega^{-2}\Big\langle$$  $$\Big\rangle\Big)$$

$$= \frac{1}{[2]^2[4]^3}\cdot\big(-[2]^3\,\omega^{-4}(-[2][3][4]) + \omega^{-2}\cdot[2]^2[3][4]\big) = \frac{\omega^{-1}\,[2]^2[3]}{[4]^2} + \frac{\omega\,[3]}{[4]^2},$$

by Lemmas 7.4, 3.3 and 7.3. When $(A,B,C) = (\circ, \circ, \circ)$, the sum of (7.12) is equal to

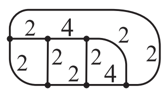
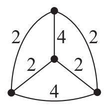
$$\frac{d_2^3}{\theta(2,2,2,\circ)^3}\Big\langle$$  $$\Big\rangle = \Big(\frac{[3]}{[2]^2[3][4]}\Big)^3\cdot\omega^{-2}\,[2]^4\Big\langle$$  $$\Big\rangle$$

$$= \frac{1}{[2]^6[4]^3}\cdot\omega^{-2}\,[2]^4\cdot[2]^2[3][4] = \frac{\omega\,[3]}{[4]^2},$$

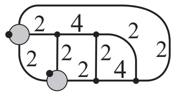
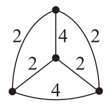
by Lemmas 7.4, 3.3 and 7.3. Hence, the sum of (7.12) for $(i,j,k) = (2,2,2)$ is equal to

$$\frac{4\,[2]^2}{[3][4]^2} - 2\cdot\frac{2\,\omega\,[2]^2}{[4]^2} + \frac{\omega\,[2]^2[3]}{[4]^2} + \Big(\frac{\omega^{-1}\,[2]^2[3]}{[4]^2} + \frac{\omega\,[3]}{[4]^2}\Big) + \frac{\omega\,[3]}{[4]^2}$$
$$= \frac{4\,[2]^2}{[3][4]^2} - \frac{4\,\omega\,[2]^2}{[4]^2} - \frac{[2]^2[3]}{[4]^2} + \frac{2\,\omega\,[3]}{[4]^2}.$$

When $(i,j,k) = (2,2,4)$, the sum of (7.12) is equal to the sum of the following two formulas,

$$\frac{d_2^2 d_4}{\theta(2,2,2,\bullet)\,\theta(2,2,4,\bullet)^4}\Big\langle$$  $$\Big\rangle = \frac{[3]^2}{([3][4]/[2]^2)\cdot 1^2}\Big(-\frac{1}{[2]}\Big)\Big\langle$$  $$\Big\rangle$$

$$= \frac{[2]^2[3]}{[4]}\Big(-\frac{1}{[2]}\Big)\Big(-\frac{1}{[3]}\Big) = \frac{[2]}{[4]},$$

$$\frac{d_2^2 d_4}{\theta(2,2,2,\circ)\,\theta(2,2,4,\bullet)^2}\Big\langle$$  $$\Big\rangle = \frac{[3]^2}{[2]^2[3][4]\cdot 1^2}\cdot\omega^{-2}\,[2]^3\Big\langle$$  $$\Big\rangle$$

$$= \frac{[3]}{[2]^2[4]}\cdot\omega^{-2}\,[2]^3\Big(-\frac{1}{[3]}\Big) = -\frac{\omega\,[2]}{[4]},$$

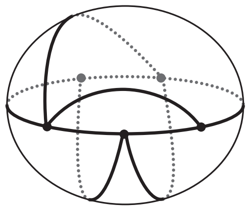
by Lemmas 7.2, 7.4, 3.3 and 7.3.

