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On the asymptotic expansions of the Kashaev invariant of hyperbolic knots with 7 crossings

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Abstract

We give presentations of the asymptotic expansions of the Kashaev invariant of hyperbolic knots with 7 crossings. As the volume conjecture states, the leading terms of the expansions present the hyperbolic volume and the Chern-Simons invariant of the complements of the knots. As coefficients of the expansions, we obtain a series of new invariants of the knots.

This paper is a continuation of the previous papers [20, 21], where the asymptotic expansions of the Kashaev invariant are calculated for hyperbolic knots with 5 and 6 crossings. A technical difficulty of this paper is to use 4-variable saddle point method, whose concrete calculations are far more complicated than the previous papers.

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1 Introduction

This paper is a continuation of the previous papers [20, 21]. We review the background of this paper; for details, see [20, 21]. Kashaev [12, 13, 14] defined the Kashaev invariant $\langle L \rangle_N \in \mathbb{C}$ of a link L for $N = 2, 3, \cdots$, and conjectured that, for any hyperbolic link L, $\frac{2\pi}{N} \log |\langle L \rangle_N|$ goes to the hyperbolic volume of $S^3 - L$ as $N \to \infty$. Further, H. Murakami and J. Murakami [18] proved that the Kashaev invariant $\langle L \rangle_N$ of any link L is equal to the N-colored Jones polynomial $J_N(L; e^{2\pi\sqrt{-1}/N})$ of L evaluated at $e^{2\pi\sqrt{-1}/N}$, and conjectured that, for any knot K, $\frac{2\pi}{N} \log |J_N(K; e^{2\pi\sqrt{-1}/N})|$ goes to the (normalized) simplicial volume of $S^3 - K$. This is called *the volume conjecture*. As a complexification of the volume conjecture, it is conjectured in [19] that, for a hyperbolic link L,

$$J_N(L; e^{2\pi\sqrt{-1}/N}) \underset{N\to\infty}{\sim} e^{N\varsigma(L)},$$

where we put

$$\varsigma(L) = \frac{1}{2\pi\sqrt{-1}} \big(\mathrm{cs}(S^3 - L) + \sqrt{-1}\,\mathrm{vol}(S^3 - L) \big),$$

and "cs" and "vol" denote the Chern-Simons invariant and the hyperbolic volume. Furthermore, it is conjectured [9] (see also [3, 10, 34]) that, for a hyperbolic knot K,

$$J_N(K; e^{2\pi\sqrt{-1}/k}) \underset{\substack{N,k\to\infty \\ u=N/k:\ \text{fixed}}}{\sim} e^{N\varsigma} N^{3/2}\, \omega \cdot \Big(1 + \sum_{i=1}^{\infty} \kappa_i \cdot \big(\frac{2\pi\sqrt{-1}}{N} \big)^i \Big)$$

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for some scalars $\varsigma, \omega, \kappa_i$ depending on K and u. Recently, it is shown in [20, 21] that, when K is a hyperbolic knot with up to 6 crossings, the asymptotic expansions of the Kashaev invariant is presented by the following form,

$$\langle K \rangle_N = e^{N\varsigma(K)} N^{3/2} \omega(K) \cdot \Big(1 + \sum_{i=1}^{d} \kappa_i(K) \cdot \big(\frac{2\pi\sqrt{-1}}{N}\big)^i + O\big(\frac{1}{N^{d+1}}\big)\Big), \qquad (1)$$

for any d, where $\omega(K)$ and $\kappa_i(K)$'s are some scalars.

The volume conjecture has been rigorously proved for some particular knots and links such as torus knots [15] (see also [4]), the figure-eight knot (by Ekholm, see also [1]) Whitehead doubles of $(2,p)$-torus knots [35], positive iterated torus knots [27], the 5_2 knot [16, 20], the knots with 6 crossings [21], and some links [8, 11, 26, 27, 28, 35]; for details see *e.g.* [17].

The aim of this paper is to extend the above formula to hyperbolic knots with 7 crossings, that is, we show the following theorem. In particular, this means that the volume conjecture holds for these knots.

Theorem 1.1. *The asymptotic expansions of the Kashaev invariant $\langle K \rangle_N$ of hyperbolic knots K with 7 crossings are presented by the form (1) for any d, where $\omega(K)$ and $\kappa_i(K)$'s are some constants depending on K.*

It is shown [22] that $2\sqrt{-1}\,\omega^2(K)$ for these knots is equal to the twisted Reidemeister torsion associated with the action on sl_2 of the holonomy representation of the hyperbolic structure. We also remark that Dimofte and Garoufalidis [2] define a formal power series from an ideal tetrahedral decomposition of a knot complement, which is expected to be equal to the asymptotic expansion of the Kashaev invariant of the knot.

There are six hyperbolic knots with 7 crossings: the 7_2, 7_3, $\cdots$, 7_7 knots. We show proofs of the theorem for the 7_2, 7_3, 7_4, $\cdots$, 7_7 knots in Sections 8, 3, 4, $\cdots$, 7 respectively; the proof of the 7_2 knot is relatively long, because of some technical difficulty unlike the proofs of the other knots. We show the proofs following the proof for the 5_2 knot in [20]. An outline of the proofs is as follows. From the definition of the Kashaev invariant, the Kashaev invariant of K is presented by a sum. We rewrite the sum as an integral via the Poisson summation formula (Proposition 2.2). When we apply the Poisson summation formula, the right-hand side of the Poisson summation formula consists of infinitely many summands, and we show that we can ignore them except for the one at 0 in the sense that they are of sufficiently small order at $N \to \infty$. Further, by the saddle point method (Proposition 2.4), we calculate the asymptotic expansion of the integral, and obtain the presentation of the theorem.

A non-trivial part of the proof is to apply the saddle point method, whose concrete calculations are far more complicated than the case of knots with up to 6 crossings in [20, 21]. In this part, we need to calculate the asymptotic behavior of an integral of the following form as $N \to \infty$,

$$\int_{\Delta'} \exp\Big(N\big(\hat{V}(t,s,u,v) - \varsigma\big)\Big)\, dt\, ds\, du\, dv,$$

where $\hat{V}(t,s,u,v)$ is the potential function of the hyperbolic structure of the knot complement, and ς is a critical value of $\hat{V}(t,s,u,v)$. The domain Δ' of the integral is a compact domain in $\mathbb{R}^4$, and its boundary is included in the following domain

$$\big\{(t,s,u,v)\in\mathbb{C}^4 \;\big|\; \mathrm{Re}\,\big(\hat{V}(t,s,u,v)-\varsigma\big)<0\big\}. \tag{2}$$

The critical value ς is given by a critical point (t_0,s_0,u_0,v_0), and it is located near Δ' in $\mathbb{C}^4$. In order to apply the saddle point method, we need to show that we can move Δ' in the imaginary direction by a homotopy in such a way that the new domain Δ'_1 contains (t_0,s_0,u_0,v_0), and $\Delta'_1-\{(t_0,s_0,u_0,v_0)\}$ is included in (2), and the boundary of Δ' always stays in (2) when we apply the homotopy. We note that, when we restrict the domain (2) to a sufficiently small neighborhood of (t_0,s_0,u_0,v_0), the resulting space is homotopy equivalent to a 3-sphere. The existence of the above homotopy means that the boundary of Δ' is homotopic to this 3-sphere in the domain (2). It is a non-trivial task to see that they are homotopic in the domain (2), since it is not easy to see the topological type of the domain (2) directly. We give such a homotopy concretely in Sections 3.5, 4.5, $\cdots$, 7.5, for the 7_3, 7_4, $\cdots$, 7_7 knots respectively. Further, in the case of the 7_2 knot, we have an additional difficulty; in this case, the boundary of Δ' is not included in the domain (2), and we need many additional calculations in this case.

By the method of this paper, the asymptotic behavior of the Kashaev invariant is discussed for some hyperbolic knots with 8 crossings in [24].

The paper is organized as follows. In Section 2, we review definitions and basic properties of the notation used in this paper. In Sections 3, 4, $\cdots$, 7, 8, we show proofs of Theorem 1.1 for the 7_3, 7_4, $\cdots$, 7_7, 7_2 knots respectively.

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2 Preliminaries

In this section, we review definitions and basic properties of the notation used in this paper.

2.1 Integral presentation of $(q)_n$

In this section, we review $(q)_n$ and its integral presentation and their basic properties.

Let N be an integer ≥ 2. We put $q=\exp(2\pi\sqrt{-1}/N)$, and put

$$(x)_n \;=\; (1-x)(1-x^2)\cdots(1-x^n)$$

for $n\geq 0$. It is known [18] (see also [20]) that for any n,m with $n\leq m$,

$$(q)_n(\overline{q})_{N-n-1}=N, \tag{3}$$

$$\sum_{n\leq k\leq m}\frac{1}{(q)_{m-k}(\overline{q})_{k-n}}=1. \tag{4}$$

Following Faddeev [5], we define a holomorphic function $\varphi(t)$ on $\{t \in \mathbb{C} \mid 0 < \operatorname{Re} t < 1\}$ by

$$\varphi(t) \;=\; \int_{-\infty}^{\infty} \frac{e^{(2t-1)x} dx}{4x \sinh x \sinh(x/N)} ,$$

noting that this integrand has poles at $n\pi\sqrt{-1}$ ($n \in \mathbb{Z}$), where, to avoid the pole at 0, we choose the following contour of the integral,

$$\gamma \;=\; (-\infty, -1] \;\cup\; \big\{ z \in \mathbb{C} \;\big|\; |z| = 1,\ \operatorname{Im} z \geq 0 \big\} \;\cup\; [1, \infty).$$

It is known [7, 30] that

$$\begin{aligned} (q)_n &= \exp\Big(\varphi\big(\frac{1}{2N}\big) - \varphi\big(\frac{2n+1}{2N}\big) \Big), \\ (\overline{q})_n &= \exp\Big(\varphi\big(1 - \frac{2n+1}{2N}\big) - \varphi\big(1 - \frac{1}{2N}\big) \Big). \end{aligned} \tag{5}$$

We put $\hbar = 2\pi\sqrt{-1}/N$, and put

$$\Phi_d(z) \;=\; \operatorname{Li}_2\big(z\big) + \sum_{1 \leq k \leq d} \hbar^{2k} c_{2k} \cdot \Big(z \frac{d}{dz} \Big)^{2k-2} \frac{z}{1-z} ,$$

where we define c_{2k} by

$$\frac{t/2}{\sinh(t/2)} \;=\; \sum_{k \geq 0} c_{2k}\, t^{2k}.$$

Then, it is known [7, 30] (see also [20]) that, for any $d \geq 0$,

$$\varphi(t) \;=\; \frac{N}{2\pi\sqrt{-1}} \, \Phi_d(e^{2\pi\sqrt{-1}\,t}) \; + O(\hbar^{2d+1}), \tag{6}$$

$$\varphi^{(k)}(t) \;=\; \frac{N}{2\pi\sqrt{-1}} \big(\frac{d}{dt}\big)^k \Phi_d(e^{2\pi\sqrt{-1}\,t}) \; + O(\hbar^{2d+1}), \tag{7}$$

for each $k > 0$. More precisely, as for the convergence of $\frac{1}{N}\varphi(t)$ as $N \to \infty$, we recall the following proposition.

Proposition 2.1 (See [20])**.** *We fix any sufficiently small $\delta > 0$ and any $M > 0$. Let d be any non-negative integer. Then, in the domain*

$$\big\{ t \in \mathbb{C} \;\big|\; \delta \leq \operatorname{Re} t \leq 1 - \delta, \quad |\operatorname{Im} t| \leq M \big\}, \tag{8}$$

$\varphi(t)$ is presented by

$$\varphi(t) \;=\; \frac{N}{2\pi\sqrt{-1}} \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t}) \; + O\big(\frac{1}{N}\big),$$

where $O(1/N)$ means the error term whose absolute value is bounded by C/N for some $C > 0$, which is independent of t (but possibly dependent on δ). In particular, $\frac{1}{N}\varphi(t)$ uniformly converges to $\frac{1}{2\pi\sqrt{-1}}\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t})$ in the domain (8).

As for properties of $\varphi(t)$, it is a consequence of (3) and (5) (see [20]) that, for any $t \in \mathbb{C}$ with $0 < \mathrm{Re}\, t < 1$,

$$\varphi(t) + \varphi(1-t) \;=\; 2\pi\sqrt{-1}\Big(-\frac{N}{2}\big(t^2 - t + \frac{1}{6}\big) + \frac{1}{24N}\Big). \tag{9}$$

Further, the following formulas are known (due to Kashaev, see [20]),

$$\begin{aligned} \varphi\big(\frac{1}{2N}\big) &= \frac{N}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6} + \frac{1}{2}\log N + \frac{\pi\sqrt{-1}}{4} - \frac{\pi\sqrt{-1}}{12N}\,, \\ \varphi\big(1-\frac{1}{2N}\big) &= \frac{N}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6} - \frac{1}{2}\log N + \frac{\pi\sqrt{-1}}{4} - \frac{\pi\sqrt{-1}}{12N}\,. \end{aligned} \tag{10}$$

2.2 Some behaviors of the dilogarithm function

In this section, we show some behaviors of the dilogarithm function.

We put

$$\Lambda(t) \;=\; \mathrm{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\,\mathrm{Li}_2\big(e^{2\pi\sqrt{-1}\,t}\big)\Big).$$

Since

$$\Lambda'(t) \;=\; -\log 2\sin \pi t, \qquad \Lambda''(t) \;=\; -\pi\cot \pi t,$$

the behavior of $\Lambda(t)$ is as follows.

t	0	$\cdots$	$\frac{1}{6}$	$\cdots$	$\frac{1}{2}$	$\cdots$	$\frac{5}{6}$	$\cdots$	1
$\Lambda(t)$	0	$\nearrow$	$\Lambda(\frac{1}{6})$	$\searrow$	0	$\searrow$	$-\Lambda(\frac{1}{6})$	$\nearrow$	0
$\Lambda'(t)$		$+$	0	$-$	$-$	$-$	0	$+$	
$\Lambda''(t)$		$-$	$-$	$-$	0	$+$	$+$	$+$	

Here, $\Lambda(\frac{1}{6}) = 0.161533...$.

Further, the behavior of $\mathrm{Li}_2\big(e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}\big)$ fixing t is presented by the following formula. It is known [21] that for any real number t with $0 < t < 1$, there exists $C > 0$ such that

$$\begin{aligned} \left(\begin{cases} 0 & \text{if } X \ge 0 \\ 2\pi\big(t-\frac{1}{2}\big)X & \text{if } X < 0 \end{cases}\right) - C \;&<\; \mathrm{Re}\Big(\tfrac{1}{2\pi\sqrt{-1}}\,\mathrm{Li}_2\big(e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}\big)\Big) \\ &<\; \left(\begin{cases} 0 & \text{if } X \ge 0 \\ 2\pi\big(t-\frac{1}{2}\big)X & \text{if } X < 0 \end{cases}\right) + C \end{aligned} \tag{11}$$

for any $X \in \mathbb{R}$.

2.3 Definition of the Kashaev invariant

In this section, we review the definition of the Kashaev invariant of oriented knots.

Following Yokota [32],[1] we review the definition of the Kashaev invariant. We put

$$\mathcal{N} \;=\; \{0, 1, \cdots, N-1\}.$$

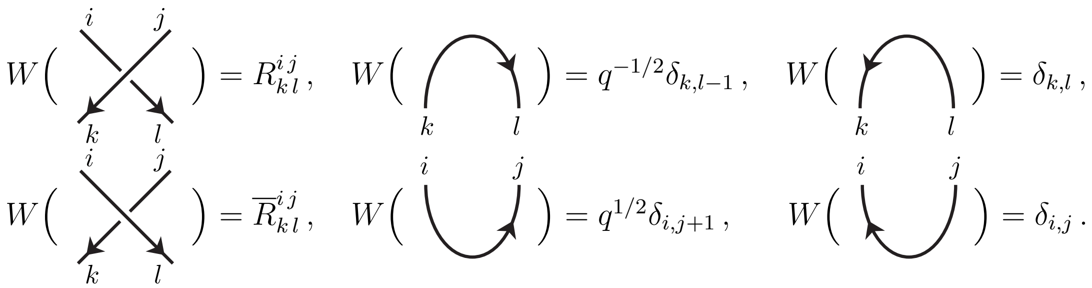
For $i, j, k, l \in \mathcal{N}$, we put

$$R^{i\,j}_{k\,l} \;=\; \frac{N\, q^{-\frac{1}{2}+i-k}\, \theta^{i\,j}_{k\,l}}{(q)_{[i-j]}(\overline{q})_{[j-l]}(q)_{[l-k-1]}(\overline{q})_{[k-i]}}\,, \qquad \overline{R}^{i\,j}_{k\,l} \;=\; \frac{N\, q^{\frac{1}{2}+j-l}\, \theta^{i\,j}_{k\,l}}{(\overline{q})_{[i-j]}(q)_{[j-l]}(\overline{q})_{[l-k-1]}(q)_{[k-i]}}\,,$$

where $[m] \in \mathcal{N}$ denotes the residue of m modulo N, and we put

$$\theta^{i\,j}_{k\,l} \;=\; \begin{cases} 1 & \text{if } \; [i-j]+[j-l]+[l-k-1]+[k-i] = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

Let K be an oriented knot. We consider a 1-tangle whose closure is isotopic to K such that its string is oriented downward at its end points. Let D be a diagram of the 1-tangle. We present D by a union of elementary tangle diagrams shown in (12). We decompose the string of D into edges by cutting it at crossings and critical points with respect to the height function of $\mathbb{R}^2$. A *labeling* is an assignment of an element of $\mathcal{N}$ to each edge. Here, we assign 0 to the two edges adjacent to the end points of D. For example, see (23). We define the *weights* of labeled elementary tangle diagrams by



(12)

Then, *the Kashaev invariant* $\langle K \rangle_N$ of K is defined by

$$\langle K \rangle_N \;=\; \sum_{\text{labelings}} \; \prod_{\substack{\text{crossings} \\ \text{of } D}} W(\text{crossings}) \prod_{\substack{\text{critical} \\ \text{points of } D}} W(\text{critical points}) \;\in \mathbb{C}.$$

2.4 The Poisson summation formula

In this section, we review a proposition obtained from the Poisson summation formula.

Proposition 2.2 (see [20]). *For* $(c_1, c_2, c_3, c_4) \in \mathbb{C}^3$ *and an oriented* 3*-ball* D' *in* $\mathbb{R}^4$*, we put*

$$\Lambda \;=\; \Big\{ \Big(\frac{i}{N}+c_1,\, \frac{j}{N}+c_2,\, \frac{k}{N}+c_3,\, \frac{l}{N}+c_4\Big) \in \mathbb{C}^4 \;\Big|\; i,j,k,l \in \mathbb{Z},\; \Big(\frac{i}{N}, \frac{j}{N}, \frac{k}{N}, \frac{l}{N}\Big) \in D' \Big\},$$

$$D \;=\; \big\{ (t+c_1,\, s+c_2,\, u+c_3,\, v+c_4) \in \mathbb{C}^4 \;\big|\; (t,s,u,v) \in D' \subset \mathbb{R}^4 \big\}.$$

[1] We make a minor modification of the definition of weights of critical points from the definition in [32], in order to make $\langle K \rangle_N$ invariant under Reidemeister moves.

Let $\psi(t,s,u,v)$ be a holomorphic function defined in a neighborhood of $\mathbf{0} \in \mathbb{C}^4$ including D. We assume that ∂D is included in the domain

$$\big\{(t,s,u,v) \in \mathbb{C}^4 \ \big| \ \operatorname{Re}\psi(t,s,u,v) < -\varepsilon_0\big\}$$

for some $\varepsilon_0 > 0$. Further, we assume that ∂D is null-homotopic in each of the following domains,

$$\big\{(t+\delta\sqrt{-1},\ s,\ u,\ v) \in \mathbb{C}^4 \ \big| \ (t,s,u,v) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t+\delta\sqrt{-1},\ s,\ u,\ v) < 2\pi\delta\big\}, \quad (13)$$
$$\big\{(t-\delta\sqrt{-1},\ s,\ u,\ v) \in \mathbb{C}^4 \ \big| \ (t,s,u,v) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t-\delta\sqrt{-1},\ s,\ u,\ v) < 2\pi\delta\big\}, \quad (14)$$
$$\big\{(t,\ s+\delta\sqrt{-1},\ u,\ v) \in \mathbb{C}^4 \ \big| \ (t,s,u,v) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t,\ s+\delta\sqrt{-1},\ u,\ v) < 2\pi\delta\big\}, \quad (15)$$
$$\big\{(t,\ s-\delta\sqrt{-1},\ u,\ v) \in \mathbb{C}^4 \ \big| \ (t,s,u,v) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t,\ s-\delta\sqrt{-1},\ u,\ v) < 2\pi\delta\big\}, \quad (16)$$
$$\big\{(t,\ s,\ u+\delta\sqrt{-1},\ v) \in \mathbb{C}^4 \ \big| \ (t,s,u,v) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t,\ s,\ u+\delta\sqrt{-1},\ v) < 2\pi\delta\big\}, \quad (17)$$
$$\big\{(t,\ s,\ u-\delta\sqrt{-1},\ v) \in \mathbb{C}^4 \ \big| \ (t,s,u,v) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t,\ s,\ u-\delta\sqrt{-1},\ v) < 2\pi\delta\big\}, \quad (18)$$
$$\big\{(t,\ s,\ u,\ v+\delta\sqrt{-1}) \in \mathbb{C}^4 \ \big| \ (t,s,u,v) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t,\ s,\ u,\ v+\delta\sqrt{-1}) < 2\pi\delta\big\}, \quad (19)$$
$$\big\{(t,\ s,\ u,\ v-\delta\sqrt{-1}) \in \mathbb{C}^4 \ \big| \ (t,s,u,v) \in D',\ \delta \ge 0,\ \operatorname{Re}\psi(t,\ s,\ u,\ v-\delta\sqrt{-1}) < 2\pi\delta\big\}. \quad (20)$$

Then,

$$\frac{1}{N^4} \sum_{(t,s,u,v) \in \Lambda} e^{N\,\psi(t,s,u,v)} \ = \ \int_D e^{N\,\psi(t,s,u,v)}\, dt\, ds\, du\, dv \ + O(e^{-N\varepsilon}),$$

for some $\varepsilon > 0$.

For a proof of the proposition, see [20].

Remark 2.3. Similarly as in [20, Remark 4.8], Proposition 2.2 can naturally be extended to the case where the holomorphic function $\psi(t,s,u,v)$ depends on N, if $\psi(t,s,u,v)$ uniformly converges to $\psi_0(t,s,u,v)$ as $N \to \infty$, and $\psi_0(t,s,u,v)$ satisfies the assumption of the proposition, and $|\Psi(t,s,u,v)|$ is bounded by a constant which is independent of N, where $\Psi(t,s,u,v)$ is some polynomial in (at most the 6th) derivatives of $\psi(t,s,u,v)$. We note that we can choose ε of the proposition independently of N in this case.

2.5 The saddle point method

In this section, we review a proposition obtained from the saddle point method.

Proposition 2.4 (see [20])**.** *Let A be a non-singular symmetric complex 4×4 matrix, and let $\psi(z_1,z_2,z_3,z_4)$ and $r(z_1,z_2,z_3,z_4)$ be holomorphic functions of the forms,*

$$\begin{aligned} \psi(z_1,z_2,z_3,z_4) &= \mathbf{z}^T A\,\mathbf{z} + r(z_1,z_2,z_3,z_4), \\ r(z_1,z_2,z_3,z_4) &= \textstyle\sum_{i,j,k} b_{ijk} z_i z_j z_k + \sum_{i,j,k,l} c_{ijkl} z_i z_j z_k z_l + \cdots, \end{aligned} \quad (21)$$

defined in a neighborhood of $\mathbf{0} \in \mathbb{C}^4$. The restriction of the domain

$$\big\{(z_1,z_2,z_3,z_4) \in \mathbb{C}^4 \ \big| \ \operatorname{Re}\psi(z_1,z_2,z_3,z_4) < 0\big\} \quad (22)$$

to a neighborhood of $\mathbf{0} \in \mathbb{C}^4$ *is homotopy equivalent to* S^3*. Let* D *be an oriented* 4*-ball embedded in* $\mathbb{C}^4$ *such that* ∂D *is included in the domain* (22) *whose inclusion is homotopic to a homotopy equivalence to the above* S^3 *in the domain* (22)*. Then,*

$$\int_D e^{N\,\psi(z_1,z_2,z_3,z_4)} dz_1\, dz_2\, dz_3\, dz_4 \;=\; \frac{\pi^2}{N^2\sqrt{\det(-A)}}\Big(1+\sum_{i=1}^{d}\frac{\lambda_i}{N^i} \;+O\big(\frac{1}{N^{d+1}}\big)\Big),$$

for any d*, where we choose the sign of* $\sqrt{\det(-A)}$ *as explained in* [20]*, and* λ_i*'s are constants presented by using coefficients of the expansion of* $\psi(z_1,z_2,z_3,z_4)$*; such presentations are obtained by formally expanding the following formula,*

$$1+\sum_{i=1}^{\infty}\frac{\lambda_i}{N^i} \;=\; \exp\Big(N\,r\big(\frac{\partial}{\partial w_1},\cdots,\frac{\partial}{\partial w_4}\big)\Big)\,\exp\left(-\frac{1}{4N}\begin{pmatrix}w_1\\ \vdots\\ w_4\end{pmatrix}^{T} A^{-1}\begin{pmatrix}w_1\\ \vdots\\ w_4\end{pmatrix}\right)\Bigg|_{w_1=\cdots=w_4=0}.$$

For a proof of the proposition, see [20].

Remark 2.5. As mentioned in [20, Remark 3.6], we can extend Proposition 2.4 to the case where $\psi(z_1,z_2,z_3,z_4)$ depends on N in such a way that $\psi(z_1,z_2,z_3,z_4)$ is of the form

$$\begin{aligned}\psi(z_1,z_2,z_3,z_4) \;=\; & \psi_0(z_1,z_2,z_3,z_4)+\psi_1(z_1,z_2,z_3,z_4)\frac{1}{N}+\psi_2(z_1,z_2,z_3,z_4)\frac{1}{N^2}\\ & +\cdots+\psi_m(z_1,z_2,z_3,z_4)\frac{1}{N^m}+r_m(z_1,z_2,z_3,z_4)\frac{1}{N^{m+1}}\,,\end{aligned}$$

where $\psi_i(z_1,z_2,z_3,z_4)$'s are holomorphic functions independent of N, and we assume that $\psi_0(z_1,z_2,z_3,z_4)$ satisfies the assumption of the proposition and $|r_m(z_1,z_2,z_3,z_4)|$ is bounded by a constant which is independent of N.

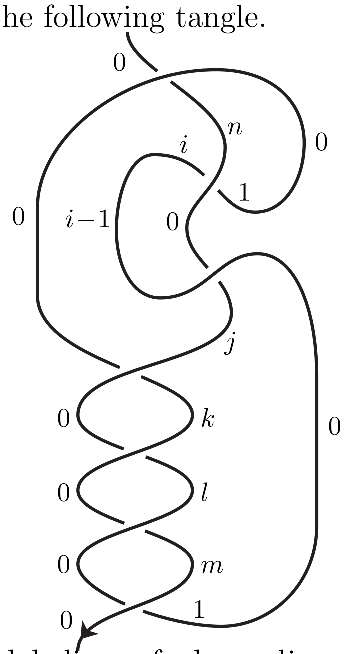
3 The 7_3 knot

In this section, we show Theorem 1.1 for the 7_3 knot. We give a proof of the theorem in Section 3.1, using lemmas shown in Sections 3.2–3.5.

3.1 Proof of Theorem 1.1 for the 7_3 knot

In this section, we show a proof of Theorem 1.1 for the 7_3 knot.

The 7_3 *knot* is the closure of the following tangle.



(23)

As shown in [32], we can put the labelings of edges adjacent to the unbounded regions as shown above. Hence, from the definition of the Kashaev invariant, the Kashaev invariant of the 7_3 knot is presented by

$$\begin{aligned}
\langle 7_3 \rangle_N \; &= \sum q^{1/2} \times \frac{N\,q^{-\frac{1}{2}}}{(\overline{q})_{N-n}(q)_{n-1}} \times \frac{N\,q^{-\frac{1}{2}+i}}{(q)_{i-n}(\overline{q})_{n-1}(\overline{q})_{N-i}} \times \frac{N\,q^{-\frac{1}{2}-i+1}}{(\overline{q})_{N-j}(q)_{j-i}(\overline{q})_{i-1}} \\
&\quad \times \frac{N\,q^{-\frac{1}{2}}}{(q)_{N-j}(\overline{q})_{j-k}(q)_{k-1}} \times \frac{N\,q^{-\frac{1}{2}}}{(q)_{N-k}(\overline{q})_{k-l}(q)_{l-1}} \times \frac{N\,q^{-\frac{1}{2}}}{(q)_{N-l}(\overline{q})_{l-m}(q)_{m-1}} \times \frac{N\,q^{-\frac{1}{2}}}{(q)_{N-m}(\overline{q})_{m-1}} \\
&= \sum_{\substack{0<i\leq j\leq N \\ 0<l\leq k\leq j}} \frac{N^5\,q^{-2}}{(\overline{q})_{i-1}(\overline{q})_{N-i}(q)_{j-i}(q)_{N-j}(\overline{q})_{N-j}(\overline{q})_{j-k}(q)_{k-1}(q)_{N-k}(\overline{q})_{k-l}(q)_{l-1}(q)_{N-l}} \\
&= \sum_{\substack{0\leq i\leq j<N \\ 0\leq l\leq k\leq j}} \frac{N^5\,q^{-2}}{(\overline{q})_{i}(\overline{q})_{N-i-1}(q)_{j-i}(q)_{N-j-1}(\overline{q})_{N-j-1}(\overline{q})_{j-k}(q)_{k}(q)_{N-k-1}(\overline{q})_{k-l}(q)_{l}(q)_{N-l-1}} \\
&= \sum_{\substack{0\leq i\leq j<N \\ 0\leq k,l \\ k+l\leq j}} \frac{N^5\,q^{-2}}{(\overline{q})_{j-i}(\overline{q})_{N-j+i-1}(q)_{i}(q)_{N-j-1}(\overline{q})_{N-j-1}(\overline{q})_{k}(q)_{j-k}(q)_{N-j+k-1}(\overline{q})_{l}(q)_{j-k-l}(q)_{N-j+k+l-1}}
\end{aligned}$$

where we obtain the third equality by replacing i, j, k, l with $i+1, j+1, k+1, l+1$ respectively, and obtain the last equality by replacing $j-i$, $j-k$, $k-l$ with i, k, l respectively.

Proof of Theorem 1.1 for the 7_3 *knot.* By (5), the above presentation of $\langle 7_3 \rangle_N$ is rewritten

$$\langle 7_3 \rangle_N \;=\; N^5 q^{-2} \sum_{\substack{0\leq i\leq j<N \\ 0\leq k,l \\ k+l\leq j}} \exp\Big(N\,\tilde{V}\big(\frac{2i+1}{2N},\ \frac{2j+1}{2N},\ \frac{2k+1}{2N},\ \frac{2l+1}{2N}\big)\Big),$$

where we put

$$\begin{aligned}
\tilde V(t,s,u,v) \;&=\; \frac{1}{N}\Big(-\varphi\big(s-t+\frac{1}{2N}\big)-\varphi\big(1-s+t-\frac{1}{2N}\big)+\varphi(t)-\varphi(s)+\varphi(1-s) \\
&\quad -\varphi(1-u)+\varphi\big(s-u+\frac{1}{2N}\big)+\varphi\big(1-s+u-\frac{1}{2N}\big)-\varphi(1-v) \\
&\quad +\varphi\big(s-u-v+\frac{1}{N}\big)+\varphi\big(1-s+u+v-\frac{1}{N}\big)-6\,\varphi\big(\frac{1}{2N}\big)+5\,\varphi\big(1-\frac{1}{2N}\big)\Big) \\
&= \frac{1}{N}\Big(\varphi(t)-2\varphi(s)+\varphi(u)+\varphi(v)\Big)-\frac{1}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6}-\frac{11}{2N}\log N-\frac{\pi\sqrt{-1}}{4N}+\frac{\pi\sqrt{-1}}{12N^2} \\
&\quad +2\pi\sqrt{-1}\cdot\frac{1}{2}\Big(\big(s-t+\frac{1}{2N}\big)^2-s^2+u^2-\big(s-u+\frac{1}{2N}\big)^2+v^2 \\
&\quad -\big(s-u-v+\frac{1}{N}\big)^2+t+2s-3u-2v+\frac{1}{N}\Big).
\end{aligned}$$

Here, we obtain the last equality by (9) and (10). Hence, by putting

$$V(t,s,u,v) \;=\; \tilde V(t,s,u,v)+\frac{11}{2N}\log N,$$

the presentation of $\langle\, 7_3\,\rangle_N$ is rewritten

$$\langle\, 7_3\,\rangle_N \;=\; N^{-1/2}q^{-2}\sum_{\substack{i,j,k,l\,\in\mathbb{Z}\\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})\in\Delta}}\exp\Big(N\,V\big(\frac{2i+1}{2N},\;\frac{2j+1}{2N},\;\frac{2k+1}{2N},\;\frac{2l+1}{2N}\big)\Big),$$

where the range of $(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})$ of the sum is given by the following domain,

$$\Delta \;=\; \big\{(t,s,u,v)\in\mathbb{R}^4 \;\big|\; 0\le t\le s\le 1,\quad 0\le u,v,\quad u+v\le s\big\}.$$

By Proposition 2.1, as $N\to\infty$, $V(t,s,u,v)$ converges to the following $\hat V(t,s,u,v)$ in the interior of Δ,

$$\begin{aligned}
\hat V(t,s,u,v) \;&=\; \frac{1}{2\pi\sqrt{-1}}\Big(\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t})-2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,s})+\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,u})+\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,v})-\frac{\pi^2}{6}\Big) \\
&\quad +2\pi\sqrt{-1}\cdot\frac{1}{2}\big((s-t)^2-s^2+u^2-(s-u)^2+v^2-(s-u-v)^2+t+2s-3u-2v\big).
\end{aligned}$$

By concrete calculation, we can check that the boundary of Δ is included in the domain

$$\big\{(t,s,u,v)\in\Delta \;\big|\; \mathrm{Re}\,\hat V(t,s,u,v)<\varsigma_R-\varepsilon\big\} \tag{24}$$

for some sufficiently small $\varepsilon>0$, where we put $\varsigma_R=0.730861...$ as in (29); we will know later that this value is equal to the real part of the critical value of $\hat V$ at the critical point of Lemma 3.2. Since the sum of the problem is of the order $O(e^{N\varsigma_R})$, we can ignore the sum of the problem restricted in the above domain, and hence, we can remove this domain from Δ. Therefore, we can choose a new domain Δ' in the interior of Δ such that

$\Delta - \Delta' \subset$ (24); more concretely, we can choose Δ' as

$$\Delta' = \left\{ (t,s,u,v) \in \Delta \;\middle|\; \begin{array}{l} 0.03 \le t \le 0.38, \quad 0.68 \le s \le 0.95, \quad 0.03 \le u \le 0.38, \\ 0.03 \le v \le 0.38, \quad 0.4 \le s-t \le 0.85, \quad 0.4 \le s-u \le 0.85, \\ -t+2u+v \le 0.85, \quad t+2s-2u-v \le 1.8 \end{array} \right\}, \tag{25}$$

where we calculate the concrete values of the bounds of these inequalities in Section 3.2. Hence, since $\Delta - \Delta' \subset$ (24), we obtain the second equality of the following formula,

$$\begin{aligned} \langle\, 7_3 \,\rangle_N &= e^{N\varsigma} N^{-1/2} q^{-2} \sum_{\substack{i,j,k,l \in \mathbb{Z} \\ (\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N}) \in \Delta}} \exp\Big(N \cdot V\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N}\big) - N\varsigma \Big) \\ &= e^{N\varsigma} \Bigg(N^{-1/2} q^{-2} \sum_{\substack{i,j,k,l \in \mathbb{Z} \\ (\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N}) \in \Delta'}} \exp\Big(N \cdot V\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N}\big) - N\varsigma \Big) + O(e^{-N\varepsilon}) \Bigg), \end{aligned}$$

for some $\varepsilon > 0$.

Further, by Proposition 2.2 (Poisson summation formula), the above sum is presented by

$$\langle\, 7_3 \,\rangle_N = e^{N\varsigma} \Bigg(N^{7/2} q^{-2} \int_{\Delta'} \exp\big(N \cdot V(t,s,u,v) - N\varsigma \big)\, dt\, ds\, du\, dv + O(e^{-N\varepsilon}) \Bigg), \tag{26}$$

noting that we verify the assumption of Proposition 2.2 in Lemma 3.3. Furthermore, by Proposition 2.4 (saddle point method), there exist some κ_i''s such that

$$\langle\, 7_3 \,\rangle_N = N^{7/2} \exp\big(N \cdot V(t_c, s_c, u_c, v_c) \big) \cdot \frac{(2\pi)^2}{N^2} \big(\det(-H) \big)^{-1/2} \Big(1 + \sum_{i=1}^{d} \kappa_i' \hbar^i + O(\hbar^{d+1}) \Big),$$

for any $d > 0$, noting that we verify the assumption of Proposition 2.4 in Lemma 3.9. Here, (t_c, s_c, u_c, v_c) is the critical point of V which corresponds to the critical point (t_0, s_0, u_0, v_0) of $\hat{V}$ of Lemma 3.2, where $\hat{V}$ is the limit of V at $N \to \infty$ whose concrete presentation is given in Section 3.2, and H is the Hesse matrix of V at (t_c, s_c, u_c, v_c).

We calculate the right-hand side of the above formula. Since $t_c = t_0 + O(\hbar)$, $s_c = s_0 + O(\hbar)$, $u_c = u_0 + O(\hbar)$ and $v_c = v_0 + O(\hbar)$, we have that $V(t_c, s_c, u_c, v_c) = V(t_0, s_0, u_0, v_0) + O(\hbar^2)$. Hence, by comparing $V(t_0, s_0, u_0, v_0)$ and $\hat{V}(t_0, s_0, u_0, v_0) = \varsigma$, we have that

$$V(t_0, s_0, u_0, v_0) = \varsigma + O(\hbar).$$

Therefore, there exist some κ_i's such that

$$\langle\, 7_3 \,\rangle_N = e^{N\varsigma} N^{3/2}\, \omega \cdot \Big(1 + \sum_{i=1}^{d} \kappa_i \hbar^i + O(\hbar^{d+1}) \Big),$$

for any $d > 0$. Hence, we obtain the theorem for the 7_3 knot. □

3.2 Estimate of the range of Δ'

In this section, we calculate the concrete values of the bounds of the inequalities in (25) so that they satisfy that $\Delta - \Delta' \subset$ (24).

Putting Λ as in Section 2.2, we have that

$$\mathrm{Re}\,\hat{V}(t,s,u,v) \;=\; \Lambda(t) - 2\,\Lambda(s) + \Lambda(u) + \Lambda(v).$$

We consider the domain

$$\big\{(t,s,u,v) \in \Delta \;\big|\; \Lambda(t) - 2\,\Lambda(s) + \Lambda(u) + \Lambda(v) \;\geq\; \varsigma_{R}\big\}, \tag{27}$$

where we put $\varsigma_R = 0.730861...$ as in (29). We note that this domain is symmetric with respect to the exchanges of t, u and v. The aim of this section is to show that this domain is included in the interior of the domain Δ' of (25). For this purpose, we show the estimates of the defining inequalities of (25) for (t,s,u,v) in (27).

We calculate the minimal value $t_{\min}$ and the maximal value $t_{\max}$ of t. Since $|\Lambda(\,\cdot\,)| \leq \Lambda(\frac{1}{6})$,

$$\Lambda(t) \;\geq\; \varsigma_R - 4\,\Lambda\big(\frac{1}{6}\big) \;=\; 0.084729...\,.$$

The minimal and maximal values of t are solutions of the equality of the above formula. By calculating a solution of the equality by Newton's method from $t = 0.03$, we obtain $t_{\min} = 0.0328657...$, and from $t = 0.4$, we obtain $t_{\max} = 0.372797...$. Therefore, we obtain an estimate of t in Δ' as

$$0.03 \;\leq\; t \;\leq\; 0.38.$$

Remark 3.1. To be precise, the above argument is not partially rigorous, since we do not estimate the error terms of the numerical solutions of Newton's method, though the above argument is practically useful, since we can guess that such error terms would be sufficiently small for the above purpose. We can obtain rigorous proofs of such estimates (the above one and the following ones) by concrete calculations (see [20, 21]), though such calculation might often be far longer than calculations by Newton's method.

We obtain the estimates of u and v in Δ' in the same way as above.

We calculate the minimal value $s_{\min}$ and the maximal value $s_{\max}$ of s. Since $\Lambda(\,\cdot\,) \leq \Lambda(\frac{1}{6})$,

$$-2\,\Lambda(s) \;\geq\; \varsigma_R - 3\,\Lambda\big(\frac{1}{6}\big) \;=\; 0.246262...\,.$$

The minimal and maximal values of s are solutions of the equality of the above formula. By calculating a solution of the equality by Newton's method from $s = 0.65$, we obtain $s_{\min} = 0.69634...$, and from $s = 0.95$, we obtain $s_{\max} = 0.935251...$. Therefore, we obtain an estimate of s in Δ' as

$$0.68 \;\leq\; s \;\leq\; 0.95.$$

Before calculating other estimates, we note that the domain (27) is a convex domain such that the boundary is a smooth closed hypersurface whose sectional curvatures are positive everywhere, which we show in Appendix A. In the following of this section, we consider the maximal and minimal value of some linear function $L(t,s,u,v)$ on this domain. The maximal and minimal values of $L(t,s,u,v)$ are obtained when the hyperplane $L(t,s,u,v)=c$ (where c is a constant) is tangent to such a domain. Such tangent points are given by solutions of a certain system of equations, and there are exactly two solutions of such a system of equations, since the domain is of the shape mentioned above. We calculate such solutions in the following of this section.

We calculate the minimal value $(s-t)_{\min}$ and the maximal value $(s-t)_{\max}$ of $s-t$. Since $\Lambda(\,\cdot\,)\le\Lambda(\frac{1}{6})$,

$$\Lambda(t)-2\,\Lambda(s)\ \ge\ \varsigma_{R}-2\,\Lambda\Big(\frac{1}{6}\Big)\ =\ 0.407795...\ .$$

We note that the domain $\big\{\Lambda(t)-2\,\Lambda(s)\ \ge\ \varsigma_{R}-2\,\Lambda(\frac{1}{6})\big\}$ is a convex domain in $\mathbb{R}^2$ such that its boundary is a closed curve whose curvatures are non-zero everywhere. Putting $w=s-t$, its minimal and maximal values are solutions of the following equations,

$$\begin{cases}\Lambda(s-w)-2\,\Lambda(s)\ =\ \varsigma_{R}-2\,\Lambda\big(\frac{1}{6}\big),\\ \dfrac{\partial}{\partial s}\big(\Lambda(s-w)-2\,\Lambda(s)\big)\ =\ 0.\end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the minimal and maximal values of $s-t$. By calculating a solution of these equations by Newton's method from $(w,s)=(0.4,0.8)$, we obtain $(s-t)_{\min}=0.429457...$, and from $(w,s)=(0.85,0.9)$, we obtain $(s-t)_{\max}=0.844926...$. Therefore, we obtain an estimate of $s-t$ in Δ' as

$$0.4\ \le\ s-t\ \le\ 0.85.$$

We obtain the estimate of $s-u$ in Δ' in the same way as above.

We calculate the maximal value $(-t+2u+v)_{\max}$ of $-t+2u+v$. Putting $w'=-t+2u+v$, its maximal value is a solution of the system of the following equations,

$$\begin{cases}\Lambda(-w'+2u+v)-2\,\Lambda(s)+\Lambda(u)+\Lambda(v)\ =\ \varsigma_{R}\,,\\ \dfrac{\partial}{\partial s}\big(\Lambda(-w'+2u+v)-2\,\Lambda(s)+\Lambda(u)+\Lambda(v)\big)\ =\ 0,\\ \dfrac{\partial}{\partial u}\big(\Lambda(-w'+2u+v)-2\,\Lambda(s)+\Lambda(u)+\Lambda(v)\big)\ =\ 0,\\ \dfrac{\partial}{\partial v}\big(\Lambda(-w'+2u+v)-2\,\Lambda(s)+\Lambda(u)+\Lambda(v)\big)\ =\ 0.\end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of $-t+2u+v$. By calculating a solution of these equations by Newton's method from $(w',s,u,v)=(0.8,0.85,0.35,0.2)$, we obtain $(-t+2u+v)_{\max}=0.801018...$. Therefore, we obtain an estimate of $-t+2u+v$ in Δ' as

$$-t+2u+v\ \le\ 0.85.$$

We calculate the maximal value $(t+2s-2u-v)_{\max}$ of $t+2s-2u-v$. Putting $w''=t+2s-2u-v$, its maximal value is a solution of the system of the following equations,

$$\begin{cases} \Lambda(w''-2s+2u+v)-2\,\Lambda(s)+\Lambda(u)+\Lambda(v) \;=\; \varsigma_{_R}\,, \\ \dfrac{\partial}{\partial s}\big(\Lambda(w''-2s+2u+v)-2\,\Lambda(s)+\Lambda(u)+\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(\Lambda(w''-2s+2u+v)-2\,\Lambda(s)+\Lambda(u)+\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial v}\big(\Lambda(w''-2s+2u+v)-2\,\Lambda(s)+\Lambda(u)+\Lambda(v)\big) \;=\; 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of $t+2s-2u-v$. By calculating a solution of these equations by Newton's method from $(w'',s,u,v)=(1.8,0.9,0.1,0.1)$, we obtain $(t+2s-2u-v)_{\max}=1.77226...$. Therefore, we obtain an estimate of $t+2s-2u-v$ in Δ' as

$$t+2s-2u-v \;\le\; 1.8.$$

3.3 Calculation of the critical value

In this section, we calculate the concrete value of a critical point of $\hat{V}$.

The differentials of $\hat{V}$ are presented by

$$\begin{aligned} \frac{\partial}{\partial t}\hat{V}(t,s,u,v) &\;=\; -\log(1-x)+2\pi\sqrt{-1}\,\big(t-s+\frac{1}{2}\big), \\ \frac{\partial}{\partial s}\hat{V}(t,s,u,v) &\;=\; 2\,\log(1-y)+2\pi\sqrt{-1}\,\big(-t-2s+2u+v+1\big), \\ \frac{\partial}{\partial u}\hat{V}(t,s,u,v) &\;=\; -\log(1-z)+2\pi\sqrt{-1}\,\big(2s-u-v-\frac{3}{2}\big), \\ \frac{\partial}{\partial v}\hat{V}(t,s,u,v) &\;=\; -\log(1-w)+2\pi\sqrt{-1}\,\big(s-u-1\big), \end{aligned}$$

where $x=e^{2\pi\sqrt{-1}\,t}$, $y=e^{2\pi\sqrt{-1}\,s}$, $z=e^{2\pi\sqrt{-1}\,u}$ and $w=e^{2\pi\sqrt{-1}\,v}$.

Lemma 3.2. *$\hat{V}$ has a unique critical point (t_0,s_0,u_0,v_0) in $P^{-1}(\Delta')$, where $P:\mathbb{C}^4\to\mathbb{R}^4$ is the projection to the real parts of the entries.*

Proof. Any critical point of $\hat{V}$ is given by a solution of $\frac{\partial}{\partial t}\hat{V}=\frac{\partial}{\partial s}\hat{V}=\frac{\partial}{\partial u}\hat{V}=\frac{\partial}{\partial v}\hat{V}=0$, and these equations are rewritten,

$$1-x \;=\; -\frac{x}{y}\,, \qquad (1-y)^2 \;=\; \frac{xy^2}{z^2w}\,, \qquad 1-z \;=\; -\frac{y^2}{zw}\,, \qquad 1-w \;=\; \frac{y}{z}\,.$$

From the first formula, we have that $x=y/(y-1)$. Further, from the third formula, we have that $w=-y^2/\big(z(1-z)\big)$. By substituting them into the second formula, we have that $z=-y/(y^3-3y^2+2y-1)$. Further, by substituting them into the fourth formula, we have that

$$y^6-7y^5+19y^4-28y^3+26y^2-13y+3 \;=\; 0.$$

Its solutions are

$$y \;=\; 0.49542...\pm\sqrt{-1}\cdot 0.342767...\;,\quad 0.537981...\pm\sqrt{-1}\cdot 1.04357...\;,\quad 2.17069...\;,\quad 2.7625...\;.$$

Among these, the solution $0.537981...-\sqrt{-1}\cdot 1.04357...$ gives a solution in Δ', from which we have that

$$\begin{aligned} x_0 &= 0.645284...+\sqrt{-1}\cdot 0.801205...\,, & t_0 &= 0.14209...-\sqrt{-1}\cdot 0.00451074...\,,\\ y_0 &= 0.537981...-\sqrt{-1}\cdot 1.04357...\,, & s_0 &= 0.825756...-\sqrt{-1}\cdot 0.0255418...\,,\\ z_0 &= 0.363612...+\sqrt{-1}\cdot 0.565801...\,, & u_0 &= 0.159092...+\sqrt{-1}\cdot 0.0631297...\,,\\ w_0 &= 1.87287...+\sqrt{-1}\cdot 1.51178...\,, & v_0 &= 0.108085...-\sqrt{-1}\cdot 0.139791...\,, \end{aligned}$$

where $x_0 = e^{2\pi\sqrt{-1}\,t_0}$, $y_0 = e^{2\pi\sqrt{-1}\,s_0}$, $z_0 = e^{2\pi\sqrt{-1}\,u_0}$ and $w_0 = e^{2\pi\sqrt{-1}\,v_0}$. These give a unique critical point in $P^{-1}(\Delta')$. □

The critical value of $\hat{V}$ at the critical point of Lemma 3.2 is presented by

$$\begin{aligned} \varsigma &= \hat{V}(t_0,s_0,u_0,v_0)\\ &= \frac{1}{2\pi\sqrt{-1}}\Big(\mathrm{Li}_2(x_0)-2\,\mathrm{Li}_2(y_0)+\mathrm{Li}_2(z_0)+\mathrm{Li}_2(w_0)-\frac{\pi^2}{6}\Big)\\ &\quad + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big((s_0-t_0)^2-s_0^2+u_0^2-(s_0-u_0)^2+v_0^2-(s_0-u_0-v_0)^2\\ &\qquad + t_0+2s_0-3u_0-2v_0\big) \qquad (28)\\ &= 0.730861...+\sqrt{-1}\cdot 0.588168...\;. \end{aligned}$$

Further, we put its real part to be ς_R,

$$\varsigma_R \;=\; \mathrm{Re}\,\varsigma \;=\; 0.730861...\;. \qquad (29)$$

3.4 Verifying the assumption of the Poisson summation formula

In this section, we verify the assumption of the Poisson summation formula in Lemma 3.3, which is used in the proof of Theorem 1.1 for the 7_3 knot in Section 3.1.

By computer calculation, we can see that the maximal value of $\mathrm{Re}\,\hat{V}-\varsigma_R$ is about 0.08. Therefore, in the proof of Lemma 3.3, it is sufficient to decrease, say, $\mathrm{Re}\,\hat{V}(t+\delta\sqrt{-1},\, s,\, u,\, v)-2\pi\delta$ by 0.08, by moving δ (though we do not use this value in the proof of the lemma).

We put

$$f(X,Y,Z,W) \;=\; \mathrm{Re}\,\hat{V}(t+X\sqrt{-1},\; s+Y\sqrt{-1},\; u+Z\sqrt{-1},\; v+W\sqrt{-1})-\varsigma_R\,.$$

Then, we have that

$$\begin{aligned} \frac{\partial f}{\partial X} &= \mathrm{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial t}\hat{V}(t+X\sqrt{-1},\; s+Y\sqrt{-1},\; u+Z\sqrt{-1},\; v+W\sqrt{-1})\Big)\\ &= -\mathrm{Im}\Big(-\log(1-x)+2\pi\sqrt{-1}\,\big(t-s+\frac{1}{2}\big)\Big) \end{aligned}$$

$$= \ \operatorname{Arg}(1-x) + 2\pi\big(s - t - \frac{1}{2}\big). \tag{30}$$

Similarly, we have that

$$\frac{\partial f}{\partial Y} \ = \ -2\operatorname{Arg}(1-y) + 2\pi\big(t + 2s - 2u - v - 1\big), \tag{31}$$

$$\frac{\partial f}{\partial Z} \ = \ \operatorname{Arg}(1-z) - 2\pi\big(2s - u - v - \frac{3}{2}\big), \tag{32}$$

$$\frac{\partial f}{\partial W} \ = \ \operatorname{Arg}(1-w) - 2\pi\big(s - u - 1\big). \tag{33}$$

Lemma 3.3. *$V(t,s,u,v) - \varsigma_R$ satisfies the assumption of Proposition 2.2.*

Proof. Since $V(t,s,u,v)$ converges uniformly to $\hat{V}(t,s,u,v)$ on Δ', we show the proof for $\hat{V}(t,s,u,v)$ instead of $V(t,s,u,v)$. We show that $\partial\Delta'$ is null-homotopic in each of (13)–(20).

As for (13), we show that we can move Δ' into the following domain,

$$\big\{(t+\delta\sqrt{-1},\ s,\ u,\ v) \in \mathbb{C}^4 \ \big|\ (t,s,u,v) \in \Delta',\ \delta \geq 0,\ \operatorname{Re}\hat{V}(t+\delta\sqrt{-1},\ s,\ u,\ v) - \varsigma_R < 2\pi\delta\big\}.$$

Hence, putting

$$F(\delta) \ = \ \operatorname{Re}\hat{V}(t+\delta\sqrt{-1},\ s,\ u,\ v) - \varsigma_R - 2\pi\delta \ = \ f(\delta,0,0,0) - 2\pi\delta,$$

it is sufficient to show that there exists $\delta_0 > 0$ such that

$$\begin{aligned} F(\delta_0) \ &< \ 0 \quad \text{for any } (t,s,u,v) \in \Delta', \text{ and} \\ F(\delta) \ &< \ 0 \quad \text{for any } (t,s,u,v) \in \partial\Delta' \text{ and } \delta \in [0,\delta_0]. \end{aligned} \tag{34}$$

Therefore, it is sufficient to show that

$$\frac{d}{d\delta}F(\delta) \ = \ \frac{\partial f}{\partial X}(\delta,0,0,0) - 2\pi \ < \ -\varepsilon',$$

for some $\varepsilon' > 0$ (because, if the above formula holds, then (34) holds for a sufficiently large δ_0). Hence, it is sufficient to show that

$$\frac{\partial f}{\partial X}(X,0,0,0) \ < \ 2\pi - \varepsilon'.$$

Further, as for (14), similarly as above, it is sufficient to show that

$$\frac{\partial f}{\partial X}(-X,0,0,0) \ < \ 2\pi - \varepsilon'$$

for some $\varepsilon' > 0$.

Hence, as for (13) and (14), it is sufficient to show that

$$-(2\pi - \varepsilon') \ < \ \frac{\partial f}{\partial X}(X,0,0,0) \ < \ 2\pi - \varepsilon' \tag{35}$$

for some $\varepsilon' > 0$. Since $0.03 \le t \le 0.38$,

$$-2\pi\big(\frac{1}{2} - t\big) \ < \ \mathrm{Arg}\,(1-x) \ < \ 0.$$

Hence, by (30),

$$-2\pi\big(1-s\big) \ < \ \frac{\partial f}{\partial X} \ < \ 2\pi\big(s - t - \frac{1}{2}\big).$$

Since $0.68 \le s$ and $s - t \le 0.85$,

$$-2\pi \cdot 0.32 \ < \ \frac{\partial f}{\partial X} \ < \ 2\pi \cdot 0.35.$$

Therefore, (35) is satisfied, as required.

As for (15) and (16), similarly as above, it is sufficient to show that

$$-(2\pi - \varepsilon') \ \le \ \frac{\partial f}{\partial Y}(0, Y, 0, 0) \ \le \ 2\pi - \varepsilon' \tag{36}$$

for some $\varepsilon' > 0$. Since $0.68 \le s \le 0.95$,

$$0 \ < \ \mathrm{Arg}\,(1-y) \ < \ 2\pi\big(s - \frac{1}{2}\big).$$

Hence, by (31),

$$-2\pi\big(-t + 2u + v\big) \ < \ \frac{\partial f}{\partial Y} \ < \ 2\pi\big(t + 2s - 2u - v - 1\big).$$

Since $-t + 2u + v \le 0.85$ and $t + 2s - 2u - v \le 1.8$,

$$-2\pi \cdot 0.85 \ < \ \frac{\partial f}{\partial Y} \ < \ 2\pi \cdot 0.8.$$

Therefore, (36) is satisfied, as required.

As for (17) and (18), similarly as above, it is sufficient to show that

$$-(2\pi - \varepsilon') \ \le \ \frac{\partial f}{\partial Z}(0, 0, Z, 0) \ \le \ 2\pi - \varepsilon' \tag{37}$$

for some $\varepsilon' > 0$. Since $0.03 \le u \le 0.38$,

$$-2\pi\big(\frac{1}{2} - u\big) \ < \ \mathrm{Arg}\,(1-z) \ < \ 0.$$

Hence, by (32),

$$-2\pi\big(2s - 2u - v - 1\big) \ < \ \frac{\partial f}{\partial Z} \ < \ 2\pi\big(\frac{3}{2} - 2s + u + v\big).$$

Since $2s - 2u - v = 2(s-u) - v \le 2 \cdot 0.85 = 1.7$ and $-2s + u + v \le -2 \cdot 0.68 + 0.38 + 0.38 \le -0.6$,

$$-2\pi \cdot 0.7 \ < \ \frac{\partial f}{\partial Z} \ < \ 2\pi \cdot 0.9.$$

Therefore, (37) is satisfied, as required.

