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Frobenius-projective Structures on Curves in Positive Characteristic

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Abstract. — In the present paper, we study *Frobenius-projective structures* on projective smooth curves in positive characteristic. The notion of Frobenius-projective structures may be regarded as an analogue, in positive characteristic, of the notion of *complex projective structures* in the classical theory of Riemann surfaces. By means of the notion of Frobenius-projective structures, we obtain a relationship between a certain rational function, i.e., a *pseudo-coordinate*, and a certain collection of data which may be regarded as an analogue, in positive characteristic, of the notion of *indigenous bundles* in the classical theory of Riemann surfaces, i.e., a *Frobenius-indigenous structure*. As an application of this relationship, we also prove the existence of certain *Frobenius-destabilized* locally free coherent sheaves of rank two.

Contents

Introduction

In the present paper, we study *Frobenius-projective structures* on projective smooth curves in positive characteristic. Let p be a prime number, k an algebraically closed field of characteristic p, g a nonnegative integer, and

$$X$$

a projective smooth curve over k of genus g. Throughout the present paper, let us fix a positive integer

$$N.$$

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Write X^F for the base change of X via the [not p-th if $N \neq 1$ but] p^N-th power Frobenius endomorphism of k and $\Phi\colon X \to X^F$ for the relative p^N-th power Frobenius morphism over k. Thus, by pulling back the "PGL_2" on X^F via Φ, we obtain a sheaf $\mathcal{G}$ of groups on X. Then a *Frobenius-projective structure of level* N on X is defined to be a subsheaf of the sheaf on X of étale morphisms over k to $\mathbb{P}^1_k$ which forms a $\mathcal{G}$-torsor, i.e., relative to the natural action of $\mathcal{G}$ on the sheaf on X of morphisms over k to $\mathbb{P}^1_k$ [cf. Definition 2.1]. One may find easily that the notion of Frobenius-projective structures may be regarded as an analogue, in positive characteristic, of the notion of *complex projective structures* [cf., e.g., [1], §2] in the classical theory of Riemann surfaces. The main result of the present paper discusses a relationship between a certain rational function on X — i.e., a *pseudo-coordinate* — and a certain pair of a $\mathbb{P}^1$-bundle $P \to X^F$ over X^F and a section of the pull-back $\Phi^*P \to X$ — i.e., a *Frobenius-indigenous structure* — obtained by considering a Frobenius-projective structure.

A *pseudo-coordinate of level* N on X is defined to be a [necessarily generically étale] morphism $X \to \mathbb{P}^1_k$ over k such that, for each closed point $x \in X$ of X, the result of the action of an element, that may depend on x, of the stalk $\mathcal{G}_{\mathrm{rtn}}$ of $\mathcal{G}$ at the generic point of X on the morphism is étale at x [cf. Definition 1.3]. For instance, if $p \in \{2, 3\}$, then every generically étale morphism to $\mathbb{P}^1_k$ over k is a pseudo-coordinate of level 1 [cf. Proposition 1.8, (ii)]. Moreover, if $p = 2$, then, for a morphism to $\mathbb{P}^1_k$ over k, it holds that the morphism is a *pseudo-tame* rational function in the sense of [8] if and only if the morphism is a pseudo-coordinate of level 2 [cf. Remark 1.3.2]. In [8], *Y. Sugiyama* and *S. Yasuda* studied pseudo-tame rational functions in order to prove an analogue in characteristic two of *Belyi's theorem*. In the proof of their main theorem, i.e., *Belyi's theorem in characteristic two*, they proved that [if $p = 2$, then] X always admits a pseudo-tame rational function [cf. [8], Corollary 3.8]. The main motivation of the study of the present paper is in fact to understand, in more conceptual terms, the notion of pseudo-tame defined in [8].

A *Frobenius-indigenous structure of level* N on X is defined to be a pair consisting of a $\mathbb{P}^1$-bundle $P \to X^F$ over X^F and a section σ of the pull-back $\Phi^*P \to X$ such that the Kodaira-Spencer section of the connection ∇_{Φ^*P} on Φ^*P [cf. Definition 3.2] at σ [i.e., the global section of the invertible sheaf $\omega_{X/k} \otimes_{\mathcal{O}_X} \sigma^*\tau_{\Phi^*P/X}$ on X obtained by differentiating σ via the connection ∇_{Φ^*P} — cf. Definition 3.3] is nowhere vanishing [cf. Definition 3.4]. One may find easily that the notion of Frobenius-indigenous structures may be regarded as an analogue, in positive characteristic, of the notion of *indigenous bundles* [cf., e.g., [1], §2] in the classical theory of Riemann surfaces. If $p \neq 2$, and $g \geq 2$, then the notion of Frobenius-indigenous structures of level 1 is essentially the same as the notion of *dormant indigenous bundles* studied in *p-adic Teichmüller theory*; moreover, there is a certain direct relationship between Frobenius-indigenous structures of level N and objects studied in *p-adic Teichmüller theory* even if $N \neq 1$ [cf. Remark 3.4.1].

The main result of the present paper is as follows [cf. Theorem 3.13].

THEOREM A. — *Suppose that* $(p, N) \neq (2, 1)$. *Then there exist* **bijections** *between the following three sets:*

(1) *the set of* $\mathcal{G}_{\mathrm{rtn}}$*-orbits of* **pseudo-coordinates of level** $\boldsymbol{N}$ *on* X

(2) *the set of* **Frobenius-projective structures of level** $\boldsymbol{N}$ *on* X

(3) *the set of isomorphism classes of* **Frobenius-indigenous structures of level $\boldsymbol{N}$** *on* X

Note that a result in the case where $(p, N) = (2, 1)$ similar to Theorem A is discussed in Corollary 5.7.

Theorem A has some applications. For instance, by applying Theorem A and a result in p-adic Teichmüller theory, one may verify that if $g \geq 2$, then X always admits a pseudo-coordinate of level 1 [cf. Corollary 1.9; Corollary 4.9, (i)]. Moreover, Theorem A yields a [*fifth* — cf. Remark 3.12.1] proof of the *uniqueness* of the isomorphism class of dormant indigenous bundles on a projective smooth curve of genus ≥ 2 in characteristic three [cf. Remark 3.4.1, (ii); Corollary 3.12].

Another application of Theorem A is as follows. Write $\mathrm{Fr}_X : X \to X$ for the p-th power Frobenius endomorphism of X. Then it is immediate [cf. Remark 4.2.3, Proposition 4.7] that if $g \geq 2$, then there exists a *bijection* between the set of (3) of Theorem A and the set of $\mathbb{P}$-equivalence [cf. Definition 4.1] classes of locally free coherent $\mathcal{O}_X$-modules $\mathcal{E}$ of rank two which satisfy the following condition: If, for a nonnegative integer i, we write

$$\mathcal{E}_i \stackrel{\mathrm{def}}{=} \overbrace{\mathrm{Fr}_X^* \circ \cdots \circ \mathrm{Fr}_X^*}^{i} \mathcal{E},$$

then

- the locally free coherent $\mathcal{O}_X$-module $\mathcal{E}_{N-1}$, hence also $\mathcal{E}$, is *stable*, but

- there exist an invertible sheaf $\mathcal{L}$ on X of degree $\frac{p^N}{2} \cdot \deg(\mathcal{E}) + g - 1 = \frac{1}{2} \cdot \deg(\mathcal{E}_N) + g - 1$ and a locally split injection $\mathcal{L} \hookrightarrow \mathcal{E}_N$ of $\mathcal{O}_X$-modules. [In particular, the locally free coherent $\mathcal{O}_X$-module $\mathcal{E}_N$ is *not semistable*.]

Thus, by applying Theorem A and the above existence of pseudo-tame rational functions proved in [8], we obtain the following application [cf. Remark 6.2.1].

THEOREM B. — *Suppose that $p = 2$, and that $g \geq 2$. Then there exists a locally free coherent $\mathcal{O}_X$-module $\mathcal{E}$ of rank two such that*

- *the locally free coherent $\mathcal{O}_X$-module $\mathrm{Fr}_X^* \mathcal{E}$, hence also $\mathcal{E}$, is* **stable**, *but*

- *the locally free coherent $\mathcal{O}_X$-module $\mathrm{Fr}_X^* \mathrm{Fr}_X^* \mathcal{E}$ admits an invertible subsheaf $\mathcal{L} \subseteq \mathrm{Fr}_X^* \mathrm{Fr}_X^* \mathcal{E}$ of degree $\frac{1}{2} \deg(\mathrm{Fr}_X^* \mathrm{Fr}_X^* \mathcal{E}) + g - 1$, which thus implies that $\mathrm{Fr}_X^* \mathrm{Fr}_X^* \mathcal{E}$ is* **not semistable**.

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1. Pseudo-coordinates

In the present §1, we introduce and discuss the notion of *pseudo-coordinates* on curves [cf. Definition 1.3 below], which may be regarded as a generalization of the notion of *pseudo-tame* rational functions studied in [8] [cf. Remark 1.3.2 below].

In the present §1, let p be a prime number, k an algebraically closed field of characteristic p, g a nonnegative integer, and

$$X$$

a projective smooth curve over k of genus g. We shall write K_X for the function field of X. Throughout the present paper, let us fix a positive integer

$$N.$$

If "$(-)$" is an object over k, then we shall write "$(-)^F$" for the object over k obtained by base changing "$(-)$" via the [not p-th if $N \neq 1$ but] p^N-th power Frobenius endomorphism of k. We shall write

$$W\colon\ X^F \longrightarrow X$$

for the morphism obtained by base changing the p^N-th power Frobenius endomorphism of $\mathrm{Spec}(k)$ via the structure morphism $X \to \mathrm{Spec}(k)$. Thus, the p^N-th power Frobenius endomorphism of X factors as the composite

$$X \longrightarrow X^F \xrightarrow{W} X.$$

We shall write

$$\Phi\colon\ X \longrightarrow X^F$$

for the first arrow in this composite, i.e., the *relative p^N-th power Frobenius morphism* over k. Note that X^F is a *projective smooth curve over k of genus g*, and Φ is a *finite flat* morphism over k *of degree p^N*.

Definition 1.1. — We shall write

$$\mathcal{P}$$

for the sheaf of sets on X that assigns, to an open subscheme $U \subseteq X$, the set of morphisms from U to $\mathbb{P}^1_k$ over k,

$$\mathcal{P}^{\text{gét}} \subseteq \mathcal{P}$$

for the subsheaf of $\mathcal{P}$ that assigns, to an open subscheme $U \subseteq X$, the set of generically étale morphisms from U to $\mathbb{P}^1_k$ over k, and

$$\mathcal{P}^{\text{ét}} \subseteq \mathcal{P}^{\text{gét}}$$

for the subsheaf of $\mathcal{P}^{\text{gét}}$ that assigns, to an open subscheme $U \subseteq X$, the set of étale morphisms from U to $\mathbb{P}^1_k$ over k.

Remark 1.1.1. — One verifies easily that both $\mathcal{P}$ and $\mathcal{P}^{\text{gét}}$ are [isomorphic to] *constant sheaves.*

Remark 1.1.2. — One verifies easily that $\mathcal{P}$ may be naturally identified with the sheaf of sets on X that assigns, to an open subscheme $U \subseteq X$, the set of sections of the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1_U \to U$.

DEFINITION 1.2.

(i) Let S be a scheme. Then we shall write

$$\mathrm{PGL}_{2,S}$$

for the sheaf of groups on S that assigns, to an open subscheme $T \subseteq S$, the group $\mathrm{Aut}_T(\mathbb{P}^1_T)$ of automorphisms over T of the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1_T \to T$.

(ii) We shall write

$$\mathcal{G} \stackrel{\mathrm{def}}{=} \Phi^{-1}\mathrm{PGL}_{2,X^F}$$

and

$$\mathcal{G}_{\mathrm{rtn}}$$

for the group obtained by forming the stalk of $\mathcal{G}$ at the generic point of X [i.e., the "PGL_2" for the function field of X^F].

