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The energy-capacity inequality on convex symplectic manifolds

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Abstract

Usher proved so called sharp energy-capacity inequality of Hofer-Zehnder capacity for closed symplectic manifolds. In this paper, we consider Floer homology on symplectic manifolds with boundary (not symplectic homology) and its spectral invariants. Then we extend the sharp energy-capacity inequalities for convex symplectic manifolds.

1 Introduction

In this section, we explain Hofer-Zehnder capacity and the sharp energy-capacity inequality. Let (M, ω) be a symplectic manifold. For any compact supported Hamiltonian function $H : S^1 \times M \to \mathbb{R}$, we define Hamiltonian vector field X_{H_t} as follows.

$$\omega(X_{H_t}, \cdot) = -dH_t$$

The time t map of this vector field defines a diffeomorphism ϕ^t_H. We denote $\phi^1_H$by ϕ_H. Such a diffeomorphism is called Hamiltonian diffeomorphism and we denote the set of Hamiltonian diffeomorphisms by $\mathrm{Ham}^c(M, \omega)$. Hofer's norm of a Hamiltonian function is defined as follows.

$$||H|| = \int_0^1 \max H_t - \min H_t dt$$

This norm also defines Hofer's norm on $\mathrm{Ham}^c(M, \omega)$ by

$$||\phi|| = \inf\{||H|| \mid \phi_H = \phi, H \in C^\infty_c(S^1 \times M)\}$$

In [4], Lalonde and McDuff proved that $||\phi|| = 0$ holds if and only if $\phi = id$ holds. In other words, Hofer's norm is non-degenerate. By using Hofer's norm, we define the displacement energy of $A \subset M$ as follows.

$$e(A, M) = \inf\{||\phi|| \mid \phi(A) \cap A, \phi \in \mathrm{Ham}^c(M, \omega)\}$$

Another important symplectic invariant of $A \subset M$ is Hofer-Zehnder capacity ([3]). We consider the following family of Hamiltonian functions.

$$\mathcal{H}(A, M) = \left\{ H \in C^\infty_c(M) \;\middle|\; \begin{array}{c} \mathrm{supp} H \subset A \backslash \partial M, H \geq 0, H^{-1}(0) \text{ and} \\ H^{-1}(\max H) \text{ contain non-empty open subset} \end{array} \right\}$$

Definition 1.1 *(1) $H \in \mathcal{H}(A, M)$ is called HZ-admissible if the flow ϕ_H^t has no non-constant periodic orbit whose period is less than 1.*

(2) $H \in \mathcal{H}(A, M)$ is called HZ$^\circ$-admissible if the flow ϕ_H^t has no non-constant contractible periodic orbit whose period is less than 1.

Hofer-Zehnder capacity $c_{HZ}(A)$ and π_1 sensitive Hofer-Zehnder capacity $c_{HZ}^\circ(A, M)$ are defined as follows.

$$c_{HZ}(A) = \{\max H \mid H \in \mathcal{H}(A, M), H \text{ is HZ-admissible}\}$$
$$c_{HZ}^\circ(A, M) = \{\max H \mid H \in \mathcal{H}(A, M), H \text{ is HZ}^\circ\text{-admissible}\}$$

There are several attempts to relate $c_{HZ}(A)$ (or $c_{HZ}^\circ(A, M)$) and $e(A, M)$. This can be written in the form

$$c_{HZ}(A) \leq C \times e(A, M)$$

or

$$c_{HZ}^\circ(A, M) \leq C \times e(A, M)$$

where C is some constant. Inequalities of these types are called energy-capacity inequalities. The most general result is the sharp energy-capacity inequality which was proved by Usher ([6]).

Theorem 1.1 (Usher) *Let (M, ω) be a closed symplectic manifold and let $A \subset M$ be any subset in M. Then the following inequality holds.*

$$c_{HZ}^\circ(A, M) \leq e(A, M)$$

In this paper, we generalize the sharp energy capacity inequality for general convex symplectic manifolds.