As for (19) and (20), similarly as above, it is sufficient to show that

$$-(2\pi - \varepsilon') \ \leq \ \frac{\partial f}{\partial W}(0,0,0,W) \ \leq \ 2\pi - \varepsilon' \tag{38}$$

for some $\varepsilon' > 0$. Since $0.03 \leq v \leq 0.38$,

$$-2\pi\big(\frac{1}{2} - v\big) \ < \ \mathrm{Arg}\,(1-w) \ < \ 0.$$

Hence, by (33),

$$-2\pi\big(s - u - v - \frac{1}{2}\big) \ < \ \frac{\partial f}{\partial W} \ < \ 2\pi\big(1 - s + u\big).$$

Since $0.4 \leq s - u \leq 0.85$ and $v > 0$,

$$-2\pi \cdot 0.35 \ < \ \frac{\partial f}{\partial W} \ < \ 2\pi \cdot 0.6.$$

Therefore, (38) is satisfied, as required. □

3.5 Verifying the assumption of the saddle point method

In this section, we verify the assumption of the saddle point method in Lemma 3.9. In order to show this lemma, we show Lemmas 3.4–3.8 in advance.

Lemma 3.4. *Fixing Y, Z and W, we regard f as a function of X.*
(1) If $s - t \leq \frac{1}{2}$, then f is monotonically decreasing as a function of X.
(2) If $s - t > \frac{1}{2}$, then f has a unique minimal point as a function of X. In particular, this minimal point goes to ∞ as $s - t \to \frac{1}{2} + 0$.

Proof. Since $x = e^{2\pi\sqrt{-1}\,(t + X\sqrt{-1})}$ and $0.03 \leq t \leq 0.38$,

$$-2\pi\big(\frac{1}{2} - t\big) \ < \ \mathrm{Arg}\,(1-x) \ < \ 0,$$

and $\mathrm{Arg}\,(1-x)$ is monotonically increasing as a function of X. Hence, by (30), $\frac{\partial f}{\partial X}$ is also monotonically increasing as a function of X. Further,

$$\begin{aligned} \frac{\partial f}{\partial X}\Big|_{X\to\infty} &= \ 2\pi\big(s - t - \frac{1}{2}\big), \\ \frac{\partial f}{\partial X}\Big|_{X\to-\infty} &= \ -2\pi\big(1 - s\big) \ < \ 0. \end{aligned}$$

If $s - t \leq \frac{1}{2}$, then $\frac{\partial f}{\partial X}$ is always negative, and (1) holds.

If $s - t > \frac{1}{2}$, then there is a unique zero of $\frac{\partial f}{\partial X}$, which gives a unique minimal point of f, and (2) holds. □

Lemma 3.5. *Fixing X, Z and W, we regard f as a function of Y.*
(1) *If $t-2u-v \geq 0$, then f is monotonically increasing as a function of Y.*
(2) *If $t+2s-2u-v \leq 1$, then f is monotonically decreasing as a function of Y.*
(3) *If $t-2u-v < 0$ and $t+2s-2u-v > 1$, then f has a unique minimal point as a function of Y. In particular, this minimal point goes to $-\infty$ as $t-2u-v \to -0$, and goes to ∞ as $t+2s-2u-v \to 1+0$.*

Proof. Since $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$ and $0.68 \leq s \leq 0.95$,

$$0 \;<\; \operatorname{Arg}(1-y) \;<\; 2\pi\big(s-\frac{1}{2}\big),$$

and $\operatorname{Arg}(1-y)$ is monotonically decreasing as a function of Y. Hence, by (31), $\frac{\partial f}{\partial Y}$ is monotonically increasing as a function of Y. Further,

$$\begin{aligned} \frac{\partial f}{\partial Y}\Big|_{Y\to\infty} &= 2\pi\big(t+2s-2u-v-1\big), \\ \frac{\partial f}{\partial Y}\Big|_{Y\to-\infty} &= -2\pi\big(-t+2u+v\big). \end{aligned}$$

If $t-2u-v \geq 0$, then $\frac{\partial f}{\partial Y}$ is always positive, and (1) holds.

If $t+2s-2u-v \leq 1$, then $\frac{\partial f}{\partial Y}$ is always negative, and (2) holds.

If $t-2u-v < 0$ and $t+2s-2u-v > 1$, then there is a unique zero of $\frac{\partial f}{\partial Y}$, which gives a unique minimal point of f, and (3) holds. □

Lemma 3.6. *Fixing X, Y and W, we regard f as a function of Z.*
(1) *If $2s-2u-v \leq 1$, then f is monotonically increasing as a function of Z.*
(2) *If $2s-u-v \geq \frac{3}{2}$, then f is monotonically decreasing as a function of Z.*
(3) *If $2s-2u-v > 1$ and $2s-u-v < \frac{3}{2}$, then f has a unique minimal point as a function of Z. In particular, this minimal point goes to $-\infty$ as $2s-2u-v \to 1+0$, and goes to ∞ as $2s-u-v \to \frac{3}{2}-0$.*

Proof. Since $z = e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})}$ and $0.03 \leq u \leq 0.38$,

$$-2\pi\big(\frac{1}{2}-u\big) \;<\; \operatorname{Arg}(1-z) \;<\; 0,$$

and $\operatorname{Arg}(1-z)$ is monotonically increasing as a function of Z. Hence, by (32), $\frac{\partial f}{\partial Z}$ is also monotonically increasing as a function of Z. Further,

$$\begin{aligned} \frac{\partial f}{\partial Z}\Big|_{Z\to\infty} &= 2\pi\big(\frac{3}{2}-2s+u+v\big), \\ \frac{\partial f}{\partial Z}\Big|_{Z\to-\infty} &= -2\pi\big(2s-2u-v-1\big). \end{aligned}$$

If $2s-2u-v \leq 1$, then $\frac{\partial f}{\partial Z}$ is always positive, and (1) holds.

If $2s-u-v \geq \frac{3}{2}$, then $\frac{\partial f}{\partial Z}$ is always negative, and (2) holds.

If $2s-2u-v > 1$ and $2s-u-v < \frac{3}{2}$, then there is a unique zero of $\frac{\partial f}{\partial Z}$, which gives a unique minimal point of f, and (3) holds. □

Lemma 3.7. *Fixing X, Y and Z, we regard f as a function of W.*
(1) If $s-u-v \le \frac{1}{2}$, then f is monotonically increasing as a function of W.
(2) If $s-u-v > \frac{1}{2}$, then f has a unique minimal point as a function of W. In particular, this minimal point goes to $-\infty$ as $s-u-v \to \frac{1}{2}+0$.

Proof. Since $w = e^{2\pi\sqrt{-1}\,(v+W\sqrt{-1})}$ and $0.03 \le v \le 0.38$,

$$-2\pi\big(\frac{1}{2}-v\big) \;<\; \mathrm{Arg}\,(1-w) \;<\; 0,$$

and $\mathrm{Arg}\,(1-w)$ is monotonically increasing as a function of W. Hence, by (33), $\frac{\partial f}{\partial W}$ is also monotonically increasing as a function of W. Further,

$$\begin{aligned} \frac{\partial f}{\partial W}\Big|_{W\to\infty} &= 2\pi\big(1-s+u\big) \;>\; 0, \\ \frac{\partial f}{\partial W}\Big|_{W\to-\infty} &= -2\pi\big(s-u-v-\frac{1}{2}\big). \end{aligned}$$

If $s-u-v \le \frac{1}{2}$, then $\frac{\partial f}{\partial W}$ is always positive, and (1) holds.

If $s-u-v > \frac{1}{2}$, then there is a unique zero of $\frac{\partial f}{\partial W}$, which gives a unique minimal point of f, and (2) holds. □

Lemma 3.8. *In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$.*
(1) If $s-t > \frac{1}{2}$, $t-2u-v<0$, $t+2s-2u-v>1$, $2s-2u-v>1$, $2s-u-v<\frac{3}{2}$ and $s-u-v>\frac{1}{2}$, then f has a unique minimal point, and the flow goes to this minimal point.
(2) Otherwise, the flow goes to infinity.

Proof. From the definition of f, we have that

$$\begin{aligned} f(X,Y,Z,W) \;=\; & \mathrm{Re}\,\frac{1}{2\pi\sqrt{-1}}\Big(\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})})\Big) + 2\pi\big(s-t-\frac{1}{2}\big)X \\ & - 2\,\mathrm{Re}\,\frac{1}{2\pi\sqrt{-1}}\Big(\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})})\Big) + 2\pi\big(2s+t-2u-v-1\big)Y \\ & + \mathrm{Re}\,\frac{1}{2\pi\sqrt{-1}}\Big(\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})})\Big) + 2\pi\big(-2s+u+v+\frac{3}{2}\big)Z \\ & + \mathrm{Re}\,\frac{1}{2\pi\sqrt{-1}}\Big(\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(v+W\sqrt{-1})})\Big) + 2\pi\big(1-s+u\big)W \;-\varsigma_R\,, \end{aligned}$$

and the contributions from X, Y, Z, W to f are independent. Hence, by Lemmas 3.4, 3.5, 3.6 and 3.7, if the assumption of (1) holds, then f has a unique minimal point, and (1) holds. Otherwise, at least one of X, Y, Z, W goes to infinity by the flow, and (2) holds. □

Lemma 3.9. *When we apply Proposition 2.4 to (26), the assumption of Proposition 2.4 holds.*

Proof. We show that there exists a homotopy Δ'_δ $(0 \le \delta \le 1)$ between $\Delta'_0 = \Delta'$ and Δ'_1 such that

$$(t_0, s_0, u_0, v_0) \in \Delta'_1, \tag{39}$$

$$\Delta'_1 - \{(t_0, s_0, u_0, v_0)\} \ \subset \ \big\{(t,s,u,v) \in \mathbb{C}^4 \ \big| \ \mathrm{Re}\, \hat{V}(t,s,u,v) < \varsigma_{_R}\big\}, \tag{40}$$

$$\partial\Delta'_\delta \ \subset \ \big\{(t,s,u,v) \in \mathbb{C}^4 \ \big| \ \mathrm{Re}\, \hat{V}(t,s,u,v) < \varsigma_{_R}\big\}. \tag{41}$$

We put $g_1(t,s,u,v)$, $g_2(t,s,u,v)$, $g_3(t,s,u,v)$, $g_4(t,s,u,v)$ to be the minimal points of Lemmas 3.4, 3.5, 3.6, 3.7 respectively. For a sufficiently large $R > 0$, we put

$$\begin{aligned}
\hat{g}_1(t,s,u,v) &= \begin{cases} R & \text{if } \ s-t \le \frac{1}{2}, \\ \min\{R, \ g_1(t,s,u,v)\} & \text{if } \ s-t > \frac{1}{2}, \end{cases} \\
\hat{g}_2(t,s,u,v) &= \begin{cases} R & \text{if } \ t+2s-2u-v \le 1, \\ \min\big\{R, \max\{-R, \ g_2(t,s,u,v)\}\big\} & \text{if } \ t-2u-v < 0, \ t+2s-2u-v > 1, \\ -R & \text{if } \ t-2u-v \ge 0, \end{cases} \\
\hat{g}_3(t,s,u,v) &= \begin{cases} R & \text{if } \ 2s-u-v \ge \frac{3}{2}, \\ \min\big\{R, \max\{-R, \ g_3(t,s,u,v)\}\big\} & \text{if } \ 2s-2u-v > 1, \ 2s-u-v < \frac{3}{2}, \\ -R & \text{if } \ 2s-2u-v \le 1, \end{cases} \\
\hat{g}_4(t,s,u,v) &= \begin{cases} \max\{-R, \ g_4(t,s,u,v)\} & \text{if } \ s-u-v > \frac{1}{2}, \\ -R & \text{if } \ s-u-v \le \frac{1}{2}, \end{cases} \\
\mathbf{g}(t,s,u,v) &= \big(g_1(t,s,u,v), \ g_2(t,s,u,v), \ g_3(t,s,u,v), \ g_4(t,s,u,v)\big), \\
\hat{\mathbf{g}}(t,s,u,v) &= \big(\hat{g}_1(t,s,u,v), \ \hat{g}_2(t,s,u,v), \ \hat{g}_3(t,s,u,v), \ \hat{g}_4(t,s,u,v)\big).
\end{aligned}$$

We note that, since $g_1(t,s,u,v) \to \infty$ as $s - t \to \frac{1}{2} + 0$, $\hat{g}_1(t,s,u,v)$ is continuous, and similarly, we can check that $\hat{g}_2(t,s,u,v)$, $\hat{g}_3(t,s,u,v)$, $\hat{g}_4(t,s,u,v)$ and $\hat{\mathbf{g}}(t,s,u,v)$ are also continuous. We set the ending of the homotopy by

$$\Delta'_1 \ = \ \big\{(t,s,u,v) + \hat{\mathbf{g}}(t,s,u,v)\sqrt{-1} \in \mathbb{C}^4 \ \big| \ (t,s,u,v) \in \Delta'\big\}.$$

Further, we define the internal part Δ'_δ $(0 < \delta < 1)$ of the homotopy by setting it along the flow from $\big(t, \ s, \ u, \ v\big)$ determined by the vector field $\big(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W}\big)$.

We show (41), as follows. From the definition of Δ',

$$\partial\Delta' \ \subset \ \big\{(t,s,u,v) \in \mathbb{C}^4 \ \big| \ \mathrm{Re}\, \hat{V}(t,s,u,v) < \varsigma_{_R}\big\}.$$

Further, by the construction of the homotopy, $\mathrm{Re}\, \hat{V}$ monotonically decreases by the homotopy. Hence, (41) holds.

We show (39) and (40), as follows. Consider the following functions

$$\begin{aligned}
F(t,s,u,v,X,Y,Z,W) &= \mathrm{Re}\, \hat{V}\big(t + X\sqrt{-1}, \ s + Y\sqrt{-1}, \ u + Z\sqrt{-1}, \ v + W\sqrt{-1}\big), \\
h(t,s,u,v) &= F\big(t, \ s, \ u, \ v, \ \hat{\mathbf{g}}(t,s,u,v)\big).
\end{aligned}$$

When the assumption of (1) of Lemma 3.8 does not hold, the flow goes to infinity by Lemma 3.8 (2), and $-h(t,s,u,v)$ is sufficiently large (because we let R be sufficiently large), and hence, (40) holds in this case. The remaining case is the case where $\hat{\mathbf{g}}(t,s,u,v) = \mathbf{g}(t,s,u,v)$. In this case, we show (40), as follows. It is shown from the definition of $\mathbf{g}(t,s,u,v)$ that

$$\frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Z} = \frac{\partial F}{\partial W} = 0 \quad \text{at } (X,Y,Z,W) = \mathbf{g}(t,s,u,v).$$

Hence,

$$\operatorname{Im}\frac{\partial \hat{V}}{\partial t} = \operatorname{Im}\frac{\partial \hat{V}}{\partial s} = \operatorname{Im}\frac{\partial \hat{V}}{\partial u} = \operatorname{Im}\frac{\partial \hat{V}}{\partial v} = 0, \qquad \text{at } (t,s,u,v) + \mathbf{g}(t,s,u,v)\sqrt{-1}\,.$$

Further,

$$\frac{\partial h}{\partial t} = \operatorname{Re}\frac{\partial \hat{V}}{\partial t}, \quad \frac{\partial h}{\partial s} = \operatorname{Re}\frac{\partial \hat{V}}{\partial s}, \quad \frac{\partial h}{\partial u} = \operatorname{Re}\frac{\partial \hat{V}}{\partial u}, \quad \frac{\partial h}{\partial v} = \operatorname{Re}\frac{\partial \hat{V}}{\partial v}, \quad \text{at } (t,s,u,v) + \mathbf{g}(t,s,u,v)\sqrt{-1}\,.$$

Therefore, when (t,s,u,v) is a critial point of $h(t,s,u,v)$, $\big((t,s,u,v) + \mathbf{g}(t,s,u,v)\sqrt{-1}\big)$ is a critical point of $\hat{V}$. Hence, by Lemma 3.2, $h(t,s,u,v)$ has a unique maximal point at $(t,s,u,v) = (\operatorname{Re} t_0, \operatorname{Re} s_0, \operatorname{Re} u_0, \operatorname{Re} v_0)$. Therefore, (39) and (40) hold. □

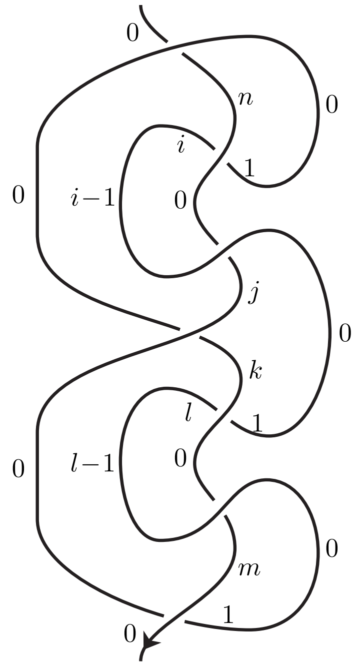
4 The 7_4 knot

In this section, we show Theorem 1.1 for the 7_4 knot. We give a proof of the theorem in Section 4.1, using lemmas shown in Sections 4.2–4.5.

4.1 Proof of Theorem 1.1 for the 7_4 knot

In this section, we show a proof of Theorem 1.1 for the 7_4 knot.

The 7_4 *knot* is the closure of the following tangle.



As shown in [32], we can put the labelings of edges adjacent to the unbounded regions as shown above. Hence, from the definition of the Kashaev invariant, the Kashaev invariant of the 7_4 knot is presented by

$$\begin{aligned}
\langle\, 7_4 \,\rangle_N \;&=\; \sum q^{1/2} \times \frac{N\, q^{-\frac{1}{2}}}{(\overline{q})_{N-n}(q)_{n-1}} \times \frac{N\, q^{-\frac{1}{2}+i}}{(q)_{i-n}(\overline{q})_{n-1}(\overline{q})_{N-i}} \times \frac{N\, q^{-\frac{1}{2}-i+1}}{(\overline{q})_{N-j}(q)_{j-i}(\overline{q})_{i-1}} \\
&\quad \times \frac{N\, q^{-\frac{1}{2}}}{(q)_{N-j}(\overline{q})_{j-k}(q)_{k-1}} \times \frac{N\, q^{-\frac{1}{2}+l}}{(q)_{l-k}(\overline{q})_{k-l}(\overline{q})_{N-l}} \times \frac{N\, q^{-\frac{1}{2}-l+1}}{(\overline{q})_{N-m}(q)_{m-l}(\overline{q})_{l-1}} \times \frac{N\, q^{-\frac{1}{2}}}{(q)_{N-m}(\overline{q})_{m-1}} \\
&= \sum_{\substack{0<i\le j\le N \\ 0<k\le l\le N \\ k\le j}} \frac{N^5\, q^{-1}}{(\overline{q})_{i-1}(\overline{q})_{N-i}(q)_{j-i}(q)_{N-j}(\overline{q})_{N-j}(\overline{q})_{j-k}(q)_{k-1}(\overline{q})_{k-1}(q)_{l-k}(\overline{q})_{l-1}(\overline{q})_{N-l}} \\
&= \sum_{\substack{0\le i\le j<N \\ 0\le k\le l<N \\ k\le j}} \frac{N^5\, q^{-1}}{(\overline{q})_{i}(\overline{q})_{N-i-1}(q)_{j-i}(q)_{N-j-1}(\overline{q})_{N-j-1}(\overline{q})_{j-k}(q)_{k}(\overline{q})_{k}(q)_{l-k}(\overline{q})_{l}(\overline{q})_{N-l-1}} \\
&= \sum_{\substack{0\le i<N-j \\ 0\le j,k,\;\; j+k<N \\ 0\le l<N-k}} \frac{N^5\, q^{-1}}{(\overline{q})_{i+j}(\overline{q})_{N-i-j-1}(q)_{i}(q)_{j}(\overline{q})_{j}(\overline{q})_{N-j-k-1}(q)_{k}(\overline{q})_{k}(q)_{l}(\overline{q})_{k+l}(\overline{q})_{N-k-l-1}}\,,
\end{aligned}$$

where we obtain the third equality by replacing i, j, k, l with $i+1, j+1, k+1, l+1$ respectively, and obtain the last equality by replacing i, j and l with $N-i-j-1$, $N-j-1$ and $k+l$ respectively.

Proof of Theorem 1.1 for the 7_4 knot. By (5), the above presentation of $\langle\, 7_4 \,\rangle_{N}$ is rewritten

$$\langle\, 7_4 \,\rangle_{N} \;=\; N^5 q^{-1} \sum_{\substack{0\le i<N-j\\ 0\le j,k,\;\; j+k<N\\ 0\le l<N-k}} \exp\Big(N\,\tilde V\big(\frac{2i+1}{2N},\;\frac{2j+1}{2N},\;\frac{2k+1}{2N},\;\frac{2l+1}{2N}\big)\Big),$$

where we put

$$\begin{aligned}
\tilde V(t,s,u,v) \;=\; & \frac{1}{N}\Big(-\varphi\big(t+s-\frac{1}{2N}\big) - \varphi\big(1-t-s+\frac{1}{2N}\big) + \varphi(t) + \varphi(s) - \varphi(1-s) \\
& - \varphi\big(s+u-\frac{1}{2N}\big) + \varphi(u) - \varphi(1-u) + \varphi(v) - \varphi\big(u+v-\frac{1}{2N}\big) \\
& - \varphi\big(1-u-v+\frac{1}{2N}\big) - 4\,\varphi\big(\frac{1}{2N}\big) + 7\,\varphi\big(1-\frac{1}{2N}\big)\Big) \\
=\; & \frac{1}{N}\Big(\varphi(t) + 2\,\varphi(s) - \varphi\big(s+u-\frac{1}{2N}\big) + 2\,\varphi(u) + \varphi(v)\Big) \\
& + \frac{1}{2\pi\sqrt{-1}}\,\frac{\pi^2}{2} - \frac{11}{2N}\log N + \frac{3\pi\sqrt{-1}}{4N} - \frac{\pi\sqrt{-1}}{4N^2} \\
& + 2\pi\sqrt{-1}\cdot\frac{1}{2}\Big(\big(t+s-\frac{1}{2N}\big)^2 + s^2 + u^2 + \big(u+v-\frac{1}{2N}\big)^2 - t - 2s - 2u - v + \frac{2}{3} + \frac{1}{N}\Big).
\end{aligned}$$

Here, we obtain the last equality by (9) and (10). Hence, by putting

$$V(t,s,u,v) \;=\; \tilde V(t,s,u,v) + \frac{11}{2N}\log N,$$

the presentation of $\langle\, 7_4 \,\rangle_{N}$ is rewritten

$$\langle\, 7_4 \,\rangle_{N} \;=\; N^{-1/2} q^{-1} \sum_{\substack{i,j,k,l\,\in\mathbb{Z}\\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})\in\Delta}} \exp\Big(N\,V\big(\frac{2i+1}{2N},\;\frac{2j+1}{2N},\;\frac{2k+1}{2N},\;\frac{2l+1}{2N}\big)\Big),$$

where the range of $(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})$ of the sum is given by the following domain,

$$\Delta \;=\; \big\{(t,s,u,v)\in\mathbb{R}^4 \;\big|\; 0\le t\le 1-s,\quad 0\le s,u,\quad s+u\le 1,\quad 0\le v\le 1-u\big\}.$$

By Proposition 2.1, as $N\to\infty$, $V(t,s,u,v)$ converges to the following $\hat V(t,s,u,v)$ in the interior of Δ,

$$\begin{aligned}
\hat V(t,s,u,v) \;=\; & \frac{1}{2\pi\sqrt{-1}}\Big(\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t}) + 2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,s}) - \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s+u)}) \\
& \qquad + 2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,u}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,v}) + \frac{\pi^2}{2}\Big) \\
& + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big((t+s)^2 + s^2 + u^2 + (u+v)^2 - t - 2s - 2u - v + \frac{2}{3}\big).
\end{aligned}$$

By concrete calculation, we can check that the boundary of Δ is included in the domain

$$\big\{(t,s,u,v)\in\Delta \;\big|\; \mathrm{Re}\,\hat V(t,s,u,v) < \varsigma_{R} - \varepsilon\big\} \tag{42}$$

for some sufficiently small $\varepsilon > 0$, where we put $\varsigma_{R} = 0.817729...$ as in (48); we will know later that this value is equal to the real part of the critical value of $\hat{V}$ at the critical point of Lemma 4.1. Hence, similarly as in Section 3.1, we choose a new domain Δ', which satisfies that $\Delta - \Delta' \subset (42)$, as

$$\Delta' = \left\{ (t,s,u,v) \in \Delta \;\middle|\; \begin{array}{l} 0.02 \le t \le 0.45, \quad 0.1 \le s \le 0.45 \\ 0.1 \le u \le 0.45, \quad 0.02 \le v \le 0.45 \end{array} \right\}, \tag{43}$$

where we calculate the concrete values of the bounds of these inequalities in Section 4.2. Hence, since $\Delta - \Delta' \subset (42)$, we obtain the second equality of the following formula,

$$\begin{aligned} \langle\, 7_4 \,\rangle_{N} &= e^{N\varsigma} N^{-1/2} q^{-1} \sum_{\substack{i,j,k,l \in \mathbb{Z} \\ (\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N}) \in \Delta}} \exp\Big(N \cdot V\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N}\big) - N\varsigma \Big) \\ &= e^{N\varsigma} \Bigg(N^{-1/2} q^{-1} \sum_{\substack{i,j,k,l \in \mathbb{Z} \\ (\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N}) \in \Delta'}} \exp\Big(N \cdot V\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N}\big) - N\varsigma \Big) + O(e^{-N\varepsilon}) \Bigg), \end{aligned}$$

for some $\varepsilon > 0$.

Further, by Proposition 2.2 (Poisson summation formula), the above sum is presented by

$$\langle\, 7_4 \,\rangle_{N} = e^{N\varsigma} \Bigg(N^{7/2} q^{-1} \int_{\Delta'} \exp\big(N \cdot V(t,s,u,v) - N\varsigma \big)\, dt\, ds\, du\, dv + O(e^{-N\varepsilon}) \Bigg), \tag{44}$$

noting that we verify the assumption of Proposition 2.2 in Lemma 4.2. Furthermore, by Proposition 2.4 (saddle point method), there exist some κ_i''s such that

$$\langle\, 7_4 \,\rangle_{N} = N^{7/2} \exp\big(N \cdot V(t_c, s_c, u_c, v_c) \big) \cdot \frac{(2\pi)^2}{N^2} \big(\det(-H) \big)^{-1/2} \Big(1 + \sum_{i=1}^{d} \kappa_i' \hbar^i + O(\hbar^{d+1}) \Big),$$

for any $d > 0$, noting that we verify the assumption of Proposition 2.4 in Lemma 4.7. Here, (t_c, s_c, u_c, v_c) is the critical point of V which corresponds to the critical point (t_0, s_0, u_0, v_0) of $\hat{V}$ of Lemma 4.1, where $\hat{V}$ is the limit of V at $N \to \infty$ whose concrete presentation is given in Section 4.2, and H is the Hesse matrix of V at (t_c, s_c, u_c, v_c).

We calculate the right-hand side of the above formula. Similarly as in Section 3.1, we have that

$$V(t_0, s_0, u_0, v_0) = \varsigma + O(\hbar).$$

Therefore, there exist some κ_i's such that

$$\langle\, 7_4 \,\rangle_{N} = e^{N\varsigma} N^{3/2}\, \omega \cdot \Big(1 + \sum_{i=1}^{d} \kappa_i \hbar^i + O(\hbar^{d+1}) \Big),$$

for any $d > 0$. Hence, we obtain the theorem for the 7_4 knot. □

4.2 Estimate of the range of Δ'

In this section, we calculate the concrete values of the bounds of the inequalities in (43) so that they satisfy that $\Delta - \Delta' \subset$ (42).

Putting Λ as in Section 2.2, we have that

$$\mathrm{Re}\, \hat{V}(t,s,u,v) \;=\; \Lambda(t) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) + \Lambda(v).$$

We consider the domain

$$\big\{(t,s,u,v) \in \Delta \;\big|\; \Lambda(t) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) + \Lambda(v) \;\geq\; \varsigma_{R}\big\}, \tag{45}$$

where we put $\varsigma_R = 0.817729...$ as in (48). The aim of this section is to show that this domain is included in the interior of the domain Δ' of (43). For this purpose, we show the estimates of the defining inequalities of (43) for (t,s,u,v) in (45).

Since $\Lambda(v) \leq \Lambda\big(\frac{1}{6}\big)$,

$$\Lambda(t) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) \;\geq\; \varsigma_R - \Lambda\Big(\frac{1}{6}\Big).$$

Further, since $\Lambda(t) \leq \Lambda\big(\frac{1}{6}\big)$,

$$2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) \;\geq\; \varsigma_R - 2\,\Lambda\Big(\frac{1}{6}\Big).$$

We calculate the minimal value $t_{\min}$ and the maximal value $t_{\max}$ of t. They are solutions of the system of the following equations,

$$\begin{cases} \Lambda(t) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) \;=\; \varsigma_R - \Lambda\big(\frac{1}{6}\big), \\ \dfrac{\partial}{\partial s}\big(\Lambda(t) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(\Lambda(t) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u)\big) \;=\; 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of t (see Appendix A). By calculating a solution of these equations by Newton's method from $(t,s,u) = (0.03, 0.2, 0.2)$, we obtain $t_{\min} = 0.0259764...$, and from $(t,s,u) = (0.4, 0.2, 0.2)$, we obtain $t_{\max} = 0.391511...$. Therefore, we obtain an estimate of t in Δ' as

$$0.02 \;\leq\; t \;\leq\; 0.45.$$

We calculate the minimal value $s_{\min}$ and the maximal value $s_{\max}$ of s. They are solutions of the following equations,

$$\begin{cases} 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) \;=\; \varsigma_R - 2\,\Lambda\big(\frac{1}{6}\big), \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u)\big) \;=\; 0. \end{cases}$$

By calculating a solution of these equations by Newton's method from $(s,u)=(0.1,0.2)$, we obtain $s_{\min}=0.104088...$, and from $(s,u)=(0.4,0.2)$, we obtain $s_{\max}=0.441784...$. Therefore, we obtain an estimate of s in Δ' as

$$0.1 \;\le\; s \;\le\; 0.45.$$

We obtain the estimates of u and v from the above estimates by the symmetry (46).

4.3 Calculation of the critical value

In this section, we calculate the concrete value of a critical point of $\hat{V}$.

The differentials of $\hat{V}$ are presented by

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{V}(t,s,u,v) &= -\log(1-x)+2\pi\sqrt{-1}\,\big(t+s-\frac{1}{2}\big),\\
\frac{\partial}{\partial s}\hat{V}(t,s,u,v) &= -2\,\log(1-y)+\log(1-yz)+2\pi\sqrt{-1}\,(t+2s-1),\\
\frac{\partial}{\partial u}\hat{V}(t,s,u,v) &= -2\,\log(1-z)+\log(1-yz)+2\pi\sqrt{-1}\,(2u+v-1),\\
\frac{\partial}{\partial v}\hat{V}(t,s,u,v) &= -\log(1-w)+2\pi\sqrt{-1}\,\big(u+v-\frac{1}{2}\big),
\end{aligned}$$

where $x=e^{2\pi\sqrt{-1}\,t}$, $y=e^{2\pi\sqrt{-1}\,s}$, $z=e^{2\pi\sqrt{-1}\,u}$ and $w=e^{2\pi\sqrt{-1}\,v}$.

Lemma 4.1. *$\hat{V}$ has a unique critical point (t_0,s_0,u_0,v_0) in $P^{-1}(\Delta')$, where $P:\mathbb{C}^4\to\mathbb{R}^4$ is the projection to the real parts of the entries.*

Proof. Any critical point of $\hat{V}$ is given by a solution of $\frac{\partial}{\partial t}\hat{V}=\frac{\partial}{\partial s}\hat{V}=\frac{\partial}{\partial u}\hat{V}=\frac{\partial}{\partial v}\hat{V}=0$, and these equations are rewritten,

$$1-x \;=\; -xy, \quad (1-y)^2 \;=\; xy^2(1-yz), \quad (1-z)^2 \;=\; z^2w(1-yz), \quad 1-w \;=\; -zw.$$

From the first formula, we have that $x=1/(1-y)$. By substituting this into the second formula, we have that $z=(y^3-2y^2+3y-1)/y^3$. Further, by the fourth formula, we have that $w=1/(1-z)$. By substituting these into the third formula, we have that

$$(y^3+2y-1)(y^4-4y^3+8y^2-5y+1) \;=\; 0.$$

Its solutions are

$$\begin{aligned}
y \;=\; &-0.226699...\pm\sqrt{-1}\cdot 1.46771...\,,\quad 0.453398...\,,\\
&0.429304...\pm\sqrt{-1}\cdot 0.10728...\,,\quad 1.5707...\pm\sqrt{-1}\cdot 1.62477...\,.
\end{aligned}$$

Among these, the first solution gives a solution in Δ', from which we have that

$$\begin{aligned}
x_0 &= 0.335258...+\sqrt{-1}\cdot 0.401127...\,, & t_0 &= 0.139198...+\sqrt{-1}\cdot 0.103226...\,,\\
y_0 &= -0.226699...+\sqrt{-1}\cdot 1.46771...\,, & s_0 &= 0.27439...-\sqrt{-1}\cdot 0.0629445...\,,\\
z_0 &= y_0, & u_0 &= s_0,\\
w_0 &= x_0, & v_0 &= t_0,
\end{aligned}$$

where $x_0 = e^{2\pi\sqrt{-1}\,t_0}$, $y_0 = e^{2\pi\sqrt{-1}\,s_0}$, $z_0 = e^{2\pi\sqrt{-1}\,u_0}$ and $w_0 = e^{2\pi\sqrt{-1}\,v_0}$. These give a unique critical point in $P^{-1}(\Delta')$. □

We note that $\hat{V}$ and the set of critical points of $\hat{V}$ have the following symmetry,

$$(t, s, u, v) \longmapsto (v, u, s, t). \tag{46}$$

The critical value of $\hat{V}$ at the critical point of Lemma 4.1 is presented by

$$\begin{aligned} \varsigma &= \hat{V}(t_0, s_0, u_0, v_0) \\ &= \frac{1}{2\pi\sqrt{-1}}\Big(\mathrm{Li}_2(x_0) + 2\,\mathrm{Li}_2(y_0) - \mathrm{Li}_2(y_0 z_0) + 2\,\mathrm{Li}_2(z_0) + \mathrm{Li}_2(w_0) + \frac{\pi^2}{2}\Big) \\ &\quad + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big((t_0+s_0)^2 + s_0^2 + u_0^2 + (u_0+v_0)^2 - t_0 - 2s_0 - 2u_0 - v_0 + \frac{2}{3}\big) \\ &= 0.817729... - \sqrt{-1}\cdot 1.50254... \,. \end{aligned} \tag{47}$$

Further, we put its real part to be ς_R,

$$\varsigma_R = \mathrm{Re}\,\varsigma = 0.817729... \,. \tag{48}$$

4.4 Verifying the assumption of the Poisson summation formula

In this section, we verify the assumption of the Poisson summation formula in Lemma 4.2, which is used in the proof of Theorem 1.1 for the 7_4 knot in Section 4.1.

By computer calculation, we can see that the maximal value of $\mathrm{Re}\,\hat{V} - \varsigma_R$ is about 0.09. Therefore, in the proof of Lemma 4.2, it is sufficient to decrease, say, $\mathrm{Re}\,\hat{V}(t + \delta\sqrt{-1},\, s,\, u,\, v) - 2\pi\delta$ by 0.09, by moving δ (though we do not use this value in the proof of the lemma).

We put

$$f(X, Y, Z, W) = \mathrm{Re}\,\hat{V}(t + X\sqrt{-1},\, s + Y\sqrt{-1},\, u + Z\sqrt{-1},\, v + W\sqrt{-1}) - \varsigma_R \,.$$

Then, we have that

$$\frac{\partial f}{\partial X} = \mathrm{Arg}\,(1 - x) - 2\pi\big(t + s - \frac{1}{2}\big), \tag{49}$$

$$\frac{\partial f}{\partial Y} = 2\,\mathrm{Arg}\,(1 - y) - \mathrm{Arg}\,(1 - yz) - 2\pi\big(t + 2s - 1\big), \tag{50}$$

$$\frac{\partial f}{\partial Z} = 2\,\mathrm{Arg}\,(1 - z) - \mathrm{Arg}\,(1 - yz) - 2\pi\big(2u + v - 1\big), \tag{51}$$

$$\frac{\partial f}{\partial W} = \mathrm{Arg}\,(1 - w) - 2\pi\big(u + v - \frac{1}{2}\big). \tag{52}$$

Lemma 4.2. *$V(t, s, u, v) - \varsigma_R$ satisfies the assumption of Proposition 2.2.*

Proof. Since $V(t,s,u,v)$ converges uniformly to $\hat{V}(t,s,u,v)$ on Δ', we show the proof for $\hat{V}(t,s,u,v)$ instead of $V(t,s,u,v)$. We show that $\partial\Delta'$ is null-homotopic in each of (13)–(20).

As for (13) and (14), similarly as in the proof of Lemma 3.3, it is sufficient to show that

$$-(2\pi-\varepsilon') \;<\; \frac{\partial f}{\partial X}(X,0,0,0) \;<\; 2\pi-\varepsilon' \tag{53}$$

for some $\varepsilon'>0$. Since $0.02\le t\le 0.45$,

$$-2\pi\big(\frac{1}{2}-t\big) \;<\; \operatorname{Arg}(1-x) \;<\; 0.$$

Hence, by (49),

$$-2\pi\cdot s \;<\; \frac{\partial f}{\partial X} \;<\; 2\pi\big(\frac{1}{2}-t-s\big).$$

Since $s\le 0.45$ and $t,s>0$,

$$-2\pi\cdot 0.45 \;<\; \frac{\partial f}{\partial X} \;<\; 2\pi\cdot 0.5.$$

Therefore, (53) is satisfied, as required.

As for (15) and (16), similarly as above, it is sufficient to show that

$$-(2\pi-\varepsilon') \;<\; \frac{\partial f}{\partial Y}(0,Y,0,0) \;<\; 2\pi-\varepsilon' \tag{54}$$

for some $\varepsilon'>0$. Since $0.1\le s\le 0.45$,

$$-2\pi\big(\frac{1}{2}-s\big) \;<\; \operatorname{Arg}(1-y) \;<\; 0.$$

Further, since $0.2\le s+u\le 0.9$,

$$\min\big\{-2\pi\big(\frac{1}{2}-s-u\big),\; 0\big\} \;\le\; \operatorname{Arg}(1-yz) \;\le\; \max\big\{0,\; 2\pi\big(s+u-\frac{1}{2}\big)\big\}.$$

Hence, by (50),

$$\begin{aligned}\frac{\partial f}{\partial Y} \;&>\; \min\big\{-2\pi(1-2s),\; -2\pi\big(u-s+\frac{1}{2}\big)\big\}\\ &\ge\; \min\big\{-2\pi\cdot 0.8,\; -2\pi\cdot 0.85\big\} \;=\; -2\pi\cdot 0.85,\\ \frac{\partial f}{\partial Y} \;&<\; \max\big\{2\pi\big(\frac{1}{2}-s-u\big),\; 0\big\} \;\le\; \max\big\{2\pi\cdot 0.3,\; 0\big\} \;=\; 2\pi\cdot 0.3,\end{aligned}$$

since $0.1\le s$ and $0.1\le u\le 0.45$. Therefore, (54) is satisfied, as required.

We obtain (17), (18), (19) and (20) from the above cases by the symmetry (46). $\square$

4.5 Verifying the assumption of the saddle point method

In this section, we verify the assumption of the saddle point method in Lemma 4.7. In order to show this lemma, we show Lemmas 4.3–4.6 in advance.

From the definition of f, we have that

$$\begin{aligned} f(X,Y,Z,W) \;&=\; \operatorname{Re} \frac{1}{2\pi\sqrt{-1}} \Big(\operatorname{Li}_2(x) + 2\operatorname{Li}_2(y) - \operatorname{Li}_2(yz) + 2\operatorname{Li}_2(z) + \operatorname{Li}_2(w) \Big) \\ &\quad - 2\pi\big(s+t-\frac{1}{2}\big)X - 2\pi(t+2s-1)Y - 2\pi(2u+v-1)Z - 2\pi\big(u+v-\frac{1}{2}\big)W, \end{aligned}$$

where $x = e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$, $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$, $z = e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})}$ and $w = e^{2\pi\sqrt{-1}\,(v+W\sqrt{-1})}$. Hence, since the contributions to f from X, W and (Y,Z) are independent, we consider each of the contributions independently.

Lemma 4.3. *Fixing Y, Z and W, we regard f as a function of X.*
(1) If $t+s \ge \frac{1}{2}$, then f is monotonically decreasing as a function of X.
(2) If $t+s < \frac{1}{2}$, then f has a unique minimal point as a function of X. In particular, this minimal point goes to ∞ as $t+s \to \frac{1}{2}-0$.

Proof. Since $x = e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$ and $0.02 \le t \le 0.45$,

$$-2\pi\big(\frac{1}{2}-t\big) \;<\; \operatorname{Arg}(1-x) \;<\; 0,$$

and $\operatorname{Arg}(1-x)$ is monotonically increasing as a function of X. Hence, by (49), $\frac{\partial f}{\partial X}$ is also monotonically increasing as a function of X. Further,

$$\begin{aligned} \frac{\partial f}{\partial X}\Big|_{X\to\infty} \;&=\; 2\pi\big(\frac{1}{2}-t-s\big), \\ \frac{\partial f}{\partial X}\Big|_{X\to-\infty} \;&=\; -2\pi\cdot s \;<\; 0. \end{aligned}$$

If $t+s \ge \frac{1}{2}$, then $\frac{\partial f}{\partial X}$ is always negative, and (1) holds.

If $t+s < \frac{1}{2}$, then there is a unique zero of $\frac{\partial f}{\partial X}$, which gives a unique minimal point of f, and (2) holds. □

Lemma 4.4. *Fixing X, Y and Z, we regard f as a function of W.*
(1) If $u+v \ge \frac{1}{2}$, then f is monotonically decreasing as a function of W.
(2) If $u+v < \frac{1}{2}$, then f has a unique minimal point as a function of W. In particular, this minimal point goes to ∞ as $u+v \to \frac{1}{2}-0$.

Proof. The lemma is obtained from Lemma 4.3 by the symmetry (46). □

In order to calculate the contribution to f from (Y,Z), we put

$$\hat f(Y,Z) \;=\; \operatorname{Re} \frac{1}{2\pi\sqrt{-1}} \Big(2\operatorname{Li}_2(y) - \operatorname{Li}_2(yz) + 2\operatorname{Li}_2(z) \Big) - 2\pi(t+2s-1)Y - 2\pi(2u+v-1)Z,$$

where $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$ and $z = e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})}$. By (50) and (51),

$$\frac{\partial\hat f}{\partial Y} = 2\,\mathrm{Arg}\,(1-y) - \mathrm{Arg}\,(1-yz) - 2\pi\big(t+2s-1\big), \tag{55}$$

$$\frac{\partial\hat f}{\partial Z} = 2\,\mathrm{Arg}\,(1-z) - \mathrm{Arg}\,(1-yz) - 2\pi\big(2u+v-1\big). \tag{56}$$

We consider the behavior of $\hat f$ in the following two lemmas, depending on the sign of $s+u-\frac{1}{2}$.

Lemma 4.5. *In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$ satisfying that $s+u \ge \frac{1}{2}$, $t+s<\frac{1}{2}$ and $u+v<\frac{1}{2}$, we consider the flow from $(Y,Z)=(0,0)$ determined by the vector field $(-\frac{\partial\hat f}{\partial Y}, -\frac{\partial\hat f}{\partial Z})$. Then, $\hat f$ has a unique minimal point, and the flow goes there.*

Proof. Since $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$, $\frac{d}{dY} = -2\pi y\frac{d}{dy}$. Hence,

$$\begin{aligned}\frac{\partial^2\hat f}{\partial Y^2} &= -2\pi\, y\,\frac{\partial}{\partial y}\Big(\mathrm{Im}\,\big(2\,\log(1-y) - \log(1-yz)\big)\Big)\\ &= 2\pi\,\mathrm{Im}\,\Big(\frac{2y}{1-y} - \frac{yz}{1-yz}\Big) = 2\pi\,\mathrm{Im}\,\Big(\frac{2}{1-y} - \frac{1}{1-yz}\Big).\end{aligned}$$

Therefore, by calculating other entries similarly, the Hesse matrix of $\hat f$ is presented by

$$2\pi\begin{pmatrix} 2a_1+b & b \\ b & 2a_2+b\end{pmatrix} \tag{57}$$

where we put

$$a_1 = \mathrm{Im}\,\frac{1}{1-y}, \quad a_2 = \mathrm{Im}\,\frac{1}{1-z}, \quad b = \mathrm{Im}\,\frac{-1}{1-yz},$$

noting that there numbers are positive. Since we can verify that the trace and the determinant of this matrix are positive, the Hesse matrix of $\hat f$ is positive definite, and $\hat f$ is a convex function.

We consider the behavior of $\hat f$ at infinity. By (11), $\frac{1}{2\pi}\hat f$ is approximated by the following function,

$$\begin{aligned} F(Y,Z) &= 2\left(\begin{cases}0 & \text{if } Y\ge 0\\ \big(s-\frac{1}{2}\big)Y & \text{if } Y<0\end{cases}\right) - \left(\begin{cases}0 & \text{if } Y+Z\ge 0\\ \big(s+u-\frac{1}{2}\big)(Y+Z) & \text{if } Y+Z<0\end{cases}\right)\\ &\quad + 2\left(\begin{cases}0 & \text{if } Z\ge 0\\ \big(u-\frac{1}{2}\big)Z & \text{if } Z<0\end{cases}\right) - (t+2s-1)Y - (2u+v-1)Z\\ &= \left(\begin{cases}(1-t-2s)Y & \text{if } Y\ge 0\\ -tY & \text{if } Y<0\end{cases}\right) - \left(\begin{cases}0 & \text{if } Y+Z\ge 0\\ \big(s+u-\frac{1}{2}\big)(Y+Z) & \text{if } Y+Z<0\end{cases}\right)\\ &\quad + \left(\begin{cases}(1-2u-v)Z & \text{if } Z\ge 0\\ -vZ & \text{if } Z<0\end{cases}\right).\end{aligned}$$

Since $t > 0$, $t + 2s < \frac{1}{2} + 0.45 = 0.95$, $s + u \geq \frac{1}{2}$, $2u + v < \frac{1}{2} + 0.45 = 0.95$ and $v > 0$, $F(Y, Z) \to \infty$ as $Y^2 + Z^2 \to \infty$. Since $\hat{f}$ is convex, $\hat{f}$ has a unique minimal point, and the flow goes there, as required. □

Lemma 4.6. *In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t, s, u, v) \in \Delta'$ satisfying that $s + u < \frac{1}{2}$, $t + s < \frac{1}{2}$ and $u + v < \frac{1}{2}$, we consider the flow from $(Y, Z) = (0, 0)$ determined by the vector field $(-\frac{\partial \hat{f}}{\partial Y}, -\frac{\partial \hat{f}}{\partial Z})$.*
(1) *If $t + s + u > \frac{1}{2}$ and $s + u + v > \frac{1}{2}$, then $\hat{f}$ has a unique minimal point, and the flow goes there.*
(2) *Otherwise, the flow goes to infinity.*

Proof. When $Y > 0$, we show that the flow goes in a direction decreasing Y, as follows. Since $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$, $0.1 \leq s \leq 0.45$ and $Y > 0$,

$$-\pi\big(\frac{1}{2} - s\big) \;<\; \mathrm{Arg}\,(1 - y).$$

Further, since $s + u < \frac{1}{2}$,

$$\mathrm{Arg}\,(1 - yz) < 0.$$

Hence, by (55),

$$\frac{\partial \hat{f}}{\partial Y} \;>\; -2\pi\big(\frac{1}{2} - s\big) - 2\pi(t + 2s - 1) \;=\; 2\pi\big(\frac{1}{2} - t - s\big) \;>\; 0.$$

Therefore, the flow goes in a direction decreasing Y.

When $Z > 0$, it follows from the above case by the symmetry (46) that the flow goes in a direction decreasing Z.

Hence, we can assume that the flow is in the domain that $Y \leq 0$ and $Z \leq 0$. In this domain, by (11), $\frac{1}{2\pi}\hat{f}$ is approximated by

$$\begin{aligned} &2\big(s - \frac{1}{2}\big)Y - \big(s + u - \frac{1}{2}\big)(Y + Z) + 2\big(u - \frac{1}{2}\big)Z - (t + 2s - 1)Y - (2u + v - 1)Z \\ &\;=\; \big(\frac{1}{2} - t - s - u\big)Y + \big(\frac{1}{2} - s - u - v\big)Z. \end{aligned}$$

Hence, since we will show below that $\hat{f}$ is convex in this domain, we obtain the lemma similarly as the proof of Lemma 4.5.

Therefore, it is sufficient to show that the Hesse matrix of $\hat{f}$ is positive definite when $Y \leq 0$ and $Z \leq 0$. Similarly as the proof of Lemma 4.5, the Hesse matrix of $\hat{f}$ is presented by

$$2\pi \begin{pmatrix} 2a_1 - b' & -b' \\ -b' & 2a_2 - b' \end{pmatrix} \tag{58}$$

where we put

$$a_1 \;=\; \mathrm{Im}\,\frac{1}{1 - y}\,, \quad a_2 \;=\; \mathrm{Im}\,\frac{1}{1 - z}\,, \quad b' \;=\; \mathrm{Im}\,\frac{1}{1 - yz}\,,$$

noting that there numbers are positive. It is sufficient to show that

$$\big(\text{the trace of (58)}\big) \;=\; 4\pi(a_1+a_2-b') \;>\; 0, \tag{59}$$

$$\begin{aligned}\big(\text{the determinant of (58)}\big) \;&=\; 4\pi^2\big((2a_1-b')(2a_2-b')-b'^2\big)\\ &=\; 8\pi^2 a_1 a_2 b'\big(\frac{2}{b'}-\frac{1}{a_1}-\frac{1}{a_2}\big) \;>\; 0. \end{aligned}\tag{60}$$

We show that (60) $\Rightarrow$ (59), as follows. Suppose that (60) holds. Then, $\frac{1}{b'} > \frac{1}{2}\big(\frac{1}{a_1}+\frac{1}{a_2}\big)$. Since a_1, a_2 and b' are positive, $\frac{1}{b'} > \frac{1}{a_1}$ or $\frac{1}{b'} > \frac{1}{a_2}$. Hence, $b' < a_1$ or $b' < a_2$. Therefore, (59) holds.

We show (60), as follows. Since $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$, a_1 is presented by

$$\begin{aligned} a_1 \;&=\; \operatorname{Im}\frac{1}{1-y} \;=\; \operatorname{Im}\frac{1-\overline{y}}{\big|1-y\big|^2} \;=\; \frac{e^{-2\pi Y}\sin 2\pi s}{(1-e^{2\pi\sqrt{-1}\,s}e^{-2\pi Y})(1-e^{-2\pi\sqrt{-1}\,s}e^{-2\pi Y})}\\ &=\; \frac{\sin 2\pi s}{e^{2\pi Y}+e^{-2\pi Y}-2\,\cos 2\pi s}\,. \end{aligned}$$

Hence,

$$\frac{1}{a_1} \;=\; \frac{e^{2\pi Y}+e^{-2\pi Y}-2\,\cos 2\pi s}{\sin 2\pi s}\,.$$

Similarly, we have that

$$\frac{1}{a_2} \;=\; \frac{e^{2\pi Z}+e^{-2\pi Z}-2\,\cos 2\pi u}{\sin 2\pi u}\,, \qquad \frac{1}{b'} \;=\; \frac{e^{2\pi(Y+Z)}+e^{-2\pi(Y+Z)}-2\,\cos 2\pi(s+u)}{\sin 2\pi(s+u)}\,.$$

Therefore, the differential of $\frac{2}{b'}-\frac{1}{a_1}-\frac{1}{a_2}$ with respect to Y is given by

$$\frac{1}{2\pi}\cdot\frac{\partial}{\partial Y}\big(\frac{2}{b'}-\frac{1}{a_1}-\frac{1}{a_2}\big) \;=\; 2\cdot\frac{e^{2\pi(Y+Z)}-e^{-2\pi(Y+Z)}}{\sin 2\pi(s+u)} - \frac{e^{2\pi Y}-e^{-2\pi Y}}{\sin 2\pi s}\,.$$

Since $0.1 \le s < \frac{1}{2}-u \le 0.4$, $\sin 2\pi s \ge \sin(2\pi\cdot 0.1) = 0.587785...$. Hence, $2/\sin 2\pi(s+u) \ge 2 > 1/\sin 2\pi s$. Further, since $Y \le 0$ and $Z \le 0$, $e^{2\pi(Y+Z)} - e^{-2\pi(Y+Z)} \le e^{2\pi Y} - e^{-2\pi Y} \le 0$. Hence, the above formula is non-positive. Therefore, it is sufficient to show (60) when $Y=0$. Further, by the symmetry (46), it is sufficient to show (60) when $Z=0$. When $Y=Z=0$,

$$\frac{1}{2}\Big(\frac{2}{b'}-\frac{1}{a_1}-\frac{1}{a_2}\Big) \;=\; 2\cdot\frac{1-\cos 2\pi(s+u)}{\sin 2\pi(s+u)} - \frac{1-\cos 2\pi s}{\sin 2\pi s} - \frac{1-\cos 2\pi u}{\sin 2\pi u}\,.$$

Further, since $\frac{1-\cos 2\pi\alpha}{\sin 2\pi\alpha} = \frac{2\,\sin^2\pi\alpha}{2\,\sin\pi\alpha\,\cos\pi\alpha} = \tan\pi\alpha$,

$$\frac{1}{2}\Big(\frac{2}{b'}-\frac{1}{a_1}-\frac{1}{a_2}\Big) \;=\; 2\,\tan\pi(s+u) - \tan\pi s - \tan\pi u \;>\; 0.$$

Hence, we obtain (60), as required. □

Lemma 4.7. *When we apply Proposition 2.4 to* (44)*, the assumption of Proposition 2.4 holds.*

Proof. We show that there exists a homotopy Δ'_δ $(0 \le \delta \le 1)$ between $\Delta'_0 = \Delta'$ and Δ'_1 such that

$$(t_c, s_c, u_c, v_c) \in \Delta'_1, \tag{61}$$

$$\Delta'_1 - \{(t_c, s_c, u_c, v_c)\} \;\subset\; \big\{(t,s,u,v) \in \mathbb{C}^4 \;\big|\; \operatorname{Re} \hat{V}(t,s,u,v) < \varsigma_R\big\}, \tag{62}$$

$$\partial\Delta'_\delta \;\subset\; \big\{(t,s,u,v) \in \mathbb{C}^4 \;\big|\; \operatorname{Re} \hat{V}(t,s,u,v) < \varsigma_R\big\}. \tag{63}$$

In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$. We put a neighborhood U of (t_0, s_0, u_0, v_0) by

$$U \;=\; \Big\{\; (t,s,u,v) \in \Delta' \;\Big|\; \frac{1}{2} - u \;<\; t+s \;<\; \frac{1}{2},\;\; \frac{1}{2} - s \;<\; u+v \;<\; \frac{1}{2} \;\Big\}.$$

Then, by Lemmas 4.3, 4.4, 4.5 and 4.6, the following (1) and (2) holds.

(1) If $(t,s,u,v) \in U$, then f has a unique minimal point, and the flow goes there.

(2) If $(t,s,u,v) \notin U$, then the flow goes to infinity.

We put $\mathbf{g}(t,s,u,v)$ to be the minimal point of (1). In particular, $|\,\mathbf{g}(t,s,u,v)\,| \to \infty$, as (t,s,u,v) goes to ∂U. Further, for a sufficiently large $R > 0$, we stop the flow when $|\,\mathbf{g}(t,s,u,v)\,| = R$. We put $\hat{\mathbf{g}}(t,s,u,v)$ to be the destination of this revised flow. In particular, when $|\,\mathbf{g}(t,s,u,v)\,| < R$, $\hat{\mathbf{g}}(t,s,u,v) = \mathbf{g}(t,s,u,v)$. We define the ending of the homotopy to be the set of the destinations of these revised flows,

$$\Delta'_1 \;=\; \big\{(t,s,u,v) + \hat{\mathbf{g}}(t,s,u,v)\sqrt{-1} \;\big|\; (t,s,u,v) \in \Delta'\big\}.$$

Further, we define the internal part of the homotopy by setting it along the flows.

We can show (61), (62) and (62) by using Lemma 4.1 in a similar way as the proof of Lemma 3.9. □

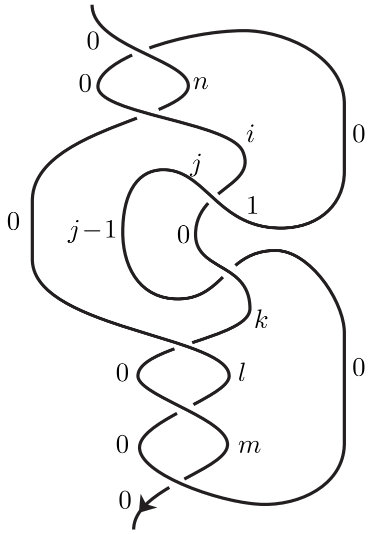
5 The 7_5 knot

In this section, we show Theorem 1.1 for the 7_5 knot. We give a proof of the theorem in Section 5.1, using lemmas shown in Sections 5.2–5.5.

5.1 Proof of Theorem 1.1 for the 7_5 knot

In this section, we show a proof of Theorem 1.1 for the 7_5 knot.

The 7_5 *knot* is the closure of the following tangle.



