RIMS-1808

An elementary proof of a non-triviality of the E_8 subfactor planar algebra

By

Kenta OKAZAKI

July 2014

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

AN ELEMENTARY PROOF OF A NON-TRIVIALITY OF THE E_8 SUBFACTOR PLANAR ALGEBRA

KENTA OKAZAKI

Abstract. In this paper, we show in a combinatorial way that the E_8 subfactor planar algebra is 1-dimensional. In the proof, we improve the Bigelow's relation of the E_8 subfactor planar algebra and give an efficient algorithm to reduce any planar diagram to the empty diagram.

1. Introduction

As a graphical approach to subfactors, Jones [2] introduced planar algebras, which are a kind of algebras given graphically in the plane. As the Kuperberg program says (see Morrison, Peters and Snyder [5]), it is a problem to

(i) give a presentation by generators and relations for each planar algebra, and

(ii) show basic properties of the planar algebra based on such a presentation,

where basic properties mean non-triviality, being positive, being spherical, being semisimple, etc. For the D_{2n} planar algebra, (i) and (ii) have been done in [5]. For the E_6 and E_8 planar algebras, Bigelow [1] has done (i), and partially done (ii), though his proof of the non-triviality of the planar algebra depends on the existence of the E_6 and E_8 subfactors. The author [6] has done (i) and (ii) for the E_6 planar algebra.

In this paper, we give a combinatorial proof of a non-triviality of the E_8 planar algebra [1]. Our proof will be a refinement of the proof of [6, Proposition 2.2]. We introduce the E_8 linear skein, motivated by Bigelow's generators and relations of the E_8 planar algebra. We define the E_8 linear skein $\mathcal{S}(\mathbb{R}^2)$ of $\mathbb{R}^2$ to be the vector space spanned by certain 10-valent graphs (which we call planar diagrams) subject to certain relations (Definition 2.1). Our relations are a modification of Bigelow's relations; we show that they are equivalent in Section 3. We show that $\mathcal{S}(\mathbb{R}^2)$ is 1-dimensional (Theorem 2.2). We give a self-contained combinatorial proof of them. To show them, it is important to give an efficient algorithm to reduce any planar diagram to the empty diagram. Such a reduction is done by decreasing the number of 10-valent vertices of a planar diagram. To do this, we use the relation (2.4) (one of our relations), which can reduce two vertices connected by two parallel edges, while the corresponding relation (3.1) (one of Bigelow's relations) reduces two vertices connected by five parallel edges. In fact, to reduce planar diagrams, our relations are more efficient than Bigelow's relations, and this is a reason why we

[1] We note that Ellie Grano announced that she independently gave the proof of a non-triviality of the E_8 planar algebra by using the Jerryfish algorithm.

define the E_8 linear skein by our relations, instead of Bigelow's relations. We show (1) by decreasing the number of vertices of any planar diagram by using (2.4). To show (2), we show that the resulting value does not depend on the choice of a process of decreasing the number of vertices; we consider all such processes and show the independence on them concretely.

The paper is organized as follows. In Section 2, we introduce the E_8 linear skein $\mathcal{S}(\mathbb{R}^2)$ of $\mathbb{R}^2$, and show that $\mathcal{S}(\mathbb{R}^2)$ is 1-dimensional. In Section 3, we show that the defining relations of our E_8 linear skein are equivalent to Bigelow's relations. In Appendix A, we show some technical formulae of the E_8 linear skein.

The author would like to thank Tomotada Ohtsuki for his encouragement and advices. The author is also grateful to Ellie Grano for her helpful comments.

Notation. Throughout the paper, the scalar field for every vector space is the complex field $\mathbb{C}$. We put

$$\omega = \exp(6\pi\sqrt{-1}/5), \quad q = \exp(\pi\sqrt{-1}/30), \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$

$$[n]! = [n][n-1]\cdots[1], \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!},$$

$$[n]' = (-1)^n[n], \qquad [n]!' = [n]'[n-1]'\cdots[1]' = (-1)^{n(n+1)/2}[n]!,$$

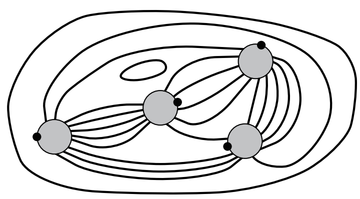
$$\begin{bmatrix} n \\ k \end{bmatrix}' = \frac{[n]!'}{[k]!'[n-k]!'} = (-1)^{(n+1)k}\begin{bmatrix} n \\ k \end{bmatrix},$$

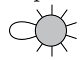
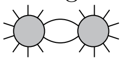
$$\alpha^{(m)} = \begin{bmatrix} m \\ 5 \end{bmatrix}^2 \Big/ \begin{bmatrix} m+6 \\ 11 \end{bmatrix}, \quad \beta^{(m)} = [2]^2[3][6]/[m+1] \qquad (5 \le m \le 8).$$

2. The E_8 linear skein

In this section, we introduce the E_8 linear skein $\mathcal{S}(\mathbb{R}^2)$ of $\mathbb{R}^2$ as a vector space spanned by certain planar graphs in Definition 2.1, and show that $\mathcal{S}(\mathbb{R}^2)$ is a 1-dimensional vector space spanned by the empty diagram in Theorem 2.2.

We define a *planar diagram* to be a 10-valent graph (possibly containing closed curves) embedded in $\mathbb{R}^2$ such that each vertex is depicted by a disk whose boundary has a base point, as shown in the following picture.

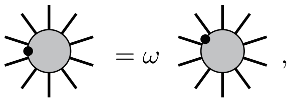


We regard isotopic planar diagrams as equivalent planar diagrams. A planar diagram is said to be *connected* if it is connected as a graph. A *cap* of a planar diagram is an edge bounding a region of the shape of a disk as shown in  . A *digon* of a planar graph is a region of the shape of a disk bounded by two edges and two vertices as shown in  .

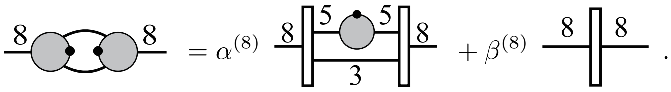
Definition 2.1. We define the E_8 *linear skein* of $\mathbb{R}^2$, denoted by $\mathcal{S}(\mathbb{R}^2)$, to be the vector space spanned by planar diagrams subject to the following local relations,

$$D \cup (\text{a closed curve}) \ = \ [2]\, D \quad \text{for any planar diagram } D, \tag{2.1}$$

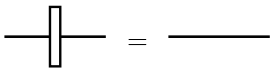
$$(\text{A planar diagram containing a cap}) = 0, \tag{2.2}$$



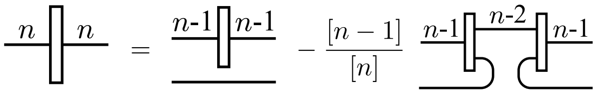
(2.3)



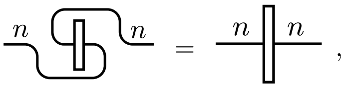
(2.4)

Here a strand attached with an integer n means n parallel strands. The white boxes, called the *Jones-Wenzl idempotents*, are inductively defined by  ,

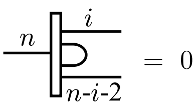
and

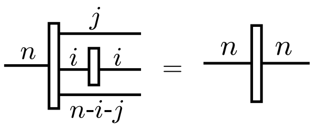
 $(2 \leq n \leq 29)$. (2.5)

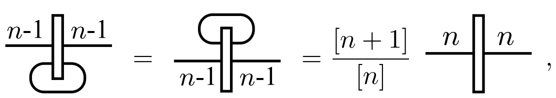
It is known, see for example [3, 4], that the Jones-Wenzl idempotents satisfy the following properties in the linear skein,



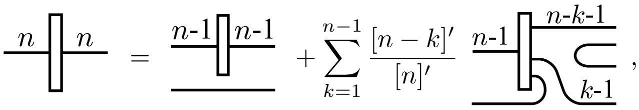
(2.6)

 $(0 \leq i \leq n-2)$, (2.7)

 $(0 \leq i \leq n,\ 0 \leq j \leq n-i)$, (2.8)



(2.9)



(2.10)

for $1 \leq n \leq 29$.

The aim of this paper is to show the following theorem, which implies that $\mathcal{S}(\mathbb{R}^2)$ is 1-dimensional.

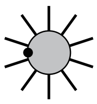
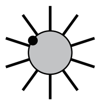
Theorem 2.2. *There exists an isomorphism* $\langle\,\rangle : \mathcal{S}(\mathbb{R}^2) \to \mathbb{C}$ *which takes the empty diagram* $\emptyset$ *to* 1*.*

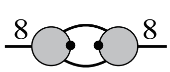
Proof. We show that $\mathcal{S}(\mathbb{R}^2)$ is spanned by the empty diagram $\emptyset$, *i.e.*, at most 1-dimensional, as follows. Let D be a planar diagram. We show that D is equal to a scalar multiple of $\emptyset$ in $\mathcal{S}(\mathbb{R}^2)$. By considering an innermost connected component of D, we can reduce the proof to the case where D is connected. If D has no vertices, then D is the empty diagram or a closed curve. Thus, by (2.1), D is equal to $\emptyset$ or $[2]\emptyset$. If D has just one vertex, then D must have a cap, and thus $D = 0$ by (2.2). Hence, we can assume that D is a connected planar diagram with at least two vertices and no caps. Then, by Lemma 2.3 below, D has a digon. By using (2.3), we move the base points of the vertices of this digon as shown in the left-hand side of (2.4). Further, by applying the left-hand side of (2.4) to this digon, D is presented by a linear sum of planar diagrams with fewer vertices. By repeating this argument, D can be presented by a scalar multiple of $\emptyset$ in $\mathcal{S}(\mathbb{R}^2)$. Hence, $\mathcal{S}(\mathbb{R}^2)$ is spanned by the empty diagram $\emptyset$.