Therefore, from (7.12), we obtain that

$$Z^{E_6}\big(L(5,1)\big)$$
$$= w^{-1}\Big(1 + 2\big(1+\omega\big) + \Big(\frac{4\,[2]^2}{[3][4]^2} - \frac{4\,\omega\,[2]^2}{[4]^2} - \frac{[2]^2[3]}{[4]^2} + \frac{2\,\omega\,[3]}{[4]^2}\Big) + 2\Big(\frac{[2]}{[4]} - \frac{\omega\,[2]}{[4]}\Big)\Big)$$
$$= w^{-1}\Big(3 + \frac{4\,[2]^2}{[3][4]^2} - \frac{[2]^2[3]}{[4]^2} + \frac{2[2]}{[4]} + \frac{2\,\omega}{[4]^2}\big([4]^2 - 2[2]^2 + [3] - [2][4]\big)\Big)$$
$$= w^{-1}\Big(3 + \frac{4\,[2]^2}{[3][4]^2} - \frac{[2]^2[3]}{[4]^2} + \frac{2[2]}{[4]} + \frac{2\,\omega}{[4]^2}\cdot 0\Big) = \frac{3+\sqrt{3}}{12},$$

by (1.4), as required. □

We note that the graph in (7.12) is dual to the following (singular) triangulation of the 2-sphere,



and we obtain $L(5,1)$ from a 3-ball by gluing faces of this triangulation to each other. Further, like (7.12), the value of our state sum invariant for any lens space can be presented by using such a graph.

In general, any closed oriented 3-manifold M can be obtained from a triangulated 3-ball by gluing faces of the boundary 2-sphere to each other. The value of our state sum invariant of M can be presented by using the dual graph of such a triangulation of the 2-sphere; see [KL] for a similar statement for the Turaev–Viro invariant.

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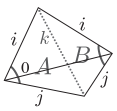
E-mail address: `junes@kurims.kyoto-u.ac.jp`

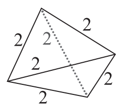
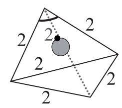
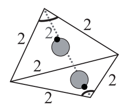
Appendix A. The values of the weights

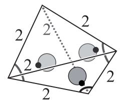
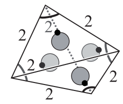
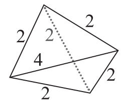
From Lemmas 3.3 and 7.3, we obtain the following table of the weights.

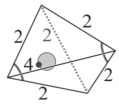
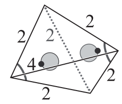
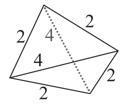
Proposition A.1. *The weights of colored tetrahedra are given as follows, where we omit to draw the face color* $\bullet$.



$= \delta_{AB} \cdot \dfrac{1}{\sqrt{d_i d_j}} \qquad (i, j, k \in \{0, 2, 4\},\ A, B \in \{\bullet, \odot\}),$

 $= \dfrac{1}{[2][4]}$,  $= 0$,  $= -\dfrac{[2]}{[3][4]}$,

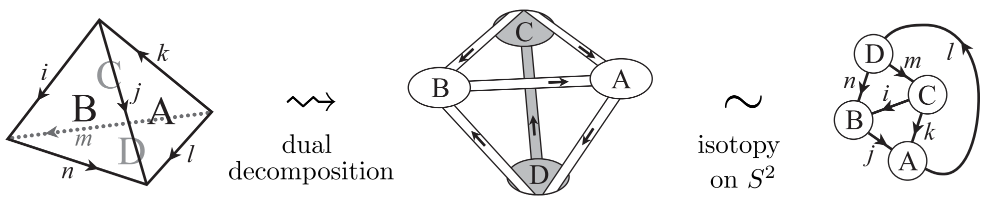
 $= \dfrac{1}{[3][4]}$,  $= 0$,  $= -\dfrac{[2]}{[3][4]}$,

 $= -\dfrac{1}{[4]}$,  $= \dfrac{[2]}{[3][4]}$,  $= -\dfrac{1}{[3]}$.