As shown in [32], we can put the labelings of edges adjacent to the unbounded regions as shown above. Hence, from the definition of the Kashaev invariant, the Kashaev invariant of the 7_5 knot is presented by

$$\begin{aligned}
\langle\, 7_5 \,\rangle_N \;&=\; \sum q^{1/2} \times \frac{N\, q^{\frac{1}{2}-n}}{(q)_{N-n}(\overline{q})_{n-1}} \times \frac{N\, q^{\frac{1}{2}+n-i}}{(\overline{q})_{N-n}(q)_{n-i}(\overline{q})_{i-1}} \times \frac{N\, q^{\frac{1}{2}+i-1}}{(\overline{q})_{j-i}(q)_{i-1}(q)_{N-j}} \\
&\quad \times \frac{N\, q^{\frac{1}{2}-k}}{(q)_{N-k}(\overline{q})_{k-j}(q)_{j-1}} \times \frac{N\, q^{\frac{1}{2}+k-l}}{(\overline{q})_{N-k}(q)_{k-l}(\overline{q})_{l-1}} \times \frac{N\, q^{\frac{1}{2}+l-m}}{(\overline{q})_{N-l}(q)_{l-m}(\overline{q})_{m-1}} \times \frac{N\, q^{\frac{1}{2}+m-1}}{(\overline{q})_{N-m}(q)_{m-1}} \\
&= \sum_{\substack{0<i\le j\le k \\ 0<l\le k\le N}} \frac{N^5\, q^2}{(q)_{i-1}(\overline{q})_{i-1}(\overline{q})_{j-i}(q)_{j-1}(q)_{N-j}(\overline{q})_{k-j}(q)_{N-k}(\overline{q})_{N-k}(q)_{k-l}(\overline{q})_{l-1}(\overline{q})_{N-l}} \\
&= \sum_{\substack{0\le i\le j\le k \\ 0\le l\le k< N}} \frac{N^5\, q^2}{(q)_{i}(\overline{q})_{i}(\overline{q})_{j-i}(q)_{j}(q)_{N-j-1}(\overline{q})_{k-j}(q)_{N-k-1}(\overline{q})_{N-k-1}(q)_{k-l}(\overline{q})_{l}(\overline{q})_{N-l-1}} \\
&= \sum_{\substack{0\le i\le j\le k \\ 0\le l\le k< N}} \frac{N^5\, q^2}{(q)_{i}(\overline{q})_{i}(\overline{q})_{j-i}(q)_{j}(q)_{N-j-1}(\overline{q})_{k-j}(q)_{N-k-1}(\overline{q})_{N-k-1}(q)_{l}(\overline{q})_{k-l}(\overline{q})_{N-k+l-1}}\,,
\end{aligned}$$

where we obtain the third equality by replacing i, j, k, l with $i+1, j+1, k+1, l+1$ respectively, and obtain the last equality by replacing l with $k-l$.

Proof of Theorem 1.1 for the 7_5 *knot.* By (5), the above presentation of $\langle\, 7_5 \,\rangle_N$ is rewritten

$$\langle\, 7_5 \,\rangle_N \;=\; N^5 q^2 \sum_{\substack{0\le i\le j\le k \\ 0\le l\le k< N}} \exp\Big(N\, \tilde{V}\big(\frac{2i+1}{2N},\ \frac{2j+1}{2N},\ \frac{2k+1}{2N},\ \frac{2l+1}{2N}\big)\Big),$$

where we put

$$\begin{aligned}\tilde{V}(t,s,u,v) &= \frac{1}{N}\Big(\varphi(t)-\varphi(1-t)-\varphi\big(1-s+t-\frac{1}{2N}\big)+\varphi(s)+\varphi(1-s)\\&\quad -\varphi\big(1-u+s-\frac{1}{2N}\big)-\varphi(u)+\varphi(1-u)+\varphi(v)-\varphi\big(u-v+\frac{1}{2N}\big)\\&\quad -\varphi\big(1-u+v-\frac{1}{2N}\big)-5\,\varphi\big(\frac{1}{2N}\big)+6\,\varphi\big(1-\frac{1}{2N}\big)\Big)\\&= \frac{1}{N}\Big(2\,\varphi(t)-\varphi\big(1-s+t-\frac{1}{2N}\big)-\varphi\big(1-u+s-\frac{1}{2N}\big)-2\,\varphi(u)+\varphi(v)\Big)\\&\quad +\frac{1}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6}-\frac{11}{2N}\log N+\frac{\pi\sqrt{-1}}{4N}-\frac{\pi\sqrt{-1}}{12N^2}\\&\quad +2\pi\sqrt{-1}\cdot\frac{1}{2}\Big(t^2-s^2-u^2+\big(u-v+\frac{1}{2N}\big)^2-t+s+v-\frac{1}{2N}\Big).\end{aligned}$$

Here, we obtain the last equality by (9) and (10). Hence, by putting

$$V(t,s,u,v) \;=\; \tilde{V}(t,s,u,v)+\frac{11}{2N}\log N,$$

the presentation of $\langle\, 7_5\,\rangle_{N}$ is rewritten

$$\langle\, 7_5\,\rangle_{N} \;=\; N^{-1/2}q^2 \sum_{\substack{i,j,k,l\,\in\mathbb{Z}\\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})\in\Delta}} \exp\Big(N\,V\big(\frac{2i+1}{2N},\;\frac{2j+1}{2N},\;\frac{2k+1}{2N},\;\frac{2l+1}{2N}\big)\Big),$$

where the range of $(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})$ of the sum is given by the following domain,

$$\Delta \;=\; \big\{(t,s,u,v)\in\mathbb{R}^4 \;\big|\; 0\le t\le s\le u,\quad 0\le v\le u\le 1\big\}.$$

By Proposition 2.1, as $N\to\infty$, $V(t,s,u,v)$ converges to the following $\hat{V}(t,s,u,v)$ in the interior of Δ,

$$\begin{aligned}\hat{V}(t,s,u,v) &= \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t})-\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(t-s)})-\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s-u)})\\&\qquad -2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,u})+\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,v})+\frac{\pi^2}{6}\Big)\\&\quad +2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t^2-s^2-u^2+(u-v)^2-t+s+v\big).\end{aligned}$$

By concrete calculation, we can check that the boundary of Δ is included in the domain

$$\big\{(t,s,u,v)\in\Delta \;\big|\; \mathrm{Re}\,\hat{V}(t,s,u,v)<\varsigma_{R}-\varepsilon\big\} \tag{64}$$

for some sufficiently small $\varepsilon>0$, where we put $\varsigma_{R}=1.02552...$ as in (69); we will know later that this value is equal to the real part of the critical value of $\hat{V}$ at the critical point of Lemma 5.1. Hence, similarly as in Section 3.1, we choose a new domain Δ', which satisfies that $\Delta-\Delta'\subset(64)$, as

$$\Delta' \;=\; \left\{(t,s,u,v)\in\Delta \;\middle|\; \begin{array}{l}0.1\le t\le 0.33,\quad 0.67\le u\le 0.9,\quad 0.05\le v\le 0.3\\ 0.15\le s-t\le 0.45,\quad 0.15\le u-s\le 0.45\end{array}\right\}, \tag{65}$$

where we calculate the concrete values of the bounds of these inequalities in Section 5.2. Hence, since $\Delta - \Delta' \subset$ (64), we obtain the second equality of the following formula,

$$\begin{aligned}\langle\, 7_5 \,\rangle_{N} \;&=\; e^{N\varsigma} N^{-1/2} q^2 \sum_{\substack{i,j,k,l\,\in\,\mathbb{Z} \\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})\in\Delta}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N}\big) - N\varsigma\Big) \\ &=\; e^{N\varsigma}\Bigg(N^{-1/2} q^2 \sum_{\substack{i,j,k,l\,\in\,\mathbb{Z} \\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})\in\Delta'}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N}\big) - N\varsigma\Big) \;+ O(e^{-N\varepsilon})\Bigg),\end{aligned}$$

for some $\varepsilon > 0$.

Further, by Proposition 2.2 (Poisson summation formula), the above sum is presented by

$$\langle\, 7_5 \,\rangle_{N} \;=\; e^{N\varsigma}\Bigg(N^{7/2} q^2 \int_{\Delta'} \exp\big(N\cdot V(t,s,u,v) - N\varsigma\big)\, dt\, ds\, du\, dv \;\; + O(e^{-N\varepsilon})\Bigg), \tag{66}$$

noting that we verify the assumption of Proposition 2.2 in Lemma 5.2. Furthermore, by Proposition 2.4 (saddle point method), there exist some κ_i''s such that

$$\langle\, 7_5 \,\rangle_{N} \;=\; N^{7/2} \exp\big(N\cdot V(t_c,s_c,u_c,v_c)\big)\cdot \frac{(2\pi)^2}{N^2}\big(\det(-H)\big)^{-1/2}\Big(1+\sum_{i=1}^{d}\kappa_i'\hbar^i + O(\hbar^{d+1})\Big),$$

for any $d > 0$, noting that we verify the assumption of Proposition 2.4 in Lemma 5.5. Here, (t_c, s_c, u_c, v_c) is the critical point of V which corresponds to the critical point (t_0, s_0, u_0, v_0) of $\hat{V}$ of Lemma 5.1, where $\hat{V}$ is the limit of V at $N \to \infty$ whose concrete presentation is given in Section 5.2, and H is the Hesse matrix of V at (t_c, s_c, u_c, v_c).

We calculate the right-hand side of the above formula. Similarly as in Section 3.1, we have that

$$V(t_0,s_0,u_0,v_0) \;=\; \varsigma + O(\hbar).$$

Therefore, there exist some κ_i's such that

$$\langle\, 7_5 \,\rangle_{N} \;=\; e^{N\varsigma} N^{3/2}\,\omega\cdot\Big(1+\sum_{i=1}^{d}\kappa_i\hbar^i + O(\hbar^{d+1})\Big),$$

for any $d > 0$. Hence, we obtain the theorem for the 7_5 knot. □

5.2 Estimate of the range of Δ'

In this section, we calculate the concrete values of the bounds of the inequalities in (65) so that they satisfy that $\Delta - \Delta' \subset$ (64).

Putting Λ as in Section 2.2, we have that

$$\operatorname{Re}\hat{V}(t,s,u,v) \;=\; 2\,\Lambda(t) + \Lambda(s-t) + \Lambda(u-s) - 2\,\Lambda(u) + \Lambda(v).$$

We consider the domain

$$\big\{(t,s,u,v)\in\Delta \;\big|\; 2\,\Lambda(t)+\Lambda(s-t)+\Lambda(u-s)-2\,\Lambda(u)+\Lambda(v)\;\geq\;\varsigma_{R}\big\}, \tag{67}$$

where we put $\varsigma_{R} = 1.02552...$ as in (69). The aim of this section is to show that this domain is included in the interior of the domain Δ' of (65). For this purpose, we show the estimates of the defining inequalities of (65) for (t,s,u,v) in (67).

Since $\Lambda(v) \leq \Lambda\big(\frac{1}{6}\big)$,

$$2\,\Lambda(t)+\Lambda(s-t)+\Lambda(u-s)-2\,\Lambda(u)\;\geq\;\varsigma_{R}-\Lambda\big(\frac{1}{6}\big).$$

We calculate the minimal value $t_{\min}$ and the maximal value $t_{\max}$ of t. They are solutions of the system of the following equations,

$$\begin{cases} 2\,\Lambda(t)+\Lambda(s-t)+\Lambda(u-s)-2\,\Lambda(u) \;=\; \varsigma_{R}-\Lambda\big(\frac{1}{6}\big), \\ \dfrac{\partial}{\partial s}\big(2\,\Lambda(t)+\Lambda(s-t)+\Lambda(u-s)-2\,\Lambda(u)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(t)+\Lambda(s-t)+\Lambda(u-s)-2\,\Lambda(u)\big) \;=\; 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of t (see Appendix A). By calculating a solution of these equations by Newton's method from $(t,s,u)=(0.15,0.5,0.8)$, we obtain $t_{\min}=0.147538...$, and from $(t,s,u)=(0.3,0.5,0.8)$, we obtain $t_{\max}=0.291380...$. Therefore, we obtain an estimate of t in Δ' as

$$0.1\;\leq\;t\;\leq\;0.33.$$

We obtain the estimate of u in Δ' from the above estimate of t by replacing (t,s,u) with $(1-u,\,1-s,\,1-t)$.

We calculate the minimal value $(s-t)_{\min}$ and the maximal value $(s-t)_{\max}$ of $s-t$. Putting $w=s-t$, its minimal and maximal values are solutions of the system of the following equations,

$$\begin{cases} 2\,\Lambda(s-w)+\Lambda(w)+\Lambda(u-s)-2\,\Lambda(u) \;=\; \varsigma_{R}-\Lambda\big(\frac{1}{6}\big), \\ \dfrac{\partial}{\partial s}\big(2\,\Lambda(s-w)+\Lambda(w)+\Lambda(u-s)-2\,\Lambda(u)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(s-w)+\Lambda(w)+\Lambda(u-s)-2\,\Lambda(u)\big) \;=\; 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of $s-t$ (see Appendix A). By calculating a solution of these equations by Newton's method from $(w,s,u)=(0.2,0.4,0.75)$, we obtain $(s-t)_{\min}=0.184286...$, and from $(w,s,u)=(0.4,0.6,0.8)$, we obtain $(s-t)_{\max}=0.397155...$. Therefore, we obtain an estimate of $s-t$ in Δ' as

$$0.15\;\leq\;s-t\;\leq\;0.45.$$

We obtain the estimate of $u-s$ in Δ' from the above estimate of $s-t$ by replacing (t,s,u) with $(1-u,\,1-s,\,1-t)$.

We calculate the minimal value $v_{\min}$ and the maximal value $v_{\max}$ of v. They are solutions of the system of the following equations,

$$\begin{cases} 2\,\Lambda(t) + \Lambda(s-t) + \Lambda(u-s) - 2\,\Lambda(u) + \Lambda(v) \;=\; \varsigma_{R}, \\ \dfrac{\partial}{\partial t}\big(2\,\Lambda(t) + \Lambda(s-t) + \Lambda(u-s) - 2\,\Lambda(u) + \Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial s}\big(2\,\Lambda(t) + \Lambda(s-t) + \Lambda(u-s) - 2\,\Lambda(u) + \Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(t) + \Lambda(s-t) + \Lambda(u-s) - 2\,\Lambda(u) + \Lambda(v)\big) \;=\; 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of v (see Appendix A). By calculating a solution of these equations by Newton's method from $(t,s,u,v) = (0.2, 0.5, 0.75, 0.1)$, we obtain $v_{\min} = 0.0846896...$, and from $(t,s,u,v) = (0.2, 0.5, 0.8, 0.3)$, we obtain $v_{\max} = 0.26949...$. Therefore, we obtain an estimate of v in Δ' as

$$0.05 \;\le\; v \;\le\; 0.3.$$

5.3 Calculation of the critical value

In this section, we calculate the concrete value of a critical point of $\hat{V}$.

The differentials of $\hat{V}$ are presented by

$$\begin{aligned} \frac{\partial}{\partial t}\hat{V}(t,s,u,v) &= -2\,\log(1-x) + \log\big(1-\frac{x}{y}\big) + 2\pi\sqrt{-1}\,\big(t-\frac{1}{2}\big), \\ \frac{\partial}{\partial s}\hat{V}(t,s,u,v) &= -\log\big(1-\frac{x}{y}\big) + \log\big(1-\frac{y}{z}\big) - 2\pi\sqrt{-1}\,\big(s-\frac{1}{2}\big), \\ \frac{\partial}{\partial u}\hat{V}(t,s,u,v) &= 2\,\log(1-z) - \log\big(1-\frac{y}{z}\big) - 2\pi\sqrt{-1}\,v, \\ \frac{\partial}{\partial v}\hat{V}(t,s,u,v) &= -\log(1-w) + 2\pi\sqrt{-1}\,\big(v-u+\frac{1}{2}\big), \end{aligned}$$

where $x = e^{2\pi\sqrt{-1}\,t}$, $y = e^{2\pi\sqrt{-1}\,s}$, $z = e^{2\pi\sqrt{-1}\,u}$ and $w = e^{2\pi\sqrt{-1}\,v}$.

Lemma 5.1. *$\hat{V}$ has a unique critical point (t_0, s_0, u_0, v_0) in $P^{-1}(\Delta')$, where $P : \mathbb{C}^4 \to \mathbb{R}^4$ is the projection to the real parts of the entries.*

Proof. Any critical point of $\hat{V}$ is given by a solution of $\frac{\partial}{\partial t}\hat{V} = \frac{\partial}{\partial s}\hat{V} = \frac{\partial}{\partial u}\hat{V} = \frac{\partial}{\partial v}\hat{V} = 0$, and these equations are rewritten,

$$(1-x)^2 \;=\; -x\big(1-\frac{x}{y}\big), \quad 1-\frac{y}{z} \;=\; -y\big(1-\frac{x}{y}\big), \quad (1-z)^2 \;=\; w\big(1-\frac{y}{z}\big), \quad 1-w \;=\; -\frac{w}{z}.$$

From the first formula, we have that $y = x^2/(x^2-x+1)$. By substituting this into the second formula, we have that $z = -x^2/(x^3-3x^2+2x-1)$. Further, from the fourth formula, we have that $w = z/(z-1)$. By substituting these into the third formula, we have that

$$x^8 + x^7 - 12x^6 + 25x^5 - 31x^4 + 25x^3 - 14x^2 + 5x - 1 \;=\; 0.$$

Its solutions are

$$x = -4.85443... , \quad 1.57227... , \quad 0.18596... \pm \sqrt{-1}\cdot 0.689115... ,$$
$$0.39462... \pm \sqrt{-1}\cdot 0.631293... , \quad 0.560504... \pm \sqrt{-1}\cdot 0.387082... .$$

Among these, the third solution gives a solution in Δ', from which we have that

$$\begin{aligned} x_0 &= 0.18596... + \sqrt{-1}\cdot 0.689115... , & t_0 &= 0.208051... + \sqrt{-1}\cdot 0.0536673... , \\ y_0 &= -0.842429... - \sqrt{-1}\cdot 0.289836... , & s_0 &= 0.552738... + \sqrt{-1}\cdot 0.0183872... , \\ z_0 &= 0.320754... - \sqrt{-1}\cdot 0.851242... , & u_0 &= 0.807352... + \sqrt{-1}\cdot 0.0150681... , \\ w_0 &= 0.427274... + \sqrt{-1}\cdot 0.717749... , & v_0 &= 0.164541... + \sqrt{-1}\cdot 0.0286421... , \end{aligned}$$

where $x_0 = e^{2\pi\sqrt{-1}\,t_0}$, $y_0 = e^{2\pi\sqrt{-1}\,s_0}$, $z_0 = e^{2\pi\sqrt{-1}\,u_0}$ and $w_0 = e^{2\pi\sqrt{-1}\,v_0}$. These give a unique critical point in $P^{-1}(\Delta')$. $\square$

The critical value of $\hat{V}$ at the critical point of Lemma 5.1 is presented by

$$\begin{aligned} \varsigma &= \hat{V}(t_0, s_0, u_0, v_0) \\ &= \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(x_0) - \mathrm{Li}_2\big(\frac{x_0}{y_0}\big) - \mathrm{Li}_2\big(\frac{y_0}{z_0}\big) - 2\,\mathrm{Li}_2(z_0) + \mathrm{Li}_2(w_0) + \frac{\pi^2}{6}\Big) \\ &\qquad + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t_0^2 - s_0^2 - u_0^2 + (u_0 - v_0)^2 - t_0 + s_0 + v_0\big) \qquad (68) \\ &= 1.02552... - \sqrt{-1}\cdot 0.378738... . \end{aligned}$$

Further, we put its real part to be ς_R,

$$\varsigma_R = \mathrm{Re}\,\varsigma = 1.02552... . \qquad (69)$$

5.4 Verifying the assumption of the Poisson summation formula

In this section, we verify the assumption of the Poisson summation formula in Lemma 5.2, which is used in the proof of Theorem 1.1 for the 7_5 knot in Section 5.1.

By computer calculation, we can see that the maximal value of $\mathrm{Re}\,\hat{V} - \varsigma_R$ is about 0.02. Therefore, in the proof of Lemma 5.2, it is sufficient to decrease, say, $\mathrm{Re}\,\hat{V}(t + \delta\sqrt{-1},\, s,\, u,\, v) - 2\pi\delta$ by 0.02, by moving δ (though we do not use this value in the proof of the lemma).

We put

$$f(X,Y,Z,W) = \mathrm{Re}\,\hat{V}(t + X\sqrt{-1},\, s + Y\sqrt{-1},\, u + Z\sqrt{-1},\, v + W\sqrt{-1}) - \varsigma_R .$$

Then, we have that

$$\frac{\partial f}{\partial X} = 2\,\mathrm{Arg}\,(1 - x) - \mathrm{Arg}\,\big(1 - \frac{x}{y}\big) - 2\pi\big(t - \frac{1}{2}\big), \qquad (70)$$

$$\frac{\partial f}{\partial Y} = \mathrm{Arg}\,\big(1 - \frac{x}{y}\big) - \mathrm{Arg}\,\big(1 - \frac{y}{z}\big) + 2\pi\big(s - \frac{1}{2}\big), \qquad (71)$$

$$\frac{\partial f}{\partial Z} = -2\,\mathrm{Arg}\,(1-z) + \mathrm{Arg}\,\big(1-\frac{y}{z}\big) + 2\pi v, \tag{72}$$

$$\frac{\partial f}{\partial W} = \mathrm{Arg}\,(1-w) + 2\pi\big(u-v-\frac{1}{2}\big). \tag{73}$$

Lemma 5.2. *$V(t,s,u,v) - \varsigma_R$ satisfies the assumption of Proposition 2.2.*

Proof. Since $V(t,s,u,v)$ converges uniformly to $\hat{V}(t,s,u,v)$ on Δ', we show the proof for $\hat{V}(t,s,u,v)$ instead of $V(t,s,u,v)$. We show that $\partial\Delta'$ is null-homotopic in each of (13)–(20).

As for (13) and (14), similarly as in the proof of Lemma 3.3, it is sufficient to show that

$$-(2\pi-\varepsilon') \;<\; \frac{\partial f}{\partial X}(X,0,0,0) \;<\; 2\pi-\varepsilon' \tag{74}$$

for some $\varepsilon' > 0$. Since $0.1 \le t \le 0.33$,

$$-2\pi\big(\frac{1}{2}-t\big) \;<\; \mathrm{Arg}\,(1-x) \;<\; 0.$$

Further, since $0.15 \le s-t \le 0.45$,

$$0 \;<\; \mathrm{Arg}\,\big(1-\frac{x}{y}\big) \;<\; 2\pi\big(\frac{1}{2}-s+t\big).$$

Hence, by (70),

$$-2\pi(1-s) \;<\; \frac{\partial f}{\partial X} \;<\; 2\pi\big(\frac{1}{2}-t\big).$$

Further, since $s = t + (s-t) \ge 0.1 + 0.15 = 0.25$ and $0.1 \le t$,

$$-2\pi\cdot 0.75 \;<\; \frac{\partial f}{\partial X} \;<\; 2\pi\cdot 0.4.$$

Therefore, (74) is satisfied, as required.

As for (15) and (16), similarly as above, it is sufficient to show that

$$-(2\pi-\varepsilon') \;<\; \frac{\partial f}{\partial Y}(0,Y,0,0) \;<\; 2\pi-\varepsilon' \tag{75}$$

for some $\varepsilon' > 0$. Since $0.15 \le s-t \le 0.45$,

$$0 \;<\; \mathrm{Arg}\,\big(1-\frac{x}{y}\big) \;<\; 2\pi\big(\frac{1}{2}-s+t\big).$$

Further, since $0.15 \le u-s \le 0.45$,

$$0 \;<\; \mathrm{Arg}\,\big(1-\frac{y}{z}\big) \;<\; 2\pi\big(\frac{1}{2}-u+s\big).$$

Hence, by (71),

$$-2\pi(1-u) \;<\; \frac{\partial f}{\partial Y} \;<\; 2\pi t.$$

Further, since $0.67 \le u$ and $t \le 0.33$,

$$-2\pi \cdot 0.33 \;<\; \frac{\partial f}{\partial Y} \;<\; 2\pi \cdot 0.33.$$

Therefore, (75) is satisfied, as required.

As for (17) and (18), similarly as above, it is sufficient to show that

$$-(2\pi - \varepsilon') \;<\; \frac{\partial f}{\partial Z}(0,0,Z,0) \;<\; 2\pi - \varepsilon' \tag{76}$$

for some $\varepsilon' > 0$. Since $0.67 \le u \le 0.9$,

$$0 \;<\; \mathrm{Arg}\,(1-z) \;<\; 2\pi\big(u - \frac{1}{2}\big).$$

Further, since $0.15 \le u - s \le 0.45$,

$$0 \;<\; \mathrm{Arg}\,\big(1 - \frac{y}{z}\big) \;<\; 2\pi\big(\frac{1}{2} - u + s\big).$$

Hence, by (72),

$$-2\pi(2u - v - 1) \;<\; \frac{\partial f}{\partial Z} \;<\; 2\pi\big(s - u + v + \frac{1}{2}\big).$$

Since $2u - v < 2u \le 2 \cdot 0.9 = 1.8$ and $s - u + v = v - (u-s) \le 0.3 - 0.15 = 0.15$,

$$-2\pi \cdot 0.8 \;<\; \frac{\partial f}{\partial Z} \;<\; 2\pi \cdot 0.65.$$

Therefore, (76) is satisfied, as required.

As for (19) and (20), similarly as above, it is sufficient to show that

$$-(2\pi - \varepsilon') \;<\; \frac{\partial f}{\partial W}(0,0,0,W) \;<\; 2\pi - \varepsilon' \tag{77}$$

for some $\varepsilon' > 0$. Since $0.05 \le v \le 0.3$,

$$-2\pi\big(\frac{1}{2} - v\big) \;<\; \mathrm{Arg}\,(1-w) \;<\; 0.$$

Hence, by (73),

$$-2\pi\big(1-u\big) \;<\; \frac{\partial f}{\partial W} \;<\; 2\pi\big(u - v - \frac{1}{2}\big).$$

Since $0.67 \le u$ and $u - v \le 0.9 - 0.05 = 0.85$,

$$-2\pi \cdot 0.33 \;<\; \frac{\partial f}{\partial W} \;<\; 2\pi \cdot 0.35.$$

Therefore, (77) is satisfied, as required. $\square$

5.5 Verifying the assumption of the saddle point method

In this section, we verify the assumption of the saddle point method in Lemma 5.5. In order to show this lemma, we show Lemmas 5.3 and 5.4 in advance.

From the definition of f, we have that

$$\begin{aligned} f(X,Y,Z,W) &= \operatorname{Re}\frac{1}{2\pi\sqrt{-1}}\Big(2\operatorname{Li}_2(x)-\operatorname{Li}_2\big(\frac{x}{y}\big)-\operatorname{Li}_2\big(\frac{y}{z}\big)-2\operatorname{Li}_2(z)+\operatorname{Li}_2(w)\Big)\\ &\quad -2\pi\big(t-\frac{1}{2}\big)X+2\pi\big(s-\frac{1}{2}\big)Y+2\pi vZ+2\pi\big(u-v-\frac{1}{2}\big)W, \end{aligned}$$

where $x=e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$, $y=e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$, $z=e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})}$ and $w=e^{2\pi\sqrt{-1}\,(v+W\sqrt{-1})}$. Hence, since the contributions to f from (X,Y,Z) and W are independent, we consider each of the contributions independently.

Lemma 5.3. *Fixing X, Y and Z, we regard f as a function of W.*
(1) If $u-v\le\frac{1}{2}$, then f is monotonically decreasing as a function of W.
(2) If $u-v>\frac{1}{2}$, then f has a unique minimal point as a function of W. In particular, this minimal point goes to ∞ as $u-v\to\frac{1}{2}+0$.

Proof. Since $w=e^{2\pi\sqrt{-1}\,(v+W\sqrt{-1})}$ and $0.05\le v\le 0.3$,

$$-2\pi\big(\frac{1}{2}-v\big) \;<\; \operatorname{Arg}(1-w) \;<\; 0,$$

and $\operatorname{Arg}(1-w)$ is monotonically increasing as a function of W. Hence, by (73), $\frac{\partial f}{\partial W}$ is also monotonically increasing as a function of W. Further,

$$\begin{aligned} \frac{\partial f}{\partial W}\Big|_{W\to\infty} &= 2\pi\big(u-v-\frac{1}{2}\big),\\ \frac{\partial f}{\partial W}\Big|_{W\to-\infty} &= -2\pi\big(1-u\big) \;<\; 0. \end{aligned}$$

If $u-v\le\frac{1}{2}$, then $\frac{\partial f}{\partial W}$ is always negative, and (1) holds.

If $u-v>\frac{1}{2}$, then there is a unique zero of $\frac{\partial f}{\partial W}$, which gives a unique minimal point of f, and (2) holds. □

In order to consider the contribution to f from (X,Y,Z), we put

$$\begin{aligned} \hat f(X,Y,Z) &= \operatorname{Re}\frac{1}{2\pi\sqrt{-1}}\Big(2\operatorname{Li}_2(x)-\operatorname{Li}_2\big(\frac{x}{y}\big)-\operatorname{Li}_2\big(\frac{y}{z}\big)-2\operatorname{Li}_2(z)\Big)\\ &\quad -2\pi\big(t-\frac{1}{2}\big)X+2\pi\big(s-\frac{1}{2}\big)Y+2\pi vZ, \end{aligned}$$

where $x=e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$, $y=e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$ and $z=e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})}$.

Lemma 5.4. *In the fiber of the projection $\mathbb{C}^4\to\mathbb{R}^4$ at $(t,s,u,v)\in\Delta'$, we consider the flow from $(X,Y,Z)=(0,0,0)$ determined by the vector field $(-\frac{\partial\hat f}{\partial X},-\frac{\partial\hat f}{\partial Y},-\frac{\partial\hat f}{\partial Z})$. Then,*

there exists a convex neighborhood U of (t_0, s_0, u_0, v_0) such that the following (1) *and* (2) *holds.*
(1) *If $(t,s,u,v) \in U$, then $\hat{f}$ has a unique minimal point, and the flow goes there.*
(2) *If $(t,s,u,v) \notin U$, then the flow goes to infinity.*

Proof. It is shown by concrete calculation that the Hesse matrix of $\hat{f}$ is presented by

$$2\pi\,\mathrm{Im}\begin{pmatrix} \dfrac{1+x}{1-x} - \dfrac{\frac{x}{y}}{1-\frac{x}{y}} & \dfrac{\frac{x}{y}}{1-\frac{x}{y}} & 0 \\ \dfrac{\frac{x}{y}}{1-\frac{x}{y}} & -\dfrac{\frac{x}{y}}{1-\frac{x}{y}} - \dfrac{\frac{y}{z}}{1-\frac{y}{z}} - 1 & \dfrac{\frac{y}{z}}{1-\frac{y}{z}} \\ 0 & \dfrac{\frac{y}{z}}{1-\frac{y}{z}} & \dfrac{-2z}{1-z} - \dfrac{\frac{y}{z}}{1-\frac{y}{z}} \end{pmatrix}$$
$$= 2\pi \begin{pmatrix} a_1+b_1 & -b_1 & 0 \\ -b_1 & b_1+b_2 & -b_2 \\ 0 & -b_2 & a_2+b_2 \end{pmatrix},$$

where we put

$$a_1 = \mathrm{Im}\,\frac{2}{1-x}, \quad a_2 = \mathrm{Im}\,\frac{-2}{1-z}, \quad b_1 = \mathrm{Im}\,\frac{-1}{1-\frac{x}{y}}, \quad b_2 = \mathrm{Im}\,\frac{-1}{1-\frac{y}{z}},$$

noting that these numbers are positive. Further, the above matrix is equivalent, as a quadratic form, to

$$2\pi \begin{pmatrix} a_1+b_1 & 0 & 0 \\ 0 & \dfrac{a_1 b_1}{a_1+b_1} + \dfrac{a_2 b_2}{a_2+b_2} & 0 \\ 0 & 0 & a_2+b_2 \end{pmatrix}.$$

Hence, the Hesse matrix of $\hat{f}$ is always positive definite, and $\hat{f}$ is a convex function.

We consider the behavior of $\hat{f}$ at infinity. By (11), $\frac{1}{2\pi}\hat{f}$ is approximated by the following $F(X,Y,Z)$,

$$\begin{aligned} F(X,Y,Z) \;=\; & \left(\begin{cases} 0 & \text{if } X \ge 0 \\ -(1-2t)X & \text{if } X < 0 \end{cases}\right) + \left(\begin{cases} 0 & \text{if } X \ge Y \\ -\big(\frac{1}{2} - s + t\big)(X-Y) & \text{if } X < Y \end{cases}\right) \\ & + \left(\begin{cases} 0 & \text{if } Y \ge Z \\ -\big(\frac{1}{2} - u + s\big)(Y-Z) & \text{if } Y < Z \end{cases}\right) + \left(\begin{cases} 0 & \text{if } Z \ge 0 \\ -(2u-1)Z & \text{if } Z < 0 \end{cases}\right) \\ & + \big(\frac{1}{2} - t\big)X + \big(s - \frac{1}{2}\big)Y + v\,Z. \end{aligned}$$

Hence, if

$$F(X,Y,Z) \to \infty \quad \text{as } X^2+Y^2+Z^2 \to \infty, \tag{78}$$

then the conclusion of (1) holds, since $\hat{f}$ is convex. Otherwise, the conclusion of (2) holds. Therefore, we consider the condition of (78). Since F is piecewise linear, this condition can be presented by a system of inequalities of t, s, u, v of degree 1. Hence, U of the lemma is convex.

In order to complete the proof of the lemma, it is sufficient to show (78) in a neighborhood of (t_0, s_0, u_0, v_0). We show that, if $s > \frac{1}{2}$, then (78) holds, as follows. We rewrite $F(X,Y,Z)$ as

$$F(X,Y,Z) = \left(\begin{cases} \big(\frac{1}{2}-t\big) & \text{if } X \geq 0 \\ -\big(\frac{1}{2}-t\big)X & \text{if } X < 0 \end{cases}\right) + \left(\begin{cases} 0 & \text{if } X \geq Y \\ -\big(\frac{1}{2}-s+t\big)(X-Y) & \text{if } X < Y \end{cases}\right)$$
$$+ \left(\begin{cases} \big(s-\frac{1}{2}\big)(Y-Z) & \text{if } Y \geq Z \\ -\big(1-u\big)(Y-Z) & \text{if } Y < Z \end{cases}\right) + \left(\begin{cases} \big(s+v-\frac{1}{2}\big) & \text{if } Z \geq 0 \\ -\big(2u-s-v-\frac{1}{2}\big)Z & \text{if } Z < 0 \end{cases}\right).$$

Since $s > \frac{1}{2}$ by the assumption and $t \leq 0.33$, $s-t \leq 0.45$, $u \leq 0.9$, $2u-s-v = (u-s)+u-v \geq 0.15+0.67-0.3 = 0.52$, each of the summands of the right-hand side of the above formula is positive. Hence, there exists some constant C such that

$$F(X,Y,Z) \geq C\big(|X|+|X-Y|+|Y-Z|+|Z|\big).$$

Therefore, (78) holds in a neighborhood of (t_0, s_0, u_0, v_0), as required. □

Lemma 5.5. *When we apply Proposition 2.4 to* (66)*, the assumption of Proposition 2.4 holds.*

Proof. We show that there exists a homotopy Δ'_δ $(0 \leq \delta \leq 1)$ between $\Delta'_0 = \Delta'$ and Δ'_1 such that

$$(t_c, s_c, u_c, v_c) \in \Delta'_1, \tag{79}$$
$$\Delta'_1 - \{(t_c, s_c, u_c, v_c)\} \subset \big\{(t,s,u,v) \in \mathbb{C}^4 \bigm| \operatorname{Re}\hat{V}(t,s,u,v) < \varsigma_R\big\}, \tag{80}$$
$$\partial\Delta'_\delta \subset \big\{(t,s,u,v) \in \mathbb{C}^4 \bigm| \operatorname{Re}\hat{V}(t,s,u,v) < \varsigma_R\big\}. \tag{81}$$

In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$. As mentioned at the beginning of this section, the contributions to f from (X,Y,Z) and W are independent. Hence, by Lemmas 5.3 and 5.4, there exists a convex neighborhood U' of (t_0, s_0, u_0, v_0) such that the following (1) and (2) holds.

(1) If $(t,s,u,v) \in U'$, then f has a unique minimal point, and the flow goes there.

(2) If $(t,s,u,v) \notin U'$, then the flow goes to infinity.

We put the homotopy Δ'_δ in a similar way as in the proof of Lemma 4.7.

We can show (79), (80) and (80) by using Lemma 5.1 in a similar way as the proof of Lemma 3.9. □

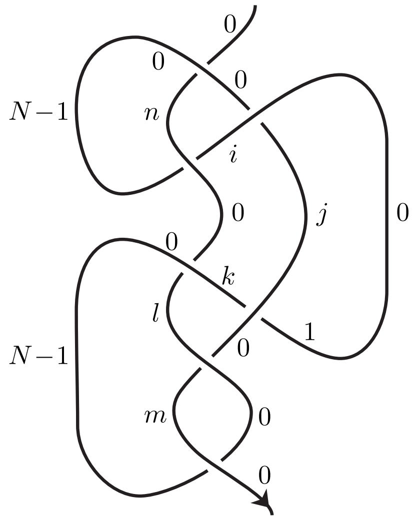
6 The 7_6 knot

In this section, we show Theorem 1.1 for the 7_6 knot. We give a proof of the theorem in Section 6.1, using lemmas shown in Sections 6.2–6.5.

6.1 Proof of Theorem 1.1 for the 7_6 knot

In this section, we show a proof of Theorem 1.1 for the 7_6 knot.

The 7_6 *knot* is the closure of the following tangle.



As shown in [32], we can put the labelings of edges adjacent to the unbounded regions as shown above. Hence, from the definition of the Kashaev invariant, the Kashaev invariant of the 7_6 knot is presented by

$$\begin{aligned}\langle\, 7_6 \,\rangle_N \;&=\; \sum q^{-1/2} \times \frac{N\, q^{\frac{1}{2}}}{(\overline{q})_{N-n-1}(q)_n} \times \frac{N\, q^{-\frac{1}{2}-i}}{(\overline{q})_{N-j}(q)_{j-i-1}(\overline{q})_i} \times \frac{N\, q^{\frac{1}{2}+i}}{(\overline{q})_{n-i}(q)_i(q)_{N-n-1}} \\ &\quad \times \frac{N\, q^{\frac{1}{2}-k}}{(q)_{N-k}(\overline{q})_{k-l-1}(q)_l} \times \frac{N\, q^{-\frac{1}{2}+k}}{(q)_{k-j}(\overline{q})_{j-1}(\overline{q})_{N-k}} \times \frac{N\, q^{\frac{1}{2}}}{(\overline{q})_l(\overline{q})_{N-m-1}(q)_{m-l}} \times \frac{N\, q^{\frac{1}{2}}}{(\overline{q})_m(q)_{N-m-1}} \\ &= \sum_{\substack{0\le i<j\le k \\ 0\le l<k\le N}} \frac{N^5\, q}{(q)_i(\overline{q})_i(q)_{j-i-1}(\overline{q})_{j-1}(\overline{q})_{N-j}(q)_{k-j}(q)_{N-k}(\overline{q})_{N-k}(\overline{q})_{k-l-1}(q)_l(\overline{q})_l} \\ &= \sum_{\substack{0\le i\le j\le k \\ 0\le l\le k<N}} \frac{N^5\, q}{(q)_i(\overline{q})_i(q)_{j-i}(\overline{q})_j(\overline{q})_{N-j-1}(q)_{N-j-k-1}(q)_k(\overline{q})_k(\overline{q})_{N-k-l-1}(q)_l(\overline{q})_l}\,,\end{aligned}$$

where we obtain the last equality by replacing j with $j+1$ and replacing k with $N-k$.

Proof of Theorem 1.1 for the 7_6 *knot.* By (5), the above presentation of $\langle\, 7_6 \,\rangle_N$ is rewritten

$$\langle\, 7_6 \,\rangle_N \;=\; N^5 q^{-1} \sum_{\substack{0\le i<N-j \\ 0\le j,k,\;\; j+k<N \\ 0\le l<N-k}} \exp\Big(N\, \tilde{V}\big(\frac{2i+1}{2N},\; \frac{2j+1}{2N},\; \frac{2k+1}{2N},\; \frac{2l+1}{2N} \big)\Big),$$

where we put

$$\begin{aligned}\tilde V(t,s,u,v) &= \frac{1}{N}\Big(\varphi(t)-\varphi(1-t)+\varphi\big(s-t+\frac{1}{2N}\big)-\varphi(s)-\varphi(1-s)\\ &\quad+\varphi\big(1-s-u+\frac{1}{2N}\big)+\varphi(u)-\varphi(1-u)-\varphi\big(u+v-\frac{1}{2N}\big)+\varphi(v)\\ &\quad-\varphi(1-v)-5\,\varphi\big(\frac{1}{2N}\big)+6\,\varphi\big(1-\frac{1}{2N}\big)\Big)\\ &= \frac{1}{N}\Big(2\,\varphi(t)+\varphi\big(s-t+\frac{1}{2N}\big)+\varphi\big(1-s-u+\frac{1}{2N}\big)+2\,\varphi(u)\\ &\quad-\varphi\big(u+v-\frac{1}{2N}\big)+2\,\varphi(v)\Big)+\frac{1}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6}-\frac{11}{2N}\log N+\frac{\pi\sqrt{-1}}{4N}-\frac{\pi\sqrt{-1}}{12N^2}\\ &\quad+2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t^2+s^2+u^2+v^2-t-s-u-v+\frac{2}{3}\big).\end{aligned}$$

Here, we obtain the last equality by (9) and (10). Hence, by putting

$$V(t,s,u,v) \;=\; \tilde V(t,s,u,v)+\frac{11}{2N}\log N,$$

the presentation of $\langle\, 7_6\,\rangle_N$ is rewritten

$$\langle\, 7_6\,\rangle_N \;=\; N^{-1/2}q^{-1}\sum_{\substack{i,j,k,l\,\in\mathbb{Z}\\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})\in\Delta}}\exp\Big(N\,V\big(\frac{2i+1}{2N},\ \frac{2j+1}{2N},\ \frac{2k+1}{2N},\ \frac{2l+1}{2N}\big)\Big),$$

where the range of $(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})$ of the sum is given by the following domain,

$$\Delta \;=\; \big\{(t,s,u,v)\in\mathbb{R}^4 \;\big|\; 0\le t\le s\le 1-u,\quad 0\le u,v,\quad u+v\le 1\big\}.$$

By Proposition 2.1, as $N\to\infty$, $V(t,s,u,v)$ converges to the following $\hat V(t,s,u,v)$ in the interior of Δ,

$$\begin{aligned}\hat V(t,s,u,v) &= \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t})+\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s-t)})+\mathrm{Li}_2(e^{-2\pi\sqrt{-1}\,(s+u)})\\ &\qquad+2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,u})-\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(u+v)})+2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,v})+\frac{\pi^2}{6}\Big)\\ &\quad+2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t^2+s^2+u^2+v^2-t-s-u-v+\frac{2}{3}\big).\end{aligned}$$

By concrete calculation, we can check that the boundary of Δ is included in the domain

$$\big\{(t,s,u,v)\in\Delta \;\big|\; \mathrm{Re}\,\hat V(t,s,u,v)<\varsigma_R-\varepsilon\big\} \tag{82}$$

for some sufficiently small $\varepsilon>0$, where we put $\varsigma_R=1.1276...$ as in (87); we will know later that this value is equal to the real part of the critical value of $\hat V$ at the critical point of Lemma 6.1. Hence, similarly as in Section 3.1, we choose a new domain Δ', which satisfies that $\Delta - \Delta' \subset (82)$, as

$$\Delta' \;=\; \left\{ (t,s,u,v) \in \Delta \;\middle|\; \begin{array}{l} 0.05 \le t \le 0.34, \quad 0.25 \le s \le 0.64, \quad 0.15 \le u \le 0.48 \\ 0.1 \le v \le 0.45, \quad 0.1 \le s-t \le 0.45, \quad 0.55 \le s+u \le 0.9 \end{array} \right\}, \tag{83}$$

where we calculate the concrete values of the bounds of these inequalities in Section 6.2. Hence, since $\Delta - \Delta' \subset (82)$, we obtain the second equality of the following formula,

$$\begin{aligned} \langle\, 7_6 \,\rangle_N \;&=\; e^{N\varsigma} N^{-1/2} q \sum_{\substack{i,j,k,l \,\in\, \mathbb{Z} \\ (\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N}) \in \Delta}} \exp\Big(N \cdot V\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N} \big) - N\varsigma \Big) \\ &=\; e^{N\varsigma} \Bigg(N^{-1/2} q \sum_{\substack{i,j,k,l \,\in\, \mathbb{Z} \\ (\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N}) \in \Delta'}} \exp\Big(N \cdot V\big(\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N} \big) - N\varsigma \Big) \; + O(e^{-N\varepsilon}) \Bigg), \end{aligned}$$

for some $\varepsilon > 0$.

Further, by Proposition 2.2 (Poisson summation formula), the above sum is presented by

$$\langle\, 7_6 \,\rangle_N \;=\; e^{N\varsigma} \Bigg(N^{7/2} q \int_{\Delta'} \exp\big(N \cdot V(t,s,u,v) - N\varsigma \big)\, dt\, ds\, du\, dv \;\; + O(e^{-N\varepsilon}) \Bigg), \tag{84}$$

noting that we verify the assumption of Proposition 2.2 in Lemma 6.2. Furthermore, by Proposition 2.4 (saddle point method), there exist some κ_i''s such that

$$\langle\, 7_6 \,\rangle_N \;=\; N^{7/2} \exp\big(N \cdot V(t_c, s_c, u_c, v_c) \big) \cdot \frac{(2\pi)^2}{N^2} \big(\det(-H) \big)^{-1/2} \Big(1 + \sum_{i=1}^{d} \kappa_i' \hbar^i + O(\hbar^{d+1}) \Big),$$

for any $d > 0$, noting that we verify the assumption of Proposition 2.4 in Lemma 6.6. Here, (t_c, s_c, u_c, v_c) is the critical point of V which corresponds to the critical point (t_0, s_0, u_0, v_0) of $\hat{V}$ of Lemma 6.1, where $\hat{V}$ is the limit of V at $N \to \infty$ whose concrete presentation is given in Section 6.2, and H is the Hesse matrix of V at (t_c, s_c, u_c, v_c).

We calculate the right-hand side of the above formula. Similarly as in Section 3.1, we have that

$$V(t_0, s_0, u_0, v_0) \;=\; \varsigma + O(\hbar).$$

Therefore, there exist some κ_i's such that

$$\langle\, 7_6 \,\rangle_N \;=\; e^{N\varsigma} N^{3/2}\, \omega \cdot \Big(1 + \sum_{i=1}^{d} \kappa_i \hbar^i + O(\hbar^{d+1}) \Big),$$

for any $d > 0$. Hence, we obtain the theorem for the 7_6 knot. □

6.2 Estimate of the range of Δ'

In this section, we calculate the concrete values of the bounds of the inequalities in (83) so that they satisfy that $\Delta - \Delta' \subset$ (82).

Putting Λ as in Section 2.2, we have that

$$\operatorname{Re}\hat{V}(t,s,u,v) \;=\; 2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v).$$

We consider the domain

$$\big\{(t,s,u,v)\in\Delta \;\big|\; 2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\;\geq\;\varsigma_{_R}\big\}, \quad (85)$$

where we put $\varsigma_{_R} = 1.1276...$ as in (87). The aim of this section is to show that this domain is included in the interior of the domain Δ' of (83). For this purpose, we show the estimates of the defining inequalities of (83) for (t,s,u,v) in (85).

We calculate the minimal value $t_{\min}$ and the maximal value $t_{\max}$ of t. They are solutions of the system of the following equations,

$$\begin{cases} 2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v) \;=\; \varsigma_{_R}, \\ \dfrac{\partial}{\partial s}\big(2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial v}\big(2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of t (see Appendix A). By calculating a solution of these equations by Newton's method from $(t,s,u,v)=(0.1,0.35,0.35,0.25)$, we obtain $t_{\min}=0.103312...$, and from $(t,s,u,v)=(0.3,0.5,0.3,0.25)$, we obtain $t_{\max}=0.319994...$. Therefore, we obtain an estimate of t in Δ' as

$$0.05 \;\leq\; t \;\leq\; 0.34.$$

We calculate the minimal value $s_{\min}$ and the maximal value $s_{\max}$ of s. They are solutions of the system of the following equations,

$$\begin{cases} 2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v) \;=\; \varsigma_{_R}, \\ \dfrac{\partial}{\partial t}\big(2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial v}\big(2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of s (see Appendix A). By calculating a solution of these equations by Newton's method from $(t,s,u,v)=(0.15,0.3,0.35,0.25)$, we obtain $s_{\min}=$

0.286689... , and from $(t, s, u, v) = (0.25, 0.6, 0.25, 0.25)$, we obtain $s_{\max} = 0.610894...$. Therefore, we obtain an estimate of s in Δ' as

$$0.25 \ \le \ s \ \le \ 0.64.$$

We calculate the minimal value $u_{\min}$ and the maximal value $u_{\max}$ of u. They are solutions of the system of the following equations,

$$\begin{cases} 2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v) \ = \ \varsigma_{R}, \\ \dfrac{\partial}{\partial t}\big(2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \ = \ 0, \\ \dfrac{\partial}{\partial s}\big(2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \ = \ 0, \\ \dfrac{\partial}{\partial v}\big(2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \ = \ 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of u (see Appendix A). By calculating a solution of these equations by Newton's method from $(t, s, u, v) = (0.2, 0.5, 0.2, 0.25)$, we obtain $u_{\min} = 0.182665...$, and from $(t, s, u, v) = (0.2, 0.35, 0.45, 0.2)$, we obtain $u_{\max} = 0.455212...$. Therefore, we obtain an estimate of u in Δ' as

$$0.15 \ \le \ u \ \le \ 0.48.$$

We calculate the minimal value $v_{\min}$ and the maximal value $v_{\max}$ of v. They are solutions of the system of the following equations,

$$\begin{cases} 2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v) \ = \ \varsigma_{R}, \\ \dfrac{\partial}{\partial t}\big(2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \ = \ 0, \\ \dfrac{\partial}{\partial s}\big(2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \ = \ 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \ = \ 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of v (see Appendix A). By calculating a solution of these equations by Newton's method from $(t, s, u, v) = (0.2, 0.45, 0.3, 0.1)$, we obtain $v_{\min} = 0.13126...$, and from $(t, s, u, v) = (0.2, 0.45, 0.3, 0.4)$, we obtain $v_{\max} = 0.390199...$. Therefore, we obtain an estimate of v in Δ' as

$$0.1 \ \le \ v \ \le \ 0.45.$$

We calculate the minimal value $(s-t)_{\min}$ and the maximal value $(s-t)_{\max}$ of $s-t$. Putting $w = s - t$, its minimal and maximal values are solutions of the system of the following equations,

$$\begin{cases} 2\,\Lambda(s-w)+\Lambda(w)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v) \;=\; \varsigma_{_R}, \\ \dfrac{\partial}{\partial s}\big(2\,\Lambda(s-w)+\Lambda(w)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(s-w)+\Lambda(w)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial v}\big(2\,\Lambda(s-w)+\Lambda(w)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of $s-t$ (see Appendix A). By calculating a solution of these equations by Newton's method from $(w,s,u,v)=(0.1,0.3,0.35,0.25)$, we obtain $(s-t)_{\min}=0.105664...$, and from $(w,s,u,v)=(0.4,0.6,0.25,0.25)$, we obtain $(s-t)_{\max}=0.411943...$. Therefore, we obtain an estimate of $s-t$ in Δ' as

$$0.1 \;\leq\; s-t \;\leq\; 0.45.$$

We calculate the minimal value $(s+u)_{\min}$ and the maximal value $(s+u)_{\max}$ of $s+u$. Putting $w'=s+u$, its minimal and maximal values are solutions of the system of the following equations,

$$\begin{cases} 2\,\Lambda(t)+\Lambda(w'-u-t)-\Lambda(w')+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v) \;=\; \varsigma_{_R}, \\ \dfrac{\partial}{\partial t}\big(2\,\Lambda(t)+\Lambda(w'-u-t)-\Lambda(w')+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(t)+\Lambda(w'-u-t)-\Lambda(w')+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial v}\big(2\,\Lambda(t)+\Lambda(w'-u-t)-\Lambda(w')+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\big) \;=\; 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of $s+u$ (see Appendix A). By calculating a solution of these equations by Newton's method from $(w',t,u,v)=(0.6,0.15,0.25,0.25)$, we obtain $(s+u)_{\min}=0.588057...$, and from $(w',t,u,v)=(0.9,0.2,0.35,0.25)$, we obtain $(s+u)_{\max}=0.894336...$. Therefore, we obtain an estimate of $s+u$ in Δ' as

$$0.55 \;\leq\; s+u \;\leq\; 0.9.$$

6.3 Calculation of the critical value

In this section, we calculate the concrete value of a critical point of $\hat{V}$.

The differentials of $\hat{V}$ are presented by

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{V}(t,s,u,v) &= -2\,\log(1-x)+\log\big(1-\frac{y}{x}\big)+2\pi\sqrt{-1}\,\big(t-\frac{1}{2}\big),\\
\frac{\partial}{\partial s}\hat{V}(t,s,u,v) &= -\log\big(1-\frac{y}{x}\big)+\log\big(1-\frac{1}{yz}\big)+2\pi\sqrt{-1}\,\big(s-\frac{1}{2}\big),\\
\frac{\partial}{\partial u}\hat{V}(t,s,u,v) &= -2\,\log(1-z)+\log\big(1-\frac{1}{yz}\big)+\log(1-zw)+2\pi\sqrt{-1}\,\big(u-\frac{1}{2}\big),\\
\frac{\partial}{\partial v}\hat{V}(t,s,u,v) &= -2\,\log(1-w)+\log(1-zw)+2\pi\sqrt{-1}\,\big(v-\frac{1}{2}\big),
\end{aligned}$$

where $x=e^{2\pi\sqrt{-1}\,t}$, $y=e^{2\pi\sqrt{-1}\,s}$, $z=e^{2\pi\sqrt{-1}\,u}$ and $w=e^{2\pi\sqrt{-1}\,v}$.

Lemma 6.1. *$\hat{V}$ has a unique critical point (t_0,s_0,u_0,v_0) in $P^{-1}(\Delta')$, where $P:\mathbb{C}^4\to\mathbb{R}^4$ is the projection to the real parts of the entries.*

Proof. Any critical point of $\hat{V}$ is given by a solution of $\frac{\partial}{\partial t}\hat{V}=\frac{\partial}{\partial s}\hat{V}=\frac{\partial}{\partial u}\hat{V}=\frac{\partial}{\partial v}\hat{V}=0$, and these equations are rewritten,

$$\begin{aligned}
(1-x)^2 &= -x\big(1-\frac{y}{x}\big),\\
1-\frac{y}{x} &= -y\big(1-\frac{1}{yz}\big),\\
(1-z)^2 &= -z\big(1-\frac{1}{yz}\big)(1-zw),\\
(1-w)^2 &= -w(1-zw).
\end{aligned}$$

From the first formula, we have that $y=x^2-x+1$. Hence, from the second formula, we have that $z=x/(x^3-2x^2+3x-1)$. Further, from the third formula, we have that $w=x(x^2-2x+2)(x^2-x+2)$. By substituting these into the fourth formula, we have that

$$x^9-6x^8+20x^7-43x^6+65x^5-69x^4+50x^3-23x^2+5x-1 \;=\; 0.$$

Its solutions are

$$\begin{aligned}
x \;&= 0.0848864...\pm\sqrt{-1}\cdot 0.271383...\ ,\quad 0.558614...\pm\sqrt{-1}\cdot 1.43795...\ ,\\
&\quad 0.629127...\pm\sqrt{-1}\cdot 1.09993...\ ,\quad 1.09612...\pm\sqrt{-1}\cdot 1.16718...\ ,\quad 1.26251...\ .
\end{aligned}$$

Among these, the solution $0.558614...+\sqrt{-1}\cdot 1.43795...$ gives a solution in Δ', from which we have that

$$\begin{aligned}
x_0 &= 0.558614...+\sqrt{-1}\cdot 1.43795...\ , & t_0 &= 0.191027...-\sqrt{-1}\cdot 0.0689933...\ ,\\
y_0 &= -1.31426...+\sqrt{-1}\cdot 0.168567...\ , & s_0 &= 0.479698...-\sqrt{-1}\cdot 0.0447913...\ ,\\
z_0 &= -0.23704...+\sqrt{-1}\cdot 1.46509...\ , & u_0 &= 0.275529...-\sqrt{-1}\cdot 0.0628402...\ ,\\
w_0 &= -0.0892864...+\sqrt{-1}\cdot 0.842785...\ , & v_0 &= 0.266799...+\sqrt{-1}\cdot 0.0263342...\ ,
\end{aligned}$$

where $x_0=e^{2\pi\sqrt{-1}\,t_0}$, $y_0=e^{2\pi\sqrt{-1}\,s_0}$, $z_0=e^{2\pi\sqrt{-1}\,u_0}$ and $w_0=e^{2\pi\sqrt{-1}\,v_0}$. These give a unique critical point in $P^{-1}(\Delta')$. □

The critical value of $\hat{V}$ at the critical point of Lemma 6.1 is presented by

$$\begin{aligned}
\varsigma &= \hat{V}(t_0, s_0, u_0, v_0) \\
&= \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(x_0) + \mathrm{Li}_2\big(\frac{y_0}{x_0}\big) + \mathrm{Li}_2\big(\frac{1}{y_0 z_0}\big) + 2\,\mathrm{Li}_2(z_0) - \mathrm{Li}_2(z_0 w_0) + 2\,\mathrm{Li}_2(w_0) + \frac{\pi^2}{6}\Big) \\
&\qquad + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t_0^2 + s_0^2 + u_0^2 + v_0^2 - t_0 - s_0 - u_0 - v_0 + \frac{2}{3}\big) \qquad (86) \\
&= 1.1276... - \sqrt{-1}\cdot 0.57266... \,.
\end{aligned}$$

Further, we put its real part to be $\varsigma_{_R}$,

$$\varsigma_{_R} = \mathrm{Re}\,\varsigma = 1.1276... \,. \qquad (87)$$

6.4 Verifying the assumption of the Poisson summation formula

In this section, we verify the assumption of the Poisson summation formula in Lemma 6.2, which is used in the proof of Theorem 1.1 for the 7_6 knot in Section 6.1.