REMARK 1.2.1.

(i) It follows immediately from Remark 1.1.2 that $\mathcal{G}$ naturally acts, via Φ, on $\mathcal{P}$. Moreover, one verifies easily that the subsheaves $\mathcal{P}^{\text{ét}} \subseteq \mathcal{P}^{\text{gét}} \subseteq \mathcal{P}$ of $\mathcal{P}$ are *preserved* by this action of $\mathcal{G}$ on $\mathcal{P}$.

(ii) It follows from Remark 1.1.1 that the actions of $\mathcal{G}$ on $\mathcal{P}$, $\mathcal{P}^{\text{gét}}$ of (i) determine actions of $\mathcal{G}_{\mathrm{rtn}}$ on $\mathcal{P}(X)$, $\mathcal{P}^{\text{gét}}(X)$, respectively.

DEFINITION 1.3. — We shall say that a global section $f \in \mathcal{P}^{\text{gét}}(X)$ of $\mathcal{P}^{\text{gét}}$ is a *pseudo-coordinate of level* N if, for each closed point $x \in X$ of X, there exist an open subscheme $U \subseteq X$ of X and an element $g \in \mathcal{G}_{\mathrm{rtn}}$ of $\mathcal{G}_{\mathrm{rtn}}$ such that $x \in U$, and, moreover, the restriction $g(f)|_U \in \mathcal{P}^{\text{gét}}(U)$ to U of the result $g(f) \in \mathcal{P}^{\text{gét}}(X)$ of the action of $g \in \mathcal{G}_{\mathrm{rtn}}$ on $f \in \mathcal{P}^{\text{gét}}(X)$ [cf. Remark 1.2.1, (ii)] is contained in the subset $\mathcal{P}^{\text{ét}}(U) \subseteq \mathcal{P}^{\text{gét}}(U)$ of $\mathcal{P}^{\text{gét}}(U)$.

We shall write

$$\mathfrak{pcd}_N(X) \subseteq \mathcal{P}^{\text{gét}}(X)$$

for the subset of pseudo-coordinates of level N.

REMARK 1.3.1. — One verifies easily that if a global section of $\mathcal{P}^{\text{gét}}$ is a *pseudo-coordinate of level* N, then every element of the $\mathcal{G}_{\mathrm{rtn}}$-orbit ($\subseteq \mathcal{P}^{\text{gét}}(X)$) of the global section is a *pseudo-coordinate of level* N.

REMARK 1.3.2. — Suppose that $(p, N) = (2, 2)$. Then one verifies easily that, for a global section $f \in \mathcal{P}^{\text{gét}}(X)$ of $\mathcal{P}^{\text{gét}}$, it holds that $f \in \mathcal{P}^{\text{gét}}(X)$ is a *pseudo-coordinate of level* N in the sense of Definition 1.3 if and only if $f \in \mathcal{P}^{\text{gét}}(X)$ [i.e., the generically étale morphism $f \colon X \to \mathbb{P}^1_k$ over k] is *pseudo-tame* in the sense of [8], Definition 2.1 [cf. also [8], Remark 2.6].

DEFINITION 1.4. — Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a global section of $\mathcal{P}^{\text{gét}}$ and $x \in X$ a closed point of X. Let us identify $A \stackrel{\text{def}}{=} k[[t]]$ with the completion $\widehat{\mathcal{O}}_{X,x}$ of $\mathcal{O}_{X,x}$ by means of a fixed isomorphism $A \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ over k. Write $F \in \mathcal{O}_{X,x}$ for the image in $\mathcal{O}_{X,x}$ of a fixed uniformizer of the discrete valuation ring $\mathcal{O}_{\mathbb{P}^1_k, f(x)}$ and

$$F \;=\; \sum_{i \geq 1} a_i t^i \;\in\; A$$

for the expansion of F in A. [Thus, the positive integer

$$\mathrm{ind}_x(f) \;\stackrel{\text{def}}{=}\; \deg(F) \;=\; \min\{\, i \in \mathbb{Z}_{\geq 1} \mid a_i \neq 0 \,\}$$

coincides with the ramification index of the dominant morphism $f\colon X \to \mathbb{P}^1_k$ at $x \in X$.] Then we shall write

$$\mathrm{ind}_x^{\notin p^N}(f) \;\stackrel{\text{def}}{=}\; \min\{\, i \in \mathbb{Z}_{\geq 1} \mid a_i \neq 0 \text{ and } i \notin p^N\mathbb{Z} \,\}$$

and

$$\underline{\mathrm{ind}}_x^{\notin p^N}(f)$$

for the uniquely determined positive integer such that $1 \leq \underline{\mathrm{ind}}_x^{\notin p^N}(f) \leq p^N - 1$, and, moreover, $\mathrm{ind}_x^{\notin p^N}(f) - \underline{\mathrm{ind}}_x^{\notin p^N}(f) \in p^N\mathbb{Z}$.

Note that one verifies easily that since f is a global section of $\mathcal{P}^{\text{gét}}$, it holds that $\mathrm{ind}_x^{\notin p^N}(f) < \infty$. Moreover, one also verifies easily that both $\mathrm{ind}_x^{\notin p^N}(f)$ and $\underline{\mathrm{ind}}_x^{\notin p^N}(f)$ do *not depend* on the choices of the fixed isomorphism $A \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ and the fixed uniformizer of $\mathcal{O}_{\mathbb{P}^1_k, f(x)}$.

LEMMA 1.5. — *Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a global section of $\mathcal{P}^{\text{gét}}$ and $x \in X$ a closed point of X. Then the following hold.*

(i) *There exists an element $g \in \mathcal{G}_{\text{rtn}}$ of $\mathcal{G}_{\text{rtn}}$ such that*

$$\mathrm{ind}_x(g(f)) \;=\; \mathrm{ind}_x^{\notin p^N}(f)$$

[*which thus implies that* $\mathrm{ind}_x(g(f)) = \mathrm{ind}_x^{\notin p^N}(g(f))$].

(ii) *Suppose that* $\mathrm{ind}_x(f) = \mathrm{ind}_x^{\notin p^N}(f)$. *Then there exist elements g_+, $g_- \in \mathcal{G}_{\text{rtn}}$ of $\mathcal{G}_{\text{rtn}}$ such that*

$$\mathrm{ind}_x(g_+(f)) \;=\; \underline{\mathrm{ind}}_x^{\notin p^N}(f), \quad \mathrm{ind}_x(g_-(f)) \;=\; p^N - \underline{\mathrm{ind}}_x^{\notin p^N}(f)$$

[*which thus implies that* $\mathrm{ind}_x(g_+(f)) = \underline{\mathrm{ind}}_x^{\notin p^N}(g_+(f))$, $\mathrm{ind}_x(g_-(f)) = p^N - \underline{\mathrm{ind}}_x^{\notin p^N}(g_-(f))$].

PROOF. — Let us identify the scheme $\mathrm{Proj}(k[u,v])$ with $\mathbb{P}^1_k$ by means of a fixed isomorphism $\mathrm{Proj}(k[u,v]) \xrightarrow{\sim} \mathbb{P}^1_k$ over k. Thus, the global section $f \in \mathcal{P}^{\text{gét}}(X)$ determines and is determined by an element $F \in K_X \setminus K_X^p$ of $K_X \setminus K_X^p$ [i.e., the image of $u/v \in k(u/v)$ in K_X via f]. To verify Lemma 1.5, we may assume without loss of generality, by replacing f by the composite of f and a suitable element of $\mathrm{Aut}_k(\mathbb{P}^1_k)$, that the image of $x \in X$ via f is the point "$(u,v) = (0,1)$", i.e., that $F \in \mathcal{O}_{X,x} \setminus \mathcal{O}_{X,x}^{\times}$.

We verify assertion (i). Write r for the uniquely determined nonnegative integer such that $rp^N < \mathrm{ind}_x^{\notin p^N}(f) < (r+1)p^N$. Then it is immediate that there exists an element

$a \in \mathcal{O}_{X,x}$ of $\mathcal{O}_{X,x}$ such that $F - a^{p^N} \in \mathfrak{m}_{X,x}^{rp^N+1}$. Thus, if we take $g \in \mathcal{G}_{\mathrm{rtn}}$ to be an element which maps F to $F - a^{p^N}$, e.g.,

$$g \ = \ \text{``}\begin{pmatrix} 1 & -a^{p^N} \\ 0 & 1 \end{pmatrix}\text{''} \ \in \ \mathcal{G}_{\mathrm{rtn}},$$

then g satisfies the condition of assertion (i). This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $\pi \in \mathcal{O}_{X,x}$ be a uniformizer of $\mathcal{O}_{X,x}$. Write r_+ for the uniquely determined nonnegative integer such that $r_+ p^N = \mathrm{ind}_x^{\not\in p^N}(f) - \underline{\mathrm{ind}}_x^{\not\in p^N}(f)$ and $r_- \stackrel{\mathrm{def}}{=} r_+ + 1$. Then if we take g_+, $g_- \in \mathcal{G}_{\mathrm{rtn}}$ to be elements which map F to $F/\pi^{r_+ p^N}$, $\pi^{r_- p^N}/F$, e.g.,

$$g_+ \ = \ \text{``}\begin{pmatrix} 1 & 0 \\ 0 & \pi^{r_+ p^N} \end{pmatrix}\text{''} \ \in \ \mathcal{G}_{\mathrm{rtn}}, \quad g_- \ = \ \text{``}\begin{pmatrix} 0 & \pi^{r_- p^N} \\ 1 & 0 \end{pmatrix}\text{''} \ \in \ \mathcal{G}_{\mathrm{rtn}},$$

then g_+, g_- satisfy the conditions of assertion (ii), respectively. This completes the proof of assertion (ii), hence also of Lemma 1.5. □

LEMMA 1.6. — *Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a global section of $\mathcal{P}^{\text{gét}}$ and $x \in X$ a closed point of X. Suppose that $\underline{\mathrm{ind}}_x^{\not\in p^N}(f) \notin \{1, p^N - 1\}$. Then, for each $g \in \mathcal{G}_{\mathrm{rtn}}$, the result $g(f) \in \mathcal{P}^{\text{gét}}(X)$ of the action of $g \in \mathcal{G}_{\mathrm{rtn}}$ on $f \in \mathcal{P}^{\text{gét}}(X)$ is* **not étale** *at x.*

PROOF. — Let us first observe that it follows immediately from Lemma 1.5, (i), (ii), that we may assume without loss of generality, by replacing f by the result of the action of a suitable element of $\mathcal{G}_{\mathrm{rtn}}$ on f, that

(a) $\mathrm{ind}_x(f) = d_0 \stackrel{\mathrm{def}}{=} \underline{\mathrm{ind}}_x^{\not\in p^N}(f)$ $(\notin \{1, p^N - 1\})$.