Appendix B. Equivalence to the E_6 $6j$-symbols

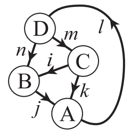
In this section, we review the values of the $6j$-symbols of the E_6 subfactor given in [SuW, W]. Further, we review that our $6j$-symbols can be transformed into the $6j$-symbols of the E_6 subfactor in Lemma B.1 (due to T. Ohtsuki).

Similarly as in Section 5, we relate an oriented tetrahedron with oriented edges to a planar trivalent graph, as follows.



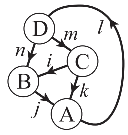
We review the values of the $6j$-symbols of the E_6 subfactor given in [SuW, W], as the weight of the tetrahedron, in terms of the above planar graph. When none of

i, j, k, l, m, n is equal to 2,



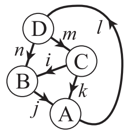
$$= 1\,.$$

When one or two of i, j, k, l, m, n are equal to 2, there are no admissible colorings of the tetrahedron. When three of i, j, k, l, m, n are equal to 2,



$$= \frac{1}{[3]}\,.$$

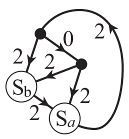
When four of i, j, k, l, m, n are equal to 2,



$$= \begin{cases} \dfrac{1}{[3]} & \text{if the remaining two are equal to 4,} \\ -\dfrac{1}{[3]} & \text{otherwise.} \end{cases}$$

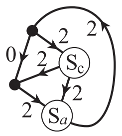
When five of i, j, k, l, m, n are equal to 2,

(B.1)



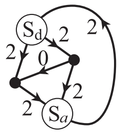
$$= \begin{cases} \dfrac{q^3}{[3]} & \text{if } (a,b) = (3,3), \\ \dfrac{q^{-3}}{[3]} & \text{if } (a,b) = (4,4), \\ 0 & \text{if } (a,b) = (3,4),\ (4,3), \end{cases}$$

(B.2)



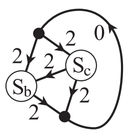
$$= \begin{cases} \dfrac{1}{[3]} & \text{if } (a,c) = (3,3),\ (4,4), \\ 0 & \text{if } (a,c) = (3,4),\ (4,3), \end{cases}$$

(B.3)



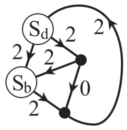
$$= \begin{cases} \dfrac{1}{[3]} & \text{if } (a,d) = (3,3),\ (4,4), \\ 0 & \text{if } (a,d) = (3,4),\ (4,3), \end{cases}$$

(B.4)

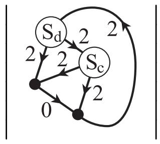


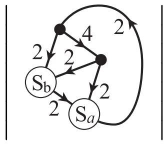
$$= \begin{cases} \dfrac{q^7}{\sqrt{2}\,[3]} & \text{if } (b,c) = (3,3),\ (4,3), \\ \dfrac{q}{\sqrt{2}\,[3]} & \text{if } (b,c) = (4,4), \\ -\dfrac{q}{\sqrt{2}\,[3]} & \text{if } (b,c) = (3,4), \end{cases}$$

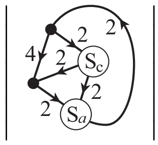
(B.5)

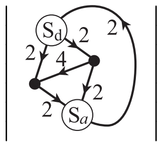


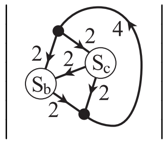
$$= \begin{cases} \dfrac{1}{[3]} & \text{if } (b,d) = (3,3),\ (4,4), \\ 0 & \text{if } (b,d) = (3,4),\ (4,3), \end{cases}$$

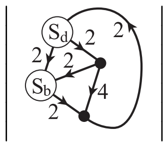
(B.6) 
$$= \begin{cases} -\dfrac{q^2}{\sqrt{2}\,[3]} & \text{if } (c,d) = (3,3),\ (3,4),\ (4,3), \\ \dfrac{q^2}{\sqrt{2}\,[3]} & \text{if } (c,d) = (4,4), \end{cases}$$

