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The Combinatorial Mono-anabelian Geometry of Curves over Algebraically Closed Fields of Characteristic $p > 0$

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Abstract

In the present paper, we develop a theory of the combinatorial anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ from the point of view of mono-anabelian geometry. We prove that the semi-graphs of anabelioids associated to pointed stable curves over algebraically closed fields of characteristic $p > 0$ can be mono-anabelian reconstructed from their admissible fundamental groups; moreover, we prove that the mono-anabelian reconstruction algorithm of two pointed stable curves with same type are compatible with open continuous homomorphisms of the admissible fundamental groups under certain assumptions. These results can be regarded as mono-anabelian versions of the combinatorial Grothendieck conjecture of curves over algebraically closed fields of characteristic $p > 0$. As an application, under certain assumptions, we obtain that two pointed stable curves with same type over an algebraic closure of $\mathbb{F}_p$ are isomorphic as schemes if and only the set of open continuous homomorphisms between the admissible fundamental groups of the pointed stable curves are not empty.

Keywords: pointed stable curve, fundamental group, combinatorial Grothendieck conjecture, mono-anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14G32; Secondary 14H30.

Contents

Introduction

In the present paper, we develop a theory of the combinatorial anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$. Before we explain the main problem that motivated the theory developed in the present paper, let us recall some general facts concerning the combinatorial anabelian geometry of curves.

Frequently, in the theory of the anabelian geometry of curves, one observes that, before starting to reconstruct the scheme structure of a curve, it necessary to reconstruct the cusps (cf. [N, Theorem 3.4], [M3, Lemma 1.3.9]) or the entire dual semi-graph associated to a pointed stable curve group-theoretically from some associated fundamental group (cf. [M2, §1 $\sim$ §5]). The techniques for doing this is various diverse situations are quite similar and only require much weaker assumptions than the assumptions that ofter hold in particular situations of interest. In order to give a unified theory concerning this topic, S. Mochizuki developed the theory of semi-graphs of anabelioids and the theory of the combinatorial anabelian geoemtry of curves (cf. [M5], [M6]). We do not recall the theory of semi-graphs of anabelioids in the present paper. Roughly speaking, a semi-graph of anabelioids (cf. [M5, Definition 2.1]) is a semi-graph (cf. [M5, Section 1]) which is equipped with a Galois category at each vertex and each edge, together with gluing isomorphisms that satisfy certain conditions; a semi-graph of anabelioids of PSC-type (cf. [M6, Definition 1.1]) is a semi-graph of anabelioids that is isomorphic to the semi-graph of anabelioids associated a pointed stable curve defined over an algebraically closed field. Let

$$X_i^\bullet := (X_i, D_{X_i}),\ i \in \{1, 2\}$$

be a pointed stable curve of type (g, n) over an algebraically closed field k_i and $\Pi_{X_i^\bullet}$ the admissible fundamental group (note that the admissible fundamental group is naturally isomorphic to the tame fundamental group if $X_i^\bullet$ is smooth over k_i) of $X_i^\bullet$ by choosing a base point (cf. Definition 1.2). Here, $X_i,\ i \in \{1, 2\}$ denotes the underlying scheme of $X_i^\bullet$, and D_{X_i} denotes the set of marked points of $X_i^\bullet$. For each $i \in \{1, 2\}$, write

$$\mathcal{G}_{X_i^\bullet}$$

for the semi-graph of anabelioids of PSC-type associated $X_i^\bullet$, $\Gamma_{X_i^\bullet}$ for the dual semi-graph of $X_i^\bullet$, $v(\Gamma_{X_i^\bullet})$ for the set of vertices of $\Gamma_{X_i^\bullet}$, and $e(\Gamma_{X_i^\bullet})$ for the set of edges of $\Gamma_{X_i^\bullet}$. By choosing a base point, we obtain the fundamental group $\Pi_{\mathcal{G}_{X_i^\bullet}}$ of $\mathcal{G}_{X_i^\bullet}$ which is naturally isomorphic to $\Pi_{X_i^\bullet}$; moreover, by choosing a suitable base point, we have $\Pi_{\mathcal{G}_{X_i^\bullet}} = \Pi_{X_i^\bullet}$.

On the other hand, for each $v \in v(\Gamma_{X_i^\bullet})$, write $\widetilde{X}_{i,v}$ for the normalization of the irreducible component of X_i corresponding to v and

$$\widetilde{X}_{i,v}^\bullet := (\widetilde{X}_{i,v}, D_{\widetilde{X}_{i,v}})$$

for the smooth pointed stable curve over k_i determined by $\widetilde{X}_{i,v}$ and the divisor of marked points $D_{\widetilde{X}_{i,v}}$ determined by the inverse images (via the natural morphism $\widetilde{X}_{i,v} \to X_i$) in $\widetilde{X}_{i,v}$ of the nodes and marked points of $X_i^\bullet$; $(g_{i,v}, n_{i,v})$ for the type of $\widetilde{X}_{i,v}^\bullet$. Then $\mathcal{G}_{X_i^\bullet}$, $i \in \{1,2\}$, contains the following information of the pointed stable curve $X_i^\bullet$:

- g_{X_i}, n_{X_i}, and $\Gamma_{X_i^\bullet}$;
- the conjugacy class of the inertia group of every marked point of $X_i^\bullet$ in $\Pi_{X_i^\bullet}$;
- the conjugacy class of the inertia group of every node of $X_i^\bullet$ in $\Pi_{X_i^\bullet}$;
- for each $v \in v(\Gamma_{X_i^\bullet})$, $g_{i,v}$, $n_{i,v}$, and the conjugacy class of the admissible fundamental group of $\widetilde{X}_{i,v}^\bullet$ in $\Pi_{X_i^\bullet}$.

The combinatorial anabelian geometry of curves is a theory which studying how much information about the isomorphism class of a semi-graph of anabelioids of PSC-type is contained already in its fundamental group. The main question of interest in the theory of the combinatorial anabelian geometry of curves is as follows:

Question 0.1. *Can we reconstruct the isomorphism class of the semi-graph of anabelioids of PSC-type associated to a pointed stable curve over an algebraically closed field group-theoretically from the isomorphism class of the admissible fundamental group of the pointed stable curve with a certain outer Galois action (i.e., reconstruct an isomorphism of the semi-graphs of anabelioids of PSC-type associated to given pointed stable curves group-theoretically from a continuous isomorphism of the admissible fundamental groups of the pointed stable curves over algebraically closed fields with certain out Galois actions)?*

The formulation of Question 0.1 is called the **combinatorial Grothendieck conjecture** for semi-graphs of anabelioids of PSC-type or, simply, the combinatorial Grothendieck conjecture, for short.

The combinatorial Grothendieck conjecture was first proved by Mochizuki in the case of outer Galois representations of IPSC-type (i.e., an outer Galois representation induced by the fundamental exact sequence of log étale fundamental groups arising from a stable log curve over a log point whose underlying scheme is an algebraically closed field, and whose log structure is $\mathbb{N}$ (cf. [M6, Example 2.5 and Corollary 2.8])). Essentially, Mochizuki proved the combinatorial Grothendieck conjecture as follows:

Theorem 0.2. *Suppose that* $\mathrm{char}(k_1) = \mathrm{char}(k_2) = 0$. *Let* $\alpha : \Pi_{\mathcal{G}_{X_1^\bullet}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{X_2^\bullet}}$ *a continuous isomorphism of profinite groups,* I_1 *and* I_2 *profinite groups,* $\rho_{I_1} : I_1 \to \mathrm{Out}(\Pi_{\mathcal{G}_{X_1^\bullet}})$ *and* $\rho_{I_2} : I_2 \to \mathrm{Out}(\Pi_{\mathcal{G}_{X_2^\bullet}})$ *outer Galois representations, and* $\beta : I_1 \xrightarrow{\sim} I_2$ *a continuous isomorphism* *of profinite groups. Suppose that ρ_{I_1} and ρ_{I_2} are outer Galois representations of IPSC-type, and that the diagram*

$$\begin{array}{ccc} I_1 & \xrightarrow{\rho_{I_1}} & \mathrm{Out}(\Pi_{\mathcal{G}_{X_1^\bullet}}) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\mathrm{Out}(\alpha)} \\ I_2 & \xrightarrow{\rho_{I_2}} & \mathrm{Out}(\Pi_{\mathcal{G}_{X_2^\bullet}}), \end{array}$$

is commutative, where $\mathrm{Out}(\Pi_{\mathcal{G}_{X_i^\bullet}})$, $i \in \{1, 2\}$, *denotes* $\mathrm{Aut}(\Pi_{\mathcal{G}_{X_i^\bullet}})/\mathrm{Inn}(\Pi_{\mathcal{G}_{X_i^\bullet}})$, *and* $\mathrm{Out}(\alpha)$ *denotes the isomorphism induced by* α. *Then we have*

$$\mathcal{G}_{X_1^\bullet} \cong \mathcal{G}_{X_2^\bullet}.$$

Remark 0.2.1. Suppose that $\mathrm{char}(k_i) = p \geq 0$, $i \in \{1, 2\}$. Let Σ be a set of prime numbers such that $p \notin \Sigma$, $i \in \{1, 2\}$. In fact, Theorem 0.2 also holds if, for each $i \in \{1, 2\}$, we replace Π_{X_i} by the maximal pro-ℓ quotients Π_{X_i} and replace $\mathcal{G}_{X_i^\bullet}$ by the semi-graph of anabelioids of pro-Σ PSC-type associated to $X_i^\bullet$.

Remark 0.2.2. Y. Hoshi and Mochizuki generalized Theorem 0.2 to the case of certain outer Galois representations of NN-type (i.e., an outer Galois representation induced by the fundamental exact sequence of log étale fundamental groups arising from a stable log curve over a log point whose underlying scheme is an algebraically closed field, and whose log structure induced by the log structure of a node of a stable log curves (cf. [HM1, Definition 2.4 and Theorem A]), [HM3, Theorem 1.9]). The proof of Mochizuki (or Hoshi and Mochizuki) requires the use of **the highly non-trivial outer Galois representations** (e.g. by using weight-monodromy conjecture for curves). For more details on the theory of combinatorial anabelian geometry of curves **in characteristic** 0 (or the theory of **prime-to-p** combinatorial anabelian geometry of curves) and its applications, see [HM1], [HM2], [HM3], [HM4], [HM5], [M6], [M7].

On the other hand, some developments of F. Pop, M. Raynaud, M. Saïdi, and A. Tamagawa (cf. [PS], [R], [T1], [T2], [T3]) from the 1990's showed evidence for very strong anabelian phenomena for curves over **algebraically closed fields of characteristic** $p > 0$. One of the main steps of the establishing a theory of anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ is reconstructing the semi-graphs of anabelioids of PSC-type from their admissible fundamental groups. When the base fields are algebraically closed fields of characteristic $p > 0$, **the Galois groups of the base fields are trivial**, and the tame (or étale) fundamental groups coincide with the geometric fundamental groups, thus in a total absence of a Galois action of the base field. In this situation, the reconstructions of the semi-graphs of anabelioids of PSC-type are quite non-trivial even the pointed stable curves are smooth.

In the case of smooth pointed stable curves, Tamagawa proved that we can reconstruct an isomorphism of semi-graphs of anabelioids of PSC-type associated to given smooth pointed stable curves (i.e., the genus, the cardinality of the set of marked points, the conjugacy class of inertia subgroups of each marked points) over algebraically closed fields of characteristic $p > 0$ group-theoretically from a continuous isomorphism of the tame (or étale) fundamental group of the smooth pointed stable curves (cf. [T2, Theorem

0.1, Lemma 5.1, and Theorem 5.2] (or [T1, Theorem 1.9, Theorem 2.5, and Theorem 2.7])).

On the other hand, at the present, almost all the results concerning the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ (i.e., Grothendieck's anabelian conjecture, or simply, the Grothendieck conjecture, for curves over algebraically closed fields of characteristic $p > 0$) were proved only in the case where the base fields are algebraic closures of $\mathbb{F}_p$. One of main goals of the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ is extending [T2, Theorem 0.2] and [T3, Theorem 0.1] to the case where the base fields are arbitrary algebraically closed fields of characteristic $p > 0$. In [Y2], the author established a relationship between the Grothendieck conjecture for curves over an algebraic closure of $\mathbb{F}_p$ and the Grothendieck conjecture for curves over arbitrary algebraically closed fields of characteristic $p > 0$ (cf. [Y2, Conjecture 7.8 and Theorem 7.9]), and observed that,

> to establish the relationship, we should not only prove that we can reconstruct an isomorphism of semi-graphs of anabelioids of PSC-type associated to given smooth pointed stable curves group-theoretically from a continuous isomorphism of the tame fundamental group of the smooth pointed stable curves, but also should prove that we can reconstruct a π_1-**epimorphism** of semi-graphs of anabelioids of PSC-type (cf. [M4, Definition 1.1.12]) associated to given smooth pointed stable curves of same type group-theoretically from an **open continuous surjective homomorphism** of the tame fundamental group of the smooth pointed stable curves of same type.

In order to extend the main results of [T2], [T3], and [Y2] to the case of (possibly singular) pointed stable curves over algebraically closed fields of characteristic $p > 0$, we may consider a similar question of Question 0.1 in positive characteristic as follows:

Question 0.3. *Can we reconstruct the isomorphism class of the semi-graph of anabelioids of PSC-type associated to an arbitrary pointed stable curve over an algebraically closed field of characteristic $p > 0$ group-theoretically from the isomorphism class of the admissible fundamental group of the pointed stable curve without any outer Galois actions? Moreover, can we reconstruct a unramified π_1-epimorphism (cf. Definition 9.2) of the semi-graphs of anabelioids of PSC-type associated to given pointed stable curves of same type over algebraically closed fields of characteristic $p > 0$ group-theoretically from an open continuous homomorphism of the admissible fundamental groups of the pointed stable curves without any outer Galois actions?*

Remark 0.3.1. Note that, in the case of algebraically closed fields of characteristic 0, then the isomorphism class of the admissible fundamental group of a pointed stable curve depends only on the genus and the cardinality of the set of marked points. Thus, no anabelian geometry exists in this situation.

In the present paper, we develop a theory of the combinatorial anabelian geometry of curves over **algebraically closed fields of characteristic** $p > 0$ from the point of view of **mono-anabelian geometry** and solve Question 0.3. The classical point of view of anabelian geometry (i.e., the anabelian geometry considered in [G1], [G2]) focuses on a comparison between two geometric objects via their fundamental groups. Moreover, the term "group-theoretic", in the classical point of view, means that "preserved by an arbitrary isomorphism between the fundamental groups under consideration". The classical point of view is referred to as **bi-anabelian geometry**. On the other hand, mono-anabelian geometry focuses on the establishing a **group-theoretic algorithm** whose input datum is an abstract topological group which is isomorphic to the fundamental group of a given geometric object of interest (resp. a continuous homomorphism of abstract topological groups which are isomorphic to the fundamental groups of given geometric objects of interest), and whose output datum is a geometry object which is isomorphic to the given geometric object (resp. a morphism of geometric objects which is isomorphic to the given geometric objects of interest). In the point of view of mono-anabelian geometry, the term "group-theoretic algorithm" is used to mean that "the algorithm in a discussion is phrased in language that only depends on the topological group structure of the fundamental groups under consideration" (cf. [H] for more details concerning the philosophy of mono-anabelian geometry). Note that, in general, we have

$$\text{mono-anabelian-type results} \Rightarrow \text{bi-anabelian-type results}.$$

From now on, we suppose that $\mathrm{char}(k_i) = p > 0$, $i \in \{1, 2\}$. The first main result of the present paper is as follows, which can be regarded as a mono-anabelian version of combinatorial Grothendieck conjecture for isomorphisms (cf. Theorem 9.1):

Theorem 0.4. *There exists a group-theoretic algorithm whose input datum is* Π_{X_i}, $i \in \{1, 2\}$, *and whose output datum is* $\mathcal{G}_{X_i^\bullet}$.

Remark 0.4.1. The bi-anabelian version of Theorem 0.4 has been proven by the author (cf. [Y1]). This means that, if $\Pi_{X_1^\bullet} \cong \Pi_{X_2^\bullet}$, then we have $\mathcal{G}_{X_1^\bullet} \cong \mathcal{G}_{X_2^\bullet}$.

Remark 0.4.2. If $X_i^\bullet$, $i \in \{1, 2\}$, are smooth over k_i, then Theorem 0.4 has been obtained by Tamagawa (cf. [T2, Theorem 0.5 and Theorem 5.2]).

Moreover, unlike the case of characteristic 0, there exists an open continuous surjective homomorphism

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

which **is not an isomorphism** even X_1 and X_2 are same type (g, n) (e.g. a specialization map (cf. [T3, Theorem 0.3])). Note that all the open continuous homomorphism between $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$ are surjections. The "moreover" part of Question 0.3 means whether or not the group-theoretic algorithm associated to $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$ obtained in Theorem 0.4 are compatible with ϕ. In other words, we have the following question:

> Does there exist a group-theoretic algorithm whose input datum is ϕ, and whose output datum is a morphism of semi-graph of anabelioids of PSC-type $\mathcal{G}_{X_1^\bullet} \to \mathcal{G}_{X_2^\bullet}$?