Let us identify $A \stackrel{\mathrm{def}}{=} k[[t]]$ with the completion $\widehat{\mathcal{O}}_{X,x}$ of $\mathcal{O}_{X,x}$ by means of a fixed isomorphism $A \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ over k. Then it is immediate that, to verify Lemma 1.6, it suffices to verify that

($*_1$): for each $g \in \mathcal{G}_{\mathrm{rtn}}$, the composite of the natural morphism $\mathrm{Spec}(A) \to X$ with $g(f)\colon X \to \mathbb{P}^1_k$ is *not formally étale.*

Let $g \in \mathcal{G}_{\mathrm{rtn}}$ be an element of $\mathcal{G}_{\mathrm{rtn}}$. Next, let us identify the scheme $\mathrm{Proj}(k[u,v])$ with $\mathbb{P}^1_k$ by means of a fixed isomorphism $\mathrm{Proj}(k[u,v]) \xrightarrow{\sim} \mathbb{P}^1_k$ over k. Write K for the field of fractions of A and

$$\mathrm{Proj}(k[u,v]) \longleftarrow \mathrm{Spec}(A); \quad (u,v) \ \mapsto \ (f_u, f_v)$$

— where f_u, $f_v \in A$ — for the composite of the natural morphism $\mathrm{Spec}(A) \to X$ with $f\colon X \to \mathbb{P}^1_k$. Thus, there exist a_g, b_g, c_g, $d_g \in k[[t^{p^N}]] = A^{p^N} \subseteq A$ [which thus implies that $\deg(a_g)$, $\deg(b_g)$, $\deg(c_g)$, $\deg(d_g) \in p^N\mathbb{Z}$] such that $a_g d_g - b_g c_g \neq 0$, and, moreover, the composite of the natural morphism $\mathrm{Spec}(A) \to X$ with $g(f)\colon X \to \mathbb{P}^1_k$ *coincides* with the morphism determined by the composite

$$\begin{array}{ccccccc}
\mathrm{Proj}(k[u,v]) & \longleftarrow & \mathrm{Proj}(K[u,v]) & \longleftarrow & \mathrm{Proj}(K[u,v]) & \longleftarrow & \mathrm{Spec}(K) \\
(u,v) & \mapsto & (u,v) & & & & \\
 & & (u,v) & \mapsto & (a_g u + b_g v, c_g u + d_g v) & & \\
 & & & & (u,v) & \mapsto & (f_u, f_v).
\end{array}$$

Next, let us observe that, to verify ($*_1$), we may assume without loss of generality, by replacing f by the composite of f and a suitable element of $\mathrm{Aut}_k(\mathbb{P}^1_k)$, that the image of $x \in X$ via f is the point "$(u, v) = (0, 1)$", i.e., that [cf. (a)]

(b) $\deg(f_u) = d_0$, and $f_v = 1$. [Recall that $2 \leq d_0 \leq p^N - 2$ — cf. (a).]

Next, let us observe that, to verify ($*_1$), we may assume without loss of generality, by replacing g by the product of g and a suitable element of $\mathrm{Aut}_k(\mathbb{P}^1_k)$, that the image of $x \in X$ via $g(f)$ is the point "$(u, v) = (0, 1)$", i.e., that [cf. (b)]

(c) if write

$$F \stackrel{\text{def}}{=} \frac{a_g f_u + b_g}{c_g f_u + d_g} \in K,$$

then $F \in A$, and, moreover, $\deg(F) \geq 1$.

Thus, it is immediate that, to verify ($*_1$), it suffices to verify that

($*_2$): $\deg(F) \neq 1$.

Next, let us observe that, to verify ($*_2$), we may assume without loss of generality, by replacing (a_g, b_g, c_g, d_g) by $t^{-\min\{\deg(a_g), \deg(b_g), \deg(c_g), \deg(d_g)\}} \cdot (a_g, b_g, c_g, d_g)$, that

(d) $0 \in \{\deg(a_g), \deg(b_g), \deg(c_g), \deg(d_g)\}$.

Here, let us verify that

(e) $\deg(b_g) \geq p^N$.

Indeed, if $\deg(b_g) = 0$, then it follows from (b) that $\deg(a_g f_u + b_g) = 0$, which thus implies that $\deg(F) \leq 0$ — in *contradiction* to (c). This completes the proof of (e).

Next, suppose that $\deg(d_g) = 0$. Then it follows from (b) that $\deg(c_g f_u + d_g) = 0$. In particular, it follows from (b) and (e) that $\deg(F) = \deg(a_g f_u + b_g) \geq 2$, as desired. Thus, to verify ($*_2$), we may assume without loss of generality that

(f) $\deg(d_g) \geq p^N$.

Next, let us verify that

(g) $\deg(a_g) \geq p^N$.

Indeed, if $\deg(a_g) = 0$, then it follows from (b) and (e) that $\deg(a_g f_u + b_g) = d_0$. In particular, it follows from (c) that $1 \leq \deg(F) = d_0 - \deg(c_g f_u + d_g)$, which thus implies that $\deg(c_g f_u + d_g) \leq d_0 - 1$. Assume that $\deg(c_g) = 0$; then it follows from (b) and (f) that $\deg(c_g f_u + d_g) = d_0$ — in *contradiction* to the above inequality $\deg(c_g f_u + d_g) \leq d_0 - 1$. Assume that $\deg(c_g) \geq p^N$; then it follows from (f) that $\deg(c_g f_u + d_g) \geq p^N$ — in *contradiction* to the above inequality $\deg(c_g f_u + d_g) \leq d_0 - 1$. This completes the proof of (g).

It follows from (d), (e), (f), (g) that $\deg(c_g) = 0$. Thus, it follows from (b) and (f) that $\deg(c_g f_u + d_g) = d_0 \leq p^N - 2$. In particular, since $\deg(a_g f_u + b_g) \geq p^N$ [cf. (e), (g)], it holds that $\deg(F) \geq 2$, as desired. This completes the proof of Lemma 1.6. □

PROPOSITION 1.7. — *Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a global section of $\mathcal{P}^{\text{gét}}$. Then it holds that f is a* **pseudo-coordinate of level N** *if and only if, for each closed point $x \in X$ of X, it* *holds that*

$$\underline{\mathrm{ind}}_x^{\notin p^N}(f) \in \{1, p^N - 1\}.$$

PROOF. — The sufficiency follows immediately from Lemma 1.5, (i), (ii). The necessity follows immediately from Lemma 1.6. □

PROPOSITION 1.8. — *Suppose that* $p \in \{2,3\}$*, and that* $N = 1$*. Then the following hold:*

(i) *It holds that* $\sharp(\mathcal{P}^{\text{gét}}(X)/\mathcal{G}_{\text{rtn}}) = 1$.

(ii) *It holds that* $\mathcal{P}^{\text{gét}}(X) = \mathfrak{pcd}_N(X)$.

PROOF. — First, we verify assertion (i). Let $F \in K_X$ be an element of K_X such that $\mathrm{ord}_x(F) = 1$ for some closed point $x \in X$ of X. [It is immediate that such an $F \in K_X$ always exists.] Then one verifies easily that $\{F^i\}_{i=0}^{p-1}$ forms a basis of the vector space K_X over K_X^p. Thus, it is immediate that, to verify assertion (i), it suffices to verify the following assertion:

> For each $a_0, \ldots, a_{p-1} \in K_X^p$ such that $(a_1, \ldots, a_{p-1}) \neq (0, \ldots, 0)$, there exist a, b, c, $d \in K_X^p$ such that
>
> $$a_0 + a_1 F + \cdots + a_{p-1} F^{p-1} = \frac{aF + b}{cF + d}, \quad ad - bc \neq 0.$$

On the other hand, since $p \in \{2,3\}$, this assertion may be easily verified. This completes the proof of assertion (i). Assertion (ii) follows from Proposition 1.7. This completes the proof of Proposition 1.8. □

REMARK 1.8.1. — Proposition 1.8 in the case where $p \notin \{2,3\}$ does *not hold* as follows:

(i) Let us discuss Proposition 1.8, (i), in the case where $p \notin \{2,3\}$. Suppose that we are in the situation of the first paragraph of the proof of Lemma 1.5. Suppose, moreover, that the element $F \in K_X$ satisfies that $\mathrm{ord}_x(F) = 1$. [It is immediate that such a pair "(f, x)" always exists.] Then it follows immediately from Lemma 1.6 that [if $p \neq 3$, then] $F \in K_X$ is *not contained* in the $\mathrm{PGL}_2(K_X^p)$-orbit of $F^2 \in K_X$. One verifies easily from this observation that Proposition 1.8, (i), in the case where $p \notin \{2,3\}$ does *not hold*.

(ii) Let us discuss Proposition 1.8, (ii), in the case where $p \notin \{2,3\}$. Suppose that we are in the situation of the discussion of (i). Then it follows from Proposition 1.7 that [if $p \notin \{2,3\}$, then] $F^2 \in K_X \setminus K_X^p$ determines a global section of $\mathcal{P}^{\text{gét}}$ which is *not a pseudo-coordinate of level* N. In particular, Proposition 1.8, (ii), in the case where $p \notin \{2,3\}$ does *not hold*.

COROLLARY 1.9. — *Suppose that* $p \in \{2,3\}$*, and that* $N = 1$*. Then* $\sharp(\mathfrak{pcd}_N(X)/\mathcal{G}_{\text{rtn}}) = 1$.

PROOF. — This assertion follows from Proposition 1.8, (i), (ii). □

2. Frobenius-projective Structures

In the present §2, we introduce and discuss the notion of *Frobenius-projective structures* on curves [cf. Definition 2.1 below]. Moreover, we also discuss a relationship between Frobenius-projective structures and pseudo-coordinates [cf. Proposition 2.7 below]. In the present §2, we maintain the notational conventions introduced at the beginning of §1.

Definition 2.1. — We shall say that a subsheaf $\mathcal{S} \subseteq \mathcal{P}^{\text{ét}}$ of $\mathcal{P}^{\text{ét}}$ is a *Frobenius-projective structure of level* N on X if $\mathcal{S}$ is preserved by the action of $\mathcal{G}$ on $\mathcal{P}^{\text{ét}}$ [cf. Remark 1.2.1, (i)], and, moreover, the sheaf $\mathcal{S}$ forms, by the resulting action of $\mathcal{G}$ on $\mathcal{S}$, a $\mathcal{G}$-torsor on X.

We shall write

$$\mathfrak{Fps}_N(X)$$

for the set of Frobenius-projective structures of level N on X.

Remark 2.1.1.

(i) One may find easily that the notion of *Frobenius-projective structures* may be regarded as an analogue, in positive characteristic, of the notion of *complex projective structures* [cf., e.g., [1], §2] in the classical theory of Riemann surfaces.

(ii) One may also find another algebraic analogue of the notion of *complex projective structures* in [5], i.e., the notion of *Schwarz structures* defined in [5], Chapter I, Definition 1.2.

(iii) As discussed in Proposition 3.11 below, a *Frobenius-projective structure* is related to a *Frobenius-indigenous structure* [cf. Definition 3.4 below]. On the other hand, a suitable *Schwarz structure* is related to an *indigenous bundle* [cf. [5], Chapter I, Corollary 2.9].

Lemma 2.2. — *Let* $\mathcal{S} \subseteq \mathcal{P}^{\text{ét}}$ *be a* **Frobenius-projective structure of level** $\boldsymbol{N}$ *on* X*. Then the following hold:*

(i) *Let* U, $V \subseteq X$ *be open subschemes of* X, $f_U \in \mathcal{S}(U)$, *and* $f_V \in \mathcal{S}(V)$*. Then the global section of* $\mathcal{P}^{\text{gét}}$ *determined by* $f_U \in \mathcal{S}(U)$ [*cf. Remark* 1.1.1] *is* **contained** *in the* $\mathcal{G}_{\text{rtn}}$*-orbit of the global section of* $\mathcal{P}^{\text{gét}}$ *determined by* $f_V \in \mathcal{S}(V)$*.*

(ii) *The global section of* $\mathcal{P}^{\text{gét}}$ *determined by a local section of* $\mathcal{S}$ *is a* **pseudo-coordinate of level** $\boldsymbol{N}$*.*

Proof. — Since X is *irreducible*, assertion (i) follows from the fact that $\mathcal{S}$ is a *$\mathcal{G}$-torsor*. Assertion (ii) follows from assertion (i), together with the fact that $\mathcal{S}$ is *contained* in $\mathcal{P}^{\text{ét}}$. □

Definition 2.3. — Let $\mathcal{S} \subseteq \mathcal{P}^{\text{ét}}$ be a Frobenius-projective structure of level N on X. Then it follows from Lemma 2.2, (i), (ii), that $\mathcal{S}$ determines a $\mathcal{G}_{\text{rtn}}$-orbit of pseudo-coordinates of level N. We shall refer to this $\mathcal{G}_{\text{rtn}}$-orbit as the *pseudo-coordinate-orbit of* *level N associated to $\mathcal{S}$.* Thus, we obtain a map

$$\mathfrak{Fps}_N(X) \longrightarrow \mathfrak{pcd}_N(X)/\mathcal{G}_{\mathrm{rtn}}.$$