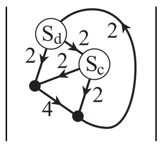
(B.7) 
$$= \begin{cases} 0 & \text{if } (a,b) = (3,3),\ (4,4), \\ \dfrac{q^3}{[3]} & \text{if } (a,b) = (3,4), \\ \dfrac{q^{-3}}{[3]} & \text{if } (a,b) = (4,3), \end{cases}$$

(B.8) 
$$= \begin{cases} \dfrac{1}{[3]} & \text{if } (a,c) = (3,3), \\ -\dfrac{1}{[3]} & \text{if } (a,c) = (4,4), \\ 0 & \text{if } (a,c) = (3,4),\ (4,3), \end{cases}$$

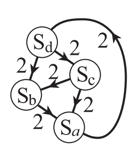
(B.9) 
$$= \begin{cases} \dfrac{1}{[3]} & \text{if } (a,d) = (3,4),\ (4,3), \\ 0 & \text{if } (a,d) = (3,3),\ (4,4), \end{cases}$$

(B.10) 
$$= \begin{cases} \dfrac{q^7}{\sqrt{2}\,[3]} & \text{if } (b,c) = (3,3), \\ -\dfrac{q^7}{\sqrt{2}\,[3]} & \text{if } (b,c) = (4,3), \\ -\dfrac{q}{\sqrt{2}\,[3]} & \text{if } (b,c) = (3,4),\ (4,4), \end{cases}$$

(B.11) 
$$= \begin{cases} \dfrac{q^6}{[3]} & \text{if } (b,d) = (3,4), \\ -\dfrac{q^6}{[3]} & \text{if } (b,d) = (4,3), \\ 0 & \text{if } (b,d) = (3,3),\ (4,4), \end{cases}$$

(B.12) 
$$= \begin{cases} -\dfrac{q^2}{\sqrt{2}\,[3]} & \text{if } (c,d) = (3,3),\ (4,3),\ (4,4), \\ \dfrac{q^2}{\sqrt{2}\,[3]} & \text{if } (c,d) = (3,4), \end{cases}$$

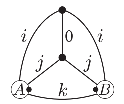
When all of i, j, k, l, m, n are equal to 2,

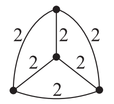
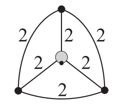
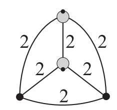
(B.13) $\left|\right.$  $\left.\right|$

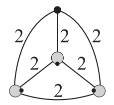
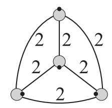
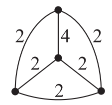
$$= \begin{cases} -\dfrac{1}{[3]^2} & \text{if } (a,b,c,d) = (3,3,3,3),\ (3,4,3,4), \\ \dfrac{q^3}{\sqrt{2}\,[3]} & \text{if } (a,b,c,d) = (3,3,4,3), \\ -\dfrac{q^3}{\sqrt{2}\,[3]} & \text{if } (a,b,c,d) = (3,4,4,4), \\ \dfrac{q^{-3}}{\sqrt{2}\,[3]} & \text{if } (a,b,c,d) = (4,3,3,4),\ (4,4,3,3), \\ \dfrac{q^{-6}}{[3]^2} & \text{if } (a,b,c,d) = (4,3,4,4), \\ \dfrac{q^{6}}{[3]^2} & \text{if } (a,b,c,d) = (4,4,4,3), \\ 0 & \text{otherwise.} \end{cases}$$

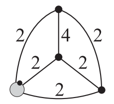
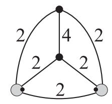
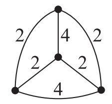
We rewrite the list of Proposition A.1, in terms of the dual planar graph, as follows,

(B.14)

$\left|\right.$  $\left.\right| = \delta_{AB} \cdot \dfrac{1}{\sqrt{d_i d_j}} \quad (i, j, k \in \{0, 2, 4\},\ A, B \in \{\bullet, \circ\}),$