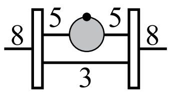
We show the proposition by improving the above argument. Let $\tilde{S}_N(\mathbb{R}^2)$ be the vector space freely spanned by planar diagrams with at most N vertices. We will inductively define the linear map $\langle\,\rangle_N : \tilde{S}_N(\mathbb{R}^2) \to \mathbb{C}$ for $N = 0, 1, 2, \cdots$, extending $\langle\,\rangle_{N-1}$, satisfying that $\langle \emptyset \rangle_N = 1$ and

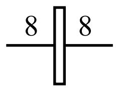
$$\langle D \cup (\text{a closed curve}) \rangle_N \;=\; [2]\,\langle D \rangle_N \;\text{ for any planar diagram } D, \tag{2.11}$$

$$\langle (\text{A planar diagram containing a cap}) \rangle_N = 0, \tag{2.12}$$

$\Big\langle$  $\Big\rangle_N = \omega \Big\langle$  $\Big\rangle_N$, (2.13)

$\Big\langle$ 

$\Big\rangle_N = \alpha^{(8)} \Big\langle$ 

$\Big\rangle_N + \beta^{(8)} \Big\langle$ 

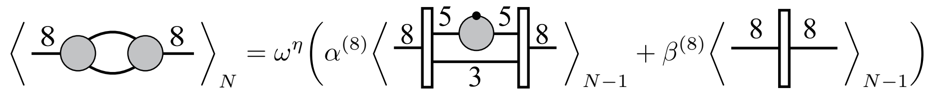
$\Big\rangle_N$. (2.14)

If such linear maps exist, we obtain a non-trivial linear map $\langle\,\rangle : \mathcal{S}(\mathbb{R}^2) \to \mathbb{C}$ as the inductive limit of them, and such a linear map $\langle\,\rangle$ must be isomorphic, since $\mathcal{S}(\mathbb{R}^2)$ is at most 1-dimensional as shown above. In the following of this proof, we define $\langle\,\rangle_N$ for $N = 0, 1, \cdots$ by induction on N showing (2.11)–(2.14).

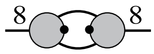
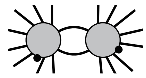
When $N = 0$, we define $\langle\,\rangle_0$, as follows. Let D be a planar diagram with no vertices. Then, D is a union of closed curves. We define $\langle D \rangle_0 = [2]^m$, where m is the number of closed curves of D. We can verify (2.11) for $N = 0$ by definition, and the conditions (2.12)–(2.14)are trivial in this case.

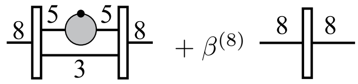
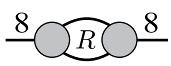
When $N = 1$, we define $\langle\,\rangle_1$, as follows. For a planar diagram D with no vertices, we put $\langle D \rangle_1 = \langle D \rangle_0$. For a planar diagram D with just one vertex, we put $\langle D \rangle_1 = 0$, noting that D must have a cap. We can verify (2.11)–(2.13)for $N = 1$ by definition, and the condition (2.14)is trivial in this case.

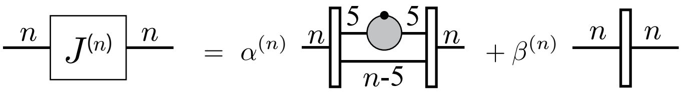
When $N \geq 2$, assuming that there exists a linear map $\langle\,\rangle_{N-1} : \tilde{S}_{N-1} \to \mathbb{C}$ satisfying (2.11)–(2.14) for $N-1$, we define a map $\langle\,\rangle_N$, as follows. For a planar diagram D with at most $N-1$ vertices, we put $\langle D\rangle_N = \langle D\rangle_{N-1}$. For a planar diagram D with just N vertices, we define $\langle D\rangle_N$, as follows. When D is disconnected, we put $\langle D\rangle_N$ to be the product of $\langle$connected component of $D\rangle_N$. If D contains a cap, we put $\langle D\rangle_N = 0$. Hence, it is sufficient to define $\langle D\rangle_N$ for a connected planar diagram D with no caps. By Lemma 2.3 below, such a planar diagram has a digon. By applying the left-hand side of the following formula to this digon, we define $\langle D\rangle_N$ by



(2.15)

where the integer η is defined by the position of the base points of the diagram in the left-hand side, as follows. We move each of the base points around the vertices clockwisely until the diagram becomes  , and η is the number of times the base points pass the edges. For example, if the diagram in the left-hand side is  , then $\eta = 7+4 = 11$. We note that the planar diagram in the left-hand side has N vertices, and the planar diagrams in the right-hand side have $N-1$ and $N-2$ vertices. We also note that the definition (2.15) is well defined independently of the π-rotation of this substitution, since the right-hand side of (2.15) is invariant under the π-rotation by the $N-1$ case of Lemma 2.6 below. Further, in order to complete the proof, we must show that $\langle D\rangle_N$ does not depend on the choice of a digon, and that this $\langle\,\rangle_N$ satisfies (2.11)–(2.14).

We show that $\langle D\rangle_N$ does not depend on the choice of a digon, as follows. For a planar diagram D with a digon R, we put D_R to be the linear sum of planar diagrams obtained from D by substituting $\omega^\eta\Big(\alpha^{(8)}$  $\Big)$ into  of this digon. We define $J^{(n)}$ for $5 \leq n \leq 8$ by



Let D be a planar diagram with two digons R_1 and R_2. Then, we have the following three cases of the mutual positions of R_1 and R_2; see Figure 1.

(a) The vertices of R_1 and R_2 are distinct.

(b) R_1 and R_2 have one common vertex.

(c) The vertices of R_1 and R_2 are equal.

We assume that the base points of vertices of R_1 and R_2 are as shown in Figure 1, since the other cases are reduced to this case from the definition of η. It is sufficient to show that $\langle D_{R_1}\rangle_{N-1} = \langle D_{R_2}\rangle_{N-1}$ in each of Cases (a)–(c).

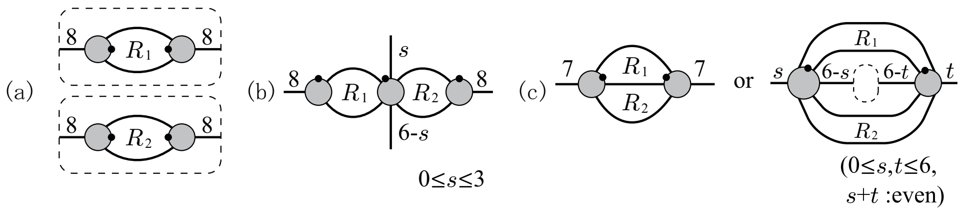
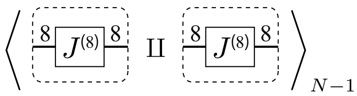
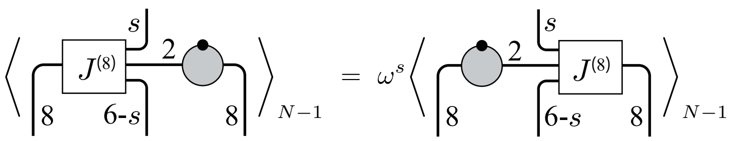


FIGURE 1. Possible positions of two digons R_1 and R_2.

Case (a). $\langle D_{R_1}\rangle_{N-1} = \langle D_{R_2}\rangle_{N-1}$, since they are equal to  by (2.14)for $N-1$, completing this case.

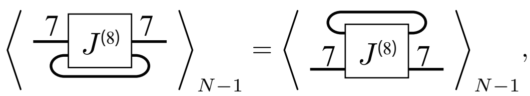
Case (b). The equation $\langle D_{R_1}\rangle_{N-1} = \langle D_{R_2}\rangle_{N-1}$ is rewritten as



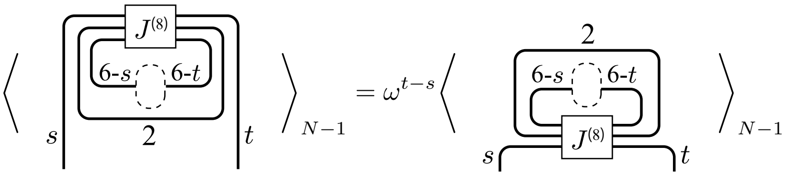
$(0 \leq s \leq 3)$, (2.16)

and we show this formula in Lemma 2.12 below, completing this case.

Case (c). When R_1 and R_2 have one common edge, it is enough to show that



and this follows from Lemma 2.7 below. When the edges of R_1 and R_2 are distinct, it is enough to show that



$(0 \leq s, t \leq 6)$

(2.17)

with $s+t$ being even, and we show this formula in Lemma 2.11 below, completing this case.

Therefore, we showed that $\langle D\rangle_N$ does not depend on the choice of a digon, and hence, we obtain a well-defined linear map $\langle\,\rangle_N : \tilde{S}_N(\mathbb{R}^2) \to \mathbb{C}$.

Finally, we show that $\langle\,\rangle_N$ satisfies (2.11)–(2.14), as follows. We recall that $\langle\,\rangle_N$ is defined by

$$\langle D\rangle_N = \begin{cases} \prod \langle \text{connected component of } D\rangle_N & \text{if } D \text{ is disconnected,} \\ 0 & \text{if } D \text{ is a connected planar diagram with a cap,} \\ \langle D_R\rangle_{N-1} & \text{if } D \text{ is a connected planar diagram with no cap.} \end{cases}$$

For any planar diagram D with N vertices, we have that

$$\langle D \cup (\text{a closed curve})\rangle_N = \langle D\rangle_N \,\langle(\text{a closed curve})\rangle_N = [2]\,\langle D\rangle_N\,,$$

from the definition of $\langle\,\rangle_N$ for disconnected planar diagrams, and hence, we obtain (2.11). From the definition of $\langle\,\rangle_N$, we obtain (2.12). From the definition of $\langle\,\rangle_N$ and (2.13)for $N-1$, we obtain (2.13). The remaining case is to show (2.14). Let D be the planar diagram in the left-hand side of (2.14). It is sufficient to show (2.14)when D is connected. If D does not have a cap, (2.14)is obtained from (2.15). We assume that D has a cap. If the cap is on a vertex outside the picture of the left-hand side of (2.14), both sides of (2.14)are 0 by definition. Otherwise, the cap is on a vertex in the picture of the left-hand side of (2.14). In this case, the left-hand side of (2.14)is 0 by definition, and the right-hand side of (2.14)is also 0 by (2.7). Hence, we obtain (2.14). Therefore, we showed that $\langle\,\rangle_N$ satisfies (2.11)–(2.14), completing the proof. □

In the proof of Theorem 2.2, we used Lemmas 2.3, 2.6, 2.8, 2.11 and 2.12 below. We show them in the following of this section.

Lemma 2.3. *A connected planar diagram with at least two vertices and no caps has a digon.*

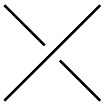
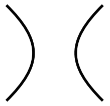
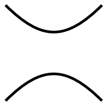
Proof. Let D be a planar diagram with no caps. In this proof, we regard D as on $\mathbb{R}^2 \cup \{\infty\} = S^2$. Assume that D has no digon. It is sufficient to show that D has at least two digons in S^2.