Definition 1.2 *Let (M, ω) be a symplectic manifold. (M, ω) is called convex if there is a sequence of codimention 0 submanifolds $\{M_n\}_{n \in \mathbb{N}}$ such that the following conditions are satisfied.*

- $M_{n-1} \subset M_n$
- $M = \cup_n M_n$
- *∂M_n is a contact type hypersurface. In other words, there exists a outward pointing Liouville vector field X_n which is defined in a neighborhood of ∂M_n. Liouville vector field means that X_n satisfies $\mathcal{L}_{X_n}\omega = \omega$.*

We prove the following theorem.

Theorem 1.2 *Let (M, ω) be a convex symplectic manifold and $A \subset M$ be a subset in M. Then, the following inequality holds.*

$$c_{HZ}^\circ(A, M) \leq e(A, M)$$

2 Floer homology on symplectic manifolds with contact type boundaries

Let (M, ω) be a symplectic manifold with a boundary. We call ∂M a contact type boundary if there exists a vector field X which satisfies the following conditions.

- X is defined in a neighborhood of ∂M
- $\mathcal{L}_X \omega = \omega$ (X is a Liouville vector field)
- X is outward pointing on ∂M

In this section, we assume that (M, ω) be a symplectic manifold with a contact type boundary. In this case, $\alpha = \iota_X \omega|_{\partial M}$ is a contact form on ∂M. Then, a neighborhood of ∂M can be identified with $(1-\epsilon, 1] \times \partial M$ whose symplectic form on $(r, y) \in (1-\epsilon, 1] \times \partial M$ is $d(r\alpha)$. We define the symplectic completion $(\widehat{M}, \widehat{\omega})$ as follows.

- $\widehat{M} = M \cup_{\partial M} [1, \infty) \times \partial M$
- $$\widehat{\omega} = \begin{cases} \omega & \text{on} \quad M \\ d(r\alpha) & \text{on} \quad (r, y) \in [1, \infty) \times \partial M \end{cases}$$

An almost complex structure J on $\widehat{M}$ is contact type if it satisfies the following properties.

- J preserves $\mathrm{Ker}(r\alpha) \subset T(\{r\} \times \partial M)$ on $\{r\} \times \partial M$
- Let X be a Liouville vector field on $[1, \infty) \times \partial M$ and Let R be a Reeb vector field of $\{r\} \times \partial M$. Then $J(X) = R$ and $J(R) = -X$ hold.

Let $T > 0$ be the smallest period of periodic Reeb orbit of contact form α on ∂M. We fix $0 < \epsilon < T$. We consider the following family of pairs of a Hamiltonian function and a contact type almost complex structure on $(\widehat{M}, \widehat{\omega})$.

$$\mathcal{H}_\epsilon = \left\{ (H, J) \,\middle|\, \begin{array}{c} J \text{ is a } S^1\text{-dependent contact type almost complex structure} \\ H : S^1 \times \widehat{M} \to \mathbb{R} \\ H(t, (r, y)) = -\epsilon r + \beta, (r, y) \in [1, \infty) \times \partial M \end{array} \right\}$$

$$P(H) = \{\text{contractible periodic orbits of } X_H\}$$

We consider Novikov covering of $P(H)$ as follows.

$$\widetilde{P}(H) = \{(r, w) | r \in P(H), w : D^2 \to M, \partial w = r\} / \backsim$$

where equivalence relation $\backsim$ is defined by

$$(r_1, w_1) \backsim (r_2, w_2) \Longleftrightarrow \begin{cases} r_1 = r_2 \\ c_1(w_1 \sharp \overline{w_2}) = 0 \\ \omega(w_1 \sharp \overline{w_2}) = 0 \end{cases}$$

The action functional $A_H : \widetilde{P}(H) \to \mathbb{R}$ is defined as follows.

$$A_H([r,w]) = -\int_{D^2} w^*\omega + \int_{S^1} H(t, r(t))dt$$

By using this action functional, we define the Floer chain complex for $(H, J) \in \mathcal{H}_\epsilon$ by

$$CF(H,J) = \left\{ \sum_{x \in \widetilde{P}(H), a_x \in \mathbb{Q}} a_x \cdot x \;\middle|\; \forall c \in \mathbb{R}, \sharp\{y | a_y \neq 0, A_H(y) > c\} < \infty \right\}$$