LEMMA 2.4. — *Suppose that $(p, N) \neq (2, 1)$. Let $U \subseteq X$ be an open subscheme of X, $f \in \mathcal{P}^{\text{ét}}(U)$, and $g \in \mathcal{G}_{\mathrm{rtn}}$. Then it holds that the result $g(f) \in \mathcal{P}^{\text{gét}}(U)$ of the action of $g \in \mathcal{G}_{\mathrm{rtn}}$ on $f \in \mathcal{P}^{\text{ét}}(U) \subseteq \mathcal{P}^{\text{gét}}(U)$ [cf. Remark* 1.1.1*; Remark* 1.2.1, (i)] *is* **contained** *in the subset $\mathcal{P}^{\text{ét}}(U) \subseteq \mathcal{P}^{\text{gét}}(U)$ if and only if $g \in \mathcal{G}_{\mathrm{rtn}}$ is* **contained** *in the subgroup $\mathcal{G}(U) \subseteq \mathcal{G}_{\mathrm{rtn}}$.*

PROOF. — The sufficiency follows from Remark 1.2.1, (i). To verify the necessity, suppose that $g \notin \mathcal{G}(U)$. Let $x \in X$ be a closed point of X such that $x \in U$, and, moreover, $g \notin \mathrm{PGL}_2(\mathcal{O}_{X,x})$ [if we regard g as an element of $\mathrm{PGL}_2(K_X)$]. Let us identify $A \stackrel{\text{def}}{=} k[[t]]$ with the completion $\widehat{\mathcal{O}}_{X,x}$ of $\mathcal{O}_{X,x}$ by means of a fixed isomorphism $A \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ over k. Then it is immediate that, to verify the necessity, it suffices to verify that

$(*_1)$: the composite of the natural morphism $\mathrm{Spec}(A) \to X$ with $g(f)\colon X \to \mathbb{P}^1_k$ is *not formally étale.*

Next, let us identify the scheme $\mathrm{Proj}(k[u, v])$ with $\mathbb{P}^1_k$ by means of a fixed isomorphism $\mathrm{Proj}(k[u, v]) \xrightarrow{\sim} \mathbb{P}^1_k$ over k. Write K for the field of fractions of A and

$$\mathrm{Proj}(k[u, v]) \longleftarrow \mathrm{Spec}(A); \quad (u, v) \mapsto (f_u, f_v)$$

— where f_u, $f_v \in A$ — for the composite of the natural morphism $\mathrm{Spec}(A) \to X$ with $f\colon X \to \mathbb{P}^1_k$. Thus, there exist a_g, b_g, c_g, $d_g \in k[[t^{p^N}]] = A^{p^N} \subseteq A$ [which thus implies that $\deg(a_g)$, $\deg(b_g)$, $\deg(c_g)$, $\deg(d_g) \in p^N\mathbb{Z}$] such that $a_g d_g - b_g c_g \neq 0$, and, moreover, the composite of the natural morphism $\mathrm{Spec}(A) \to X$ with $g(f)\colon X \to \mathbb{P}^1_k$ *coincides* with the morphism determined by the composite

$$\begin{array}{ccccccc} \mathrm{Proj}(k[u,v]) & \longleftarrow & \mathrm{Proj}(K[u,v]) & \longleftarrow & \mathrm{Proj}(K[u,v]) & \longleftarrow & \mathrm{Spec}(K) \\ (u,v) & \mapsto & (u,v) & & & & \\ & & (u,v) & \mapsto & (a_g u + b_g v, c_g u + d_g v) & & \\ & & & & (u,v) & \mapsto & (f_u, f_v). \end{array}$$

Now let us observe that, to verify $(*_1)$, we may assume without loss of generality, by replacing (a_g, b_g, c_g, d_g) by $t^{-\min\{\deg(a_g),\deg(b_g),\deg(c_g),\deg(d_g)\}} \cdot (a_g, b_g, c_g, d_g)$, that

(a) $0 \in \{\deg(a_g), \deg(b_g), \deg(c_g), \deg(d_g)\}$.

Moreover, let us observe that since $g \notin \mathrm{PGL}_2(\mathcal{O}_{X,x})$, it holds that

(b) $\deg(a_g d_g - b_g c_g) \geq p^N$.

Next, let us observe that, to verify $(*_1)$, we may assume without loss of generality, by replacing f by the composite of f and a suitable element of $\mathrm{Aut}_k(\mathbb{P}^1_k)$, that the image of $x \in X$ via f is the point "$(u, v) = (0, 1)$", i.e., that

(c) $\deg(f_u) = 1$ [cf. our assumption that $f \in \mathcal{P}^{\text{ét}}(U)$], and $f_v = 1$.

Moreover, let us observe that, to verify $(*_1)$, we may assume without loss of generality, by replacing g by the product of g and a suitable element of $\mathrm{Aut}_k(\mathbb{P}^1_k)$, that the image of $x \in X$ via $g(f)$ is the point "$(u, v) = (0, 1)$", i.e., that [cf. (c)]

(d) if write

$$F \overset{\text{def}}{=} \frac{a_g f_u + b_g}{c_g f_u + d_g} \in K,$$

then $F \in A$, and, moreover, $\deg(F) \geq 1$.

Thus, it is immediate that, to verify $(*_1)$, it suffices to verify that

$(*_2)$: $\deg(F) \neq 1$.

Here, let us verify that

(e) $\deg(b_g) \geq p^N$.

Indeed, if $\deg(b_g) = 0$, then it follows from (c) that $\deg(a_g f_u + b_g) = 0$, which thus implies that $\deg(F) \leq 0$ — in *contradiction* to (d). This completes the proof of (e).

Next, suppose that $\deg(d_g) = 0$. Then it follows from (b) and (e) that $\deg(a_g) \geq p^N$. On the other hand, since $\deg(d_g) = 0$, it follows from (c) that $\deg(c_g f_u + d_g) = 0$. In particular, it follows from (e) that $\deg(F) = \deg(a_g f_u + b_g) \geq p^N \geq 2$, as desired. Thus, to verify $(*_2)$, we may assume without loss of generality that

(f) $\deg(d_g) \geq p^N$.

Next, let us verify that

(g) $\deg(a_g) \geq p^N$.

Indeed, if $\deg(a_g) = 0$, then it follows from (c) and (e) that $\deg(a_g f_u + b_g) = 1$. In particular, it follows from (c) and (f) that $\deg(F) = 1 - \deg(c_g f_u + d_g) \leq 1 - 1 = 0$ — in *contradiction* to (d). This completes the proof of (g).

It follows from (a), (e), (f), (g) that $\deg(c_g) = 0$. Thus, it follows from (c) and (f) that $\deg(c_g f_u + d_g) = 1$. In particular, it follows from (e) and (g) that $\deg(F) = \deg(a_g f_u + b_g) - 1 \geq p^N - 1$. Now since [we have assumed that] $(p, N) \neq (2, 1)$, it holds that $p^N - 1 \geq 2$, as desired. This completes the proof of Lemma 2.4. □

REMARK 2.4.1. — The necessity of Lemma 2.4 in the case where $(p, N) = (2, 1)$ does *not hold*. Indeed, suppose that we are in the situation of the first paragraph of the proof of Lemma 1.5. Suppose, moreover, that the element $F \in K_X$ satisfies that $\mathrm{ord}_x(F) = 1$, which thus implies that $f \in \mathcal{P}^{\text{ét}}(U)$ for some open subscheme $U \subseteq X$ of X such that $x \in U$. [It is immediate that such a pair "(f, x)" always exists.] Let us consider the element $g \in \mathcal{G}_{\text{rtn}}$ of $\mathcal{G}_{\text{rtn}}$ determined by the matrix

$$\begin{pmatrix} F^2 & F^2 \\ 1 & F^2 \end{pmatrix} \in \mathrm{GL}_2(K_X^2).$$

Then since the determinant of this matrix is $F^4 - F^2 \notin \mathcal{O}_{X,x}^{\times}$, it holds that $g \in \mathcal{G}_{\text{rtn}}$ is *not contained* in $\mathcal{G}(V) \subseteq \mathcal{G}_{\text{rtn}}$ for every open subscheme $V \subseteq X$ of X such that $x \in V$. On the other hand, since

$$\frac{F \cdot F^2 + F^2}{F \cdot 1 + F^2} = F,$$

the result $g(f)$ of the action of g on f is given by f, which thus implies that $g(f) \in \mathcal{P}^{\text{ét}}(U)$.

LEMMA 2.5. — *Suppose that* $(p, N) \neq (2, 1)$. *Let* $f \in \mathcal{P}^{\text{gét}}(X)$ *be a* **pseudo-coordinate of level** $\boldsymbol{N}$. *Then the following hold:*

(i) *Write* $\mathcal{S}_f \subseteq \mathcal{P}^{\text{ét}}$ *for the subsheaf of* $\mathcal{P}^{\text{ét}}$ *that assigns, to an open subscheme* $U \subseteq X$, *the subset of* $\mathcal{P}^{\text{ét}}(U)$ *obtained by forming the intersection of* $\mathcal{P}^{\text{ét}}(U)$ *and the* $\mathcal{G}_{\text{rtn}}$*-orbit* $(\subseteq \mathcal{P}^{\text{gét}}(U))$ *of* $f|_U$ [*cf. Remark* 1.1.1*; Remark* 1.2.1, (i)]*:*

$$\mathcal{S}_f(U) \stackrel{\text{def}}{=} \mathcal{P}^{\text{ét}}(U) \cap (\mathcal{G}_{\text{rtn}} \cdot f|_U).$$

Then the subsheaf $\mathcal{S}_f$ *is a* **Frobenius-projective structure of level** $\boldsymbol{N}$ *on* X.

(ii) *Let* $g \in \mathcal{P}^{\text{gét}}(X)$ *be a global section of* $\mathcal{P}^{\text{gét}}$ *which is* **contained** *in the* $\mathcal{G}_{\text{rtn}}$*-orbit of* $f \in \mathcal{P}^{\text{gét}}(X)$. [*So* g *is a* **pseudo-coordinate of level** $\boldsymbol{N}$ — *cf. Remark* 1.3.1.] *Then* $\mathcal{S}_f = \mathcal{S}_g$ [*cf.* (i)].

PROOF. — Assertion (i) follows immediately from Lemma 2.4, together with the definition of a pseudo-coordinate of level N. Assertion (ii) follows immediately from the definition of "$\mathcal{S}_f$". □

DEFINITION 2.6. — Suppose that $(p, N) \neq (2, 1)$. Let $f \in \mathcal{P}^{\text{gét}}(X)$ be a pseudo-coordinate of level N. Then it follows from Lemma 2.5, (i), that f determines a Frobenius-projective structure of level N on X. We shall refer to this Frobenius-projective structure of level N as the *Frobenius-projective structure of level* N *associated to* f. Thus, we obtain a map

$$\mathfrak{pcd}_N(X)/\mathcal{G}_{\text{rtn}} \longrightarrow \mathfrak{Fps}_N(X)$$

[cf. Lemma 2.5, (ii)].

PROPOSITION 2.7. — *Suppose that* $(p, N) \neq (2, 1)$. *Then the assignments of Definition* 2.3 *and Definition* 2.6 *determine a* **bijection**

$$\mathfrak{Fps}_N(X) \xrightarrow{\sim} \mathfrak{pcd}_N(X)/\mathcal{G}_{\text{rtn}}.$$

PROOF. — This assertion follows immediately from the constructions of Lemma 2.2 and Lemma 2.5. □

REMARK 2.7.1. — If $(p, N) = (2, 1)$, then, as discussed in Corollary 5.7, (ii), below, the map

$$\mathfrak{Fps}_N(X) \longrightarrow \mathfrak{pcd}_N(X)/\mathcal{G}_{\text{rtn}}$$

of Definition 2.3 is *surjective* but *not injective*.