$\left|\right.$  $\left.\right| = \dfrac{1}{[2][4]}$, $\left|\right.$  $\left.\right| = 0$, $\left|\right.$  $\left.\right| = -\dfrac{[2]}{[3][4]}$,

$\left|\right.$  $\left.\right| = \dfrac{1}{[3][4]}$, $\left|\right.$  $\left.\right| = 0$, $\left|\right.$  $\left.\right| = -\dfrac{[2]}{[3][4]}$,

$\left|\right.$  $\left.\right| = -\dfrac{1}{[4]}$, $\left|\right.$  $\left.\right| = \dfrac{[2]}{[3][4]}$, $\left|\right.$  $\left.\right| = -\dfrac{1}{[3]}$.

We review a proof of the following lemma, which was shown by T. Ohtsuki.

Lemma B.1 (T. Ohtsuki)**.** *The $6j$-symbols given in Section 5 can be transformed into the $6j$-symbols of the E_6 subfactor.*

Proof. We put

$$u_3 = \exp\Big(\frac{\pi\sqrt{-1}}{8}\Big) \cdot \sqrt{\frac{3-\sqrt{3}}{6}}, \qquad u_4 = -\exp\Big(-\frac{\pi\sqrt{-1}}{8}\Big) \cdot \sqrt{\frac{3+\sqrt{3}}{6}},$$

$$v_3 = \exp\Big(-\frac{5\pi\sqrt{-1}}{24}\Big) \cdot \sqrt{\frac{3+\sqrt{3}}{6}}, \quad v_4 = \exp\Big(-\frac{11\pi\sqrt{-1}}{24}\Big) \cdot \sqrt{\frac{3-\sqrt{3}}{6}}.$$

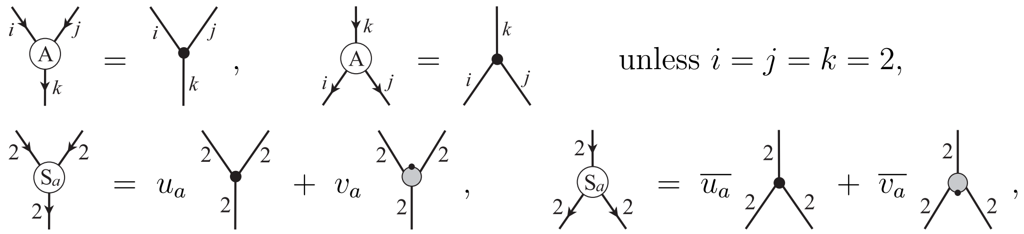
Then,

$$
\begin{aligned}
u_3\overline{u_3} &= \frac{3-\sqrt{3}}{6}, \quad & u_4\overline{u_4} &= \frac{3+\sqrt{3}}{6}, \quad & u_3\overline{v_3} &= \frac{q^4}{\sqrt{6}}, \\
v_3\overline{v_3} &= \frac{3+\sqrt{3}}{6}, \quad & v_4\overline{v_4} &= \frac{3-\sqrt{3}}{6}, \quad & u_4\overline{v_4} &= -\frac{q^4}{\sqrt{6}},
\end{aligned}
$$

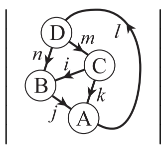
$$u_3\overline{u_3} + u_4\overline{u_4} = 1, \quad v_3\overline{v_3} + v_4\overline{v_4} = 1, \quad u_3\overline{v_3} + u_4\overline{v_4} = 0.$$

Hence, $\begin{pmatrix} u_3 & v_3 \\ u_4 & v_4 \end{pmatrix}$ is a unitary matrix.