By computer calculation, we can see that the maximal value of $\mathrm{Re}\,\hat{V} - \varsigma_{_R}$ is about 0.06. Therefore, in the proof of Lemma 6.2, it is sufficient to decrease, say, $\mathrm{Re}\,\hat{V}(t + \delta\sqrt{-1},\, s,\, u,\, v) - 2\pi\delta$ by 0.06, by moving δ (though we do not use this value in the proof of the lemma).

We put

$$f(X,Y,Z,W) = \mathrm{Re}\,\hat{V}(t + X\sqrt{-1},\, s + Y\sqrt{-1},\, u + Z\sqrt{-1},\, v + W\sqrt{-1}) - \varsigma_{_R}\,.$$

Then, we have that

$$\frac{\partial f}{\partial X} = 2\,\mathrm{Arg}\,(1-x) - \mathrm{Arg}\,\big(1 - \frac{y}{x}\big) - 2\pi\big(t - \frac{1}{2}\big), \qquad (88)$$

$$\frac{\partial f}{\partial Y} = \mathrm{Arg}\,\big(1 - \frac{y}{x}\big) - \mathrm{Arg}\,\big(1 - \frac{1}{yz}\big) - 2\pi\big(s - \frac{1}{2}\big), \qquad (89)$$

$$\frac{\partial f}{\partial Z} = 2\,\mathrm{Arg}\,(1-z) - \mathrm{Arg}\,\big(1 - \frac{1}{yz}\big) - \mathrm{Arg}\,(1 - zw) - 2\pi\big(u - \frac{1}{2}\big), \qquad (90)$$

$$\frac{\partial f}{\partial W} = 2\,\mathrm{Arg}\,(1-w) - \mathrm{Arg}\,(1 - zw) - 2\pi\big(v - \frac{1}{2}\big), \qquad (91)$$

where $x = e^{2\pi\sqrt{-1}\,(t + X\sqrt{-1})}$, $y = e^{2\pi\sqrt{-1}\,(s + Y\sqrt{-1})}$, $z = e^{2\pi\sqrt{-1}\,(u + Z\sqrt{-1})}$ and $w = e^{2\pi\sqrt{-1}\,(v + W\sqrt{-1})}$.

Lemma 6.2. *$V(t,s,u,v) - \varsigma_{_R}$ satisfies the assumption of Proposition 2.2.*

Proof. Since $V(t,s,u,v)$ converges uniformly to $\hat{V}(t,s,u,v)$ on Δ', we show the proof for $\hat{V}(t,s,u,v)$ instead of $V(t,s,u,v)$. We show that $\partial\Delta'$ is null-homotopic in each of (13)–(20).

As for (13) and (14), similarly as in the proof of Lemma 3.3, it is sufficient to show that

$$-(2\pi - \varepsilon') < \frac{\partial f}{\partial X}(X,0,0,0) < 2\pi - \varepsilon' \qquad (92)$$

for some $\varepsilon' > 0$. Since $0.05 \le t \le 0.34$,

$$-2\pi\big(\frac{1}{2} - t\big) \ < \ \mathrm{Arg}\,(1-x) \ < \ 0.$$

Further, since $0.1 \le s - t \le 0.45$,

$$-2\pi\big(\frac{1}{2} - s + t\big) \ < \ \mathrm{Arg}\,\big(1 - \frac{y}{x}\big) \ < \ 0.$$

Hence, by (88),

$$-2\pi\big(\frac{1}{2} - t\big) \ < \ \frac{\partial f}{\partial X} \ < \ 2\pi(1-s).$$

Since $0.05 \le t$ and $0.25 \le s$,

$$-2\pi \cdot 0.45 \ < \ \frac{\partial f}{\partial X} \ < \ 2\pi \cdot 0.75.$$

Therefore, (92) is satisfied, as required.

As for (15) and (16), similarly as above, it is sufficient to show that

$$-(2\pi - \varepsilon') \ < \ \frac{\partial f}{\partial Y}(0, Y, 0, 0) \ < \ 2\pi - \varepsilon' \tag{93}$$

for some $\varepsilon' > 0$. Since $0.1 \le s - t \le 0.45$,

$$-2\pi\big(\frac{1}{2} - s + t\big) \ < \ \mathrm{Arg}\,\big(1 - \frac{y}{x}\big) \ < \ 0.$$

Further, since $0.55 \le s + u \le 0.9$,

$$-2\pi\big(s + u - \frac{1}{2}\big) \ < \ \mathrm{Arg}\,\big(1 - \frac{1}{yz}\big) \ < \ 0.$$

Hence, by (89),

$$-2\pi t \ < \ \frac{\partial f}{\partial Y} \ < \ 2\pi u.$$

Since $t \le 0.34$ and $u \le 0.48$,

$$-2\pi \cdot 0.34 \ < \ \frac{\partial f}{\partial Y} \ < \ 2\pi \cdot 0.48.$$

Therefore, (93) is satisfied, as required.

As for (17) and (18), similarly as above, it is sufficient to show that

$$-(2\pi - \varepsilon') \ < \ \frac{\partial f}{\partial Z}(0, 0, Z, 0) \ < \ 2\pi - \varepsilon' \tag{94}$$

for some $\varepsilon' > 0$. Since $0.15 \le u \le 0.48$,

$$-2\pi\big(\frac{1}{2} - u\big) \ < \ \mathrm{Arg}\,(1-z) \ < \ 0.$$

Further, since $0.55 \le s+u \le 0.9$,

$$-2\pi\big(s+u-\frac{1}{2}\big) \ < \ \mathrm{Arg}\,\big(1-\frac{1}{yz}\big) \ < \ 0.$$

Furthermore, since $0.25 \le u+v \le 0.93$,

$$\min\big\{-2\pi\big(\frac{1}{2}-u-v\big),\ 0\big\} \ < \ \mathrm{Arg}\,(1-zw) \ < \ \max\big\{0,\ 2\pi\big(u+v-\frac{1}{2}\big)\big\}.$$

Hence, by (90),

$$\begin{aligned}
\frac{\partial f}{\partial Z} \ &> \ \min\big\{-2\pi\big(\frac{1}{2}-u\big),\ -2\pi v\big\} \ \ge \ \min\big\{-2\pi\cdot 0.35,\ -2\pi\cdot 0.45\big\} \ = \ -2\pi\cdot 0.45, \\
\frac{\partial f}{\partial Z} \ &< \ \max\big\{2\pi\big(\frac{1}{2}+s-u-v\big),\ 2\pi s\big\} \ \le \ \max\big\{2\pi\cdot 0.89,\ 2\pi\cdot 0.64\big\} \ = \ 2\pi\cdot 0.89,
\end{aligned}$$

since $0.15 \le u,\ \ 0.1 \le v \le 0.45$ and $s \le 0.64$. Therefore, (94) is satisfied, as required.

As for (19) and (20), similarly as above, it is sufficient to show that

$$-(2\pi-\varepsilon') \ < \ \frac{\partial f}{\partial W}(0,0,0,W) \ < \ 2\pi-\varepsilon' \tag{95}$$

for some $\varepsilon' > 0$. Since $0.15 \le u \le 0.48$,

$$-2\pi\big(\frac{1}{2}-u\big) \ < \ \mathrm{Arg}\,(1-z) \ < \ 0.$$

Further, since $0.25 \le u+v \le 0.93$,

$$\min\big\{-2\pi\big(\frac{1}{2}-u-v\big),\ 0\big\} \ < \ \mathrm{Arg}\,(1-zw) \ < \ \max\big\{0,\ 2\pi\big(u+v-\frac{1}{2}\big)\big\}.$$

Hence, by (91),

$$\begin{aligned}
\frac{\partial f}{\partial W} \ &> \ \min\big\{-2\pi\big(\frac{1}{2}-v\big),\ -2\pi u\big\} \ \ge \ \min\big\{-2\pi\cdot 0.4,\ -2\pi\cdot 0.48\big\} \ = \ -2\pi\cdot 0.48, \\
\frac{\partial f}{\partial W} \ &< \ \max\big\{2\pi\big(1-u-2v\big),\ 2\pi\big(\frac{1}{2}-v\big)\big\} \ \le \ \max\big\{2\pi\cdot 0.65,\ 2\pi\cdot 0.4\big\} \ = \ 2\pi\cdot 0.65,
\end{aligned}$$

since $0.1 \le v$ and $0.15 \le u \le 0.48$. Therefore, (95) is satisfied, as required. □

6.5 Verifying the assumption of the saddle point method

In this section, we verify the assumption of the saddle point method in Lemma 6.6. In order to show this lemma, we show Lemmas 6.3–6.5 in advance.

Lemma 6.3. *In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$ satisfying that $s < \frac{1}{2}$ and $u+v > \frac{1}{2}$, $f \to \infty$ as $X^2+Y^2+Z^2+W^2 \to \infty$.*

Proof. By (11), $\frac{1}{2\pi} f$ is approximated by the following function,

$$\begin{aligned}
F(X,Y,Z,W) \ &= \ \left(\begin{cases} \big(\frac{1}{2}-t\big)X & \text{if } X \geq 0 \\ -\big(\frac{1}{2}-t\big)X & \text{if } X < 0 \end{cases} \right) \ + \ \left(\begin{cases} 0 & \text{if } Y \geq X \\ \big(s-t-\frac{1}{2}\big)(Y-X) & \text{if } Y < X \end{cases} \right) \\
&+ \big(\frac{1}{2}-s\big)Y \ - \left(\begin{cases} 0 & \text{if } Y+Z \geq 0 \\ \big(s+u-\frac{1}{2}\big)(Y+Z) & \text{if } Y+Z < 0 \end{cases} \right) \ + \left(\begin{cases} \big(\frac{1}{2}-u\big)Z & \text{if } Z \geq 0 \\ -\big(\frac{1}{2}-u\big)Z & \text{if } Z < 0 \end{cases} \right) \\
&- \ \left(\begin{cases} 0 & \text{if } Z+W \geq 0 \\ \big(u+v-\frac{1}{2}\big)(Z+W) & \text{if } Z+W < 0 \end{cases} \right) \ + \ \left(\begin{cases} \big(\frac{1}{2}-v\big)W & \text{if } W \geq 0 \\ -\big(\frac{1}{2}-v\big)W & \text{if } W < 0 \end{cases} \right).
\end{aligned}$$

We note that all summands of the right-hand side except for the third summand are non-negative. Further, the sum of the first three summands are rewritten,

$$\left(\begin{cases} \big(1-t-s\big)X & \text{if } X \geq 0 \\ -\big(s-t\big)X & \text{if } X < 0 \end{cases} \right) \ + \ \left(\begin{cases} \big(\frac{1}{2}-s\big)(Y-X) & \text{if } Y \geq X \\ -t\,(Y-X) & \text{if } Y < X \end{cases} \right).$$

Hence,

$$\begin{aligned}
F(X,Y,Z,W) \ &\geq \ \left(\begin{cases} \big(1-t-s\big)X & \text{if } X \geq 0 \\ -\big(s-t\big)X & \text{if } X < 0 \end{cases} \right) \ + \ \left(\begin{cases} \big(\frac{1}{2}-s\big)(Y-X) & \text{if } Y \geq X \\ -t\,(Y-X) & \text{if } Y < X \end{cases} \right) \\
&\quad + \ \left(\begin{cases} \big(\frac{1}{2}-u\big)Z & \text{if } Z \geq 0 \\ -\big(\frac{1}{2}-u\big)Z & \text{if } Z < 0 \end{cases} \right) \ + \ \left(\begin{cases} \big(\frac{1}{2}-v\big)W & \text{if } W \geq 0 \\ -\big(\frac{1}{2}-v\big)W & \text{if } W < 0 \end{cases} \right) \\
&\geq \ C\big(\,|X| + |X-Y| + |Z| + |W|\,\big),
\end{aligned}$$

for some constant $C > 0$. Therefore, we obtain the lemma. □

In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$ in the following two lemmas, depending on the sign of $u+v-\frac{1}{2}$.

Lemma 6.4. *In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$. There exists a convex neighborhood U of (t_0,s_0,u_0,v_0) such that the following* (1) *and* (2) *holds.*
(1) *If $(t,s,u,v) \in U$ and $u+v \geq \frac{1}{2}$, then f has a unique minimal point, and the flow goes there.*
(2) *If $(t,s,u,v) \notin U$ and $u+v \geq \frac{1}{2}$, then the flow goes to infinity.*

Proof. Similarly as the proof of Lemma 4.5, the Hesse matrix of f is calculated as

$$2\pi \begin{pmatrix} 2a_1+b_1 & -b_1 & 0 & 0 \\ -b_1 & b_1+b_2 & b_2 & 0 \\ 0 & b_2 & 2a_3+b_2+b_3 & b_3 \\ 0 & 0 & b_3 & 2a_4+b_3 \end{pmatrix},$$

where we put

$$\begin{aligned} a_1 &= \operatorname{Im}\frac{1}{1-x}\,, & a_3 &= \operatorname{Im}\frac{1}{1-z}\,, & a_4 &= \operatorname{Im}\frac{1}{1-w}\,, \\ b_1 &= \operatorname{Im}\frac{1}{1-\frac{y}{x}}\,, & b_2 &= \operatorname{Im}\frac{1}{1-\frac{1}{yz}}\,, & b_3 &= \operatorname{Im}\frac{-1}{1-zw}\,, \end{aligned}$$

noting that these numbers are positive. The above matrix is equivalent, as a quadratic form, to

$$2\pi \begin{pmatrix} 2a_1+b_1 & 0 & 0 & 0 \\ 0 & 2a_2+b_2 & b_2 & 0 \\ 0 & b_2 & 2a_3+b_2+b_3 & b_3 \\ 0 & 0 & b_3 & 2a_4+b_3 \end{pmatrix},$$

where we put $a_2 = a_1 b_1/(2a_1+b_1)$. Further, this matrix is equivalent, as a quadratic form, to

$$2\pi \begin{pmatrix} 2a_1+b_1 & 0 & 0 & 0 \\ 0 & 2a_2+b_2 & 0 & 0 \\ 0 & 0 & 2a_3'+2a_3+b_3 & b_3 \\ 0 & 0 & b_3 & 2a_4+b_3 \end{pmatrix},$$

where we put $a_3' = a_2 b_2/(2a_2+b_2)$. Furthermore, the following matrix is positive definite,

$$\begin{pmatrix} 2a_3+b_3 & b_3 \\ b_3 & 2a_4+b_3 \end{pmatrix},$$

since we can verify that its trace and determinant are positive. Therefore, the Hesse matrix of f is positive definite, and f is a convex function.

Hence, since $\frac{1}{2\pi}f$ is approximated by F as in the proof of Lemma 6.3, if

$$F(X,Y,Z,W) \to \infty \quad \text{as} \quad X^2+Y^2+Z^2+W^2 \to \infty, \tag{96}$$

then the conclusion of (1) holds, since f is convex. Otherwise, the conclusion of (2) holds. Therefore, we consider the condition of (96). Since F is piecewise linear, this condition can be presented by a system of inequalities of t, s, u, v of degree 1. Hence, U of the lemma is convex. Further, by Lemma 6.3, (96) holds in a neighborhood of (t_0, s_0, u_0, v_0). Therefore, we obtain the lemma. □

Lemma 6.5. *In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$. There exists a convex neighborhood U of (t_0, s_0, u_0, v_0) such that the following* (1) *and* (2) *holds.*
(1) *If $(t,s,u,v) \in U$ and $u+v < \frac{1}{2}$, then f has a unique minimal point, and the flow goes there.*
(2) *If $(t,s,u,v) \notin U$ and $u+v < \frac{1}{2}$, then the flow goes to infinity.*

Proof. Similarly as the proof of Lemma 6.4, the Hesse matrix of f is calculated as

$$2\pi \begin{pmatrix} 2a_1+b_1 & -b_1 & 0 & 0 \\ -b_1 & b_1+b_2 & b_2 & 0 \\ 0 & b_2 & 2a_3+b_2-b_3' & -b_3' \\ 0 & 0 & -b_3' & 2a_4-b_3' \end{pmatrix},$$

where we put a_1, a_3, a_4, b_1, b_2 as in the proof of Lemma 6.4, and put

$$b_3' = \operatorname{Im} \frac{1}{1-zw},$$

noting that these numbers are positive. We will show below that the flow of the lemma always goes to the domain that $Z \le 0$ and $W \le 0$, and that

$$\begin{pmatrix} 2a_3-b_3' & -b_3' \\ -b_3' & 2a_4-b_3' \end{pmatrix} \text{ is positive definite in this domain.} \tag{97}$$

Then, we can show the lemma, in the same way as the proof of Lemma 6.4.

We show that the flow of the lemma goes to the domain that $Z \le 0$ and $W \le 0$, as follows. When $Z > 0$, since $0.15 \le u \le 0.48$ and $0.25 \le u+v < \frac{1}{2}$, $\operatorname{Arg}(1-z) > -\pi\big(\frac{1}{2}-u\big)$ and $\operatorname{Arg}(1-zw) < 0$. Hence, by (90), $\frac{\partial f}{\partial Z} > 0$, and the flow goes in a direction decreasing Z. Further, when $W > 0$, we can similarly show that $\frac{\partial f}{\partial W} > 0$, and the flow goes in a direction decreasing W. Therefore, the flow goes to the domain that $Z \le 0$ and $W \le 0$.

We show (97), as follows. It is sufficient to show that

$$\big(\text{the trace of the matrix of (97)}\big) = 2\pi(a_3+a_3-b_3') > 0, \tag{98}$$

$$\big(\text{the determinant of the matrix of (97)}\big) = 2a_3a_4b_3'\big(\frac{2}{b_3'}-\frac{1}{a_3}-\frac{1}{a_4}\big) > 0. \tag{99}$$

We can show that (99) $\Rightarrow$ (98), in the same way as in the proof of Lemma 4.6. We show (99), as follows. Similarly as in the proof of Lemma 4.6, we have that

$$\frac{1}{a_3} = \frac{e^{2\pi Z}+e^{-2\pi Z}-2\cos 2\pi u}{\sin 2\pi u}, \qquad \frac{1}{a_4} = \frac{e^{2\pi W}+e^{-2\pi W}-2\cos 2\pi v}{\sin 2\pi v},$$
$$\frac{1}{b_3'} = \frac{e^{2\pi(Z+W)}+e^{-2\pi(Z+W)}-2\cos 2\pi(u+v)}{\sin 2\pi(u+v)}.$$

Hence, the differential of $\frac{2}{b_3'}-\frac{1}{a_3}-\frac{1}{a_4}$ with respect to Z is given by

$$\frac{1}{2\pi}\cdot\frac{\partial}{\partial Z}\big(\frac{2}{b_3'}-\frac{1}{a_3}-\frac{1}{a_4}\big) = 2\cdot\frac{e^{2\pi(Z+W)}-e^{-2\pi(Z+W)}}{\sin 2\pi(u+v)} - \frac{e^{2\pi Z}-e^{-2\pi Z}}{\sin 2\pi u}.$$

Since $0.15 \le u < \frac{1}{2}-v \le 0.4$, $\sin 2\pi u \le \sin(2\pi\cdot 0.1) = 0.587785...$. Hence, $2/\sin 2\pi(u+v) \ge 2 > 1/\sin 2\pi u$. Further, since $Z \le 0$ and $W \le 0$, $e^{2\pi(Z+W)}-e^{-2\pi(Z+W)} \le e^{2\pi Z}-e^{-2\pi Z} \le 0$. Hence, the above formula is non-positive. Therefore, it is sufficient to show (99) when $Z=0$. In a similar way, we can show that it is sufficient to show (99) when $W=0$. When $Z=W=0$, similarly as in the proof of Lemma 4.6, we have that

$$\frac{1}{2}\Big(\frac{2}{b'_3}-\frac{1}{a_3}-\frac{1}{a_4}\Big) \;=\; 2\tan\pi(u+v)-\tan\pi u-\tan\pi v \;>\; 0,$$

since $0<u<u+v<\frac{1}{2}$ and $0<v<u+v<\frac{1}{2}$. Hence, we obtain (99), as required. □

Lemma 6.6. *When we apply Proposition 2.4 to* (84)*, the assumption of Proposition 2.4 holds.*

Proof. We show that there exists a homotopy Δ'_δ $(0\le\delta\le 1)$ between $\Delta'_0=\Delta'$ and Δ'_1 such that

$$(t_c,s_c,u_c,v_c)\in\Delta'_1, \tag{100}$$

$$\Delta'_1-\{(t_c,s_c,u_c,v_c)\} \;\subset\; \big\{(t,s,u,v)\in\mathbb{C}^4 \;\big|\; \operatorname{Re}\hat{V}(t,s,u,v)<\varsigma_{R}\big\}, \tag{101}$$

$$\partial\Delta'_\delta \;\subset\; \big\{(t,s,u,v)\in\mathbb{C}^4 \;\big|\; \operatorname{Re}\hat{V}(t,s,u,v)<\varsigma_{R}\big\}. \tag{102}$$

In the fiber of the projection $\mathbb{C}^4\to\mathbb{R}^4$ at $(t,s,u,v)\in\Delta'$, we consider the flow from $(X,Y,Z,W)=(0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X},-\frac{\partial f}{\partial Y},-\frac{\partial f}{\partial Z},-\frac{\partial f}{\partial W})$. By Lemmas 6.3, 6.4 and 6.5, there exists a convex neighborhood U' of (t_0,s_0,u_0,v_0) such that the following (1) and (2) holds.

(1) If $(t,s,u,v)\in U'$, then f has a unique minimal point, and the flow goes there.

(2) If $(t,s,u,v)\notin U'$, then the flow goes to infinity.

We put the homotopy Δ'_δ in a similar way as in the proof of Lemma 4.7.

We can show (100), (101) and (101) by using Lemma 6.1 in a similar way as the proof of Lemma 3.9. □

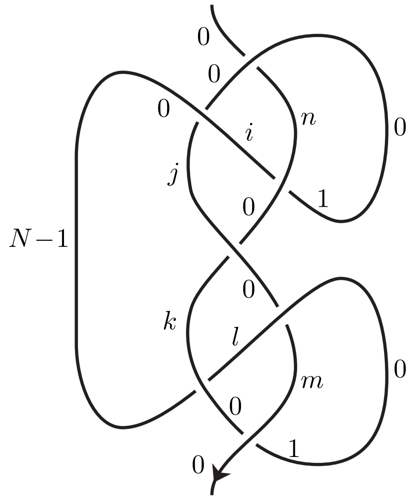
7 The 7_7 knot

In this section, we show Theorem 1.1 for the 7_7 knot. We give a proof of the theorem in Section 7.1, using lemmas shown in Sections 7.2–7.5.

7.1 Proof of Theorem 1.1 for the 7_7 knot

In this section, we show a proof of Theorem 1.1 for the 7_7 knot.

The 7_7 *knot* is the closure of the following tangle.



As shown in [32], we can put the labelings of edges adjacent to the unbounded regions as shown above. Hence, from the definition of the Kashaev invariant, the Kashaev invariant of the 7_7 knot is presented by

$$\begin{aligned}\langle\, 7_7 \,\rangle_{N} \;&=\; \sum q^{1/2} \times \frac{N\, q^{-\frac{1}{2}}}{(\overline{q})_{N-n}(q)_{n-1}} \times \frac{N\, q^{\frac{1}{2}-i}}{(q)_{N-i}(\overline{q})_{i-j-1}(q)_j} \times \frac{N\, q^{-\frac{1}{2}+i}}{(q)_{i-n}(\overline{q})_{n-1}(\overline{q})_{N-i}} \\ &\quad \times \frac{N\, q^{\frac{1}{2}}}{(\overline{q})_j(\overline{q})_{N-k-1}(q)_{k-j}} \times \frac{N\, q^{-\frac{1}{2}-l}}{(\overline{q})_{N-m}(q)_{m-l-1}(\overline{q})_l} \times \frac{N\, q^{\frac{1}{2}+l}}{(\overline{q})_{k-l}(q)_l(q)_{N-k-1}} \times \frac{N\, q^{-\frac{1}{2}}}{(q)_{N-m}(\overline{q})_{m-1}} \\ &= \sum_{\substack{0\le j<i\le N \\ 0\le l\le k<N \\ j\le k}} \frac{N^5}{(q)_{N-i}(\overline{q})_{N-i}(\overline{q})_{i-j-1}(q)_j(\overline{q})_j(q)_{k-j}(q)_{N-k-1}(\overline{q})_{N-k-1}(\overline{q})_{k-l}(q)_l(\overline{q})_l} \\ &= \sum_{\substack{0\le i,j,k,l \;\; i+j<N \\ j+k<N,\;\; k+l<N}} \frac{N^5}{(q)_i(\overline{q})_i(\overline{q})_{N-i-j-1}(q)_j(\overline{q})_j(q)_{N-j-k-1}(q)_k(\overline{q})_k(\overline{q})_{N-k-l-1}(q)_l(\overline{q})_l}\,,\end{aligned}$$

where we obtain the last equality by replacing i and k with $N-i$ and $N-k-1$ respectively.

Proof of Theorem 1.1 for the 7_7 *knot.* By (5), the above presentation of $\langle\, 7_7 \,\rangle_N$ is rewritten

$$\langle\, 7_7 \,\rangle_{N} \;=\; N^5 \sum_{\substack{0\le i,j,k,l \;\; i+j<N \\ j+k<N,\;\; k+l<N}} \exp\Big(N\, \tilde{V}\big(\frac{2i+1}{2N},\ \frac{2j+1}{2N},\ \frac{2k+1}{2N},\ \frac{2l+1}{2N}\big)\Big),$$

where we put

$$\begin{aligned}\tilde{V}(t,s,u,v) \;&=\; \frac{1}{N}\Big(\varphi(t) - \varphi(1-t) - \varphi\big(t+s-\frac{1}{2N}\big) + \varphi(s) - \varphi(1-s) \\ &\quad + \varphi\big(1-s-u+\frac{1}{2N}\big) + \varphi(u) - \varphi(1-u) - \varphi\big(u+v-\frac{1}{2N}\big) + \varphi(v) \\ &\quad - \varphi(1-v) - 5\,\varphi\big(\frac{1}{2N}\big) + 6\,\varphi\big(1-\frac{1}{2N}\big)\Big)\end{aligned}$$

$$= \frac{1}{N}\Big(2\,\varphi(t) - \varphi\big(t+s-\frac{1}{2N}\big) + 2\,\varphi(s) + \varphi\big(1-s-u+\frac{1}{2N}\big) + 2\,\varphi(u)$$
$$- \varphi\big(u+v-\frac{1}{2N}\big) + 2\,\varphi(v)\Big) + \frac{1}{2\pi\sqrt{-1}}\,\frac{\pi^2}{6} - \frac{11}{2N}\log N + \frac{\pi\sqrt{-1}}{4N} - \frac{\pi\sqrt{-1}}{12N^2}$$
$$+ 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t^2+s^2+u^2+v^2-t-s-u-v+\frac{2}{3}\big).$$

Here, we obtain the last equality by (9) and (10). Hence, by putting

$$V(t,s,u,v) \;=\; \tilde V(t,s,u,v) + \frac{11}{2N}\log N,$$

the presentation of $\langle\, 7_7 \,\rangle_N$ is rewritten

$$\langle\, 7_7 \,\rangle_N \;=\; N^{-1/2} \sum_{\substack{i,j,k,l\,\in\mathbb{Z} \\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})\in\Delta}} \exp\Big(N\,V\big(\frac{2i+1}{2N},\ \frac{2j+1}{2N},\ \frac{2k+1}{2N},\ \frac{2l+1}{2N}\big)\Big),$$

where the range of $(\frac{2i+1}{2N}, \frac{2j+1}{2N}, \frac{2k+1}{2N}, \frac{2l+1}{2N})$ of the sum is given by the following domain,

$$\Delta \;=\; \big\{(t,s,u,v)\in\mathbb{R}^4 \;\big|\; 0\le t,s,u,v,\quad t+s\le 1,\quad s+u\le 1,\quad u+v\le 1\big\}.$$

By Proposition 2.1, as $N\to\infty$, $V(t,s,u,v)$ converges to the following $\hat V(t,s,u,v)$ in the interior of Δ,

$$\hat V(t,s,u,v) \;=\; \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t}) - \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(t+s)}) + 2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,s})$$
$$+ \mathrm{Li}_2(e^{-2\pi\sqrt{-1}\,(s+u)}) + 2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,u}) - \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(u+v)}) + 2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,v}) + \frac{\pi^2}{6}\Big)$$
$$+ 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t^2+s^2+u^2+v^2-t-s-u-v+\frac{2}{3}\big).$$

By concrete calculation, we can check that the boundary of Δ is included in the domain

$$\big\{(t,s,u,v)\in\Delta \;\big|\; \mathrm{Re}\,\hat V(t,s,u,v) < \varsigma_R - \varepsilon\big\} \tag{103}$$

for some sufficiently small $\varepsilon>0$, where we put $\varsigma_R = 1.21648...$ as in (109); we will know later that this value is equal to the real part of the critical value of $\hat V$ at the critical point of Lemma 7.1. Hence, similarly as in Section 3.1, we choose a new domain Δ', which satisfies that $\Delta-\Delta'\subset(103)$, as

$$\Delta' \;=\; \left\{(t,s,u,v)\in\Delta \;\middle|\; \begin{array}{l} 0.1\le t\le 0.4,\quad 0.26\le s\le 0.45 \\ 0.26\le u\le 0.45,\quad 0.1\le v\le 0.4 \end{array}\right\}, \tag{104}$$

where we calculate the concrete values of the bounds of these inequalities in Section 7.2. Hence, since $\Delta-\Delta'\subset(103)$, we obtain the second equality of the following formula,

$$\langle\, 7_7 \,\rangle_N \;=\; e^{N\varsigma} N^{-1/2} \sum_{\substack{i,j,k,l\,\in\mathbb{Z} \\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})\in\Delta}} \exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N}\big) - N\varsigma\Big)$$

$$= e^{N\varsigma}\Bigg(N^{-1/2}\sum_{\substack{i,j,k,l\in\mathbb{Z}\\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})\in\Delta'}}\exp\Big(N\cdot V\big(\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N}\big)-N\varsigma\Big)\;+O(e^{-N\varepsilon})\Bigg)$$

for some $\varepsilon>0$.

Further, by Proposition 2.2 (Poisson summation formula), the above sum is presented by

$$\langle\, 7_7\,\rangle_N \;=\; e^{N\varsigma}\Bigg(N^{7/2}\int_{\Delta'}\exp\big(N\cdot V(t,s,u,v)-N\varsigma\big)\,dt\,ds\,du\,dv\;+O(e^{-N\varepsilon})\Bigg), \tag{105}$$

noting that we verify the assumption of Proposition 2.2 in Lemma 7.2. Furthermore, by Proposition 2.4 (saddle point method), there exist some κ_i''s such that

$$\langle\, 7_7\,\rangle_N \;=\; N^{7/2}\exp\big(N\cdot V(t_c,s_c,u_c,v_c)\big)\cdot\frac{(2\pi)^2}{N^2}\big(\det(-H)\big)^{-1/2}\Big(1+\sum_{i=1}^{d}\kappa_i'\hbar^i+O(\hbar^{d+1})\Big),$$

for any $d>0$, noting that we verify the assumption of Proposition 2.4 in Lemma 7.8. Here, (t_c,s_c,u_c,v_c) is the critical point of V which corresponds to the critical point (t_0,s_0,u_0,v_0) of $\hat{V}$ of Lemma 7.1, where $\hat{V}$ is the limit of V at $N\to\infty$ whose concrete presentation is given in Section 7.2, and H is the Hesse matrix of V at (t_c,s_c,u_c,v_c).

We calculate the right-hand side of the above formula. Similarly as in Section 3.1, we have that

$$V(t_0,s_0,u_0,v_0) \;=\; \varsigma+O(\hbar).$$

Therefore, there exist some κ_i's such that

$$\langle\, 7_7\,\rangle_N \;=\; e^{N\varsigma}N^{3/2}\,\omega\cdot\Big(1+\sum_{i=1}^{d}\kappa_i\hbar^i+O(\hbar^{d+1})\Big),$$

for any $d>0$. Hence, we obtain the theorem for the 7_7 knot. □

7.2 Estimate of the range of Δ'

In this section, we calculate the concrete values of the bounds of the inequalities in (104) so that they satisfy that $\Delta-\Delta'\subset$ (103).

Putting Λ as in Section 2.2, we have that

$$\operatorname{Re}\hat{V}(t,s,u,v) \;=\; 2\,\Lambda(t)-\Lambda(t+s)+2\,\Lambda(s)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v).$$

We consider the domain

$$\big\{(t,s,u,v)\in\Delta \;\big|\; 2\,\Lambda(t)-\Lambda(t+s)+2\,\Lambda(s)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)\;\geq\;\varsigma_R\big\}, \tag{106}$$

where we put $\varsigma_R=1.21648...$ as in (109). The aim of this section is to show that this domain is included in the interior of the domain Δ' of (104). For this purpose, we show the estimates of the defining inequalities of (104) for (t,s,u,v) in (106).

We calculate the minimal value $t_{\min}$ and the maximal value $t_{\max}$ of t. They are solutions of the system of the following equations,

$$\begin{cases} 2\,\Lambda(t) - \Lambda(t+s) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v) \;=\; \varsigma_{R}, \\ \dfrac{\partial}{\partial s}\big(2\,\Lambda(t) - \Lambda(t+s) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(t) - \Lambda(t+s) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial v}\big(2\,\Lambda(t) - \Lambda(t+s) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \;=\; 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of t (see Appendix A). By calculating a solution of these equations by Newton's method from $(t,s,u,v) = (0.2, 0.35, 0.35, 0.25)$, we obtain $t_{\min} = 0.173218...$, and from $(t,s,u,v) = (0.3, 0.35, 0.35, 0.25)$, we obtain $t_{\max} = 0.322858...$. Therefore, we obtain an estimate of t in Δ' as

$$0.1 \;\leq\; t \;\leq\; 0.4.$$

We calculate the minimal value $s_{\min}$ and the maximal value $s_{\max}$ of s. They are solutions of the system of the following equations,

$$\begin{cases} 2\,\Lambda(t) - \Lambda(t+s) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v) \;=\; \varsigma_{R}, \\ \dfrac{\partial}{\partial t}\big(2\,\Lambda(t) - \Lambda(t+s) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial u}\big(2\,\Lambda(t) - \Lambda(t+s) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \;=\; 0, \\ \dfrac{\partial}{\partial v}\big(2\,\Lambda(t) - \Lambda(t+s) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big) \;=\; 0. \end{cases}$$

We note that there are exactly two solutions of this system of equations corresponding to the maximal and minimal values of s (see Appendix A). By calculating a solution of these equations by Newton's method from $(t,s,u,v) = (0.25, 0.25, 0.35, 0.25)$, we obtain $s_{\min} = 0.264013...$, and from $(t,s,u,v) = (0.2, 0.4, 0.3, 0.25)$, we obtain $s_{\max} = 0.436051...$. Therefore, we obtain an estimate of s in Δ' as

$$0.26 \;\leq\; s \;\leq\; 0.45.$$

We obtain the estimates of u and v from the above estimates by the symmetry (107).

7.3 Calculation of the critical value

In this section, we calculate the concrete value of a critical point of $\hat{V}$.

The differentials of $\hat{V}$ are presented by

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{V}(t,s,u,v) &= -2\,\log(1-x)+\log(1-xy)+2\pi\sqrt{-1}\,\big(t-\frac{1}{2}\big),\\
\frac{\partial}{\partial s}\hat{V}(t,s,u,v) &= -2\,\log(1-y)+\log(1-xy)+\log\big(1-\frac{1}{yz}\big)+2\pi\sqrt{-1}\,\big(s-\frac{1}{2}\big),\\
\frac{\partial}{\partial u}\hat{V}(t,s,u,v) &= -2\,\log(1-z)+\log(1-zw)+\log\big(1-\frac{1}{yz}\big)+2\pi\sqrt{-1}\,\big(u-\frac{1}{2}\big),\\
\frac{\partial}{\partial v}\hat{V}(t,s,u,v) &= -2\,\log(1-w)+\log(1-zw)+2\pi\sqrt{-1}\,\big(v-\frac{1}{2}\big),
\end{aligned}$$

where $x=e^{2\pi\sqrt{-1}\,t}$, $y=e^{2\pi\sqrt{-1}\,s}$, $z=e^{2\pi\sqrt{-1}\,u}$ and $w=e^{2\pi\sqrt{-1}\,v}$.

Lemma 7.1. *$\hat{V}$ has a unique critical point (t_0,s_0,u_0,v_0) in $P^{-1}(\Delta')$, where $P:\mathbb{C}^4\to\mathbb{R}^4$ is the projection to the real parts of the entries.*

Proof. Any critical point of $\hat{V}$ is given by a solution of $\frac{\partial}{\partial t}\hat{V}=\frac{\partial}{\partial s}\hat{V}=\frac{\partial}{\partial u}\hat{V}=\frac{\partial}{\partial v}\hat{V}=0$, and these equations are rewritten,

$$\begin{aligned}
(1-x)^2 &= -x(1-xy),\\
(1-y)^2 &= -y(1-xy)\big(1-\frac{1}{yz}\big),\\
(1-z)^2 &= -z(1-zw)\big(1-\frac{1}{yz}\big),\\
(1-w)^2 &= -w(1-zw).
\end{aligned}$$

From the first formula, we have that $y=(x^2-x+1)/x^2$. Hence, from the second formula, we have that $z=x^3/\big((x-1)(x^2+1)\big)$. Further, from the third formula, we have that $w=(x^6-2x^5+5x^4-6x^3+5x^2-3x+1)/x^5$. By substituting these into the fourth formula, we have that

$$\big(2x^4-3x^3+3x^2-2x+1\big)\big(x^6-x^5+3x^4-4x^3+4x^2-3x+1\big) \;=\; 0.$$

Its solutions are

$$\begin{aligned}
x \;=\;& 0.0287264...\pm\sqrt{-1}\cdot 0.813859...\;,\quad 0.721274...\pm\sqrt{-1}\cdot 0.48342...\;,\\
&-0.377439...\pm\sqrt{-1}\cdot 1.47725...\;,\quad 0.232606...\pm\sqrt{-1}\cdot 0.943705...\;,\\
&0.644833...\pm\sqrt{-1}\cdot 0.198843...\;.
\end{aligned}$$

Among these, the first solution gives a solution in Δ', from which we have that

$$\begin{aligned}
x_0 &= 0.0287264...+\sqrt{-1}\cdot 0.813859...\;, & t_0 &= 0.244385...+\sqrt{-1}\cdot 0.0326818...\;,\\
y_0 &= -0.547424...+\sqrt{-1}\cdot 1.12087...\;, & s_0 &= 0.322307...-\sqrt{-1}\cdot 0.0351838...\;,\\
z_0 &= y_0, & u_0 &= s_0,\\
w_0 &= x_0, & v_0 &= t_0,
\end{aligned}$$

where $x_0=e^{2\pi\sqrt{-1}\,t_0}$, $y_0=e^{2\pi\sqrt{-1}\,s_0}$, $z_0=e^{2\pi\sqrt{-1}\,u_0}$ and $w_0=e^{2\pi\sqrt{-1}\,v_0}$. These give a unique critical point in $P^{-1}(\Delta')$. $\square$

We note that $\hat{V}$ and the set of critical points of $\hat{V}$ have the following symmetry,

$$(t,s,u,v) \;\longmapsto\; (v,u,s,t). \tag{107}$$

The critical value of $\hat{V}$ at the critical point of Lemma 7.1 is presented by

$$\begin{aligned}
\varsigma \;&=\; \hat{V}(t_0,s_0,u_0,v_0) \\
&=\; \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(x_0) - \mathrm{Li}_2(x_0y_0) + 2\,\mathrm{Li}_2(y_0) + \mathrm{Li}_2\big(\frac{1}{y_0z_0}\big) + 2\,\mathrm{Li}_2(z_0) - \mathrm{Li}_2(z_0w_0) \\
&\qquad + 2\,\mathrm{Li}_2(w_0) + \frac{\pi^2}{6}\Big) + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(t_0^2+s_0^2+u_0^2+v_0^2-t_0-s_0-u_0-v_0+\frac{2}{3}\big) \qquad (108)\\
&=\; 1.21648... - \sqrt{-1}\cdot 0.417787...\,.
\end{aligned}$$

Further, we put its real part to be ς_R,

$$\varsigma_R \;=\; \mathrm{Re}\,\varsigma \;=\; 1.21648... \tag{109}$$

7.4 Verifying the assumption of the Poisson summation formula

In this section, we verify the assumption of the Poisson summation formula in Lemma 7.2, which is used in the proof of Theorem 1.1 for the 7_7 knot in Section 7.1.

By computer calculation, we can see that the maximal value of $\mathrm{Re}\,\hat{V} - \varsigma_R$ is about 0.02. Therefore, in the proof of Lemma 7.2, it is sufficient to decrease, say, $\mathrm{Re}\,\hat{V}(t + \delta\sqrt{-1},\, s,\, u,\, v) - 2\pi\delta$ by 0.02, by moving δ (though we do not use this value in the proof of the lemma).

We put

$$f(X,Y,Z,W) \;=\; \mathrm{Re}\,\hat{V}(t+X\sqrt{-1},\, s+Y\sqrt{-1},\, u+Z\sqrt{-1},\, v+W\sqrt{-1}) - \varsigma_R\,.$$

Then, we have that

$$\frac{\partial f}{\partial X} \;=\; 2\,\mathrm{Arg}\,(1-x) - \mathrm{Arg}\,(1-xy) - 2\pi\big(t-\frac{1}{2}\big), \tag{110}$$

$$\frac{\partial f}{\partial Y} \;=\; 2\,\mathrm{Arg}\,(1-y) - \mathrm{Arg}\,(1-xy) - \mathrm{Arg}\,\big(1-\frac{1}{yz}\big) - 2\pi\big(s-\frac{1}{2}\big), \tag{111}$$

$$\frac{\partial f}{\partial Z} \;=\; 2\,\mathrm{Arg}\,(1-z) - \mathrm{Arg}\,(1-zw) - \mathrm{Arg}\,\big(1-\frac{1}{yz}\big) - 2\pi\big(u-\frac{1}{2}\big), \tag{112}$$

$$\frac{\partial f}{\partial W} \;=\; 2\,\mathrm{Arg}\,(1-w) - \mathrm{Arg}\,(1-zw) - 2\pi\big(v-\frac{1}{2}\big), \tag{113}$$

where $x = e^{2\pi\sqrt{-1}\,(t+X\sqrt{-1})}$, $y = e^{2\pi\sqrt{-1}\,(s+Y\sqrt{-1})}$, $z = e^{2\pi\sqrt{-1}\,(u+Z\sqrt{-1})}$ and $w = e^{2\pi\sqrt{-1}\,(v+W\sqrt{-1})}$.

Lemma 7.2. *$V(t,s,u,v) - \varsigma_R$ satisfies the assumption of Proposition 2.2.*

Proof. Since $V(t,s,u,v)$ converges uniformly to $\hat{V}(t,s,u,v)$ on Δ', we show the proof for $\hat{V}(t,s,u,v)$ instead of $V(t,s,u,v)$. We show that $\partial\Delta'$ is null-homotopic in each of (13)–(20).

As for (13) and (14), similarly as in the proof of Lemma 3.3, it is sufficient to show that

$$-(2\pi-\varepsilon') \ < \ \frac{\partial f}{\partial X}(X,0,0,0) \ < \ 2\pi-\varepsilon' \tag{114}$$

for some $\varepsilon' > 0$. Since $0.1 \le t \le 0.4$,

$$-2\pi\big(\frac{1}{2}-t\big) \ < \ \mathrm{Arg}\,(1-x) \ < \ 0.$$

Further, since $0.36 \le t+s \le 0.85$,

$$\min\big\{-2\pi\big(\frac{1}{2}-t-s\big),\ 0\big\} \ < \ \mathrm{Arg}\,(1-xy) \ < \ \max\big\{0,\ 2\pi\big(t+s-\frac{1}{2}\big)\big\}.$$

Hence, by (110),

$$\begin{aligned}\frac{\partial f}{\partial X} \ &> \ \min\big\{-2\pi\big(\frac{1}{2}-t\big),\ -2\pi\cdot s\big\} \ \ge \ \min\big\{-2\pi\cdot 0.4,\ -2\pi\cdot 0.45\big\} \ = \ -2\pi\cdot 0.45,\\ \frac{\partial f}{\partial X} \ &< \ \max\big\{2\pi(1-2t-s),\ 2\pi\big(\frac{1}{2}-t\big)\big\} \ \le \ \max\big\{2\pi\cdot 0.54,\ 2\pi\cdot 0.4\big\} \ = \ 2\pi\cdot 0.54,\end{aligned}$$

since $0.1 \le t$ and $0.26 \le s \le 0.45$. Therefore, (114) is satisfied, as required.

As for (15) and (16), similarly as above, it is sufficient to show that

$$-(2\pi-\varepsilon') \ < \ \frac{\partial f}{\partial Y}(0,Y,0,0) \ < \ 2\pi-\varepsilon' \tag{115}$$

for some $\varepsilon' > 0$. Since $0.26 \le s \le 0.45$,

$$-2\pi\big(\frac{1}{2}-s\big) \ < \ \mathrm{Arg}\,(1-y) \ < \ 0.$$

Further, since $0.36 \le t+s \le 0.85$,

$$\min\big\{-2\pi\big(\frac{1}{2}-t-s\big),\ 0\big\} \ < \ \mathrm{Arg}\,(1-xy) \ < \ \max\big\{0,\ 2\pi\big(t+s-\frac{1}{2}\big)\big\}.$$

Furthermore, since $0.52 \le s+u \le 0.9$,

$$-2\pi\big(s+u-\frac{1}{2}\big) \ < \ \mathrm{Arg}\,\big(1-\frac{1}{yz}\big) \ < \ 0.$$

Hence, by (111),

$$\begin{aligned}\frac{\partial f}{\partial Y} \ &> \ \min\big\{-2\pi\cdot u,\ -2\pi\big(t+s+u-\frac{1}{2}\big)\big\}\\ &\qquad\qquad \ge \ \min\big\{-2\pi\cdot 0.45,\ -2\pi\cdot 0.8\big\} \ = \ -2\pi\cdot 0.8,\\ \frac{\partial f}{\partial Y} \ &< \ \max\big\{2\pi(1-t-2s),\ 2\pi\big(\frac{1}{2}-s\big)\big\} \ \le \ \max\big\{2\pi\cdot 0.5,\ 2\pi\cdot 0.3\big\} \ = \ 2\pi\cdot 0.5,\end{aligned}$$

since $0.1 \le t$ and $0.26 \le s \le 0.45$. Therefore, (115) is satisfied, as required.

We obtain (17), (18), (19) and (20) from the above cases by the symmetry (107). □

7.5 Verifying the assumption of the saddle point method

In this section, we verify the assumption of the saddle point method in Lemma 7.8. In order to show this lemma, we show Lemmas 7.3–7.7 in advance.

Lemma 7.3. *In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$, $f \to \infty$ as $X^2+Y^2+Z^2+W^2 \to \infty$.*

Proof. By (11), $\frac{1}{2\pi} f$ is approximated by the following function,

$$F_1(X,Y) \;+\; \left(\begin{cases} 0 & \text{if } Y+Z \le 0 \\ \big(s+u-\frac{1}{2}\big)(Y+Z) & \text{if } Y+Z > 0 \end{cases} \right) \;+\; F_2(Z,W), \tag{116}$$

where we put

$$\begin{aligned} F_1(X,Y) \;&=\; \left(\begin{cases} \big(\frac{1}{2}-t\big)X & \text{if } X \ge 0 \\ -\big(\frac{1}{2}-t\big)X & \text{if } X < 0 \end{cases} \right) + \left(\begin{cases} \big(\frac{1}{2}-s\big)Y & \text{if } Y \ge 0 \\ -\big(\frac{1}{2}-s\big)Y & \text{if } Y < 0 \end{cases} \right) \\ &\quad - \left(\begin{cases} 0 & \text{if } X+Y \ge 0 \\ \big(t+s-\frac{1}{2}\big)(X+Y) & \text{if } X+Y < 0 \end{cases} \right), \\ F_2(Z,W) \;&=\; \left(\begin{cases} \big(\frac{1}{2}-u\big)Z & \text{if } Z \ge 0 \\ -\big(\frac{1}{2}-u\big)Z & \text{if } Z < 0 \end{cases} \right) + \left(\begin{cases} \big(\frac{1}{2}-v\big)W & \text{if } W \ge 0 \\ -\big(\frac{1}{2}-v\big)W & \text{if } W < 0 \end{cases} \right) \\ &\quad - \left(\begin{cases} 0 & \text{if } Z+W \ge 0 \\ \big(u+v-\frac{1}{2}\big)(Z+W) & \text{if } Z+W < 0 \end{cases} \right). \end{aligned}$$

Since the middle term of (116) is non-negative, it is sufficient to show that $F_1(X,Y) \to \infty$ as $X^2+Y^2 \to \infty$, and $F_2(Z,W) \to \infty$ as $Z^2+W^2 \to \infty$. By the symmetry (107), it is sufficient to show that

$$F_1(X,Y) \to \infty \quad \text{as } \; X^2+Y^2 \to \infty. \tag{117}$$

We show (117), as follows. When $X+Y \ge 0$,

$$F_1(X,Y) \;=\; \left(\begin{cases} \big(\frac{1}{2}-t\big)X & \text{if } X \ge 0 \\ -\big(\frac{1}{2}-t\big)X & \text{if } X < 0 \end{cases} \right) + \left(\begin{cases} \big(\frac{1}{2}-s\big)Y & \text{if } Y \ge 0 \\ -\big(\frac{1}{2}-s\big)Y & \text{if } Y < 0 \end{cases} \right),$$

and hence, (117) holds. When $X+Y < 0$,

$$\begin{aligned} F_1(X,Y) \;&=\; \left(\begin{cases} \big(\frac{1}{2}-t\big)X & \text{if } X \ge 0 \\ -\big(\frac{1}{2}-t\big)X & \text{if } X < 0 \end{cases} \right) + \left(\begin{cases} \big(\frac{1}{2}-s\big)Y & \text{if } Y \ge 0 \\ -\big(\frac{1}{2}-s\big)Y & \text{if } Y < 0 \end{cases} \right) \\ &\quad - \big(t+s-\frac{1}{2}\big)(X+Y) \\ &= \left(\begin{cases} (1-2t-s)X & \text{if } X \ge 0 \\ -s\,X & \text{if } X < 0 \end{cases} \right) + \left(\begin{cases} (1-t-2s)Y & \text{if } Y \ge 0 \\ -t\,Y & \text{if } Y < 0 \end{cases} \right). \end{aligned}$$

Hence, when $X < 0$ and $Y < 0$, (117) holds. Further, when $X > 0$ and $X + Y < 0$, $F_1(X,Y) = (1-t-s)X - t(X+Y)$, and hence, (117) holds. Furthermore, when $Y > 0$ and $X + Y < 0$, similarly, (117) holds. Therefore, (117) holds, as required. □

In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$ in the following four lemmas, depending on the signs of $t+s-\frac{1}{2}$ and $u+v-\frac{1}{2}$.

Lemma 7.4. *In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$ satisfying that $t+s \geq \frac{1}{2}$ and $u+v \geq \frac{1}{2}$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$. Then, f has a unique minimal point, and the flow goes there.*

Proof. Similarly as the proof of Lemma 4.5, the Hesse matrix of f is calculated as

$$2\pi \begin{pmatrix} 2a_1+b_1 & b_1 & 0 & 0 \\ b_1 & 2a_2+b_1+b_2 & b_2 & 0 \\ 0 & b_2 & 2a_3+b_2+b_3 & b_3 \\ 0 & 0 & b_3 & 2a_4+b_3 \end{pmatrix},$$

where we put

$$\begin{aligned} a_1 &= \operatorname{Im}\frac{1}{1-x}, \quad a_2 = \operatorname{Im}\frac{1}{1-y}, \quad a_3 = \operatorname{Im}\frac{1}{1-z}, \quad a_4 = \operatorname{Im}\frac{1}{1-w}, \\ b_1 &= \operatorname{Im}\frac{-1}{1-xy}, \quad b_2 = \operatorname{Im}\frac{1}{1-\frac{1}{yz}}, \quad b_3 = \operatorname{Im}\frac{-1}{1-zw}, \end{aligned}$$

noting that these numbers are positive. The above matrix is equivalent, as a quadratic form, to

$$2\pi \begin{pmatrix} 2a_1+b_1 & 0 & 0 & 0 \\ 0 & \frac{4a_1a_2+2a_1b_1+2a_2b_1}{2a_1+b_1}+b_2 & b_2 & 0 \\ 0 & b_2 & \frac{4a_3a_4+2a_3b_3+2a_4b_3}{2a_4+b_3}+b_2 & 0 \\ 0 & 0 & 0 & 2a_4+b_3 \end{pmatrix}.$$

Since we can verify that the trace and the determinant of the middle 2×2 submatrix are positive, the above matrix is positive definite. Hence, the Hesse matrix of f is positive definite, and f is a convex function. Further, since $f \to \infty$ at infinity by Lemma 7.3, f has a unique minimal point, and the flow of the lemma goes there, as required. □

Lemma 7.5. *In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$ satisfying that $t+s < \frac{1}{2}$ and $u+v \geq \frac{1}{2}$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$. Then, f has a unique minimal point, and the flow goes there.*

Proof. Similarly as the proof of Lemma 7.4, the Hesse matrix of f is presented by

$$2\pi \begin{pmatrix} 2a_1 - b_1' & -b_1' & 0 & 0 \\ -b_1' & 2a_2 - b_1' + b_2 & b_2 & 0 \\ 0 & b_2 & 2a_3 + b_2 + b_3 & b_3 \\ 0 & 0 & b_3 & 2a_4 + b_3 \end{pmatrix},$$

where we put $a_1, \cdots, a_4$, b_2 and b_3 as in the proof of Lemma 7.4, and put

$$b_1' \;=\; \mathrm{Im}\,\frac{1}{1 - xy},$$

noting that these numbers are positive. We will show below that the flow of the lemma always goes to the domain that $X \le 0$ and $Y \le 0$, and that

$$\begin{pmatrix} 2a_1 - b_1' & -b_1' \\ -b_1' & 2a_2 - b_1' \end{pmatrix} \text{ is positive definite in this domain.} \tag{118}$$

Then, we can show the lemma, in the same way as the proof of Lemma 7.4.

We show that the flow of the lemma goes to the domain that $X \le 0$ and $Y \le 0$, as follows. When $X > 0$, since $0.1 \le t \le 0.4$ and $0.36 \le t + s < \frac{1}{2}$, $\mathrm{Arg}\,(1 - x) > -\pi\big(\frac{1}{2} - t\big)$ and $\mathrm{Arg}\,(1 - xy) < 0$. Hence, by (110), $\frac{\partial f}{\partial X} > 0$, and the flow goes in a direction decreasing X. Further, when $Y > 0$, we can similarly show that $\frac{\partial f}{\partial Y} > 0$, and the flow goes in a direction decreasing Y. Therefore, the flow goes to the domain that $X \le 0$ and $Y \le 0$.