COROLLARY 2.8. — *Suppose that* $(p, N) = (3, 1)$. *Then* $\sharp\mathfrak{Fps}_N(X) = 1$.

PROOF. — This assertion follows from Proposition 2.7, together with Corollary 1.9. □

3. Frobenius-indigenous Structures

In the present §3, we introduce and discuss the notion of *Frobenius-indigenous structures* on curves [cf. Definition 3.4 below], which may be regarded as a generalization of the notion of *dormant indigenous bundles* studied in [6] [cf. Remark 3.4.1, (ii), below]. Moreover, we also discuss a relationship between Frobenius-indigenous structures and Frobenius-projective structures [cf. Proposition 3.11 below].

In the present §3, we maintain the notational conventions introduced at the beginning of §1. Moreover, if $U \subseteq X$ is an open subscheme of X, and $i \in \{1, 2\}$, then write $\mathcal{J}_U \subseteq \mathcal{O}_{U\times_k U}$ for the ideal of $\mathcal{O}_{U\times_k U}$ which defines the diagonal morphism with respect to U/k, $U_{(1)} \subseteq U \times_k U$ for the closed subscheme of $U \times_k U$ defined by the ideal $\mathcal{J}_U^2 \subseteq \mathcal{O}_{U\times_k U}$, and $\mathrm{pr}_i \colon U_{(1)} \to U$ for the i-th projection morphism.

We use the notation "ω" (respectively, "τ") to denote the relative cotangent (respectively, tangent) sheaf. Thus, it holds that $\omega_{U/k} = \mathcal{J}_U/\mathcal{J}_U^2$ and $\tau_{U/k} = \mathcal{H}om_{\mathcal{O}_U}(\mathcal{J}_U/\mathcal{J}_U^2, \mathcal{O}_U)$.

Lemma 3.1. — *Let $U \subseteq X$ be an open subscheme of X. Then the natural morphism*

$$U \times_{U^F} U \longrightarrow U \times_k U$$

is a **closed immersion** *whose image* **contains** *the closed subscheme $U_{(1)} \subseteq U \times_k U$.*

Proof. — This assertion follows from a straightforward computation. □

Definition 3.2. — Let $U \subseteq X$ be an open subscheme of X and S an object over U^F. Then it follows from Lemma 3.1 that we have a natural identification of $\mathrm{pr}_1^*\Phi^*S$ with $\mathrm{pr}_2^*\Phi^*S$ over $U_{(1)}$. We shall write

$$\nabla_{\Phi^*S}$$

for the connection on Φ^*S relative to U/k obtained by forming this identification.

Definition 3.3. — Let $U \subseteq X$ be an open subscheme of X, $P \to U$ a $\mathbb{P}^1$-bundle over U, ∇ a connection on P relative to U/k, and σ a section of $P \to U$. Then, by considering the difference between $\mathrm{pr}_1^*\sigma$ and $\mathrm{pr}_2^*\sigma$ relative to ∇, we have a global section of the invertible sheaf on U

$$\omega_{U/k} \otimes_{\mathcal{O}_U} \sigma^*\tau_{P/U}.$$

We shall refer to this global section as the *Kodaira-Spencer section* of ∇ at σ.

Definition 3.4. — We shall say that a pair $(P \to X^F, \sigma)$ consisting of a $\mathbb{P}^1$-bundle $P \to X^F$ over X^F and a section σ of the pull-back $\Phi^*P \to X$ is a *Frobenius-indigenous structure of level* N on X if the Kodaira-Spencer section of the connection ∇_{Φ^*P} at σ is nowhere vanishing.

For two Frobenius-indigenous structures $\mathcal{I}_1 = (P_1 \to X^F, \sigma_1)$, $\mathcal{I}_2 = (P_2 \to X^F, \sigma_2)$ of level N on X, we shall say that $\mathcal{I}_1$ is *isomorphic* to $\mathcal{I}_2$ if there exists an isomorphism $P_1 \xrightarrow{\sim} P_2$ over X^F compatible with σ_1 and σ_2.

We shall write

$$\mathfrak{Fis}_N(X)$$

for the set of isomorphism classes of Frobenius-indigenous structures of level N on X.

REMARK 3.4.1. — Suppose that $p \neq 2$, and that $g \geq 2$.

(i) Write Π_{N+1} for the *VF-pattern of pure tone* $N+1$ [cf. [6], Chapter IV, Definition 2.6]. Then it is immediate that there is a certain direct relationship between *Frobenius-indigenous structures of level* N and objects parametrized by the stack "$\mathcal{X}$" defined in [6], Chapter III, §1.3, in the case where we take the VF-pattern "Π" to be Π_{N+1}. In the notation of [6], Chapter III, §1, by taking the VF-pattern "Π" to be Π_{N+1} and the triple "(p, g, r)" to be $(p, g, 0)$, we have natural functors

$$\mathcal{X} \longrightarrow \mathcal{W} \longrightarrow \mathcal{R}_{g,0}^{\Pi_{N+1}(0),\Pi_{N+1}(-1)} \longrightarrow \overline{\mathcal{M}}_{g,0}.$$

Then one verifies easily that if the classifying morphism $\mathrm{Spec}(k) \to \overline{\mathcal{M}}_{g,0}$ of the projective smooth curve X over k *factors through* the stack $\mathcal{X}$ [relative to the above functors], then there exists a *Frobenius-indigenous structure of level* N on X.

(ii) Suppose, moreover, that $N = 1$. Then one verifies easily that the notion of *Frobenius-indigenous structures of level* N in the sense of Definition 3.4 is essentially the same as the notion of *dormant indigenous bundles* in the sense of [5], Chapter I, Definition 2.2; [6], Chapter II, Definition 1.1.

LEMMA 3.5. — *Let* $(P \to X^F, \sigma)$ *be a* **Frobenius-indigenous structure of level** $\boldsymbol{N}$ *on* X*,* $\mathcal{E}$ *a locally free coherent* $\mathcal{O}_{X^F}$*-module of rank two whose projectivization* $\mathbb{P}(\mathcal{E})$ *is* **isomorphic** *to* P *over* X^F*, and* $\Phi^*\mathcal{E} \twoheadrightarrow \mathcal{Q}$ *a surjection of* $\mathcal{O}_X$*-modules onto an invertible sheaf* $\mathcal{Q}$ *on* X *which defines, relative to an isomorphism of* P *with* $\mathbb{P}(\mathcal{E})$ *over* X^F*, the section* σ*. Write* $\mathcal{L}$ *for the kernel of the surjection* $\Phi^*\mathcal{E} \twoheadrightarrow \mathcal{Q}$*. Then it holds that* $2 \cdot \deg(\mathcal{Q}) = p^N \cdot \deg(\mathcal{E}) - 2g + 2$*, hence also that* $2 \cdot \deg(\mathcal{L}) = p^N \cdot \deg(\mathcal{E}) + 2g - 2$*.*

PROOF. — This assertion follows immediately from our assumption that the Kodaira-Spencer section of the connection ∇_{Φ^*P} at σ is *nowhere vanishing*, i.e., that the homomorphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$\mathcal{L} \hookrightarrow \Phi^*\mathcal{E} \overset{\nabla_{\Phi^*\mathcal{E}}}{\to} \omega_{X/k} \otimes_{\mathcal{O}_X} \Phi^*\mathcal{E} \twoheadrightarrow \omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{Q}$$

is an *isomorphism.* □

LEMMA 3.6. — *Suppose that* $g \geq 2$*. Let* $P \to X^F$ *be a* $\mathbb{P}^1$*-bundle over* X^F *and* σ_1*,* σ_2 *sections of* $\Phi^*P \to X$*. Then if both* $(P \to X^F, \sigma_1)$ *and* $(P \to X^F, \sigma_2)$ *are* **Frobenius-indigenous structures of level** $\boldsymbol{N}$ *on* X*, then* $\sigma_1 = \sigma_2$*.*

PROOF. — Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_{X^F}$-module of rank two whose projectivization $\mathbb{P}(\mathcal{E})$ is *isomorphic* to P over X^F. Let us fix an isomorphism of P with $\mathbb{P}(\mathcal{E})$ over X^F. For each $i \in \{1, 2\}$, let $\Phi^*\mathcal{E} \twoheadrightarrow \mathcal{Q}_i$ be a surjection of $\mathcal{O}_X$-modules onto an invertible sheaf $\mathcal{Q}_i$ on X which defines, relative to the fixed isomorphism $P \xrightarrow{\sim} \mathbb{P}(\mathcal{E})$, the section σ_i. Write $\mathcal{L}_i$ for the kernel of the surjection $\Phi^*\mathcal{E} \twoheadrightarrow \mathcal{Q}_i$. Then since [we have assumed that] $g \geq 2$, it follows from Lemma 3.5 that $\deg(\mathcal{Q}_i) < \deg(\mathcal{L}_j)$ for each i, $j \in \{1, 2\}$, which thus implies that $\mathcal{L}_1 = \mathcal{L}_2$, as desired. This completes the proof of Lemma 3.6. □

LEMMA 3.7. — *Let $\mathcal{S} \subseteq \mathcal{P}^{\text{ét}}$ be a* **Frobenius-projective structure of level** $\boldsymbol{N}$ *on X. Thus, the sheaf $\Phi_*\mathcal{S}$ is a PGL_{2,X^F}-torsor on X^F. Write $P_\mathcal{S} \to X^F$ for the $\mathbb{P}^1$-bundle associated to the PGL_{2,X^F}-torsor $\Phi_*\mathcal{S}$ [i.e., the quotient of $\Phi_*\mathcal{S} \times_{X^F} \mathbb{P}^1_{X^F}$ by the diagonal action of PGL_{2,X^F}]. For each local section s of $\Phi_*\mathcal{S}$, write σ_s for the local section of the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1_X \to X$ [cf. Remark 1.1.2]. Then the pair consisting of*

- *the $\mathbb{P}^1$-bundle $P_\mathcal{S} \to X^F$ over X^F*

and

- *the section of $\Phi^* P_\mathcal{S} \to X$ determined by the various pairs "(s, σ_s)" — where "s" ranges over the local sections of $\Phi_*\mathcal{S}$ —*

is a **Frobenius-indigenous structure of level** $\boldsymbol{N}$ *on X.*

PROOF. — This assertion follows immediately from the fact that $\mathcal{S}$ is *contained* in $\mathcal{P}^{\text{ét}}$. □

DEFINITION 3.8. — Let $\mathcal{S} \subseteq \mathcal{P}^{\text{ét}}$ be a Frobenius-projective structure of level N on X. Then it follows from Lemma 3.7 that $\mathcal{S}$ determines a Frobenius-indigenous structure of level N on X. We shall refer to this Frobenius-indigenous structure of level N as the *Frobenius-indigenous structure of level N associated to $\mathcal{S}$*. Thus, we obtain a map

$$\mathfrak{Fps}_N(X) \longrightarrow \mathfrak{Fis}_N(X).$$

LEMMA 3.9. — *Let $(P \to X^F, \sigma)$ be a* **Frobenius-indigenous structure of level** $\boldsymbol{N}$ *on X. Then the following hold:*

(i) *Let $U \subseteq X$ be an open subscheme of X such that the restriction $P|_{U^F}$ is* **isomorphic** *to the trivial $\mathbb{P}^1$-bundle over U^F and $\iota_U \colon P|_{U^F} \xrightarrow{\sim} \mathbb{P}^1_k \times_k U^F$ an isomorphism over U^F. Write $f_{U,\iota_U} \in \mathcal{P}(U)$ for the section of $\mathcal{P}$ obtained by forming the composite*

$$U \xrightarrow{\sigma|_U} (\Phi^*P)|_U \xrightarrow[\sim]{\Phi^*\iota_U} \mathbb{P}^1_k \times_k U \xrightarrow{\mathrm{pr}_1} \mathbb{P}^1_k.$$

Then $f_{U,\iota_U} \in \mathcal{P}^{\text{ét}}(U)$.