Let v, e, and f be the numbers of vertices, edges, and faces of D respectively. Let C_k be the number of k-gons of D. By definition, $f = \sum_{k\geq 2} C_k$. Further, $10v = 2e = \sum_{k\geq 2} kC_k$, since D is 10-valent. From these equations and Euler's formula $v - e + f = 2$, we obtain

$$\begin{aligned} 10 = 5v - 5e + 5f = -4e + 5f &= -2\sum_{k\geq 2} kC_k + 5\sum_{k\geq 2} C_k \\ &= C_2 - \sum_{k\geq 3}(2k-5)C_k \leq C_2. \end{aligned}$$

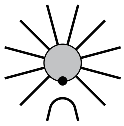
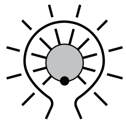
Hence, D has at least two digons in S^2, as required. □

In order to show Lemmas 2.6, 2.8, 2.11 and 2.12, we show Lemma 2.4 below, which says that an edge can "pass-over" a vertex. It is known, see for example [3], that a tangle diagram is regarded as in the linear skein by putting

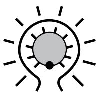
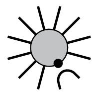
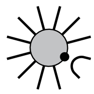
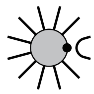
 $= A$  $+ A^{-1}$ 

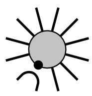
with $A = \sqrt{-1}\, q^{1/2} = \sqrt{-1}\exp(\pi\sqrt{-1}/60)$, noting that $[2] = -A^2 - A^{-2}$. Further, it is known, see [3], that the value of a tangle diagram in the linear skein is invariant under Reidemeister moves II and III.

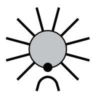
Lemma 2.4. *For an integer $N \geq 2$, let $\langle\ \rangle_N$ be a linear map $\tilde{\mathcal{S}}_N(\mathbb{R}^2) \longrightarrow \mathbb{C}$ satisfying* (2.11)–(2.14). *Then,*

$\Big\langle$  $\Big\rangle_N = \Big\langle$  $\Big\rangle_N .$

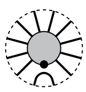
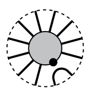
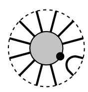
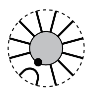
Proof. In the proof, we will omit to write $\langle\ \rangle_N$ for each of the diagrams. By expanding all crossings of the right-hand side of the required formula and by (2.13), we have that

 $= A^{-10}$  $+\,\omega^{-1}A^{-8}$  $+\,\omega^{-2}A^{-6}$ 

$+ \cdots + A^{10}$  $= \displaystyle\sum_{j=1}^{11} \omega^{-j+1} A^{2j-12} D_j$

$= -\displaystyle\sum_{j=1}^{11} \zeta^j D_j =$  $- \displaystyle\sum_{j=0}^{11} \zeta^j D_j .$

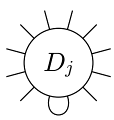
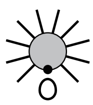
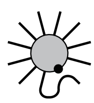
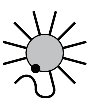
Here we put $\zeta = \exp(-\pi\sqrt{-1}/6)$ and

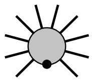
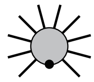
$D_0 =$  , $D_1 =$  , $D_2 =$  , $\ldots,$ $D_{11} =$  .

Hence, It is sufficient to show that

$$\sum_{j=0}^{11} \zeta^j D_j = 0.$$

Let Γ be a planar graph obtained from D_0 by replacing the disk with an 12-valent vertex. If Γ has a cap on a 10-valent vertex, then $\langle D_i \rangle_N$ for each i is equal to 0 by (2.12). If Γ has a cap on the 12-valent vertex, then the required formula holds, because

$\displaystyle\sum_{j=0}^{11} \zeta^j$  $=$  $+\,\zeta$  $+\,\zeta^{11}$ 

$= ([2] + \zeta\omega + \zeta^{-1}\omega^{-1})$  $= ([2] - q - q^{-1})$  $= 0,$

where the first equality is obtained by (2.12), and the second one is obtained by (2.11)and (2.13). We note that if a cap is on another position, a similar calculation shows the required formula. If Γ has no caps and has at least two 10-valent vertices, then, by Lemma 2.5 below, Γ has a digon whose vertices are 10-valent. By applying (2.14)to this digon, we can decrease the number of vertices of D_i for each i, keeping the required formula unchanged. Hence, repeating this argument, we can reduce the proof of the lemma to the case where Γ has at most one 10-valent vertex. Because such Γ must have a cap on the 12-valent vertex, we obtain the required formula. □

Lemma 2.5. *Let Γ be a connected planar graph with no caps, whose vertices are one 12-valent vertex and at least two 10-valent vertices. Then, Γ has a digon whose vertices are 10-valent.*

Proof. In this proof, we regard Γ as on $\mathbb{R}^2 \cup \{\infty\} = S^2$. We put v, e, f and C_k ($k = 2, 3, \dots$) of Γ in the same way as in the proof of Lemma 2.3. Let C_2' be the number of digons of Γ whose vertices are 10-valent, and let C_2'' be the number of digons of Γ which have the 12-valent vertex. By definition, $C_2 = C_2' + C_2''$. It is sufficient to show that $2 \leq C_2'$.

Let m be the number of the vertices adjacent to the 12-valent vertex. Then, since the 12-valent vertex has no cap, $m \geq 2$. Further, there exist m faces which contain the 12-valent vertex and at least two 10-valent vertices. Since such faces are not digons, we have that $C_2'' \leq 12 - m$ (see Figure 2).

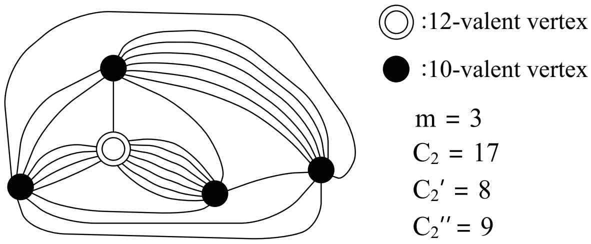


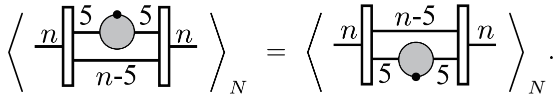
FIGURE 2. An example of a graph Γ.

Thus, we have $C_2'' \leq 10$. In a similar way as in the proof of Lemma 2.3, we have that $10(v-1) + 12 = 2e = \sum_{k \geq 2} kC_k$ and $f = \sum_{k \geq 2} C_k$. From these equations and Euler's formula $v - e + f = 2$, we have that

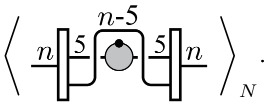
$$\begin{aligned} 10 = 5v - 5e + 5f = -1 - 4e + 5f \;&=\; -1 - 2\sum_{k \geq 2} kC_k + 5\sum_{k \geq 2} C_k \\ &= -1 + C_2' + C_2'' - \sum_{k \geq 3}(2k-5)C_k \leq C_2' + 9 - \sum_{k \geq 3}(2k-5)C_k. \end{aligned}$$

Since at least one face of Γ is not a digon, $\sum_{k\geq 3}(2k-5)C_k \geq 1$. Hence, we obtain $2 \leq C'_2$, as required. □

Lemma 2.6. *For an integer $N \geq 2$, let $\langle\ \rangle_N$ be a linear map $\tilde{\mathcal{S}}_N(\mathbb{R}^2) \longrightarrow \mathbb{C}$ satisfying (2.11)–(2.14). Then, for an integer $n = 6, 7, 8$,*

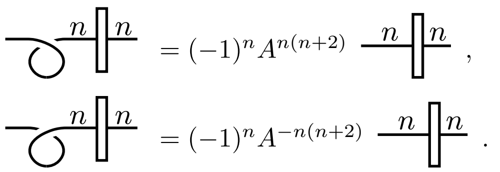


Proof. By Lemma 2.4, the left-hand side of the required formula is equal to



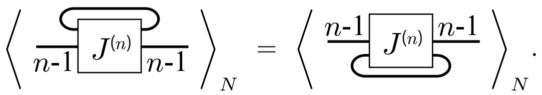
By expanding the crossings and by using (2.13), the above is equal to the right-hand side of the required formula. □

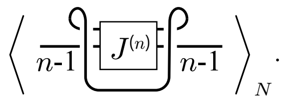
It is known, see for example [3], that



$(1 \leq n \leq 29)$ (2.18)

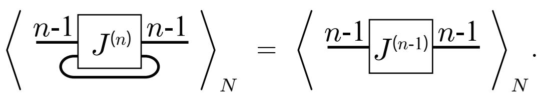
Lemma 2.7. *For an integer $N \geq 2$, let $\langle\ \rangle_N$ be a linear map $\tilde{\mathcal{S}}_N(\mathbb{R}^2) \longrightarrow \mathbb{C}$ satisfying (2.11)–(2.14). Then, for an integer $n = 6, 7, 8$,*



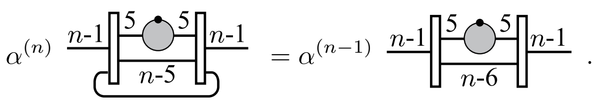
Proof. By Lemma 2.4, the left-hand side of the required formula is equal to 

By (2.18) and by expanding the crossings, this is equal to the right-hand of the required formula. □

Lemma 2.8. *For an integer $N \geq 2$, let $\langle\ \rangle_N$ be a linear map $\tilde{\mathcal{S}}_N(\mathbb{R}^2) \longrightarrow \mathbb{C}$ satisfying (2.11)–(2.13). Then,*

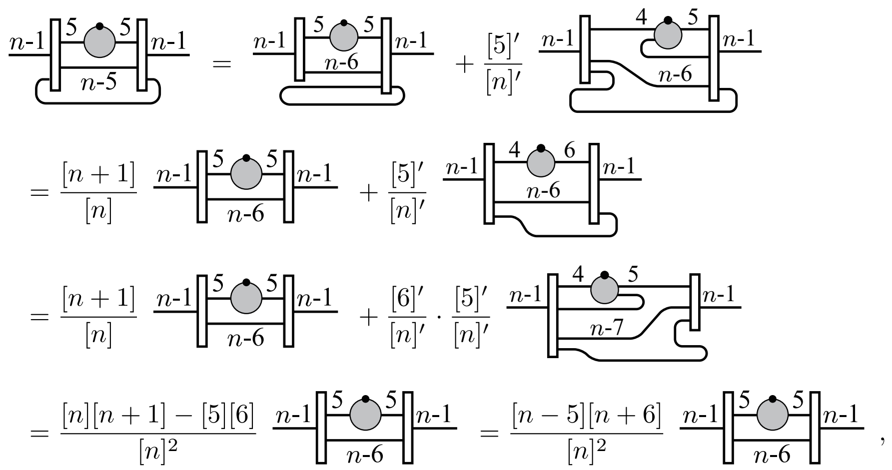


Proof. In the proof, we will omit to write $\langle\ \rangle_N$ for each of the diagrams. By (2.9) and by the definition of $J^{(n)}$, it is sufficient to show that



(2.19)

We have that



where the first equality is obtained by (2.10), the second one is obtained by (2.9), the third one is obtained by (2.7) and (2.10), and the last one is obtained by (A.12). Since

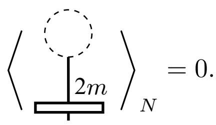
$$\frac{\alpha^{(n-1)}}{\alpha^{(n)}} = \frac{\begin{bmatrix} n-1 \\ 5 \end{bmatrix}^2}{\begin{bmatrix} n+5 \\ 11 \end{bmatrix}} \cdot \frac{\begin{bmatrix} n+6 \\ 11 \end{bmatrix}}{\begin{bmatrix} n \\ 5 \end{bmatrix}^2} = \frac{[n-1]!^2}{[5]!^2[n-6]!^2} \cdot \frac{[11]![n-6]!}{[n+5]!}$$
$$\cdot \frac{[n+6]!}{[11]![n-5]!} \cdot \frac{[5]!^2[n-5]!^2}{[n]!^2} = \frac{[n-5][n+6]}{[n]^2},$$

we obtain (2.19), as required.