We consider the moduli space of pseudo-holomorphic cylinders. For $x = [r_1, w_1]$ and $y = [r_2, w_2]$ in $\widetilde{P}(H)$,

$$\widetilde{\mathcal{M}}(x,y,H,J) = \left\{ u : \mathbb{R} \times S^1 \to \widehat{M} \;\middle|\; \begin{array}{c} \partial_s u + J_t(\partial_t u - X_{H_t}) = 0 \\ \lim_{s\to-\infty} u(s,t) = r_1(t), \lim_{s\to\infty} u(s,t) = r_2(t) \\ (r_2, w_1 \sharp u) \backsim (r_2, w_2) \end{array} \right\}$$

Above moduli space has a natural $\mathbb{R}$ action.

$$\mathcal{M}(x,y,H,J) = \widetilde{\mathcal{M}}(x,y,H,J)/\mathbb{R}$$

We call a Hamiltonian function $H : S^1 \times \widehat{M} \to \mathbb{R}$ non-degenerate if

$$d\phi_H : T_p\widehat{M} \to T_p\widehat{M}$$

does not have 1 as an eigenvalue for all one periodic point $p \in \widehat{M}$. We define a subset of $\mathcal{H}_\epsilon$ as follows.

$$\mathcal{H}_\epsilon^{\mathrm{reg}} = \{(H,J) \in \mathcal{H}_\epsilon \mid H \text{ is non-degenerate}\}$$

In order to define a boundary operator, we need the following lemma ([1], [7]). Let $(V, d\theta)$ be a exact symplectic manifold such that its Liouville vector field X points inward on ∂V. We fix a Riemann surface with a boundary S and a 1-form γ such that $\gamma|_{\partial S} = 0$ and $d\gamma \leq 0$ hold. Let J be a S-dependent almost complex structure such that J is contact type near ∂S. Then, the following lemma holds.

Lemma 2.1 ([1], [7]) *Let H be a Hamiltonian function such that $H|_{\partial V} \equiv C$ holds. Let u be a map*

$$u : S \to V$$

which satisfies the following properties.

- $u(\partial S) \subset \partial V$
- $(du - X_H \otimes \gamma)^{0,1} = 0$

Then, $u(S) \subset \partial V$ holds.

This lemma implies that we can ignore $\widehat{M}\backslash M$. We use this lemma implicitly not only for boundary operators and connecting homomorphism but also for pair of pants products which we will define later. Then, by counting 0-dimentional part of $\mathcal{M}(x, y, H, J)$, we can define the boundary operator ∂ on the Floer chain complex for any $(H, J) \in \mathcal{H}_\epsilon^{\text{reg}}$ ([2]).

$$\partial(x) = \sum_{y \in \widetilde{P}(H)} \sharp\mathcal{M}(s, y, H, J)y$$

∂ satisfies $\partial \circ \partial = 0$ and we denote its homology by $HF(H, J)$. This boundary operator decreases the values of the action functional A_H. In other words, if

$$\widetilde{\mathcal{M}}(x, y, H, J) \neq \phi$$

holds, then $A_H(x) \geq A_H(y)$ holds. This implies that we have a filtration on the Floer chain complex as follows. For any $a \in \mathbb{R}$,

$$CF(H, J)^{<a} = \left\{ \sum_{x \in \widetilde{P}(H), a_x \in \mathbb{Q}, A_H(x) < a} a_x \cdot x \right\}$$

We denote the homology of $(CF^{<a}(H, J), \partial)$ by $HF^{<a}(H, J)$.

For $(H_1, J_1), (H_2, J_2) \in \mathcal{H}_\epsilon^{\text{reg}}$, we consider a $\mathbb{R}$ dependent smooth family $\{(H_s, J_s)\}_{s \in \mathbb{R}}$ of $\mathcal{H}_\epsilon$ which satisfies the following properties.