We show (118), as follows. It is sufficient to show that

$$\big(\text{the trace of the matrix of (118)}\big) \;=\; 2\pi(a_1 + a_2 - b') \;>\; 0, \tag{119}$$

$$\big(\text{the determinant of the matrix of (118)}\big) \;=\; 2a_1a_2b'\big(\frac{2}{b'} - \frac{1}{a_1} - \frac{1}{a_2}\big) \;>\; 0. \tag{120}$$

We can show that (120) $\Rightarrow$ (119), in the same way as in the proof of Lemma 4.6. We show (120), as follows. Similarly as in the proof of Lemma 4.6, we have that

$$\frac{1}{a_1} \;=\; \frac{e^{2\pi X} + e^{-2\pi X} - 2\,\cos 2\pi t}{\sin 2\pi t}, \qquad \frac{1}{a_2} \;=\; \frac{e^{2\pi Y} + e^{-2\pi Y} - 2\,\cos 2\pi s}{\sin 2\pi s},$$
$$\frac{1}{b'} \;=\; \frac{e^{2\pi(X+Y)} + e^{-2\pi(X+Y)} - 2\,\cos 2\pi(t+s)}{\sin 2\pi(t+s)}.$$

Hence, the differential of $\frac{2}{b'} - \frac{1}{a_1} - \frac{1}{a_2}$ with respect to X is given by

$$\frac{1}{2\pi} \cdot \frac{\partial}{\partial X}\big(\frac{2}{b'} - \frac{1}{a_1} - \frac{1}{a_2}\big) \;=\; 2 \cdot \frac{e^{2\pi(X+Y)} - e^{-2\pi(X+Y)}}{\sin 2\pi(t+s)} - \frac{e^{2\pi X} - e^{-2\pi X}}{\sin 2\pi t}.$$

Since $0.1 \le t \le 0.4$, $\sin 2\pi t \le \sin(2\pi \cdot 0.1) = 0.587785...$. Hence, $2/\sin 2\pi(t+s) \ge 2 > 1/\sin 2\pi t$. Further, since $X \le 0$ and $Y \le 0$, $e^{2\pi(X+Y)} - e^{-2\pi(X+Y)} \le e^{2\pi X} - e^{-2\pi X} \le 0$. Hence, the above formula is non-positive. Therefore, it is sufficient to show (120) when $X = 0$. In a similar way, we can show that it is sufficient to show (120) when $Y = 0$. When $X = Y = 0$, similarly as in the proof of Lemma 4.6, we have that

$$\frac{1}{2}\Big(\frac{2}{b'} - \frac{1}{a_1} - \frac{1}{a_2}\Big) \;=\; 2\,\tan \pi(t+s) - \tan \pi t - \tan \pi s \;>\; 0,$$

since $0 < t < t+s < \frac{1}{2}$ and $0 < s < t+s < \frac{1}{2}$. Hence, we obtain (120), as required. □

Lemma 7.6. *In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$ satisfying that $t+s \geq \frac{1}{2}$ and $u+v < \frac{1}{2}$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$. Then, f has a unique minimal point, and the flow goes there.*

Proof. We obtain the lemma from Lemma 7.5 by the symmetry (107). □

Lemma 7.7. *In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$ satisfying that $t+s < \frac{1}{2}$ and $u+v < \frac{1}{2}$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$. Then, f has a unique minimal point, and the flow goes there.*

Proof. Similarly as the proof of Lemma 7.4, the Hesse matrix of f is presented by

$$2\pi \begin{pmatrix} 2a_1 - b_1' & -b_1' & 0 & 0 \\ -b_1' & 2a_2 - b_1' + b_2 & b_2 & 0 \\ 0 & b_2 & 2a_3 + b_2 - b_3' & -b_3' \\ 0 & 0 & -b_3' & 2a_4 - b_3' \end{pmatrix},$$

where we put $a_1, \cdots, a_4$, b_1' and b_2 as in the proofs of Lemmas 7.4 and 7.5, and put

$$b_3' \;=\; \operatorname{Im} \frac{1}{1-zw},$$

noting that these numbers are positive. In a similar way as the proof of Lemma 7.5, we can show that the flow of the lemma always goes to the domain that $Z \leq 0$ and $W \leq 0$, and that

$$\begin{pmatrix} 2a_3 - b_3' & -b_3' \\ -b_3' & 2a_4 - b_3' \end{pmatrix} \text{ is positive definite in this domain.}$$

Hence, we can show the lemma, in the same way as the proofs of Lemmas 7.4 and 7.5. □

Lemma 7.8. *When we apply Proposition 2.4 to* (105)*, the assumption of Proposition 2.4 holds.*

Proof. We show that there exists a homotopy Δ'_δ $(0 \leq \delta \leq 1)$ between $\Delta'_0 = \Delta'$ and Δ'_1 such that

$$(t_c, s_c, u_c, v_c) \in \Delta'_1, \tag{121}$$

$$\Delta'_1 - \{(t_c, s_c, u_c, v_c)\} \;\subset\; \big\{(t,s,u,v) \in \mathbb{C}^4 \;\big|\; \operatorname{Re} \hat{V}(t,s,u,v) < \varsigma_R\big\}, \tag{122}$$

$$\partial \Delta'_\delta \;\subset\; \big\{(t,s,u,v) \in \mathbb{C}^4 \;\big|\; \operatorname{Re} \hat{V}(t,s,u,v) < \varsigma_R\big\}. \tag{123}$$

In the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at $(t,s,u,v) \in \Delta'$, we consider the flow from $(X,Y,Z,W) = (0,0,0,0)$ determined by the vector field $(-\frac{\partial f}{\partial X}, -\frac{\partial f}{\partial Y}, -\frac{\partial f}{\partial Z}, -\frac{\partial f}{\partial W})$. Then, by Lemmas 7.4, 7.5, 7.6 and 7.7, f has a unique minimal point, and the flow goes there.

We put $\mathbf{g}(t,s,u,v)$ to be this minimal point. We define the ending of the homotopy to be the set of these minimal points,

$$\Delta'_1 \;=\; \big\{(t,s,u,v) + \mathbf{g}(t,s,u,v)\sqrt{-1} \;\big|\; (t,s,u,v) \in \Delta'\big\}.$$

Further, we define the internal part of the homotopy by setting it along the flows.

We can show (121), (122) and (122) by using Lemma 7.1 in a similar way as the proof of Lemma 3.9. □

8 The 7_2 knot

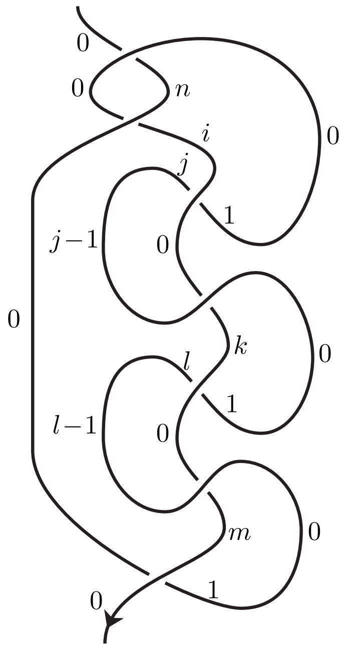
In this section, we show Theorem 1.1 for the 7_2 knot. We give a proof of the theorem in Section 8.1, using propositions and lemmas shown in Sections 8.2–8.6.

Unlike the cases of other knots, the boundary of the domain Δ of the integral is not included in the domain that $\operatorname{Re} V < \varsigma_R$ in this case. By this reason, we need many additional calculations when we use the Poisson summation formula and the saddle point method in the proof of the theorem in this section.

8.1 Proof of Theorem 1.1 for the 7_2 knot

In this section, we show a proof of Theorem 1.1 for the 7_2 knot.

Since the Kashaev invariant of the mirror image of a knot is equal to the complex conjugate of the Kashaev invariant of the original knot, it is sufficient to show the theorem for the mirror image $\overline{7_2}$ of the 7_2 knot. *The* $\overline{7_2}$ *knot* is the closure of the following tangle.



As shown in [32], we can put the labelings of edges adjacent to the unbounded regions as shown above. Hence, from the definition of the Kashaev invariant, the Kashaev invariant of the $\overline{7_2}$ knot is presented by

$$\begin{aligned}
\langle\, \overline{7_2}\,\rangle_{\scriptscriptstyle N} &= \sum q^{1/2} \times \frac{N\, q^{-\frac{1}{2}}}{(\overline{q})_{N-n}(q)_{n-1}} \times \frac{N\, q^{-\frac{1}{2}}}{(q)_{N-n}(\overline{q})_{n-i}(q)_{i-1}} \times \frac{N\, q^{-\frac{1}{2}+j}}{(q)_{j-i}(\overline{q})_{i-1}(\overline{q})_{N-j}} \\
&\quad \times \frac{N\, q^{-\frac{1}{2}-j+1}}{(\overline{q})_{N-k}(q)_{k-j}(\overline{q})_{j-1}} \times \frac{N\, q^{-\frac{1}{2}+l}}{(q)_{l-k}(\overline{q})_{k-1}(\overline{q})_{N-l}} \times \frac{N\, q^{-\frac{1}{2}-l+1}}{(\overline{q})_{N-m}(q)_{m-l}(\overline{q})_{l-1}} \times \frac{N\, q^{-\frac{1}{2}}}{(q)_{N-m}(\overline{q})_{m-1}} \\
&= \sum_{0<i\le j\le k\le l\le N} \frac{N^5\, q^{-1}}{(q)_{i-1}(\overline{q})_{i-1}(q)_{j-i}(\overline{q})_{j-1}(\overline{q})_{N-j}(q)_{k-j}(\overline{q})_{k-1}(\overline{q})_{N-k}(q)_{l-k}(\overline{q})_{l-1}(\overline{q})_{N-l}} \\
&= \sum_{0\le i\le j\le k\le l< N} \frac{N^5\, q^{-1}}{(q)_{i}(\overline{q})_{i}(q)_{j-i}(\overline{q})_{j}(\overline{q})_{N-j-1}(q)_{k-j}(\overline{q})_{k}(\overline{q})_{N-k-1}(q)_{l-k}(\overline{q})_{l}(\overline{q})_{N-l-1}} \\
&= \sum_{\substack{0\le i_1,\cdots,i_4 \\ i_1+\cdots+i_4<N}} \frac{N^5\, q^{-1}}{(q)_{i_1}(\overline{q})_{i_1}(q)_{i_2}(\overline{q})_{j_2}(\overline{q})_{N-j_2-1}(q)_{i_3}(\overline{q})_{j_3}(\overline{q})_{N-j_3-1}(q)_{i_4}(\overline{q})_{j_4}(\overline{q})_{N-j_4-1}}
\end{aligned}$$

where we obtain the third equality by replacing i, j, k, l with $i+1,\ j+1,\ k+1,\ l+1$ respectively, and obtain the last equality by putting $i_1 = i,\ \ i_2 = j,\ \ i_3 = k,\ \ i_4 = l,$ $j_1 = i_1,\ \ j_2 = i_1 + i_2,\ \ j_3 = i_1 + i_2 + i_3,\ \ j_4 = i_1 + \cdots + i_4$.

Proof of Theorem 1.1 for the $\overline{7_2}$ knot. By (5), the above presentation of $\langle\, \overline{7_2}\,\rangle_{\scriptscriptstyle N}$ is rewritten

$$\langle\, \overline{7_2}\,\rangle_{\scriptscriptstyle N} \ = \ N^5 q^{-1} \sum_{\substack{0\le i_1,\cdots,i_4 \\ i_1+\cdots+i_4<N}} \exp\Big(N\, \tilde{V}\big(\frac{2i_1+1}{2N},\ \frac{2i_2+1}{2N},\ \frac{2i_3+1}{2N},\ \frac{2i_4+1}{2N}\big)\Big),$$

where we put $\mathbf{t} = (t_1, \cdots, t_4),\ \ s_1 = t_1,\ \ s_2 = t_1 + t_2,\ \ s_3 = t_1 + t_2 + t_3,\ \ s_4 = t_1 + \cdots + t_4$ and

$$\begin{aligned}
\tilde{V}(\mathbf{t}) \ &= \ \frac{1}{N}\Big(\varphi(t_1) - \varphi(1-t_1) + \varphi(t_2) + \varphi(t_3) + \varphi(t_4) \\
&\quad - \varphi\big(s_2 - \frac{1}{2N}\big) - \varphi\big(1 - s_2 + \frac{1}{2N}\big) - \varphi\big(s_3 - \frac{1}{N}\big) - \varphi\big(1 - s_3 + \frac{1}{N}\big) \\
&\quad - \varphi\big(s_4 - \frac{3}{2N}\big) - \varphi\big(1 - s_4 + \frac{3}{2N}\big) - 4\,\varphi\big(\frac{1}{2N}\big) + 7\,\varphi\big(1 - \frac{1}{2N}\big)\Big) \\
&= \frac{1}{N}\Big(2\,\varphi(t_1) + \varphi(t_2) + \varphi(t_3) + \varphi(t_4)\Big) + \frac{1}{2\pi\sqrt{-1}} \cdot \frac{\pi^2}{2} - \frac{11}{2N}\log N + \frac{3\pi\sqrt{-1}}{4N} - \frac{\pi\sqrt{-1}}{4N^2} \\
&\quad + 2\pi\sqrt{-1} \cdot \frac{1}{2}\,\Big(s_1^2 + \big(s_2 - \frac{1}{2N}\big)^2 + \big(s_3 - \frac{1}{N}\big)^2 + \big(s_4 - \frac{3}{2N}\big)^2 - s_1 - s_2 - s_3 - s_4 + \frac{3}{N} + \frac{2}{3}\Big).
\end{aligned}$$

Here, we obtain the last equality by (9) and (10). Hence, by putting

$$V(\mathbf{t}) \ = \ \tilde{V}(\mathbf{t}) + \frac{11}{2N}\log N,$$

the presentation of $\langle\, \overline{7_2}\,\rangle_{\scriptscriptstyle N}$ is rewritten

$$\langle\, \overline{7_2}\,\rangle_{\scriptscriptstyle N} \ = \ N^{-1/2} q^{-1} \sum_{\substack{i,j,k,l\,\in\,\mathbb{Z} \\ (\frac{2i+1}{2N},\frac{2j+1}{2N},\frac{2k+1}{2N},\frac{2l+1}{2N})\in\Delta}} \exp\Big(N\, V\big(\frac{2i_1+1}{2N},\ \frac{2i_2+1}{2N},\ \frac{2i_3+1}{2N},\ \frac{2i_4+1}{2N}\big)\Big), \qquad (124)$$

where the range of $(\frac{2i_1+1}{2N}, \frac{2i_2+1}{2N}, \frac{2i_3+1}{2N}, \frac{2i_4+1}{2N})$ of the sum is given by the following domain,

$$\Delta \;=\; \big\{\mathbf{t} \in \mathbb{R}^4 \;\big|\; 0 \le\, t_1, \cdots, t_4 \,\le 1, \qquad t_1 + \cdots + t_4 \le 1 + \frac{1}{N}\big\}.$$

By Proposition 2.1, (6), as $N \to \infty$, $V(\mathbf{t})$ converges to the following $\hat{V}(\mathbf{t})$ in the interior of Δ,

$$\hat{V}(\mathbf{t}) \;=\; \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_1}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_2}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_3}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_4}) + \frac{\pi^2}{2}\Big)$$
$$+\, 2\pi\sqrt{-1}\cdot\frac{1}{2}\,\big(s_1^2 + s_2^2 + s_3^2 + s_4^2 - s_1 - s_2 - s_3 - s_4 + \frac{2}{3}\big).$$

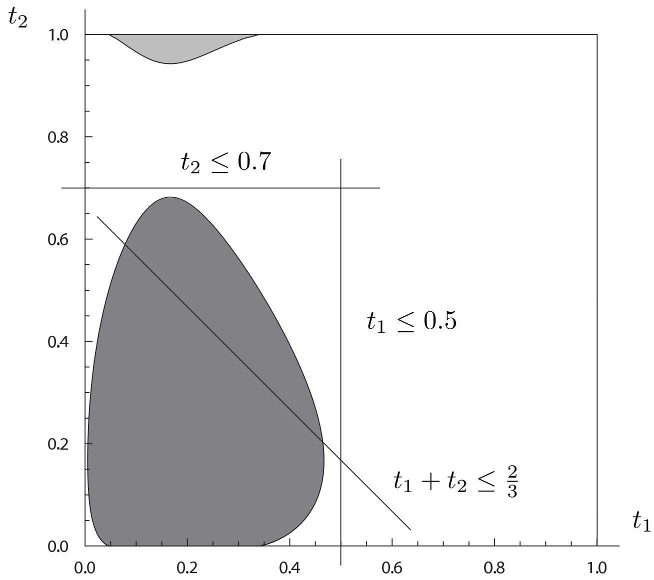


Figure 1: The dark gray domain is the image of the projection of the domain (126) to $\{(t_1, t_2) \in \mathbb{R}^2\}$

We change Δ in the following 4 steps; see Remark 8.1 below for the reason why we change Δ in such a way.

Step 1: We note that, unlike the cases of other hyperbolic knots with 7 crossings, some parts of the boundary of Δ is not included in the domain

$$\big\{\mathbf{t} \in \Delta \;\big|\; \mathrm{Re}\,\hat{V}(\mathbf{t}) < \varsigma_R - \varepsilon\big\} \tag{125}$$

for a sufficiently small $\varepsilon > 0$, where we put $\varsigma_R = 0.530263...$ as in (147), which is the real part of the critical value of $\hat{V}$ at the critical point of Lemma 8.18. To check this, we consider the domain

$$\big\{\mathbf{t} \in \Delta \;\big|\; \mathrm{Re}\,\hat{V}(\mathbf{t}) \ge \varsigma_R\big\}. \tag{126}$$

Putting $\Lambda(t)$ as in Section 2.2,

$$\mathrm{Re}\,\hat{V}(\mathbf{t}) \;=\; 2\,\Lambda(t_1) + \Lambda(t_2) + \Lambda(t_3) + \Lambda(t_4).$$

Hence, the domain (126) is symmetric with respect to the exchanges of t_2, t_3 and t_4. In Figure 1, we graphically show the image of the projection of the domain (126) to $\{(t_1, t_2) \in \mathbb{R}^2\}$ as the dark gray domain in the figure. Further, for each part of the boundary of Δ, we put

$$\partial\Delta \;=\; \partial_1\Delta \cup \partial_2\Delta \cup \partial_3\Delta \cup \partial_4\Delta \cup \partial_5\Delta,$$

where we put

$$\begin{aligned}\partial_k\Delta \;&=\; \big\{\mathbf{t}\in\partial\Delta \;\big|\; t_k = 0\big\} \qquad (k=1,2,3,4),\\ \partial_5\Delta \;&=\; \big\{\mathbf{t}\in\partial\Delta \;\big|\; t_1+t_2+t_3+t_4 = 1+\frac{1}{N}\big\}.\end{aligned}$$

By concrete calculation, we can check that $\partial_1\Delta$ is included in the domain (125) for a sufficiently small $\varepsilon > 0$, *i.e.*, we can graphically observe in Figure 1 that the line $\{t_1 = 0\}$ does not intersect with the dark gray domain in the figure. Hence, we can restrict the range of t_1 to $t_1 \geq \delta_1$ for a sufficiently small $\delta_1 > 0$; as shown in Section 8.2, we can choose δ_1 as $\delta_1 = 0.003$. Further, by concrete calculation, we can check that $\partial_2\Delta$ is not included in the domain (125), *i.e.*, we can graphically observe in Figure 1 that the line $\{t_2 = 0\}$ intersects with the dark gray domain in the figure. However, we note that the boundary of $\partial_2\Delta$ is included in the domain (125) by Lemma 8.2. In a neighborhood of $\partial_2\Delta$, by Proposition 8.3 in Section 8.2.1, the restriction of the sum (124) to the domain $t_2 < 0.003$ is sufficiently small. That is, though the summand of (124) itself is not sufficiently small on $\partial_2\Delta$, we can show by the Poisson summation formula and the saddle point method that the sum of the summand of (124) over $\partial_2\Delta$ is sufficiently small. Hence, we can restrict the range of t_2 to $t_2 \geq 0.003$. Similarly, in neighborhoods of $\partial_3\Delta$ and $\partial_4\Delta$, by Propositions 8.7 and 8.11 in Sections 8.2.2 and 8.2.3, we can restrict the ranges of t_3 and t_4 to $t_3 \geq 0.003$ and $t_4 \geq 0.003$. Further, we can check that $\partial_5\Delta$ is not included in the domain (125), *i.e.*, we can graphically observe in Figure 1 that the line $\{t_1 + t_2 = \frac{2}{3}\}$ intersects with the dark gray domain in the figure, noting that the maximal points of $\Lambda(t_3)$ and $\Lambda(t_4)$ are $t_3 = \frac{1}{6}$ and $t_4 = \frac{1}{6}$ respectively. $\big($We will extend the range of $t_1+t_2+t_3+t_4$ to $t_1+t_2+t_3+t_4 \leq 1.45$ by Proposition 8.15, later in Step 3, so that the new boundary of the extended domain is included in (125).$\big)$ Therefore, putting

$$\Delta'' \;=\; \big\{\mathbf{t}\in\mathbb{R}^4 \;\big|\; 0.003 \leq\, t_1,\cdots,t_4 \,\leq 1 \qquad t_1+\cdots+t_4 \leq 1+\frac{1}{N}\big\},$$

we have that

$$\langle\, \overline{7_2}\,\rangle_N \;=\; e^{N\varsigma}\Bigg(N^{-1/2}q^{-1}\sum_{\substack{i,j,k,l\,\in\,\mathbb{Z}\\ (\frac{2i_1+1}{2N},\frac{2i_2+1}{2N},\frac{2i_3+1}{2N},\frac{2i_4+1}{2N})\in\Delta''}}\exp\Big(N\cdot V\big(\frac{2i_1+1}{2N},\frac{2i_2+1}{2N},\frac{2i_3+1}{2N},\frac{2i_4+1}{2N}\big)-N\varsigma\Big)+O(e^{-N\varepsilon})\Bigg),$$

for a sufficiently small $\varepsilon > 0$. We note that, by Proposition 2.1, on Δ'', $V(\mathbf{t})$ uniformly converges to $\hat{V}(\mathbf{t})$ as $N \to \infty$.

Step 2: As shown in Section 8.2.4, we can restrict Δ'' to the domain $t_1 \leq 0.5$ and

$t_2, t_3, t_4 \leq 0.7$ in such a way that the removed part is included in (125). Hence, putting

$$\Delta''' = \left\{ \mathbf{t} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} 0.003 \leq t_1 \leq 0.5, \quad 0.003 \leq t_2, t_3, t_4 \leq 0.7 \\ t_1 + t_2 + t_3 + t_4 \leq 1 + \frac{1}{N} \end{array} \right\},$$

we have that

$$\langle \overline{7_2} \rangle_N = e^{N\varsigma} \Bigg(N^{-1/2} q^{-1} \sum_{\substack{i,j,k,l \in \mathbb{Z} \\ (\frac{2i_1+1}{2N}, \frac{2i_2+1}{2N}, \frac{2i_3+1}{2N}, \frac{2i_4+1}{2N}) \in \Delta'''}} \exp\Big(N \cdot V\big(\frac{2i_1+1}{2N}, \frac{2i_2+1}{2N}, \frac{2i_3+1}{2N}, \frac{2i_4+1}{2N}\big) - N\varsigma \Big) + O(e^{-N\varepsilon}) \Bigg) \tag{127}$$

for a sufficiently small $\varepsilon > 0$.

Step 3: By Proposition 8.15 in Section 8.2.5, we can extend Δ''' to the domain $t_1 + t_2 + t_3 + t_4 \leq 1.45$ so that the new boundary of the extended domain is included in (125). Hence, putting

$$\Delta'''' = \left\{ \mathbf{t} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} 0.003 \leq t_1 \leq 0.5, \quad 0.003 \leq t_2, t_3, t_4 \leq 0.7 \\ t_1 + t_2 + t_3 + t_4 \leq 1.45 \end{array} \right\},$$

we have that

$$\langle \overline{7_2} \rangle_N = e^{N\varsigma} \Bigg(N^{-1/2} q^{-1} \sum_{\substack{i,j,k,l \in \mathbb{Z} \\ (\frac{2i_1+1}{2N}, \frac{2i_2+1}{2N}, \frac{2i_3+1}{2N}, \frac{2i_4+1}{2N}) \in \Delta''''}} \exp\Big(N \cdot V\big(\frac{2i_1+1}{2N}, \frac{2i_2+1}{2N}, \frac{2i_3+1}{2N}, \frac{2i_4+1}{2N}\big) - N\varsigma \Big) + O(e^{-N\varepsilon}) \Bigg),$$

for a sufficiently small $\varepsilon > 0$. We note that, now, the domain (126) is included in the interior of Δ'''', and the boundary of Δ'''' is included in the domain (125), except for neighborhoods of $\partial_2\Delta$, $\partial_3\Delta$ and $\partial_4\Delta$.

Step 4: As shown in Section 8.2.6, we can restrict this domain to the domain $t_1 + t_2 \leq 0.9$, $t_1 + t_2 + t_3 \leq 1.2$ and $t_1 + t_2 + t_4 \leq 1.2$ in such a way that the removed part is included in (125). Therefore, putting

$$\Delta' = \left\{ \mathbf{t} \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} 0.003 \leq t_1 \leq 0.5, \quad 0.003 \leq t_2, t_3, t_4 \leq 0.7 \\ t_1 + t_2 + t_3 + t_4 \leq 1.45 \\ t_1 + t_2 \leq 0.9, \quad t_1 + t_2 + t_3 \leq 1.2, \quad t_1 + t_2 + t_4 \leq 1.2 \end{array} \right\},$$

we have that

$$\langle \overline{7_2} \rangle_N = e^{N\varsigma} \Bigg(N^{-1/2} q^{-1} \sum_{\substack{i,j,k,l \in \mathbb{Z} \\ (\frac{2i_1+1}{2N}, \frac{2i_2+1}{2N}, \frac{2i_3+1}{2N}, \frac{2i_4+1}{2N}) \in \Delta'}} \exp\Big(N \cdot V\big(\frac{2i_1+1}{2N}, \frac{2i_2+1}{2N}, \frac{2i_3+1}{2N}, \frac{2i_4+1}{2N}\big) - N\varsigma \Big) + O(e^{-N\varepsilon}) \Bigg),$$

for some sufficiently small $\varepsilon > 0$. We note that we need the restriction of this step when we apply the Poisson summation formula and the saddle point method later.

Further, by Proposition 8.19 (Poisson summation formula), the above sum is presented by

$$\langle\, \overline{7_2}\,\rangle_{N} \;=\; e^{N\varsigma}\Bigg(N^{7/2} q^{-1} \int_{\Delta'} \exp\big(N\cdot V(\mathbf{t}) - N\varsigma\big)\, d\mathbf{t} \;+ O(e^{-N\varepsilon})\Bigg). \tag{128}$$

Furthermore, by Proposition 2.4 (saddle point method), there exist some κ_i''s such that

$$\langle\, \overline{7_2}\,\rangle_{N} \;=\; N^{7/2} \exp\big(N\cdot V(\mathbf{t}_c)\big)\cdot \frac{(2\pi)^2}{N^2}\big(\det(-H)\big)^{-1/2}\Big(1+\sum_{i=1}^{d}\kappa_i'\hbar^i + O(\hbar^{d+1})\Big),$$

for any $d>0$, noting that we verify the assumption of Proposition 2.4 in Proposition 8.28. Here, $\mathbf{t}_c$ is the critical point of V which corresponds to the critical point $\mathbf{t}_0$ of $\hat{V}$ of Lemma 8.18, where $\hat{V}$ is the limit of V at $N\to\infty$ whose concrete presentation is given in Section 8.2, and H is the Hesse matrix of V at $\mathbf{t}_c$.

We calculate the right-hand side of the above formula. Since $\mathbf{t}_c = \mathbf{t}_0 + O(\hbar)$, we have that $V(\mathbf{t}_c) = V(\mathbf{t}_0) + O(\hbar^2)$. Hence, by comparing $V(\mathbf{t}_0)$ and $\hat{V}(\mathbf{t}_0)=\varsigma$, we have that

$$V(\mathbf{t}_0) \;=\; \varsigma + O(\hbar).$$

Therefore, there exist some κ_i's such that

$$\langle\, \overline{7_2}\,\rangle_{N} \;=\; e^{N\varsigma} N^{3/2}\,\omega\cdot\Big(1+\sum_{i=1}^{d}\kappa_i\hbar^i + O(\hbar^{d+1})\Big),$$

for any $d>0$. Hence, we obtain the theorem for the $\overline{7_2}$ knot. □

Remark 8.1. In the above proof of Theorem 1.1, we changed Δ in the 4 steps. In this remark, we explain the reason why we changed Δ in such a way.

In Step 1, we changed Δ to Δ'' in order that we can use Proposition 2.1 on Δ'' to show that $V(\mathbf{t})$ uniformly converges to $\hat{V}(\mathbf{t})$. Further, to apply the saddle point method later, it is a problem that the boundary $\{t_1+t_2+t_3+t_4 = 1+\frac{1}{N}\}$ of Δ'' intersects with the domain $\{\mathrm{Re}\,\hat{V}(\mathbf{t}) \ge \varsigma_R\}$, *i.e.*, we can graphically observe that the line $\{t_1+t_2=\frac{2}{3}\}$ intersects with the dark gray domain in Figure 1. To avoid this intersection, we want to move the boundary $\{t_1+t_2+t_3+t_4 = 1+\frac{1}{N}\}$ to $\{t_1+t_2+t_3+t_4 = 1.45\}$. If we moved this boundary from Δ'', the above mentioned line would intersect with the light gray domain in Figure 1 during we move it. Hence, before that, we restrict Δ'' to Δ''' in Step 2, and extend Δ''' to Δ'''' in Step 3 moving the boundary $\{t_1+t_2+t_3+t_4 = 1+\frac{1}{N}\}$ to $\{t_1+t_2+t_3+t_4 = 1.45\}$. After that, we restrict Δ'''' to Δ' in order that we can use the new defining inequality of Δ' in the proofs of the Poisson summation formula and the saddle point method later. This is the reason why we changed Δ in the above 4 steps.

8.2 Changing Δ to Δ'

In this section, we show that the change of the sum (124) is sufficiently small when we change the range of the sum from Δ to Δ'. We show that we can change Δ to Δ'' in

Sections 8.2.1–8.2.3, and can change Δ'' to Δ''' in Section 8.2.4, and can change Δ''' to Δ'''' in Section 8.2.5, and can change Δ'''' to Δ' in Section 8.2.6.

To consider the difference between Δ and Δ', we put

$$\Delta - \Delta' \;=\; \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$$

where we put

$$\Delta_k \;=\; \{\mathbf{t} \in \Delta \mid t_k < 0.003\} \quad (k = 1, 2, 3, 4).$$

Lemma 8.2.

$$\begin{aligned} \Delta_1 \;&\subset\; \{\mathbf{t} \in \Delta \mid \operatorname{Re} \hat{V}(\mathbf{t}) < \varsigma_R\}, \\ \Delta_i \cap \Delta_j \;&\subset\; \{\mathbf{t} \in \Delta \mid \operatorname{Re} \hat{V}(\mathbf{t}) < \varsigma_R\} \qquad i, j \in \{2, 3, 4\}. \end{aligned}$$

Proof. As mentioned before, $\operatorname{Re} \hat{V}(\mathbf{t})$ is presented by

$$\operatorname{Re} \hat{V}(\mathbf{t}) \;=\; 2\,\Lambda(t_1) + \Lambda(t_2) + \Lambda(t_3) + \Lambda(t_4).$$

We recall that the behavior of $\Lambda(t)$ is as mentioned in Section 2.2.

For any $\mathbf{t} \in \Delta_1$,

$$\begin{aligned} \operatorname{Re} \hat{V}(\mathbf{t}) - \varsigma_R \;&\le\; 3\,\Lambda\big(\frac{1}{6}\big) + 2\,\Lambda(0.003) - \varsigma_R \\ &=\; 3 \cdot 0.1615329... + 2 \cdot 0.0149138... - 0.530263... \\ &=\; -0.0158375... \;<\; 0. \end{aligned}$$

Hence, the first formula of the lemma holds.

For any $\mathbf{t} \in \Delta_i \cap \Delta_j$ $(i, j \in \{2, 3, 4\})$,

$$\begin{aligned} \operatorname{Re} \hat{V}(\mathbf{t}) - \varsigma_R \;&\le\; 3\,\Lambda\big(\frac{1}{6}\big) + 2\,\Lambda(0.003) - \varsigma_R \\ &=\; -0.0158375... \;<\; 0. \end{aligned}$$

Hence, the second formula of the lemma holds. □

Lemma 8.2 guarantees that, when we consider the change of the sum (124) on Δ and Δ'', it is sufficient to consider the partial sums of (124) on Δ_2, Δ_3, Δ_4 respectively. We show that such partial sums are sufficiently small in the following three subsections.

8.2.1 Restriction of the sum to $t_2 \le 0.003$

In this section, we show that we can restrict Δ to the domain $t_2 \le 0.003$. That is, the aim of this section is to show the following proposition.

Proposition 8.3. *The restriction of the sum* (124) *to the range* $i_2/N < 0.003$ *is estimated as follows,*

$$\sum_{\substack{0 \le i_1, \cdots, i_4 \\ i_1 + \cdots + i_4 < N \\ i_2/N < 0.003}} \exp\Big(N\,V\big(\frac{2i_1+1}{2N}, \frac{2i_2+1}{2N}, \frac{2i_3+1}{2N}, \frac{2i_4+1}{2N}\big)\Big) \;=\; O(e^{N(\varsigma_R - \varepsilon)}).$$

Proof. Fixing $i_2 < N \cdot 0.003$, it is sufficient to show that

$$\sum_{\substack{0 \le i_1, i_3, i_4 \\ i_1+i_3+i_4 < N(1-0.003)}} \exp\Big(N\, V\big(\frac{2i_1+1}{2N},\ \frac{2i_2+1}{2N},\ \frac{2i_3+1}{2N},\ \frac{2i_4+1}{2N}\big)\Big) \tag{129}$$

is of order $O(e^{N(\varsigma_R-\varepsilon)})$.

We can calculate (129) in a similar way as the proof of Theorem 1.1 in Section 8.1, by using the Poisson summation formula and the saddle point method, noting that the boundary of the range of the sum (129) is included in the domain $\{\mathbf{t} \mid \operatorname{Re} \hat{V}(\mathbf{t}) < \varsigma_R\}$ by Lemma 8.2. We show a sketch proof of this calculation in the following of this subsection. We fix $t_2 < 0.003$, and put

$$\begin{aligned} U_2(t_1,t_3,t_4) &= \hat{V}(t_1,t_2,t_3,t_4) \\ &= \frac{1}{2\pi\sqrt{-1}}\Big(2\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_1}) + \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_2}) + \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_3}) + \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_4}) + \frac{\pi^2}{2}\Big) \\ &\quad + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(s_1^2+s_2^2+s_3^2+s_4^2 - s_1 - s_2 - s_3 - s_4 + \frac{2}{3}\big), \end{aligned}$$

where we put $s_1 = t_1,\ \ s_2 = t_1+t_2,\ \ s_3 = t_1+t_2+t_3,\ \ s_4 = t_1+t_2+t_3+t_4$. Similarly as the proof of Theorem 1.1, we can restrict the domain of the sum (129) to

$$\Delta_2' = \left\{ (t_1,t_3,t_4) \ \middle| \ \begin{array}{l} 0.03 \le t_1 \le 0.4,\quad 0.005 \le t_3 \le 0.47,\quad 0.005 \le t_4 \le 0.47 \\ 0.1 \le t_1+t_3 \le 0.7,\quad 0.15 \le t_1+t_3+t_4 \le 0.95 \end{array} \right\},$$

where we omit concrete computations of the bounds of these inequalities. Hence, by the Poisson summation formula and Lemma 8.4, (129) is approximated by

$$\int_{\Delta_2'} e^{N\,U_2(t_1,t_3,t_4)}\, dt_1\, dt_3\, dt_4 \ + \ \int_{\Delta_2'} e^{N\,\big(U_2(t_1,t_3,t_4)+2\pi\sqrt{-1}\,t_1\big)}\, dt_1\, dt_3\, dt_4.$$

Further, by Lemmas 8.6 and 8.5, the first and second summands are of order $O(e^{N(\varsigma_R-\varepsilon)})$. Hence, we obtain the proposition. □

Lemma 8.4.

$$\sum_{(m_1,m_3,m_4)} \int_{\Delta_2'} \exp\Big(N\big(U_2(t_1,t_3,t_4) - 2\pi\sqrt{-1}\,(m_1t_1+m_3t_3+m_4t_4)\big)\Big)\, dt_1\, dt_3\, dt_4 \ = \ O(e^{N(\varsigma_R-\varepsilon)}),$$

where the sum runs over $(m_1,m_3,m_4) \in \mathbb{Z}^3 - \{(0,0,0),(-1,0,0)\}$.

Proof. We can show the lemma similarly as the proof of Proposition 2.2 (see [20]). In the case of this lemma, it is sufficient to show that

$$\text{when } m_4 \ne 0, \qquad -(2\pi-\varepsilon) \ < \ \operatorname{Re}\Big(\frac{\partial}{\partial\delta}\, U_2(t_1,t_3,t_4+\delta\sqrt{-1})\Big) \ < \ 2\pi-\varepsilon, \tag{130}$$

$$\text{when } m_3 \ne 0, \qquad -(2\pi-\varepsilon) \ < \ \operatorname{Re}\Big(\frac{\partial}{\partial\delta}\, U_2(t_1,t_3+\delta\sqrt{-1},t_4)\Big) \ < \ 2\pi-\varepsilon, \tag{131}$$

$$\text{when } m_1 \neq 0, -1, \quad -(2\pi - \varepsilon) \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\, U_2(t_1 + \delta\sqrt{-1}, t_3, t_4)\Big) \;<\; 4\pi - \varepsilon, \qquad (132)$$

for some $\varepsilon > 0$.

We show (130), as follows. The middle term is calculated as

$$\begin{aligned} \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\, U_2(t_1, t_3, t_4 + \delta\sqrt{-1})\Big) \;&=\; \mathrm{Re}\Big(\sqrt{-1}\cdot\frac{\partial}{\partial t_4}\, U_2(t_1, t_3, t_4 + \delta\sqrt{-1})\Big) \\ &=\; -\mathrm{Im}\Big(-\log(1 - x_4) + 2\pi\sqrt{-1}\big(s_4 - \frac{1}{2}\big)\Big) \\ &=\; \mathrm{Arg}\,(1 - x_4) - 2\pi\big(s_4 - \frac{1}{2}\big), \end{aligned}$$

where $x_4 = e^{2\pi\sqrt{-1}\,(t_4 + \delta\sqrt{-1})}$. Since $0 < t_4 < 0.5$,

$$-2\pi\big(\frac{1}{2} - t_4\big) \;<\; \mathrm{Arg}\,(1 - x_4) \;<\; 0.$$

Hence,

$$-2\pi s_3 \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\, U_2(t_1, t_3, t_4 + \delta\sqrt{-1})\Big) \;<\; 2\pi\big(\frac{1}{2} - s_4\big).$$

Therefore, since $s_3 \leq 0.7 + 0.003$ and $s_4 \geq 0.15$, (130) is satisfied.

We show (131), as follows. The middle term is calculated as

$$\mathrm{Re}\Big(\frac{\partial}{\partial\delta}\, U_2(t_1, t_3 + \delta\sqrt{-1}, t_4)\Big) \;=\; \mathrm{Arg}\,(1 - x_3) - 2\pi(s_3 + s_4 - 1),$$

where $x_3 = e^{2\pi\sqrt{-1}\,(t_3 + \delta\sqrt{-1})}$. Since $0 < t_3 < 0.5$,

$$-2\pi\big(\frac{1}{2} - t_3\big) \;<\; \mathrm{Arg}\,(1 - x_3) \;<\; 0.$$

Hence,

$$-2\pi\big(s_2 + s_4 - \frac{1}{2}\big) \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\, U_2(t_1, t_3 + \delta\sqrt{-1}, t_4)\Big) \;<\; 2\pi(1 - s_3 - s_4).$$

Therefore, since $s_2 \leq 0.4 + 0.003, \;\; s_4 \leq 0.95 + 0.003$ and $s_3 \geq 0.1, \;\; s_4 \geq 0.15$, (131) is satisfied.

We show (132), as follows. The middle term is calculated as

$$\mathrm{Re}\Big(\frac{\partial}{\partial\delta}\, U_2(t_1 + \delta\sqrt{-1}, t_3, t_4)\Big) \;=\; 2\,\mathrm{Arg}\,(1 - x_1) - 2\pi(s_1 + s_2 + s_3 + s_4 - 2),$$

where $x_1 = e^{2\pi\sqrt{-1}\,(t_1 + \delta\sqrt{-1})}$. Since $0 < t_1 < 0.5$,

$$-2\pi\big(\frac{1}{2} - t_1\big) \;<\; \mathrm{Arg}\,(1 - x_1) \;<\; 0.$$

Hence,

$$-2\pi(s_3 + s_4 - 1 + 0.003) \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\, U_2(t_1 + \delta\sqrt{-1}, t_3, t_4)\Big) \;<\; 2\pi(2 - s_1 - s_2 - s_3 - s_4).$$

Therefore, since $s_3 \leq 0.703, \;\; s_4 \leq 0.953$ and $s_i \geq 0, \;\;$ (132) is satisfied. □

Lemma 8.5.

$$\int_{\Delta_2'} e^{N\left(U_2(t_1,t_3,t_4)+2\pi\sqrt{-1}\,t_1\right)} dt_1\,dt_3\,dt_4 \;=\; O(e^{N\,\mathrm{Re}\,v_2}),$$

where $v_2 = 0.464948... - \sqrt{-1}\cdot 0.753795...$ *.*

Proof. We can show the lemma similarly as the proof of Theorem 1.1, by using the saddle point method. We show a sketch proof in this proof.

We put

$$U_2'(t_1,t_3,t_4) \;=\; U_2(t_1,t_3,t_4) + 2\pi\sqrt{-1}\,t_1,$$

and we fix $t_2 = 0.003$. The differentials of U_2' are presented by

$$\begin{aligned}\frac{\partial}{\partial t_1}U_2'(t_1,t_3,t_4) &= -2\log(1-x_1) + 2\pi\sqrt{-1}\,\big(s_1+s_2+s_3+s_4-1\big),\\ \frac{\partial}{\partial t_3}U_2'(t_1,t_3,t_4) &= -\log(1-x_3) + 2\pi\sqrt{-1}\,\big(s_3+s_4-1\big),\\ \frac{\partial}{\partial t_4}U_2'(t_1,t_3,t_4) &= -\log(1-x_4) + 2\pi\sqrt{-1}\,\big(s_4-\frac{1}{2}\big),\end{aligned}$$

where $x_k = e^{2\pi\sqrt{-1}\,t_k}$. Hence, any critical point of U_2' is a solution of the following equations,

$$\begin{aligned}(1-x_1)^2 &= e^{2\pi\sqrt{-1}\,(s_1+s_2+s_3+s_4-2)} = x_1^4\,x_2^3\,x_3^2\,x_4\,,\\ 1-x_3 &= e^{2\pi\sqrt{-1}\,(s_3+s_4-1)} = x_1^2\,x_2^2\,x_3^2\,x_4\,,\\ 1-x_4 &= e^{2\pi\sqrt{-1}\,(s_4-\frac{1}{2})} = -x_1\,x_2\,x_3\,x_4\,.\end{aligned}$$

By concrete calculation, it is shown that they have a unique solution on Δ_2', which is given by

$$\begin{aligned}x_1 &= 0.386143... + \sqrt{-1}\cdot 0.407062...\,, & t_1 &= 0.129196... + \sqrt{-1}\cdot 0.0919756...\,,\\ x_3 &= 2.6314... + \sqrt{-1}\cdot 0.555382...\,, & t_3 &= 0.0331053... - \sqrt{-1}\cdot 0.157453...\,,\\ x_4 &= 0.134251... + \sqrt{-1}\cdot 0.74488...\,, & t_4 &= 0.22162... + \sqrt{-1}\cdot 0.0443324...\,.\end{aligned}$$

Hence, the critical value of U_2' at this critical point is given by

$$v_2 \;=\; 0.464948... - \sqrt{-1}\cdot 0.753795...\,.$$

Therefore, by the saddle point method, we can show that the left-hand side of the formula of the lemma is of order $e^{N\,\mathrm{Re}\,v_2}$, and we obtain the lemma. □

Lemma 8.6.

$$\int_{\Delta_2'} e^{N\,U_2(t_1,t_3,t_4)}\,dt_1\,dt_3\,dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}).$$

Proof. By concrete calculation, we can show that, unlike Lemma 8.5, there is no critical point of U_2 on Δ_2'. Similarly as the proof of the saddle point method, we can show the lemma by moving Δ_2' in each fiber of the projection $\mathbb{C}^3 \to \mathbb{R}^3 \supset \Delta_2'$. We show a sketch proof in this proof. In the fiber at (t_1, t_3, t_4), we put

$$f(X, Z, W) \;=\; \operatorname{Re} U_2(t_1 + X\sqrt{-1}, t_3 + Z\sqrt{-1}, t_4 + W\sqrt{-1}) - \varsigma_R\,.$$

It is sufficient to show that $f \to -\infty$ as we move (X, Z, W) appropriately.

Regarding f as a function of X,

$$\frac{\partial f}{\partial X} \;=\; 2\operatorname{Arg}(1 - x_1) - 2\pi(s_1 + s_2 + s_3 + s_4 - 2),$$

where $x_1 = e^{2\pi\sqrt{-1}\,(t_1 + X\sqrt{-1})}$. Since $0 < t_1 < 0.5$,

$$-2\pi\big(\frac{1}{2} - t_1\big) \;<\; \operatorname{Arg}(1 - x_1) \;<\; 0.$$

Hence,

$$-2\pi(s_3 + s_4 - 1 + 0.003) \;<\; \frac{\partial f}{\partial X} \;<\; 2\pi(2 - s_1 - s_2 - s_3 - s_4).$$

Therefore, when $s_3 + s_4 < 1 - 0.003$, $\frac{\partial f}{\partial X} \geq \varepsilon'$ for some $\varepsilon' > 0$, and hence, $f \to -\infty$ as $X \to -\infty$.

Regarding f as a function of W,

$$\frac{\partial f}{\partial W} \;=\; \operatorname{Arg}(1 - x_4) - 2\pi\big(s_4 - \frac{1}{2}\big),$$

where $x_4 = e^{2\pi\sqrt{-1}\,(t_4 + W\sqrt{-1})}$. Since $0 < t_4 < 0.5$,

$$-2\pi\big(\frac{1}{2} - t_4\big) \;<\; \operatorname{Arg}(1 - x_4) \;<\; 0.$$

Hence,

$$-2\pi s_3 \;<\; \frac{\partial f}{\partial W} \;<\; 2\pi\big(\frac{1}{2} - s_4\big).$$

Therefore, when $s_4 > 0.5$, $\frac{\partial f}{\partial W} \leq -\varepsilon''$ for some $\varepsilon'' > 0$, and hence, $f \to -\infty$ as $W \to \infty$.

The remaining case is the case where $1 - 0.003 \leq s_3 + s_4$ and $s_4 \leq 0.5$. Since $s_3 + s_4 = 2s_4 - t_4 \leq 1 - t_4 \leq 1 - 0.005$, this case is the empty case. Therefore, we obtain the lemma. □

8.2.2 Restriction of the sum to $t_3 \leq 0.003$

In this section, we show that we can restrict Δ to the domain $t_3 \leq 0.003$. That is, the aim of this section is to show the following proposition.

Proposition 8.7. *The restriction of the sum* (124) *to the range* $i_3/N < 0.003$ *is estimated as follows,*

$$\sum_{\substack{0 \leq i_1, \cdots, i_4 \\ i_1 + \cdots + i_4 < N \\ i_3/N < 0.003}} \exp\Big(N\,V\big(\frac{2i_1+1}{2N}, \frac{2i_2+1}{2N}, \frac{2i_3+1}{2N}, \frac{2i_4+1}{2N}\big)\Big) \;=\; O(e^{N(\varsigma_R - \varepsilon)}).$$

Proof. Fixing $i_3 < N \cdot 0.003$, it is sufficient to show that

$$\sum_{\substack{0 \le i_1, i_2, i_4 \\ i_1+i_2+i_4 < N(1-0.003)}} \exp\Big(N\, V\big(\frac{2i_1+1}{2N},\ \frac{2i_2+1}{2N},\ \frac{2i_3+1}{2N},\ \frac{2i_4+1}{2N} \big) \Big) \tag{133}$$

is of order $O(e^{N(\varsigma_R - \varepsilon)})$.

We can calculate (133) in a similar way as the proof of Theorem 1.1 in Section 8.1, by using the Poisson summation formula and the saddle point method, noting that the boundary of the range of the sum (133) is included in the domain $\{ \mathbf{t} \mid \operatorname{Re} \hat{V}(\mathbf{t}) < \varsigma_R \}$ by Lemma 8.2. We show a sketch proof of this calculation in the following of this subsection. We fix $t_3 < 0.003$, and put

$$\begin{aligned} U_3(t_1,t_2,t_4) &= \hat{V}(t_1,t_2,t_3,t_4) \\ &= \frac{1}{2\pi\sqrt{-1}} \Big(2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_1}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_2}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_3}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_4}) + \frac{\pi^2}{2} \Big) \\ &\quad + 2\pi\sqrt{-1} \cdot \frac{1}{2} \big(s_1^2 + s_2^2 + s_3^2 + s_4^2 - s_1 - s_2 - s_3 - s_4 + \frac{2}{3} \big), \end{aligned}$$

where we put $s_1 = t_1,\ \ s_2 = t_1 + t_2,\ \ s_3 = t_1 + t_2 + t_3,\ \ s_4 = t_1 + t_2 + t_3 + t_4$. Similarly as the proof of Theorem 1.1, we can restrict the domain of the sum (133) to

$$\Delta_3' = \left\{ (t_1,t_2,t_4) \ \middle| \ \begin{array}{l} 0.03 \le t_1 \le 0.4,\quad 0.005 \le t_2 \le 0.47,\quad 0.005 \le t_4 \le 0.47 \\ 0.1 \le s_2 \le 0.7,\quad 0.15 \le s_4 \le 0.95,\quad t_1 + s_3 + s_4 \le 1.9 \end{array} \right\},$$

where we omit concrete computations of the bounds of these inequalities. Hence, by the Poisson summation formula and Lemma 8.8, (133) is approximated by

$$\int_{\Delta_3'} e^{N\, U_3(t_1,t_2,t_4)}\, dt_1\, dt_2\, dt_4 \ + \int_{\Delta_3'} e^{N\, \left(U_3(t_1,t_2,t_4) + 2\pi\sqrt{-1}\,(t_1+t_2) \right)}\, dt_1\, dt_2\, dt_4\,.$$

Further, by Lemmas 8.9 and 8.10, the first and second summands are bounded by $O(e^{N(\varsigma_R - \varepsilon)})$. Hence, we obtain the proposition. □

Lemma 8.8.

$$\sum_{(m_1,m_2,m_4)} \int_{\Delta_3'} \exp\Big(N\big(U_3(t_1,t_2,t_4) - 2\pi\sqrt{-1}\,(m_1 t_1 + m_2 t_2 + m_4 t_4) \big) \Big)\, dt_1\, dt_2\, dt_4 \ = \ O(e^{N(\varsigma_R - \varepsilon)}),$$

where the sum runs over $(m_1,m_2,m_4) \in \mathbb{Z}^3 - \{(0,0,0), (-1,-1,0)\}$.

Proof. We can show the lemma similarly as the proof of Lemma 8.4. In the case of this lemma, it is sufficient to show that

$$\text{when } m_4 \ne 0, \qquad -(2\pi - \varepsilon) \ < \ \operatorname{Re}\Big(\frac{\partial}{\partial \delta} U_3(t_1,t_2,t_4 + \delta\sqrt{-1}) \Big) \ < \ 2\pi - \varepsilon, \tag{134}$$

$$\text{when } m_2 \ne 0, -1, \quad -(2\pi - \varepsilon) \ < \ \operatorname{Re}\Big(\frac{\partial}{\partial \delta} U_3(t_1,t_2 + \delta\sqrt{-1},t_4) \Big) \ < \ 4\pi - \varepsilon, \tag{135}$$

$$\text{when } m_1 \neq m_2, \quad -(2\pi-\varepsilon) \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta} U_3(t_1+\delta\sqrt{-1}, t_2-\delta\sqrt{-1}, t_4)\Big) \;<\; 2\pi-\varepsilon, \tag{136}$$

for some $\varepsilon > 0$.

We can show (134) in the same way as in the proof of Lemma 8.4.

We show (135), as follows. The middle term is calculated as

$$\mathrm{Re}\Big(\frac{\partial}{\partial\delta} U_3(t_1, t_2+\delta\sqrt{-1}, t_4)\Big) \;=\; \mathrm{Arg}\,(1-x_2) - 2\pi\big(s_2+s_3+s_4-\frac{3}{2}\big),$$

where $x_2 = e^{2\pi\sqrt{-1}\,(t_2+\delta\sqrt{-1})}$. Since $0 < t_2 < 0.5$,

$$-2\pi\big(\frac{1}{2}-t_2\big) \;<\; \mathrm{Arg}\,(1-x_2) \;<\; 0.$$

Hence,

$$-2\pi(t_1+s_3+s_4-1) \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta} U_3(t_1, t_2+\delta\sqrt{-1}, t_4)\Big) \;<\; 2\pi\big(\frac{3}{2}-s_2-s_3-s_4\big).$$

Therefore, since $t_1+s_3+s_4 \leq 1.9$ and $0 \leq s_i$, (135) is satisfied.

We show (136), as follows. The middle term is calculated as

$$\mathrm{Re}\Big(\frac{\partial}{\partial\delta} U_3(t_1+\delta\sqrt{-1}, t_2-\delta\sqrt{-1}, t_4)\Big) \;=\; 2\,\mathrm{Arg}\,(1-x_1) - \mathrm{Arg}\,(1-x_2) - 2\pi\big(s_1-\frac{1}{2}\big),$$

where $x_1 = e^{2\pi\sqrt{-1}\,(t_1+\delta\sqrt{-1})}$ and $x_2 = e^{2\pi\sqrt{-1}\,(t_2-\delta\sqrt{-1})}$. Since $0 < t_1 < 0.5$,

$$-2\pi\big(\frac{1}{2}-t_1\big) \;<\; \mathrm{Arg}\,(1-x_1) \;<\; 0.$$

Hence,

$$-2\pi\big(\frac{1}{2}-t_1\big) \;<\; 2\,\mathrm{Arg}\,(1-x_1) - 2\pi\big(s_1-\frac{1}{2}\big) \;<\; 2\pi\big(\frac{1}{2}-t_1\big).$$

Further, since $-\pi < \mathrm{Arg}\,(1-x_2) < \pi$,

$$-2\pi(1-t_1) \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta} U_3(t_1+\delta\sqrt{-1}, t_2-\delta\sqrt{-1}, t_4)\Big) \;<\; 2\pi(1-t_1).$$

Therefore, since $t_1 \geq 0.03$, (136) is satisfied.

The remaining case is the case where $m_1 = m_2 = 0, -1$ and $m_4 = 0$. The concrete values of (m_1, m_2, m_4) are $(0,0,0), (-1,-1,0)$, which are excluded from the range of the sum of the formula of the lemma. Hence, we obtain the lemma. □

Lemma 8.9.

$$\int_{\Delta_3'} e^{N\,U_3(t_1,t_2,t_4)}\, dt_1\, dt_2\, dt_4 \;=\; O(e^{N\,\mathrm{Re}\, v_3}),$$

where $v_3 = 0.479418... - \sqrt{-1}\cdot 2.13181...$.

Proof. We can show the lemma similarly as the proof of Lemma 8.5 by using the saddle point method. We show a sketch proof in this proof.

We fix $t_3 = 0.003$. The differentials of U_3 are presented by

$$\begin{aligned}
\frac{\partial}{\partial t_1} U_3(t_1,t_2,t_4) &= -2\log(1-x_1) + 2\pi\sqrt{-1}\,\big(s_1+s_2+s_3+s_4-2\big), \\
\frac{\partial}{\partial t_2} U_3(t_1,t_2,t_4) &= -\log(1-x_2) + 2\pi\sqrt{-1}\,\big(s_2+s_3+s_4-\frac{3}{2}\big), \\
\frac{\partial}{\partial t_4} U_3(t_1,t_2,t_4) &= -\log(1-x_4) + 2\pi\sqrt{-1}\,\big(s_4-\frac{1}{2}\big),
\end{aligned}$$

where $x_k = e^{2\pi\sqrt{-1}\,t_k}$. Hence, any critical point of U_3 is a solution of the following equations,

$$\begin{aligned}
(1-x_1)^2 &= e^{2\pi\sqrt{-1}\,(s_1+s_2+s_3+s_4-2)} = x_1^4\,x_2^3\,x_3^2\,x_4\,, \\
1-x_2 &= e^{2\pi\sqrt{-1}\,(s_2+s_3+s_4-\frac{3}{2})} = -x_1^3\,x_2^3\,x_3^2\,x_4\,, \\
1-x_4 &= e^{2\pi\sqrt{-1}\,(s_4-\frac{1}{2})} = -x_1\,x_2\,x_3\,x_4\,.
\end{aligned}$$

By concrete calculation, it is shown that they have a unique solution on Δ_3', which is given by

$$\begin{aligned}
x_1 &= 0.65433... + \sqrt{-1}\cdot 1.49642...\,, & t_1 &= 0.184394... - \sqrt{-1}\cdot 0.0780744...\,, \\
x_2 &= -0.100364... + \sqrt{-1}\cdot 0.935416...\,, & t_2 &= 0.267011... + \sqrt{-1}\cdot 0.00971496... \\
x_4 &= 0.392143... + \sqrt{-1}\cdot 0.0688233...\,, & t_4 &= 0.027651... + \sqrt{-1}\cdot 0.146575...\,.
\end{aligned}$$

Hence, the critical value of U_3 at this critical point is given by

$$v_3 = 0.479418... - \sqrt{-1}\cdot 2.13181...\,.$$

Therefore, by the saddle point method, we can show that the left-hand side of the formula of the lemma is of order $e^{N\,\mathrm{Re}\,v_3}$, and we obtain the lemma. □

Lemma 8.10.

$$\int_{\Delta_3'} e^{N\,\big(U_3(t_1,t_2,t_4)+2\pi\sqrt{-1}\,(t_1+t_2)\big)}\,dt_1\,dt_2\,dt_4 = O(e^{N\,\mathrm{Re}\,v_3'}),$$

where $v_3' = 0.478116... - \sqrt{-1}\cdot 0.490192...\,.$

Proof. We can show the lemma similarly as the proof of Lemma 8.5 by using the saddle point method. We show a sketch proof in this proof.