For this question, we have the second main result of the present paper as follows, which can be regarded as a mono-anabelian version of the combinatorial Grothendieck conjecture for surjections (cf. Theorem 9.3 for more precise form):

Theorem 0.5. *For each* $i \in \{1, 2\}$, *suppose that* X_i *satisfied the following conditions* *(i) the genus of the normalization of each irreducible component of $X_i^\bullet$ is positive;*

(ii) $\Gamma_{X_i^\bullet}$ is 2-connected (cf. Definition 1.1 (b));

(iii) $\#(v(\Gamma^{\mathrm{cpt}}_{X_i^\bullet})^{b\leq 1}) = 0$ (cf. Definition 1.1 (c) (d));

(iv) $\#e(\Gamma_{X_1^\bullet}) = \#e(\Gamma_{X_2^\bullet})$ and $\#v(\Gamma_{X_1^\bullet}) = \#v(\Gamma_{X_2^\bullet})$.

Then there exists a group-theoretic algorithm whose input datum is an open continuous homomorphism $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$, and whose output datum is a unramified π_1-epimorphism (cf. Definition 9.2) of semi-graphs of anabelioids of PSC-type $\Phi : \mathcal{G}_{X_1^\bullet} \to \mathcal{G}_{X_2^\bullet}$.

Let $\overline{\mathbb{F}}_{p,i} \subseteq k_i$, $i \in \{1, 2\}$ be the algebraic closure of $\mathbb{F}_p$ in k_i. By combining [T3, Theorem 0.1] and [Y2, Theorem 0.4], we obtain the following result concerning the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$, which is the third main result of the present paper and generalizes [T2, Theorem 0.2], [T3, Theorem 0.1], and [Y2, Theorem 0.4] to certain pointed stable curves (possibly singular) (cf. Theorem 10.1):

Theorem 0.6. *(a) Suppose that, for each $i \in \{1, 2\}$ and each $v \in v(\Gamma_{X_i^\bullet})$, $(g_{i,v}, n_{i,v})$ is equal to either $(0, n_{i,v})$ or $(1, 1)$. Moreover, suppose that $p \neq 2$ when there exits $v \in v(\Gamma_{X_i^\bullet})$ such that $(g_{i,v}, n_{i,v}) = (1, 1)$.*

(a-i) Suppose that $k_1 = \overline{\mathbb{F}}_{p,1}$ and $k_2 = \overline{\mathbb{F}}_{p,2}$, and that $X_1^\bullet$ is an irreducible pointed stable curve over $\overline{\mathbb{F}}_p$. Then we can detect whether or not $X_1^\bullet$ is isomorphic to a pointed irreducible component (cf. Section 10) of $X_2^\bullet$ as schemes group-theoretically from $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$.

(a-ii) Suppose that $k_1 = \overline{\mathbb{F}}_{p,1}$, that $(g, n) = (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$, that

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

an open continuous surjective homomorphism, and that there exists an isomorphism of dual semi-graphs

$$\rho : \Gamma_{X_1^\bullet} \xrightarrow{\sim} \Gamma_{X_2^\bullet}$$

such that, for each $v \in v(\Gamma_{X_1^\bullet})$, $(g_{1,v}, n_{1,v}) = (g_{2,\rho(v)}, n_{2,\rho(v)})$. Let $X^\bullet_{q_{X_2}}$ be a minimal model $X^\bullet_{q_{X_2}}$ of $X_2^\bullet$. Then $X^\bullet_{q_{X_2}}$ is a pointed stable curve over $\overline{\mathbb{F}}_{p,2}$; moreover, if we suppose that $X^\bullet_{q_{X_2}} = X_2^\bullet$ (i.e., $k_2 = \overline{\mathbb{F}}_{p,2}$), then, for each $v \in v(\Gamma_{X_1^\bullet})$, $X^\bullet_{1,v}$ is isomorphic to $X^\bullet_{2,\rho(v)}$ as schemes. In particular, if $X_i^\bullet$, $i \in \{1, 2\}$, is irreducible, then $X_1^\bullet$ is isomorphic to $X^\bullet_{q_{X_2}}$ as schemes if and only if

$$\mathrm{Hom}^{\mathrm{open}}(\Pi_{X_1^\bullet}, \Pi_{X_2^\bullet}) \neq \emptyset,$$

where $\mathrm{Hom}^{\mathrm{open}}(-, -)$ denotes the set of open continuous homomorphisms of profinite groups.

(b) Suppose that $k_i = \overline{\mathbb{F}}_{p,i}$, $i \in \{1, 2\}$. Then there are at most finitely many $\overline{\mathbb{F}}_{p,i}$-isomorphism classes of irreducible pointed stable curves over $\overline{\mathbb{F}}_{p,i}$ whose admissible fundamental groups are isomorphic to the admissible fundamental group of a pointed irreducible component of $X_i^\bullet$.

Finally, let us explain the ideas of the proofs of Theorem 0.4 and Theorem 0.5. Let $i \in \{1, 2\}$. For simplicity, we assume that $X_i^\bullet$ satisfies the conditions (i)$\sim$(iv) of Theorem 0.5, and that the p-rank (cf. Definition 1.3) of the normalization of each irreducible component of $X_i^\bullet$ are positive. For each open subgroup $H_i \subseteq \Pi_{X_i}$, write $X_{H_i}^\bullet$ for the pointed stable curve of type $(g_{X_{H_i}}, n_{X_{H_i}})$ over k_i corresponding to H_i and $\Gamma_{X_{H_i}^\bullet}$ for the dual semi-graph of $X_{H_i}^\bullet$.

Our method of proving Theorem 0.4 is as follows. The main difficult is, for each open subgroup $H_i \subseteq \Pi_{X_i}$, proving that the profinite completion of the topological fundamental group of $\Gamma_{X_{H_i}^\bullet}$ and the étale fundamental group of the underlying curve of $X_{H_i}^\bullet$ (or the weight-monodromy filtration of the first ℓ-adic étale cohomology group of $X_{H_i}^\bullet$, where $\ell \neq p$) can be **mono-anabelian reconstructed** (cf. Definition 3.1) from H_i. Moreover, by applying the general theory of admissible coverings of pointed stable curves, it is sufficient to prove that $(g_{X_{H_i}}, n_{X_{H_i}})$ and the Betti number $r_{X_{H_i}}$ of $\Gamma_{X_{H_i}^\bullet}$ can be mono-anabelian reconstructed from H_i. In order to do that, we have the following key observation:

> Tamagawa's theorem concerning the limit of p-average
>
> $$\mathrm{Arv}_p(H_i)$$
>
> of H_i (cf. Definition 1.4 and Theorem 1.5) plays a role of (outer) Galois representations in the theory of the combinatorial anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

By using the p-Galois admissible coverings (i.e., Galois admissible coverings whose Galois groups are isomorphic to p-groups), the Betti number $r_{X_{H_i}}$ can be mono-anabelian reconstructed from H_i. Thus, Theorem 1.5 implies that the $(g_{X_{H_i}}, n_{X_{H_i}})$ can be mono-anabelian reconstructed from H_i.

On the other hand, our method of proving Theorem 0.5 is as follows. To verify that the group-theoretic algorithm associated to $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$ obtained in Theorem 0.4 are compatible with a given open continuous surjective homomorphism $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$, we need to prove that, for each $H_2 \subseteq \Pi_{X_2^\bullet}$, the profinite completion of the topological fundamental group of $\Gamma_{X_{H_2}^\bullet}$ and the étale fundamental group of the underlying curve $X_{H_2}^\bullet$ induces the profinite completion of the topological fundamental group of $\Gamma_{X_{H_1}^\bullet}$ and the étale fundamental group of the underlying curve $X_{H_1}^\bullet$ (or the weight-monodromy filtration of the first ℓ-adic étale cohomology group of $X_{H_2}^\bullet$ induces the weight-monodromy filtration of the first ℓ-adic étale cohomology group of $X_{H_1}^\bullet$, where $\ell \neq p$) group-theoretically from the natural surjection $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$, where $H_1 := \phi^{-1}(H_2)$. In order to do that, we prove that $(g_{X_{H_1}}, n_{X_{H_1}}) = (g_{X_{H_2}}, n_{X_{H_2}})$, and that X_{H_i}, $i \in \{1, 2\}$, satisfies the conditions (i)$\sim$(iv) of Theorem 0.5. Then Theorem 0.5 follows from the computations of admissible coverings of pointed stable curves by applying the following key observation:

> The inequality of the limit of p-averages (cf. Remark 1.5.3)
>
> $$\mathrm{Arv}_p(H_1) \geq \mathrm{Arv}_p(H_2)$$
>
> of H_1 and H_2 induced by the surjection $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$ plays a role of the comparability of (outer) Galois representations in the theory of the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ (in fact, we have $\mathrm{Arv}_p(H_1) = \mathrm{Arv}_p(H_2)$ (cf. Corollary 9.5)).

The present paper is organized as follows. In Section 1, we review some definitions and results which will be used in the present paper. In Section 2, we establish a correspondence between a subset of line bundles and the set of vertices (resp. the set of edges, the set of genera of irreducible components) of a pointed stable curve. In Section 3, by applying the results obtained in Section 2, we give a mono-anabelian reconstruction algorithm for dual semi-graph of a pointed stable curve from its admissible fundamental group. In Section 4$\sim$6, we reconstruct the sets of vertices (resp. the sets of edges, the sets of genera of irreducible components, the sets of p-ranks of irreducible components) via surjections of the admissible fundamental groups of pointed stable curves. In Section 7, we give a mono-anabelian reconstruction algorithm for the isomorphisms of dual semi-graphs of pointed stable curves from surjections of the admissible fundamental groups of pointed stable curves. In Section 8, we prove that, there exists cofinal systems of open subgroups of the admissible fundamental groups of pointed stable curves such that the pointed stable curves corresponding to the open subgroups contained in the cofinal systems satisfy the conditions (i)$\sim$(iv) of Theorem 0.5. In Section 9, by using the results obtained in previous sections, we prove Theorem 0.4 and Theorem 0.5. In Section 10, we apply Theorem 0.5 to the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ and obtain Theorem 10.1.

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1 Preliminaries

In this section, we recall some definitions and results which will be used in the present paper.

Definition 1.1. Let $\mathbb{G} := (v(\mathbb{G}), e(\mathbb{G}), \{\zeta_e^{\mathbb{G}}\}_{e\in e(\mathbb{G})})$ be a semi-graph (cf. [M5, Section 1]). Here, $v(\mathbb{G})$, $e(\mathbb{G})$, and $\{\zeta_e^{\mathbb{G}}\}_{e\in e(\mathbb{G})}$ denote the set of vertices of $\mathbb{G}$, the set of edges of $\mathbb{G}$, and the set of coincidence maps of $\mathbb{G}$, respectively.

(a) We shall write $e(\mathbb{G})$ for the set of edges, $e^{\mathrm{op}}(\mathbb{G}) \subseteq e(\mathbb{G})$ (resp. $e^{\mathrm{cl}}(\mathbb{G}) \subseteq e(\mathbb{G})$) for the set of **open** (resp. **closed**) edges of $\mathbb{G}$.

(b) Let $v \in v(\mathbb{G})$. We shall call $\mathbb{G}$ **2-connected at** v if $\mathbb{G} \setminus \{v\}$ is either empty or connected.

(c) We define an **one-point compactification** $\mathbb{G}^{\mathrm{cpt}}$ of $\mathbb{G}$ as follows: if $e^{\mathrm{op}}(\mathbb{G}) = \emptyset$, we set $\mathbb{G}^{\mathrm{cpt}} = \mathbb{G}$; otherwise, the set of vertices of $\mathbb{G}^{\mathrm{cpt}}$ is $v(\mathbb{G}^{\mathrm{cpt}}) := v(\mathbb{G}) \coprod \{v_\infty\}$, the set of edges of $\mathbb{G}^{\mathrm{cpt}}$ is $e(\mathbb{G}^{\mathrm{cpt}}) := e(\mathbb{G})$, and each edge $e \in e^{\mathrm{op}}(\mathbb{G}) \subseteq e(\mathbb{G}^{\mathrm{cpt}})$ connects v_∞ with the vertex that is abutted by e.

(d) For each $v \in v(\mathbb{G})$, we set

$$b(v) := \sum_{e\in e(\mathbb{G})} b_e(v),$$

where $b_e(v) \in \{0, 1, 2\}$ denotes the number of times that e meets v. Moreover, we set

$$v(\mathbb{G}^{\mathrm{cpt}})^{b\leq 1} := \{v \in v(\mathbb{G}) \subseteq v(\mathbb{G}^{\mathrm{cpt}}) \mid b(v) \leq 1\}.$$

Next, we fix some notations. Let k be an algebraically closed field and

$$X^\bullet = (X, D_X)$$

a pointed stable curve of type (g_X, n_X) over k. Here, X denotes the underlying scheme of $X^\bullet$, and D_X denotes the set of marked points of $X^\bullet$. Write

$$\Gamma_{X^\bullet}$$

for the dual semi-graph of $X^\bullet$, and Γ_X for the dual graph of X. Note that, by the definitions of $\Gamma_{X^\bullet}$ and Γ_X, we have a natural embedding $\Gamma_X \hookrightarrow \Gamma_{X^\bullet}$; then we may identify $v(\Gamma_X)$ and $e(\Gamma_X)$ with $v(\Gamma_{X^\bullet})$ and $e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, respectively, via the natural embedding $\Gamma_X \hookrightarrow \Gamma_{X^\bullet}$. We denote by

$$\Pi^{\mathrm{top}}_{X^\bullet}$$

for the profinite completion of the topological fundamental group of $\Gamma_{X^\bullet}$, and denotes r_X the Betti number $\dim_{\mathbb{C}}(\mathrm{H}^1(\Gamma_{X^\bullet}, \mathbb{C}))$ of the semi-graph $\Gamma_{X^\bullet}$.

Definition 1.2. Let $Y^\bullet := (Y, D_Y)$ be a pointed stable curve over k and $f^\bullet : Y^\bullet \to X^\bullet$ a morphism of pointed stable curves over $\operatorname{Spec} k$.

We shall call $f^\bullet$ a **Galois admissible covering** over $\operatorname{Spec} k$ (or Galois admissible covering for short) if the following conditions hold:

(i) there exists a finite group $G \subseteq \mathrm{Aut}_k(Y^\bullet)$ such that $Y^\bullet/G = X^\bullet$, and $f^\bullet$ is equal to the quotient morphism $Y^\bullet \to Y^\bullet/G$;

(ii) for each $y \in Y^{\mathrm{sm}} \setminus D_Y$, $f^\bullet$ is étale at y, where $(-)^{\mathrm{sm}}$ denotes the smooth locus of $(-)$;

(iii) for any $y \in Y^{\mathrm{sing}}$, the image $f^\bullet(y)$ is contained in X^{sing}, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$;

(iv) for each $y \in Y^{\mathrm{sing}}$, the local morphism between two nodes induced by $f^\bullet$ may be described as follows:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,f^\bullet(y)} \cong k[[u,v]]/uv & \to & \hat{\mathcal{O}}_{Y,y} \cong k[[s,t]]/st \\ u & \mapsto & s^n \\ v & \mapsto & t^n, \end{array}$$

where $(n, \mathrm{char}(k)) = 1$ if $\mathrm{char}(k) > 0$; moreover, write $D_y \subseteq G$ for the decomposition group of y and $\#D_y$ for the cardinality of D_y; then $\tau(s) = \zeta_{\#D_y} s$ and $\tau(t) = \zeta^{-1}_{\#D_y} t$ for each $\tau \in D_y$, where $\zeta_{\#D_y}$ is a primitive $\#D_y$-th root of unit;

(v) the local morphism between two marked points induced by $f^\bullet$ may be described as follows:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,f^\bullet(y)} \cong k[[a]] & \to & \hat{\mathcal{O}}_{Y,y} \cong k[[b]] \\ a & \mapsto & b^m, \end{array}$$

where $(m, \mathrm{char}(k)) = 1$ if $\mathrm{char}(k) > 0$ (i.e., a tamely ramified extension).

Moreover, we shall call $f^\bullet$ an **admissible covering** if there exists a morphism of pointed stable curves $(f^\bullet)' : (Y^\bullet)' \to Y^\bullet$ over $\mathrm{Spec}\, k$ such that the composite morphism $f^\bullet \circ (f^\bullet)' : (Y^\bullet)' \to X^\bullet$ is a Galois admissible covering over $\mathrm{Spec}\, k$.