- $(H_s, J_s) = (H_1, J_1)$ for $s \lll 0$
- $(H_s, J_s) = (H_2, J_2)$ for $s \ggg 0$

Then, by counting the 0 dimentional part of the moduli space,

$$\mathcal{M}(x, y, H_s, J_s) = \left\{ u : \mathbb{R} \times S^1 \to \mathbb{R} \;\middle|\; \begin{array}{c} \partial_s u + J(s, t)(\partial_t u - X_{H(s,t)}) = 0 \\ u(-\infty) = x, u(+\infty) = y \end{array} \right\}$$

we obtain a chain map

$$CF(H_1, J_1) \to CF(H_2, J_2)$$

and induced map

$$HF(H_1, J_1) \to HF(H_2, J_2)$$

As in the closed case, we can see that there is an isomorphism

$$HF(H, J) \cong H^*(M : \Lambda)$$

where Λ is the $\mathbb{Q}$ coefficient Novikov ring of (M, ω). If $\epsilon_1 \geq \epsilon_2$ holds, there is a canonical map

$$HF(H_1, J_1) \to HF(H_2, J_2)$$

for $(H_1, J_1) \in \mathcal{H}^{\text{reg}_{\epsilon_1}}$ and $(H_2, J_2) \in \mathcal{H}^{\text{reg}_{\epsilon_2}}$. (This canonical map appears when we treat symplectic homology theory.) This map is an isomorphism.

3 Pair of pants product

In this section, we define pair of pants product

$$* : HF(H_1, J_1) \otimes HF(H_2, J_2) \to HF(H_3, J_3)$$

for $(H_i, J_i) \in \mathcal{H}^{\mathrm{reg}}_\epsilon$ $(i = 1, 2, 3)$. We define the following Riemann surface Σ.

$$\Sigma = (\mathbb{R} \times [-1, 0] \sqcup \mathbb{R} \times [0, 1]) / \sim$$

where $\sim$ is defined as follows.

- $[0, \infty) \times \{0_-\}$ is identified with $[0, \infty) \times \{0_+\}$
- $[0, \infty) \times \{-1\}$ is identified with $[0, \infty) \times \{1\}$
- $(-\infty, 0] \times \{-1\}$ is identified with $(-\infty, 0] \times \{0_-\}$
- $(-\infty, 0] \times \{1\}$ is identified with $(-\infty, 0] \times \{0_+\}$

For $0 < \epsilon_1 < \frac{1}{2}\epsilon$ and $(K_1, J_1'), (K_2, J_2') \in \mathcal{H}^{\mathrm{reg}}_{\epsilon_1}$and (H_3, J_3), we fix a $z \subset \Sigma$ dependent smooth family (H_z, J_z) so that it satisfies the following properties.

- $H_z : \widehat{M} \to \mathbb{R}$ and J_z is a contact type almost complex structure
- $H_z((r, y)) = -\epsilon_z r + \beta_z$, $(r, y) \in [1, \infty) \times \partial M$
- $\partial_t \epsilon_z = 0$ and $\partial_s \epsilon_z \geq 0$
- $(H_z, J_z) = (K_1, J_1')$ for $z = (s, t) \in \mathbb{R} \times [0, 1]$ and $s \lll 0$
- $(H_z, J_z) = (K_2, J_2')$ for $z = (s, t) \in \mathbb{R} \times [-1, 0]$ and $s \lll 0$
- $(H_z, J_z) = (\frac{1}{2} H_3(\frac{1}{2}(t + 1), \cdot), J_3)$ for $z = (s, t) \in \mathbb{R} \times [-1, 1]$ and $s \ggg 0$

For $x_i \in \widetilde{P}(K_i)$ $(i = 1, 2)$ and $y \in \widetilde{P}(H_3)$ we consider the following moduli space.

$$\mathcal{M}(s_1, x_2, y, H_z, J_z) = \left\{ u : \Sigma \to \widetilde{M} \;\middle|\; \begin{array}{c} \partial_s u(z) + J_z(\partial_t u(z) - X_{H_z}) = 0 \\ u(-\infty \times [0, 1]) = x_1, u(-\infty \times [-1, 0]) = x_2 \\ u(+\infty) = y \end{array} \right\}$$

By counting 0 dimentional part of this moduli space in an obvious way, we obtain the following pairing.

$$\widetilde{*} : HF(K_1, J_1') \otimes HF(K_2, J_2') \to HF(H_3, J_3)$$

The standard cobordims argument implies that this pairing does not depend on the choice of a family (H_z, J_z).