(ii) *The collection of sections $f_{U,\iota_U} \in \mathcal{P}^{\text{ét}}(U)$ [cf. (i)] — where (U, ι_U) ranges over the pairs as in (i) — determines a* **Frobenius-projective structure of level** $\boldsymbol{N}$ *on X.*

PROOF. — Assertion (i) follows from our assumption that the Kodaira-Spencer section of the connection ∇_{Φ^*P} at σ is nowhere vanishing. Assertion (ii) follows immediately from assertion (i). □

DEFINITION 3.10. — Let $\mathcal{I}$ be a Frobenius-indigenous structure of level N on X. Then it follows from Lemma 3.9, (ii), that $\mathcal{I}$ determines a Frobenius-projective structure of level N on X. We shall refer to this Frobenius-projective structure of level N as the *Frobenius-projective structure of level N associated to $\mathcal{I}$*. Thus, we obtain a map

$$\mathfrak{Fis}_N(X) \longrightarrow \mathfrak{Fps}_N(X).$$

PROPOSITION 3.11. — *The assignments of Definition* 3.8 *and Definition* 3.10 *determine a* **bijection**

$$\mathfrak{Fps}_N(X) \xrightarrow{\sim} \mathfrak{Fis}_N(X).$$

PROOF. — This assertion follows immediately from the constructions of Lemma 3.7 and Lemma 3.9. □

REMARK 3.11.1. — Proposition 3.11 may be regarded as an analogue, in positive characteristic, of [1], Theorem 3.

COROLLARY 3.12. — *Suppose that* $(p, N) = (3, 1)$. *Then* $\sharp\mathfrak{Fis}_N(X) = 1$.

PROOF. — This assertion follows from Proposition 3.11, together with Corollary 2.8. □

REMARK 3.12.1. — Suppose that $g \geq 2$. Then it follows from Remark 3.4.1, (ii), that the conclusion of Corollary 3.12 is equivalent to the following assertion:

> $(*)$: If $p = 3$, then the set of isomorphism classes of *dormant indigenous bundles* on X is *of cardinality one*.

Now let us recall that we already have *four* proofs of the assertion $(*)$ as follows:

(1) the proof essentially obtained by the theory of *molecules* established by *S. Mochizuki* [cf. [2], Remark 2.1.1]

(2) the proof essentially obtained by the *formula of the number* of isomorphism classes of dormant indigenous bundles on a sufficiently general curve established by *Y. Wakabayashi* [cf. the proof of [2], Theorem 2.1]

(3) the proof obtained by an *explicit local computation* of the p-curvatures of indigenous bundles in characteristic three established by the author of the present paper [cf. [2], Remark 3.1.1]

(4) the proof obtained by the *uniqueness* of the isomorphism class of dormant opers of rank $p - 1$ established by the author of the present paper [cf. [3], Theorem A]

Thus, we conclude that the proof of the assertion $(*)$ given in the proof of Corollary 3.12, i.e.,

(5) the proof essentially obtained by the *uniqueness* of the $\mathcal{G}_{\mathrm{rtn}}$-orbit of generically étale rational functions in the case where $(p, N) = (3, 1)$ [cf. Proposition 1.8, (i)],

may be regarded as the *fifth* proof of the assertion $(*)$.

THEOREM 3.13. — *Suppose that* $(p, N) \neq (2, 1)$. *Then there exist* **bijections**

$$\mathfrak{pcd}_N(X)/\mathcal{G}_{\mathrm{rtn}} \xrightarrow{\sim} \mathfrak{Fps}_N(X) \xrightarrow{\sim} \mathfrak{Fis}_N(X).$$

PROOF. — This assertion follows from Proposition 2.7 and Proposition 3.11. □

4. Relationship Between Certain Frobenius-destabilized Bundles

In the present §4, we discuss a relationship between Frobenius-indigenous structures and certain *Frobenius-destabilized bundles* over X^F [cf. Proposition 4.7 below].

In the present §4, we maintain the notational conventions introduced at the beginnings of §1 and §3. Write, moreover,

$$X^f$$

for the "X^F" in the case where $N = 1$ and

$$\phi\colon X \longrightarrow X^f$$

for the "Φ" in the case where $N = 1$. Thus, the morphism $\Phi\colon X \to X^F$ factors as the composite

$$X \xrightarrow{\phi} X^f \longrightarrow X^F.$$

We shall write

$$\Phi_{f\to F}\colon\ X^f \longrightarrow X^F$$

for the second arrow in this composite [i.e., the "Φ" in the case where we take the pair "(X, N)" to be $(X^f, N-1)$].

In the present §4, suppose, moreover, that

$$g \ \geq\ 2.$$

Definition 4.1. — Let S be a scheme and $\mathcal{E}_1$, $\mathcal{E}_2$ two $\mathcal{O}_S$-modules. Then we shall say that $\mathcal{E}_1$ is $\mathbb{P}$-*equivalent* to $\mathcal{E}_2$ if there exist an invertible sheaf $\mathcal{L}$ on S and an isomorphism $\mathcal{E}_1 \otimes_{\mathcal{O}_S} \mathcal{L} \xrightarrow{\sim} \mathcal{E}_2$ of $\mathcal{O}_S$-modules. We shall write

$$\mathcal{E}_1 \ \sim_{\mathbb{P}}\ \mathcal{E}_2$$

if $\mathcal{E}_1$ is $\mathbb{P}$-equivalent to $\mathcal{E}_2$.

Definition 4.2. — Let d be a positive integer and $\mathcal{E}$ a locally free coherent $\mathcal{O}_{X^F}$-module of rank two. Then we shall say that $\mathcal{E}$ is (N, d)-*Frobenius-destabilized* if the following conditions are satisfied:

(1) The locally free coherent $\mathcal{O}_{X^f}$-module $\Phi_{f\to F}^*\mathcal{E}$ of rank two is stable. [In particular, the locally free coherent $\mathcal{O}_{X^F}$-module $\mathcal{E}$ of rank two is stable.]

(2) There exist an invertible sheaf $\mathcal{L}$ on X of degree $\frac{p^N}{2} \cdot \deg(\mathcal{E}) + d$ and a locally split injection $\mathcal{L} \hookrightarrow \Phi^*\mathcal{E}$ of $\mathcal{O}_X$-modules. [In particular, the locally free coherent $\mathcal{O}_X$-module $\Phi^*\mathcal{E} = \phi^*\Phi_{f\to F}^*\mathcal{E}$ of rank two is not semistable.] Note that one verifies easily that the quotient, which is an invertible sheaf on X, of $\Phi^*\mathcal{E}$ by $\mathcal{L}$ is of degree $\frac{p^N}{2} \cdot \deg(\mathcal{E}) - d = \deg(\mathcal{L}) - 2d$.

We shall write

$$\mathfrak{Fds}_N(X)$$

for the set of $\mathbb{P}$-equivalence classes [cf. Remark 4.2.1 below] of $(N, g-1)$-Frobenius-destabilized locally free coherent $\mathcal{O}_{X^F}$-modules of rank two.

REMARK 4.2.1. — Let $\mathcal{E}_1$, $\mathcal{E}_2$ be locally free coherent $\mathcal{O}_{X^F}$-modules of rank two. Suppose that $\mathcal{E}_1 \sim_{\mathbb{P}} \mathcal{E}_2$. Then one verifies easily that it holds that $\mathcal{E}_1$ is $(N, g-1)$*-Frobenius-destabilized* if and only if $\mathcal{E}_2$ is $(N, g-1)$*-Frobenius-destabilized.*

REMARK 4.2.2.

(i) Let $\mathcal{E}$ be a locally free coherent $\mathcal{O}_{X^F}$-module of rank two. Then one verifies easily that it holds that $\mathcal{E}$ is $(N, g-1)$*-Frobenius-destabilized* if and only if $\Phi^*_{f\to F}\mathcal{E}$ is $(1, g-1)$*-Frobenius-destabilized.*

(ii) Let $P \to X^F$ be a $\mathbb{P}^1$-bundle over X and σ a section of $\Phi^*P \to X$. Then one verifies easily that it holds that $(P \to X^F, \sigma)$ is a *Frobenius-indigenous structure of level* N on X if and only if $(\Phi^*_{f\to F}P \to X^f, \sigma)$ is a *Frobenius-indigenous structure of level* 1 on X.

REMARK 4.2.3. — Write $\mathrm{Fr}_X \colon X \to X$ for the p-th power Frobenius endomorphism of X. Then one verifies easily that the assignment "$\mathcal{E} \mapsto W_*\mathcal{E}$" determines a *bijection* of the set $\mathfrak{FIs}_N(X)$ with the set of $\mathbb{P}$-equivalence classes of locally free coherent $\mathcal{O}_X$-modules $\mathcal{F}$ of rank two which satisfy the following condition: If, for a nonnegative integer i, we write

$$\mathcal{F}_i \stackrel{\mathrm{def}}{=} \overbrace{\mathrm{Fr}_X^* \circ \cdots \circ \mathrm{Fr}_X^*}^{i} \mathcal{F},$$

then

- the locally free coherent $\mathcal{O}_X$-module $\mathcal{F}_{N-1}$, hence also $\mathcal{F}$, is *stable*, but
- there exist an invertible sheaf $\mathcal{L}$ on X of degree $\frac{p^N}{2}\cdot\deg(\mathcal{F})+g-1 = \frac{1}{2}\cdot\deg(\mathcal{F}_N)+g-1$ and a locally split injection $\mathcal{L} \hookrightarrow \mathcal{F}_N$ of $\mathcal{O}_X$-modules. [In particular, the locally free coherent $\mathcal{O}_X$-module $\mathcal{F}_N$ is *not semistable.*]

LEMMA 4.3. — *Let* $\mathcal{E}$ *be an* $\boldsymbol{(N, g-1)}$**-Frobenius-destabilized** *locally free coherent* $\mathcal{O}_{X^F}$*-module of rank two. Write* $\mathbb{P}(\mathcal{E}) \to X^F$ *for the projectivization of* $\mathcal{E}$*. Then there exists a* [*uniquely determined — cf. Lemma* 3.6] *section* σ *of* $\Phi^*\mathbb{P}(\mathcal{E}) \to X$ *such that the pair* $(\mathbb{P}(\mathcal{E}) \to X^F, \sigma)$ *is a* **Frobenius-indigenous structure of level** $\boldsymbol{N}$ *on* X*.*

PROOF. — Let us first observe that it follows from Remark 4.2.2, (i), (ii), that we may assume without loss of generality, by replacing $\mathcal{E}$ by $\Phi^*_{f\to F}\mathcal{E}$, that $N = 1$. Let $\mathcal{L} \subseteq \Phi^*\mathcal{E}$ be as in condition (2) of Definition 4.2. Then it is immediate that, to verify Lemma 4.3, it suffices to verify that the homomorphism of $\mathcal{O}_X$-modules obtained by forming the composite

$$\mathcal{L} \hookrightarrow \Phi^*\mathcal{E} \stackrel{\nabla_{\Phi^*\mathcal{E}}}{\to} \omega_{X/k} \otimes_{\mathcal{O}_X} \Phi^*\mathcal{E} \twoheadrightarrow \omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{Q}$$

— where we write $\mathcal{Q}$ for the invertible sheaf on X obtained by forming the quotient of $\Phi^*\mathcal{E}$ by $\mathcal{L}$ — is an *isomorphism.* Thus, since $\deg(\mathcal{L}) = \deg(\omega_{X/k}) + \deg(\mathcal{Q})$ [cf. condition (2) of Definition 4.2], one verifies easily that, to verify Lemma 4.3, it suffices to verify that the image of the above composite is *nonzero*, i.e., that the submodule $\mathcal{L} \subseteq \Phi^*\mathcal{E}$ is *not horizontal* [with respect to the connection $\nabla_{\Phi^*\mathcal{E}}$]. On the other hand, if the submodule