Let $Z^\bullet$ be the disjoint union of finitely many pointed stable curves over $\mathrm{Spec}\, k$. We shall call a morphism $Z^\bullet \to X^\bullet$ over $\mathrm{Spec}\, k$ **multi-admissible covering** if the restriction of $Z^\bullet \to X^\bullet$ to each connected component of $Z^\bullet$ is admissible. We use the notation $\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)$ to denote the category which consists of (empty object and) all the multi-admissible coverings of $X^\bullet$. It is well-known that $\mathrm{Cov}^{\mathrm{adm}}(X^\bullet)$ is a Galois category. Thus, by choosing a base point $x \in X^{\mathrm{sm}} \setminus D_X$, we obtain a fundamental group $\pi_1^{\mathrm{adm}}(X^\bullet, x)$ which is called the **admissible fundamental group** of $X^\bullet$. For simplicity of notation, we omit the base point and denote the admissible fundamental group by $\Pi_{X^\bullet}$. Write

$$\Pi_{X^\bullet}^{\text{ét}}$$

for the étale fundamental group of $X^\bullet$ (i.e., the étale fundamental group of X). Note that we have natural surjections (for suitable choices of base points)

$$\Pi_{X^\bullet} \twoheadrightarrow \Pi_{X^\bullet}^{\text{ét}} \twoheadrightarrow \Pi_{X^\bullet}^{\mathrm{top}}.$$

For more details on admissible coverings and the admissible fundamental groups for pointed stable curves, see [M1], [M2].

Remark 1.2.1. Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack of pointed stable curves of type (g,n) over $\mathrm{Spec}\, \mathbb{Z}$ and $\mathcal{M}_{g,n}$ the open substack of $\overline{\mathcal{M}}_{g,n}$ parametrizing pointed smooth curves. Write $\overline{\mathcal{M}}_{g,n}^{\log}$ for the log stack obtained by equipping $\overline{\mathcal{M}}_{g,n}$ with the natural log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ relative to $\mathrm{Spec}\, \mathbb{Z}$.

The pointed stable curve $X^\bullet \to \mathrm{Spec}\, k$ induces a morphism $\mathrm{Spec}\, k \to \overline{\mathcal{M}}_{g_X,n_X}$. Write $s_X^{\log}$ for the log scheme whose underlying scheme is $\mathrm{Spec}\, k$, and whose log structure is the pulling-back log structure induced by the morphism $\mathrm{Spec}\, k \to \overline{\mathcal{M}}_{g_X,n_X}$. We obtain a natural morphism $s_X^{\log} \to \overline{\mathcal{M}}_{g_X,n_X}^{\log}$ induced by the morphism $\mathrm{Spec}\, k \to \overline{\mathcal{M}}_{g_X,n_X}$ and a stable log curve $X^{\log} := s_X^{\log} \times_{\overline{\mathcal{M}}_{g_X,n_X}^{\log}} \overline{\mathcal{M}}_{g_X,n_X+1}^{\log}$ over $s_X^{\log}$ whose underlying scheme is X. Then the admissible fundamental group $\Pi_{X^\bullet}$ of $X^\bullet$ is naturally isomorphic to the geometric log étale fundamental group of $X^{\log}$ (i.e., $\mathrm{Ker}(\pi_1(X^{\log}) \to \pi_1(s_X^{\log}))$).

Remark 1.2.2. If $X^\bullet$ is smooth over k, by the definition of admissible fundamental groups, then the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the tame fundamental group of $X \setminus D_X$.

In the remainder of the present paper, we suppose that the characteristic of k is $p > 0$.

Definition 1.3. We define the p**-rank** of $X^\bullet$ to be

$$\sigma(X^\bullet) := \dim_{\mathbb{F}_p}(\Pi_{X^\bullet}^{\mathrm{ab}} \otimes \mathbb{F}_p) = \dim_{\mathbb{F}_p}(\Pi_{X^\bullet}^{\text{ét},\mathrm{ab}} \otimes \mathbb{F}_p),$$

where $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$.

Remark 1.3.1. For each $v \in v(\Gamma_{X^\bullet})$, write X_v for the irreducible components of X corresponding to v. Then it is easy to prove that

$$\sigma(X^\bullet) = \sigma(X) = \sum_{v \in v(\Gamma_{X^\bullet})} \sigma(\widetilde{X_v}) + r_X,$$

where $\widetilde{(-)}$ denotes the normalization of $(-)$.

Definition 1.4. Let Π be a profinite group, n a natural number, and ℓ a prime number.

(i) We denote by $\Pi(n)$ the topological closure of the subgroup $[\Pi, \Pi]\Pi^n$ of Π. Note that $\Pi/\Pi(n) = \Pi^{\mathrm{ab}} \otimes (\mathbb{Z}/n\mathbb{Z})$.

(ii) We set $\gamma_\ell(\Pi(n)) := \dim_{\mathbb{F}_\ell}(\Pi/\Pi(n)) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

(iii) Let n be a natural number such that $[\Pi : \Pi(n)] < \infty$. We define **ℓ-average** of Π to be

$$\gamma_\ell^{\mathrm{av}}(n)(\Pi) := \gamma_\ell(\Pi(n))/[\Pi : \Pi(n)] \in \mathbb{Q}_{\geq 0} \cup \{\infty\}.$$

Morever, suppose that $[\Pi : \Pi(\ell^t - 1)] < \infty$ for each natural number $t \in \mathbb{N}$. We denote by

$$\mathrm{Arv}_\ell(\Pi) := \lim_{t \to \infty} \gamma_\ell^{\mathrm{av}}(\ell^t - 1)(\Pi) \in \mathbb{Q}_{\geq 0} \cup \{\infty\},$$

and we shall call $\mathrm{Arv}_\ell(\Pi)$ **the limit of ℓ-average of H**.

The following highly nontrivial result concerning the limit of p-average of $X^\bullet$ was proved by Tamagawa (cf. [T4, Theorem 3.10]), which plays a fundamental role in the theory of combinatorial anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

Theorem 1.5. *Suppose that, for any $v \in v(\Gamma_{X^\bullet}) \subseteq v(\Gamma^{\mathrm{cpt}}_{X^\bullet})$, $\Gamma^{\mathrm{cpt}}_{X^\bullet}$ is 2-connected at v. Then we have*

$$\mathrm{Arv}_p(\Pi_{X^\bullet}) = g_X - r_X - \# v(\Gamma^{\mathrm{cpt}}_{X^\bullet})^{b \leq 1}.$$

Remark 1.5.1. Tamagawa proved Theorem 1.5 as a main theorem of [T2] in the case of smooth pointed stable curves by developing a general theory of Raynaud's theta divisor. This result means that, if $X^\bullet$ is a smooth pointed stable curve, then there exists a group-theoretic algorithm whose input datum is $\Pi_{X^\bullet}$, and the output datum is (g_X, n_X). Afterwards, in order to compare the admissible fundamental groups of the generic fiber and the special fiber of a pointed stable curve over a complete discrete valuation ring with positive characteristic residue field, Tamagawa extended the result to the case of arbitrary pointed stable curves by using a result concerning the abelian injectivity of admissible fundamental groups (cf. [T4]).

Remark 1.5.2. Let $Z^\bullet$ be a pointed stable curve over k. Then there exists a prime-to-p solvable Galois admissible covering (i.e, the Galois group of the admissible covering is solvable) $W^\bullet \to Z^\bullet$ such that the genus of the normalization of each irreducible component of $W^\bullet$ is positive, that the dual semi-graph $\Gamma_{W^\bullet}$ of $W^\bullet$ is 2-connected, and that $\#(v(\Gamma^{\mathrm{cpt}}_{W^\bullet})^{b \leq 1}) = 0$.

Remark 1.5.3. Let $X_i^\bullet$, $i \in \{1,2\}$, be pointed stable curves of type (g_{X_i}, n_{X_i}) over an algebraically closed fields k_i of characteristic p and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Suppose that

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

is an open continuous surjective homomorphism, and that $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Since ϕ induces an isomorphism of the maximal prime-to-p quotients of $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$, we have

$$\mathrm{Arv}_p(\Pi_{X_1^\bullet}) \geq \mathrm{Arv}_p(\Pi_{X_2^\bullet}).$$

Definition 1.6. Let $f^\bullet : Y^\bullet \to X^\bullet$ be an admissible covering over k of degree $\deg(f^\bullet)$. For any $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ (resp. $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$), write x_e for the node (resp. marked point) corresponding to e. We define the following sets:

$$e_{f^\bullet}^{\mathrm{cl,ra}} := \{e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \mid \#(f^\bullet)^{-1}(x_e) = 1\},$$

$$e_{f^\bullet}^{\mathrm{cl,\acute{e}t}} := \{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \mid \#(f^\bullet)^{-1}(x_e) = \deg(f^\bullet)\},$$

$$e_{f^\bullet}^{\mathrm{op,ra}} := \{e \in e^{\mathrm{op}}(\Gamma_{X^\bullet}) \mid \#(f^\bullet)^{-1}(x_e) = 1\},$$

$$v_{f^\bullet}^{\mathrm{ra}} := \{v \in v(\Gamma_{X^\bullet}) \mid \text{the number of the irreducible components of } (f^\bullet)^{-1}(X_v) \text{ is } 1\},$$

and

$$v_{f^\bullet}^{\mathrm{sp}} := \{v \in v(\Gamma_{X^\bullet}) \mid \text{the number of the irreducible components of } (f^\bullet)^{-1}(X_v) \text{ is } \deg(f^\bullet)\}.$$

Note that, if the Galois closure of $f^\bullet$ is a p-Galois admissible covering (i.e., the Galois group is a p-group), then the definition of admissible covering implies that

$$\#e_{f^\bullet}^{\mathrm{cl,ra}} = \#e_{f^\bullet}^{\mathrm{op,ra}} = 0.$$

Lemma 1.7. *Let k_i, $i \in \{1,2\}$, be an algebraically closed field of characteristic $p > 0$, ℓ a prime number, $X_i^\bullet$ a pointed stable curve over k_i of type (g,n). Let $f_i^\bullet : Y_i^\bullet \to X_i^\bullet$, $i \in \{1,2\}$, be a Galois* **étale** *covering over k_i of degree ℓ, $\Gamma_{X_i^\bullet}$ and $\Gamma_{Y_i^\bullet}$ the dual semi-graphs of $X_i^\bullet$ and $Y_i^\bullet$, r_{X_i} and r_{Y_i} for the Betti numbers of $\Gamma_{X_i^\bullet}$ and $\Gamma_{Y_i^\bullet}$, respectively. Suppose that $r_{X_1} = r_{X_2}$, that $\#v(\Gamma_{X_1}) = \#v(\Gamma_{X_2})$, and that $\#e(\Gamma_{X_1}) = \#e(\Gamma_{X_2})$. Then we have*

$$\#v_{f_1^\bullet}^{\mathrm{sp}} \geq \#v_{f_2^\bullet}^{\mathrm{sp}} \text{ if and only if } r_{Y_1} \leq r_{Y_2}.$$

Moreover, we have

$$\#v_{f_1^\bullet}^{\mathrm{sp}} = \#v_{f_2^\bullet}^{\mathrm{sp}} \text{ if and only if } r_{Y_1} = r_{Y_2}.$$

Proof. Since $X_1^\bullet$ and $X_2^\bullet$ are same type, we have $\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) = \#e^{\mathrm{cl}}(\Gamma_{X_2^\bullet})$. Moreover, since $f_1^\bullet$ and $f_2^\bullet$ are étale coverings, we have

$$r_{Y_1} = \ell\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) - \#v(\Gamma_{X_1^\bullet}) - (\ell-1)\#v_{f_1^\bullet}^{\mathrm{sp}} + 1$$

and

$$r_{Y_2} = \ell\#e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) - \#v(\Gamma_{X_2^\bullet}) - (\ell-1)\#v_{f_2^\bullet}^{\mathrm{sp}} + 1.$$

Then we obtain that $r_{Y_1} \leq r_{Y_2}$ if and only if $\#v_{f_1^\bullet}^{\mathrm{sp}} \geq \#v_{f_2^\bullet}^{\mathrm{sp}}$, and that $r_{Y_1} = r_{Y_2}$ if and only if $\#v_{f_1^\bullet}^{\mathrm{sp}} = \#v_{f_2^\bullet}^{\mathrm{sp}}$. □

2 Line bundles, sets of vertices, and sets of edges

We maintain the notations introduced in Section 1. Let ℓ be a prime number. We define a subset of $v(\Gamma_{X^\bullet})$ to be

$$v(\Gamma_{X^\bullet})^{>0,\ell} := \{v \in v(\Gamma_{X^\bullet}) \mid \dim_{\mathbb{F}_\ell} \mathrm{H}^1_{\text{ét}}(\widetilde{X_v}, \mathbb{F}_\ell) > 0\}.$$

Write $M^{\text{ét}}_{X^\bullet}$ and $M^{\text{top}}_{X^\bullet}$ for $\mathrm{H}^1_{\text{ét}}(X^\bullet, \mathbb{F}_\ell)$ and $\mathrm{H}^1(\Gamma_{X^\bullet}, \mathbb{F}_\ell)$, respectively. Note that there is a natural injection $M^{\text{top}}_{X^\bullet} \hookrightarrow M^{\text{ét}}_{X^\bullet}$ induced by the natural surjection $\Pi_{X^\bullet} \twoheadrightarrow \Pi^{\text{top}}_{X^\bullet}$. Moreover, we take

$$M^{\text{ntop}}_{X^\bullet} := \operatorname{coker}(M^{\text{top}}_{X^\bullet} \hookrightarrow M^{\text{ét}}_{X^\bullet}).$$

The elements of $M^{\text{ét}}_{X^\bullet}$ correspond to étale, Galois abelian coverings of $X^\bullet$ of degree ℓ. Let $V^*_{\ell,X^\bullet} \subseteq M^{\text{ét}}_{X^\bullet}$ be the subset of elements whose image in $M^{\text{ntop}}_{X^\bullet}$ is not 0, and $\alpha \in V^*_{\ell,X^\bullet}$. We denote by

$$X^\bullet_\alpha \to X^\bullet$$

for the étale covering correspond to the line bundle α and denote by $\Gamma_{X^\bullet_\alpha}$ the dual semi-graph of $X^\bullet_\alpha$. Then we obtain a map

$$\iota : V^*_{\ell,X^\bullet} \to \mathbb{Z}$$

that maps $\alpha \mapsto \#(v(\Gamma_{X^\bullet_\alpha}))$. We define

$$V_{\ell,X^\bullet} \subseteq V^*_{\ell,X^\bullet}$$

to be the subset of elements α which ι attains its maximum (i.e., $\iota(\alpha) = \ell(\#v(\Gamma_{X^\bullet})-1)+1$) and define a pre-equivalence relation $\sim$ on $V_{\ell,X^\bullet}$ as follows:

> let $\alpha, \beta \in V_{\ell,X^\bullet}$; then $\alpha \sim \beta$ if , for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \in V^*_{\ell,X^\bullet}$, we have $\lambda\alpha + \mu\beta \in V_{\ell,X^\bullet}$.

Then we have the following result (see also [Y1, Section 2]).

Theorem 2.1. *The pre-equivalence relation $\sim$ on $V_{\ell,X^\bullet}$ defined above is an equivalence relation. Moreover, we have a natural bijection*

$$\kappa_{\ell,X^\bullet} : V_{\ell,X^\bullet}/\sim \xrightarrow{\sim} v(\Gamma_{X^\bullet})^{>0,\ell}.$$

Proof. For any $\delta \in V_{\ell,X^\bullet}$, $\iota(\delta)$ attains its maximum implies that there exists a unique irreducible component $I^\delta_{X^\bullet_\delta} \subseteq X^\bullet_\delta$ whose decomposition group is not trivial. We write $I^\delta_{X^\bullet} \subseteq X^\bullet$ for the image of $I^\delta_{X^\bullet_\delta}$ of the covering morphism $X^\bullet_\delta \to X^\bullet$. Note that $I^\delta_{X^\bullet} \in v(\Gamma_{X^\bullet})^{>0,\ell}$. Then $V_{\ell,X^\bullet} = \emptyset$ if and only if $v(\Gamma_{X^\bullet})^{>0,\ell} = \emptyset$.

We suppose that $v(\Gamma_{X^\bullet})^{>0,\ell} \neq \emptyset$. Let $\alpha, \beta \in V_{\ell,X^\bullet}$. If $I^\alpha_{X^\bullet} = I^\beta_{X^\bullet}$, then, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \neq 0$, we have $I^{\lambda\alpha+\mu\beta}_{X^\bullet} = I^\alpha_{X^\bullet} = I^\beta_{X^\bullet}$. Thus, $\alpha \sim \beta$. On the other hand, if $\alpha \sim \beta$, we have $I^\alpha_{X^\bullet} = I^\beta_{X^\bullet}$; otherwise, there exist two irreducible components of $X^\bullet_{\alpha+\beta}$ whose decomposition groups are not trivial. Thus, $\alpha \sim \beta$ if and only if $I^\alpha_{X^\bullet} = I^\beta_{X^\bullet}$. This means that $\sim$ is an equivalence relation on $V_{\ell,X^\bullet}$.