We take the composition of this pairing $\widetilde{*}$ and the inverse of canonical isomorphisms

$$HF(K_i, J_i') \to HF(H_i, J_i)$$

and obtain a desired pairing

$$*: HF(H_1, J_1) \otimes HF(H_2, J_2) \to HF(H_3, J_3)$$

for $(H_i, J_i) \in \mathcal{H}_\epsilon^{\mathrm{reg}}$. $*$ does not depend on the choice of $\epsilon_1 < \epsilon$. This follows from the following argument. We choose $(L_i, J_i'') \in \mathcal{H}_{\epsilon_2}^{\mathrm{reg}}$ for $\epsilon_1 \le \epsilon_2 < \epsilon$. Then we have the following commutative diagram. Commutativity implies independence of the choice.

$$\begin{array}{ccccc} HF(H_1,J_1)\otimes HF(H_2,J_2) & \xleftarrow{\cong} & HF(K_1,J_1')\otimes HF(K_2,J_2') & \longrightarrow & HF(H_3,J_3) \\ \| & & \downarrow & & \| \\ HF(H_1,J_1)\otimes HF(H_2,J_2) & \xleftarrow{\cong} & HF(L_1,J_1'')\otimes HF(L_2,J_2'') & \longrightarrow & HF(H_3,J_3) \end{array}$$

The fact that $*$ does not depend on the choice of ϵ_1 and (K_i, J_i') also implies that $*$ is associative.

4 Spectral invariants

We generalize spectral invariants of Floer homology for non compact case. In this section, we assume that (M, ω) is a symplectic manifold with a contact type boundary. What we have to check is that this spectral invariants also satisfy triangle inequality. First, we introduce some notations about Hamiltonian functions.

$$C_c^\infty(S^1 \times M) = \{H \in C^\infty(S^1 \times M) \mid \mathrm{supp} H \in \mathrm{Int} M\}$$
$$H \sharp K(t, x) = H(t, x) + K(t, (\phi_H^t)^{-1}(x))$$
$$\overline{H}(H)(t, x) = -H(t, \phi_H^t(x))$$

Then, Hamiltonian diffeomorphisms gemerated by $H\sharp K$ and $\overline{H}$ satisfy the following properties.

$$\phi_{H\sharp K}^t(x) = \phi_H^t(\phi_K^t(x))$$
$$\phi_{\overline{H}}^t(x) = (\phi_H^t)^{-1}(x)$$

For $(H, J) \in \mathcal{H}_\epsilon^{\mathrm{reg}}$ and $e \in HF(H, J)$, we define "pre" spectral invariant $\widehat{\rho}(H, e)$ by

$$\widehat{\rho}(H, e) = \inf\{a \mid e \in \mathrm{Im}(HF^{<a}(H, J) \to HF(H, J))\}$$

As in the closed case, this does not depend on J and the following inequality holds.

$$|\widehat{\rho}(H, e) - \widehat{\rho}(K, e)| \le ||H - K||$$

This inequality enable us to extend $\widehat{\rho}(\cdot, e)$ for continuous function $H \in C(S^1 \times \widehat{M})$ such that

$$H(t, (r, y)) = -\epsilon r + C, \quad (r, y) \in [1, \infty) \times \partial M$$

holds. For compact supported continuous function $H \in C_c(S^1 \times M)$, we define the canonical extension H_ϵ by

$$H_\epsilon(t,x) = \begin{cases} H(t,x) & x \in M \\ -\epsilon(r-1) & x = (r,y) \in [1,\infty) \times \partial M \end{cases}$$

Then, we define spectral invariant of H by

$$\rho(H,e) = \widehat{\rho}(H_\epsilon, e)$$

We prove the following triangle inequality.