By putting

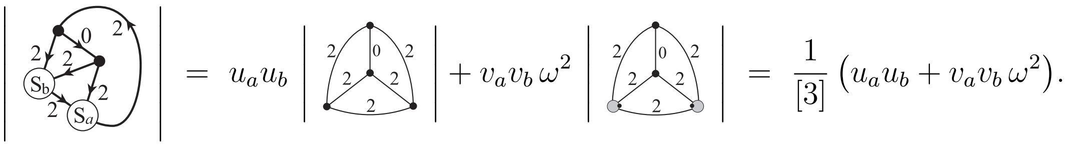


we calculate



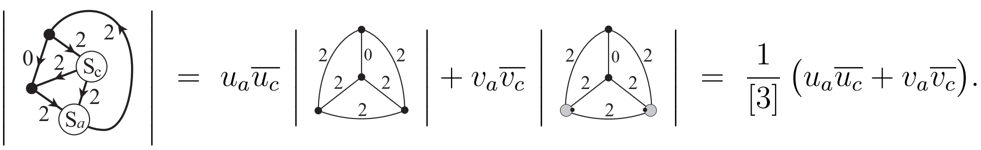
by using (B.14), and verify that it is equal to the above mentioned value. When all vertices are colored by $\bullet$, it is easy to check the proof. Hence, we consider the case where there are vertices colored by S_3, S_4, *i.e.*, we verify the values of (B.1)–(B.13) in the following of this proof.

As for (B.1),



Hence, we can verify (B.1) by concrete calculation for each (a, b).

As for (B.2),

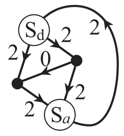
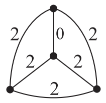
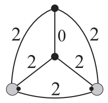


Hence, we can verify (B.2) for each (a, c), since

$$u_3\overline{u_3} + v_3\overline{v_3} = 1, \quad u_4\overline{u_4} + v_4\overline{v_4} = 1, \quad u_3\overline{u_4} + v_3\overline{v_4} = 0,$$

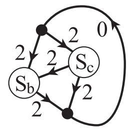
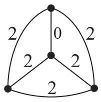
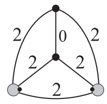
which can be checked concretely.

As for (B.3),

$\Bigg|$  $\Bigg| = u_a\overline{u_d}\,\Bigg|$  $\Bigg| + v_a\overline{v_d}\,\Bigg|$  $\Bigg| = \dfrac{1}{[3]}\big(u_a\overline{u_d}+v_a\overline{v_d}\big),$

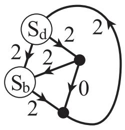
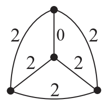
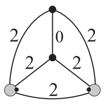
and we can verify (B.3) in the same way as the case of (B.2).

As for (B.4),

$\Bigg|$  $\Bigg| = u_b\overline{u_c}\,\Bigg|$  $\Bigg| + v_b\overline{v_c}\,\omega^2\,\Bigg|$  $\Bigg| = \dfrac{1}{[3]}\big(u_b\overline{u_c}+v_b\overline{v_c}\,\omega^2\big).$

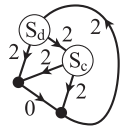
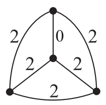
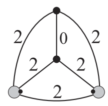
Hence, we can verify (B.4) by concrete calculation for each (b,c).

As for (B.5),

$\Bigg|$  $\Bigg| = u_b\overline{u_d}\,\Bigg|$  $\Bigg| + v_b\overline{v_d}\,\Bigg|$  $\Bigg| = \dfrac{1}{[3]}\big(u_b\overline{u_d}+v_b\overline{v_d}\big),$

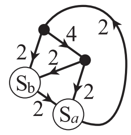
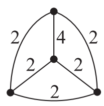
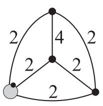
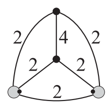
and we can verify (B.5) in the same way as the case of (B.2).