Moreover, we obtain a natural morphism

$$\kappa_{\ell,X^\bullet} : V_{\ell,X^\bullet}/\sim \to v(\Gamma_{X^\bullet})^{>0,\ell}$$

that maps $[\delta] \mapsto I^{\delta}_{X^\bullet}$, where $[\delta]$ denotes the image of δ in $V_{\ell,X^\bullet}/\sim$. Let us prove that $\kappa_{\ell,X^\bullet}$ is a bijection. It is easy to see that $\kappa_{\ell,X^\bullet}$ is an injection. For any irreducible component $X_v \in v(\Gamma_{X^\bullet})^{>0,\ell}$, we may construct an étale, Galois abelian covering $f^\bullet : Y^\bullet \to X^\bullet$ of degree ℓ such that X_v is the unique irreducible component of $X^\bullet$ whose inverse image $(f^\bullet)^{-1}(X^\bullet_v)$ is connected. Then the cardinality of the set of irreducible components of $Y^\bullet$ is equal to $\ell(\#v(\Gamma_{X^\bullet}) - 1) + 1$. Thus, we obtain an element of $V_{\ell,X^\bullet}$ corresponding to $Y^\bullet$. This means that $\kappa_{\ell,X^\bullet}$ is a surjection. We complete the proof of the theorem. $\square$

Remark 2.1.1. Let $\mathfrak{Primes}$ be the set of prime numbers and $\ell, \ell' \in \mathfrak{Primes}$ prime numbers distinct from each other. Write

$$V_{\ell,X^\bullet}/\sim \ \text{ and } V_{\ell',X^\bullet}/\sim$$

for the sets induced by ℓ and ℓ' defined as above, respectively. Suppose that $v(\Gamma_{X^\bullet})^{>0,\ell} \subseteq v(\Gamma_{X^\bullet})^{>0,\ell'}$. Note that $v(\Gamma_{X^\bullet})^{>0,\ell} = v(\Gamma_{X^\bullet})^{>0,\ell'}$ if ℓ and ℓ' are not equal to p. Then we claim that there is a natural injection

$$V_{\ell,X^\bullet}/\sim \hookrightarrow V_{\ell',X^\bullet}/\sim .$$

For each $\alpha \in V_{\ell,X^\bullet}$ and each $\alpha' \in V_{\ell',X^\bullet}$, we write $Y^\bullet_\alpha \to X^\bullet$ and $Y^\bullet_{\alpha'} \to X^\bullet$ for the Galois admissible coverings corresponding to α and α', respectively. Consider the connected Galois admissible covering

$$Y^\bullet_\alpha \times_{X^\bullet} Y^\bullet_{\alpha'} \to X^\bullet$$

over k with degree $\ell\ell'$. Then it is easy to see that α and α' correspond to same irreducible component if and only if the cardinality of the set of irreducible components of $Y^\bullet_\alpha \times_{X^\bullet} Y^\bullet_{\alpha'} \to X^\bullet$ is equal to

$$\ell\ell'(\#v(\Gamma_{X^\bullet}) - 1) + 1.$$

Then we obtain a natural injection $V_{\ell,X^\bullet}/\sim \hookrightarrow V_{\ell',X^\bullet}/\sim$. In particular, if ℓ and ℓ' are not equal to p, then we have a natural bijection

$$V_{\ell,X^\bullet}/\sim \xrightarrow{\sim} V_{\ell',X^\bullet}/\sim .$$

Remark 2.1.2. Let $g^\bullet : Z^\bullet \to X^\bullet$ be a Galois admissible covering over k with degree $\deg(g^\bullet)$, $\Gamma_{Z^\bullet}$ the dual semi-graph of $Z^\bullet$, and ℓ a prime number such that $(\ell, \deg(g^\bullet)) = 1$. We denote by

$$\gamma^{\mathrm{vex},>0,\ell}_{g^\bullet} : v(\Gamma_{Z^\bullet})^{>0,\ell} \to v(\Gamma_{X^\bullet})^{>0,\ell}$$

the morphism of sets of vertices induced by $g^\bullet$. Write $V_{\ell,Z^\bullet}$ and $V_{\ell,X^\bullet}$ for the sets of line bundles defined as above.

We have a natural map

$$\gamma^{\mathrm{vex},\ell}_{g^\bullet} : V_{\ell,Z^\bullet}/\sim \to V_{\ell,X^\bullet}/\sim$$

defined as follows. For each $\alpha \in V_{\ell,Z^\bullet}$, we may define

$$\gamma^{\mathrm{vex},\ell}_{g^\bullet}([\alpha]) = [\alpha_{X^\bullet}],$$

where $\alpha_{X^\bullet} \in V_{\ell,X^\bullet}$ such that the following conditions are satisfied:

(i) $\alpha_{X^\bullet}$ induced a line bundle $\alpha_{Z^\bullet} = \sum_{\beta \in L_{\alpha_{X^\bullet}}} c_\beta \beta$ via the pull-back morphism induced by $g^\bullet$, where $L_{\alpha_{X^\bullet}}$ is a subset of $V_{\ell, Z^\bullet}$ such that, for any $\beta_1, \beta_2 \in L_{\alpha_{X^\bullet}}$ distinct from each other, then $[\beta_1] \neq [\beta_2]$, $c_{\beta_1} \neq 0$, and $c_{\beta_2} \neq 0$;

(ii) there exists $\beta \in L_{\alpha_{X^\bullet}}$ such that $\beta \sim \alpha$.

It is easy to check that $\gamma_{g^\bullet}^{\mathrm{vex}, \ell}$ is well-defined, and that the following diagram

$$\begin{array}{ccc} V_{\ell, Z^\bullet}/\sim & \xrightarrow{\kappa_{\ell, Z^\bullet}} & v(\Gamma_{Z^\bullet})^{>0, \ell} \\ {\scriptstyle \gamma_{g^\bullet}^{\mathrm{vex}, \ell}} \downarrow & & {\scriptstyle \gamma_{g^\bullet}^{\mathrm{vex}, >0, \ell}} \downarrow \\ V_{\ell, X^\bullet}/\sim & \xrightarrow{\kappa_{\ell, X^\bullet}} & v(\Gamma_{X^\bullet})^{>0, \ell} \end{array}$$

is commutative.

In the remainder of this section, suppose that the genus of the normalization of each irreducible component of $X^\bullet$ is **positive**, that $\Gamma_{X^\bullet}$ is 2-**connected**. We shall call that

$$(\ell, d, f^\bullet : Y^\bullet \to X^\bullet)$$

is a triple associated to $X^\bullet$ if

(i) ℓ and d are prime numbers distinct from each other and from p;

(ii) $\ell \equiv 1 \pmod{d}$; this means that all d^{th} roots of unity are contained in $\mathbb{F}_\ell$; moreover, we write $G_d \subseteq \mathbb{F}_\ell^\times$ for the subgroup of d^{th} roots of unity;

(iii) $f^\bullet : Y^\bullet := (Y, D_Y) \to X^\bullet$ is a Galois **étale** covering whose Galois group is isomorphic to G_d (note that, since the genus of the normalization of each irreducible component of $X^\bullet$ is positive, $f^\bullet$ exists);

(iv) $\# v_{f^\bullet}^{\mathrm{sp}} = 0$.

We fix a triple

$$(\ell, d, f^\bullet : Y^\bullet \to X^\bullet)$$

associated to $X^\bullet$. Write $M_{Y^\bullet}^{\text{ét}}$ and $M_{Y^\bullet}$ for $\mathrm{H}^1_{\text{ét}}(Y^\bullet, \mathbb{F}_\ell) = \mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_\ell)$ and $\mathrm{Hom}(\Pi_{Y^\bullet}, \mathbb{F}_\ell)$, respectively, where $\Pi_{Y^\bullet}$ denotes the admissible fundamental group of $Y^\bullet$. Note that there is a natural injection $M_{Y^\bullet}^{\text{ét}} \hookrightarrow M_{Y^\bullet}$ induced by the natural surjection $\Pi_{Y^\bullet} \twoheadrightarrow \Pi_{Y^\bullet}^{\text{ét}}$. Then we obtain an exact sequence

$$0 \to M_{Y^\bullet}^{\text{ét}} \to M_{Y^\bullet} \to M_{Y^\bullet}^{\mathrm{ra}} := \mathrm{coker}(M_{Y^\bullet}^{\text{ét}} \hookrightarrow M_{Y^\bullet}) \to 0$$

with a natural action of G_d.

Let $M_{Y^\bullet, G_d}^{\mathrm{ra}} \subseteq M_{Y^\bullet}^{\mathrm{ra}}$ be the subset of elements on which G_d acts via the character $G_d \hookrightarrow \mathbb{F}_\ell^\times$, and $U_{\ell, Y^\bullet}^* \subseteq M_{Y^\bullet}$ the subset of elements that map to nonzero elements of $M_{Y^\bullet, G_n}^{\mathrm{ra}}$. Let $\alpha \in U_{\ell, Y^\bullet}^*$. Write

$$g_\alpha^\bullet : Y_\alpha^\bullet \to Y^\bullet$$

for the admissible covering corresponding to the line bundle α. Then we obtain a morphism

$$\epsilon : U^{*}_{\ell,Y^{\bullet}} \to \mathbb{Z}$$

that maps α to $\#e(\Gamma_{Y^{\bullet}_{\alpha}})$, where $\Gamma_{Y^{\bullet}_{\alpha}}$ denotes the dual semi-graph of $Y^{\bullet}_{\alpha}$. We define two subsets of $U^{*}_{\ell,Y^{\bullet}}$ to be

$$U^{\mathrm{nd}}_{\ell,Y^{\bullet}} := \{\alpha \in U^{*}_{\ell,Y^{\bullet}} \mid \#e^{\mathrm{cl,ra}}_{g^{\bullet}_{\alpha}} = d\},$$

and

$$U^{\mathrm{mp}}_{\ell,Y^{\bullet}} := \{\alpha \in U^{*}_{\ell,Y^{\bullet}} \mid \#e^{\mathrm{op,ra}}_{g^{\bullet}_{\alpha}} = d\}.$$

Note that $U^{\mathrm{nd}}_{\ell,Y^{\bullet}}$ (resp. $U^{\mathrm{mp}}_{\ell,Y^{\bullet}}$) is not empty. Moreover, we define a pre-equivalence relation $\sim$ on $U^{\mathrm{nd}}_{\ell,Y^{\bullet}}$ (resp. $U^{\mathrm{mp}}_{\ell,Y^{\bullet}}$) as follows:

> let $\alpha, \beta \in U^{\mathrm{nd}}_{\ell,Y^{\bullet}}$ (resp. $\alpha, \beta \in U^{\mathrm{mp}}_{\ell,Y^{\bullet}}$), then $\alpha \sim \beta$ if, for each $\lambda, \mu \in \mathbb{F}^{\times}_{\ell}$ for which $\lambda\alpha + \mu\beta \in U^{*}_{\ell,Y^{\bullet}}$, we have $\lambda\alpha + \mu\beta \in U^{\mathrm{nd}}_{\ell,Y_{\bullet}}$ (resp. $U^{\mathrm{mp}}_{\ell,Y^{\bullet}}$).

Then we have the following result.

Theorem 2.2. *The pre-equivalence relation $\sim$ on $U^{\mathrm{nd}}_{\ell,Y^{\bullet}}$ (resp. $U^{\mathrm{mp}}_{\ell,Y^{\bullet}}$) defined above is an equivalence relation, and, moreover, the quotient set $U^{\mathrm{nd}}_{\ell,Y^{\bullet}}/\sim$ (resp. $U^{\mathrm{mp}}_{\ell,Y^{\bullet}}/\sim$) is naturally isomorphic to $e^{\mathrm{cl}}(\Gamma_{X^{\bullet}})$ (resp. $e^{\mathrm{op}}(\Gamma_{X^{\bullet}})$).*

Proof. For each $\alpha \in U^{\mathrm{nd}}_{\ell,Y^{\bullet}}$, since the image of α is contained in $M^{\mathrm{ra}}_{Y^{\bullet},G_d}$, we obtain that the action of G_d on the set $\{y_e\}_{e\in e^{\mathrm{cl,ra}}_{g^{\bullet}_{\alpha}}} \subseteq \mathrm{Nod}(Y^{\bullet})$ is transitive, where $\mathrm{Nod}(-)$ denotes the set of nodes of $(-)$. Thus, there exists a unique node x_{α} of $X^{\bullet}$ such that $f^{\bullet}(y_e) = x_{\alpha}$ for each $e \in e^{\mathrm{cl,ra}}_{g^{\bullet}_{\alpha}}$. Write $e_{x_{\alpha}} \in \Gamma_{X^{\bullet}}$ for the edge corresponding to x_{α}.

Let $\beta, \gamma \in U^{\mathrm{nd}}_{\ell,Y^{\bullet}}$. If $e^{\mathrm{cl,ra}}_{g^{\bullet}_{\beta}} = e^{\mathrm{cl,ra}}_{g^{\bullet}_{\gamma}}$, then, for each $\lambda, \mu \in \mathbb{F}^{\times}_{\ell}$ for which $\lambda\beta + \mu\gamma \neq 0$, we have $e^{\mathrm{cl,ra}}_{g^{\bullet}_{\lambda\beta+\mu\gamma}} = e^{\mathrm{cl,ra}}_{g^{\bullet}_{\beta}} = e^{\mathrm{cl,ra}}_{g^{\bullet}_{\gamma}}$. Thus, $\beta \sim \gamma$. On the other hand, if $\beta \sim \gamma$, then we have $e^{\mathrm{cl,ra}}_{g^{\bullet}_{\beta}} = e^{\mathrm{cl,ra}}_{g^{\bullet}_{\gamma}}$; otherwise, we obtain $\#e^{\mathrm{cl,ra}}_{g^{\bullet}_{\beta+\gamma}} = 2d$. Thus, $\beta \sim \gamma$ if and only if $e^{\mathrm{cl,ra}}_{g^{\bullet}_{\beta}} = e^{\mathrm{cl,ra}}_{g^{\bullet}_{\gamma}}$. This means that $\sim$ is an equivalence relation on $U^{\mathrm{nd}}_{\ell,Y^{\bullet}}$.

We define a map

$$\vartheta^{\mathrm{nd}}_{\ell,X^{\bullet}} : U^{\mathrm{nd}}_{\ell,Y^{\bullet}}/\sim \to e^{\mathrm{cl}}(\Gamma_{X^{\bullet}})$$

that maps $[\alpha] \mapsto e_{x_{\alpha}}$, where $[\alpha]$ denotes the image of α in $U^{\mathrm{nd}}_{\ell,Y^{\bullet}}/\sim$. Let us prove that $\vartheta^{\mathrm{nd}}_{\ell,X^{\bullet}}$ is a bijection. It is easy to see that $\vartheta^{\mathrm{nd}}_{\ell,X^{\bullet}}$ is an injection. On the other hand, for each $e \in e^{\mathrm{cl}}(\Gamma_{X^{\bullet}})$, the structure of the maximal pro-ℓ admissible fundamental groups implies that we may construct a Galois covering of $h^{\bullet} : Z^{\bullet} \to Y^{\bullet}$ such that the line bundle corresponding to $h^{\bullet}$ is contained in $U^{\mathrm{nd}}_{\ell,Y^{\bullet}}$. Then $\vartheta^{\mathrm{nd}}_{\ell,X^{\bullet}}$ is a surjection.

Similar arguments to the arguments given in the proof above imply that the "resp" part holds. This completes the proof of the theorem. □

Remark 2.2.1. In this remark, we prove that the sets

$$U^{\mathrm{nd}}_{\ell,Y^{\bullet}}/\sim \ \text{and}\ U^{\mathrm{mp}}_{\ell,Y^{\bullet}}/\sim$$

do not depend on the choices of the triples associated to $X^{\bullet}$.