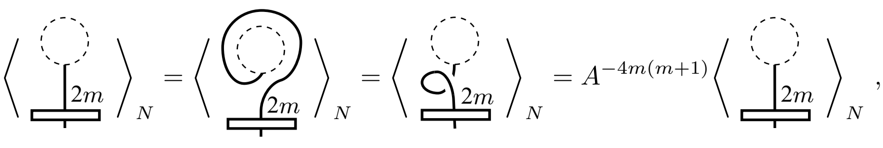
$\square$

In order to show Lemmas 2.11 and 2.12, we show Lemmas 2.9 and 2.10, as follows.

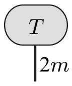
Lemma 2.9. *For an integer $N \geq 2$, let $\langle\ \rangle_N$ be a linear map $\tilde{\mathcal{S}}_N(\mathbb{R}^2) \longrightarrow \mathbb{C}$ satisfying* (2.11)–(2.14). *Let m be an integer such that $1 \leq m \leq 13$, $m \neq 5, 9$. Then,*

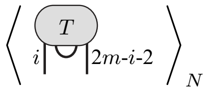


Proof. We have that

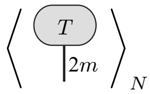


where the second equality is obtained by Lemma 2.4 and the third one is obtained by (2.18). Since $A^{-4m(m+1)} = \exp(-2\pi\sqrt{-1} \cdot m(m+1)/30) \neq 1$, we obtain the required formula. $\square$

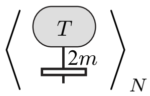
Lemma 2.10. *Under the assumption of Lemma 2.9, we consider a planar diagram of the form* 

. If 

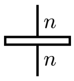
$= 0$ *for any* $i = 0, 1, \ldots, 2m-2$*, then*

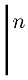


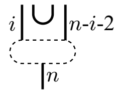
$= 0.$

Proof. By Lemma 2.9, 

$= 0$. Further, we have that



$=$ 

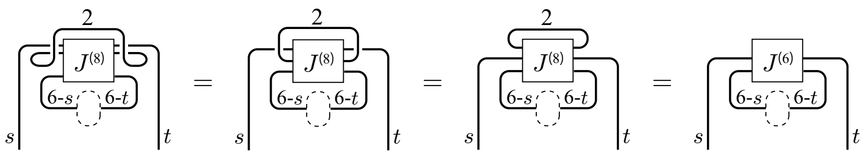
$+$ (a linear sum of planar diagrams of the form 

)

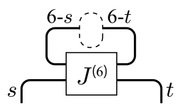
in the linear skein, which can be shown by induction on n from the definition of the Jones-Wenzl idempotents. Hence, by the assumption of the lemma putting $n = 2m$, we obtain the required formula. □

Lemma 2.11. *The formula* (2.17) *holds for integers* s, t *satifying that* $0 \leq s, t \leq 6$ *and* $s + t$ *is even.*

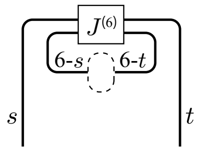
Proof. In the proof, we will omit to write $\langle\ \rangle_N$ for each of the diagrams. By Lemma 2.4, the left-hand side of (2.17) is equal to

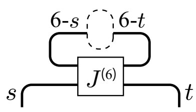


,

where the first equality is obtained by (2.18), the second one is obtained by expanding the crossings, and the last one is obtained by Lemma 2.8. In the same way, we can verify that the right-hand side of (2.17) is equal to 

. Hence, it is sufficient to show that

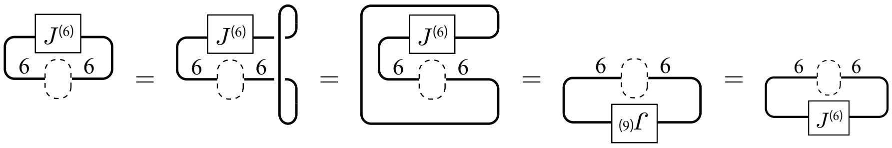


$=$ 

. (2.20)

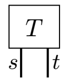
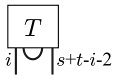
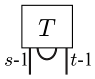
We show (2.20), as follows.

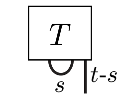
The case $\mathbf{s+t=0 \Leftrightarrow s=t=0}$: We have that

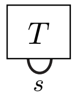


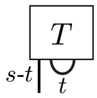
,

where the first equality is obtained by (2.18), the second one is obtained by Lemma 2.4, and the last one is obtained by Lemma 2.6 and (2.6), completing this case.

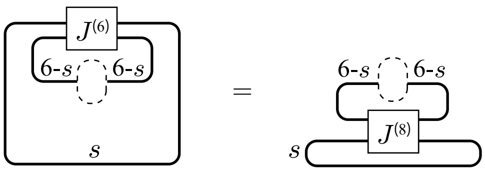
The case $\mathbf{0 < s+t < 10}$: We denote (LHS of (2.20)) $-$ (RHS of (2.20)) by . By (2.7),  $= 0$ for $0 \le i \le s+t-2$, $i \ne s-1$. Hence, by Lemma 2.10, it is sufficient to show that  $= 0$. By repeating this argument if necessary, it is sufficient to show that

 $= 0$ if $s < t$, (2.21)

 $= 0$ if $s = t$, (2.22)

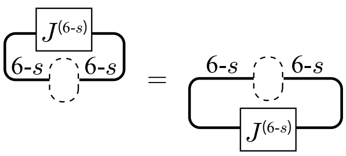
 $= 0$ if $s > t$. (2.23)

(2.21) and (2.23) are obtained by (2.7) and Lemma 2.10, noting that $0 < |t-s| < 6$. (2.22) is rewritten as



.

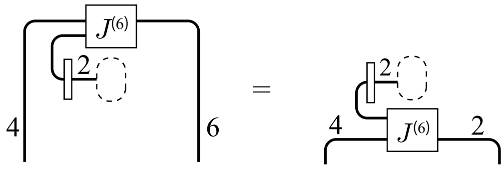
In a similar way as the beginning of the proof, we see that the above formula is equivalent to



.

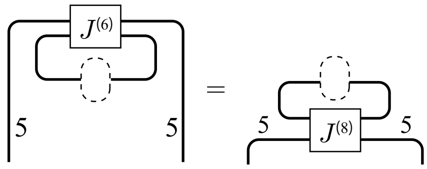
Further, this can be shown in the same way as the proof of the case $s+t=0$, completing this case.

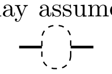
The case $\mathbf{s+t=10}$: When $(s,t)=(4,6)$, by using (2.8), (2.20) is rewritten as



By Lemma 2.9, the both sides of the above formula are equal to 0.

When $(s,t)=(5,5)$, we show (2.20), that is,



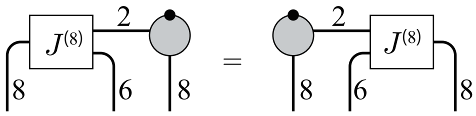
If there is a cap in the disk  , then the both sides of the required formula are equal to 0 by (2.12). Thus, we assume that there are no caps in the disk. If there are at least two vertices in the disk, then, in the same way as the proof of Lemma 2.3, we can verify that there is a digon in the disk. Thus, by applying (2.14)to this digon, we can reduce the number of vertices in the disk keeping the required formula unchanged. If there is one vertex in the disk, then this vertex must have a cap. Hence, we may assume that there are no vertices in the disk. Thus, by (2.11), we may assume that  is equal to . Thus, in a similar way as the beginning of the proof, we obtain the required formula. When $(s,t)=(6,4)$, we obtain (2.20) in the same way as the proof of the case $(s,t)=(4,6)$, completing this case.

The case $\mathbf{s+t=12 \Leftrightarrow s=t=6}$: We obtain (2.20) because of the definition of $\langle\ \rangle_N$ for disconnected planar diagrams, completing this case. ☐

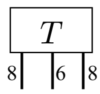
Lemma 2.12. *The formula* (2.16) *holds for* $s=0,1,2,3$.

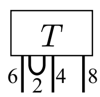
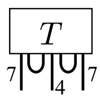
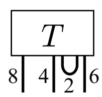
Proof. In the proof, we omit to write $\langle\ \rangle_{N-1}$ for each of the diagrams.

We first show (2.16) for $s=0$, that is,

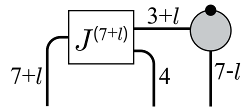


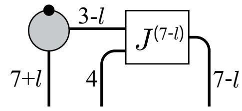
(2.24)

We denote (LHS of (2.24)) $-$ (RHS of (2.24)) by  . In a similar way as the proof of Lemma 2.11 in the case $0<s+t<10$, it is sufficient to show that

 $=0,$  $=0,$  $=0.$

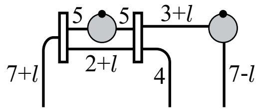
By Lemma 2.8, these are rewritten as

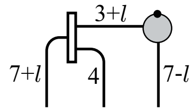


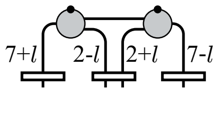
$=$


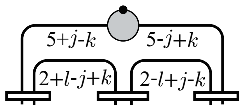
$$(l=-1,0,1). \tag{2.25}$$

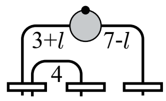
By Lemma A.3 below, the left-hand side of (2.25) is equal to

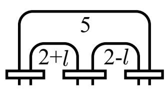
$\alpha^{(7+l)}$ 

$+\,\beta^{(7+l)}$ 

$= \alpha^{(7+l)} \dfrac{[3+l]'}{\begin{bmatrix} 7+l \\ 2+l \end{bmatrix}'}$ 

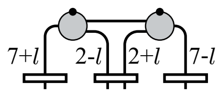
$+\,\alpha^{(7+l)} \displaystyle\sum_{i=0}^{1+l}\sum_{j=0}^{i}\sum_{k=0}^{2-l} A(i,j,k,l)$ 

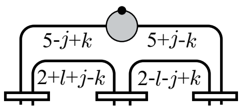
$+\,\beta^{(7+l)}$ 

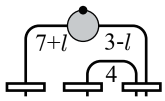
$+\,\alpha^{(7+l)} \displaystyle\sum_{i=0}^{1+l} B(i,l)$ 

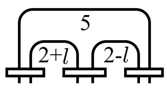
$$, \tag{2.26}$$

where $A(i,j,k,l)$ and $B(i,l)$ are given in (A.3) and (A.4). By considering the mirror image of the left-hand side of (2.25) and by replacing l with $-l$, the right-hand side of (2.25) is equal to

$\alpha^{(7-l)} \dfrac{[3-l]'}{\begin{bmatrix} 7-l \\ 2-l \end{bmatrix}'}$ 

$+\,\alpha^{(7-l)} \displaystyle\sum_{i=0}^{1-l}\sum_{j=0}^{i}\sum_{k=0}^{2+l} A(i,j,k,-l)$ 

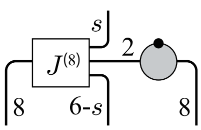
$+\,\beta^{(7-l)}$ 

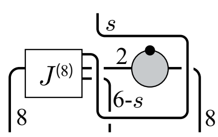
$+\,\alpha^{(7-l)} \displaystyle\sum_{i=0}^{1-l} B(i,-l)$ 

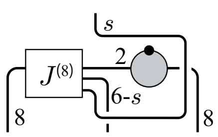
$$. \tag{2.27}$$

By Lemma A.5 below, each of the coefficients of the diagrams of (2.26) and (2.27) are equal. Hence, (2.25) was shown, and thus, (2.16) for $s=0$ was shown, as required.