Lemma 4.1 *For any e_1, e_2 and $H, K \in C_c(S^1 \times M)$, the following inequality holds.*

$$\rho(H \sharp K, e_1 * e_2) \leq \rho(H, e_1) + \rho(K, e_2)$$

Proof We fix $\delta > 0$ and two functions

$$f_\epsilon, f_{\frac{1}{2}\epsilon} : [0,\infty) \to \mathbb{R}$$

such that

- $f_\epsilon(r) = f_{\frac{1}{2}\epsilon}(r)$ on $r \in [0,1]$
- $f'_\epsilon \leq f'_{\frac{1}{2}\epsilon} \leq 0$
- $f''_\epsilon, f''_{\frac{1}{2}\epsilon} < 0$
- $|f_\epsilon(r) + \epsilon(r-1)| \leq \delta$, $|f_{\frac{1}{2}\epsilon}(r) + \frac{1}{2}\epsilon(r-1)| \leq \delta$ on $r \in [1,\infty)$
- $f'_\epsilon(r) = -\epsilon$, $f'_{\frac{1}{2}\epsilon}(r) = -\frac{1}{2}\epsilon$ for $r \ggg 0$

Then we can take four non-degenerate Hamiltonian functions

$$\widetilde{H}_\epsilon, \widetilde{H}_{\frac{1}{2}\epsilon}, \widetilde{K}_\epsilon, \widetilde{K}_{\frac{1}{2}\epsilon} \in C^\infty(S^1 \times \widehat{M})$$

which satisfy the following conditions.

- $|\widetilde{H}_\tau - H_\tau| \leq \delta$, $|\widetilde{K}_\tau - K_\tau| \leq \delta$ ($\tau = \epsilon$ or $\frac{1}{2}\epsilon$)
- $\widetilde{H}_\tau(t,(r,y)) = f_\tau(r)$ ($r \in [1-\kappa,\infty)$ for some $\kappa > 0$)
- $\widetilde{H}_\epsilon(t,x) = \widetilde{H}_{\frac{1}{2}\epsilon}(t,x)$ ($x \in M \backslash [1-\kappa,1] \times \partial M$)
- $\widetilde{K}_\tau(t,(r,y)) = f_\tau(r)$ ($r \in [1-\kappa,\infty)$ for some $\kappa > 0$)
- $\widetilde{K}_\epsilon(t,x) = \widetilde{K}_{\frac{1}{2}\epsilon}(t,x)$ ($x \in M \backslash [1-\kappa,1] \times \partial M$)

By definition, $*$ is decomposed as follows.

$$\begin{array}{ccc} HF(\widetilde{H}_\epsilon, J_1) \otimes HF(\widetilde{K}_\epsilon, J_2) & \xrightarrow{\cong} & HF(\widetilde{H}_{\frac{1}{2}\epsilon}, J_1) \otimes HF(\widetilde{K}_{\frac{1}{2}\epsilon}, J_2) \\ & & \downarrow \widetilde{*} \\ & & HF(\widetilde{H}_{\frac{1}{2}\epsilon} \sharp \widetilde{K}_{\frac{1}{2}\epsilon}, J_3) \end{array}$$

What we want to prove is that $*$ preserves the energy filtration. In othe words, we want to prove that

$$*(HF^{<a}(\widetilde{H}_\epsilon, J_1) \otimes HF^{<b}(\widetilde{K}_\epsilon, J_2)) \subset HF^{a+b}(\widetilde{H}_\epsilon \sharp \widetilde{K}_\epsilon, J_3)$$

holds for any $a, b \in \mathbb{R}$. As in the closed case, we can see that

$$\widetilde{*}(HF^{<a}(\widetilde{H}_{\frac{1}{2}\epsilon}, J_1) \otimes HF^{<b}(\widetilde{K}_{\frac{1}{2}\epsilon}, J_2)) \subset HF^{a+b}(\widetilde{H}_\epsilon \sharp \widetilde{K}_\epsilon, J_3)$$