Let

$$(\ell^*, d^*, f^{\bullet,*} : Y^{\bullet,*} \to X^\bullet)$$

be any triple associated to X. Hence we obtain a resulting set $U^{\mathrm{nd}}_{\ell^*,Y^{\bullet,*}}/\sim$ and a natural bijection

$$\vartheta^{\mathrm{nd}}_{\ell^*,X^\bullet} : U^{\mathrm{nd}}_{\ell^*,Y^{\bullet,*}}/\sim \to e^{\mathrm{cl}}(\Gamma_{X^\bullet}).$$

First, suppose that $\ell \neq \ell^*$, and that $d \neq d^*$. Then there exists a natural bijection

$$U^{\mathrm{nd}}_{\ell^*,Y^{\bullet,*}}/\sim \xrightarrow{\sim} U^{\mathrm{nd}}_{\ell,Y^\bullet}/\sim$$

which compatible with the bijections $\vartheta^{\mathrm{nd}}_{\ell^*,X^\bullet}$ and $\vartheta^{\mathrm{nd}}_{\ell,X^\bullet}$ as follows. Let $\alpha \in U^{\mathrm{nd}}_{\ell,Y^\bullet}$ and $\alpha^* \in U^{\mathrm{nd}}_{\ell^*,Y^{\bullet,*}}$. Write $Y^\bullet_\alpha \to Y^\bullet$ and $Y^{\bullet,*}_{\alpha^*} \to Y^{\bullet,*}$ for the Galois admissible coverings corresponding to α and α^*, respectively. Let us consider

$$Y^\bullet_\alpha \times_{X^\bullet} Y^{\bullet,*}_{\alpha^*}.$$

Thus, we obtain a connected Galois admissible covering $Y^\bullet_\alpha \times_{X^\bullet} Y^{\bullet,*}_{\alpha^*} \to X^\bullet$ of degree $dd^*\ell\ell^*$. Then it is easy to check that α and α^* correspond to same nodes if and only if the cardinality of the set of nodes of $Y^\bullet \times_{X^\bullet} Y^{\bullet,*}$ is equal to

$$dd^*(\ell\ell^*\#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) - 1) + 1).$$

In general case, for any two given triples $(\ell, d, f^\bullet : Y^\bullet \to X^\bullet)$ and $(\ell^*, d^*, f^{\bullet,*} : Y^{\bullet,*} \to X^\bullet)$ associated to $X^\bullet$, we may choose a triple

$$(\ell^{**}, d^{**}, f^{\bullet,**} : Y^{\bullet,**} \to X^\bullet)$$

associated to $X^\bullet$ such that $\ell^{**} \neq \ell$, $\ell^{**} \neq \ell^*$, $d^{**} \neq d$, and $d^{**} \neq d^*$. Hence we obtain a resulting set $U^{\mathrm{nd}}_{\ell^{**},Y^{\bullet,**}}/\sim$ and a natural bijection $\vartheta^{\mathrm{nd}}_{\ell^{**},X^\bullet} : U^{\mathrm{nd}}_{\ell^{**},Y^{\bullet,**}}/\sim \to e^{\mathrm{cl}}(\Gamma_{X^\bullet})$. Then the proof above implies that there are two natural bijections

$$U^{\mathrm{nd}}_{\ell^{**},Y^{\bullet,**}}/\sim \cong U^{\mathrm{nd}}_{\ell,Y^\bullet}/\sim \ \text{ and } U^{\mathrm{nd}}_{\ell^{**},Y^{\bullet,**}}/\sim \cong U^{\mathrm{nd}}_{\ell^*,Y^{\bullet,*}}/\sim .$$

Thus, we obtain $U^{\mathrm{nd}}_{\ell^*,Y^{\bullet,*}}/\sim \cong U^{\mathrm{nd}}_{\ell,Y^\bullet}/\sim$.

Remark 2.2.2. Let $g^\bullet : Z^\bullet \to X^\bullet$ be a Galois admissible covering over k with degree $\deg(g^\bullet)$ and $\Gamma_{Z^\bullet}$ the dual semi-graph of $Z^\bullet$. Let

$$(\ell, d, f^\bullet_X : Y^\bullet_X \to X^\bullet)$$

be a triple associated to $X^\bullet$ such that $(\ell, \deg(g^\bullet)) = (d, \deg(g^\bullet)) = 1$. Then we obtain a triple

$$(\ell, d, f^\bullet_Z : Y^\bullet_Z := Y^\bullet_X \times_{X^\bullet} Z^\bullet \to Z^\bullet)$$

associated to $Z^\bullet$ induced by $(\ell, d, f^\bullet_X : Y^\bullet_X \to X^\bullet)$. Moreover, we obtain two natural maps

$$\gamma^{\mathrm{cl,edge}}_{g^\bullet} : e^{\mathrm{cl}}(\Gamma_{Z^\bullet}) \to e^{\mathrm{cl}}(\Gamma_{X^\bullet})$$

and

$$\gamma^{\mathrm{op,edge}}_{g^\bullet} : e^{\mathrm{op}}(\Gamma_{Z^\bullet}) \to e^{\mathrm{op}}(\Gamma_{X^\bullet})$$

induced by $g^\bullet$. Write $U^{\mathrm{nd}}_{\ell,Y^\bullet_Z}$ and $U^{\mathrm{nd}}_{\ell,Y^\bullet}$ for the sets of line bundles defined in above. Then we have a natural map

$$\gamma^{\mathrm{nd}}_{g^\bullet} : U^{\mathrm{nd}}_{\ell,Y^\bullet_Z}/\sim \to U^{\mathrm{nd}}_{\ell,Y^\bullet}/\sim$$

defined as follows. For each $\alpha \in U^{\mathrm{nd}}_{\ell,Y^\bullet_Z}$, we define

$$\gamma^{\mathrm{nd},\ell}_{g^\bullet}([\alpha]) = [\alpha_{X^\bullet}],$$

where $\alpha_{X^\bullet} \in U^{\mathrm{nd}}_{\ell,Y^\bullet_Z}$ such that the following conditions are satisfied:

(i) $\alpha_{X^\bullet}$ induced a line bundle $\alpha_{Z^\bullet} = \sum_{\beta \in J_{\alpha_{X^\bullet}}} c_\beta \beta$ via the pull-back morphism induced by $g^\bullet$, where $J_{\alpha_{X^\bullet}}$ is a subset of $U^{\mathrm{nd}}_{\ell,Y^\bullet_Z}$ such that, for any $\beta_1, \beta_2 \in J_{\alpha_{X^\bullet}}$ distinct from each other, then $[\beta_1] \neq [\beta_2]$, $c_{\beta_1} \neq 0$, and $c_{\beta_2} \neq 0$;

(ii) there exists $\beta \in J_{\alpha_X}$ such that $\beta \sim \alpha$.

By applying similar arguments to the arguments given above, we obtain a natural map

$$\gamma^{\mathrm{mp}}_{g^\bullet} : U^{\mathrm{mp}}_{\ell,Y^\bullet_Z}/\sim \to U^{\mathrm{mp}}_{\ell,Y^\bullet}/\sim .$$

It is easy to check that $\gamma^{\mathrm{nd}}_{g^\bullet}$ and $\gamma^{\mathrm{mp}}_{g^\bullet}$ are well-defined, and that the following diagrams

$$\begin{array}{ccc} U^{\mathrm{nd}}_{\ell,Y^\bullet_Z}/\sim & \xrightarrow{\vartheta^{\mathrm{nd}}_{\ell,Z^\bullet}} & e^{\mathrm{cl}}(\Gamma_{Z^\bullet}) \\ \gamma^{\mathrm{nd}}_{g^\bullet}\Big\downarrow & & \Big\downarrow \gamma^{\mathrm{cl,edge}}_{g^\bullet} \\ U^{\mathrm{nd}}_{\ell,Y^\bullet}/\sim & \xrightarrow{\vartheta^{\mathrm{nd}}_{\ell,X^\bullet}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}), \end{array}$$

and

$$\begin{array}{ccc} U^{\mathrm{mp}}_{\ell,Y^\bullet_Z}/\sim & \xrightarrow{\vartheta^{\mathrm{mp}}_{\ell,Z^\bullet}} & e^{\mathrm{op}}(\Gamma_{Z^\bullet}) \\ \gamma^{\mathrm{mp}}_{g^\bullet}\Big\downarrow & & \Big\downarrow \gamma^{\mathrm{op,edge}}_{g^\bullet} \\ U^{\mathrm{mp}}_{\ell,Y^\bullet}/\sim & \xrightarrow{\vartheta^{\mathrm{mp}}_{\ell,X^\bullet}} & e^{\mathrm{op}}(\Gamma_{X^\bullet}). \end{array}$$

are commutative.

Next, let us calculate the cardinality $\#U^{\mathrm{nd}}_{\ell,Y^\bullet}$ (resp. $\#U^{\mathrm{mp}}_{\ell,Y^\bullet}$) of the set $U^{\mathrm{nd}}_{\ell,Y^\bullet}$ (resp. $U^{\mathrm{mp}}_{\ell,Y^\bullet}$). We define

$$U^{\mathrm{nd}}_{\ell,Y^\bullet,e} := \{\alpha \in U^{\mathrm{nd}}_{\ell,Y^\bullet} \mid g^\bullet_\alpha \text{ is ramified over } (f^\bullet)^{-1}(x_e)\}$$

$$(\text{resp. } U^{\mathrm{mp}}_{\ell,Y^\bullet,e} := \{\alpha \in U^{\mathrm{nd}}_{\ell,Y^\bullet} \mid g^\bullet_\alpha \text{ is ramified over } (f^\bullet)^{-1}(x_e)\})$$

for each $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ (resp. $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$), where x_e denote the node (resp. the marked point) of $X^\bullet$ corresponding to e. Then, for each $e, e' \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ (resp. $e, e' \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$) distinct from each other, we have

$$U^{\mathrm{nd}}_{\ell,Y^\bullet,e} \cap U^{\mathrm{nd}}_{\ell,Y^\bullet,e'} = \emptyset \text{ (resp. } U^{\mathrm{mp}}_{\ell,Y^\bullet,e} \cap U^{\mathrm{mp}}_{\ell,Y^\bullet,e'} = \emptyset).$$

Moreover, we have

$$U^{\mathrm{nd}}_{\ell,Y^\bullet} = \bigcup_{e\in e^{\mathrm{cl}}(\Gamma_{X^\bullet})} U^{\mathrm{nd}}_{\ell,Y^\bullet,e} \text{ (resp. } U^{\mathrm{mp}}_{\ell,Y^\bullet} = \bigcup_{e\in e^{\mathrm{op}}(\Gamma_{X^\bullet})} U^{\mathrm{mp}}_{\ell,Y^\bullet,e}\text{)}.$$

We fix a closed (resp. an open) edge $e \in e^{\mathrm{cl}}(\Gamma_{X^\bullet})$ (resp. $e \in e^{\mathrm{op}}(\Gamma_{X^\bullet})$). Write Y^{cl}_e (resp. Y^{op}_e) for the normalization of the underlying curve Y of $Y^\bullet$ at $(f^\bullet)^{-1}(x_e)$ and

$$\mathrm{nl}^{\mathrm{cl}}_e : Y^{\mathrm{cl}}_e \to Y \text{ (resp. } \mathrm{nl}^{\mathrm{op}}_e : Y^{\mathrm{op}}_e \to Y\text{)}$$

for the resulting morphism. Since the genus of the normalization of each irreducible component of $X^\bullet$ is positive, and $\Gamma_{X^\bullet}$ is 2-connected, we have that Y^{cl}_e (resp. Y^{op}_e) is connected, and that the genus of the normalization of each irreducible component of Y^{cl}_e (resp. Y^{op}_e) is also positive. Moreover, since the marked points are smooth points of Y, we have $\mathrm{nl}^{\mathrm{op}}_{\mathrm{e}}$ is an identity.

Proposition 2.3. *Write g_Y for the genus of $Y^\bullet$. We have*

$$\#U^{\mathrm{nd}}_{\ell,Y^\bullet,e} = \ell^{2(g_Y-d)+1} - \ell^{2(g_Y-d)} \text{ (resp. } \#U^{\mathrm{mp}}_{\ell,Y^\bullet,e} = \ell^{2g_Y+1} - \ell^{2g_Y}\text{)}.$$

Moreover, we have

$$\#U^{\mathrm{nd}}_{\ell,Y^\bullet} = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet})(\ell^{2(g_Y-d)+1} - \ell^{2(g_Y-d)}) \text{ (resp. } \#U^{\mathrm{mp}}_{\ell,Y^\bullet} = \#e^{\mathrm{op}}(\Gamma_{X^\bullet})(\ell^{2g_Y+1} - \ell^{2g_Y})\text{)}.$$

Proof. Write E^{cl}_e (resp. E^{op}_e) for $(f^\bullet \circ \mathrm{nl}^{\mathrm{cl}}_e)^{-1}(x_e)$ (resp. $(f^\bullet \circ \mathrm{nl}^{\mathrm{op}}_e)^{-1}(x_e)$). Then $U^{\mathrm{nd}}_{\ell,Y^\bullet,e}$ (resp. $U^{\mathrm{mp}}_{\ell,Y^\bullet,e}$) can be naturally regarded as a subset of

$$\mathrm{H}^1_{\text{ét}}(Y^{\mathrm{cl}}_e \setminus E^{\mathrm{cl}}_e, \mathbb{F}_\ell)$$

via the natural open immersion $Y^{\mathrm{cl}}_e \setminus E^{\mathrm{cl}}_e \hookrightarrow Y^{\mathrm{cl}}_e$ (resp. $Y^{\mathrm{op}}_e \setminus E^{\mathrm{op}}_e \hookrightarrow Y^{\mathrm{op}}_e$). Write

$$L^{\mathrm{cl}}_e \text{ (resp. } L^{\mathrm{op}}_e\text{)}$$

for the $\mathbb{F}_\ell$-vector space generated by $U^{\mathrm{nd}}_{\ell,Y^\bullet,e}$ (resp. $U^{\mathrm{mp}}_{\ell,Y^\bullet,e}$) in $\mathrm{H}^1_{\text{ét}}(Y^{\mathrm{cl}}_e \setminus E^{\mathrm{cl}}_e, \mathbb{F}_\ell)$ (resp. $\mathrm{H}^1_{\text{ét}}(Y^{\mathrm{op}}_e \setminus E^{\mathrm{op}}_e, \mathbb{F}_\ell)$). Then we have

$$U^{\mathrm{nd}}_{\ell,Y^\bullet,e} = L^{\mathrm{cl}}_e \setminus \mathrm{H}^1_{\text{ét}}(Y^{\mathrm{cl}}_e, \mathbb{F}_\ell) \text{ (resp. } U^{\mathrm{mp}}_{\ell,Y^\bullet,e} = L^{\mathrm{op}}_e \setminus \mathrm{H}^1_{\text{ét}}(Y^{\mathrm{op}}_e, \mathbb{F}_\ell)\text{)}.$$

Write $H^{\mathrm{cl,ra}}_e$ (resp. $H^{\mathrm{op,ra}}_e$) for $L^{\mathrm{cl}}_e/\mathrm{H}^1_{\text{ét}}(Y^{\mathrm{cl}}_e, \mathbb{F}_\ell)$ (resp. $L^{\mathrm{op}}_e/\mathrm{H}^1_{\text{ét}}(Y^{\mathrm{op}}_e, \mathbb{F}_\ell)$). We have an exact sequence as follows:

$$0 \to \mathrm{H}^1_{\text{ét}}(Y^{\mathrm{cl}}_e, \mathbb{F}_\ell) \to L^{\mathrm{cl}}_e \to H^{\mathrm{cl,ra}}_e \to 0$$

$$\text{(resp. } 0 \to \mathrm{H}^1_{\text{ét}}(Y^{\mathrm{op}}_e, \mathbb{F}_\ell) \to L^{\mathrm{op}}_e \to H^{\mathrm{op,ra}}_e \to 0\text{)}.$$

On the other hand, since the action of G_d on $(f^\bullet)^{-1}(x_e)$ is translative, the structure of the maximal prime-to-p quotient of $\Pi_{Y^\bullet}$ implies that

$$\dim_{\mathbb{F}_\ell}(H^{\mathrm{cl,ra}}_e) = 1 \text{ (resp. } \dim_{\mathbb{F}_\ell}(H^{\mathrm{op,ra}}_e) = 1\text{)}.$$

Since

$$\dim_{\mathbb{F}_\ell}(\mathrm{H}^1_{\text{ét}}(Y^{\mathrm{cl}}_e, \mathbb{F}_\ell)) = 2(g_Y - d) \text{ (resp. } \dim_{\mathbb{F}_\ell}(\mathrm{H}^1_{\text{ét}}(Y^{\mathrm{op}}_e, \mathbb{F}_\ell)) = 2g_Y\text{)},$$

we obtain

$$\#U^{\mathrm{nd}}_{\ell,Y^\bullet,e} = \ell^{2(g_Y-d)+1} - \ell^{2(g_Y-d)} \text{ (resp. } \#U^{\mathrm{mp}}_{\ell,Y^\bullet,e} = \ell^{2g_Y+1} - \ell^{2g_Y}\text{)}.$$

This completes the proof of the proposition. □

Finally, for each $e \in e^{\mathrm{cl}}(\Gamma_{Y^\bullet})$ (resp. $e \in e^{\mathrm{op}}(\Gamma_{Y^\bullet})$) and each $m \in \mathbb{Z}_{\geq 0}$, we define a subset of $U^{\mathrm{nd}}_{\ell, Y^\bullet, e}$ (resp. $U^{\mathrm{mp}}_{\ell, Y^\bullet, e}$) to be

$$U^{\mathrm{nd,sp}=m}_{\ell, Y^\bullet, e} := \{\alpha \in U^{\mathrm{nd}}_{\ell, Y^\bullet, e} \mid \# v^{\mathrm{sp}}_{g^\bullet_\alpha} = m\}$$

$$(\text{resp. } U^{\mathrm{mp,sp}=m}_{\ell, Y^\bullet, e} := \{\alpha \in U^{\mathrm{mp}}_{\ell, Y^\bullet, e} \mid \# v^{\mathrm{sp}}_{g^\bullet_\alpha} = m\}).$$

If e is a closed edge corresponding to a node which is contained in two different irreducible components of $Y^\bullet$, then

$$U^{\mathrm{nd,sp}=m}_{\ell, Y^\bullet, e} = \emptyset \text{ for } m \geq \# v(\Gamma_{Y^\bullet}) - 1.$$

If e is either an open edge or a closed edge corresponding to a node which is contained in a unique different irreducible component of $Y^\bullet$, then

$$U^{\mathrm{nd,sp}=m}_{\ell, Y^\bullet, e} = \emptyset \text{ for } m \geq \# v(\Gamma_{Y^\bullet}).$$

3 Mono-anabelian reconstruction algorithm for dual semi-graphs

We maintain the notations introduced in Section 2. First, let us define the term "mono-anabelian reconstruction".