As for (B.6),

$\Bigg|$  $\Bigg| = \overline{u_c u_d}\,\Bigg|$  $\Bigg| + \overline{v_c v_d}\,\omega^2\,\Bigg|$  $\Bigg| = \dfrac{1}{[3]}\big(\overline{u_c u_d}+\overline{v_c v_d}\,\omega^2\big).$

Hence, we can verify (B.6) by concrete calculation for each (c,d).

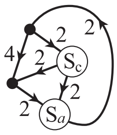
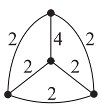
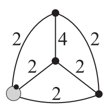
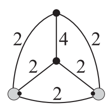
As for (B.7),

$\Bigg|$  $\Bigg| = u_a u_b\,\Bigg|$  $\Bigg| + \big(v_a u_b\,\omega^2 + u_a v_b\big)\,\Bigg|$  $\Bigg| + v_a v_b\,\omega^2\,\Bigg|$  $\Bigg|$

$$= -\frac{[2]}{[3][4]}\,u_a u_b - \frac{1}{[4]}\big(v_a u_b\,\omega^2 + u_a v_b\big) + \frac{[2]}{[3][4]}\,v_a v_b\,\omega^2.$$

Hence, we can verify (B.7) by concrete calculation for each (a,b).

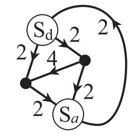
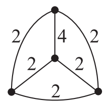
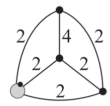
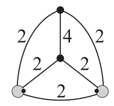
As for (B.8),

$\Bigg|$  $\Bigg| = u_a\overline{u_c}\,\Bigg|$  $\Bigg| + \big(v_a\overline{u_c}\,\omega^{-2} + u_a\overline{v_c}\,\omega^2\big)\,\Bigg|$  $\Bigg| + v_a\overline{v_c}\,\Bigg|$  $\Bigg|$

$$= -\frac{[2]}{[3][4]}\,u_a\overline{u_c} - \frac{1}{[4]}\big(v_a\overline{u_c}\,\omega^{-2} + u_a\overline{v_c}\,\omega^2\big) + \frac{[2]}{[3][4]}\,v_a\overline{v_c}\,.$$

Hence, we can verify (B.8) by concrete calculation for each (a,c).

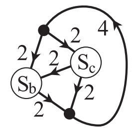
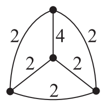
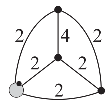
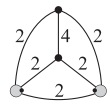
As for (B.9),

$\Bigg|$  $\Bigg| = u_a\overline{u_d}\Bigg|$  $\Bigg| + \left(v_a\overline{u_d} + u_a\overline{v_d}\right)\Bigg|$  $\Bigg| + v_a\overline{v_d}\Bigg|$  $\Bigg|$

$$= -\frac{[2]}{[3][4]}\, u_a\overline{u_d} - \frac{1}{[4]}\left(v_a\overline{u_d} + u_a\overline{v_d}\right) + \frac{[2]}{[3][4]}\, v_a\overline{v_d}\,.$$

Hence, we can verify (B.9) by concrete calculation for each (a, d).

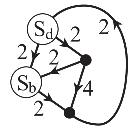
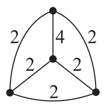
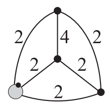
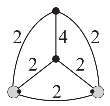
As for (B.10),

$\Bigg|$  $\Bigg| = u_b\overline{u_c}\Bigg|$  $\Bigg| + \left(v_b\overline{u_c}\,\omega^{-2} + u_b\overline{v_c}\,\omega^{-2}\right)\Bigg|$  $\Bigg| + v_b\overline{v_c}\,\omega^2\Bigg|$  $\Bigg|$

$$= -\frac{[2]}{[3][4]}\, u_b\overline{u_c} - \frac{1}{[4]}\left(v_b\overline{u_c}\,\omega^{-2} + u_b\overline{v_c}\,\omega^{-2}\right) + \frac{[2]}{[3][4]}\, v_b\overline{v_c}\,\omega^2.$$

Hence, we can verify (B.10) by concrete calculation for each (b, c).