holds. So what we have to prove is the inverse of canonical isomorphisms

$$\iota_1 : HF(\widetilde{H}_{\frac{1}{2}\epsilon}, J_1) \to HF(\widetilde{H}_\epsilon, J_1)$$
$$\iota_2 : HF(\widetilde{K}_{\frac{1}{2}\epsilon}, J_2) \to HF(\widetilde{K}_\epsilon, J_2)$$

preserve energy filtrations. In other words,

$$\iota_1^{-1}(HF^{<a}(\widetilde{H}_\epsilon, J_1)) \subset HF^{<a}(\widetilde{H}_{\frac{1}{2}\epsilon}, J_1)$$
$$\iota_2^{-1}(HF^{<b}(\widetilde{K}_\epsilon, J_2)) \subset HF^{<b}(\widetilde{K}_{\frac{2}{2}\epsilon}, J_2)$$

hold. For this purpose, we fix a monotone increasing function

$$\rho : \mathbb{R} \to [0, 1]$$

such that

$$\rho(s) = \begin{cases} 0 & s \lll 0 \\ 1 & s \ggg 0 \end{cases}$$

holds and a homotopy (H_s, J_s) from $(\widetilde{H}_{\frac{1}{2}\epsilon}, J_1)$ to $(\widetilde{H}_{\epsilon, J_1})$ by

$$H_s(t, x) = (1 - \rho(s))\widetilde{H}_{\frac{1}{2}\epsilon}(t, x) + \rho(s)\widetilde{H}_\epsilon(t, x)$$
$$J_s = J_1$$

This H_s satisfies $\frac{\partial}{\partial s} H_s \leq 0$. Then Lemma 2.1 implies that the moduli space $\mathcal{M}(x, y, H_s, J_s)$ has the natural $\mathbb{R}$ action for any $x \in P(\widetilde{H}_{\frac{1}{2}\epsilon})$ and $y \in P(\widetilde{H}_\epsilon)$. So, if the dimension of a connected component of this moduli space equals to 0, $x = y$ holds and it consists of trivial one point (s-independent cylinder $u(s, t) = x(t)$). So, by identifying $P(\widetilde{H}_{\frac{1}{2}\epsilon})$ and $P(\widetilde{H}_\epsilon)$,

$$\iota_1 : CF(\widetilde{H}_{\frac{1}{2}\epsilon}, J_1) \to CF(\widetilde{H}_\epsilon, J_1)$$

becomes an identity map. This implies that

$$\iota_1^{-1}(HF^{<a}(\widetilde{H}_\epsilon, J_1)) \subset HF^{<a}(\widetilde{H}_{\frac{1}{2}\epsilon}, J_1)$$

holds for any $a \in \mathbb{R}$. The same property also holds for ι_2^{-1}. So, we have

$$\widehat{\rho}(\widetilde{H}_{\frac{1}{2}\epsilon} \sharp \widetilde{K}_{\frac{1}{2}\epsilon}, e_1 * e_2) \leq \widehat{\rho}(\widetilde{H}_\epsilon, e_1) + \widehat{\rho}(\widetilde{K}_\epsilon, e_2)$$

holds. Then, by definition, we can see that

$$\begin{aligned} &\rho(H\sharp K, e_1 * e_2) - \rho(H, e_1) - \rho(K, e_2) \\ &= \widehat{\rho}((H\sharp K)_\epsilon, e_1 * e_2) - \widehat{\rho}(H_\epsilon, e_1) - \widehat{\rho}(K_\epsilon, e_2) \\ &\leq \widehat{\rho}(\widetilde{H}_{\frac{1}{2}\epsilon} \sharp \widetilde{K}_{\frac{1}{2}\epsilon}, e_1 * e_2) - \widehat{\rho}(\widetilde{H}_\epsilon, e_1) - \widehat{\rho}(\widetilde{K}_\epsilon) + 3\delta \leq 3\delta \end{aligned}$$

So, we proved that

$$\rho(H\sharp K, e_1 * e_2) \leq \rho(H, e_1) + \rho(K, e_2)$$

□

By using this triangle inequality, we can prove the next lemma.