$\mathcal{L} \subseteq \Phi^*\mathcal{E}$ is *horizontal*, then since $N = 1$, we obtain [cf., e.g., [4], Theorem 5.1] an invertible sheaf $\mathcal{F}$ on X^F of degree $\frac{1}{p} \cdot \deg(\mathcal{L})$ and a locally split injection $\mathcal{F} \hookrightarrow \mathcal{E}$ of $\mathcal{O}_{X^F}$-modules; thus, since $2 \cdot \deg(\mathcal{F}) = \frac{2}{p} \cdot \deg(\mathcal{L}) = \deg(\mathcal{E}) + \frac{2g-2}{p} > \deg(\mathcal{E})$ [cf. condition (2) of Definition 4.2], and $\mathcal{E}$ is *stable* [cf. condition (1) of Definition 4.2], we obtain a *contradiction*. This completes the proof Lemma 4.3. □

DEFINITION 4.4. — Let $\mathcal{E}$ be an $(N, g-1)$-Frobenius-destabilized locally free coherent $\mathcal{O}_{X^F}$-module of rank two. Then it follows from Lemma 4.3 that $\mathcal{E}$ determines a Frobenius-indigenous structure of level N on X. We shall refer to this Frobenius-indigenous structure of level N as the *Frobenius-indigenous structure of level N associated to $\mathcal{E}$*. Thus, we obtain a map

$$\mathfrak{Fds}_N(X) \longrightarrow \mathfrak{Fis}_N(X).$$

LEMMA 4.5. — *Let $(P \to X^F, \sigma)$ be a* **Frobenius-indigenous structure of level N** *on X and $\mathcal{E}$ a locally free coherent $\mathcal{O}_{X^F}$-module of rank two whose projectivization is isomorphic to P over X^F. Then $\mathcal{E}$ is* **$(N, g-1)$-Frobenius-destabilized**.

PROOF. — Let us first observe that it follows from Remark 4.2.2, (i), (ii), that we may assume without loss of generality, by replacing P by $\Phi^*_{f \to F}P$, that $N = 1$. Let $\Phi^*\mathcal{E} \twoheadrightarrow \mathcal{Q}$ be a surjection of $\mathcal{O}_X$-modules onto an invertible sheaf $\mathcal{Q}$ on X which defines, relative to an isomorphism of P with the projectivization of $\mathcal{E}$ over X^F, the section σ. Write $\mathcal{L}$ for the kernel of the surjection $\Phi^*\mathcal{E} \twoheadrightarrow \mathcal{Q}$. Then since $2 \cdot \deg(\mathcal{L}) = p \cdot \deg(\mathcal{E}) + 2g - 2$ [cf. Lemma 3.5], we conclude that condition (2) of Definition 4.2 holds.

Assume that condition (1) of Definition 4.2 does *not hold*, i.e., that there exist an invertible sheaf $\mathcal{F}$ on X^F and a locally split injection $\mathcal{F} \hookrightarrow \mathcal{E}$ of $\mathcal{O}_{X^F}$-modules such that $2 \cdot \deg(\mathcal{F}) \geq \deg(\mathcal{E})$, which thus implies that $2 \cdot \deg(\Phi^*\mathcal{E}/\Phi^*\mathcal{F}) < \deg(\Phi^*\mathcal{E})$. Then since

$$2 \cdot \deg(\Phi^*\mathcal{E}/\Phi^*\mathcal{F}) \ < \ \deg(\Phi^*\mathcal{E}) \ = \ p \cdot \deg(\mathcal{E}) \ < \ 2 \cdot \deg(\mathcal{L})$$

[cf. Lemma 3.5], we obtain that $\mathcal{L} \subseteq \Phi^*\mathcal{F}$, which thus implies that $\mathcal{L} = \Phi^*\mathcal{F}$. In particular, the invertible subsheaf $\mathcal{L} \subseteq \Phi^*\mathcal{E}$ of $\Phi^*\mathcal{E}$ is *horizontal*. On the other hand, since [we have assumed that] the Kodaira-Spencer section of the connection ∇_{Φ^*P} at σ is *nowhere vanishing*, we obtain a *contradiction*. This completes the proof of Lemma 4.5. □

REMARK 4.5.1. — The contents of Lemma 4.3 and Lemma 4.5 may be considered to be essentially contained in [7], Proposition 4.2.

DEFINITION 4.6. — Let $\mathcal{I}$ be a Frobenius-indigenous structure of level N on X. Then it follows from Lemma 4.5 that $\mathcal{I}$ determines a $\mathbb{P}$-equivalence class of $(N, g-1)$-Frobenius-destabilized locally free coherent $\mathcal{O}_{X^F}$-modules of rank two. We shall refer to this $\mathbb{P}$-equivalence class as the *$(N, g-1)$-Frobenius-destabilized class associated to $\mathcal{I}$*. Thus, we obtain a map

$$\mathfrak{Fis}_N(X) \longrightarrow \mathfrak{Fds}_N(X).$$

PROPOSITION 4.7. — *The assignments of Definition* 4.4 *and Definition* 4.6 *determine a* **bijection**

$$\mathfrak{Fds}_N(X) \xrightarrow{\sim} \mathfrak{Fis}_N(X).$$

PROOF. — This assertion follows immediately from the constructions of Lemma 4.3 and Lemma 4.5. □

COROLLARY 4.8. — *Suppose that* $(p, N) \neq (2, 1)$*, and that* $g \geq 2$*. Then there exist* **bijections**

$$\mathfrak{pcd}_N(X)/\mathcal{G}_{\mathrm{rtn}} \xrightarrow{\sim} \mathfrak{Fps}_N(X) \xrightarrow{\sim} \mathfrak{Fis}_N(X) \xrightarrow{\sim} \mathfrak{Fds}_N(X).$$

PROOF. — This assertion follows from Theorem 3.13 and Proposition 4.7. □

COROLLARY 4.9. — *Suppose that* $p \neq 2$*, that* $g \geq 2$*, and that* $N = 1$*. Then the following hold:*

(i) *The four sets*

$$\mathfrak{pcd}_N(X)/\mathcal{G}_{\mathrm{rtn}}, \quad \mathfrak{Fps}_N(X), \quad \mathfrak{Fis}_N(X), \quad \mathfrak{Fds}_N(X)$$

are **nonempty**.

(ii) *There exists a nonempty open subscheme of the coarse moduli space of projective smooth curves over* k *of genus* g *such that if the curve* X *is parametrized by the open subscheme, then it holds that*

$$\sharp(\mathfrak{pcd}_N(X)/\mathcal{G}_{\mathrm{rtn}}) = \sharp\mathfrak{Fps}_N(X) = \sharp\mathfrak{Fis}_N(X) = \sharp\mathfrak{Fds}_N(X)$$
$$= \frac{p^{g-1}}{2^{2g-1}} \cdot \sum_{\theta=1}^{p-1} \sin\left(\frac{\pi \cdot \theta}{p}\right)^{2-2g} = \frac{(-p)^{g-1}}{2} \cdot \sum_{\zeta^p=1,\, \zeta\neq 1} \zeta^{g-1} \cdot (1-\zeta)^{2-2g}.$$

PROOF. — First, we verify assertion (i). Let us first observe that it follows from Corollary 4.8 that, to verify assertion (i), it suffices to verify that $\mathfrak{Fis}_N(X) \neq \emptyset$. On the other hand, this follows immediately, in light of Remark 3.4.1, (ii), from [6], Chapter II, Theorem 2.8. This completes the proof of assertion (i).

Assertion (ii) follows immediately, in light of Remark 3.4.1, (ii), and Corollary 4.8, from [9], Theorem A [cf. also the discussion given in the second paragraph of the proof of [2], Theorem 2.1]. This completes the proof of assertion (ii), hence also of Corollary 4.9. □

REMARK 4.9.1. — Note that a result in the case where $p = 2$ similar to Corollary 4.9 will be discussed in Corollary 5.7, (iii), below.

COROLLARY 4.10. — *Suppose that* $p \neq 2$*, and that* $g \geq 2$*. Then there exists a nonempty open subscheme of the coarse moduli space of projective smooth curves over* k *of genus* g *such that if the curve* X *is parametrized by the open subscheme, then the four sets*

$$\mathfrak{pcd}_N(X)/\mathcal{G}_{\mathrm{rtn}}, \quad \mathfrak{Fps}_N(X), \quad \mathfrak{Fis}_N(X), \quad \mathfrak{Fds}_N(X)$$

are **nonempty**.

PROOF. — Let us first observe that it follows immediately, in light of Corollary 4.8, from Remark 3.4.1, (i), that, to verify Corollary 4.10, it suffices to verify that the image of the composite "$\mathcal{X} \to \overline{\mathcal{M}}_{g,0}$" of the displayed functors of Remark 3.4.1, (i), *contains* a nonempty open substack of $\overline{\mathcal{M}}_{g,0}$. In particular, it follows immediately from the various definitions involved that, to verify Corollary 4.10, it suffices to verify that the image of the natural functor from the VF-stack $\overline{\mathcal{N}}_{g,0}^{\Pi_{N+1}}$ of pure tone $N+1$ [cf. [6], Chapter IV, Definition 2.6] to $\overline{\mathcal{M}}_{g,0}$ *contains* a nonempty open substack of $\overline{\mathcal{M}}_{g,0}$. On the other hand, this follows immediately from [6], Chapter IV, Remark following Theorem 2.9, together with [6], Chapter III, Lemma 1.8. This competes the proof of Corollary 4.10. □

5. FROBENIUS-INDIGENOUS STRUCTURES OF LEVEL ONE IN CHARACTERISTIC TWO

In the present §5, we discuss *Frobenius-indigenous structures of level one in characteristic two* [cf. Proposition 5.6, Corollary 5.7 below]. In the present §5, we maintain the notational conventions introduced at the beginnings of §1 and §3. Suppose, moreover, that

$$(p, N) \;=\; (2, 1).$$

DEFINITION 5.1. — Let $\mathcal{L}$ be an invertible sheaf on X. Then we shall write

$$\pi_{\mathcal{L}} \colon \; \Phi^*\Phi_*\mathcal{L} \;\twoheadrightarrow\; \mathcal{L}$$

for the [necessarily surjective] homomorphism of $\mathcal{O}_X$-modules obtained by considering restrictions.

LEMMA 5.2. — *Let $\mathcal{L}$ be an invertible sheaf on X. Write $P \to X^F$ for the $\mathbb{P}^1$-bundle over X^F determined by the locally free coherent $\mathcal{O}_{X^F}$-module $\Phi_*\mathcal{L}$ of rank two and σ for the section of $\Phi^*P \to X$ determined by $\pi_{\mathcal{L}}$. Then the pair $(P \to X^F, \sigma)$ is a* **Frobenius-indigenous structure of level N** *on X.*

PROOF. — This assertion follows immediately from an explicit local computation of the Kodaira-Spencer section of ∇_{Φ^*P} at σ. □

DEFINITION 5.3. — Let $\mathcal{L}$ be an invertible sheaf on X. Then it follows from Lemma 5.2 that $\mathcal{L}$ determines a Frobenius-indigenous structure of level N on X. We shall write

$$\mathcal{I}(\mathcal{L})$$

for this Frobenius-indigenous structure of level N.

LEMMA 5.4. — *Let $\mathcal{L}$ be an invertible sheaf on X. Then the following hold:*

(i) *It holds that* $\deg(\Phi^*\Phi_*\mathcal{L}) = 2 \cdot \deg(\mathcal{L}) + 2g - 2$.

(ii) *It holds that* $\deg(\Phi_*\mathcal{L}) = \deg(\mathcal{L}) + g - 1$.

Proof. — Assertion (i) follows from Lemma 3.5 and Lemma 5.2. Assertion (ii) follows from assertion (i). □

Lemma 5.5. — *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be invertible sheaves on X. Then the following conditions are equivalent:*

(1) *It holds that $\Phi_*\mathcal{L}_1 \sim_{\mathbb{P}} \Phi_*\mathcal{L}_2$.*

(2) *It holds that* $\deg(\mathcal{L}_1) - \deg(\mathcal{L}_2)$ *is* **even**.