We put

$$U_3'(t_1,t_2,t_4) = U_3(t_1,t_2,t_4) + 2\pi\sqrt{-1}\,(t_1+t_2),$$

and we fix $t_3 = 0.003$. The differentials of U'_3 are presented by

$$\begin{aligned}
\frac{\partial}{\partial t_1} U'_3(t_1,t_2,t_4) &= -2\log(1-x_1) + 2\pi\sqrt{-1}\,\big(s_1+s_2+s_3+s_4-2\big), \\
\frac{\partial}{\partial t_2} U'_3(t_1,t_2,t_4) &= -\log(1-x_2) + 2\pi\sqrt{-1}\,\big(s_2+s_3+s_4-\frac{3}{2}\big), \\
\frac{\partial}{\partial t_4} U'_3(t_1,t_2,t_4) &= -\log(1-x_4) + 2\pi\sqrt{-1}\,\big(s_4-\frac{1}{2}\big),
\end{aligned}$$

where $x_k = e^{2\pi\sqrt{-1}\,t_k}$. Hence, any critical point of U'_3 is a solution of the following equations,

$$\begin{aligned}
(1-x_1)^2 &= e^{2\pi\sqrt{-1}\,(s_1+s_2+s_3+s_4-2)} = x_1^4\,x_2^3\,x_3^2\,x_4\,, \\
1-x_2 &= e^{2\pi\sqrt{-1}\,(s_2+s_3+s_4-\frac{3}{2})} = -x_1^3\,x_2^3\,x_3^2\,x_4\,, \\
1-x_4 &= e^{2\pi\sqrt{-1}\,(s_4-\frac{1}{2})} = -x_1\,x_2\,x_3\,x_4\,.
\end{aligned}$$

By concrete calculation, it is shown that they have a unique solution on Δ'_3, which is given by

$$\begin{aligned}
x_1 &= 0.850268... + \sqrt{-1}\cdot 0.628312...\,, & t_1 &= 0.101286... - \sqrt{-1}\cdot 0.00885708...\,, \\
x_2 &= 0.610976... + \sqrt{-1}\cdot 0.0661802...\,, & t_2 &= 0.0171725... + \sqrt{-1}\cdot 0.077487...\,, \\
x_4 &= 1.09811... + \sqrt{-1}\cdot 0.929638...\,, & t_4 &= 0.111807... - \sqrt{-1}\cdot 0.0578993...\,.
\end{aligned}$$

Hence, the critical value of U'_3 at this critical point is given by

$$v'_3 = 0.478116... - \sqrt{-1}\cdot 0.490192...\,.$$

Therefore, by the saddle point method, we can show that the left-hand side of the formula of the lemma is of order $e^{N\,\mathrm{Re}\,v'_3}$, and we obtain the lemma. □

8.2.3 Restriction of the sum to $t_4 \le 0.003$

In this section, we show that we can restrict Δ to the domain $t_4 \le 0.003$. That is, the aim of this section is to show the following proposition.

Proposition 8.11. *The restriction of the sum* (124) *to the range $i_4/N < 0.003$ is estimated as follows,*

$$\sum_{\substack{0\le i_1,\cdots,i_4 \\ i_1+\cdots+i_4<N \\ i_4/N<0.003}} \exp\Big(N\,V\big(\frac{2i_1+1}{2N},\ \frac{2i_2+1}{2N},\ \frac{2i_3+1}{2N},\ \frac{2i_4+1}{2N}\big)\Big) = O(e^{N(\varsigma_R-\varepsilon)}).$$

Proof. Fixing $i_4 < N\cdot 0.003$, it is sufficient to show that

$$\sum_{\substack{0\le i_1,i_2,i_3 \\ i_1+i_2+i_3<N(1-0.003)}} \exp\Big(N\,V\big(\frac{2i_1+1}{2N},\ \frac{2i_2+1}{2N},\ \frac{2i_3+1}{2N},\ \frac{2i_4+1}{2N}\big)\Big) \tag{137}$$

is of order $O(e^{N(\varsigma_R-\varepsilon)})$.

We can calculate (137) in a similar way as the proof of Theorem 1.1 in Section 8.1, by using the Poisson summation formula and the saddle point method, noting that the boundary of the range of the sum (137) is included in the domain $\{\mathbf{t} \mid \operatorname{Re}\hat{V}(\mathbf{t}) < \varsigma_R\}$ by Lemma 8.2. We show a sketch proof of this calculation in the following of this subsection. We fix $t_4 < 0.003$, and put

$$\begin{aligned} U_4(t_1,t_2,t_3) &= \hat{V}(t_1,t_2,t_3,t_4) \\ &= \frac{1}{2\pi\sqrt{-1}}\Big(2\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_1}) + \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_2}) + \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_3}) + \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_4}) + \frac{\pi^2}{2}\Big) \\ &\quad + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big(s_1^2+s_2^2+s_3^2+s_4^2-s_1-s_2-s_3-s_4+\frac{2}{3}\big), \end{aligned}$$

where we put $s_1 = t_1,\ \ s_2 = t_1+t_2,\ \ s_3 = t_1+t_2+t_3,\ \ s_4 = t_1+t_2+t_3+t_4$. Similarly as the proof of Theorem 1.1, we can restrict the domain of the sum (137) to

$$\Delta_4' = \left\{ (t_1,t_2,t_3) \;\middle|\; \begin{array}{l} 0.03 \le t_1 \le 0.4,\quad 0.005 \le t_2 \le 0.47,\quad 0.005 \le t_3 \le 0.47 \\ 0.1 \le s_2 \le 0.7,\quad 0.15 \le s_3 \le 0.95,\quad s_2+s_3 \le 1.7 \end{array} \right\},$$

where we omit concrete computations of the bounds of these inequalities. Hence, by the Poisson summation formula and Lemma 8.12, (137) is approximated by

$$\int_{\Delta_4'} e^{N\,U_4(t_1,t_2,t_3)}\,dt_1\,dt_2\,dt_3 \;+\; \int_{\Delta_4'} e^{N\,\big(U_4(t_1,t_2,t_3)-2\pi\sqrt{-1}\,(t_1+t_2+t_3)\big)}\,dt_1\,dt_2\,dt_3\,.$$

Further, by Lemmas 8.13 and 8.14, the first and second summands are bounded by $O(e^{N(\varsigma_R-\varepsilon)})$. Hence, we obtain the proposition. □

Lemma 8.12.

$$\sum_{(m_1,m_2,m_3)} \int_{\Delta_4'} \exp\Big(N\big(U_4(t_1,t_2,t_3) - 2\pi\sqrt{-1}\,(m_1t_1+m_2t_2+m_3t_3)\big)\Big)\,dt_1\,dt_2\,dt_3 \;=\; O(e^{N(\varsigma_R-\varepsilon)}),$$

where the sum runs over $(m_1,m_2,m_3) \in \mathbb{Z}^3 - \{(0,0,0),(1,1,1)\}$.

Proof. We can show the lemma similarly as the proof of Lemma 8.4. In the case of this lemma, it is sufficient to show that

$$\text{when } m_3 \ne 0,1, \quad -(4\pi-\varepsilon) \;<\; \operatorname{Re}\Big(\frac{\partial}{\partial\delta}U_4(t_1,t_2,t_3+\delta\sqrt{-1})\Big) \;<\; 2\pi-\varepsilon, \tag{138}$$

$$\text{when } m_1 \ne m_2, \quad -(2\pi-\varepsilon) \;<\; \operatorname{Re}\Big(\frac{\partial}{\partial\delta}U_4(t_1+\delta\sqrt{-1},t_2-\delta\sqrt{-1},t_3)\Big) \;<\; 2\pi-\varepsilon \tag{139}$$

$$\text{when } m_2 \ne m_3, \quad -(2\pi-\varepsilon) \;<\; \operatorname{Re}\Big(\frac{\partial}{\partial\delta}U_4(t_1,t_2+\delta\sqrt{-1},t_3-\delta\sqrt{-1})\Big) \;<\; 2\pi-\varepsilon, \tag{140}$$

for some $\varepsilon > 0$.

We show (138), as follows. The middle term is calculated as

$$\operatorname{Re}\Big(\frac{\partial}{\partial\delta}U_4(t_1,t_2,t_3+\delta\sqrt{-1})\Big) \;=\; \operatorname{Arg}(1-x_3)-2\pi(s_3+s_4-1),$$

where $x_3=e^{2\pi\sqrt{-1}\,(t_3+\delta\sqrt{-1})}$. Since $0<t_3<0.5$,

$$-2\pi\big(\frac{1}{2}-t_3\big) \;<\; \operatorname{Arg}(1-x_3) \;<\; 0.$$

Hence,

$$-2\pi\big(s_2+s_4-\frac{1}{2}\big) \;<\; \operatorname{Re}\Big(\frac{\partial}{\partial\delta}U_4(t_1,t_2,t_3+\delta\sqrt{-1})\Big) \;<\; 2\pi(1-s_3-s_4).$$

Therefore, since $s_2+s_4\le 1.7+0.003$ and $0\le s_i,$ (138) is satisfied.

We can show (139) in the same way as in the proof of Lemma 8.8.

We show (140), as follows. The middle term is calculated as

$$\operatorname{Re}\Big(\frac{\partial}{\partial\delta}U_4(t_1,t_2+\delta\sqrt{-1},t_3-\delta\sqrt{-1})\Big) \;=\; \operatorname{Arg}(1-x_2)-\operatorname{Arg}(1-x_3)-2\pi\big(s_2-\frac{1}{2}\big),$$

where $x_2=e^{2\pi\sqrt{-1}\,(t_2+\delta\sqrt{-1})}$ and $x_3=e^{2\pi\sqrt{-1}\,(t_3-\delta\sqrt{-1})}$. Since $0<t_2<0.5$,

$$-2\pi\big(\frac{1}{2}-t_2\big) \;<\; \operatorname{Arg}(1-x_2) \;<\; 0.$$

Hence,

$$-2\pi\cdot 0.4 \;\le\; -2\pi\cdot t_1 \;<\; \operatorname{Arg}(1-x_2)-2\pi\big(s_2-\frac{1}{2}\big) \;<\; 2\pi\big(\frac{1}{2}-s_2\big) \;\le\; 2\pi\cdot 0.4.$$

Further, since $-\pi<\operatorname{Arg}(1-x_3)<\pi$,

$$-2\pi\cdot 0.9 \;<\; \operatorname{Re}\Big(\frac{\partial}{\partial\delta}U_4(t_1,t_2+\delta\sqrt{-1},t_3-\delta\sqrt{-1})\Big) \;<\; 2\pi\cdot 0.9.$$

Therefore, (140) is satisfied.

The remaining case is the case where $m_1=m_2=m_3=0,1$. The concrete values of (m_1,m_2,m_3) are $(0,0,0),(1,1,1)$, which are excluded from the range of the sum of the lemma. Hence, we obtain the lemma. □

Lemma 8.13.

$$\int_{\Delta'_4} e^{N\,U_4(t_1,t_2,t_3)}\,dt_1\,dt_2\,dt_3 \;=\; O(e^{N\,\operatorname{Re}v_4}),$$

where $v_4=0.525499...-\sqrt{-1}\cdot 2.12671...$.

Proof. We can show the lemma similarly as the proof of Lemma 8.5 by using the saddle point method. We show a sketch proof in this proof.

We fix $t_4 = 0.003$. The differentials of U_4 are presented by

$$\begin{aligned}
\frac{\partial}{\partial t_1} U_4(t_1, t_2, t_3) &= -2\log(1 - x_1) + 2\pi\sqrt{-1}\,\big(s_1 + s_2 + s_3 + s_4 - 2\big),\\
\frac{\partial}{\partial t_2} U_4(t_1, t_2, t_3) &= -\log(1 - x_2) + 2\pi\sqrt{-1}\,\big(s_2 + s_3 + s_4 - \frac{3}{2}\big),\\
\frac{\partial}{\partial t_3} U_4(t_1, t_2, t_3) &= -\log(1 - x_3) + 2\pi\sqrt{-1}\,\big(s_3 + s_4 - 1\big),
\end{aligned}$$

where $x_k = e^{2\pi\sqrt{-1}\,t_k}$. Hence, any critical point of U_4 is a solution of the following equations,

$$\begin{aligned}
(1 - x_1)^2 &= e^{2\pi\sqrt{-1}\,(s_1+s_2+s_3+s_4-2)} = x_1^4\, x_2^3\, x_3^2\, x_4\,,\\
1 - x_2 &= e^{2\pi\sqrt{-1}\,(s_2+s_3+s_4-\frac{3}{2})} = -x_1^3\, x_2^3\, x_3^2\, x_4\,,\\
1 - x_3 &= e^{2\pi\sqrt{-1}\,(s_3+s_4-1)} = x_1^2\, x_2^2\, x_3^2\, x_4\,.
\end{aligned}$$

By concrete calculation, it is shown that they have a unique solution on Δ_4', which is given by

$$\begin{aligned}
x_1 &= 0.889267... + \sqrt{-1}\cdot 1.60022...\,, & t_1 &= 0.169273... - \sqrt{-1}\cdot 0.0962416...\,,\\
x_2 &= 0.154601... + \sqrt{-1}\cdot 1.12276...\,, & t_2 &= 0.228222... - \sqrt{-1}\cdot 0.0199232...\,,\\
x_3 &= 0.349254... + \sqrt{-1}\cdot 0.18807...\,, & t_3 &= 0.0786168... + \sqrt{-1}\cdot 0.147162...\,.
\end{aligned}$$

Hence, the critical value of U_4 at this critical point is given by

$$v_4 = 0.525499... - \sqrt{-1}\cdot 2.12671...\,.$$

Therefore, by the saddle point method, we can show that the left-hand side of the formula of the lemma is of order $e^{N\,\mathrm{Re}\,v_4}$, and we obtain the lemma. □

Lemma 8.14.

$$\int_{\Delta_4'} e^{N\,\big(U_4(t_1,t_2,t_3) - 2\pi\sqrt{-1}\,(t_1+t_2+t_3)\big)}\,dt_1\,dt_2\,dt_3 = O(e^{N\,\mathrm{Re}\,v_4'}),$$

where $v_4' = 0.488837... - \sqrt{-1}\cdot 6.50435...$.

Proof. We can show the lemma similarly as the proof of Lemma 8.5 by using the saddle point method. We show a sketch proof in this proof.

We put

$$U_4'(t_1, t_2, t_3) = U_4(t_1, t_2, t_3) - 2\pi\sqrt{-1}\,(t_1 + t_2 + t_3),$$

and we fix $t_4 = 0.003$. The differentials of U_4' are presented by

$$\begin{aligned}
\frac{\partial}{\partial t_1} U_4(t_1, t_2, t_3) &= -2\log(1 - x_1) + 2\pi\sqrt{-1}\,\big(s_1 + s_2 + s_3 + s_4 - 2\big),\\
\frac{\partial}{\partial t_2} U_4(t_1, t_2, t_3) &= -\log(1 - x_2) + 2\pi\sqrt{-1}\,\big(s_2 + s_3 + s_4 - \frac{3}{2}\big),\\
\frac{\partial}{\partial t_3} U_4(t_1, t_2, t_3) &= -\log(1 - x_3) + 2\pi\sqrt{-1}\,\big(s_3 + s_4 - 1\big),
\end{aligned}$$

where $x_k = e^{2\pi\sqrt{-1}\,t_k}$. Hence, any critical point of U_4' is a solution of the following equations,

$$\begin{aligned}(1-x_1)^2 &= e^{2\pi\sqrt{-1}\,(s_1+s_2+s_3+s_4-2)} = x_1^4\,x_2^3\,x_3^2\,x_4\,,\\ 1-x_2 &= e^{2\pi\sqrt{-1}\,(s_2+s_3+s_4-\frac{3}{2})} = -x_1^3\,x_2^3\,x_3^2\,x_4\,,\\ 1-x_3 &= e^{2\pi\sqrt{-1}\,(s_3+s_4-1)} = x_1^2\,x_2^2\,x_3^2\,x_4\,.\end{aligned}$$

By concrete calculation, it is shown that they have a unique solution on Δ_4', which is given by

$$\begin{aligned}x_1 &= 0.215852... + \sqrt{-1}\cdot 1.09531...\,, & t_1 &= 0.219032... - \sqrt{-1}\cdot 0.0175211...\,,\\ x_2 &= -0.610953... + \sqrt{-1}\cdot 0.216459...\,, & t_2 &= 0.445808... + \sqrt{-1}\cdot 0.0690111...\,,\\ x_3 &= 0.122125... + \sqrt{-1}\cdot 2.06768...\,, & t_3 &= 0.240611... - \sqrt{-1}\cdot 0.115892...\,.\end{aligned}$$

Hence, the critical value of U_4' at this critical point is given by

$$v_4' = 0.488837... - \sqrt{-1}\cdot 6.50435...\,.$$

Therefore, by the saddle point method, we can show that the left-hand side of the formula of the lemma is of order $e^{N\,\mathrm{Re}\,v_4'}$, and we obtain the lemma. □

8.2.4 Restriction of the sum to $t_1 \le 0.5$ and $t_2, t_3, t_4 \le 0.7$

In this section, we show that we can restrict Δ'' to the domain $t_1 \le 0.5$ and $t_2, t_3, t_4 \le 0.7$ in such a way that the removed part is included in the domain (125). That is, assuming that

$$2\,\Lambda(t_1) + \Lambda(t_2) + \Lambda(t_3) + \Lambda(t_4) \ \ge\ \varsigma_R\,,$$

we show that $t_1 \le 0.5$ and $t_2, t_3, t_4 \le 0.7$ in this section.

We calculate an upper bound of t_1. Since $\Lambda(t)$ has a maximal value at $t = 1/6$,

$$2\,\Lambda(t_1) \ \ge\ \varsigma_R - 3\,\Lambda\big(\frac{1}{6}\big) \ =\ 0.530263... - 3\cdot 0.161533... \ =\ 0.045664... \ >\ 0.$$

Hence, noting that the behavior of $\Lambda(t)$ is as mentioned in Section 2.2, we have that

$$t_1 \ \le\ 0.5.$$

We calculate the maximal value $t_{2\,\max}$ of t_2. Since $\Lambda(t)$ has a maximal value at $t = 1/6$,

$$\Lambda(t_2) \ \ge\ \varsigma_R - 4\,\Lambda\big(\frac{1}{6}\big) \ =\ 0.530263... - 4\cdot 0.161533... \ =\ -0.115869...\,.$$

Hence, $t_{2\,\max}$ is a solution of the following equation,

$$\Lambda(t_2) \ =\ \varsigma_R - 4\,\Lambda\big(\frac{1}{6}\big).$$

By calculating a solution of this equation by Newton's method from $t_2 = 0.7$, we obtain $t_{2\,\max} = 0.681959...$. In fact,

$$\Lambda(0.7) \;=\; -0.124907 \;<\; -0.115869... \;=\; \varsigma_R - 4\,\Lambda\big(\frac{1}{6}\big).$$

Therefore, since the behavior of $\Lambda(t)$ is as mentioned in Section 2.2, we obtain an estimate of t_2 as

$$t_2 \;\le\; 0.7.$$

We obtain

$$t_3 \;\le\; 0.7 \quad \text{and} \quad t_4 \;\le\; 0.7$$

in the same way as above.

8.2.5 Extension of the sum to $t_1 + t_2 + t_3 + t_4 \le 1.45$

In this section, we show that we can extend Δ''' to the domain $t_1 + t_2 + t_3 + t_4 \le 1.45$. That is, the aim of this section is to show the following proposition.

Proposition 8.15. *The extension of the sum* (127) *to the range* $1 \le (i_1+i_2+i_3+i_4)/N \le 1.45$ *is estimated as follows,*

$$\sum_{\substack{(\frac{2i_1+1}{2N},\frac{2i_2+1}{2N},\frac{2i_3+1}{2N},\frac{2i_4+1}{2N})\in\Delta' \\ N\le i_1+\dots+i_4\le N\cdot 1.45}} \exp\Big(N\,V\big(\frac{2i_1+1}{2N},\ \frac{2i_2+1}{2N},\ \frac{2i_3+1}{2N},\ \frac{2i_4+1}{2N}\big)\Big) \;=\; O(e^{N(\varsigma_R-\varepsilon)}).$$

Proof. Fixing $j_4 = i_1 + i_2 + i_3 + i_4$ with $N \le j_4 \le N \cdot 1.45$, it is sufficient to show that

$$\sum_{\substack{(\frac{2i_1+1}{2N},\frac{2i_2+1}{2N},\frac{2i_3+1}{2N},\frac{2i_4+1}{2N})\in\Delta' \\ i_1+\dots+i_4=j_4}} \exp\Big(N\,V\big(\frac{2i_1+1}{2N},\ \frac{2i_2+1}{2N},\ \frac{2i_3+1}{2N},\ \frac{2i_4+1}{2N}\big)\Big) \tag{141}$$

is of order $O(e^{N(\varsigma_R-\varepsilon)})$.

We can calculate (141) in a similar way as the proof of Theorem 1.1 in Section 8.1, by using the Poisson summation formula and the saddle point method, noting that the boundary of the range of the sum (141) is included in the domain $\{\mathbf{t} \mid \operatorname{Re}\hat{V}(\mathbf{t}) < \varsigma_R\}$. We show a sketch proof of this calculation in the following of this subsection. We fix $s_4 = t_1 + t_2 + t_3 + t_4$ with $1 \le s_4 \le 1.45$, and put

$$\begin{aligned} U_5(t_2,t_3,t_4) \;&=\; \hat{V}(t_1,t_2,t_3,t_4) \\ =\; &\frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,(s_4-t_2-t_3-t_4)}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_2}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_3}) \\ &+ \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_4}) + \frac{\pi^2}{2}\Big) \;+ 2\pi\sqrt{-1}\cdot\frac{1}{2}\,\big(s_1^2+s_2^2+s_3^2-s_1-s_2-s_3+\frac{1}{2}\big), \end{aligned}$$

where we put $s_1 = s_4 - t_2 - t_3 - t_4,\ \ s_2 = s_4 - t_3 - t_4,\ \ s_3 = s_4 - t_4$. Similarly as the proof of Theorem 1.1, we can restrict the domain of the sum (141) to

$$\Delta_5' \;=\; \left\{ (t_2,t_3,t_4) \;\middle|\; \begin{array}{l} 0.01 \le\ t_2,t_3,t_4\ \le 0.7,\quad 0.03 \le s_1 \le 0.5 \\ 0.15 \le s_2 \le 0.9,\quad 0.3 \le s_3 \le 1.2,\quad 1 \le s_4 \le 1.45 \\ s_2 + t_4 \le 1.2 \end{array} \right\},$$

where we omit concrete computations of the bounds of these inequalities. Hence, by the Poisson summation formula and Lemma 8.16, (141) is approximated by

$$\int_{\Delta'_5} \exp\big(N\big(U_5(t_2,t_3,t_4)\big)\big)\, dt_2\, dt_3\, dt_4\,.$$

Further, by Lemma 8.17, this integral is of order $O(e^{N(\varsigma_R-\varepsilon)})$. Hence, we obtain the proposition. □

The differentials of U_5 are presented by

$$\begin{aligned}
\frac{\partial}{\partial t_2} U_5 &= 2\,\log(1-x_1) - \log(1-x_2) - 2\pi\sqrt{-1}\,\big(s_1 - \frac{1}{2}\big),\\
\frac{\partial}{\partial t_3} U_5 &= 2\,\log(1-x_1) - \log(1-x_3) - 2\pi\sqrt{-1}\,(s_1 + s_2 - 1),\\
\frac{\partial}{\partial t_4} U_5 &= 2\,\log(1-x_1) - \log(1-x_4) - 2\pi\sqrt{-1}\,\big(s_1 + s_2 + s_3 - \frac{3}{2}\big),
\end{aligned}$$

where $x_k = e^{2\pi\sqrt{-1}\,t_k}$.

Lemma 8.16.

$$\sum_{(m_2,m_3,m_4)} \int_{\Delta'_5} \exp\Big(N\big(U_5(t_2,t_3,t_4) - 2\pi\sqrt{-1}\,(m_2t_2 + m_3t_3 + m_4t_4)\big)\Big)\, dt_2\, dt_3\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}),$$

where the sum runs over $(m_2, m_3, m_4) \in \mathbb{Z}^3 - \{(0,0,0)\}$.

Proof. We can show the lemma similarly as the proof of Proposition 2.2 (see [20]). In the case of this lemma, it is sufficient to show that

$$\text{when } m_2 \neq 0, \quad -(2\pi-\varepsilon) \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta} U_5(t_2+\delta\sqrt{-1}, t_3, t_4)\Big) \;<\; 2\pi-\varepsilon, \tag{142}$$

$$\text{when } m_2 \neq m_3, \quad -(2\pi-\varepsilon) \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta} U_5(t_2+\delta\sqrt{-1}, t_3-\delta\sqrt{-1}, t_4)\Big) \;<\; 2\pi-\varepsilon, \tag{143}$$

$$\text{when } m_3 \neq m_4, \quad -(2\pi-\varepsilon) \;<\; \mathrm{Re}\Big(\frac{\partial}{\partial\delta} U_5(t_2, t_3+\delta\sqrt{-1}, t_4-\delta\sqrt{-1})\Big) \;<\; 2\pi-\varepsilon, \tag{144}$$

for some $\varepsilon > 0$.

We show (142), as follows. The middle term is calculated as

$$\begin{aligned}
&\mathrm{Re}\Big(\frac{\partial}{\partial\delta} U_5(t_2+\delta\sqrt{-1}, t_3, t_4)\Big)\\
&= \mathrm{Re}\Big(\sqrt{-1}\cdot\frac{\partial}{\partial t_2} U_5(t_2+\delta\sqrt{-1}, t_3, t_4)\Big)\\
&= -\mathrm{Im}\Big(2\,\log(1-x_1) - \log(1-x_2) - 2\pi\sqrt{-1}\big(s_1-\frac{1}{2}\big)\Big)\\
&= -2\,\mathrm{Arg}\,(1-x_1) + \mathrm{Arg}\,(1-x_2) + 2\pi\big(s_1-\frac{1}{2}\big),
\end{aligned}$$

where $x_1 = e^{2\pi\sqrt{-1}\,(s_1 - \delta\sqrt{-1})}$ and $x_2 = e^{2\pi\sqrt{-1}\,(t_2+\delta\sqrt{-1})}$. Since $0 < s_1 < 0.5$,

$$0 \;<\; -2\,\mathrm{Arg}\,(1-x_1) \;\le\; 2\pi(1-2s_1).$$

Hence,

$$-2\pi\big(\frac{1}{2}-s_1\big) \;<\; -2\,\mathrm{Arg}\,(1-x_1) + 2\pi\big(s_1-\frac{1}{2}\big) \;<\; 2\pi\big(\frac{1}{2}-s_1\big).$$

Further,

$$\min\Big\{-2\pi\big(\frac{1}{2}-t_2\big),\ 0\Big\} \;\le\; \mathrm{Arg}\,(1-x_2) \;\le\; \max\Big\{0,\ 2\pi\big(t_2-\frac{1}{2}\big)\Big\}.$$

Hence,

$$\begin{aligned}\min\Big\{-2\pi(1-s_2),\ -2\pi\big(\frac{1}{2}-s_1\big)\Big\} \;&\le\; \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\,U_5(t_2+\delta\sqrt{-1},t_3,t_4)\Big)\\ &\le\; \max\Big\{2\pi\big(\frac{1}{2}-s_1\big),\ 2\pi(t_2-s_1)\Big\}.\end{aligned}$$

Therefore, since $s_1 \ge 0$, $s_2 \ge 0.15$ and $t_2 \le 0.7$,

$$-2\pi\cdot 0.85 \;\le\; \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\,U_5(t_2+\delta\sqrt{-1},t_3,t_4)\Big) \;\le\; 2\pi\cdot 0.7,$$

and hence, (142) is satisfied.

We show (143), as follows. The middle term is calculated as

$$\begin{aligned}&\mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\,U_5(t_2+\delta\sqrt{-1},t_3-\delta\sqrt{-1},t_4)\Big)\\ &=\; \mathrm{Re}\,\Big(\sqrt{-1}\cdot\big(\frac{\partial}{\partial t_2}-\frac{\partial}{\partial t_3}\big)\,U_5(t_2+\delta\sqrt{-1},t_3-\delta\sqrt{-1},t_4)\Big)\\ &=\; -\mathrm{Im}\,\Big(-\log(1-x_2)+\log(1-x_3)+2\pi\sqrt{-1}\big(s_2-\frac{1}{2}\big)\Big)\\ &=\; \mathrm{Arg}\,(1-x_2)-\mathrm{Arg}\,(1-x_3)-2\pi\big(s_2-\frac{1}{2}\big),\end{aligned}$$

where $x_2 = e^{2\pi\sqrt{-1}\,(t_2+\delta\sqrt{-1})}$ and $x_3 = e^{2\pi\sqrt{-1}\,(t_3-\delta\sqrt{-1})}$. Since

$$\min\Big\{-2\pi\big(\frac{1}{2}-t_2\big),\ 0\Big\} \;\le\; \mathrm{Arg}\,(1-x_2) \;\le\; \max\Big\{0,\ 2\pi\big(t_2-\frac{1}{2}\big)\Big\},$$

we have that

$$\begin{aligned}\min\Big\{-2\pi\cdot s_1,\ -2\pi\big(s_2-\frac{1}{2}\big)\Big\} \;&\le\; \mathrm{Arg}\,(1-x_2)-2\pi\big(s_2-\frac{1}{2}\big)\\ &\le\; \max\Big\{2\pi\big(\frac{1}{2}-s_2\big),\ -2\pi\cdot s_1\Big\}.\end{aligned}$$

Hence, since $s_1 \le 0.5$ and $0.15 \le s_2 \le 0.9$,

$$-2\pi\cdot 0.5 \;\le\; \mathrm{Arg}\,(1-x_2)-2\pi\big(s_2-\frac{1}{2}\big) \;\le\; 2\pi\cdot 0.35.$$

Further,

$$\min\Big\{0,\ -2\pi\big(t_3-\frac{1}{2}\big)\Big\}\ \le\ -\mathrm{Arg}\,(1-x_3)\ \le\ \max\Big\{2\pi\big(\frac{1}{2}-t_3\big),\ 0\Big\}.$$

Hence, since $0 \le t_3 \le 0.7$,

$$-2\pi\cdot 0.2\ \le\ -\mathrm{Arg}\,(1-x_3)\ \le\ 2\pi\cdot 0.5.$$

Therefore,

$$-2\pi\cdot 0.7\ \le\ \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\,U_5(t_2+\delta\sqrt{-1},t_3-\delta\sqrt{-1},t_4)\Big)\ \le\ 2\pi\cdot 0.85,$$

and hence, (143) is satisfied.

We show (144), as follows. The middle term is calculated as

$$\begin{aligned}
&\mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\,U_5(t_2,t_3+\delta\sqrt{-1},t_4-\delta\sqrt{-1})\Big)\\
&\quad=\ \mathrm{Re}\,\Big(\sqrt{-1}\cdot\big(\frac{\partial}{\partial t_3}-\frac{\partial}{\partial t_4}\big)\,U_5(t_2,t_3+\delta\sqrt{-1},t_4-\delta\sqrt{-1})\Big)\\
&\quad=\ -\mathrm{Im}\,\Big(-\log(1-x_3)+\log(1-x_4)+2\pi\sqrt{-1}\big(s_3-\frac{1}{2}\big)\Big)\\
&\quad=\ \mathrm{Arg}\,(1-x_3)-\mathrm{Arg}\,(1-x_4)-2\pi\big(s_3-\frac{1}{2}\big),
\end{aligned}$$

where $x_3=e^{2\pi\sqrt{-1}\,(t_3+\delta\sqrt{-1})}$ and $x_4=e^{2\pi\sqrt{-1}\,(t_4-\delta\sqrt{-1})}$. Since

$$\min\Big\{-2\pi\big(\frac{1}{2}-t_3\big),\ 0\Big\}\ \le\ \mathrm{Arg}\,(1-x_3)\ \le\ \max\Big\{0,\ 2\pi\big(t_3-\frac{1}{2}\big)\Big\},$$

we have that

$$\begin{aligned}
\min\Big\{-2\pi\cdot s_2,\ -2\pi\big(s_3-\frac{1}{2}\big)\Big\}\ &\le\ \mathrm{Arg}\,(1-x_3)-2\pi\big(s_3-\frac{1}{2}\big)\\
&\le\ \max\Big\{2\pi\big(\frac{1}{2}-s_3\big),\ -2\pi\cdot s_2\Big\}\ \le\ 2\pi\cdot 0.2.
\end{aligned}$$

Further, since

$$\min\Big\{0,\ -2\pi\big(t_4-\frac{1}{2}\big)\Big\}\ \le\ -\mathrm{Arg}\,(1-x_4)\ \le\ \max\Big\{2\pi\big(\frac{1}{2}-t_4\big),\ 0\Big\}\ \le\ \pi,$$

we have that

$$\begin{aligned}
&\mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\,U_5(t_2,t_3+\delta\sqrt{-1},t_4-\delta\sqrt{-1})\Big)\\
&\qquad\ge\ \min\Big\{-2\pi\cdot s_2,\ -2\pi\big(s_3-\frac{1}{2}\big),\ -2\pi\big(s_2+t_4-\frac{1}{2}\big),\ -2\pi(s_4-1)\Big\}\\
&\qquad\ge\ \min\Big\{-2\pi\cdot 0.9,\ -2\pi\cdot 0.7,\ -2\pi\cdot 0.7,\ -2\pi\cdot 0.45\Big\}\ \le\ -2\pi\cdot 0.9,\\
&\mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\,U_5(t_2,t_3+\delta\sqrt{-1},t_4-\delta\sqrt{-1})\Big)\ \le\ 2\pi\cdot 0.7,
\end{aligned}$$

since $s_2\le 0.9,\ \ s_3\le 1.2,\ \ s_2+t_4\le 1.2,\ \ s_4\le 1.45$. Therefore, (144) is satisfied. □

Lemma 8.17.

$$\int_{\Delta_5'} \exp\big(N\big(U_5(t_2,t_3,t_4)\big)\,dt_2\,dt_3\,dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}).$$

Proof. Similarly as the proof of the saddle point method, we can show the lemma by moving Δ_5' in each fiber of the projection $\mathbb{C}^3 \to \mathbb{R}^3 \supset \Delta_5'$. We show a sketch proof in this proof. In the fiber at (t_2,t_3,t_4), we put

$$F(\delta) \;=\; \operatorname{Re} U_5(t_2, t_3+\delta\sqrt{-1}, t_4-\delta\sqrt{-1}) - \varsigma_R.$$

It is sufficient to show that

$$F(\delta) \;<\; 0 \quad \text{for any sufficiently large } \delta. \tag{145}$$

We have that

$$\frac{dF}{d\delta} \;=\; \operatorname{Arg}(1-x_3) - \operatorname{Arg}(1-x_4) - 2\pi\big(s_3-\frac{1}{2}\big),$$

where $x_3 = e^{2\pi\sqrt{-1}\,(t_3+\delta\sqrt{-1})}$ and $x_4 = e^{2\pi\sqrt{-1}\,(t_4-\delta\sqrt{-1})}$. Hence,

$$\begin{aligned}
\frac{dF}{d\delta}\Big|_{\delta\to\infty} &\;=\; 0 - 2\pi\big(t_4-\frac{1}{2}\big) - 2\pi\big(s_3-\frac{1}{2}\big) \;=\; -2\pi(s_4-1) \;\le\; 0,\\
\frac{dF}{d\delta}\Big|_{\delta\to-\infty} &\;=\; 2\pi\big(t_3-\frac{1}{2}\big) - 0 - 2\pi\big(s_3-\frac{1}{2}\big) \;=\; -2\pi\cdot s_2 \;<\; 0.
\end{aligned}$$

Since $\operatorname{Arg}(1-x_3)-\operatorname{Arg}(1-x_4)$ is a monotonic function of δ, $\frac{dF}{d\delta}<0$ for any $\delta\in\mathbb{R}$, and F is monotonically decreasing. Recall that $s_4\ge 1$ in this section. If $s_4>1$, $\frac{dF}{d\delta}<-\varepsilon$ for some $\varepsilon>0$, and hence, (145) is satisfied. If $s_4=0$, we have that

$$\begin{aligned}
\lim_{\delta\to\infty} F(\delta) &\;=\; \lim_{\delta\to\infty} \operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\big(2\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_1}) + \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_2}) + \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,(t_3+\delta\sqrt{-1})})\\
&\qquad + \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,(t_4-\delta\sqrt{-1})})\big) + 2\pi\sqrt{-1}\cdot\frac{1}{2}\big((s_3+\delta\sqrt{-1})^2 - (s_3+\delta\sqrt{-1})\big)\Big) - \varsigma_R\\
&\;=\; \operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\big(2\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_1}) + \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_2})\big)\Big) - \varsigma_R\\
&\qquad + \lim_{\delta\to\infty} \operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,(t_4-\delta\sqrt{-1})})\Big) + 2\pi\big(t_4-\frac{1}{2}\big)\,\delta\\
&\;=\; \operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\big(2\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_1}) + \operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_2})\big)\Big) - \varsigma_R.
\end{aligned}$$

Since $\operatorname{Im}\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t})$ has a maximal value at $t=1/6$,

$$\begin{aligned}
\lim_{\delta\to\infty} F(\delta) &\;\le\; \operatorname{Re}\Big(\frac{1}{2\pi\sqrt{-1}}\big(2\operatorname{Li}_2(e^{\pi\sqrt{-1}/3}) + \operatorname{Li}_2(e^{\pi\sqrt{-1}/3})\big)\Big) - \varsigma_R\\
&\;=\; 3\cdot 0.161533... - 0.530263... \;=\; -0.045664... \;<\; 0,
\end{aligned}$$

and hence, (145) is satisfied. ∎

8.2.6 Restriction of the sum to $t_1 + t_2 \leq 0.9,\ t_1 + t_2 + t_3 \leq 1.2$ and $t_1 + t_2 + t_4 \leq 1.2$

In this section, we show that we can restrict the domain Δ'''' to the domain $t_1 + t_2 \leq 0.9$, $t_1 + t_2 + t_3 \leq 1.2$ and $t_1 + t_2 + t_4 \leq 1.2$ in such a way that the removed part is included in the domain (125). That is, assuming that $\mathbf{t} \in \Delta''$, $t_1 \leq 0.5$, $t_2, t_3, t_4 \leq 0.7$ and

$$2\,\Lambda(t_1) + \Lambda(t_2) + \Lambda(t_3) + \Lambda(t_4) \ \geq\ \varsigma_{R}\,,$$

we show that $t_1 + t_2 \leq 0.9$, $t_1 + t_2 + t_3 \leq 1.2$ and $t_1 + t_2 + t_4 \leq 1.2$ in this section.

We calculate the maximal value $s_{2\,\max}$ of $s_2 = t_1 + t_2$. Since $\Lambda(t)$ has a maximal value at $t = 1/6$,

$$2\,\Lambda(t_1) + \Lambda(t_2) \ \geq\ \varsigma_{R} - 2\,\Lambda\big(\frac{1}{6}\big) \ =\ 0.207197...\,.$$

Hence, $s_{2\,\max}$ is a solution of the following equations,

$$\begin{cases} 2\,\Lambda(s_2 - t_2) + \Lambda(t_2) \ =\ \varsigma_{R} - 2\,\Lambda\big(\frac{1}{6}\big), \\ \dfrac{\partial}{\partial t_2}\big(2\,\Lambda(s_2 - t_2) + \Lambda(t_2)\big) \ =\ 0. \end{cases}$$

By calculating a solution of these equations by Newton's method from $(s_2, t_2) = (0.9, 0.6)$, we obtain $s_{2\,\max} = 0.877703...$. Therefore, we obtain an estimate of $s_2 = t_1 + t_2$ as

$$t_1 + t_2 \ \leq\ 0.9.$$

To be precise, the above argument is not partially rigorous, since we do not show the uniqueness of the solution, though the above argument is practically useful, since we can obtain a concrete estimate of $t_1 + t_2$. We give a rigorous proof that $t_1 + t_2 \leq 0.9$ in Section A.7.

We calculate the maximal value $s_{3\,\max}$ of $s_3 = t_1 + t_2 + t_3$. Since $\Lambda(t)$ has a maximal value at $t = 1/6$,

$$2\,\Lambda(t_1) + \Lambda(t_2) + \Lambda(t_3) \ \geq\ \varsigma_{R} - \Lambda\big(\frac{1}{6}\big) \ =\ 0.36873....$$

Hence, $s_{3\,\max}$ is a solution of the following equations,

$$\begin{cases} 2\,\Lambda(s_3 - t_2 - t_3) + \Lambda(t_2) + \Lambda(t_3) \ =\ \varsigma_{R} - \Lambda\big(\frac{1}{6}\big), \\ \dfrac{\partial}{\partial t_2}\big(2\,\Lambda(s_3 - t_2 - t_3) + \Lambda(t_2) + \Lambda(t_3)\big) \ =\ 0, \\ \dfrac{\partial}{\partial t_3}\big(2\,\Lambda(s_3 - t_2 - t_3) + \Lambda(t_2) + \Lambda(t_3)\big) \ =\ 0. \end{cases}$$

By calculating a solution of these equations by Newton's method from $(s_3, t_2, t_3) = (1.2, 0.4, 0.4)$, we obtain $s_{3\,\max} = 1.13786...$. Therefore, we obtain an estimate of $s_3 = t_1 + t_2 + t_3$ as

$$t_1 + t_2 + t_3 \ \leq\ 1.2.$$

To be precise, the above argument is not partially rigorous similarly as the above case. We give a rigorous proof that $t_1 + t_2 + t_3 \leq 1.2$ in Section A.7.

We obtain that

$$t_1 + t_2 + t_4 \ \le \ 1.2$$

in the same way as above.

8.3 Calculation of the critical value

In this section, we calculate the concrete value of a critical point of $\hat{V}$.

The differentials of $\hat{V}$ are presented by

$$\begin{aligned}
\frac{\partial}{\partial t_1}\hat{V}(\mathbf{t}) &= -2\log(1-x_1) + 2\pi\sqrt{-1}\,\big(s_1+s_2+s_3+s_4-2\big),\\
\frac{\partial}{\partial t_2}\hat{V}(\mathbf{t}) &= -\log(1-x_2) + 2\pi\sqrt{-1}\,\big(s_2+s_3+s_4-\frac{3}{2}\big),\\
\frac{\partial}{\partial t_3}\hat{V}(\mathbf{t}) &= -\log(1-x_3) + 2\pi\sqrt{-1}\,\big(s_3+s_4-1\big),\\
\frac{\partial}{\partial t_4}\hat{V}(\mathbf{t}) &= -\log(1-x_4) + 2\pi\sqrt{-1}\,\big(s_4-\frac{1}{2}\big),
\end{aligned}$$

where $x_k = e^{2\pi\sqrt{-1}\,t_k}$ $\ (k=1,2,3,4)$.

Lemma 8.18. *$\hat{V}$ has a unique critical point $\mathbf{t}_0$ in $P^{-1}(\Delta')$, where $P:\mathbb{C}^4\to\mathbb{R}^4$ is the projection to the real parts of the entries.*

Proof. Any critical point of $\hat{V}$ is given by a solution of $\frac{\partial}{\partial t_1}\hat{V} = \frac{\partial}{\partial t_2}\hat{V} = \frac{\partial}{\partial t_3}\hat{V} = \frac{\partial}{\partial t_4}\hat{V} = 0$, and these equations are rewritten,

$$\begin{aligned}
(1-x_1)^2 &= e^{2\pi\sqrt{-1}\,(s_1+s_2+s_3+s_4-2)} &= x_1^4\,x_2^3\,x_3^2\,x_4,\\
1-x_2 &= e^{2\pi\sqrt{-1}\,(s_2+s_3+s_4-\frac{3}{2})} &= -x_1^3\,x_2^3\,x_3^2\,x_4,\\
1-x_3 &= e^{2\pi\sqrt{-1}\,(s_3+s_4-1)} &= x_1^2\,x_2^2\,x_3^2\,x_4,\\
1-x_4 &= e^{2\pi\sqrt{-1}\,(s_4-\frac{1}{2})} &= -x_1\,x_2\,x_3\,x_4.
\end{aligned}$$

Putting $y_2 = x_1x_2,\ \ y_3 = x_1x_2x_3,\ \ y_4 = x_1x_2x_3x_4$, the above equations are rewritten,

$$(1-x_1)^2 \ = \ x_1y_2y_3y_4, \quad 1-\frac{y_2}{x_1} \ = \ -y_2y_3y_4, \quad 1-\frac{y_3}{y_2} \ = \ y_3y_4, \quad 1-\frac{y_4}{y_3} \ = \ -y_4.$$

From the fourth equation, we have that $y_3 = y_4/(1+y_4)$. Further, from the third equation, we have that $y_2 = -y_4/(y_4^2-y_4-1)$. Furthermore, from the second equation, we have that $x_1 = y_4(y_4+1)/(2y_4+1)$. By substituting them into the first equation, we obtain that

$$y_4^5 - 2y_4^4 + 3y_4^3 + 2y_4^2 - 2y_4 - 1 \ = \ 0.$$

Its solutions are

$$y_4 \ = \ -0.532511...\pm\sqrt{-1}\cdot 0.0564334..., \quad 1.10636...\pm\sqrt{-1}\cdot 1.69341..., \quad 0.852303... \,.$$

Among these, the solution $-0.532511... + \sqrt{-1}\cdot 0.0564334...$ gives a solution in Δ', from which we have that

$$\begin{aligned} x_1 &= 0.941819... + \sqrt{-1}\cdot 1.69128... , & t_1 &= 0.169133... - \sqrt{-1}\cdot 0.105128... , \\ x_2 &= 0.193141... + \sqrt{-1}\cdot 1.23996... , & t_2 &= 0.225407... - \sqrt{-1}\cdot 0.0361386... , \\ x_3 &= 0.424148... + \sqrt{-1}\cdot 0.19808... , & t_3 &= 0.0695358... + \sqrt{-1}\cdot 0.120803... , \\ x_4 &= 0.467489... + \sqrt{-1}\cdot 0.0564334... , & t_4 &= 0.01912... + \sqrt{-1}\cdot 0.119867... . \end{aligned}$$

These give a unique critical point in $P^{-1}(\Delta')$. $\square$

The critical value of $\hat{V}$ at the critical point of Lemma 8.18 is presented by

$$\varsigma \;=\; \hat{V}(\mathbf{t}_0) \;=\; 0.530263... - \sqrt{-1}\cdot 1.74407... . \tag{146}$$

Further, we put its real part to be ς_R,

$$\varsigma_R \;=\; \operatorname{Re}\varsigma \;=\; 0.530263... . \tag{147}$$

8.4 Calculation by the Poisson summation formula

In this section, we show Proposition 8.19 below, which is used in the proof of Theorem 1.1 for the 7_2 knot in Section 8.1.

Proposition 8.19. *For the notation in Section 8.1,*

$$\sum_{\substack{i_1,i_2,i_3,i_4\in\mathbb{Z} \\ (i_1/N,i_2/N,i_3/N,i_4/N)\in\Delta'}} \exp\Big(N\cdot V\big(\frac{2i_1+1}{2N},\frac{2i_2+1}{2N},\frac{2i_3+1}{2N},\frac{2i_4+1}{2N}\big)\Big)$$
$$= \int_{\Delta'} \exp\big(N\cdot V(\mathbf{t})\big)\,d\mathbf{t} \;+O(e^{N(\varsigma_R-\varepsilon)})$$

for some $\varepsilon>0$.

Proof. We put a function $g:\mathbb{R}^4\to\mathbb{R}$ by

$$g(\mathbf{t}) \;=\; \begin{cases} 1 & \text{if } \mathbf{t}\in\Delta', \\ 0 & \text{if } \mathbf{t}\notin N(\Delta'), \end{cases}$$
$$0\le g(\mathbf{t})\le 1 \quad \text{if } \mathbf{t}\in N(\Delta')-\Delta',$$

for $\mathbf{t}=(t_1,t_2,t_3,t_4)\in\mathbb{R}^4$, such that g depends only on t_k (independently of other t_j $(j\ne k)$) in a neighborhood of $\partial_k\Delta$ for each $k=2,3,4$. By applying the Poisson summation formula to $g(\mathbf{t})V(\mathbf{t})$, the sum of the proposition is presented by

$$\sum_{m_1,m_2,m_3,m_4\in\mathbb{Z}} \int_{\Delta'} \exp\Big(N\big(\hat{V}(t_1,t_2,t_3,t_4)-2\pi\sqrt{-1}\,(m_1t_1+m_2t_2+m_3t_3+m_4t_4)\big)\Big)\,dt_1\,dt_2\,dt_3\,dt_4 .$$

By Lemmas 8.20–8.23 below, the summand at (m_1,m_2,m_3,m_4) is of the order $O(e^{N(\varsigma_R-\varepsilon)})$ for some $\varepsilon>0$ in the cases where $m_4\ne 0,1,\; m_1\ne m_2,\; m_2\ne m_3$ or $m_3\ne m_4$. Namely, the summand is of the order $O(e^{N(\varsigma_R-\varepsilon)})$ when $(m_1,m_2,m_3,m_4)\neq(0,0,0,0),(1,1,1,1)$. (To be precise, it is necessary to show that the sum of such summands is of this order; we can show it in a similar way as in [20].) Further, by Proposition 8.32, the summand at $(m_1,m_2,m_3,m_4)=(1,1,1,1)$ is of the order $O(e^{-N\varepsilon})$ for some $\varepsilon>0$. Hence, we obtain the proposition. □

Lemma 8.20. *When* $m_4\neq 0,1$,

$$\int_{\Delta'}\exp\Big(N\big(\hat V(t_1,t_2,t_3,t_4)-2\pi\sqrt{-1}\,(m_1t_1+m_2t_2+m_3t_3+m_4t_4)\big)\Big)\,dt_1\,dt_2\,dt_3\,dt_4 \\ = O(e^{N(\varsigma_R-\varepsilon)}).$$

Proof. We put

$$\hat V'(t_1,t_2,t_3,t_4) \;=\; \hat V(t_1,t_2,t_3,t_4)-2\pi\sqrt{-1}\,(m_1t_1+m_2t_2+m_3t_3+m_4t_4).$$

We show the lemma by moving Δ' into the imaginary direction of t_4. When $m_4>1$, it is sufficient to show that

$$\mathrm{Re}\,\big(\hat V'(t_1,t_2,t_3,t_4-\delta_0\sqrt{-1})\big) \;<\; \varsigma_R \quad \text{for any } (t_1,t_2,t_3,t_4)\in\Delta', \tag{148}$$

$$\int_{\partial\Delta'\times[0,\delta_0]} e^{N\,\hat V'(t_1,t_2,t_3,t_4-\delta\sqrt{-1})}\,dt_1\,dt_2\,dt_3\,dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}), \tag{149}$$

for some $\delta_0>0$. When $m_4<0$, it is sufficient to show that

$$\mathrm{Re}\,\big(\hat V'(t_1,t_2,t_3,t_4+\delta_0\sqrt{-1})\big) \;<\; \varsigma_R \quad \text{for any } (t_1,t_2,t_3,t_4)\in\Delta', \tag{150}$$

$$\int_{\partial\Delta'\times[0,\delta_0]} e^{N\,\hat V'(t_1,t_2,t_3,t_4+\delta\sqrt{-1})}\,dt_1\,dt_2\,dt_3\,dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}), \tag{151}$$

for some $\delta_0>0$.

We show (148) and (150) for some sufficiently large δ_0, as follows. It is sufficient to show that

$$\text{if } m_4>1,\quad \frac{\partial}{\partial\delta}\mathrm{Re}\,\big(\hat V'(t_1,t_2,t_3,t_4-\delta\sqrt{-1})\big) \;<\; -\varepsilon' \quad \text{for any } \delta\geq 0, \tag{152}$$

$$\text{if } m_4<0,\quad \frac{\partial}{\partial\delta}\mathrm{Re}\,\big(\hat V'(t_1,t_2,t_3,t_4+\delta\sqrt{-1})\big) \;<\; -\varepsilon' \quad \text{for any } \delta\geq 0, \tag{153}$$

for some $\varepsilon'>0$. Hence, since $\mathrm{Re}\,\big(\hat V'(t_1,t_2,t_3,t_4+\delta\sqrt{-1})\big)=\mathrm{Re}\,\big(\hat V(t_1,t_2,t_3,t_4+\delta\sqrt{-1})\big)+2\pi m_4\delta$, it is sufficient to show that

$$-(4\pi-\varepsilon') \;<\; \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\hat V(t_1,t_2,t_3,t_4+\delta\sqrt{-1})\Big) \;<\; 2\pi-\varepsilon' \tag{154}$$

for some $\varepsilon'>0$. The middle term is calculated as

$$\mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\hat V(t_1,t_2,t_3,t_4+\delta\sqrt{-1})\Big) \;=\; \mathrm{Re}\,\Big(\sqrt{-1}\cdot\frac{\partial}{\partial t_4}\hat V(t_1,t_2,t_3,t_4+\delta\sqrt{-1})\Big)$$

$$= -\mathrm{Im}\Big(-\log(1-x_4) + 2\pi\sqrt{-1}\,\big(s_4 - \frac{1}{2}\big)\Big)$$
$$= \mathrm{Arg}\,(1-x_4) - 2\pi\big(s_4 - \frac{1}{2}\big),$$

where $x_4 = e^{2\pi\sqrt{-1}\,(t_4+\delta\sqrt{-1})}$. If $0 < t_4 \le \frac{1}{2}$,

$$-2\pi\big(\frac{1}{2} - t_4\big) \ \le\ \mathrm{Arg}\,(1-x_4) \ \le\ 0.$$

Hence,

$$-2\pi s_3 \ \le\ \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\hat{V}(t_1,t_2,t_3,t_4+\delta\sqrt{-1})\Big) \ \le\ 2\pi\big(\frac{1}{2} - s_4\big).$$

Therefore, since $s_3 \le 1.2$ and $s_4 \ge 0$, (154) is satisfied. If $\frac{1}{2} < t_4 < 1$,

$$0 \ <\ \mathrm{Arg}\,(1-x_4) \ <\ 2\pi\big(t_4 - \frac{1}{2}\big).$$

Hence,

$$-2\pi\big(s_4 - \frac{1}{2}\big) \ <\ \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\hat{V}(t_1,t_2,t_3,t_4+\delta\sqrt{-1})\Big) \ <\ -2\pi s_3.$$

Therefore, since $s_4 \le 1.45$ and $s_3 \ge 0$, (154) is satisfied.

We show (149) and (151), as follows. Since $\partial\Delta'$ is the union of $\partial_1\Delta'$, $\cdots$, $\partial_5\Delta'$, it is sufficient to show that

$$\text{if } m_4 > 1, \quad \int_{\partial_i\Delta'\times[0,\delta_0]} e^{N\,\hat{V}'(t_1,t_2,t_3,t_4-\delta\sqrt{-1})}\,dt_1\,dt_2\,dt_3\,dt_4 \ =\ O(e^{N(\varsigma_R-\varepsilon)}), \tag{155}$$
$$\text{if } m_4 < 0, \quad \int_{\partial_i\Delta'\times[0,\delta_0]} e^{N\,\hat{V}'(t_1,t_2,t_3,t_4+\delta\sqrt{-1})}\,dt_1\,dt_2\,dt_3\,dt_4 \ =\ O(e^{N(\varsigma_R-\varepsilon)}), \tag{156}$$

for any $\delta_0 \ge 0$ and any $i = 1,\cdots,5$.

When $i = 1$, $\partial_1\Delta'$ is included in the domain that $\mathrm{Re}\,\hat{V} < \varsigma_R$. When $m_4 > 1$, $\mathrm{Re}\,\hat{V}'(t_1,t_2,t_3,t_4-\delta\sqrt{-1})$ is monotonically decreasing with respect to δ by (152), and hence, (155) holds. When $m_4 < 0$, $\mathrm{Re}\,\hat{V}'(t_1,t_2,t_3,t_4+\delta\sqrt{-1})$ is monotonically decreasing with respect to δ by (153), and hence, (156) holds.

When $i = 5$, $\partial_5\Delta'$ is included in the domain that $\mathrm{Re}\,\hat{V} < \varsigma_R$. Hence, (155) and (156) hold in the same way as the above case.

When $i = 2$, since t_2 is fixed to be 0.003, the 4-form $dt_1\,dt_2\,dt_3\,dt_4$ vanishes on $\partial_2\Delta' \times [0,\delta_0]$. Hence, (155) and (156) hold.

When $i = 3$, since t_3 is fixed to be 0.003, (155) and (156) hold in the same way as the above case.

When $i = 4$, in order to show (155) and (156), it is sufficient to show that

$$\int_{\partial_4\Delta'} e^{N\,\hat{V}'(t_1,t_2,t_3,0.003+\delta\sqrt{-1})}\,dt_1\,dt_2\,dt_3 \ =\ O(e^{N(\varsigma_R-\varepsilon)}),$$

where we fix δ such that $\delta \le 0$ when $m_4 > 1$, and $\delta \ge 0$ when $m_4 < 0$. Noting that $\operatorname{Re}\hat{V}'(t_1,t_2,t_3,0.003) > \operatorname{Re}\hat{V}'(t_1,t_2,t_3,0.003+\delta\sqrt{-1})$ by (152) and (153), it is sufficient to show that

$$\int_{\Delta'_4} e^{N\hat{V}'(t_1,t_2,t_3,0.003+\delta\sqrt{-1})}\,dt_1\,dt_2\,dt_3 \;=\; O(e^{N(\varsigma_R-\varepsilon)}), \tag{157}$$

where we recall that Δ'_4 is defined in Section 8.2.3. We put

$$\begin{aligned}\hat{U}_4(t_1,t_2,t_3) \;=\; & \frac{1}{2\pi\sqrt{-1}}\Big(2\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_1})+\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_2})+\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t_3})\Big)\\ & + 2\pi\sqrt{-1}\cdot\frac{1}{2}\left(s_1^2+s_2^2+s_3^2+s_4^2-s_1-s_2-s_3-s_4+\frac{2}{3}\right),\end{aligned}$$

where $s_1=t_1,\;\; s_2=t_1+t_2,\;\; s_3=t_1+t_2+t_3,\;\; s_4=t_1+t_2+t_3+0.003$. It is sufficient to show that, for any $(m_1,m_2,m_3)\in\mathbb{Z}^3$,

$$\int_{\Delta'_4}\exp\Big(N\big(\hat{U}_4(t_1,t_2,t_3)-2\pi\sqrt{-1}\,(m_1t_1+m_2t_2+m_3t_3)\big)\Big)\,dt_1\,dt_2\,dt_3 \;=\; O(e^{N(\varsigma_R-\varepsilon)}).$$

We can show this formula by Lemmas 8.12, 8.13 and 8.14. Therefore, (157) holds. □

Lemma 8.21. *When* $m_1 \ne m_2$,

$$\begin{aligned}\int_{\Delta'}\exp\Big(N\big(\hat{V}(t_1,t_2,t_3,t_4)-2\pi\sqrt{-1}\,(m_1t_1+m_2t_2+m_3t_3+m_4t_4)\big)\Big)\,dt_1\,dt_2\,dt_3\,dt_4\\ =\; O(e^{N(\varsigma_R-\varepsilon)}).\end{aligned}$$

Proof. We put $\hat{V}'$ as in the proof of Lemma 8.20. When $m_1 > m_2$, it is sufficient to show that

$$\operatorname{Re}\big(\hat{V}'(t_1-\delta_0\sqrt{-1},t_2+\delta_0\sqrt{-1},t_3,t_4)\big) \;<\; \varsigma_R \quad \text{for any } (t_1,t_2,t_3,t_4)\in\Delta', \tag{158}$$

$$\int_{\partial\Delta'\times[0,\delta_0]} e^{N\hat{V}'(t_1-\delta\sqrt{-1},t_2+\delta\sqrt{-1},t_3,t_4)}\,dt_1\,dt_2\,dt_3\,dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}), \tag{159}$$

for some $\delta_0 > 0$. When $m_1 < m_2$, it is sufficient to show that

$$\operatorname{Re}\big(\hat{V}'(t_1+\delta_0\sqrt{-1},t_2-\delta_0\sqrt{-1},t_3,t_4)\big) \;<\; \varsigma_R \quad \text{for any } (t_1,t_2,t_3,t_4)\in\Delta', \tag{160}$$

$$\int_{\partial\Delta'\times[0,\delta_0]} e^{N\hat{V}'(t_1+\delta\sqrt{-1},t_2-\delta\sqrt{-1},t_3,t_4)}\,dt_1\,dt_2\,dt_3\,dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}), \tag{161}$$

for some $\delta_0 > 0$.