We next show (2.16) for $s=1,2,3$. We have that

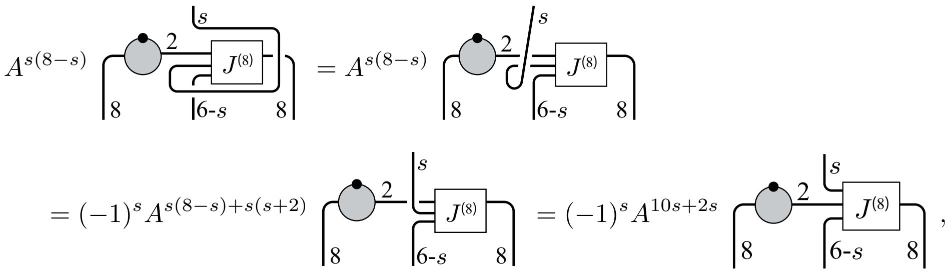


$=$


$= A^{s(8-s)}$


,

where the first equality is obtained by Lemma 2.4, the second one is obtained by (2.18) and the last one is obtained by expanding the crossings. By applying (2.24), this is equal to

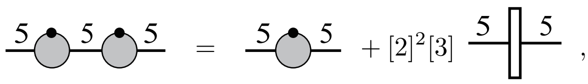


where the first equality is obtained by (2.18) and the second one is obtained by expanding the crossings. Since $(-1)^s A^{10s+2s} = (-A^{12})^s = \omega^s$, we obtain (2.16) for $s = 1, 2, 3$. $\square$

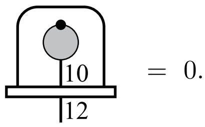
3. Equivalence to Bigelow's relations of the E_8 subfactor planar algebra

Bigelow [1] defined a planar algebra $\mathcal{S}'(\mathbb{R}^2)$ (in his paper this is denoted by $\mathcal{P}$) by giving generators and relations, and proved that its principal graph is the E_8 Dynkin diagram. However, his proof of a non-triviality of $\mathcal{S}'(\mathbb{R}^2)$ relies on the existence of the E_8 subfactor planar algebra. In this section, we show that $\mathcal{S}(\mathbb{R}^2)$ is isomorphic to $\mathcal{S}'(\mathbb{R}^2)$. We note that an S-labeled disc of [1] corresponds to a vertex of this paper.

As in [1], we define $\mathcal{S}'(\mathbb{R}^2)$ to be the vector space spanned by planar diagrams subject to the relations (2.1)–(2.3) and (3.1), (3.2) below,



(3.1)



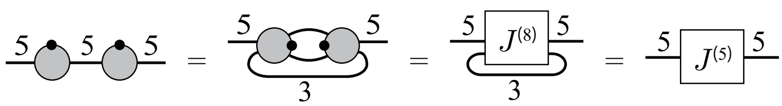
(3.2)

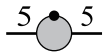
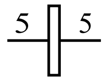
We recall that $\mathcal{S}(\mathbb{R}^2)$ is the vector space spanned by planar diagrams subject to the relations (2.1)–(2.4).

Proposition 3.1. *$\mathcal{S}(\mathbb{R}^2)$ is isomorphic to $\mathcal{S}'(\mathbb{R}^2)$.*

Proof. We assume (2.1)–(2.3) in this proof. It is enough to show that (2.4) is equivalent to (3.1) and (3.2).

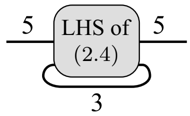
Assuming (2.4), we show (3.1) and (3.2), as follows. We obtain (3.1) from

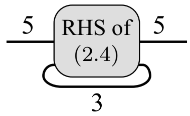


$=$  $+ [2]^2[3]$  ,

where the first equality is obtained by (2.3), the second one is obtained by (2.4), the third one is obtained by Lemma 2.8. Further, we obtain (3.2) from Lemma 2.9 in the case $m = 6$.

Assuming (3.1) and (3.2), we show (2.4), as follows. From the proof of [1, Lemma 3.1], an edge can pass-over a vertex. Hence, the formula of Lemma 2.10 holds in $\mathcal{S}'(\mathbb{R}^2)$ in a similar way as the proof of Lemma 2.9. Hence, by a similar argument as in the proof of Lemma 2.11 in the case $0 < s + t < 10$, it is sufficient to show that



$=$


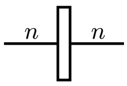
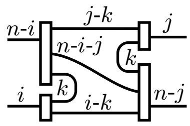
.

This is obtained from (2.2) and Lemma 2.8, noting that these equations are obtained from the relations (2.1)–(2.3). Therefore, we obtain (2.4), as required. $\square$

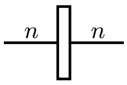
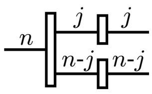
Appendix A. Technical calculations

In this appendix, we show some technical formulae of the E_8 linear skein, which are needed to show Theorem 2.2 in Case (b).

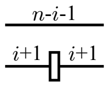
Lemma A.1. *For non-negative integers i, j, n satisfying that $i + j \leq n$ and $i \leq j$,*

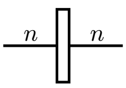
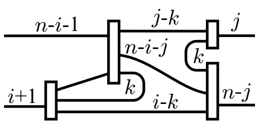
 $= \sum_{k=0}^{i} \dfrac{\begin{bmatrix} i \\ k \end{bmatrix}' \begin{bmatrix} j \\ k \end{bmatrix}'}{\begin{bmatrix} n \\ k \end{bmatrix}'}$  . (A.1)

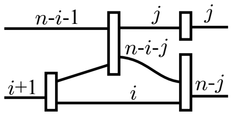
Proof. We fix non-negative integers j, n satisfying that $j \leq n$. We show the lemma by induction on i satisfying the assumption of the lemma. When $i = 0$, (A.1) is rewritten as

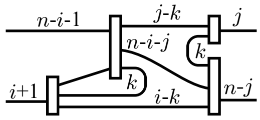
 $=$  ,

and this is obtained by (2.8).

Under the assumption that (A.1) holds for $i \geq 0$ satisfying that $i \leq n - j - 1$ and $i \leq j - 1$, we show that (A.1) holds for $i + 1$, as follows. We put $c_k^{ijn} = \begin{bmatrix} i \\ k \end{bmatrix}' \begin{bmatrix} j \\ k \end{bmatrix}' \Big/ \begin{bmatrix} n \\ k \end{bmatrix}'$ for $0 \leq k \leq i$. By connecting  to both-sides of (A.1) from the left, and by using (2.8), we have that

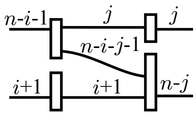
 $= \sum_{k=0}^{i} c_k^{ijn}$ 

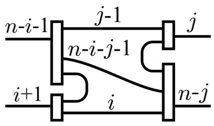
$=$ 

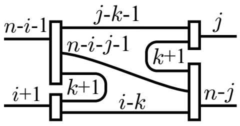
$+\sum_{k=1}^{i} c_k^{ijn}$ 

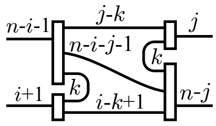
.

By applying (2.10) to the Jones-Wenzl idempotent in the center position, this is equal to

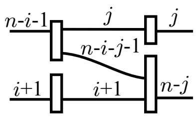


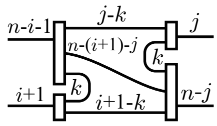
$+\dfrac{[j]'}{[n-i]'}$ 

$+\sum_{k=1}^{i} c_k^{ijn}\Bigg(\dfrac{[j-k]'}{[n-i]'}$ 

$+\dfrac{[n-i-k]'}{[n-i]'}$ 

$\Bigg)$

$=$ 

$+\sum_{k=1}^{i+1} \dfrac{c_{k-1}^{ijn}[j-k+1]' + c_k^{ijn}[n-i-k]'}{[n-i]'}$ 

.

Here we put $c_{i+1}^{ijn} = 0$. Hence, it is sufficient to show that

$$\frac{c_{k-1}^{ijn}[j-k+1]' + c_k^{ijn}[n-i-k]'}{[n-i]'} = c_k^{i+1\,jk} \qquad (1 \leq k \leq i+1). \tag{A.2}$$

This is rewritten as

$$\frac{c_{k-1}^{ijn}}{c_k^{i+1\,jn}}[j-k+1]' + \frac{c_k^{ijn}}{c_k^{i+1\,jn}}[n-i-k]' = [n-i]' \qquad (1 \leq k \leq i+1).$$

Since

$$\frac{c_{k-1}^{ijn}}{c_k^{i+1\,jn}} = \frac{\begin{bmatrix} i \\ k-1 \end{bmatrix}'}{\begin{bmatrix} i+1 \\ k \end{bmatrix}'} \cdot \frac{\begin{bmatrix} j \\ k-1 \end{bmatrix}'}{\begin{bmatrix} j \\ k \end{bmatrix}'} \cdot \frac{\begin{bmatrix} n \\ k \end{bmatrix}'}{\begin{bmatrix} n \\ k-1 \end{bmatrix}'} = \frac{[k]'}{[i+1]'} \cdot \frac{[k]'}{[j-k+1]'} \cdot \frac{[n-k+1]'}{[k]'},$$

$$\frac{c_k^{ijn}}{c_k^{i+1\,jn}} = \frac{\begin{bmatrix} i \\ k \end{bmatrix}'}{\begin{bmatrix} i+1 \\ k \end{bmatrix}'} = \frac{[i-k+1]'}{[i+1]'},$$

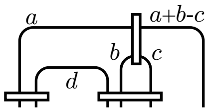
(A.2) is equivalent to

$$[k]'[n-k+1]' + [i-k+1]'[n-i-k]' = [n-i]'[i+1]' \qquad .(1 \leq k \leq i+1)$$

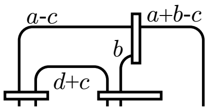
By definition, we can verify this equation. Hence, (A.1) for $i+1$ was shown, as required.