Proof. — The implication (1) ⇒ (2) follows immediately from Lemma 5.4, (ii). Next, we verify the implication (2) ⇒ (1). Suppose that condition (2) is satisfied. Then one verifies easily that there exists an invertible sheaf $\mathcal{F}$ on X^F such that $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes_{\mathcal{O}_X} \Phi^*\mathcal{F}$, which thus implies that $\Phi_*\mathcal{L}_1 \cong \Phi_*\mathcal{L}_2 \otimes_{\mathcal{O}_{X^F}} \mathcal{F}$. In particular, condition (1) is satisfied. This completes the proof of the implication (2) ⇒ (1), hence also of Lemma 5.5. □

Proposition 5.6. — *The following hold:*

(i) *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be invertible sheaves on X. Then if $\mathcal{I}(\mathcal{L}_1)$ is* **isomorphic** *to $\mathcal{I}(\mathcal{L}_2)$, then* $\deg(\mathcal{L}_1) - \deg(\mathcal{L}_2)$ *is* **even**.

(ii) *Suppose that $g \geq 2$. Then, in the situation of* (i), *if* $\deg(\mathcal{L}_1) - \deg(\mathcal{L}_2)$ *is* **even**, *then $\mathcal{I}(\mathcal{L}_1)$ is* **isomorphic** *to $\mathcal{I}(\mathcal{L}_2)$.*

(iii) *Suppose that $g \geq 2$. Let $\mathcal{L}_{\mathrm{odd}}$, $\mathcal{L}_{\mathrm{even}}$ be invertible sheaves on X of degree* **odd**, **even**, *respectively. Write $[\mathcal{I}(\mathcal{L}_{\mathrm{odd}})]$, $[\mathcal{I}(\mathcal{L}_{\mathrm{even}})]$ for the isomorphism classes of the Frobenius-indigenous structures $\mathcal{I}(\mathcal{L}_{\mathrm{odd}})$, $\mathcal{I}(\mathcal{L}_{\mathrm{even}})$ of level N on X, respectively. [So $[\mathcal{I}(\mathcal{L}_{\mathrm{odd}})] \neq [\mathcal{I}(\mathcal{L}_{\mathrm{even}})]$ by* (i).*] Then it holds that*

$$\mathfrak{Fis}_N(X) \;=\; \{[\mathcal{I}(\mathcal{L}_{\mathrm{odd}})],\, [\mathcal{I}(\mathcal{L}_{\mathrm{even}})]\}.$$

Proof. — Assertion (i) follows from Lemma 5.5. Assertion (ii) follows immediately from Lemma 5.5, together with Lemma 3.6.

Finally, we verify assertion (iii). Let us first observe that it follows from assertion (ii) that, to verify assertion (iii), it suffices to verify the following assertion:

> $(*)$: Let $\mathcal{I} = (P \to X^F, \sigma)$ be a Frobenius-indigenous structure of level N on X. Then there exists an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{I}$ is *isomorphic* to $\mathcal{I}(\mathcal{L})$.

To this end, take a locally free coherent $\mathcal{O}_{X^F}$-module $\mathcal{E}$ of rank two whose projectivization $\mathbb{P}(\mathcal{E})$ is *isomorphic* to P over X^F. Write $\Phi^*\mathcal{E} \twoheadrightarrow \mathcal{L}$ for the quotient which defines, relative to an isomorphism of P with $\mathbb{P}(\mathcal{E})$ over X^F, the section σ. Then it follows from a similar argument to the argument applied in the proof of [3], Lemma 2.2, that the homomorphism $\mathcal{E} \to \Phi_*\mathcal{L}$ of $\mathcal{O}_{X^F}$-modules determined by the surjection $\Phi^*\mathcal{E} \twoheadrightarrow \mathcal{L}$ is an *isomorphism*. Thus, it follows immediately from Lemma 3.6 that $\mathcal{I}$ is *isomorphic* to $\mathcal{I}(\mathcal{L})$. This completes the proof of assertion (iii), hence also of Proposition 5.6. □

Corollary 5.7. — *Suppose that $(p, N) = (2, 1)$. Then the following hold:*

(i) *It holds that*

$$1 = \sharp(\mathfrak{pcd}_N(X)/\mathcal{G}_{\mathrm{rtn}}) < 2 \leq \sharp\mathfrak{Fps}_N(X) = \sharp\mathfrak{Fis}_N(X).$$

(ii) *The map*

$$\mathfrak{Fps}_N(X) \longrightarrow \mathfrak{pcd}_N(X)/\mathcal{G}_{\mathrm{rtn}}$$

of Definition 2.3 *is* **surjective** *but* **not injective**.

(iii) *Suppose that* $g \geq 2$. *Then it holds that*

$$1 = \sharp(\mathfrak{pcd}_N(X)/\mathcal{G}_{\mathrm{rtn}}) < 2 = \sharp\mathfrak{Fps}_N(X) = \sharp\mathfrak{Fis}_N(X) = \sharp\mathfrak{Fds}_N(X).$$

PROOF. — Assertion (i) follows from Corollary 1.9, Proposition 3.11, and Proposition 5.6, (i). Assertion (ii) follows from assertion (i). Assertion (iii) follows from Proposition 4.7 and Proposition 5.6, (iii), together with assertion (i). □

6. APPLICATIONS OF A RESULT OF SUGIYAMA AND YASUDA

In the present §6, we discuss applications of a result of *Y. Sugiyama* and *S. Yasuda* obtained in [8]. In the present §6, let us apply the notational conventions introduced at the beginnings of §1 and §4.

Let us recall that one important result obtained in [8] is as follows.

THEOREM 6.1. — *Suppose that* $(p, N) = (2, 2)$. *Then there exists a* **pseudo-coordinate of level** $\boldsymbol{N}$ *on* X.

PROOF. — This assertion is the content of [8], Corollary 3.8 [cf. also Remark 1.3.2 of the present paper]. □

REMARK 6.1.1. — Suppose that $(p, N) = (2, 2)$. Let us recall that, for each f, $g \in K_X \setminus K_X^p$, the "*mysterious*" element "$a(f, g)$" was defined in [8], Definition 2.7, and played an important role in the proof of [8], Corollary 3.8. This "*mysterious*" element satisfies the following three conditions:

- If f is contained in the $\mathcal{G}_{\mathrm{rtn}}$-orbit of g, then $a(f, h) = a(g, h)$ [cf. [8], Proposition 2.9, (4)].
- It holds that $a(f, g) = 0$ if and only if f is contained in the $\mathcal{G}_{\mathrm{rtn}}$-orbit of g [cf. [8], Proposition 2.10].
- It holds that $a(f, g) = a(f, h) + a(h, g)$ [cf. [8], Proposition 2.8; [8], Proposition 2.9, (1)].

These three properties are reminiscent of the following three properties of the *Schwarzian derivative*

$$\theta_2(f, g) \stackrel{\text{def}}{=} \frac{2 \cdot f^{(1)} \cdot f^{(3)} - 3 \cdot (f^{(2)})^2}{2 \cdot (f^{(1)})^2} \quad (\text{where } f^{(n)} \stackrel{\text{def}}{=} \frac{d^n f}{dg^n})$$

[cf., e.g., [1], §4] for local projective coordinates f, g on a Riemann surface in the classical theory of Riemann surfaces:

- If f is contained in the $\mathrm{PGL}_2(\mathbb{C})$-orbit of g, then $\theta_2(f,h) = \theta_2(g,h)$.
- It holds that $\theta_2(f,g) = 0$ if and only if f is contained in the $\mathrm{PGL}_2(\mathbb{C})$-orbit of g.
- It holds that $\theta_2(f,g) \cdot (dg)^{\otimes 2} = \theta_2(f,h) \cdot (dh)^{\otimes 2} + \theta_2(h,g) \cdot (dg)^{\otimes 2}$.

Thus, the author of the present paper considers that, in the context of the present paper, this "*mysterious*" element "$a(f,g)$" defined in [8] should be regarded as a certain analogue *in characteristic two* of the *Schwarzian derivative* in the classical theory of Riemann surfaces. This observation was in fact a starting point of the study of the present paper.

COROLLARY 6.2. — *Suppose that* $(p,N) = (2,2)$. *Then there exist*

- *a* **Frobenius-projective structure of level** $\boldsymbol{N}$ *on* X

and

- *a* **Frobenius-indigenous structure of level** $\boldsymbol{N}$ *on* X.

If, moreover, $g \geq 2$, *then there exists*

- *an* $\boldsymbol{(N, g-1)}$**-Frobenius-destabilized** *locally free coherent* $\mathcal{O}_{X^F}$*-module of rank two.*

PROOF. — The first assertion follows from Theorem 6.1, together with Theorem 3.13. The final assertion follows from Theorem 6.1, together with Corollary 4.8. ☐

REMARK 6.2.1. — Suppose that $p = 2$, and that $g \geq 2$. Then it follows, in light of Remark 4.2.3, from Corollary 6.2 that there exists a locally free coherent $\mathcal{O}_X$-module $\mathcal{E}$ of rank two such that

- the locally free coherent $\mathcal{O}_X$-module $\mathrm{Fr}_X^*\mathcal{E}$, hence also $\mathcal{E}$, is *stable*, but
- the locally free coherent $\mathcal{O}_X$-module $\mathrm{Fr}_X^*\mathrm{Fr}_X^*\mathcal{E}$ admits an invertible subsheaf $\mathcal{L} \subseteq \mathrm{Fr}_X^*\mathrm{Fr}_X^*\mathcal{E}$ of degree $\frac{1}{2}\deg(\mathrm{Fr}_X^*\mathrm{Fr}_X^*\mathcal{E}) + g - 1$, which thus implies that $\mathrm{Fr}_X^*\mathrm{Fr}_X^*\mathcal{E}$ is *not semistable.*

COROLLARY 6.3. — *Suppose that* $(p,N) = (2,2)$*, and that* $g \geq 2$. *Let* $\mathcal{L}$ *be an invertible sheaf on* X. *Suppose, moreover, that* $\mathcal{L}$ *is of degree* **odd** (*respectively,* **even**) *if* g *is* **even** (*respectively,* **odd**). *Then there exists a* **stable** *locally free coherent* $\mathcal{O}_{X^F}$*-module* $\mathcal{E}$ *of rank two such that* $\phi_*\mathcal{L} \sim_{\mathbb{P}} \Phi^*_{f \to F}\mathcal{E}$.

PROOF. — Let $\mathcal{E}$ be an $(N, g-1)$-*Frobenius-destabilized* locally free coherent $\mathcal{O}_{X^F}$-module of rank two [cf. Corollary 6.2]. Then it follows from Lemma 4.3, together with Remark 4.2.2, (ii), that there exists a [uniquely determined — cf. Lemma 3.6] section σ of the $\mathbb{P}^1$-bundle $\Phi^*\mathbb{P}(\mathcal{E}) = \phi^*\mathbb{P}(\Phi^*_{f \to F}\mathcal{E}) \to X$ such that the pair $\mathcal{I} \stackrel{\text{def}}{=} (\mathbb{P}(\Phi^*_{f \to F}\mathcal{E}) \to X^f, \sigma)$ is a *Frobenius-indigenous structure of level* 1 on X. Here, let us observe that one verifies easily that $\deg(\Phi^*_{f \to F}\mathcal{E})$ is *even.* Thus, it follows from Proposition 5.6, (iii), together with Lemma 5.4, (ii), that $\mathcal{I}$ is *isomorphic* to $\mathcal{I}(\mathcal{L})$, which thus implies that $\Phi^*_{f \to F}\mathcal{E} \sim_{\mathbb{P}} \phi_*\mathcal{L}$, as desired. This completes the proof of Corollary 6.3. ☐

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