Definition 3.1. Let $\mathcal{F}_i, i \in \{1, 2\}$, be a geometric object and $\Pi_{\mathcal{F}_i}$ a profinite group associated to the geometric object $\mathcal{F}_i$. Given an invariant $\mathrm{Inv}_{\mathcal{F}_i}$ depending on the isomorphism class of $\mathcal{F}_i$ (in a certain category), we shall say that $\mathrm{Inv}_{\mathcal{F}_i}$ can be **mono-anabelian reconstructed** from $\Pi_{\mathcal{F}_i}$ if there exists a purely group-theoretic algorithm whose input datum is $\Pi_{\mathcal{F}_i}$, and whose output datum is $\mathrm{Inv}_{\mathcal{F}_i}$.

Suppose that we are given an additional structure $\mathrm{Add}_{\mathcal{F}_i}$ (e.g., a family of subgroups, a family of quotient groups) on the profinite group $\Pi_{\mathcal{F}_i}$ depending functorially on $\mathcal{F}_i$; then we shall say that $\mathrm{Add}_{\mathcal{F}_i}$ can be **mono-anabelian reconstructed** from $\Pi_{\mathcal{F}_i}$ if there exists a purely group-theoretic algorithm whose input datum is $\Pi_{\mathcal{F}_i}$, and whose output datum is $\mathrm{Add}_{\mathcal{F}_i}$.

We shall say that a map (or a morphism) $\mathrm{Add}_{\mathcal{F}_1} \to \mathrm{Add}_{\mathcal{F}_2}$ can be **mono-anabelian reconstructed** from $\Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$ if there exists a purely group-theoretic algorithm whose input datum is $\Pi_{\mathcal{F}_1} \to \Pi_{\mathcal{F}_2}$, and whose output datum is $\mathrm{Add}_{\mathcal{F}_1} \to \mathrm{Add}_{\mathcal{F}_2}$.

Let us fix some notations. For each open subgroup $H \subseteq \Pi_{X^\bullet}$, we write $X^\bullet_H$, $\Gamma_{X^\bullet_H}$, and r_{X_H}, for the pointed stable curve of type (g_{X_H}, n_{X_H}) over k corresponding to H, dual semi-graph of $X^\bullet_H$, and the Betti number of $\Gamma_{X^\bullet_H}$, respectively. Then we obtain an admissible covering

$$X^\bullet_H \to X^\bullet$$

and a natural morphism of dual semi-graphs

$$\Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$$

induced by the admissible covering. Moreover, if H is an open normal subgroup, then $\Gamma_{X^\bullet_H}$ admits a natural action of $\Pi_{X^\bullet}/H$ induced by the action of $\Pi_{X^\bullet}/H$ on $X^\bullet_H$. Note that we have $\Gamma_{X^\bullet_H}/(\Pi_{X^\bullet}/H) = \Gamma_{X^\bullet}$. Moreover, we introduce the following conditions for $X^\bullet$:

Condition A . We shall say that $X^\bullet$ satisfies Condition A if the following conditions are satisfied:

- the genus of the normalization of each irreducible component of $X^\bullet$ is positive;
- $\Gamma_{X^\bullet}$ is 2-connected;
- $\#(v(\Gamma^{\mathrm{cpt}}_{X^\bullet})^{b\leq 1}) = 0$.

In the remainder of the present section, we suppose that $X^\bullet$ **satisfies Condition A**. Then we have the following lemma.

Lemma 3.2. *The data $p := \mathrm{char}(k)$, g_X, $n_X = \#e^{\mathrm{op}}(\Gamma_{X^\bullet})$, r_X, and $\Pi^{\mathrm{top},p}_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$, where $\Pi^{\mathrm{top},p}_{X^\bullet}$ denotes the maximal pro-p quotient of $\Pi^{\mathrm{top}}_{X^\bullet}$.*

Proof. See [Y1, Lemma 5.4]. □

Lemma 3.3. *(i) The set $v(\Gamma_{X^\bullet})^{>0,p}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.*

(ii) Let $H \subseteq \Pi_{X^\bullet}$ be any open normal subgroup. Then the natural map

$$v(\Gamma_{X^\bullet_H})^{>0,p} \to v(\Gamma_{X^\bullet})^{>0,p}$$

can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

(iii) The cardinality $\#v(\Gamma_{X^\bullet})$ of $v(\Gamma_{X^\bullet})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

Proof. First, let us prove (i). By applying Lemma 3.1, we obtain that $V^*_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Then to verify that $v(\Gamma_{X^\bullet})^{>0,p}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$, it is sufficient to prove that $V_{p,X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Let $\alpha \in V^*_{X^\bullet}$ and $H_\alpha \subseteq \Pi_{X^\bullet}$ the open normal subgroup corresponding to α. Write $X^\bullet_{H_\alpha}$ for the étale covering corresponding to H_α and $\Gamma_{X^\bullet_{H_\alpha}}$ for the dual semi-graph of $X^\bullet_{H_\alpha}$. Then we have the following claim:

Claim:

$$\#v(\Gamma_{X^\bullet_{H_\alpha}}) = p(\#v(\Gamma_{X^\bullet}) - 1) + 1$$

if and only if

$$r_{X_{H_\alpha}} = pr_X.$$

Let us prove the claim. Since $r_{X_{H_\alpha}} = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_\alpha}}) - \#v(\Gamma_{X^\bullet_{H_\alpha}}) + 1$ and $r_X = \#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) - \#v(\Gamma_{X^\bullet}) + 1$, we have $r_{X_{H_\alpha}} = pr_X$ holds if and only if

$$\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_\alpha}}) - \#v(\Gamma_{X^\bullet_{H_\alpha}}) = p\#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) - p(\#v(\Gamma_{X^\bullet}) - 1) - 1.$$

Since $\Pi_{X^\bullet}/H_\alpha \cong \mathbb{Z}/p\mathbb{Z}$, we obtain $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_\alpha}}) = p\#e^{\mathrm{cl}}(\Gamma_{X^\bullet})$. Thus,

$$\#v(\Gamma_{X^\bullet_{H_\alpha}}) = p(\#v(\Gamma_{X^\bullet}) - 1) + 1$$

if and only if $r_{X_{H_\alpha}} = pr_X$.

By the definition of $V_{p,X^\bullet}$, the claim above implies that $V_{p,X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Then we obtain that the set $v(\Gamma_{X^\bullet})^{>0,p}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

Second, we prove (ii). The natural injection $H \hookrightarrow \Pi_{X^\bullet}$ induces a natural morphism

$$\mathrm{Hom}(\Pi^{\mathrm{ab}}_{X^\bullet}, \mathbb{F}_p) \to \mathrm{Hom}(H^{\mathrm{ab}}, \mathbb{F}_p).$$

Then it is easy to see that $v(\Gamma_{X^\bullet_H})^{>0,p} \to v(\Gamma_{X^\bullet})^{>0,p}$ can be mono-anabelian reconstructed from $H \hookrightarrow \Pi_{X^\bullet}$ follows from Remark 2.1.2. This completes the proof of (ii).

Next, we prove (iii). Since, for each open normal subgroup $H \subseteq \Pi_{X^\bullet}$, we have

$$V_{p,X^\bullet_H} \subseteq \mathrm{Hom}(H^{\mathrm{ab}}, \mathbb{F}_p),$$

$V_{p,X^\bullet_H}$ admits a natural action of $\Pi_{X^\bullet}/H$ via the natural outer representation

$$\Pi_{X^\bullet}/H \to \mathrm{Out}(H) := \mathrm{Aut}(H)/\mathrm{Inn}(H)$$

induced by the natural exact sequence

$$1 \to H \to \Pi_{X^\bullet} \to \Pi_{X^\bullet}/H \to 1.$$

By Theorem 1.5, there exits a open normal subgroup $Q \subseteq \Pi_{X^\bullet}$ such that the p-rank of the normalization of each irreducible component of the curve corresponding to Q is positive. Then we obtain that

$$\#v(\Gamma_{X^\bullet}) = \#(\varinjlim_{N \subseteq \Pi_{X^\bullet} \text{ open normal}} v(\Gamma_{X^\bullet_N})^{>0,p}/(\Pi_{X^\bullet}/N)).$$

This completes the proof of the lemma. $\square$

Lemma 3.4. *The data $\#e^{\mathrm{cl}}(\Gamma_{X^\bullet})$, $\Pi^{\mathrm{top}}_{X^\bullet}$ and $\Pi^{\text{ét}}_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.*

Proof. By Lemma 3.2 and Lemma 3.3 (iii), we have r_X and $\#v(\Gamma_{X^\bullet})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Then

$$\#e^{\mathrm{cl}}(\Gamma_{X^\bullet}) := r_X + \#v(\Gamma_{X^\bullet}) - 1$$

and

$$\#e^{\mathrm{op}}(\Gamma_{X^\bullet}) := n - \#e^{\mathrm{cl}}(\Gamma_{X^\bullet})$$

can be also mono-anabelian reconstructed from $\Pi_{X^\bullet}$. We set

$$\mathrm{Et}(\Pi_{X^\bullet}) := \{H \subseteq \Pi_{X^\bullet} \text{ open normal} \mid \#e^{\mathrm{cl}}(\Gamma_{X^\bullet_H}) = \#(G/H)\#e^{\mathrm{cl}}(\Gamma_{X^\bullet})$$

$$\text{and } \#e^{\mathrm{op}}(\Gamma_{X^\bullet_H}) = \#(G/H)\#e^{\mathrm{op}}(\Gamma_{X^\bullet})\}.$$

Then we have

$$\Pi^{\text{ét}}_{X^\bullet} = \Pi_{X^\bullet}/ \bigcap_{H\in \mathrm{Et}(\Pi_{X^\bullet})} H.$$

On the other hand, we set

$$\mathrm{Top}(\Pi_{X^\bullet}) := \{H \subseteq \Pi^{\text{ét}}_{X^\bullet} \text{ open normal} \mid g_{X_H} - r_{X_H} = \#(G/H)(g_X - r_X)\}.$$

Then we have

$$\Pi^{\mathrm{top}}_{X^\bullet} = \Pi^{\text{ét}}_{X^\bullet}/ \bigcap_{H\in \mathrm{Top}(\Pi_{X^\bullet})} H.$$

This completes the proof of the corollary. $\square$

Next, we prove the mono-anabelian version of Theorem 2.1 and Theorem 2.2.

Theorem 3.5. *(i) Let ℓ be an arbitrary prime number. Then the set*

$$V_{\ell,X^\bullet}/\sim$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

(ii) Let ℓ', ℓ'' be prime numbers distinct from each other such that $\ell'' \neq p$, then there is a natural injection

$$V_{\ell',X^\bullet}/\sim \hookrightarrow V_{\ell'',X^\bullet}/\sim$$

which fits into the following commutative diagram

$$\begin{array}{ccc} V_{\ell',X^\bullet}/\sim & \xrightarrow{\kappa_{\ell',X^\bullet}} & v(\Gamma_{X^\bullet})^{>0,\ell'} \\ \downarrow & & \downarrow \\ V_{\ell'',X^\bullet}/\sim & \xrightarrow{\kappa_{\ell'',X^\bullet}} & v(\Gamma_{X^\bullet})^{>0,\ell''}. \end{array}$$

Moreover, the injection can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

(iii) The set of vertices $v(\Gamma_{X^\bullet})$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

Proof. By applying Lemma 3.4, $\Pi^{\text{ét}}_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Then by similar arguments to the arguments given in the proof of Lemma 3.3 (i), (i) follows immediately.

Let $\alpha' \in V_{\ell',X^\bullet}$ and $\alpha'' \in V_{\ell'',X^\bullet}$. Write $Y^\bullet_{\alpha'}$ and $Y^\bullet_{\alpha''}$ for the pointed stable curves corresponding to α' and α'', H_α and $H_{\alpha''}$ for the open subgroups of $\Pi_{X^\bullet}$ corresponding to $Y^\bullet_{\alpha'}$ and $Y^\bullet_{\alpha''}$, respectively. Then we obtain that

$$Y^\bullet_{\alpha'} \times_{X^\bullet} Y^\bullet_{\alpha''}$$

is a connected pointed stable curve corresponding to the open subgroup $H_{\alpha'} \cap H_{\alpha''} \subseteq \Pi_{X^\bullet}$. Moreover, Lemma 3.3 (iii) implies that the cardinality of the set of irreducible components of $Y^\bullet_{\alpha'} \times_{X^\bullet} Y^\bullet_{\alpha''}$ can be mono-anabelian reconstructed from $H_{\alpha'} \cap H_{\alpha''} \subseteq \Pi_{X^\bullet}$. Then (ii) follows from Remark 2.1.1.

On the other hand, Lemma 3.2 implies that p can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Moreover, we note that, since $X^\bullet$ satisfies Condition A, we have

$$v(\Gamma_{X^\bullet})^{>0,\ell} = v(\Gamma_{X^\bullet})$$

when $\ell \neq p$. Then (iii) follows from (i) and (ii). $\square$

Theorem 3.6. *(i) Let*

$$(\ell, d, f^\bullet : Y^\bullet \to X^\bullet)$$

be an arbitrary triple associated to $X^\bullet$. Then

$$U^{\mathrm{nd}}_{\ell,Y^\bullet}/\sim,\ U^{\mathrm{mp}}_{\ell,Y^\bullet}/\sim$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

(ii) Let

$$(\ell', d', f^{\bullet,'} : Y^{\bullet,'} \to X^\bullet) \text{ and } (\ell'', d'', f^{\bullet,''} : Y^{\bullet,''} \to X^\bullet)$$

be triples associated to $X^\bullet$. Then there are natural bijections

$$U^{\mathrm{nd}}_{\ell',Y^{\bullet,'}}/\sim \to U^{\mathrm{nd}}_{\ell'',Y^{\bullet,''}}/\sim,\ U^{\mathrm{mp}}_{\ell',Y^{\bullet,'}}/\sim \to U^{\mathrm{mp}}_{\ell'',Y^{\bullet,''}}/\sim$$

which fits into the following commutative diagram

$$\begin{array}{ccc} U^{\mathrm{nd}}_{\ell',Y^{\bullet,'}}/\sim & \xrightarrow{\vartheta^{\mathrm{nd}}_{\ell',X^\bullet}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}) \\ \downarrow & & \| \\ U^{\mathrm{nd}}_{\ell'',Y^{\bullet,''}}/\sim & \xrightarrow{\vartheta^{\mathrm{nd}}_{\ell'',X^\bullet}} & e^{\mathrm{cl}}(\Gamma_{X^\bullet}), \end{array}$$

and

$$\begin{array}{ccc} U^{\mathrm{mp}}_{\ell',Y^{\bullet,'}}/\sim & \xrightarrow{\vartheta^{\mathrm{mp}}_{\ell',X^\bullet}} & e^{\mathrm{op}}(\Gamma_{X^\bullet}) \\ \downarrow & & \| \\ U^{\mathrm{mp}}_{\ell'',Y^{\bullet,''}}/\sim & \xrightarrow{\vartheta^{\mathrm{mp}}_{\ell'',X^\bullet}} & e^{\mathrm{op}}(\Gamma_{X^\bullet}), \end{array}$$

respectively. Moreover, the bijections can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$, respectively.

(iii) The sets of closed edges and open edges

$$e^{\mathrm{cl}}(\Gamma_{X^\bullet}),\ e^{\mathrm{op}}(\Gamma_{X^\bullet})$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$, respectively.