□

Lemma A.2. *For integers a, b, c, d such that $a, b > 0$, $c \geq 0$ and $c \leq a + b$,*

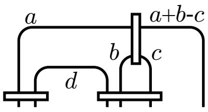


$$= \frac{\begin{bmatrix} a \\ c \end{bmatrix}'}{\begin{bmatrix} a+b \\ c \end{bmatrix}'}$$

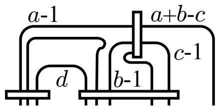


in the linear skein.

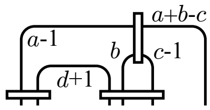
Proof. If $c = 0$, the required formula is trivial. Thus, we assume that $c > 0$. By (2.7) and (2.10),



$$= \frac{[a]'}{[a+b]'}$$

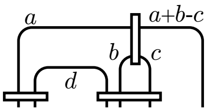


$$= \frac{[a]'}{[a+b]'}$$

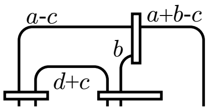


.

By repeating this argument, we have that



$$= \frac{[a]'[a-1]' \cdots [a-c+1]'}{[a+b]'[a+b-1]' \cdots [a+b-c+1]'}$$



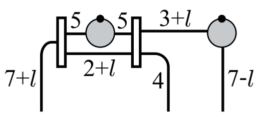
.

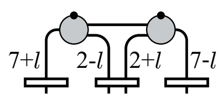
Since

$$\frac{[a]'[a-1]' \cdots [a-c+1]'}{[a+b]'[a+b-1]' \cdots [a+b-c+1]'} = \frac{[a]!'}{[a-c]!'} \cdot \frac{[a+b-c]!'}{[a+b]!'} \cdot \frac{[c]!'}{[c]!'} = \frac{\begin{bmatrix} a \\ c \end{bmatrix}'}{\begin{bmatrix} a+b \\ c \end{bmatrix}'},$$

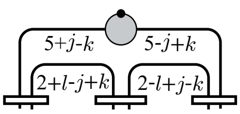
we obtain the required formula. ☐

Lemma A.3. *For an integer $N \geq 2$, let $\langle\ \rangle_N$ be a linear map $\tilde{\mathcal{S}}_N(\mathbb{R}^2) \longrightarrow \mathbb{C}$ satisfying* (2.11)–(2.14)*. Then, for $l = -1, 0, 1$,*

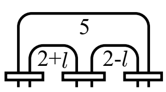
$$\Big\langle$$


$$\Big\rangle_N = \frac{[3+l]'}{\begin{bmatrix} 7+l \\ 2+l \end{bmatrix}'} \Big\langle$$


$$\Big\rangle_N$$

$$+ \sum_{i=0}^{1+l} \sum_{j=0}^{i} \sum_{k=0}^{2-l} A(i,j,k,l) \Big\langle$$


$$\Big\rangle_N$$

$$+ \sum_{i=0}^{1+l} B(i,l) \Big\langle$$


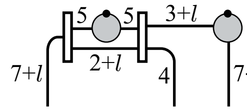
$$\Big\rangle_N .$$

Here we put

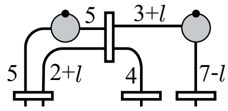
$$A(i,j,k,l) \;=\; \alpha^{(7-l+i)} \frac{{\begin{bmatrix} 2+l \\ i \end{bmatrix}}' {\begin{bmatrix} 3+l \\ i \end{bmatrix}}' {\begin{bmatrix} i \\ j \end{bmatrix}}' {\begin{bmatrix} 5 \\ j \end{bmatrix}}' {\begin{bmatrix} 5+i \\ i-j \end{bmatrix}}' {\begin{bmatrix} 2-l \\ k \end{bmatrix}}' {\begin{bmatrix} 5+j \\ k \end{bmatrix}}'}{{\begin{bmatrix} 7+l \\ i \end{bmatrix}}' {\begin{bmatrix} 7-l+i \\ j \end{bmatrix}}' {\begin{bmatrix} 7-l+i \\ i-j \end{bmatrix}}' {\begin{bmatrix} 7-l+j \\ k \end{bmatrix}}'}, \tag{A.3}$$

$$B(i,l) \;=\; \beta^{(7-l+i)} \frac{{\begin{bmatrix} 2+l \\ i \end{bmatrix}}' {\begin{bmatrix} 3+l \\ i \end{bmatrix}}' {\begin{bmatrix} 5+i \\ i \end{bmatrix}}'}{{\begin{bmatrix} 7+l \\ i \end{bmatrix}}' {\begin{bmatrix} 7-l+i \\ i \end{bmatrix}}'}. \tag{A.4}$$

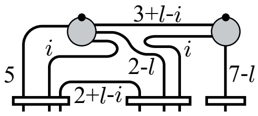
Proof. In the proof, we omit to write $\langle\ \rangle_N$ for each of the diagrams. The left-hand side of the required formula is calculated, as follows.



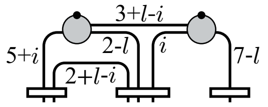
$=$



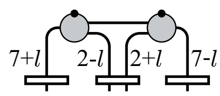
$$= \sum_{i=0}^{2+l} \frac{{\begin{bmatrix} 2+l \\ i \end{bmatrix}}' {\begin{bmatrix} 3+l \\ i \end{bmatrix}}'}{{\begin{bmatrix} 7+l \\ i \end{bmatrix}}'}$$



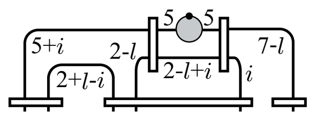
$$= \sum_{i=0}^{1+l} \frac{{\begin{bmatrix} 2+l \\ i \end{bmatrix}}' {\begin{bmatrix} 3+l \\ i \end{bmatrix}}'}{{\begin{bmatrix} 7+l \\ i \end{bmatrix}}'}$$



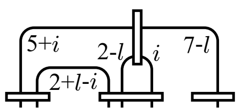
$$+ \frac{{\begin{bmatrix} 2+l \\ 2+l \end{bmatrix}}' {\begin{bmatrix} 3+l \\ 2+l \end{bmatrix}}'}{{\begin{bmatrix} 7+l \\ 2+l \end{bmatrix}}'}$$



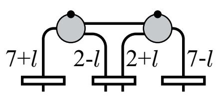
$$= \sum_{i=0}^{1+l} \frac{{\begin{bmatrix} 2+l \\ i \end{bmatrix}}' {\begin{bmatrix} 3+l \\ i \end{bmatrix}}'}{{\begin{bmatrix} 7+l \\ i \end{bmatrix}}'} \Bigg(\alpha^{(7-l+i)}$$



$$+ \beta^{(7-l+i)}$$



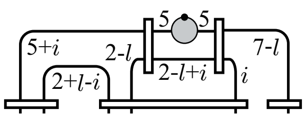
$$\Bigg) + \frac{[3+l]'}{{\begin{bmatrix} 7+l \\ 2+l \end{bmatrix}}'}$$



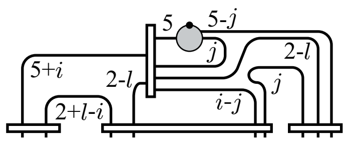
$, \qquad \text{(A.5)}$

where the first equality is obtained by (2.8), the second one is obtained by Lemma A.1 and the last one is obtained by (2.14)and Lemma 2.8.

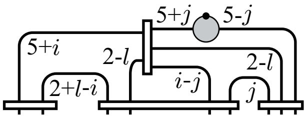
The first term of (A.5) is calculated, as follows. We have that



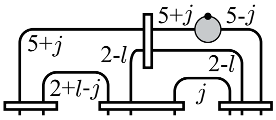
$$= \sum_{j=0}^{i} \frac{{\begin{bmatrix} i \\ j \end{bmatrix}}' {\begin{bmatrix} 5 \\ j \end{bmatrix}}'}{{\begin{bmatrix} 7-l+i \\ j \end{bmatrix}}'}$$



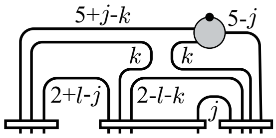
$$= \sum_{j=0}^{i} \frac{\begin{bmatrix} i \\ j \end{bmatrix}' \begin{bmatrix} 5 \\ j \end{bmatrix}'}{\begin{bmatrix} 7-l+i \\ j \end{bmatrix}'}$$



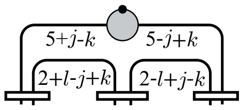
$$= \sum_{j=0}^{i} \frac{\begin{bmatrix} i \\ j \end{bmatrix}' \begin{bmatrix} 5 \\ j \end{bmatrix}'}{\begin{bmatrix} 7-l+i \\ j \end{bmatrix}'} \cdot \frac{\begin{bmatrix} 5+i \\ i-j \end{bmatrix}'}{\begin{bmatrix} 7-l+i \\ i-j \end{bmatrix}'}$$



$$= \sum_{j=0}^{i} \frac{\begin{bmatrix} i \\ j \end{bmatrix}' \begin{bmatrix} 5 \\ j \end{bmatrix}'}{\begin{bmatrix} 7-l+i \\ j \end{bmatrix}'} \cdot \frac{\begin{bmatrix} 5+i \\ i-j \end{bmatrix}'}{\begin{bmatrix} 7-l+i \\ i-j \end{bmatrix}'} \sum_{k=0}^{2-l} \frac{\begin{bmatrix} 2-l \\ k \end{bmatrix}' \begin{bmatrix} 5+j \\ k \end{bmatrix}'}{\begin{bmatrix} 7-l+j \\ k \end{bmatrix}'}$$



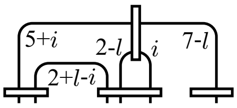
$$= \sum_{j=0}^{i} \sum_{k=0}^{2-l} \frac{\begin{bmatrix} i \\ j \end{bmatrix}' \begin{bmatrix} 5 \\ j \end{bmatrix}'}{\begin{bmatrix} 7-l+i \\ j \end{bmatrix}'} \cdot \frac{\begin{bmatrix} 5+i \\ i-j \end{bmatrix}'}{\begin{bmatrix} 7-l+i \\ i-j \end{bmatrix}'} \cdot \frac{\begin{bmatrix} 2-l \\ k \end{bmatrix}' \begin{bmatrix} 5+j \\ k \end{bmatrix}'}{\begin{bmatrix} 7-l+j \\ k \end{bmatrix}'}$$



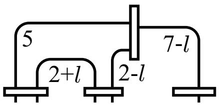
,

where the first and the fourth equalities are obtained by Lemma A.1, and the third one is obtained by Lemma A.2.