We show (158) and (160) for some sufficiently large δ_0, as follows. It is sufficient to show that

$$\text{if } m_1 > m_2, \quad \frac{\partial}{\partial\delta}\operatorname{Re}\big(\hat{V}'(t_1-\delta\sqrt{-1},t_2+\delta\sqrt{-1},t_3,t_4)\big) \;<\; -\varepsilon' \quad \text{for any } \delta \ge 0, \tag{162}$$

$$\text{if } m_1 < m_2, \quad \frac{\partial}{\partial\delta}\operatorname{Re}\big(\hat{V}'(t_1+\delta\sqrt{-1},t_2-\delta\sqrt{-1},t_3,t_4)\big) \;<\; -\varepsilon' \quad \text{for any } \delta \ge 0, \tag{163}$$

for some $\varepsilon' > 0$. Hence, since $\mathrm{Re}\,\big(\hat{V}'(t_1+\delta\sqrt{-1}, t_2-\delta\sqrt{-1}, t_3, t_4)\big) = \mathrm{Re}\,\big(\hat{V}(t_1+\delta\sqrt{-1}, t_2-\delta\sqrt{-1}, t_3, t_4)\big) + 2\pi(m_1-m_2)\delta$, it is sufficient to show that

$$-(2\pi-\varepsilon) \;<\; \mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\hat{V}(t_1+\delta\sqrt{-1}, t_2-\delta\sqrt{-1}, t_3, t_4)\Big) \;<\; 2\pi-\varepsilon \tag{164}$$

for some $\varepsilon > 0$. The middle term is calculated as

$$\mathrm{Re}\,\Big(\frac{\partial}{\partial\delta}\hat{V}(t_1+\delta\sqrt{-1}, t_2-\delta\sqrt{-1}, t_3, t_4)\Big) \;=\; 2\,\mathrm{Arg}\,(1-x_1) - \mathrm{Arg}\,(1-x_2) - 2\pi\big(s_1-\frac{1}{2}\big),$$

where $x_1 = e^{2\pi\sqrt{-1}\,(t_1+\delta\sqrt{-1})}$ and $x_2 = e^{2\pi\sqrt{-1}\,(t_2-\delta\sqrt{-1})}$. Since $0 < t_1 \le \frac{1}{2}$,

$$-2\pi\big(\frac{1}{2}-t_1\big) \;\le\; \mathrm{Arg}\,(1-x_1) \;\le\; 0.$$

Hence,

$$-2\pi\big(\frac{1}{2}-t_1\big) \;\le\; 2\,\mathrm{Arg}\,(1-x_1) - 2\pi\big(s_1-\frac{1}{2}\big) \;\le\; 2\pi\big(\frac{1}{2}-t_1\big).$$

Therefore, since $t_1 \ge 0.003$,

$$-2\pi\cdot 0.497 \;\le\; 2\,\mathrm{Arg}\,(1-x_1) - 2\pi\big(s_1-\frac{1}{2}\big) \;\le\; 2\pi\cdot 0.497.$$

Further, since $-\pi < \mathrm{Arg}\,(1-x_2) < \pi$, (164) is satisfied.

We show that (159) and (161), as follows. Since $\partial\Delta'$ is the union of $\partial_1\Delta', \cdots, \partial_5\Delta'$, it is sufficient to show that

$$\text{if } m_1 > m_2, \quad \int_{\partial_i\Delta'\times[0,\delta_0]} e^{N\,\hat{V}'(t_1-\delta\sqrt{-1}, t_2+\delta\sqrt{-1}, t_3, t_4)}\, dt_1\, dt_2\, dt_3\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}), \tag{165}$$

$$\text{if } m_1 < m_2, \quad \int_{\partial_i\Delta'\times[0,\delta_0]} e^{N\,\hat{V}'(t_1+\delta\sqrt{-1}, t_2-\delta\sqrt{-1}, t_3, t_4)}\, dt_1\, dt_2\, dt_3\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}), \tag{166}$$

for any $\delta_0 \ge 0$ and any $i = 1, \cdots, 5$.

When $i = 1, 3, 4, 5$, we can show (165) and (166) in a similar way as in the proof of Lemma 8.20.

When $i = 2$, in order to show (165) and (166), it is sufficient to show that

$$\int_{\partial_2\Delta'} e^{N\,\hat{V}'(t_1+\delta\sqrt{-1}, 0.003-\delta\sqrt{-1}, t_3, t_4)}\, dt_1\, dt_3\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}),$$

where we fix δ such that $\delta \le 0$ when $m_1 > m_2$, and $\delta \ge 0$ when $m_1 < m_2$. Noting that $\mathrm{Re}\,\hat{V}'(t_1, 0.003, t_3, t_4) > \mathrm{Re}\,\hat{V}'(t_1+\delta\sqrt{-1}, 0.003-\delta\sqrt{-1}, t_3, t_4)$ by (162) and (163), it is sufficient to show that

$$\int_{\Delta'_2} e^{N\,\hat{V}'(t_1+\delta\sqrt{-1}, 0.003-\delta\sqrt{-1}, t_3, t_4)}\, dt_1\, dt_3\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}), \tag{167}$$

where we recall that Δ'_2 is defined in Section 8.2.1. We put

$$\hat{U}_2(t_1, t_3, t_4) \;=\; \frac{1}{2\pi\sqrt{-1}}\Big(2\,\mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_1}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_3}) + \mathrm{Li}_2(e^{2\pi\sqrt{-1}\,t_4})\Big) + 2\pi\sqrt{-1}\cdot\frac{1}{2}\,\big(s_1^2+s_2^2+s_3^2+s_4^2-s_1-s_2-s_3-s_4+\frac{2}{3}\big),$$

where $s_1 = t_1$, $s_2 = t_1 + 0.003$, $s_3 = t_1 + t_3 + 0.003$, $s_4 = t_1 + t_3 + t_4 + 0.003$. It is sufficient to show that, for any $(m_1, m_3, m_4) \in \mathbb{Z}^3$,

$$\int_{\Delta_2'} \exp\Big(N\big(\hat{U}_2(t_1 + \delta\sqrt{-1}, t_3, t_4) - 2\pi\sqrt{-1}\,(m_1(t_1 + \delta\sqrt{-1}) + m_3 t_3 + m_4 t_4)\big)\Big)\, dt_1\, dt_3\, dt_4 = O(e^{N(\varsigma_R - \varepsilon)}).$$

We can show this formula by Lemmas 8.4, 8.5 and 8.6. Therefore, (167) holds. □

Lemma 8.22. *When* $m_2 \neq m_3$,

$$\int_{\Delta'} \exp\Big(N\big(\hat{V}(t_1, t_2, t_3, t_4) - 2\pi\sqrt{-1}\,(m_1 t_1 + m_2 t_2 + m_3 t_3 + m_4 t_4)\big)\Big)\, dt_1\, dt_2\, dt_3\, dt_4 = O(e^{N(\varsigma_R - \varepsilon)}).$$

Proof. We put $\hat{V}'$ as in the proof of Lemma 8.20. Similarly as the proof of Lemma 8.21, it is sufficient to show that

$$-(2\pi - \varepsilon') < \operatorname{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t_1, t_2 + \delta\sqrt{-1}, t_3 - \delta\sqrt{-1}, t_4)\Big) < 2\pi - \varepsilon' \qquad (168)$$

for some $\varepsilon' > 0$, and that

$$\text{if } m_2 > m_3, \quad \int_{\partial_i\Delta' \times [0,\delta_0]} e^{N\hat{V}'(t_1, t_2 - \delta\sqrt{-1}, t_3 + \delta\sqrt{-1}, t_4)}\, dt_1\, dt_2\, dt_3\, dt_4 = O(e^{N(\varsigma_R - \varepsilon)}), \qquad (169)$$

$$\text{if } m_2 < m_3, \quad \int_{\partial_i\Delta' \times [0,\delta_0]} e^{N\hat{V}'(t_1, t_2 + \delta\sqrt{-1}, t_3 - \delta\sqrt{-1}, t_4)}\, dt_1\, dt_2\, dt_3\, dt_4 = O(e^{N(\varsigma_R - \varepsilon)}), \qquad (170)$$

for any $\delta_0 \geq 0$ and any $i = 1, \cdots, 5$.

We show (168), as follows. The middle term of (168) is calculated as

$$\operatorname{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t_1, t_2 + \delta\sqrt{-1}, t_3 - \delta\sqrt{-1}, t_4)\Big) = \operatorname{Arg}(1 - x_2) - \operatorname{Arg}(1 - x_3) - 2\pi\big(s_2 - \frac{1}{2}\big),$$

where $x_2 = e^{2\pi\sqrt{-1}\,(t_2 + \delta\sqrt{-1})}$ and $x_3 = e^{2\pi\sqrt{-1}\,(t_3 - \delta\sqrt{-1})}$. Since

$$\min\big\{-2\pi\big(\frac{1}{2} - t_2\big),\ 0\big\} \leq \operatorname{Arg}(1 - x_2) \leq \max\big\{0,\ 2\pi\big(t_2 - \frac{1}{2}\big)\big\},$$

we have that

$$\min\big\{-2\pi\cdot t_1,\ -2\pi\big(s_2 - \frac{1}{2}\big)\big\} \leq \operatorname{Arg}(1 - x_2) - 2\pi\big(s_2 - \frac{1}{2}\big) \leq \max\big\{2\pi\big(\frac{1}{2} - s_2\big),\ -2\pi\cdot t_1\big\}.$$

Hence, since $0.003 \leq t_1 \leq 0.5$ and $0.003 \leq t_1 < s_2 \leq 0.9$,

$$-\pi \leq \operatorname{Arg}(1 - x_2) - 2\pi\big(s_2 - \frac{1}{2}\big) \leq 2\pi\cdot 0.497.$$

Further,

$$\min\{0,\ -2\pi\big(t_3 - \frac{1}{2}\big)\} \ \le\ -\mathrm{Arg}\,(1-x_3)\ \le\ \max\{2\pi\big(\frac{1}{2} - t_3\big),\ 0\}.$$

Hence, since $0 < t_3 \le 0.7$,

$$-2\pi\cdot 0.2\ \le\ -\mathrm{Arg}\,(1-x_3)\ \le\ \pi.$$

Therefore,

$$-2\pi\cdot 0.7\ \le\ \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t_1, t_2+\delta\sqrt{-1}, t_3-\delta\sqrt{-1}, t_4)\Big)\ \le\ 2\pi\cdot 0.997,$$

and hence, (168) is satisfied.

We show (169) and (170) for $i = 1, \cdots, 5$, as follows.

When $i = 1, 4, 5$, we can show (169) and (170) in a similar way as in the proof of Lemma 8.20.

When $i = 2$, in order to show (169) and (170), it is sufficient to show that

$$\int_{\partial_2\Delta'} e^{N\,\hat{V}'(t_1, 0.003+\delta\sqrt{-1}, t_3-\delta\sqrt{-1}, t_4)}\, dt_1\, dt_3\, dt_4\ =\ O(e^{N(\varsigma_R-\varepsilon)}),$$

where we fix δ such that $\delta \le 0$ when $m_2 > m_3$, and $\delta \ge 0$ when $m_2 < m_3$. We can show this formula in a similar way as in the proof of Lemma 8.21.

When $i = 3$, in order to show (169) and (170), it is sufficient to show that

$$\int_{\partial_3\Delta'} e^{N\,\hat{V}'(t_1, t_2+\delta\sqrt{-1}, 0.003-\delta\sqrt{-1}, t_4)}\, dt_1\, dt_2\, dt_4\ =\ O(e^{N(\varsigma_R-\varepsilon)}),$$

where we fix δ such that $\delta \le 0$ when $m_2 > m_3$, and $\delta \ge 0$ when $m_2 < m_3$. We can show this formula by Lemmas 8.8, 8.9 and 8.10 in a similar way as in the proof of Lemma 8.21. □

Lemma 8.23. *When $m_3 \ne m_4$,*

$$\int_{\Delta'} \exp\Big(N\big(\hat{V}(t_1,t_2,t_3,t_4) - 2\pi\sqrt{-1}\,(m_1t_1+m_2t_2+m_3t_3+m_4t_4)\big)\Big)\, dt_1\, dt_2\, dt_3\, dt_4 = O(e^{N(\varsigma_R-\varepsilon)}).$$

Proof. We put $\hat{V}'$ as in the proof of Lemma 8.20. Similarly as the proof of Lemma 8.21, it is sufficient to show that

$$-(2\pi-\varepsilon')\ <\ \mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t_1, t_2, t_3+\delta\sqrt{-1}, t_4-\delta\sqrt{-1})\Big)\ <\ 2\pi-\varepsilon' \tag{171}$$

for some $\varepsilon' > 0$, and that

$$\text{if } m_3 > m_4,\quad \int_{\partial_i\Delta'\times[0,\delta_0]} e^{N\,\hat{V}'(t_1,t_2,t_3-\delta\sqrt{-1}, t_4+\delta\sqrt{-1})}\, dt_1\, dt_2\, dt_3\, dt_4\ =\ O(e^{N(\varsigma_R-\varepsilon)}), \tag{172}$$

$$\text{if } m_3 < m_4, \quad \int_{\partial_i \Delta' \times [0,\delta_0]} e^{N\,\hat{V}'(t_1,t_2,t_3+\delta\sqrt{-1},t_4-\delta\sqrt{-1})}\, dt_1\, dt_2\, dt_3\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}), \qquad (173)$$

for any $\delta_0 \geq 0$ and any $i = 1, \cdots, 5$.

We show (171), as follows. The middle term of (171) is calculated as

$$\mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t_1,t_2,t_3+\delta\sqrt{-1},t_4-\delta\sqrt{-1})\Big) \;=\; \mathrm{Arg}\,(1-x_3) - \mathrm{Arg}\,(1-x_4) - 2\pi\big(s_3 - \frac{1}{2}\big),$$

where $x_3 = e^{2\pi\sqrt{-1}\,(t_3+\delta\sqrt{-1})}$ and $x_4 = e^{2\pi\sqrt{-1}\,(t_4-\delta\sqrt{-1})}$. Since

$$\min\big\{-2\pi\big(\frac{1}{2}-t_3\big),\ 0\big\} \;\leq\; \mathrm{Arg}\,(1-x_3) \;\leq\; \max\big\{0,\ 2\pi\big(t_3-\frac{1}{2}\big)\big\},$$

we have that

$$\begin{aligned}\min\big\{-2\pi\cdot s_2,\ -2\pi\big(s_3-\frac{1}{2}\big)\big\} \;&\leq\; \mathrm{Arg}\,(1-x_3) - 2\pi\big(s_3-\frac{1}{2}\big)\\ &\leq\; \max\big\{2\pi\big(\frac{1}{2}-s_3\big),\ -2\pi\cdot s_2\big\} \;\leq\; 2\pi\cdot 0.497,\end{aligned}$$

since $0.003 \leq t_1 < s_3$ and $0 < s_2$. Further,

$$\min\big\{0,\ -2\pi\big(t_4-\frac{1}{2}\big)\big\} \;\leq\; -\mathrm{Arg}\,(1-x_4) \;\leq\; \max\big\{2\pi\big(\frac{1}{2}-t_4\big),\ 0\big\} \;\leq\; \pi,$$

since $t_4 > 0$. Therefore,

$$\begin{aligned}&\mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t_1,t_2,t_3+\delta\sqrt{-1},t_4-\delta\sqrt{-1})\Big) \;\leq\; 2\pi\cdot 0.97,\\ &\mathrm{Re}\Big(\frac{\partial}{\partial\delta}\hat{V}(t_1,t_2,t_3+\delta\sqrt{-1},t_4-\delta\sqrt{-1})\Big) \;\geq\; -2\pi\cdot\max\Big\{s_2,\ s_3-\frac{1}{2},\ s_2+t_4-\frac{1}{2},\ s_4-1\Big\}\\ &\qquad\qquad\geq\; -2\pi\cdot\max\big\{0.9,\ 0.7,\ 0.7,\ 0.45\big\} \;\geq\; -2\pi\cdot 0.9,\end{aligned}$$

and hence, (171) is satisfied.

We show (172) and (173) for $i = 1, \cdots, 5$, as follows.

When $i = 1, 2, 5$, we can show (172) and (173) in a similar way as in the proof of Lemma 8.20.

When $i = 3$, in order to show (172) and (173), it is sufficient to show that

$$\int_{\partial_3\Delta'} e^{N\,\hat{V}'(t_1,t_2,0.003+\delta\sqrt{-1},t_4-\delta\sqrt{-1})}\, dt_1\, dt_2\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}),$$

where we fix δ such that $\delta \leq 0$ when $m_3 > m_4$, and $\delta \geq 0$ when $m_3 < m_4$. We can show this formula in a similar way as in the proof of Lemma 8.21.

When $i = 4$, in order to show (172) and (173), it is sufficient to show that

$$\int_{\partial_4\Delta'} e^{N\,\hat{V}'(t_1,t_2,t_3+\delta\sqrt{-1},0.003-\delta\sqrt{-1})}\, dt_1\, dt_2\, dt_3 \;=\; O(e^{N(\varsigma_R-\varepsilon)}),$$

where we fix δ such that $\delta \leq 0$ when $m_3 > m_4$, and $\delta \geq 0$ when $m_3 < m_4$. We can show this formula by Lemmas 8.12, 8.13 and 8.14 in a similar way as in the proof of Lemma 8.21. □

8.5 Verifying the assumption of the saddle point method

In this section, we verify the assumption of the saddle point method in Proposition 8.28. In order to show this proposition, we show Lemmas 8.24–8.27 in advance.

We put

$$f(X,Y,Z,W) \;=\; \mathrm{Re}\,\hat{V}(t_1+X\sqrt{-1}, t_2+Y\sqrt{-1}, t_3+Z\sqrt{-1}, t_4+W\sqrt{-1}) - \varsigma_R\,.$$

Then, we have that

$$\begin{aligned}
\frac{\partial f}{\partial X} &= 2\,\mathrm{Arg}\,(1-x_1) - 2\pi(s_1+s_2+s_3+s_4-2),\\
\frac{\partial f}{\partial Y} &= \mathrm{Arg}\,(1-x_2) - 2\pi\big(s_2+s_3+s_4-\frac{3}{2}\big),\\
\frac{\partial f}{\partial Z} &= \mathrm{Arg}\,(1-x_3) - 2\pi(s_3+s_4-1),\\
\frac{\partial f}{\partial W} &= \mathrm{Arg}\,(1-x_4) - 2\pi\big(s_4-\frac{1}{2}\big),
\end{aligned}$$

where $x_1 = e^{2\pi\sqrt{-1}\,(t_1+X\sqrt{-1})}$, $x_2 = e^{2\pi\sqrt{-1}\,(t_2+Y\sqrt{-1})}$, $x_3 = e^{2\pi\sqrt{-1}\,(t_3+Z\sqrt{-1})}$ and $x_4 = e^{2\pi\sqrt{-1}\,(t_4+W\sqrt{-1})}$.

Lemma 8.24. *Fixing X, Y, Z, we regard f as a function of W.*
(1) *If $s_4 \ge \frac{1}{2}$, then f is monotonically decreasing as a function of W.*
(2) *If $s_4 < \frac{1}{2}$, then f has a unique minimal point as a function of W. In particular, this minimal point goes to ∞ as $s_4 \to \frac{1}{2}-0$.*

Proof. If $0 < t_4 < \frac{1}{2}$, we have that

$$-2\pi\big(\frac{1}{2}-t_4\big) \;<\; \mathrm{Arg}\,(1-x_4) \;<\; 0,$$

and $\mathrm{Arg}\,(1-x_4)$ is monotonically increasing as a function of W. Further,

$$\begin{aligned}
\frac{\partial f}{\partial W}\Big|_{W\to\infty} &= -2\pi\big(s_4-\frac{1}{2}\big),\\
\frac{\partial f}{\partial W}\Big|_{W\to-\infty} &= -2\pi\big(\frac{1}{2}-t_4\big) - 2\pi\big(s_4-\frac{1}{2}\big) \;=\; -2\pi\cdot s_3 \;<\; 0.
\end{aligned}$$

If $s_4 < \frac{1}{2}$, there is a unique zero of $\frac{\partial f}{\partial W}$, which gives a unique minimal point of f, and hence, (2) holds. If $s_4 \ge \frac{1}{2}$, $\frac{\partial f}{\partial W}$ is always negative, and (1) holds.

If $\frac{1}{2} \le t_4 < 1$, $\mathrm{Arg}\,(1-x_4)$ is monotonically decreasing as a function of W. Since $\frac{\partial f}{\partial W}\Big|_{W\to-\infty}$ is negative, $\frac{\partial f}{\partial W}$ is always negative, and (1) holds. □

Lemma 8.25. *Fixing X, Y, W, we regard f as a function of Z.*
(1) *If $s_3+s_4 \ge 1$, then f is monotonically decreasing as a function of Z.*
(2) *If $s_3+s_4 < 1$ and $s_2+s_4 > \frac{1}{2}$, then f has a unique minimal point as a function of*

Z*. In particular, this minimal point goes to* ∞ *as* $s_3+s_4 \to 1-0$*, and goes to* $-\infty$ *as* $s_2+s_4 \to \frac{1}{2}+0$.
(3) *If* $s_2+s_4 \le \frac{1}{2}$*, then* f *is monotonically increasing as a function of* Z*.*

Proof. If $0 < t_3 < \frac{1}{2}$, we have that

$$-2\pi\big(\frac{1}{2}-t_3\big) \ < \ \mathrm{Arg}\,(1-x_3) \ < \ 0,$$

and $\mathrm{Arg}\,(1-x_3)$ is monotonically increasing as a function of Z. Further,

$$\begin{aligned} \frac{\partial f}{\partial Z}\Big|_{Z\to\infty} &= -2\pi(s_3+s_4-1), \\ \frac{\partial f}{\partial Z}\Big|_{Z\to-\infty} &= -2\pi\big(\frac{1}{2}-t_3\big) - 2\pi(s_3+s_4-1) \ = \ -2\pi\big(s_2+s_4-\frac{1}{2}\big). \end{aligned}$$

If $s_3+s_4 \ge 1$, $\frac{\partial f}{\partial Z}$ is always negative, and (1) holds. If $s_3+s_4<1$ and $s_2+s_4>\frac{1}{2}$, there is a unique zero of $\frac{\partial f}{\partial Z}$, which gives a unique minimal point of f, and hence, (2) holds. If $s_2+s_4 \le \frac{1}{2}$, then $\frac{\partial f}{\partial Z}$ is always positive, and (3) holds.

If $\frac{1}{2} \le t_3 < 1$, $\mathrm{Arg}\,(1-x_3)$ is monotonically decreasing as a function of Z. Since $s_3+s_4 > 2\,t_3 \ge 1$, $\frac{\partial f}{\partial Z}\Big|_{Z\to-\infty}$ is negative, and $\frac{\partial f}{\partial Z}$ is always negative. Hence, (1) holds. □

Lemma 8.26. *Fixing* X*,* Z*,* W*, we regard* f *as a function of* Y*.*
(1) *If* $s_2+s_3+s_4 \ge \frac{3}{2}$*, then* f *is monotonically decreasing as a function of* Y*.*
(2) *If* $s_2+s_3+s_4 < \frac{3}{2}$ *and* $t_1+s_3+s_4>1$*, then* f *has a unique minimal point as a function of* Y*. In particular, this minimal point goes to* ∞ *as* $s_2+s_3+s_4 \to \frac{3}{2}-0$*, and goes to* $-\infty$ *as* $t_1+s_3+s_4 \to 1+0$*.*
(3) *If* $t_1+s_3+s_4 \le 1$*, then* f *is monotonically increasing as a function of* Y*.*

Proof. We can prove the lemma, similarly as the proof of Lemma 8.25. □

Lemma 8.27. *Fixing* Y*,* Z*,* W*, we regard* f *as a function of* X*.*
(1) *If* $s_1+s_2+s_3+s_4 \ge 2$*, then* f *is monotonically decreasing as a function of* X*.*
(2) *If* $s_1+s_2+s_3+s_4 < 2$ *and* $t_2+s_3+s_4>1$*, then* f *has a unique minimal point as a function of* X*. In particular, this minimal point goes to* ∞ *as* $s_1+s_2+s_3+s_4 \to 2-0$*, and goes to* $-\infty$ *as* $t_2+s_3+s_4 \to 1+0$*.*
(3) *If* $t_2+s_3+s_4 \le 1$*, then* f *is monotonically increasing as a function of* X*.*

Proof. We can prove the lemma, similarly as the proof of Lemma 8.25. □

Proposition 8.28. *When we apply the saddle point method to* (128)*, the assumption of the saddle point method holds.*

Proof. We show that there exists a homotopy $\Delta'(\delta)$ $(0 \le \delta \le 1)$ between $\Delta'(0)=\Delta'$ and $\Delta'(1)$ such that

$$(t_{1c}, t_{2c}, t_{3c}, t_{4c}) \in \Delta'(1), \qquad (174)$$

$$\Delta'(1) - \{(t_{1c}, t_{2c}, t_{3c}, t_{4c})\} \;\subset\; \big\{(t_1,t_2,t_3,t_4) \in \mathbb{C}^4 \;\big|\; \mathrm{Re}\,\hat{V}(t_1,t_2,t_3,t_4) < \varsigma_R\big\}, \qquad (175)$$

$$\int_{\bigcup \partial\Delta'(\delta)} \hat{V}(t_1,t_2,t_3,t_4)\, dt_1\, dt_2\, dt_3\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}) \quad \text{for some } \varepsilon > 0. \qquad (176)$$

We make the homotopy, as follows. We note that the behavior of $f(X,Y,Z,W)$ as a function of X does not depend on Y, Z, W, and the behavior of $f(X,Y,Z,W)$ as a function of Y does not depend on X, Z, W, and the behavior of $f(X,Y,Z,W)$ as a function of Z does not depend on X, Y, W, and the behavior of $f(X,Y,Z,W)$ as a function of W does not depend on X, Y, Z. We make the homotopy by taking (t_1,t_2,t_3,t_4) to the minimal point (or infinity) of Lemmas 8.24–8.27 in each fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$. We note that we can choose any order of moving variables to the minimal points (or infinity) in a general fiber. In each fiber on $\partial_2\Delta'$, we take X, Z, W to the minimal points (or infinity) in advance, and take Y to the minimal point (or infinity) later. In each fiber on $\partial_3\Delta'$, we take X, Y, W to the minimal points (or infinity) in advance, and take Z to the minimal point (or infinity) later. In each fiber on $\partial_4\Delta'$, we take X, Y, Z to the minimal points (or infinity) in advance, and take W to the minimal point (or infinity) later. When $s_4 < \frac{1}{2}$, $s_2+s_4 > \frac{1}{2}$, $t_1+s_3+s_4 > 1$ and $t_2+s_3+s_4 > 1$, f has a unique minimal point by Lemmas 8.24–8.27; we put it to be $(X,Y,Z,W) = \big(g_1(t_1,t_2,t_3,t_4), g_2(t_1,t_2,t_3,t_4), g_3(t_1,t_2,t_3,t_4), g_4(t_1,t_2,t_3,t_4)\big)$.

We show (176), as follows. It is sufficient to show that

$$\int_{\bigcup \partial_i\Delta'(\delta)} \hat{V}(t_1,t_2,t_3,t_4)\, dt_1\, dt_2\, dt_3\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}) \qquad (177)$$

for each $i = 1, \cdots, 5$. When $i=1$, $\partial_1\Delta'$ is included in the domain that $\mathrm{Re}\,\hat{V} < \varsigma_R$. Further, by the construction of the homotopy, $\mathrm{Re}\,\hat{V}$ monotonically decreases by the homotopy. Hence, (177) holds. When $i=5$, $\partial_5\Delta'$ is also included in the domain that $\mathrm{Re}\,\hat{V} < \varsigma_R$, and hence, (177) holds, similarly as the above case. When $i=2$, in a fiber on $\partial_2\Delta'$, the homotopy moves X, Z, W in advance, fixing t_2 and Y. In this range of δ, the restriction of the 4-form $dt_1\, dt_2\, dt_3\, dt_4$ to $\bigcup \partial_2\Delta'(\delta)$ vanishes, and the integral of (177) is 0. Further, when the homotopy moves Y later, $\partial_2\Delta'(\delta)$ is included in the domain that $\mathrm{Re}\,\hat{V} < \varsigma_R$ by Lemma 8.29 below. Hence, (177) holds. When $i = 3, 4$, (177) holds similarly, by Lemmas 8.30 and 8.31 below.

We show (174) and (175), as follows. In a similar way as the cases of other knots, we can show that, when (t_1,t_2,t_3,t_4) is a critical point of $h(t_1,t_2,t_3,t_4)$, $\big(t_1+g_1(t_1,t_2,t_3,t_4)\sqrt{-1},\ t_2+g_2(t_1,t_2,t_3,t_4)\sqrt{-1},\ t_3+g_3(t_1,t_2,t_3,t_4)\sqrt{-1},\ t_4+g_4(t_1,t_2,t_3,t_4)\sqrt{-1}\big)$ is a critical point of $\hat{V}$. It follows that $h(t_1,t_2,t_3,t_4)$ has a unique maximal point at $(t_1,t_2,t_3,t_4) = (\mathrm{Re}\,t_{1c}, \mathrm{Re}\,t_{2c}, \mathrm{Re}\,t_{3c}, \mathrm{Re}\,t_{4c})$. Therefore, (174) and (175) hold. □

Lemma 8.29. *When the homotopy moves X, Z, W to the minimal points (or infinity) in each fiber on $\partial_2\Delta'$, the homotopy moves $\partial_2\Delta'$ into the domain that* $\mathrm{Re}\,\hat{V} < \varsigma_R$.

Proof. As shown in the proof of Lemma 8.6, we can take $X \to -\infty$ or $W \to \infty$ in every fiber on $\partial_2\Delta'$. Hence, $f \to -\infty$ in every fiber. Therefore, we obtain the lemma. □

Lemma 8.30. *When the homotopy moves X, Y, W to the minimal points (or infinity) in each fiber on $\partial_3\Delta'$, the homotopy moves $\partial_3\Delta'$ into the domain that* $\mathrm{Re}\,\hat{V} < \varsigma_R$.

Proof. As calculated in the proof of Lemma 8.9, when we move X, Y, W to the minimal points (or infinity), $\partial_3\Delta'$ is moved to the position of the saddle point method of Lemma 8.9. Then, f is bounded by the critical value of Lemma 8.9, and it follows that $\partial_3\Delta'$ (after the move) is included in the domain that $\mathrm{Re}\,\hat{V} < \varsigma_R$. □

Lemma 8.31. *When the homotopy moves X, Y, Z to the minimal points (or infinity) in each fiber on $\partial_4\Delta'$, the homotopy moves $\partial_4\Delta'$ into the domain that* $\mathrm{Re}\,\hat{V} < \varsigma_R$.

Proof. As calculated in the proof of Lemma 8.13, when we move X, Y, Z to the minimal points (or infinity), $\partial_4\Delta'$ is moved to the position of the saddle point method of Lemma 8.13. Then, f is bounded by the critical value of Lemma 8.13, and it follows that $\partial_4\Delta'$ (after the move) is included in the domain that $\mathrm{Re}\,\hat{V} < \varsigma_R$. □

8.6 Estimate of the integral at $(m_1, m_2, m_3, m_4) = (1, 1, 1, 1)$

In this section, we show Proposition 8.32 below, which is used in the proof of Proposition 8.19 in Section 8.4.

Proposition 8.32. *For the notation of Proposition 8.19,*

$$\int_{\Delta'} \exp\Big(N\big(\hat{V}(t_1,t_2,t_3,t_4) - 2\pi\sqrt{-1}\,(t_1+t_2+t_3+t_4)\big)\Big)\,dt_1\,dt_2\,dt_3\,dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}).$$

Before showing a proof of the proposition, we show some lemmas. We put

$$\begin{aligned}
\hat{V}'(t_1,t_2,t_3,t_4) &= \hat{V}(t_1,t_2,t_3,t_4) - 2\pi\sqrt{-1}\,(t_1+t_2+t_3+t_4),\\
f'(X,Y,Z,W) &= \mathrm{Re}\,\hat{V}'(t_1+X\sqrt{-1}, t_2+Y\sqrt{-1}, t_3+Z\sqrt{-1}, t_4+W\sqrt{-1}) - \varsigma_R\\
&= f(X,Y,Z,W) + 2\pi(X+Y+Z+W).
\end{aligned}$$

It is sufficient to show that, in the fiber of the projection $\mathbb{C}^4 \to \mathbb{R}^4$ at each $(t_1,t_2,t_3,t_4) \in \Delta'$, we can move X, Y, Z, W in such a way that f' becomes negative. We will show this actually by moving X and W, fixing Y and Z. We have that

$$\begin{aligned}
\frac{\partial f'}{\partial X} &= \frac{\partial f}{\partial X} + 2\pi \;=\; 2\,\mathrm{Arg}\,(1-x_1) - 2\pi(s_1+s_2+s_3+s_4-3),\\
\frac{\partial f'}{\partial W} &= \frac{\partial f}{\partial W} + 2\pi \;=\; \mathrm{Arg}\,(1-x_4) - 2\pi\big(s_4 - \frac{3}{2}\big),
\end{aligned}$$

where $x_1 = e^{2\pi\sqrt{-1}\,(t_1+X\sqrt{-1})}$ and $x_4 = e^{2\pi\sqrt{-1}\,(t_4+W\sqrt{-1})}$.

Lemma 8.33. *Fixing X, Y, Z, we regard f' as a function of W.*
(1) *If $s_3 \le 1$, then f' is monotonically increasing as a function of W.*
(2) *If $s_3 > 1$, then f' has a unique minimal point at $W = g_4(t_1,t_2,t_3,t_4)$, where*

$$g_4(t_1,t_2,t_3,t_4) \;=\; \frac{1}{2\pi}\,\log\frac{\sin 2\pi(s_3-1)}{\sin 2\pi\big(\frac{3}{2}-s_4\big)}\,.$$

In particular, this minimal point goes to $-\infty$ *as* $s_3 \to 1+0$.

Proof. If $0 < t_4 < \frac{1}{2}$, we have that

$$-2\pi\big(\frac{1}{2} - t_4\big) \ < \ \mathrm{Arg}\,(1-x_4) \ < \ 0,$$

and $\mathrm{Arg}\,(1-x_4)$ is monotonically increasing as a function of W. Further,

$$\begin{aligned} \frac{\partial f'}{\partial W}\Big|_{W\to\infty} &= \ 2\pi\big(\frac{3}{2} - s_4\big) \ > \ 0, \\ \frac{\partial f'}{\partial W}\Big|_{W\to-\infty} &= \ -2\pi\big(\frac{1}{2} - t_4\big) - 2\pi\big(s_4 - \frac{3}{2}\big) \ = \ -2\pi(s_3 - 1). \end{aligned}$$

If $s_3 > 1$, there is a unique zero of $\frac{\partial f'}{\partial W}$, which gives a unique minimal point of f', and hence, (2) holds. If $s_3 \le 1$, $\frac{\partial f'}{\partial W}$ is always positive, and (1) holds.

If $\frac{1}{2} \le t_4 < 1$, $\mathrm{Arg}\,(1-x_4)$ is monotonically decreasing as a function of W. Since $\frac{\partial f'}{\partial W}\Big|_{W\to\infty}$ is positive, $\frac{\partial f'}{\partial W}$ is always positive, and (1) holds. □

Lemma 8.34. *Fixing* Y, Z, W, *we regard* f' *as a function of* X.
(1) *If* $s_1+s_2+s_3+s_4 \ge 3$, *then* f' *is monotonically decreasing as a function of* X.
(2) *If* $s_1+s_2+s_3+s_4 < 3$ *and* $t_2+s_3+s_4 > 2$, *then* f' *has a unique minimal point at* $X = g_1(t_1,t_2,t_3,t_4)$, *where*

$$g_1(t_1,t_2,t_3,t_4) \ = \ \frac{1}{2\pi}\log\frac{\sin\pi(t_2+s_3+s_4-2)}{\sin\pi(3-s_1-s_2-s_3-s_4)}.$$

In particular, this minimal point goes to ∞ *as* $s_1+s_2+s_3+s_4 \to 3-0$, *and goes to* $-\infty$ *as* $t_2+s_3+s_4 \to 2+0$.
(3) *If* $t_2+s_3+s_4 \le 2$, *then* f' *is monotonically increasing as a function of* X.

Proof. Since $0 < t_1 < \frac{1}{2}$, we have that

$$-2\pi\big(\frac{1}{2} - t_1\big) \ < \ \mathrm{Arg}\,(1-x_1) \ < \ 0,$$

and $\mathrm{Arg}\,(1-x_1)$ is monotonically increasing as a function of X. Further,

$$\begin{aligned} \frac{\partial f'}{\partial X}\Big|_{X\to\infty} &= \ -2\pi(s_1+s_2+s_3+s_4-3), \\ \frac{\partial f'}{\partial X}\Big|_{X\to-\infty} &= \ -2\pi(1-2\,t_1) - 2\pi(s_1+s_2+s_3+s_4-3) \ = \ -2\pi(t_2+s_3+s_4-2). \end{aligned}$$

Hence, we can show the lemma similarly as the proof of Lemma 8.33. □

We now show a proof of Proposition 8.32 by using the above lemmas.

Proof of Proposition 8.32. We show that

$$\int_{\Delta'} e^{N\hat{V}'(t_1,t_2,t_3,t_4)}\, dt_1\, dt_2\, dt_3\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}).$$

It is sufficient to show that there exists a homotopy $\Delta'(\delta)$ $(0\le\delta\le 1)$ between $\Delta'(0)=\Delta'$ and $\Delta'(1)$ such that

$$\Delta'(1) \;\subset\; \big\{(t_1,t_2,t_3,t_4)\in\mathbb{C}^4 \;\big|\; \operatorname{Re}\hat{V}'(t_1,t_2,t_3,t_4) < \varsigma_R\big\}, \tag{178}$$

$$\int_{\bigcup \partial\Delta'(\delta)} e^{N\hat{V}'(t_1,t_2,t_3,t_4)}\, dt_1\, dt_2\, dt_3\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}) \quad \text{for some } \varepsilon>0. \tag{179}$$

We make the homotopy, as follows. We note that the behavior of $f'(X,Y,Z,W)$ as a function of X does not depend on W, and the behavior of $f'(X,Y,Z,W)$ as a function of W does not depend on X. We define the homotopy by taking X and W to the minimal points (or infinity) of Lemmas 8.33 and 8.34 in each fiber of the projection $\mathbb{C}^4\to\mathbb{R}^4$ at (t_1,t_2,t_3,t_4). We note that we can choose any order of moving X and W to the minimal points (or infinity) in a general fiber. In each fiber on $\partial_4\Delta'$, noting that $f'\to-\infty$ as $W\to-\infty$ by Lemma 8.33 since $s_3\le 0.95$, we move (t_1,t_2,t_3,t_4) to $(t_1,t_2,t_3,t_4-\delta'\sqrt{-1})$ for a sufficiently large δ' in advance, and move X later.

We show (179), as follows. It is sufficient to show that

$$\int_{\bigcup \partial_i\Delta'(\delta)} e^{N\hat{V}'(t_1,t_2,t_3,t_4)}\, dt_1\, dt_2\, dt_3\, dt_4 \;=\; O(e^{N(\varsigma_R-\varepsilon)}) \tag{180}$$

for each $i=1,\cdots,5$. When $i=1$, $\partial_1\Delta'$ is included in the domain that $\operatorname{Re}\hat{V}'<\varsigma_R$. Further, by the construction of the homotopy, $\operatorname{Re}\hat{V}$ monotonically decreases by the homotopy. Hence, (180) holds. When $i=5$, $\partial_5\Delta'$ is also included in the domain that $\operatorname{Re}\hat{V}'<\varsigma_R$, and (180) holds similarly as the above case. When $i=2$, since the homotopy fixes t_2 and Y, the restriction of the 4-form $dt_1\, dt_2\, dt_3\, dt_4$ to $\bigcup\partial_2\Delta'(\delta)$ vanishes, and the integral of (180) is 0. Hence, (180) holds. When $i=3$, (180) holds similarly as the above case. When $i=4$, we have that

$$\int_{\bigcup \partial_4\Delta'(\delta)} e^{N\hat{V}'(t_1,t_2,t_3,t_4)}\, dt_1\, dt_2\, dt_3\, dt_4 \;=\; -\sqrt{-1}\int d\delta' \int_{\Delta'_4} e^{N\hat{V}'(t_1,t_2,t_3,0.003-\delta'\sqrt{-1})}\, dt_1\, dt_2\, dt_3,$$

and (180) holds by Lemma 8.35 below.

We show (178), as follows. When $s_3\le 1$, since $f'\to-\infty$ as $W\to-\infty$ by Lemma 8.33, (178) holds. When $t_2+s_3+s_4\le 2$, since $f'\to-\infty$ as $X\to\infty$ by Lemma 8.34, (178) holds. When $s_1+s_2+s_3+s_4\ge 3$, since $f'\to-\infty$ as $X\to-\infty$ by Lemma 8.34, (178) holds. The remaining case is the case where $s_3>1$, $t_2+s_3+s_4>2$ and $s_1+s_2+s_3+s_4<3$. In this case, we show (178), as follows. We put

$$F(t_1,t_2,t_3,t_4,X,W) \;=\; \operatorname{Re}\Big(\hat{V}'\big(t_1+X\sqrt{-1},\, t_2,\, t_3,\, t_4+W\sqrt{-1}\big)\Big),$$
$$h(t_1,t_2,t_3,t_4) \;=\; F\big(t_1,t_2,t_3,t_4,g_1(t_1,t_2,t_3,t_4),g_4(t_1,t_2,t_3,t_4)\big),$$

where g_1 and g_4 are as in Lemmas 8.33 and 8.34. Let (t_1, t_2, t_3, t_4) be a critical point of $h(t_1, t_2, t_3, t_4)$, and put $X = g_1(t_1, t_2, t_3, t_4)$ and $W = g_4(t_1, t_2, t_3, t_4)$. Then, $(t_1, t_2, t_3, t_4, X, W)$ is a critical point of F. Hence,

$$\frac{\partial}{\partial t_1} \hat{V}' = 0, \quad \mathrm{Re}\Big(\frac{\partial}{\partial t_2} \hat{V}'\Big) = 0, \quad \mathrm{Re}\Big(\frac{\partial}{\partial t_3} \hat{V}'\Big) = 0, \quad \frac{\partial}{\partial t_4} \hat{V}' = 0.$$

Therefore,

$$\begin{aligned}
-2\log(1-x_1) + 2\pi\sqrt{-1}\,\big(s_1+s_2+s_3+s_4-3\big) &= 0,\\
\mathrm{Re}\Big(-\log(1-x_2) + 2\pi\sqrt{-1}\,\big(s_2+s_3+s_4-\tfrac{5}{2}\big)\Big) &= 0,\\
\mathrm{Re}\Big(-\log(1-x_3) + 2\pi\sqrt{-1}\,\big(s_3+s_4-3\big)\Big) &= 0,\\
-\log(1-x_4) + 2\pi\sqrt{-1}\,\big(s_4-\tfrac{3}{2}\big) &= 0.
\end{aligned}$$

Hence, F has a unique critical point given by

$$\begin{aligned}
x_1 &= 0.233604... + \sqrt{-1}\cdot 0.706749...\,, & t_1 &= 0.199193... + \sqrt{-1}\cdot 0.0469884...\,,\\
x_2 &= -0.0659927... + \sqrt{-1}\cdot 0.99782...\,, & t_2 &= 0.260511...\,,\\
x_3 &= -0.92395... - \sqrt{-1}\cdot 0.382514...\,, & t_3 &= 0.562471...\,,\\
x_4 &= 3.29476... + \sqrt{-1}\cdot 1.29574...\,, & t_4 &= 0.0596343... - \sqrt{-1}\cdot 0.20121...\,,
\end{aligned}$$

and its critical value is given by

$$v_0' = 0.454575...\,.$$

This critical point gives a maximal point of h, and hence, $h(t_1, t_2, t_3, t_4)$ is bounded by v_0'. Since v_0' is less than $\varsigma_R = 0.530263...$, (178) holds. □

Lemma 8.35. *For an arbitrarily fixed $\delta' \geq 0$,*

$$\int_{\Delta_4'} e^{N\,\hat{V}(t_1,t_2,t_3,0.003-\delta'\sqrt{-1})}\, dt_1\, dt_2\, dt_3 = O(e^{N(\varsigma_R-\varepsilon)}).$$

Proof. $\partial\Delta_4'$ is originally included in the domain that $\mathrm{Re}\,\hat{V}' < \varsigma_R$. Further, since $s_3 \leq 0.95$ in $\partial\Delta_4'$, $\mathrm{Re}\,\hat{V}'(t_1, t_2, t_3, 0.003 - \delta'\sqrt{-1})$ is monotonically decreasing with respect to δ' by Lemma 8.33. Hence, $\partial\Delta_4'$ is included in the domain that $\mathrm{Re}\,\hat{V}'(t_1, t_2, t_3, 0.003 - \delta'\sqrt{-1}) < \varsigma_R$. Further, since $\mathrm{Re}\,\hat{V}'(t_1, t_2, t_3, 0.003 - \delta'\sqrt{-1} - \delta''\sqrt{-1}) \to -\infty$ as $\delta'' \to \infty$ by Lemma 8.33, we can show the lemma by taking the domain of the integral as $\delta'' \to \infty$. □

A The domain $\{\mathrm{Re}\,\hat{V} \geq \varsigma_R\}$ is convex

In Sections 3.2, 4.2, 5.2, 6.2 and 7.2, we estimate the maximal and minimal values of some linear function $L(t, s, u, v)$ on the domain $\big\{(t, s, u, v) \;\big|\; \mathrm{Re}\,\hat{V}(t, s, u, v) \geq \varsigma_R\big\}$. In this section, we explain that this domain is a convex domain such that its boundary is a smooth closed hypersurface whose sectional curvatures are positive everywhere. Then, the maximal and minimal values of $L(t,s,u,v)$ are obtained when the hyperplane $L(t,s,u,v)=c$ (where c is a constant) is tangent to this domain, and there are exactly two such tangent points corresponding to the maximal and minimal values of $L(t,s,u,v)$. Such tangent points are given by solutions of a certain system of equations given in Sections 3.2, 4.2, 5.2, 6.2 and 7.2. Hence, such a system of equations has exactly two solutions; we use this fact in these sections.

So, the aim of this section is to show that the domain $\big\{(t,s,u,v) \bigm| \operatorname{Re}\hat{V}(t,s,u,v) \ge \varsigma_R\big\}$ is a convex domain such that its boundary is a smooth closed hypersurface whose sectional curvatures are positive everywhere. We show this for the 7_3, 7_4, 7_5, 7_6, 7_7 knots in Sections A.2, A.3, A.4, A.5, A.6, respectively. In Section A.1, we prepare some lemmas. In Section A.7, we give rigorous proofs of such estimates for the 7_2 knot in another way.

A.1 Some lemmas

In this section, we give some lemmas, which we use in the following sections.

Let $F(x,y,z,w)$ be a smooth concave function whose maximal value is positive. Then, the domain $\big\{(x,y,z,w)\in\mathbb{R}^4 \bigm| F(x,y,z,w)\ge 0\big\}$ is a convex domain and its boundary is a smooth surface.

Lemma A.1. *Let $F(x,y,z,w)$ be a smooth concave function whose maximal value is positive and Hesse matrix is negative definite. Then, the domain $\big\{(x,y,z,w)\in\mathbb{R}^4 \bigm| F(x,y,z,w)\ge 0\big\}$ is a convex domain and its boundary is a smooth hypersurface whose sectional curvatures are positive everywhere.*

We can show the lemma in a similar way as in [21, Appendix B].

Lemma A.2. *We put $G(t,s)=2\Lambda(t)-\Lambda(t+s)+2\Lambda(s)$.*
(1) *On the domain $\big\{(t,s)\in\mathbb{R}^2 \bigm| 0<t,s<0.5\big\}$, $G(t,s)$ is a concave function whose Hesse matrix is negative definite.*
(2) *On the domain $\big\{(t,s)\in\mathbb{R}^2 \bigm| 0\le t,s,\ \ t+s\le 1\big\}$, the upper bound of $G(t,s)$ is given by $G(t,s)\le G(\frac14,\frac14)=4\Lambda(\frac14)=0.583122\dots$.*

Proof. We show (1) of the lemma, as follows. The differentials of G are given by

$$\begin{aligned}\frac{\partial G}{\partial t} &= 2\,\Lambda'(t)-\Lambda'(t+s) = -2\log 2\sin\pi t+\log 2\sin\pi(t+s),\\ \frac{\partial G}{\partial s} &= 2\,\Lambda'(s)-\Lambda'(t+s) = -2\log 2\sin\pi s+\log 2\sin\pi(t+s).\end{aligned}$$

Further, their differentials are given by

$$\begin{aligned}\frac{\partial^2 G}{\partial t^2} &= 2\,\Lambda''(t)-\Lambda''(t+s) = -2\pi\cot\pi t+\pi\cot\pi(t+s),\\ \frac{\partial^2 G}{\partial t\,\partial s} &= -\Lambda''(t+s) = \pi\cot\pi(t+s),\\ \frac{\partial^2 G}{\partial s^2} &= 2\,\Lambda''(s)-\Lambda''(t+s) = -2\pi\cot\pi s+\pi\cot\pi(t+s).\end{aligned}$$

We put $a = \cot \pi t$ and $b = \cot \pi s$, noting that they are positive since $0 < t, s < 0.5$. Further, noting that $\cot(\alpha+\beta) = (\cot\alpha \cot\beta - 1)/(\cot\alpha + \cot\beta)$, we have that

$$\frac{1}{\pi} \cdot \frac{\partial^2 G}{\partial t^2} \;=\; -2a + \frac{ab-1}{a+b}, \qquad \frac{1}{\pi} \cdot \frac{\partial^2 G}{\partial t\, \partial s} \;=\; \frac{ab-1}{a+b}, \qquad \frac{1}{\pi} \cdot \frac{\partial^2 G}{\partial s^2} \;=\; -2b + \frac{ab-1}{a+b}.$$

We put the Hesse matrix of G to be H. Then,

$$\frac{1}{\pi} \cdot \operatorname{trace} H \;=\; -2a-2b+2\cdot \frac{ab-1}{a+b} \;=\; -\frac{2\big((a+b)^2 - ab + 1\big)}{a+b} \;=\; -\frac{2\big(a^2+b^2+ab+1\big)}{a+b} \;<\; 0.$$

Further,

$$\frac{1}{\pi^2} \cdot \det H \;=\; 4ab - (2a+2b) \cdot \frac{ab-1}{a+b} \;=\; 2(ab+1) \;>\; 0.$$

Hence, the two eigenvalues of H are negative, and H is negative definite. Therefore, G is a concave function on $\big\{(t,s) \;\big|\; 0 < t, s < 0.5\big\}$, whose Hesse matrix is negative definite, as required.

We show (2) of the lemma, as follows. A maximal point of G is whether a solution of $\frac{\partial G}{\partial t} = \frac{\partial G}{\partial s} = 0$ or a point on the boundary of the domain $\{0 \le t, s, \quad t+s \le 1\}$. We can show by concrete calculation that the solutions of the above equation are $(t,s) = (\frac{1}{4}, \frac{1}{4}), (1,0), (0,1)$, and at these points the values of G is bounded by $G(\frac{1}{4}, \frac{1}{4}) = 4\Lambda(\frac{1}{4})$. Further, the boundary of the domain $\{0 \le t, s, \quad t+s \le 1\}$ consists of $\{t=0\}$, $\{s=0\}$ and $\{t+s=1\}$. On the boundary $\{t=0\}$, the upper bound of G is given by $G = \Lambda(s) \le \Lambda(\frac{1}{6}) \le 4\Lambda(\frac{1}{4})$. On the boundary $\{s=0\}$, the upper bound of G is given by $4\Lambda(\frac{1}{4})$ in the same way. On the boundary $\{t+s=1\}$, the upper bound of G is given by $G = 0 \le 4\Lambda(\frac{1}{4})$. Therefore, the upper bound of G is given by $4\Lambda(\frac{1}{4})$ on the domain of (2), as required. □

Lemma A.3.
(1) *We put* $G_1(t,s,u) = 2\,\Lambda(t) + \Lambda(s-t) + \Lambda(u-s) - 2\,\Lambda(u)$. *On the domain*

$$\big\{(t,s,u) \in \mathbb{R}^3 \;\big|\; 0 < t, s < 0.5 < u < 1, \quad 0 < s-t < 0.5, \quad 0 < u-s < 0.5\big\}, \tag{181}$$

$G_1(t,s,u)$ *is a concave function whose Hesse matrix is negative definite.*
(2) *We put* $G_2(t,s,u) = 2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s+u) + 2\,\Lambda(u)$. *On the domain*

$$\big\{(t,s,u) \in \mathbb{R}^3 \;\big|\; 0 \le t \le s \le 1-u, \quad 0 \le u\big\}, \tag{182}$$

the upper bound of $G_2(t,s,u)$ *is given by*

$$G_2\Big(\frac{1}{\pi}\arccos\frac{\sqrt{17}-1}{4},\ \frac{1}{2},\ \frac{1}{\pi}\arccos\frac{\sqrt{17}-1}{4}\Big) \;=\; 0.887067\ldots.$$

Proof. We show (1) of the lemma, as follows. The differentials of G_1 are given by

$$\frac{\partial G_1}{\partial t} \;=\; 2\,\Lambda'(t) - \Lambda'(s-t), \qquad \frac{\partial G_1}{\partial s} \;=\; \Lambda'(s-t) - \Lambda'(u-s), \qquad \frac{\partial G_1}{\partial u} \;=\; -2\,\Lambda'(u) + \Lambda'(u-s).$$

Further, their differentials are given by

$$\frac{\partial^2 G_1}{\partial t^2} \;=\; 2\,\Lambda''(t) + \Lambda''(s-t), \qquad \frac{\partial^2 G_1}{\partial t\,\partial s} \;=\; -\Lambda''(s-t)$$

$$\frac{\partial^2 G_1}{\partial s^2} = \Lambda''(s-t) + \Lambda''(u-s), \qquad \frac{\partial^2 G_1}{\partial s\,\partial u} = -\Lambda''(u-s),$$
$$\frac{\partial^2 G_1}{\partial u^2} = -2\,\Lambda''(u) + \Lambda''(u-s).$$

We put $a_1 = -2\Lambda''(t)$, $b_1 = -\Lambda''(s-t)$, $b_2 = -\Lambda''(u-s)$ and $a_2 = 2\Lambda''(u)$, noting that they are positive on the domain of (1). The Hesse matrix of G_1 is given by

$$-\begin{pmatrix} a_1+b_1 & -b_1 & 0 \\ -b_1 & b_1+b_2 & -b_2 \\ 0 & -b_2 & a_2+b_2 \end{pmatrix}.$$

As we show in the proof of Lemma 5.4, (-1) times the above matrix is positive definite. Hence, the Hesse matrix of G_1 is negative definite on the domain of (1), as required.