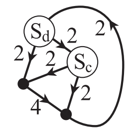
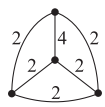
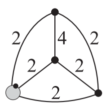
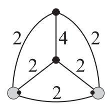
As for (B.11),

$\Bigg|$  $\Bigg| = u_b\overline{u_d}\Bigg|$  $\Bigg| + \left(v_b\overline{u_d}\,\omega^{2} + u_b\overline{v_d}\,\omega^{-2}\right)\Bigg|$  $\Bigg| + v_b\overline{v_d}\Bigg|$  $\Bigg|$

$$= -\frac{[2]}{[3][4]}\, u_b\overline{u_d} - \frac{1}{[4]}\left(v_b\overline{u_d}\,\omega^{2} + u_b\overline{v_d}\,\omega^{-2}\right) + \frac{[2]}{[3][4]}\, v_b\overline{v_d}\,.$$

Hence, we can verify (B.11) by concrete calculation for each (b, d).

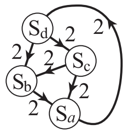
As for (B.12),

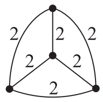
$\Bigg|$  $\Bigg| = \overline{u_c u_d}\Bigg|$  $\Bigg| + \left(\overline{v_c u_d} + \overline{u_c v_d}\,\omega^{2}\right)\Bigg|$  $\Bigg| + \overline{v_c v_d}\,\omega^2\Bigg|$  $\Bigg|$

$$= -\frac{[2]}{[3][4]}\, \overline{u_c u_d} - \frac{1}{[4]}\left(\overline{v_c u_d} + \overline{u_c v_d}\,\omega^{2}\right) + \frac{[2]}{[3][4]}\, \overline{v_c v_d}\,\omega^2.$$

Hence, we can verify (B.12) by concrete calculation for each (c, d).

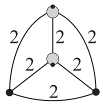
As for (B.13),

$\left|\right.$ 

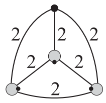
$\left.\right| = u_a u_b \overline{u_c u_d} \left|\right.$ 

$\left.\right|$

$$+ \Big(v_a v_b \overline{u_c u_d}\,\omega^2 + v_a u_b \overline{v_c u_d} + v_a u_b \overline{u_c v_d}$$

$$+ u_a v_b \overline{v_c u_d}\,\omega^2 + u_a v_b \overline{u_c v_d} + u_a u_b \overline{v_c v_d}\,\omega^2 \Big) \left|\right.$$ 

$\left.\right|$

$$+ \Big(v_a v_b \overline{v_c u_d}\,\omega^2 + v_a v_b \overline{u_c v_d}\,\omega^{-2} + v_a u_b \overline{v_c v_d}\,\omega^{-2} + u_a v_b \overline{v_c v_d}\,\omega^2 \Big) \left|\right.$$ 

$\left.\right|$

$$= \frac{1}{[2][4]}\, u_a u_b \overline{u_c u_d}$$

$$- \frac{[2]}{[3][4]} \Big(v_a v_b \overline{u_c u_d}\,\omega^2 + v_a u_b \overline{v_c u_d} + v_a u_b \overline{u_c v_d}$$

$$+ u_a v_b \overline{v_c u_d}\,\omega^2 + u_a v_b \overline{u_c v_d} + u_a u_b \overline{v_c v_d}\,\omega^2 \Big)$$

$$+ \frac{1}{[3][4]} \Big(v_a v_b \overline{v_c u_d}\,\omega^2 + v_a v_b \overline{u_c v_d}\,\omega^{-2} + v_a u_b \overline{v_c v_d}\,\omega^{-2} + u_a v_b \overline{v_c v_d}\,\omega^2 \Big).$$

Hence, we can verify (B.13) by concrete calculation for each (a, b, c, d), completing the proof. $\square$