Lemma 4.2 *For $H, K \in C_c^\infty(S^1 \times M)$, we assume that ϕ_K displaces supp$H$. Then*

$$\rho(H, e_1 * e_2) \leq \rho(K, e_1) + \rho(\overline{K}, e_2).$$

holds for any e_1, e_2

Proof We fix $\forall \delta > 0$. We can take a non-degenerate Hamiltonian $T \in C^\infty(S^1 \times \widehat{M})$ such that

- $T(t, (r, y)) = -\epsilon(r - 1)$ on $(r, y) \in [1, \infty) \times \partial M$
- $|T - K_\epsilon| \leq \delta$
- $|(H\sharp K)_\epsilon - H\sharp T| \leq \delta$
- $\phi_T(\text{supp}H) \cap \text{supp}H = \phi$

As in the closed case, we have

$$\widehat{\rho}(H\sharp K, e_1) = \widehat{\rho}(T, e_1)$$

So, we can see that

$$\begin{aligned} &\rho(H, e_1 * e_2) \leq \rho(H\sharp K, e_1) + \rho(\overline{K}, e_2) \\ &= \widehat{\rho}((H\sharp K)_\epsilon, e_1) + \rho(\overline{K}, e_2) \leq \widehat{\rho}(H\sharp T, e_1) + \rho(\overline{K}, e_2) + \delta \\ &= \widehat{\rho}(T, e_1) + \rho(\overline{K}, e_2) + \delta \leq \widehat{\rho}(K_\epsilon, e_1) + \rho(\overline{K}, e_2) + 2\delta \\ &= \rho(K, e_1) + \rho(\overline{K}, e_2) + 2\delta \end{aligned}$$

□

5 Proof of the sharp energy capacity inequality

In this section, we assume that (M, ω) is a convex symplectic manifold. In other words, there is a sequence of codimension 0 submanifolds

$$M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots$$

such that

- $\bigcup_{n \geq 1} M_n = M$
- $(M_n, \omega_n = \omega|_{M_n})$ has a contact type boundary

hold. We fix $A \subset M$. Let $H \in \mathcal{H}(A)$ and $K \in C_c^\infty(S^1 \times M)$ be two Hamiltonian functions such that

- H is HZ °-admissible
- $\phi_K(A) \cap A = \phi$

hold. Our purpose is to prove

$$\max H \leq ||K||$$

holds. From the second assumption, A is relatively compact. So, we can take sufficiently large $n \geq 1$ so that $A \subset \mathrm{Int}M_n$ and $\mathrm{supp}K \subset \mathrm{Int}M_n$ hold. From now on, we consider spectral invariants on (M_n, ω_n). We fix $\forall \delta > 0$. As in [6], we can take a Morse function $\widetilde{H} : \widehat{M_n} \to \mathbb{R}$ such that

- $|\widetilde{H}|_{M_n} - H_{M_n}| \leq \delta$
- $\widetilde{H}((r, y)) = -\epsilon(r-1)$ on $[1, \infty) \times \partial M_n$
- a period of any non-constant contractible orbit of $X_{\widetilde{H}}$ is larger than 1.

Let $1 \in HF(\widetilde{H}, J)$ be the unit. As in [5], we can see that

$$\widehat{\rho}(\widetilde{H}, 1) = \max \widetilde{H}$$

holds. This also implies that $\rho(0, 1) = \widehat{\rho}(0_\epsilon, 1) = 0$ holds. So we have

$$\rho(K, 1) + \rho(\overline{K}, 1) \leq (\rho(K, 1) - \rho(0, 1)) + (\rho(\overline{K}, 1) - \rho(0, 1)) \leq ||K||$$

Then, the following inequality holds.

$$\begin{aligned} ||K|| - \max H &\geq ||K|| - \max \widetilde{H} - \delta \\ &\geq \rho(K, 1) + \rho(\overline{K}, 1) - \widehat{\rho}(\widetilde{H}, 1) - \delta \\ &\geq \rho(K, 1) + \rho(\overline{K}, 1) - \rho(H, 1) - 2\delta \geq -2\delta \end{aligned}$$

This inequality implies that

$$\max H \leq ||K||$$

holds.

□

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