Proof. We only treat the case of nodes. First, let us prove (i). By the definition of $U^*_{\ell,Y^\bullet}$, we have that the set of line bundles $U^*_{\ell,Y^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Hence, to verify (i), it is sufficient to prove that the set of line bundles $U^{\mathrm{nd}}_{\ell,Y^\bullet} \subseteq U^*_{\ell,Y^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Write $\Pi_{Y^\bullet}$ for the admissible fundamental group of $Y^\bullet$. Note that $\Pi_{Y^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Let $\alpha \in U^*_{\ell,Y^\bullet}$ and $H_\alpha \subseteq \Pi_{Y^\bullet}$ the open normal subgroup corresponding to α. Write $Y^\bullet_{H_\alpha}$ for the étale covering corresponding to H_α and $\Gamma_{Y^\bullet_{H_\alpha}}$ for the dual semi-graph of $Y^\bullet_{H_\alpha}$. We observe that

$$\alpha \in U^{\mathrm{nd}}_{\ell,Y^\bullet}$$

if and only if

$$\#e^{\mathrm{cl}}(Y^\bullet_{H_\alpha}) = \ell(\#e^{\mathrm{cl}}(\Gamma_{Y^\bullet}) - d) + d.$$

Since $\#e^{\mathrm{cl}}(Y^\bullet_{H_\alpha})$ and $\#e^{\mathrm{cl}}(\Gamma_{Y^\bullet})$ can be mono-anabelian reconstructed from H_α and $\Pi_{Y^\bullet}$, respectively. Then we obtain that $U^{\mathrm{nd}}_{\ell,Y^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. This completes the proof of (i).

Next, we prove (ii). Let $\alpha' \in U^{\mathrm{nd}}_{\ell',Y^{\bullet,\prime}}$ and $\alpha'' \in U^{\mathrm{nd}}_{\ell'',Y^{\bullet,\prime\prime}}$. Write $Y^\bullet_{\alpha'}$ and $Y^\bullet_{\alpha''}$ for the pointed stable curves corresponding to α' and α'', H_α and $H_{\alpha''}$ for the open subgroups of $\Pi_{X^\bullet}$ corresponding to $Y^\bullet_{\alpha'}$ and $Y^\bullet_{\alpha''}$, respectively. Then we obtain that

$$Y^\bullet_{\alpha'} \times_{X^\bullet} Y^\bullet_{\alpha''}$$

is a connected pointed stable curve corresponding to the open subgroup $H_{\alpha'} \cap H_{\alpha''} \subseteq \Pi_{X^\bullet}$. Moreover, Lemma 3.3 (iii) implies that the cardinality of the set of irreducible components of $Y^\bullet_{\alpha'} \times_{X^\bullet} Y^\bullet_{\alpha''}$ can be mono-anabelian reconstructed from $H_{\alpha'} \cap H_{\alpha''} \subseteq \Pi_{X^\bullet}$. Then (ii) follow immediately from (i), Remark 2.2.1.

Next, let us prove (iii). By Lemma 3.2 and Lemma 3.4, $p := \mathrm{char}(k)$ and $\Pi^{\text{ét}}_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. Then we may choose a triple

$$(\ell''', d''', f^{\bullet,\prime\prime\prime} : Y^{\bullet,\prime\prime\prime} \to X^\bullet)$$

associated to $X^\bullet$ group-theoretically from $\Pi_{X^\bullet}$. Thus, (iii) follows from (i) and (ii). This completes the proof of the theorem. $\square$

Theorem 3.7. *Let $H \subseteq \Pi_{X^\bullet}$ be any open subgroup.*

(i) The natural maps

$$v(\Gamma_{X^\bullet_H}) \to v(\Gamma_X),\ e^{\mathrm{cl}}(\Gamma_{X^\bullet_H}) \to e^{\mathrm{cl}}(\Gamma_{X^\bullet}),\ \textit{and}\ \ e^{\mathrm{op}}(\Gamma_{X^\bullet_H}) \to e^{\mathrm{op}}(\Gamma_{X^\bullet})$$

can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$, respectively.

(ii) Suppose that H is normal. Then the natural action of $\Pi_{X^\bullet}/H$ on $v(\Gamma_{X^\bullet_H})$ (resp. $e^{\mathrm{cl}}(\Gamma_{X^\bullet_H})$, $e^{\mathrm{op}}(\Gamma_{X^\bullet_H})$) induced by the natural action of $\Pi_{X^\bullet}/H$ on $X^\bullet_H$ can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

Proof. Let us prove (i). Write N_H for the maximal open normal subgroup which is contained in H. By Lemma 3.2, we may choose a prime number ℓ group-theoretically from H and $\Pi_{X^\bullet}$ such that $\ell \neq p$ ($:= \mathrm{char}(k)$) and $(\ell, \#(\Pi_{X^\bullet}/N_H)) = 1$. Similar arguments to the arguments given in the proof of Lemma 3.3 (i), we obtain that

$$V_{\ell,X^\bullet},\ V_{\ell,X^\bullet_H}$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$, H, respectively. For each $\alpha \in V_{\ell,X^\bullet}$ and each $\alpha_H \in V_{\ell,X^\bullet_H}$, we write $Q_\alpha \subseteq \Pi_{X^\bullet}$ and $Q_{\alpha_H} \subseteq H$ for the open normal subgroups corresponding to α and α_H, respectively. Note that $Q_\alpha \cap H \neq Q_{\alpha_H}$. Then, by Remark 2.1.2, we observe that

$$[\alpha_H] \mapsto [\alpha]$$

if and only if there exists $\alpha'_H \in V_{\ell, X^\bullet_H}$ such that $\alpha_H \sim \alpha'_H$, and that

$$\#v(\Gamma_{X^\bullet_{Q_\alpha \cap Q_{\alpha_H}}}) = \ell \#v(\Gamma_{X^\bullet_{Q_\alpha \cap H}}),$$

where $[\alpha]$ and $[\alpha_H]$ denote the images of α and α_H in $V_{\ell, X^\bullet}/\sim$ and $V_{\ell, X^\bullet_H}/\sim$, respectively. Thus, the natural map $v(\Gamma_{X^\bullet_H}) \to v(\Gamma_X)$ can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

Next, let us prove that the natural maps of sets of edges can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$. We only treat the case of closed edges. By Lemma 3.2 and Lemma 3.4, we may choose a triple

$$(\ell^*, d^*, f^{\bullet,*} : Y^{\bullet,*} \to X^\bullet)$$

associated to $X^\bullet$ group-theoretically from H and $\Pi_{X^\bullet}$ such that $(\ell^*, \#\Pi_{X^\bullet}/N_H) = 1$ and $(d^*, \#\Pi_{X^\bullet}/N_H) = 1$. Write $H_{Y^{\bullet,*}} \subseteq \Pi_{X^\bullet}$ for the open normal subgroup corresponding to $Y^{\bullet,*}$. Write $f^{\bullet,*}_{X_H} : Y^{\bullet,*}_{X_H} \to X^\bullet_H$ for the étale covering corresponding to the open normal subgroup $H \cap H_{Y^{\bullet,*}}$ of H. Thus, the triple $(\ell^*, d^*, f^{\bullet,*} : Y^{\bullet,*} \to X^\bullet)$ associated to $X^\bullet$ induces group-theoretically a triple

$$(\ell^*, d^*, f^{\bullet,*}_{X_H} : Y^{\bullet,*}_{X_H} \to X^\bullet_H)$$

associated to $X^\bullet_H$. Similar arguments to the arguments given in the proof of Lemma 3.6 (i), we obtain that

$$U^{\mathrm{nd}}_{\ell, Y^{\bullet,*}},\ U^{\mathrm{nd}}_{\ell, Y^{\bullet,*}_{X_H}}$$

can be be mono-anabelian reconstructed from $\Pi_{X^\bullet}$, H, respectively. For each $\beta \in U^{\mathrm{nd}}_{\ell, Y^{\bullet,*}}$ and each $\beta_H \in U^{\mathrm{nd}}_{\ell, Y^{\bullet,*}_{X_H}}$, we write $P_\beta \subseteq H_{Y^{\bullet,*}} \subseteq \Pi_{X^\bullet}$ and $P_{\beta_H} \subseteq H \cap H_{Y^{\bullet,*}} \subseteq H$. Note that $P_\beta \cap H \cap H_{Y^{\bullet,*}} \neq P_{\beta_H}$. Then, by Remark 2.2.2, we observe that

$$[\beta_H] \mapsto [\beta]$$

if and only if there exists β'_H such that $\beta'_H \sim \beta_H$, and that

$$\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{P_\beta \cap P_{\beta_H}}}) = \ell \#e^{\mathrm{cl}}(\Gamma_{X^\bullet_{P_\beta \cap H \cap H_{Y^{\bullet,*}}}}),$$

where $[\beta]$ and $[\beta_H]$ denote the images of β and β_H in $U^{\mathrm{nd}}_{\ell, Y^{\bullet,*}}/\sim$ and $U^{\mathrm{nd}}_{\ell, Y^{\bullet,*}_{X_H}}/\sim$, respectively. Thus, the natural map $e^{\mathrm{cl}}(\Gamma_{X^\bullet_H}) \to e^{\mathrm{cl}}(\Gamma_X)$ can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$.

Next, let us prove (ii). By lemma 3.2, we may choose a prime number ℓ group-theoretically from H and $\Pi_{X^\bullet}$ such that $\ell \neq p$ $(:= \mathrm{char}(k))$ and $(\ell, \#(\Pi_{X^\bullet}/H)) = 1$. Then there is an action of $\Pi_{X^\bullet}/H$ on the set of line bundles

$$V_{\ell, X^\bullet_H} \subseteq \mathrm{H}^1_{\text{ét}}(X^\bullet_H, \mathbb{F}_\ell) = \mathrm{Hom}(H^{\mathrm{ab}}, \mathbb{F}_\ell)$$

induced by the natural outer representation

$$\Pi_{X^\bullet}/H \to \mathrm{Out}(H)$$

induced by the following natural exact sequence

$$1 \to H \to \Pi_{X^\bullet} \to \Pi_{X^\bullet}/H \to 1.$$

Let $\alpha, \alpha' \in V_{\ell, X_H^\bullet}$. We obverse that, for each $\sigma \in \Pi_{X^\bullet}/H$, $\alpha \sim \alpha'$ if and only if $\sigma(\alpha) \sim \sigma(\alpha')$. Thus, we obtain an action of $\Pi_{X^\bullet}/H$ on $v(\Gamma_{X_H^\bullet})$ group-theoretically from the injection $H \hookrightarrow \Pi_{X^\bullet}$. On the other hand, it is easy to check that the action of $\Pi_{X^\bullet}/H$ on $v(\Gamma_{X_H^\bullet})$ obtained above coincides with the action of $\Pi_{X^\bullet}/H$ on $v(\Gamma_{X_H^\bullet})$ induced by the natural action of $\Pi_{X^\bullet}/H$ on $X_H^\bullet$. This completes the proof of the "non-resp" part of (ii).

Next, let us prove the "resp" part of (ii). We only treat the case of closed edges. By Lemma 3.2 and Lemma 3.4, we may choose a triple

$$(\ell^*, d^*, f^{\bullet,*} : Y^{\bullet,*} \to X^\bullet)$$

associated to $X^\bullet$ group-theoretically from H and $\Pi_{X^\bullet}$ such that $(\ell^*, \#\Pi_{X^\bullet}/H) = 1$ and $(d^*, \#\Pi_{X^\bullet}/H) = 1$. Write $H_{Y^{\bullet,*}} \subseteq \Pi_{X^\bullet}$ for the open normal subgroup corresponding to $Y^{\bullet,*}$. Write $f_{X_H}^{\bullet,*} : Y_{X_H}^{\bullet,*} \to X_H^\bullet$ for the étale covering corresponding to the open normal subgroup $H_{Y_{X_H}^{\bullet,*}} := H \cap H_{Y^{\bullet,*}}$ of H. Thus, the triple $(\ell^*, d^*, f^{\bullet,*} : Y^{\bullet,*} \to X^\bullet)$ associated to $X^\bullet$ induces group-theoretically a triple

$$(\ell^*, d^*, f_{X_H}^{\bullet,*} : Y_{X_H}^{\bullet,*} \to X_H^\bullet)$$

associated to $X_H^\bullet$. Note that $H_{Y_{X_H}^{\bullet,*}}$ is an open normal subgroup of $\Pi_{X^\bullet}$ and

$$\Pi_{X^\bullet}/H_{Y_{X_H}^{\bullet,*}} \cong \Pi_{X^\bullet}/H \times \mathbb{Z}/d^*\mathbb{Z}.$$

Thus, we obtain an action of $\Pi_{X^\bullet}/H$ on

$$U^{\mathrm{nd}}_{\ell, Y_{X_H}^{\bullet,*}} \subseteq \mathrm{H}^1_{\text{ét}}(Y_H^{\bullet,*}, \mathbb{F}_\ell) = \mathrm{Hom}(H^{\mathrm{ab}}_{Y_{X_H}^{\bullet,*}}, \mathbb{F}_\ell)$$

induced by the natural outer representation

$$\Pi_{X^\bullet}/H \hookrightarrow \Pi_{X^\bullet}/H_{Y_{X_H}^\bullet} \to \mathrm{Out}(H_{Y_{X_H}^\bullet})$$

induced by the following natural exact sequence

$$1 \to H_{Y_{X_H}^\bullet} \to \Pi_{X^\bullet} \to \Pi_{X^\bullet}/H_{Y_{X_H}^\bullet} \to 1.$$

Let $\beta, \beta' \in U^{\mathrm{nd}}_{\ell, Y_{X_H}^{\bullet,*}}$. We obverse that, for each $\tau \in \Pi_{X^\bullet}/H$, $\beta \sim \beta'$ if and only if $\tau(\beta) \sim \tau(\beta')$. Thus, we obtain an action of $\Pi_{X^\bullet}/H$ on $e^{\mathrm{cl}}(\Gamma_{X_H^\bullet})$ group-theoretically from the injection $H \hookrightarrow \Pi_{X^\bullet}$. On the other hand, it is check to see that the action of $\Pi_{X^\bullet}/H$ on $v(\Gamma_{X_H^\bullet})$ obtained above coincides with the action of $\Pi_{X^\bullet}/H$ on $e^{\mathrm{cl}}(\Gamma_{X_H^\bullet})$ induced by the natural action of $\Pi_{X^\bullet}/H$ on $X_H^\bullet$. This completes the proof of the "resp" part of (ii). $\square$

Finally, we give a mono-anabelian reconstruction algorithm for dual semi-graphs.

Theorem 3.8. *(i) The dual semi-graph*

$$\Gamma_{X^\bullet}$$

can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$.

(ii) For each open subgroup $H \subseteq \Pi_{X^\bullet}$*, the natural map of dual semi-graphs*

$$\Gamma_{X^\bullet_H} \to \Gamma_{X^\bullet}$$

can be mono-anabelian reconstructed from the natural injection $H \hookrightarrow \Pi_{X^\bullet}$*. Moreover, if* $H \subseteq \Pi_{X^\bullet}$ *is an open normal subgroup, then the natural action of* $\Pi_{X^\bullet}/H$ *on* $\Gamma_{X^\bullet_H}$ *induced by the natural action of* $\Pi_{X^\bullet}/H$ *on* $X^\bullet_H$ *can be mono-anabelian reconstructed from the natural injection* $H \hookrightarrow \Pi_{X^\bullet}$.

Proof. Theorem 3.7 implies that, to verify (ii), we only need to prove the dual semi-graphs of $X^\bullet_H$ and $X^\bullet$ can be mono-anabelian reconstructed from H and $\Pi_{X^\bullet}$. Hence we only prove (i).

By Lemma 3.2 and Lemma 3.4, we may choose a triple

$$(\ell, d, f^\bullet : Y^\bullet \to X^\bullet)$$

associated to $X^\bullet$ group-theoretically from $\Pi_{X^\bullet}$. Write $H_{Y^\bullet}$ for the open normal subgroup of $\Pi_{X^\bullet}$ corresponding to $Y^\bullet$. Then the sets of line bundles

$$U^{\mathrm{nd}}_{\ell,Y^\bullet}/\sim,\ U^{\mathrm{mp}}_{\ell,Y^\bullet}/\sim$$

can be mono-anabelian reconstructed from $H_{Y^\bullet}$ and $\Pi_{X^\bullet}$. Let $\alpha \in U^{\mathrm{nd}}_{\ell,Y^\bullet} \cup U^{\mathrm{mp}}_{\ell,Y^\bullet}$ be any element. Write $e_\alpha := \{b_{e_\alpha,1}, b_{e_\alpha,2}\}$ for the image of α in $e(\Gamma_{X^\bullet})$, where $b_{e_\alpha,1}, b_{e_\alpha,2}$ denote the branches of e_α (cf. [M5, Section 1]). To verify (i), we only need to prove that the coincidence map $\zeta^{\Gamma_{X^\bullet}}_{e_\alpha}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. We only treat the case where $\alpha \in U^{\mathrm{nd}}_{\ell,Y^\bullet}$. Moreover, since the composition of maps

$$U^{\mathrm{nd,sp}=m}_{\ell,Y^\bullet} \hookrightarrow U^{\mathrm{nd}}_{\ell,Y^\bullet} \to U^{\mathrm{nd}}_{\ell,Y^\bullet}/\sim \xrightarrow{\sim} e^{\mathrm{cl}}(\Gamma_{X^\bullet})$$

is a surjection, to verify (i), we may assume that $\alpha \in U^{\mathrm{nd,sp}=m}_{\ell,Y^\bullet}$.