We show (2) of the lemma, as follows. We note that $G_2(t,s,u) = G_1(t,s,1-u)$. A maximal point of G_2 is whether a solution of $\frac{\partial G_2}{\partial t} = \frac{\partial G_2}{\partial s} = \frac{\partial G_2}{\partial u} = 0$ or a point on the boundary of the domain (182). We calculate a solution of the equation in the interior of the domain (182). The differentials of G_2 are given by

$$\frac{\partial G_2}{\partial t} = 2\,\Lambda'(t) - \Lambda'(s-t) = -2\log 2\sin\pi t + \log 2\sin\pi(s-t),$$
$$\frac{\partial G_2}{\partial s} = \Lambda'(s-t) - \Lambda'(s+u) = -2\log 2\sin\pi(s-t) + \log 2\sin\pi(s+u),$$
$$\frac{\partial G_2}{\partial u} = 2\,\Lambda'(u) - \Lambda'(s+u) = -2\log 2\sin\pi u + \log 2\sin\pi(s+u).$$

Hence, the above mentioned equation is rewritten

$$2\sin^2\pi t = \sin\pi(s-t) = \sin\pi(s+u) = 2\sin^2\pi u.$$

Since $\sin\pi t = \sin\pi u$, we have that $t = u$ or $t+u = 1$; we choose that $t = u$, since we consider a solution in the interior of (182). Further, since $\sin\pi(s-t) = \sin\pi(s+u)$ and $t = u$, we have that $s = \frac{1}{2}$. Hence, $2\sin^2\pi t = \sin\pi(\frac{1}{2}-t) = \cos\pi t$. Therefore, putting $x = \cos\pi t$, we have that $2(1-x^2) = x$. Hence, $x = \frac{\sqrt{17}-1}{4}$ and $t = \frac{1}{\pi}\arccos\big(\frac{\sqrt{17}-1}{4}\big) = 0.214823\ldots$. Putting $t_0 = u_0 = \frac{1}{\pi}\arccos\big(\frac{\sqrt{17}-1}{4}\big)$ and $s_0 = \frac{1}{2}$, (t_0, s_0, u_0) is a unique critical point of G_2 in the interior of (182). It follows by (1) that $(t_0, s_0, 1-u_0)$ is a unique maximal point of G_1 in the domain (181). Hence, (t_0, s_0, u_0) is a unique maximal point of G_2, and its maximal value is $G_2(t_0, s_0, u_0) = 0.887067\ldots$. Further, we see that G_2 is bounded by this value on the boundary of the domain (182). This boundary consists of $\{t=0\}$, $\{t=s\}$, $\{s+u=1\}$ and $\{u=0\}$. On the boundary $\{t=0\}$,

$$G_2(0,s,u) - G_2(t_0,s_0,u_0) = \Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) - G_2(t_0,s_0,u_0)$$
$$\le \Lambda\big(\frac{1}{6}\big) + 4\,\Lambda\big(\frac{1}{4}\big) - G_2(t_0,s_0,u_0) = -0.142413\ldots < 0,$$

by Lemma A.2. On the boundary $\{t=s\}$,

$$G_2(t,t,u) - G_2(t_0,s_0,u_0) = 2\,\Lambda(t) - \Lambda(t+u) + 2\,\Lambda(u) - G_2(t_0,s_0,u_0)$$

$$\leq\ 4\,\Lambda\big(\frac{1}{4}\big) - G_2(t_0,s_0,u_0) \ = \ -0.303946...\ <\ 0,$$

by Lemma A.2. On the boundary $\{s+u=1\}$,

$$G_2(t,1-u,u) - G_2(t_0,s_0,u_0) \ = \ 2\,\Lambda(t) - \Lambda(t+u) + 2\,\Lambda(u) - G_2(t_0,s_0,u_0) \ <\ 0,$$

in the same way as above. On the boundary $\{u=0\}$,

$$\begin{aligned} G_2(t,s,0) - G_2(t_0,s_0,u_0) \ &= \ 2\,\Lambda(t) + \Lambda(s-t) - \Lambda(s) - G_2(t_0,s_0,u_0) \\ \leq\ 4\,\Lambda\big(\frac{1}{6}\big) - G_2(t_0,s_0,u_0) \ &= \ -0.240936...\ <\ 0, \end{aligned}$$

since $|\Lambda(\,\cdot\,)| \leq \Lambda(\frac{1}{6})$. Hence, G_2 is bounded by $G_2(t_0,s_0,u_0)$ on the boundary of (182). Therefore, the upper bound of G_2 is given by $G_2(t_0,s_0,u_0)$, as required. □

A.2 The domain $\{\operatorname{Re}\hat{V} \geq \varsigma_R\}$ is convex for the 7_3 knot

For the 7_3 knot, we recall that

$$\begin{aligned} &\operatorname{Re}\hat{V}(t,s,u,v) \ = \ \Lambda(t) - 2\,\Lambda(s) + \Lambda(u) + \Lambda(v), \\ &\varsigma_R \ = \ 0.730861... \qquad \big(\text{given in (29)}\big), \\ &\Delta \ = \ \big\{(t,s,u,v)\in\mathbb{R}^4 \ \big|\ 0\leq t\leq s\leq 1, \ \ 0\leq u,v, \ \ u+v\leq s\big\}, \end{aligned}$$

as we put in Section 3. The aim of this section is to show Lemma A.4 below, without using results in Section 3.2. As we mentioned at the beginning of Appendix A, we can show by this lemma that each system of equations in Section 3.2 has exactly two solutions; we use this fact in Section 3.2.

Lemma A.4. *The domain* $\big\{(t,s,u,v)\in\Delta \ \big|\ \operatorname{Re}\hat{V}(t,s,u,v)\geq\varsigma_R\big\}$ *is a compact convex domain in the interior of* Δ *such that its boundary is a smooth closed hypersurface whose sectional curvatures are positive everywhere.*

Proof. We restrict Δ to

$$\Delta'' \ = \ \big\{(t,s,u,v)\in\Delta \ \big|\ 0\ <\ t,u,v\ <\ 0.5\ <\ s\ <\ 1\big\}$$

in such a way that Δ'' includes the domain $\big\{\operatorname{Re}\hat{V}\geq\varsigma_R\big\}$ in Step 1 below. Further, in Step 2 below, we show that, on Δ'', $\operatorname{Re}\hat{V}$ is a smooth concave function whose Hesse matrix negative definite. Furthermore, we can verify by concrete calculation that $\operatorname{Re}\hat{V}<\varsigma_R$ on the boundary of Δ''. Hence, by Lemma A.1, we obtain the lemma. So, we show Steps 1 and 2 below in the following of this proof.

Step 1: In this step, we show that the domain $\big\{\operatorname{Re}\hat{V}\geq\varsigma_R\big\}$ is included in Δ''. Assuming that $\operatorname{Re}\hat{V}(t,s,u,v)\geq\varsigma_R$ for $(t,s,u,v)\in\Delta$, we calculate ranges of t, s, u and v.

We calculate a range of t, as follows. Since $|\Lambda(\,\cdot\,)|\leq\Lambda(\frac{1}{6})$,

$$\Lambda(t) \ \geq\ \varsigma_R - 4\,\Lambda\big(\frac{1}{6}\big) \ = \ 0.084729...\ >\ 0.$$

Hence, we obtain that $0 < t < 0.5$.

We obtain ranges of u and v as $0 < u, v < 0.5$ in the same way as above.

We calculate a range of s, as follows. Since $\Lambda(\,\cdot\,) \leq \Lambda(\frac{1}{6})$,

$$-2\,\Lambda(s) \;\geq\; \varsigma_{R} - 3\,\Lambda\Big(\frac{1}{6}\Big) \;=\; 0.246262... \;>\; 0.$$

Hence, we obtain that $0.5 < s < 1$.

Therefore, the domain $\big\{\mathrm{Re}\,\hat{V} \geq \varsigma_{R}\big\}$ is included in Δ'', as required.

Step 2: In this step, we show that $\mathrm{Re}\,\hat{V}$ is a smooth concave function whose Hesse matrix is negative definite on Δ''. We note that $\Lambda(t)$, $-2\Lambda(s)$, $\Lambda(u)$ and $\Lambda(v)$ are smooth concave functions whose second derivatives are negative on Δ''. Since the Hesse matrix of $\mathrm{Re}\,\hat{V}$ is equal to their direct sum, $\mathrm{Re}\,\hat{V}$ is a smooth concave function whose Hesse matrix is negative definite on Δ'', as required. □

A.3 The domain $\{\mathrm{Re}\,\hat{V} \geq \varsigma_{R}\}$ is convex for the 7_4 knot

For the 7_4 knot, we recall that

$$\begin{aligned}
&\mathrm{Re}\,\hat{V}(t,s,u,v) \;=\; \Lambda(t) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) + \Lambda(v),\\
&\varsigma_{R} \;=\; 0.817729... \qquad \big(\text{given in (48)}\big),\\
&\Delta \;=\; \big\{(t,s,u,v) \in \mathbb{R}^4 \;\big|\; 0 \leq t \leq 1-s,\;\; 0 \leq s,u,\;\; s+u \leq 1,\;\; 0 \leq v \leq 1-u\big\},
\end{aligned}$$

as we put in Section 4. The aim of this section is to show Lemma A.5 below, without using results in Section 4.2. As we mentioned at the beginning of Appendix A, we can show by this lemma that each system of equations in Section 4.2 has exactly two solutions; we use this fact in Section 4.2.

Lemma A.5. *The domain $\big\{(t,s,u,v) \in \Delta \;\big|\; \mathrm{Re}\,\hat{V}(t,s,u,v) \geq \varsigma_{R}\big\}$ is a compact convex domain in the interior of Δ such that its boundary is a smooth closed hypersurface whose sectional curvatures are positive everywhere.*

Proof. Assuming that $\mathrm{Re}\,\hat{V}(t,s,u,v) \geq \varsigma_{R}$ for $(t,s,u,v) \in \Delta$, we calculate ranges of t, s, u and v.

We calculate a range of s, as follows. Since $|\Lambda(\,\cdot\,)| \leq \Lambda(\frac{1}{6})$,

$$\Lambda(s) \;\geq\; \varsigma_{R} - 5\,\Lambda\Big(\frac{1}{6}\Big) \;=\; 0.010064... \;>\; 0.$$

Hence, we obtain that $0 < s < 0.5$.

We obtain that $0 < u < 0.5$ in the same way as above.

We calculate a range of t, as follows. Since $2\Lambda(s) - \Lambda(s+u) + 2\Lambda(u) \leq 4\Lambda(\frac{1}{4})$ by Lemma A.2, and $\Lambda(\,\cdot\,) \leq \Lambda(\frac{1}{6})$,

$$\Lambda(t) \;\geq\; \varsigma_{R} - 4\,\Lambda\Big(\frac{1}{4}\Big) - \Lambda\Big(\frac{1}{6}\Big) \;=\; 0.0730742... \;>\; 0.$$

Hence, we obtain that $0 < t < 0.5$.

We obtain that $0 < v < 0.5$ in the same way as above.

Therefore, the domain $\{\mathrm{Re}\,\hat{V} \geq \varsigma_R\}$ is included in

$$\Delta'' \;=\; \big\{(t,s,u,v) \in \Delta \;\big|\; 0 \;<\; t,s,u,v \;<\; 0.5\big\}.$$

We note that $\Lambda(t)$, $F(s,u)$ and $\Lambda(v)$ are smooth concave functions whose Hesse matrices are negative definite on Δ''. Since the Hesse matrix of $\mathrm{Re}\,\hat{V}$ is equal to their direct sum, $\mathrm{Re}\,\hat{V}$ is a smooth concave function whose Hesse matrix is negative definite on Δ''. Therefore, by Lemma A.1, we obtain the lemma. □

A.4 The domain $\{\mathrm{Re}\,\hat{V} \geq \varsigma_R\}$ is convex for the 7_5 knot

For the 7_5 knot, we recall that

$$\begin{aligned}
&\mathrm{Re}\,\hat{V}(t,s,u,v) \;=\; 2\,\Lambda(t) + \Lambda(s-t) + \Lambda(u-s) - 2\,\Lambda(u) + \Lambda(v),\\
&\varsigma_R \;=\; 1.02552... \qquad \big(\text{given in (69)}\big),\\
&\Delta \;=\; \big\{(t,s,u,v) \in \mathbb{R}^4 \;\big|\; 0 \leq t \leq s \leq u, \;\; 0 \leq v \leq u \leq 1\big\},
\end{aligned}$$

as we put in Section 5. The aim of this section is to show Lemma A.6 below, without using results in Section 5.2. As we mentioned at the beginning of Appendix A, we can show by this lemma that each system of equations in Section 5.2 has exactly two solutions; we use this fact in Section 5.2.

Lemma A.6. *The domain $\big\{(t,s,u,v) \in \Delta \;\big|\; \mathrm{Re}\,\hat{V}(t,s,u,v) \geq \varsigma_R\big\}$ is a compact convex domain in the interior of Δ such that its boundary is a smooth closed hypersurface whose sectional curvatures are positive everywhere.*

Proof. We restrict Δ to

$$\Delta'' \;=\; \big\{(t,s,u,v) \in \Delta \;\big|\; 0 \;<\; t,v \;<\; 0.5 \;<\; u \;<\; 1, \;\; 0 \;<\; s-t \;<\; 0.5, \;\; 0 \;<\; u-s \;<\; 0.5\big\}$$

in such a way that Δ'' includes the domain $\big\{\mathrm{Re}\,\hat{V} \geq \varsigma_R\big\}$ in Step 1 below. Further, in Step 2 below, we show that, on Δ'', $\mathrm{Re}\,\hat{V}$ is a smooth concave function whose Hesse matrix negative definite. Furthermore, we can verify by concrete calculation that $\mathrm{Re}\,\hat{V} < \varsigma_R$ on the boundary of Δ''. Hence, by Lemma A.1, we obtain the lemma. So, we show Steps 1 and 2 below in the following of this proof.

Step 1: In this step, we show that the domain $\big\{\mathrm{Re}\,\hat{V} \geq \varsigma_R\big\}$ is included in Δ''. Assuming that $\mathrm{Re}\,\hat{V}(t,s,u,v) \geq \varsigma_R$ for $(t,s,u,v) \in \Delta$, we calculate ranges of t, u, v, $s-t$ and $u-s$.

We calculate a range of v, as follows. Since $|\Lambda(\,\cdot\,)| \leq \Lambda(\frac{1}{6})$,

$$\Lambda(v) \;\geq\; \varsigma_R - 6\,\Lambda\big(\frac{1}{6}\big) \;=\; 0.056322... \;>\; 0.$$

Hence, we obtain that $0 < v < 0.5$.

In the same way, we obtain that $0 < t < 0.5 < u < 1$ and $0 < s-t < 0.5$ and $0 < u-s < 0.5$.

Therefore, the domain $\big\{\mathrm{Re}\,\hat{V} \geq \varsigma_R\big\}$ is included in Δ'', as required.

Step 2: In this step, we show that $\operatorname{Re}\hat{V}$ is a smooth concave function whose Hesse matrix is negative definite on Δ''.

We note that, by Lemma A.3, $G_1(t,s,u) = 2\Lambda(t) + \Lambda(s-t) + \Lambda(u-s) - 2\Lambda(u)$ is a concave function whose Hesse matrix is negative definite on Δ''. Further, $\Lambda(v)$ is a concave function whose second derivative is negative on Δ''. Since the Hesse matrix of $\operatorname{Re}\hat{V}$ is equal to the direct sum of $\Lambda''(v)$ and the Hesse matrix of G_1, $\operatorname{Re}\hat{V}$ is a smooth concave function whose Hesse matrix is negative definite on Δ'', as required. $\square$

A.5 The domain $\{\operatorname{Re}\hat{V} \geq \varsigma_R\}$ is convex for the 7_6 knot

For the 7_6 knot, we recall that

$$\begin{aligned}
\operatorname{Re}\hat{V}(t,s,u,v) &= 2\Lambda(t) + \Lambda(s-t) - \Lambda(s+u) + 2\Lambda(u) - \Lambda(u+v) + 2\Lambda(v), \\
\varsigma_R &= 1.1276... \qquad \big(\text{given in (87)}\big), \\
\Delta &= \big\{(t,s,u,v) \in \mathbb{R}^4 \bigm| 0 \leq t \leq s \leq 1-u, \quad 0 \leq u, v, \quad u+v \leq 1\big\},
\end{aligned}$$

as we put in Section 6. The aim of this section is to show Lemma A.7 below, without using results in Section 6.2. As we mentioned at the beginning of Appendix A, we can show by this lemma that each system of equations in Section 6.2 has exactly two solutions; we use this fact in Section 6.2.

Lemma A.7. *The domain $\big\{(t,s,u,v) \in \Delta \bigm| \operatorname{Re}\hat{V}(t,s,u,v) \geq \varsigma_R\big\}$ is a compact convex domain in the interior of Δ such that its boundary is a smooth closed hypersurface whose sectional curvatures are positive everywhere.*

Proof. We restrict Δ to

$$\Delta'' = \big\{(t,s,u,v) \in \Delta \bigm| 0 < t,u,v < 0.5, \quad 0 < s-t < 0.5 < s+u < 1\big\}$$

in such a way that Δ'' includes the domain $\big\{\operatorname{Re}\hat{V} \geq \varsigma_R\big\}$ in Step 1 below. Further, in Step 2 below, we show that, on Δ'', $\operatorname{Re}\hat{V}$ is a smooth concave function whose Hesse matrix negative definite. Furthermore, we can verify by concrete calculation that $\operatorname{Re}\hat{V} < \varsigma_R$ on the boundary of Δ''. Hence, by Lemma A.1, we obtain the lemma. So, we show Steps 1 and 2 below in the following of this proof.

Step 1: In this step, we show that the domain $\big\{\operatorname{Re}\hat{V} \geq \varsigma_R\big\}$ is included in Δ''. Assuming that $\operatorname{Re}\hat{V}(t,s,u,v) \geq \varsigma_R$ for $(t,s,u,v) \in \Delta$, we calculate ranges of t, u, v, $s-t$ and $s+u$.

We calculate a range of t, as follows. Since $2\Lambda(u) - \Lambda(u+v) + 2\Lambda(v) \leq 4\Lambda(\frac{1}{4})$ by Lemma A.2, and $|\Lambda(\,\cdot\,)| \leq \Lambda(\frac{1}{6})$,

$$\Lambda(t) \geq \varsigma_R - 3\Lambda\big(\frac{1}{6}\big) - 4\Lambda\big(\frac{1}{4}\big) = 0.059879... > 0.$$

Hence, we obtain that $0 < t < 0.5$.

We obtain that $0 < s-t < 0.5 < s+u < 1$ in the same way.

We calculate a range of v, as follows. Since $G_2(t,s,u) \leq G_2(t_0,s_0,u_0)$ by Lemma A.3 (2), and $|\Lambda(\,\cdot\,)| \leq \Lambda(\frac{1}{6})$,

$$2\Lambda(v) \geq \varsigma_R - G_2(t_0,s_0,u_0) - \Lambda\big(\frac{1}{6}\big) = 0.078999... > 0.$$

Hence, we obtain that $0 < v < 0.5$.

We calculate a range of u, as follows. By Lemma A.8 below,

$$2\,\Lambda(u) \;\geq\; \varsigma_{R} - 1.12 \;=\; 0.0076... \;>\; 0.$$

Hence, we obtain that $0 < u < 0.5$.

Therefore, the domain $\big\{\mathrm{Re}\,\hat{V} \geq \varsigma_{R}\big\}$ is included in Δ'', as required.

Step 2: In this step, we show that $\mathrm{Re}\,\hat{V}$ is a smooth concave function whose Hesse matrix is negative definite on Δ''.

We put

$$F(t,s,u,v) \;=\; \mathrm{Re}\,\hat{V}(t,s,u,v) \;=\; 2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v).$$

The second derivatives of F are given by

$$\begin{aligned}
\frac{\partial^2 F}{\partial t^2} &= 2\,\Lambda''(t)+\Lambda''(s-t) \;=\; -\pi(a_1+b_1),\\
\frac{\partial^2 F}{\partial t\,\partial s} &= -\Lambda''(s-t) \;=\; \pi b_1,\\
\frac{\partial^2 F}{\partial s^2} &= \Lambda''(s-t)-\Lambda''(s+u) \;=\; -\pi(b_1+b_2),\\
\frac{\partial^2 F}{\partial s\,\partial u} &= -\Lambda''(s+u) \;=\; -\pi b_2,\\
\frac{\partial^2 F}{\partial u^2} &= -\Lambda''(s+u)+2\,\Lambda(u)-\Lambda''(u+v), \;=\; -\pi(b_2+2c-b_3),\\
\frac{\partial^2 F}{\partial u\,\partial v} &= -\Lambda''(u+v) \;=\; \pi b_3,\\
\frac{\partial^2 F}{\partial v^2} &= 2\,\Lambda''(v)-\Lambda''(u+v) \;=\; -\pi(2d+b_3),
\end{aligned}$$

where we put

$$a_1 \;=\; 2\cot\pi t, \quad b_1 \;=\; \cot\pi(s-t), \quad b_2 \;=\; -\cot\pi(s+u), \quad c \;=\; \cot\pi u, \quad d \;=\; \cot\pi v,$$

(which are positive on Δ'') and $b_3 = \cot\pi(u+v) = (cd-1)/(c+d)$. Hence, $-\frac{1}{\pi}$ times the Hesse matrix of F is given by

$$\begin{pmatrix} a_1+b_1 & -b_1 & 0 & 0 \\ -b_1 & b_1+b_2 & b_2 & 0 \\ 0 & b_2 & 2c+b_2-b_3 & -b_3 \\ 0 & 0 & -b_3 & 2d-b_3 \end{pmatrix}. \tag{183}$$

It is sufficient to show that this matrix is positive definite on Δ''.

This matrix is related by elementary transformation as a quadratic form to the direct sum of $\big(a_1+b_1\big)$ and the following matrix,

$$\begin{pmatrix} a_2+b_2 & b_2 & 0 \\ b_2 & 2c+b_2-b_3 & -b_3 \\ 0 & -b_3 & 2d-b_3 \end{pmatrix},$$

where we put

$$a_2 \;=\; b_1 - \frac{b_1^2}{a_1+b_1} \;=\; \frac{a_1 b_1}{a_1+b_1} \;>\; 0.$$

Further, the above matrix is related by elementary transformation as a quadratic form to the direct sum of $\big(a_2+b_2\big)$ and the following matrix,

$$\begin{pmatrix} a_3+2c-b_3 & -b_3 \\ -b_3 & 2d-b_3 \end{pmatrix}, \tag{184}$$

where we put

$$a_3 \;=\; b_2 - \frac{b_2^2}{a_2+b_2} \;=\; \frac{a_2 b_2}{a_2+b_2} \;>\; 0.$$

Furthermore,

$$\text{trace}\big(\text{the matrix (184)}\big) \;=\; a_3+2(c+d)-2b_3 \;=\; a_3+\frac{2}{c+d}\,(c^2+d^2+cd+1) \;>\; 0,$$

$$\det\big(\text{the matrix (184)}\big) \;=\; a_3(2d-b_3)+4cd+2(c+d)b_3 \;=\; a_3\frac{cd+2d^2+1}{c+d}+2cd+2 \;>\; 0.$$

Therefore, the two eigenvalues of the matrix (184) are positive, and the matrix (184) is positive definite. Hence, the matrix (183) is positive definite, as required. □

The following lemma is used in the above proof of Lemma A.7.

Lemma A.8. *We put* $G(t,s,u,v) = 2\,\Lambda(t)+\Lambda(s-t)-\Lambda(s+u)-\Lambda(u+v)+2\,\Lambda(v)$*. On the domain*

$$\Delta \;=\; \big\{(t,s,u,v)\in\mathbb{R}^4 \;\big|\; 0\le t\le s\le 1-u, \quad 0\le u,v, \quad u+v\le 1\big\},$$

an upper bound of G *is given by* $G(t,s,u,v)\le 1.12$*.*

Proof. A maximal point of G is whether a solution of $\frac{\partial G}{\partial t}=\frac{\partial G}{\partial s}=\frac{\partial G}{\partial u}=\frac{\partial G}{\partial v}=0$ or a point on the boundary of Δ. We show that G is bounded by 1.12 at such a critical point and at the boundary of Δ.

We show that G is bounded by 1.12 at a critical point of G in the interior of Δ, as follows. We calculate a solution of the above mentioned equation. The differentials of G are given by

$$\begin{aligned}
\frac{\partial G}{\partial t} &= 2\,\Lambda'(t)-\Lambda'(s-t) \;=\; -2\log 2\sin\pi t+\log 2\sin\pi(s-t), \\
\frac{\partial G}{\partial s} &= \Lambda'(s-t)-\Lambda'(s+u) \;=\; -2\log 2\sin\pi(s-t)+\log 2\sin\pi(s+u), \\
\frac{\partial G}{\partial u} &= -\Lambda'(s+u)-\Lambda'(u+v) \;=\; 2\log 2\sin\pi(s+u)+\log 2\sin\pi(u+v), \\
\frac{\partial G}{\partial v} &= 2\,\Lambda'(v)-\Lambda'(u+v) \;=\; -2\log 2\sin\pi v+\log 2\sin\pi(u+v).
\end{aligned}$$

Hence, the above mentioned equation is rewritten

$$\begin{aligned} 2\sin^2\pi t &= \sin\pi(s-t) = \sin\pi(s+u), \\ 4\sin\pi(s+u)\sin\pi(u+v) &= 1, \qquad 2\sin^2\pi v = \sin\pi(u+v). \end{aligned} \tag{185}$$

Since $\sin\pi(s-t) = \sin\pi(s+u)$, we have that $s-t = s+u$ or $2s-t+u=1$; we choose $2s-t+u=1$, since we consider a solution in the interior of Δ. We put

$$a = \tan\frac{\pi t}{2}, \qquad b = \tan\frac{\pi s}{2}, \qquad c = \tan\frac{\pi u}{2}, \qquad d = \tan\frac{\pi v}{2}.$$

Then, we have that $\sin\pi t = \frac{2a}{1+a^2}$, $\cos\pi t = \frac{1-a^2}{1+a^2}$, $\cdots$, and so on. Since $2\sin^2\pi t = \sin\pi(s-t)$, we have that $2\big(\frac{2a}{1+a^2}\big)^2 = \frac{2b}{1+b^2}\cdot\frac{1-a^2}{1+a^2} - \frac{1-b^2}{1+b^2}\cdot\frac{2a}{1+a^2}$. Hence,

$$4a^2(1+b^2) = \big(b(1-a^2) - a(1-b^2)\big)(1+a^2).$$

Similarly, since $2\sin^2\pi v = \sin\pi(u+v)$, we have that

$$4d^2(1+c^2) = \big(c(1-d^2) + d(1-c^2)\big)(1+d^2).$$

Further, since $16\sin^2\pi t\sin^2\pi v = 4\sin\pi(s+u)\sin\pi(u+v) = 1$, we have that $4\sin\pi t\sin\pi v = 1$. Hence,

$$16ad = (1+a^2)(1+d^2).$$

Furthermore, since $2s-t+u=1$, we have that $\frac{2b}{1+b^2} = \cot\pi s = \tan\big(\frac{\pi}{2} - \pi s\big) = \tan\big(\frac{\pi u}{2} - \frac{\pi t}{2}\big) = \frac{c-a}{1+ac}$. Hence,

$$(1-b^2)(1+ac) = 2b(c-a).$$

Therefore, the system of equations (185) is rewritten

$$\begin{aligned} 4a^2(1+b^2) &= \big(b(1-a^2) - a(1-b^2)\big)(1+a^2), \qquad (1-b^2)(1+ac) = 2b(c-a), \\ 4d^2(1+c^2) &= \big(c(1-d^2) + d(1-c^2)\big)(1+d^2), \qquad 16ad = (1+a^2)(1+d^2). \end{aligned} \tag{186}$$

By Lemma A.9 below, this system of equations has the following unique solution in the interior of Δ,

$$\begin{aligned} &a = 0.23500046..., \quad b = 0.46445467..., \quad c = 1.34645751..., \quad d = 0.30711467..., \\ &t = 0.14693973..., \quad s = 0.27680813..., \quad u = 0.59332346..., \quad v = 0.18969433.... \end{aligned}$$

Its critical value is bounded by $G = 1.11152546... \le 1.12$, as required.

We show that G is bounded by 1.12 on the boundary of Δ, as follows. The boundary of Δ consists of $\{t=0\}$, $\{t=s\}$, $\{s+u=1\}$, $\{u=0\}$, $\{v=0\}$ and $\{u+v=1\}$. On the boundary $\{t=0\}$,

$$G = \Lambda(s) - \Lambda(s+u) - \Lambda(s+v) + 2\Lambda(v) \le 5\Lambda\Big(\frac{1}{6}\Big) = 0.807665... \le 1.12.$$

On the boundary $\{t=s\}$,

$$G = 2\Lambda(t) - \Lambda(t+u) - \Lambda(u+v) + 2\Lambda(v) \le 6\Lambda\Big(\frac{1}{6}\Big) = 0.969198... \le 1.12.$$

On the other parts of the boundary, we can verify that G is bounded by 1.12 in similar ways. Therefore, G is bounded by 1.12 on the boundary of Δ, as required. □

Lemma A.9. *The system of equations* (186) *is rewritten as the following single equation,*

$$
\begin{aligned}
&\alpha^{24} - 8\,\alpha^{23} + 36\,\alpha^{22} - 120\,\alpha^{21} + 330\,\alpha^{20} - 696\,\alpha^{19} + 1156\,\alpha^{18} - 1416\,\alpha^{17} - 1561\,\alpha^{16} \\
&+ 10160\,\alpha^{15} - 28664\,\alpha^{14} + 69712\,\alpha^{13} - 84756\,\alpha^{12} + 88464\,\alpha^{11} - 187032\,\alpha^{10} \\
&- 213776\,\alpha^{9} + 512991\,\alpha^{8} - 472872\,\alpha^{7} + 3604052\,\alpha^{6} - 64088\,\alpha^{5} - 1782934\,\alpha^{4} \\
&+ 333992\,\alpha^{3} - 11718764\,\alpha^{2} + 168728\,\alpha + 9803929 \;=\; 0,
\end{aligned}
$$

where we put $\alpha = (a - \frac{1}{a})/2$*. In particular, there is a unique solution* $a = 0.23500046...$ *such that the corresponding* (t, s, u, v) *is in the interior of* Δ*.*

Proof. We rewrite the system of equations (186) as a single polynomial equation of a, as follows. The first and second equations of (186) are quadratic polynomial equations of b. So, we can remove the term b^2 as a linear sum of these equations, and we can present b by a rational function of a and c. By substituting this rational function into the first (or second) equation of (186), we obtain a polynomial equation of a and c as the numerator of the resulting equation. By calculating concretely, this polynomial equation is given by

$$c^2\big(a^8 + 3a^6 - 61a^4 + a^2\big) - c\,\big(2a^7 + 6a^5 + 6a^3 + 2a\big) + a^6 - 61a^4 + 3a^2 + 1 \;=\; 0.$$

This equation and the third equation of (186) are quadratic polynomial equations of c. Hence, we can remove c from these equation in a similar way as above, and obtain a polynomial equation of a and d. By using the fourth equation of (186) linearly, we can remove the term $a^i d^j$ for $i, j \geq 2$ from this equation. By calculating concretely, the resulting equation is given by

$$
\begin{aligned}
f(a,d) \;=\;& -3974889215 + 13107208\,a - 4160749800\,a^2 + 4192680\,a^3 - 117444612\,a^4 \\
& - 4232248\,a^5 + 66997288\,a^6 + 4307752\,a^7 - 1267066\,a^8 - 400680\,a^9 + 144424\,a^{10} \\
& - 23752\,a^{11} - 7172\,a^{12} + 2392\,a^{13} - 232\,a^{14} - 8\,a^{15} + a^{16} \\
& - 262144\,d - 3891657728\,d^2 - 262144\,d^3 + 82839040\,d^4 - 392192\,d^6 - 16384\,a\,d^7 + 256\,d^8 \\
& + d\,\big(63342362592\,a - 209715440\,a^2 + 2709523808\,a^3 + 142629984\,a^4 - 1098853344\,a^5 \\
& \quad - 74910736\,a^6 + 26818592\,a^7 + 6238528\,a^8 - 2476256\,a^9 + 392688\,a^{10} + 119456\,a^{11} \\
& \quad - 39840\,a^{12} + 3296\,a^{13} + 272\,a^{14} - 32\,a^{15}\big) \\
& + a\,\big(17301504\,d^2 - 1583398912\,d^3 + 10436608\,d^5\big) \;=\; 0.
\end{aligned}
$$

We consider to remove d from this equation and the fourth equation of (186). The fourth equation of (186) is a quadratic polynomial equation of d. Let d_1 and d_2 be its solutions. Then,

$$d_1 + d_2 \;=\; \frac{16\,a}{1 + a^2}, \qquad d_1 d_2 = 1.$$

Further, $f(a, d_1) f(a, d_2)$ is a symmetric polynomial in d_1 and d_2, *i.e.*, a polynomial in $d_1 + d_2$ and $d_1 d_2$. By substituting above formulas, we can remove $d_1 + d_2$ and $d_1 d_2$ from this polynomial, and obtain a polynomial equation of a as the numerator of the resulting formula. By calculating concretely, we obtain the equation of the lemma.

We consider a solution of this equation. By calculating concretely, the positive solutions are given by

$$\begin{aligned} a \ &= 0.23500046..., \quad 0.31673729..., \quad 0.38876234..., \quad 0.39875987..., \\ & 2.42024595..., \quad 2.90127305..., \quad 4.82771874..., \quad 5.46433094.... \end{aligned}$$

Further, by calculating the corresponding (t, s, u, v) concretely, we can verify that the corresponding (t, s, u, v) is in the interior of Δ only when $a = 0.23500046...$. Hence, we obtain the lemma. □

Remark A.10. In the above argument, we use a numerical solution of Lemma A.9 in the proof of Lemma A.8. This argument is practically useful, though it is not rigorous. To be precise, we can estimate this solution concretely as precisely as we need, and we can rewrite this argument by using such an estimate in a rigorous way.

A.6 The domain $\{\mathrm{Re}\,\hat{V} \geq \varsigma_R\}$ is convex for the 7_7 knot

For the 7_7 knot, we recall that

$$\begin{aligned} \mathrm{Re}\,\hat{V}(t,s,u,v) \ &= \ 2\,\Lambda(t) - \Lambda(t+s) + 2\,\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v), \\ \varsigma_R \ &= \ 1.21648... \qquad \big(\text{given in (109)}\big), \\ \Delta \ &= \ \big\{(t,s,u,v) \in \mathbb{R}^4 \ \big| \ 0 \leq t,s,u,v, \quad t+s \leq 1, \quad s+u \leq 1, \quad u+v \leq 1\big\}, \end{aligned}$$

as we put in Section 7. The aim of this section is to show Lemma A.11 below, without using results in Section 7.2. As we mentioned at the beginning of Appendix A, we can show by this lemma that each system of equations in Section 7.2 has exactly two solutions; we use this fact in Section 7.2.

Lemma A.11. *The domain $\big\{(t,s,u,v) \in \Delta \ \big| \ \mathrm{Re}\,\hat{V}(t,s,u,v) \geq \varsigma_R\big\}$ is a compact convex domain in the interior of Δ such that its boundary is a smooth closed hypersurface whose sectional curvatures are positive everywhere.*

Proof. We restrict Δ to

$$\Delta'' \ = \ \big\{(t,s,u,v) \in \Delta \ \big| \ 0 \ < \ t,s,u,v \ < \ 0.5 \ < \ s+u \ < \ 1\big\}$$

in such a way that Δ'' includes the domain $\big\{\mathrm{Re}\,\hat{V} \geq \varsigma_R\big\}$ in Step 1 below. Further, in Step 2 below, we show that, on Δ'', $\mathrm{Re}\,\hat{V}$ is a smooth concave function whose Hesse matrix negative definite. Furthermore, we can verify by concrete calculation that $\mathrm{Re}\,\hat{V} < \varsigma_R$ on the boundary of Δ''. Hence, by Lemma A.1, we obtain the lemma. So, we show Steps 1 and 2 below in the following of this proof.

Step 1: In this step, we show that the domain $\big\{\mathrm{Re}\,\hat{V} \geq \varsigma_R\big\}$ is included in Δ''. Assuming that $\mathrm{Re}\,\hat{V}(t,s,u,v) \geq \varsigma_R$ for $(t,s,u,v) \in \Delta$, we calculate ranges of t, s, u, v and $s+u$.

We calculate a range of $s+u$, as follows. Since $2\Lambda(t) - \Lambda(t+s) + 2\Lambda(s) \leq 4\Lambda(\frac{1}{4})$ and $2\Lambda(u) - \Lambda(u+v) + 2\Lambda(v) \leq 4\Lambda(\frac{1}{4})$ by Lemma A.2,

$$-\Lambda(s+u) \ \geq \ \varsigma_R - 2 \cdot 4\Lambda\Big(\frac{1}{4}\Big) = 0.0502364... \ > \ 0.$$

Hence, we obtain that $0.5 < s+u < 1$.

We calculate a range of v, as follows. Since $2\,\Lambda(t) - \Lambda(t+s) + \Lambda(s) \le 0.46$ and $\Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) \le 0.46$ by Lemma A.12 below and $|\Lambda(\,\cdot\,)| \le \Lambda(\frac{1}{6})$,

$$2\,\Lambda(v) \;=\; \varsigma_R - 2\cdot 0.46 - \Lambda\big(\frac{1}{6}\big) \;=\; 0.134947... \;>\; 0.$$

Hence, we obtain that $0 < v < 0.5$.

We obtain that $0 < t < 0.5$ in the same way as above.

We calculate a range of s, as follows. If s satisfied that $0.5 \le s \le 1$, then $\operatorname{Re}\hat{V}(t,s,u,v) = F_1(t,s) + F_2(s,u,v) \le 0.46 + 0.74 = 1.2 < \varsigma_R$ by Lemmas A.12 and A.13 below, and this is a contradiction. If s was 0, then

$$\operatorname{Re}\hat{V}(t,s,u,v) \;=\; \Lambda(t) + \Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v) \;\le\; 5\,\Lambda\big(\frac{1}{6}\big) \;=\; 0.807665... \;<\; \varsigma_R\,,$$

and this is a contradiction. Hence, we obtain that $0 < s < 0.5$.

We obtain that $0 < u < 0.5$ in the same way.

Therefore, the domain $\big\{\operatorname{Re}\hat{V} \ge \varsigma_R\big\}$ is included in Δ'', as required.

Step 2: In this step, we show that $\operatorname{Re}\hat{V}$ is a smooth concave function whose Hesse matrix is negative definite on Δ''.

We recall that

$$\operatorname{Re}\hat{V}(t,s,u,v) \;=\; \big(2\,\Lambda(t) - \Lambda(t+s) + 2\,\Lambda(s)\big) - \Lambda(s+u) + \big(2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)\big).$$

Its Hesse matrix is equal to the sum of the Hesse matrix of $-\Lambda(s+u)$ (whose entries are negative on Δ'') and the direct sum of the Hesse matrices of $2\,\Lambda(t) - \Lambda(t+s) + 2\,\Lambda(s)$ and $2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)$ (which are negative definite on Δ'' by Lemma A.2). Therefore, $\operatorname{Re}\hat{V}$ is a smooth concave function whose Hesse matrix is negative definite on Δ'', as required. □

The following two lemmas are used in the above proof of Lemma A.11.

Lemma A.12. *We put $F_1(t,s) = 2\,\Lambda(t) - \Lambda(t+s) + \Lambda(s)$. On the domain*

$$\Delta_1 \;=\; \big\{(t,s) \in \mathbb{R}^2 \;\big|\; 0 \le t, s, \quad t+s \le 1\big\},$$

an upper bound of F_1 is given by $F_1(t,s) \le 0.46$.

Proof. A maximal point of F_1 is whether a solution of $\frac{\partial F_1}{\partial t} = \frac{\partial F_1}{\partial s} = 0$ or a point on the boundary of Δ_1. We show that F_1 is bounded by 0.46 at such critical points and at the boundary of Δ_1.

We show that F_1 is bounded by 0.46 at critical points of F_1 in the interior of Δ_1, as follows. We calculate a solution of the above mentioned equation. The differentials of F_1 are given by

$$\frac{\partial F_1}{\partial t} \;=\; 2\,\Lambda'(t) - \Lambda'(t+s) \;=\; -2\log 2\sin \pi t + \log 2\sin\pi(t+s),$$

$$\frac{\partial F_1}{\partial s} = \Lambda'(s) - \Lambda'(t+s) = -\log 2\sin \pi s + \log 2 \sin \pi(t+s).$$

Hence, the above mentioned equation is rewritten

$$2\sin^2 \pi t = \sin \pi(t+s) = \sin \pi s.$$

Since $\sin \pi(t+s) = \sin \pi s$, we have that $t = 0$ or $t + 2s = 1$; we choose $t + 2s = 1$, since we consider solutions in the interior of Δ_1. Putting $t = 1 - 2s$,

$$\sin \pi s = 2\sin^2 \pi(1-2s) = 2\sin^2 2\pi s = 8\sin^2 \pi s \cos^2 \pi s.$$

Hence, $8 \sin \pi s \cos^2 \pi s = 1$. Further, putting $x = \sin \pi s$, we have that $8x(1-x^2) = 1$. We have the following two positive solutions of this equation,

$$x = 0.1270508..., \qquad 0.9304029... .$$

The corresponding values of (t,s) are given by

$$(t,s) = (0.9188977..., 0.0405511...), \qquad (0.2389143..., 0.3805428...).$$

Further, the corresponding values of F_1 are given by

$$F_1 = -0.0800755..., \qquad 0.4587632... .$$

They are bounded by 0.46, as required.

We show that F_1 is bounded by 0.46 on the boundary of Δ_1. The boundary of Δ_1 consists of $\{t = 0\}$, $\{s = 0\}$ and $\{t+s = 1\}$. On the boundary $\{t = 0\}$, we have that $F_1 = 0 \leq 0.46$. On the boundary $\{s = 0\}$, we have that $F_1 = \Lambda(t) \leq \Lambda(\frac{1}{6}) = 0.161533... < 0.46$. On the boundary $\{t+s = 1\}$, we have that $F_1 = \Lambda(t) < 0.46$ in the same way as above. Therefore, F_1 is bounded by 0.46 on the boundary of Δ_1, as required. □

Lemma A.13. *We put $F_2(s,u,v) = \Lambda(s) - \Lambda(s+u) + 2\,\Lambda(u) - \Lambda(u+v) + 2\,\Lambda(v)$. On the domain*

$$\Delta_2 = \big\{(s,u,v) \in \mathbb{R}^3 \bigm| 0 \leq u, v, \quad 0.5 \leq s, \quad s+u \leq 1, \quad u+v \leq 1\big\},$$

an upper bound of F_2 is given by $F_2(s,u,v) \leq 0.74$.

Proof. A maximal point of F_2 is whether a solution of $\frac{\partial F_2}{\partial s} = \frac{\partial F_2}{\partial u} = \frac{\partial F_2}{\partial v} = 0$ or a point on the boundary of Δ_2. We show that F_2 is bounded by 0.74 at such critical points and at the boundary of Δ_2.

We show that F_2 has no critical points in the interior of Δ_2, as follows. We calculate a solution of the above mentioned equation. The differential of F_2 by s is given by

$$\frac{\partial F_2}{\partial s} = \Lambda'(s) - \Lambda'(s+u) = -\log 2 \sin \pi s + \log 2 \sin \pi(s+u).$$

Hence, the equation $\frac{\partial F_2}{\partial s} = 0$ is rewritten

$$\sin \pi s = \sin \pi(s+u).$$

Since $0.5 \le s \le s+u \le 1$, the above equation has no solution in the interior of Δ_2. Therefore, F_2 has no critical points in the interior of Δ_2.

We show that F_2 is bounded by 0.74 on the boundary of Δ_2, as follows. The boundary of Δ_2 consists of the boundary $\{u=0\}$, $\{v=0\}$, $\{s=0.5\}$, $\{s+u=1\}$ and $\{u+v=1\}$. On the boundary $\{u=0\}$,

$$F_2 \;=\; \Lambda(v) \;\le\; \Lambda\big(\frac{1}{6}\big) \;=\; 0.161533... \;\le\; 0.74.$$

On the boundary $\{v=0\}$,

$$F_2 \;=\; \Lambda(s)-\Lambda(s+u)+\Lambda(u) \;\le\; 3\,\Lambda\big(\frac{1}{6}\big) \;=\; 0.484599... \;\le\; 0.74.$$

On the boundary $\{s=0.5\}$, we have that $F_2 \le 0.74$ by Lemma A.14 below. On the boundary $\{s+u=1\}$,

$$F_2 \;=\; \Lambda(u)-\Lambda(u+v)+2\,\Lambda(v) \;\le\; 4\,\Lambda\big(\frac{1}{6}\big) \;=\; 0.646132... \;\le\; 0.74.$$

On the boundary $\{u+v=1\}$,

$$F_2 \;=\; \Lambda(s)-\Lambda(s+u) \;\le\; 2\,\Lambda\big(\frac{1}{6}\big) \;=\; 0.323066... \;\le\; 0.74.$$

Therefore, F_2 is bounded by 0.74 on the boundary of Δ_2, as required. $\square$

The following lemma is used in the above proof of Lemma A.13.

Lemma A.14. *We put* $G_2(u,v) = -\Lambda(u+0.5)+2\,\Lambda(u)-\Lambda(u+v)+2\,\Lambda(v)$. *On the domain* $\big\{(u,v)\in\mathbb{R}^2 \;\big|\; 0\le u\le 0.5,\quad 0\le v\le 1-u\big\}$, *an upper bound of* G_2 *is given by* $G_2(u,v)\le 0.74$.

Proof. A maximal point of G_2 is whether a solution of $\frac{\partial F_2}{\partial u} = \frac{\partial F_2}{\partial v} = 0$ or a point on the boundary of the domain of the lemma. We can see that G_2 is bounded by 0.74 on the boundary of the domain of the lemma as a special case of the proof of Lemma A.13. Hence, it is sufficient to show that G_2 is bounded by 0.74 at critical points of G_2 in the interior of the domain of the lemma.

We show that G_2 is bounded by 0.74 at critical points of G_2 in the interior of the domain of the lemma, as follows. The differentials of G_2 are given by

$$\begin{aligned}\frac{\partial G_2}{\partial u} \;&=\; 2\,\Lambda'(u)-\Lambda'(u+0.5)-\Lambda'(u+v)\\ &=\; -2\log 2\sin\pi u+\log 2\sin\pi(u+0.5)+\log 2\sin\pi(u+v),\\ \frac{\partial G_2}{\partial v} \;&=\; 2\,\Lambda'(v)-\Lambda'(u+v) \;=\; -\log 2\sin\pi v+\log 2\sin\pi(u+v).\end{aligned}$$

Hence, the equation $\frac{\partial G_2}{\partial u} = \frac{\partial G_2}{\partial v} = 0$ is rewritten

$$\sin^2\pi u \;=\; \sin\pi\big(u+\frac{1}{2}\big)\,\sin\pi(u+v), \qquad 2\sin^2\pi v \;=\; \sin\pi(u+v). \tag{187}$$

We put $y = \cos \pi u = \sin \pi (u + \frac{1}{2})$. Then, $\sin^2 \pi u = 1 - y^2$. Hence, from the first equation of (187), we have that $\sin \pi(u+v) = \frac{1-y^2}{y}$. Therefore, from the second equation of (187), we have that $\sin^2 \pi v = \frac{1-y^2}{2y}$. Hence, from the second equation of (187), we have that

$$\begin{aligned} \frac{1-y^2}{y} &= \sin \pi(u+v) = \sin \pi u \cos \pi v + \cos \pi u \sin \pi v \\ &= \sqrt{1-y^2} \sqrt{\frac{y^2+2y-1}{2y}} + y \sqrt{\frac{1-y^2}{2y}} . \end{aligned}$$

Therefore, we can show by some concrete calculation that

$$8\, y^5 + 16\, y^4 - 16\, y^3 - 15\, y^2 + 4\, y + 4 = 0.$$

This equation has the following two positive real solutions,

$$y = 0.6198210... , \quad 0.9612566... .$$

The corresponding values of (u, v) are given by

$$(u,v) = (0.2872051... ,\ 0.2489731...), \quad (0.0888946... ,\ 0.0637065...).$$

Further, the corresponding values of G_2 are given by

$$G_2 = 0.7353326... , \quad 0.4259419... ,$$

which are bounded by 0.74. Hence, G_2 is bounded by 0.74 at critical points of G_2 in the interior of the domain of the lemma, as required. □

A.7 Proofs of estimates for the 7_2 knot in Section 8.2.6

For the 7_2 knot, in Section 8.2.6, we give estimates that $t_1 + t_2 \le 0.9$ and $t_1 + t_2 + t_3 \le 1.2$ assuming that

$$\operatorname{Re} \hat{V}(\mathbf{t}) = 2\,\Lambda(t_1) + \Lambda(t_2) + \Lambda(t_3) + \Lambda(t_4) \ \ge\ \varsigma_{_R} \tag{188}$$

in the domain

$$\big\{ \mathbf{t} \in \Delta'' \bigm| t_1 \le 0.5, \quad t_2, t_3, t_4 \le 0.7 \big\}, \tag{189}$$

where we recall that $\varsigma_{_R} = 0.530263...$ as given in (147). In this section, we show rigorous proofs of these estimates in Sections A.7.1 and A.7.2 respectively. Unlike the previous sections, we give direct proofs of them in this section.

A.7.1 Proof of the estimate that $t_1 + t_2 \le 0.9$

In this section, we show that $t_1 + t_2 \le 0.9$ assuming (188) in the domain (189). That is, we show that we can restrict the domain (189) to the domain $t_1 + t_2 \le 0.9$ in such a way that the removed part is included in the domain $\{\operatorname{Re} \hat{V}(\mathbf{t}) < \varsigma_{_R}\}$. We recall that, by (188),

$$2\,\Lambda(t_1) + \Lambda(t_2) \ \ge\ \varsigma_{_R} - 2\,\Lambda\big(\frac{1}{6}\big) = 0.207197... .$$

Hence, it is sufficient to show the following lemma.

Lemma A.15. *The domain*

$$\big\{(t_1,t_2)\in\mathbb{R}^2 \;\big|\; t_1\le 0.5,\quad t_2\le 0.7,\quad t_1+t_2\ge 0.9\big\} \tag{190}$$

is included in the domain $\{2\,\Lambda(t_1)+\Lambda(t_2)\le 0.2\}$.

Proof. It is sufficient to show that, in the domain (190), the value of $2\,\Lambda(t_1)+\Lambda(t_2)$ is bounded by 0.2. We note that $0.2\le t_1\le 0.5$ and $0.4\le t_2\le 0.7$ in the domain (190). Since the behavior of $\Lambda(t)$ is as mentioned in Section 2.2, in these ranges, $\Lambda(t_1)$ and $\Lambda(t_2)$ are monotonically decreasing with respect to t_1 and t_2 respectively. Hence, it is sufficient to show that, on the interval

$$\big\{(t_1,t_2)\in\mathbb{R}^2 \;\big|\; t_1\le 0.5,\quad t_2\le 0.7,\quad t_1+t_2= 0.9\big\},$$

the value of $2\,\Lambda(t_1)+\Lambda(t_2)$ is bounded by 0.2. Therefore, putting $f(t)=2\,\Lambda(t)+\Lambda(0.9-t)$, it is sufficient to show that $f(t)\le 0.2$ for $0.2\le t\le 0.5$.

We show that $f(t)\le 0.2$ for $0.2\le t\le 0.5$, as follows. We consider a maximal point of $f(t)$. It is given by

$$f'(t) \;=\; 2\,\Lambda'(t)-\Lambda'(0.9-t) \;=\; 0.$$

Since $\Lambda'(t)=-\log 2\sin\pi t$, we have that

$$2\log 2\sin\pi t \;=\; \log 2\sin\pi(0.9-t).$$

Hence,

$$2\sin^2\pi t \;=\; \sin\pi(0.9-t) \;=\; \sin\pi\cdot 0.9\,\cos\pi t \;-\; \cos\pi\cdot 0.9\,\sin\pi t.$$

Therefore, by putting $a=\sin\pi t$,

$$2a^2 \;=\; \sqrt{1-a^2}\,\sin\pi\cdot 0.9 \;-\; a\,\sin\pi\cdot 0.9.$$

Hence,

$$a^2\big(2a+\cos\pi\cdot 0.9\big)^2 \;=\; (1-a^2)\,\sin^2\pi\cdot 0.9.$$

This equation has the following unique positive real solution,

$$a_0 \;=\; 0.65417005\ldots.$$

The corresponding values of t and $f(t)$ are given by

$$t_0 \;=\; 0.22698193\ldots, \qquad f(t_0) \;=\; 0.19462761\ldots \;\le\; 0.2.$$

Further, we can verify this t_0 is the maximal point of $f(t)$ in $0.2\le t\le 0.5$. Therefore, $f(t)\le 0.2$ for $0.2\le t\le 0.5$, as required. □

A.7.2 Proof of the estimate that $t_1+t_2+t_3\le 1.2$

In this section, we show that $t_1+t_2+t_3\le 1.2$ assuming (188) in the domain (189). That is, we show that we can restrict the domain (189) to the domain $t_1+t_2+t_3\le 1.2$ in such a way that the removed part is included in the domain $\{\operatorname{Re}\hat V(\mathbf{t})<\varsigma_{R}\}$. We recall that, by (188),

$$2\,\Lambda(t_1)+\Lambda(t_2)+\Lambda(t_3) \;\ge\; \varsigma_{R}-\Lambda\big(\frac{1}{6}\big) \;=\; 0.36873\ldots\,.$$

Hence, it is sufficient to show the following lemma.

Lemma A.16. *The domain*

$$\big\{(t_1,t_2,t_3)\in\mathbb{R}^3 \;\big|\; t_1\le 0.5,\quad t_2,t_3\le 0.7,\quad t_1+t_2+t_3\ge 1.2\big\} \tag{191}$$

is included in the domain $\big\{2\,\Lambda(t_1)+\Lambda(t_2)+\Lambda(t_3)\le 0.35\big\}$.

Proof. It is sufficient to show that, in the domain (191), the value of $2\,\Lambda(t_1)+\Lambda(t_2)+\Lambda(t_3)$ is bounded by 0.35. A maximal point of $2\,\Lambda(t_1)+\Lambda(t_2)+\Lambda(t_3)$ is whether a maximal point of $2\,\Lambda(t_1)+\Lambda(t_2)+\Lambda(t_3)$ in the interior of the domain (191) or a point on the boundary of the domain (191). Since the maximal points of $\Lambda(t_1)$, $\Lambda(t_2)$ and $\Lambda(t_3)$ are $t_1=\frac{1}{6}$, $t_2=\frac{1}{6}$ and $t_3=\frac{1}{6}$ respectively, there is no maximal point of $2\,\Lambda(t_1)+\Lambda(t_2)+\Lambda(t_3)$ in the interior of the domain (191). Further, as shown in Section 8.2.4, the parts $\{t_1=0.5\}$, $\{t_2=0.7\}$ and $\{t_3=0.7\}$ are included in the domain $\{\operatorname{Re}\hat{V}(\mathbf{t})<\varsigma_{R}\}$. Hence, it is sufficient to show that, on the domain

$$\big\{(t_1,t_2,t_3)\in\mathbb{R}^3 \;\big|\; t_1\le 0.5,\quad t_2,t_3\le 0.7,\quad t_1+t_2+t_3=1.2\big\},$$

the value of $2\,\Lambda(t_1)+\Lambda(t_2)+\Lambda(t_3)$ is bounded by 0.35. Therefore, putting

$$g(t_2,t_3) \;=\; 2\,\Lambda(1.2-t_2-t_3)+\Lambda(t_2)+\Lambda(t_3),$$

it is sufficient to show that $g(t_2,t_3)\le 0.35$ in the domain

$$\big\{(t_2,t_3)\in\mathbb{R}^2 \;\big|\; 0\le t_2,t_3\le 0.7,\quad 0.7\le t_2+t_3\le 1.2\big\}. \tag{192}$$

We show that $g(t_2,t_3)\le 0.35$ in the domain (192), as follows. A maximal point of $g(t_2,t_3)$ is whether a maximal point of $g(t_2,t_3)$ in the interior of the domain (192), or a point on the boundary of the domain (192). The boundary of (192) consists of $\{t_2=0.7\}$, $\{t_3=0.7\}$, $\{t_2+t_3=0.7\}$, $\{t_2+t_3=1.2\}$, and we can verify by concrete calculation that $g(t_2,t_3)\le 0.35$ on this boundary. We show that $g(t_2,t_3)\le 0.35$ at any critical point of $g(t_2,t_3)$ in the interior of the domain (192), as follows. The differentials of g are given by The differentials of G_2 are given by

$$\begin{aligned}
\frac{\partial g}{\partial t_2} &= -2\,\Lambda'(1.2-t_2-t_3)+\Lambda'(t_2) \;=\; 2\log 2\sin\pi(1.2-t_2-t_3)-\log 2\sin\pi t_2\,,\\
\frac{\partial g}{\partial t_3} &= -2\,\Lambda'(1.2-t_2-t_3)+\Lambda'(t_3) \;=\; 2\log 2\sin\pi(1.2-t_2-t_3)-\log 2\sin\pi t_3\,.
\end{aligned}$$

Hence, a critical point of g is given by

$$\sin\pi t_2 \;=\; 2\sin^2\pi(1.2-t_2-t_3) \;=\; \sin\pi t_3\,.$$

Since $\sin\pi t_2=\sin\pi t_3$, we have that $t_2=t_3$ or $t_2+t_3=1$. If $t_2+t_3=1$, we have that $\sin\pi t_2=\sin\pi t_3=2\sin^2\pi\cdot 0.2$, and we can verify by concrete calculation that this equation has no solution in the domain (192). If $t_2=t_3$, we have that $2\sin^2\pi(1.2-2t_2)=\sin\pi t_2$. By putting $t=0.6-t_2$, this equation is rewritten $2\sin^2 2\pi t=\sin\pi(0.6-t)$. Hence,

$$8\sin^2\pi t\,\cos^2\pi t \;=\; \sin\pi\cdot 0.6\,\cos\pi t \;-\; \cos\pi\cdot 0.6\,\sin\pi t.$$

Putting $a = \tan\frac{\pi t}{2}$, we have that $\sin\pi t = \frac{2a}{1+a^2}$ and $\cos\pi t = \frac{1-a^2}{1+a^2}$. Therefore, the above equation is rewritten

$$8(2a)^2(1-a^2)^2 \;=\; (1+a^2)^3\big((1-a^2)\,\sin\pi\cdot 0.6 \;-\; 2a\,\cos\pi\cdot 0.6\big).$$

This equation has the following three positive real solutions,

$$a = 0.1985194\ldots, \quad 0.7542454\ldots, \quad 1.1583976\ldots.$$

We can verify by concrete calculation that, only from the first solution, we obtain a solution $t_2 = t_3 = 0.4752406...$ in the domain (192), and the corresponding critical value is given by $g(t_2, t_2) = 0.3261675... \leq 0.35$. Therefore, we obtain that $g(t_2, t_3) \leq 0.35$ in the domain (192), as required. □

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