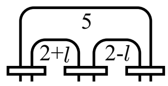
By Lemma A.2, the second term of (A.5) is calculated, as follows.



$$= \frac{\begin{bmatrix} 5+i \\ i \end{bmatrix}'}{\begin{bmatrix} 7-l+i \\ i \end{bmatrix}'}$$



$$= \frac{\begin{bmatrix} 5+i \\ i \end{bmatrix}'}{\begin{bmatrix} 7-l+i \\ i \end{bmatrix}'}$$



.

By substituting these formulae into (A.5), we obtain the required formula. □

Lemma A.4.

$$[9] = [2][4], \tag{A.6}$$

$$[14] = [2][5], \tag{A.7}$$

$$[2]^2[3][12]^2 = [5][6]^2[7], \tag{A.8}$$

$$[2][13] = [4][5], \tag{A.9}$$

$$[5][14] = [7][8], \tag{A.10}$$

$$[2][13][14] = [4][7][8]. \tag{A.11}$$

Proof. We show (A.6), as follows. By using $[n] = q^{-n+1} + q^{-n+3} + \cdots + q^{n-3} + q^{n-1}$, we have that

$$\begin{aligned} [9] - [2][4] &= (q^{-8} + q^{-6} + \cdots + q^6 + q^8) - (q + q^{-1})(q^{-3} + q^{-1} + q + q^3) \\ &= q^{-8} + q^{-6} - q^{-2} - 1 - q^2 + q^6 + q^8. \end{aligned}$$

Since $q = \exp(\pi\sqrt{-1}/30)$ satisfies $f(q) = 0$, where we put $f(x) = x^{-8} + x^{-6} - x^{-2} - 1 - x^2 + x^6 + x^8$, we obtain (A.6).

(A.7)–(A.10) are obtained by showing that (LHS) − (RHS) of each of the required formulae are divided by $f(q)$; Indeed, we can verify that

$$\begin{aligned}
[14] - [2][5] \;&=\; (q^{-1}+q)(q^{-2}+q^2)(q^{-2}-1+q^2)f(q),\\
[2][13][14] - [4][7][8] \;&=\; (q^{-1}+q)^2(q^{-2}-1+q^2)(q^{-2}-q^{-1}+1-q+q^2)\\
&\quad\times(q^{-2}+q^{-1}+1+q+q^2)(q^{-3}-q^{-2}+q^{-1}-1+q-q^2+q^3)\\
&\quad\times(q^{-3}+q^{-2}+q^{-1}+1+q+q^2+q^3)(q^{-4}-q^{-2}+1-q^2+q^4)f(q),\\
[2]^2[3][12]^2 - [5][6]^2[7] \;&=\; (q^{-1}+q)^2(q^{-1}-1+q)^2(q^{-1}+1+q)^2(q^{-2}-1+q^2)^2\\
&\quad\times(q^{-8}+2q^{-6}+2q^{-4}+q^{-2}+1+q^2+2q^4+2q^6+q^8)f(q),\\
[2][13] - [4][5] \;&= = (q^{-1}+q)(q^{-1}-1+q)(q^{-1}+1+q)(q^{-2}-1+q^2)f(q),\\
[5][14] - [7][8] \;&=\; (q^{-1}+q)(q^{-2}-1+q^2)(q^{-3}-q^{-2}+q^{-1}-1+q-q^2+q^3)\\
&\quad\times(q^{-3}+q^{-2}+q^{-1}+1+q+q^2+q^3)f(q),
\end{aligned}$$

as required.

(A.11) is obtained by (A.9) and (A.10). □

For integers m, n, k, we can verify that

$$[m+k][n+k] = [m][n] + [k][m+n+k], \tag{A.12}$$

$$[m][2] = [m-1] + [m+1], \tag{A.13}$$

$$[m][3] = [m-2] + [m] + [m+2] \tag{A.14}$$

by direct calculation. From (A.12) and (A.6), we obtain

$$[9] = [3] + [5]. \tag{A.15}$$

Lemma A.5. *For $l = -1, 0, 1$,*

$$\alpha^{(7+l)} \frac{[3+l]'}{\begin{bmatrix} 7+l \\ 2+l \end{bmatrix}'} \;=\; \alpha^{(7-l)} \frac{[3-l]'}{\begin{bmatrix} 7-l \\ 2-l \end{bmatrix}'}, \tag{A.16}$$

$$\alpha^{(7+l)} \sum_{\substack{0\le i\le 1+l,\ 0\le j\le i,\\ 0\le k\le 2-l,\, l-j+k=2}} A(i,j,k,l) + \beta^{(7+l)} = \alpha^{(7-l)} \sum_{\substack{0\le i\le 1-l,\ 0\le j\le i,\\ 0\le k\le 2+l,\, -l+j-k=2}} A(i,j,k,-l), \tag{A.17}$$

$$\alpha^{(7+l)} \sum_{\substack{0\le i\le 1+l,\ 0\le j\le i,\\ 0\le k\le 2-l,\, l-j+k=m}} A(i,j,k,l) = \alpha^{(7-l)} \sum_{\substack{0\le i\le 1-l,\ 0\le j\le i,\\ 0\le k\le 2+l,\, l+j-k=m}} A(i,j,k,-l) \quad (m=-1,0,1), \tag{A.18}$$

$$\alpha^{(7+l)} \sum_{\substack{0\le i\le 1+l,\ 0\le j\le i,\\ 0\le k\le 2-l,\, l-j+k=-2}} A(i,j,k,l) = \alpha^{(7-l)} \sum_{\substack{0\le i\le 1-l,\ 0\le j\le i,\\ 0\le k\le 2+l,\, -l+j-k=-2}} A(i,j,k,-l) + \beta^{(7-l)}, \tag{A.19}$$

$$\alpha^{(7+l)} \sum_{i=0}^{1+l} B(i,l) \;=\; \alpha^{(7-l)} \sum_{i=0}^{1-l} B(i,-l), \tag{A.20}$$

Proof. where $A(i,j,k,l)$ and $B(i,l)$ are defined in (A.3) and (A.4). We first note that each of the required formulae in the case $l=-1$ are equivalent to that in the case $l=1$, so we have only to show (A.16)–(A.20) in the cases $l=0,1$.

We show (A.16)–(A.20) in the case $l=0$, as follows.

(A.16) is trivial.

We show (A.17), as follows. We have that

$$\begin{aligned}
&\Bigg(\alpha^{(7)} \sum_{\substack{0\le i\le 1,\ 0\le j\le i,\\ 0\le k\le 2,\ -j+k=2}} A(i,j,k,0) + \beta^{(7)}\Bigg) - \alpha^{(7)} \sum_{\substack{0\le i\le 1,\ 0\le j\le i,\\ 0\le k\le 2,\ j-k=2}} A(i,j,k,0) \\
&= \alpha^{(7)}\big(A(0,0,2,0)+A(1,0,2,0)\big)+\beta^{(7)} \\
&= \frac{[6]^2[7]^2}{[2][12][13]}\Big(\frac{[4][5][6][7]}{[2][12][13]} - \frac{[4][5][6]^2[8]}{[12][13][14]}\Big) + \frac{[2][3][6]}{[8]} \\
&= \frac{[6]\big([4][5][6]^2[7]^3[8][14] - [2][4][5][6]^3[7]^2[8]^2 + [2][13]\cdot[2]^2[3][12]^2\cdot[2][13][14]\big)}{[2]^2[8][12]^2[13]^2[14]} \\
&= \frac{[6]\big([4][5][6]^2[7]^3[8]\cdot[2][5] - [2][4][5][6]^3[7]^2[8]^2 + [2][13]\cdot[4][7][8]\cdot[5][6]^2[7]\big)}{[2]^2[8][12]^2[13]^2[14]} \\
&= \frac{[4][5][6]^3[7]^2}{[2][12]^2[13]^2[14]}\cdot([5][7]-[6][8]+[13]) = 0,
\end{aligned}$$

where the fourth equality is obtained by (A.7), (A.8) and (A.11), and the last one is obtained by (A.12), completing this case.

We show (A.18) in the case $m=-1$, as follows. We have that

$$\begin{aligned}
&\alpha^{(7)} \sum_{\substack{0\le i\le 1,\ 0\le j\le i,\\ 0\le k\le 2,\ -j+k=-1}} A(i,j,k,0) - \alpha^{(7)} \sum_{\substack{0\le i\le 1,\ 0\le j\le i,\\ 0\le k\le 2,\ j-k=-1}} A(i,j,k,0) \\
&= \alpha^{(7)}A(1,1,0,0) - \alpha^{(7)}\big(A(0,0,1,0)+A(1,0,1,0)+A(1,1,2,0)\big) \\
&= \alpha^{(7)}\alpha^{(8)}\Big(\frac{[2][3][5]}{[7][8]} + \frac{\alpha^{(7)}}{\alpha^{(8)}}\cdot\frac{[2][5]}{[7]} - \frac{[2]^2[3][5][6]}{[7]^2[8]} - \frac{[2][3][5]^2[6]}{[7]^2[8]^2}\Big) \\
&= \alpha^{(7)}\alpha^{(8)}\frac{[2][3][5]}{[7]^2[8]^2}\big(([7][8]-[5][6]) + [7][14] - [2][6][8]\big) \\
&= \alpha^{(7)}\alpha^{(8)}\frac{[2][3][5]}{[7]^2[8]^2}([2][13] + [7]\cdot[2][5] - [2][6][8]) \\
&= \alpha^{(7)}\alpha^{(8)}\frac{[2]^2[3][5]}{[7]^2[8]^2}([13]+[5][7]-[6][8]) = 0,
\end{aligned}$$

where the fourth equality is obtained by (A.12) and (A.7), and the last one is obtained by (A.12), completing this case.

(A.18) in the case $m=0$ is trivial.

(A.18) in the case $m=1$ is equivalent to (A.18) in the case $m=-1$.

(A.19) is equivalent to (A.17).

(A.20) is trivial.

Therefore, (A.16)–(A.20) in the case $l = 0$ was shown.

We show (A.16)–(A.20) in the case $l = 1$, as follows.

We show (A.16), as follows. We have that

$$\alpha^{(8)} \frac{[4]'}{\begin{bmatrix} 8 \\ 3 \end{bmatrix}'} - \alpha^{(6)} \frac{[2]'}{\begin{bmatrix} 6 \\ 1 \end{bmatrix}'} = \alpha^{(8)} \Big(- \frac{[4][3][2]}{[8][7][6]} + \frac{\alpha^{(6)}}{\alpha^{(8)}} \cdot \frac{[2]}{[6]} \Big)$$
$$= \alpha^{(8)} \frac{[2][3]}{[6][7]^2[8]^2} \big(- [4][7][8] + [2][13][14] \big) = 0,$$

where the last equality is obtained by (A.8), completing this case.

We show (A.17), as follows. We have that

$$\Bigg(\alpha^{(8)} \sum_{\substack{0 \le i \le 2,\ 0 \le j \le i, \\ 0 \le k \le 1,\ 1-j+k=2}} A(i,j,k,1) + \beta^{(8)} \Bigg) - \alpha^{(6)} \sum_{\substack{0 \le i \le 0,\ 0 \le j \le i, \\ 0 \le k \le 3,\ -1+j-k=2}} A(i,j,k,-1)$$
$$= \alpha^{(8)} \big(A(0,0,1,1) + A(1,0,1,1) + A(2,0,1,1) \big) + \beta^{(8)}$$
$$= \frac{[6]^2[7]^2[8]^2}{[2][3][12][13][14]} \Big(- \frac{[5][6]}{[12]} + \frac{[3][4][5][6]^2[7]}{[2][8][12][13]} - \frac{[3][4][5][6]^2[7]}{[2][12][13][14]} \Big) + \frac{[2]^2[3][6]}{[9]}$$
$$= \frac{[2][5][6]^3[7]^2[8]^2}{[2]^2[3][12]^2 \cdot [2][13][14]} \Big(- [2] + \frac{[2][3][4][6][7]}{[8] \cdot [2][13]} - \frac{[2][3][4][6][7]}{[2][13][14]} \Big) + \frac{[2]^2[3][6]}{[9]}$$
$$= \frac{[2]^2[6][8]}{[4]} \Big(- 1 + \frac{[3][6][7]}{[5][8]} - \frac{[3][6]}{[8]} \Big) + \frac{[2][3][6]}{[4]}$$
$$= \frac{[2][6]}{[4][5]} (-[2][5][8] + [2][3][6][7] - [2][3][5][6] + [3][5]),$$

where the fourth equality is obtained by (A.8), (A.9) and (A.11). The above formula is equal to 0, because

$$[2][3][6][7] - [2][3][5][6] = [2][6]([3][7] - [3][5])$$
$$= [2][6]\{([5] + [7] + [9]) - ([3] + [5] + [7])\} = [2][6]([9] - [3]) = [2][6][5]$$
$$= ([5] + [7])[5] = ([9] - [3] + [7])[5] = ([2][8] - [3])[5]$$
$$= [2][5][8] - [3][5],$$

where the second equality is obtained by (A.14), the fouth and the sixth ones are obtained by (A.15), the fifth and the seventh ones are obtained by (A.13) completing this case.

We show (A.18), as follows. We have that

$$\alpha^{(8)} \sum_{\substack{0 \le i \le 2,\ 0 \le j \le i, \\ 0 \le k \le 1,\ 1-j+k=-1}} A(i,j,k,1) - \alpha^{(6)} \sum_{\substack{0 \le i \le 0,\ 0 \le j \le i, \\ 0 \le k \le 3,\ 1+j-k=-1}} A(i,j,k,-1)$$
$$= \alpha^{(8)} A(2,2,0,1) - \alpha^{(6)} A(0,0,2,-1)$$

$$
\begin{aligned}
&= \alpha^{(8)}\big(A(2,2,0,1) - \frac{\alpha^{(6)}}{\alpha^{(8)}}A(0,0,2,-1)\big)\\
&= \alpha^{(8)}\Big(\frac{[6]^2[7]^2[8]^2}{[2][3][12][13][14]} - \frac{[2][3][13][14]}{[7]^2[8]^2}\cdot\frac{[4][5][6]^2[7][8]}{[2][12][13][14]}\Big)\\
&= \alpha^{(8)}\frac{[3][4][5][6]^2(-[4][7][8]+[2][13][14])}{[2][7][8][12][13][14]} = 0,
\end{aligned}
$$

where the last equality is obtained by (A.11), completing this case

We show (A.18) in the case $m = 0$, as follows. We have that

$$
\begin{aligned}
&\alpha^{(8)}\sum_{\substack{0\le i\le 2,\ 0\le j\le i,\\ 0\le k\le 1,\ 1-j+k=0}} A(i,j,k,1) - \alpha^{(6)}\sum_{\substack{0\le i\le 0,\ 0\le j\le i,\\ 0\le k\le 3,\ 1+j-k=0}} A(i,j,k,-1)\\
&= \alpha^{(8)}\big(A(1,1,0,1)+A(2,1,0,1)+A(2,2,1,1) - \frac{\alpha^{(6)}}{\alpha^{(8)}}A(0,0,1,-1)\big)\\
&= \alpha^{(8)}\Big(-\frac{[3][4][5][6]^2[7]}{[2][8][12][13]} + \frac{[3][4][5][6]^2[7]^2}{[8][12][13][14]} + \frac{[3][4]^2[5][6]^2[7]}{[2][8][12][13][14]}\\
&\quad - \frac{[2][3][13][14]}{[7]^2[8]^2}\cdot\frac{[5][6]^2[7]^2[8]}{[2][12][13][14]}\Big)\\
&= \alpha^{(8)}\frac{[3][5][6]^2([4]^2[7]+[2][4][7]^2-[4][7][14]-[2][13][14])}{[8][12]\cdot[2][13][14]}\\
&= \alpha^{(8)}\frac{[3][5][6]^2}{[8]^2[12]}([4]+[2][7]-[2][5]-[8])\\
&= \alpha^{(8)}\frac{[3][5][6]^2}{[8]^2[12]}\{[4]+([6]+[8])-([4]+[6])-[8]\} = 0,
\end{aligned}
$$

where the fourth equality is obtained by (A.7) and (A.11), and the fifth one is obtained by (A.13), completing this case.

We show (A.18) in the case $m = 1$, as follows. We have that

$$
\begin{aligned}
&\alpha^{(8)}\sum_{\substack{0\le i\le 2,\ 0\le j\le i,\\ 0\le k\le 1,\ 1-j+k=1}} A(i,j,k,1) - \alpha^{(6)}\sum_{\substack{0\le i\le 0,\ 0\le j\le i,\\ 0\le k\le 3,\ 1+j-k=1}} A(i,j,k,-1)\\
&= \alpha^{(8)}\big(A(0,0,0,1)+A(1,0,0,1)+A(1,1,1,1)+A(2,0,0,1)\\
&\qquad + A(2,1,1,1) - \frac{\alpha^{(6)}}{\alpha^{(8)}}A(0,0,0,-1)\big)\\
&= \alpha^{(8)}\Big(-\frac{[6]^2}{[12]} + \frac{[3][4][6]^3[7]}{[2][8][12][13]} + \frac{[3][4][5][6]^3}{[2][8][12][13]} - \frac{[3][4][6]^3[7]}{[2][12][13][14]}\\
&\qquad - \frac{[3][4][5][6]^3[7]}{[8][12][13][14]} + \frac{[2][3][13][14]}{[7]^2[8]^2}\cdot\frac{[6]^2[7]^2[8]^2}{[2][3][12][13][14]}\Big)\\
&= \frac{[3][4][6]^3}{[2][8][12][13][14]}\big([7]([14]-[2][5])+[5][14]-[7][8]\big) = 0,
\end{aligned}
$$

where the last equality is obtained by (A.7) and (A.10), completing this case.

We show (A.19), as follows. We have that

$$
\begin{aligned}
&\alpha^{(8)} \sum_{\substack{0\le i\le 2,\ 0\le j\le i,\\ 0\le k\le 1,\ 1-j+k=-2}} A(i,j,k,1) - \Bigg(\alpha^{(6)} \sum_{\substack{0\le i\le 0,\ 0\le j\le i,\\ 0\le k\le 3,\ -1+j-k=-2}} A(i,j,k,-1) + \beta^{(6)}\Bigg) \\
&= -\alpha^{(6)} A(0,0,3,-1) + \beta^{(6)} \\
&= -\frac{[6]^2}{[12]} \cdot \frac{[4][5][6][7][8]}{[2][12][13][14]} + \frac{[2]^2[3][6]}{[7]} \\
&= \frac{[6]}{[2][7][12]^2[13][14]}(-[4][5][6]^2[7]^2[8] + [2]^2[3][12]^2 \cdot [2][13][14]) \\
&= \frac{[6]}{[2][7][12]^2[13][14]}(-[4][5][6]^2[7]^2[8] + [5][6]^2[7] \cdot [4][7][8]) = 0,
\end{aligned}
$$

where the fourth equality is obtained by (A.8) and (A.11), completing this case.

We show (A.20), as follows. We have that

$$
\begin{aligned}
&\alpha^{(8)} \sum_{i=0}^{2} B(i,1) - \alpha^{(6)} \sum_{i=0}^{0} B(i,-1) \\
&= \frac{[6]^2[7]^2[8]^2}{[2][3][12][13][14]}\Big(-\frac{[2]^2[3][6]}{[7]} + \frac{[2]^2[3]^2[4][6]^2}{[7][8]^2} - \frac{[2]^2[3]^3[4][6]^2}{[7][8]^2[9]}\Big) + \frac{[6]^2}{[12]} \cdot \frac{[2]^2[3][6]}{[9]} \\
&= \frac{[2]^2[6]^3(-[3]^2[4][6][7] + [3][4][6][7][9] - [7][8]^2 \cdot [9] + [3] \cdot [2][13][14])}{[9][12] \cdot [2][13][14]} \\
&= -\frac{[2]^2[6]^3}{[8][9][12]}([3]^2[6] - [2][3][4][6] + [2][8]^2 - [3][8]),
\end{aligned}
$$

where the third equality is obtained by (A.6) and (A.11). The above formula is equal to 0, because

$$
\begin{aligned}
[2]^2[8] - [3][8] &= ([2][8] - [3])[8] = ([7] + [9] - [3])[8] = ([7] + [5])[8] \\
&= [2][6][8] = ([7] + [9])[6] = ([3] + [5] + [7])[6] \\
&= [3][5][6] = [3][6]([3] - [2][4]) = [3]^2[6] - [2][3][4][6],
\end{aligned}
$$

where the second, the fourth, the fifth and the eighth equalities are obtained by (A.13), the third ones are obtained by (A.15), and the seventh one is obtained by (A.14), completing this case.

Therefore, (A.16)–(A.20) in the case $l = 1$ was shown. □

References

[1] S. Bigelow, *Skein theory for the ADE planar algebras*, J. Pure Appl. Algebra **214** (2010) 658–666.

[2] V. Jones, *Planar algebras, I*, `arxiv.org:math/9909027v1[math.QA]`.

[3] W. Lickorish, *An introduction to knot theory*, Graduate Texts in Math. **175**, Springer-Verlag, 1997.

[4] Scott Morrison, *A formula for the Jones-Wenzl projections*, unpublished, available at `http://tqft.net/math/JonesWenzlProjections.pdf`

[5] S. Morrison, E. Peters, N. Snyder, *Skein theory for the D_{2n} planar algebras*, J. Pure Appl. Algebra **214** (2010) 117–139.

[6] K. Okazaki, *The state sum invariant of 3-manifolds constructed from the E_6 linear skein*, Algebraic and Geometric Topology **13** (2013) 3469–3536.