Write $Y^\bullet_\alpha \to Y^\bullet$ for the $\mathbb{Z}/\ell\mathbb{Z}$-admissible covering corresponding to α, $H_{Y^\bullet_\alpha}$ for the open normal subgroup of $H_{Y^\bullet}$ corresponding to $Y^\bullet_\alpha$, and $\Gamma_{Y^\bullet_\alpha}$ for the dual semi-graph of $Y^\bullet_\alpha$. Let $m_1 = \#v(\Gamma_{X^\bullet}) - 2$ and $m_2 = \#v(\Gamma_{X^\bullet}) - 1$. We observe that $\alpha \in U^{\mathrm{nd,sp}=m_i}_{\ell,Y^\bullet}$, $i \in \{1, 2\}$, if and only if

$$\#v(\Gamma_{Y^\bullet_\alpha}) = \#v(\Gamma_{Y^\bullet}) - m_i + \ell m_i = \#v(\Gamma_{X^\bullet}) - m_i + \ell m_i.$$

Since $v(\Gamma_{Y^\bullet_\alpha})$ and $v(\Gamma_{X^\bullet})$ can be mono-anabelian reconstructed from $H_{Y^\bullet_\alpha}$ and $\Pi_{X^\bullet}$, respectively, $U^{\mathrm{nd,sp}=m_i}_{\ell,Y^\bullet}$, $i \in \{1, 2\}$, can be mono-anabelian reconstructed from $H_{Y^\bullet}$ and $\Pi_{X^\bullet}$.

We define n to be m_2 if $U^{\mathrm{nd,sp}=m_2}_{\ell,Y^\bullet} \neq \emptyset$ (i.e., α corresponds a node which is contained in a unique irreducible component of $X^\bullet$), and to be m_1 if $U^{\mathrm{nd,sp}=m_2}_{\ell,Y^\bullet} = \emptyset$ (i.e., α corresponds a node which is contained in two different irreducible components of $X^\bullet$). We may assume that $\alpha \in U_{\ell,Y^\bullet}^{\mathrm{nd,sp}=n}$. Theorem 3.7 (i) implies that the natural map

$$\gamma_\alpha : v(\Gamma_{Y_\alpha^\bullet}) \to v(\Gamma_{X^\bullet})$$

can be mono-anabelian reconstructed from $H_\alpha \hookrightarrow \Pi_{X^\bullet}$. Then we have

$$\{\zeta_{e_\alpha}^{\Gamma_{X^\bullet}}(b_{e_\alpha,1}), \zeta_{e_\alpha}^{\Gamma_{X^\bullet}}(b_{e_\alpha,2})\} = \{v \in v(\Gamma_{X^\bullet}) \mid \#\gamma_\alpha^{-1}(v) = 1\}.$$

This means that $\Gamma_{X^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X^\bullet}$. This completes the proof of (i). □

4 Reconstruction of sets of vertices, sets of edges, and sets of genera via surjections

We fix some notations. Let k_i, $i \in \{1,2\}$, be an algebraically closed field of characteristic $p > 0$ and $\ell \neq p$ a prime number. Let $X_i^\bullet, i \in \{1,2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over k_i, $\Pi_{X_i^\bullet}$ the admissible fundamental groups of $X_i^\bullet$, $\Gamma_{X_i^\bullet}$ the dual semi-graphs of $X_i^\bullet$, and r_{X_i} the Betti number of $\Gamma_{X_i^\bullet}$. Moreover, we introduce the following condition:

Condition B . We shall say that $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition B if the following conditions are satisfied:

- $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$;
- $\#v(\Gamma_{X_1^\bullet}) = \#v(\Gamma_{X_2^\bullet})$;
- $\#e(\Gamma_{X_1^\bullet}) = \#e(\Gamma_{X_2^\bullet})$.

In this section, we suppose that $X_1^\bullet$ and $X_2^\bullet$ **satisfy Condition A and Condition B**, and that

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

is an open continuous surjective homomorphism of the admissible fundamental groups of $X_1^\bullet$ and $X_2^\bullet$. Denote by

$$(g, n) := (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}).$$

We will prove that the surjection $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces bijections between the sets of vertices, the sets of closed edges, and the sets of open edges of the dual semi-graphs $\Gamma_{X_1^\bullet}$ and $\overset{\sim}{\to} \Gamma_{X_2^\bullet}$, respectively; moreover, for each $v \in v(\Gamma_{X_1^\bullet})$, the genus of the normalization of the irreducible component of $X_1^\bullet$ corresponding to v is equal to the genus of the normalization of the irreducible component of $X_2^\bullet$ corresponding to the image of v.

Since $X_1^\bullet$ and $X_2^\bullet$ are pointed stable curves of type (g,n), the sujection $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces a natural isomorphism of maximal prime-to-p quotients of the admissible fundamental groups

$$\phi^{p'} : \Pi_{X_1^\bullet}^{p'} \overset{\sim}{\to} \Pi_{X_2^\bullet}^{p'}.$$

Then, for each G-Galois admissible covering (i.e., whose Galois group of the covering is isomorphic to G)

$$f_1^\bullet : Y_1^\bullet \to X_1^\bullet \text{ (resp. } f_2^\bullet : Y_2^\bullet \to X_2^\bullet)$$

over k_1 (resp. k_2) such that $(\#G, p) = 1$ induces a G-Galois admissible covering

$$f_2^\bullet : Y_2^\bullet \to X_2^\bullet \text{ (resp. } f_1^\bullet : Y_1^\bullet \to X_1^\bullet).$$

over k_2 (resp. k_1). For $i \in \{1, 2\}$, write g_{Y_i}, for the genus of $Y_i^\bullet$, $\Gamma_{Y_i^\bullet}$ for the dual semi-graph of $Y_i^\bullet$, and r_{Y_i} for the Betti number of $\Gamma_{Y_i^\bullet}$.

Lemma 4.1. *Let $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ be a Galois* **étale** *covering of degree ℓ over k_1 and $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ the Galois admissible covering of degree ℓ over k_2 induced by $f_1^\bullet$. Suppose that $\#v_{f_1^\bullet}^{\mathrm{sp}} = m$. Then we have*

$$\#e_{f_2^\bullet}^{\mathrm{cl,ra}} + \frac{1}{2}\#e_{f_2^\bullet}^{\mathrm{op,ra}} + \#v_{f_2^\bullet}^{\mathrm{sp}} \leq m.$$

Proof. Since $f_1^\bullet$ is an étale covering, the Riemann-Hurwitz formula implies that

$$g_{Y_1} - g_{Y_2} = -\frac{1}{2}(\ell - 1)\#e_{f_2^\bullet}^{\mathrm{op,ra}}.$$

On the other hand, we have

$$r_{Y_1} = \ell\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) - \#v(\Gamma_{X_1^\bullet}) + \#v_{f_1^\bullet}^{\mathrm{sp}} - \ell\#v_{f_1^\bullet}^{\mathrm{sp}} + 1$$
$$= \ell\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) - \#v(\Gamma_{X_1^\bullet}) - (\ell - 1)m + 1$$

and

$$r_{Y_2} = \ell\#e_{f_2^\bullet}^{\mathrm{cl,\acute{e}t}} + \#e_{f_2^\bullet}^{\mathrm{cl,ra}} - \ell\#v_{f_2^\bullet}^{\mathrm{sp}} - \#v_{f_2^\bullet}^{\mathrm{ra}} + 1.$$

Since $\#e(\Gamma_{X_1^\bullet}) = \#e(\Gamma_{X_2^\bullet})$ and $\#v(\Gamma_{X_1^\bullet}) = \#v(\Gamma_{X_2^\bullet})$, we obtain that

$$r_{Y_1} - r_{Y_2} = (\ell - 1)\#e_{f_2^\bullet}^{\mathrm{cl,ra}} + (\ell - 1)(\#v_{f_2^\bullet}^{\mathrm{sp}} - m)$$

Moreover, by using Theorem 1.5, we have

$$g_{Y_1} - g_{Y_2} \geq r_{Y_1} - r_{Y_2}.$$

Thus,

$$\#e_{f_2^\bullet}^{\mathrm{cl,ra}} + \frac{1}{2}\#e_{f_2^\bullet}^{\mathrm{op,ra}} + \#v_{f_2^\bullet}^{\mathrm{sp}} \leq m.$$

This completes the proof of the lemma. □

Lemma 4.2. *Let $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ be a Galois* **étale** *covering of degree ℓ over k_1 and $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ the Galois admissible covering of degree ℓ induced by $f_1^\bullet$ over k_2. Suppose that $\#v_{f_1^\bullet}^{\mathrm{sp}} = 0$. Then $f_2^\bullet$ is an étale covering, and $\#v_{f_2^\bullet}^{\mathrm{sp}} = 0$.*

Proof. The lemma follows immediately from Lemma 1.7. □

Lemma 4.3. *Let $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ be a Galois* **étale** *covering of degree ℓ over k_1 and $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ the Galois admissible covering of degree ℓ over k_2 induced by $f_1^\bullet$. Suppose that $\#v_{f_1^\bullet}^{\mathrm{sp}} = 1$. Then $f_2^\bullet$ is an étale covering.*

Proof. In order to verify the lemma, we only need to prove that

$$\#e_{f_2^\bullet}^{\mathrm{cl,ra}} = \#e_{f_2^\bullet}^{\mathrm{op,ra}} = 0.$$

By applying Lemma 4.1, we have

$$\#e_{f_2^\bullet}^{\mathrm{cl,ra}} + \frac{1}{2}\#e_{f_2^\bullet}^{\mathrm{op,ra}} + \#v_{f_2^\bullet}^{\mathrm{sp}} \leq 1.$$

Suppose that $\#e_{f_2^\bullet}^{\mathrm{cl,ra}} \neq 0$. The structure of the maxmial prime-to-p quotient of admissible fundamental groups imply that either $\#e_{f_2^\bullet}^{\mathrm{cl,ra}} = 1$ and $\#e_{f_2^\bullet}^{\mathrm{op,ra}} \geq 1$ or $\#e_{f_2^\bullet}^{\mathrm{cl,ra}} \geq 2$ holds. Then we obtain a contradiction. Thus, we have $\#e_{f_2^\bullet}^{\mathrm{cl,ra}} = 0$.

Suppose that $\#e_{f_2^\bullet}^{\mathrm{op,ra}} \neq 0$. Since $\#e_{f_2^\bullet}^{\mathrm{cl,ra}} = 0$, we have $\#e_{f_2^\bullet}^{\mathrm{op,ra}} = 2$. Let $\ell' \neq p$ be a prime number distinct from ℓ, and let

$$g_1^\bullet : Z_1^\bullet \to X_1^\bullet$$

be a Galois étale covering of degree ℓ' over k_1 such that $\#v_{g_1^\bullet}^{\mathrm{sp}} = 0$. Then Lemma 4.2 implies that the Galois admissible covering

$$g_2^\bullet : Z_2^\bullet \to X_2^\bullet$$

of degree ℓ' over k_2 induced by $g_2^\bullet$ is an étale covering such that $\#v_{g_2^\bullet}^{\mathrm{sp}} = 0$. Write $\Gamma_{Z_1^\bullet}$ and $\Gamma_{Z_2^\bullet}$ for the dual semi-graphs of $Z_1^\bullet$ and $Z_2^\bullet$, respectively. We have

$$\#v(\Gamma_{X_1^\bullet}) = \#v(\Gamma_{Z_1^\bullet}) = \#v(\Gamma_{Z_2^\bullet}) = \#v(\Gamma_{X_2^\bullet}),$$

$$\ell'\#e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) = \#e^{\mathrm{op}}(\Gamma_{Z_1^\bullet}) = \#e^{\mathrm{op}}(\Gamma_{Z_2^\bullet}) = \ell'\#e^{\mathrm{op}}(\Gamma_{X_2^\bullet}),$$

and

$$\ell'\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) = \#e^{\mathrm{cl}}(\Gamma_{Z_1^\bullet}) = \#e^{\mathrm{cl}}(\Gamma_{Z_2^\bullet}) = \ell'\#e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}).$$

Write $W_1^\bullet$ and $W_2^\bullet$ for $Y_1^\bullet \times_{X_1^\bullet} Z_1^\bullet$ and $Y_2^\bullet \times_{X_2^\bullet} Z_2^\bullet$, respectively. Then $f_1^\bullet$ and $f_2^\bullet$ induce two Galois admissible coverings

$$h_1^\bullet : W_1^\bullet \to Z_1^\bullet$$

and

$$h_2^\bullet : W_2^\bullet \to Z_2^\bullet$$

over k_1 and k_2 of degree ℓ, respectively. We have that $h_1^\bullet$ is an étale covering such that $\#v_{h_1^\bullet}^{\mathrm{sp}} = 1$, and that $\#e_{h_2^\bullet}^{\mathrm{op,ra}} = 2\ell'$. Then Lemma 4.1 implies that

$$\#e_{h_2^\bullet}^{\mathrm{cl,ra}} + \frac{1}{2}\#e_{h_2^\bullet}^{\mathrm{op,ra}} + \#v_{h_2^\bullet}^{\mathrm{sp}} = \#e_{h_2^\bullet}^{\mathrm{cl,ra}} + \ell' + \#v_{h_2^\bullet}^{\mathrm{sp}} \leq 1.$$

This is a contradiction. Thus, we obtain $\#e_{f_2^\bullet}^{\mathrm{op,ra}} = 0$. This completes the proof of the lemma. $\square$

Proposition 4.4. *Write $M_{X_1^\bullet}$ and $M_{X_2^\bullet}$ for* $\mathrm{Hom}(\Pi_{X_1^\bullet}, \mathbb{F}_\ell)$ *and* $\mathrm{Hom}(\Pi_{X_2^\bullet}, \mathbb{F}_\ell)$*, $M^{\text{ét}}_{X_1^\bullet}$ and $M^{\text{ét}}_{X_2^\bullet}$ for* $\mathrm{H}^1_{\text{ét}}(X_1^\bullet, \mathbb{F}_\ell)$ *and* $\mathrm{H}^1_{\text{ét}}(X_2^\bullet, \mathbb{F}_\ell)$*, respectively. Then the isomorphism $\psi_\ell : M_{X_2^\bullet} \xrightarrow{\sim} M_{X_1^\bullet}$ induced by the isomorphism $\phi^{p'}$ induces an isomorphism*

$$\psi_\ell^{\text{ét}} : M^{\text{ét}}_{X_2^\bullet} \xrightarrow{\sim} M^{\text{ét}}_{X_1^\bullet}$$

which fits into a commutative diagram as follows:

$$\begin{array}{ccc} M_{X_2^\bullet} & \xrightarrow{\psi_\ell} & M_{X_1^\bullet} \\ \uparrow & & \uparrow \\ M^{\text{ét}}_{X_2^\bullet} & \xrightarrow{\psi_\ell^{\text{ét}}} & M^{\text{ét}}_{X_1^\bullet}, \end{array}$$

where the vertical arrows are the natural injections.

Proof. To verify the proposition, we only need to prove that $\psi_\ell^{-1} : M_{X_1^\bullet} \xrightarrow{\sim} M_{X_2^\bullet}$ induces an isomorphism $\psi_\ell^{-1,\text{ét}} : M^{\text{ét}}_{X_1^\bullet} \xrightarrow{\sim} M^{\text{ét}}_{X_2^\bullet}$ which fits into a commutative diagram as follows:

$$\begin{array}{ccc} M_{X_1^\bullet} & \xrightarrow{\psi_\ell^{-1}} & M_{X_2^\bullet} \\ \uparrow & & \uparrow \\ M^{\text{ét}}_{X_1^\bullet} & \xrightarrow{\psi_\ell^{-1,\text{ét}}} & M^{\text{ét}}_{X_1^\bullet}, \end{array}$$

where the vertical arrows are the natural injections.

For each line bundle $\alpha \in M^{\text{ét}}_{X_1^\bullet}$ over $X_1^\bullet$, write $f^\bullet_{1,\alpha} : Y^\bullet_{1,\alpha} \to X_1^\bullet$ for the étale covering corresponding to α. We set

$$L^{\#v^{\text{sp}}=1}_{X_1^\bullet} := \{\alpha \in M^{\text{ét}}_{X_1^\bullet} \mid \#v^{\text{sp}}_{f^\bullet_{1,\alpha}} = 1\}.$$

Then it is easy to see that $M^{\text{ét}}_{X_1^\bullet}$ is generated by $L^{\#v^{\text{sp}}=1}_{X_1^\bullet}$ as an $\mathbb{F}_\ell$-vector space.

On the other hand, Lemma 4.3 implies that, for each $\alpha \in L^{\#v^{\text{sp}}=1}_{X_1^\bullet}$, $f^\bullet_{1,\alpha}$ induces a Galois étale covering of $X_2^\bullet$ of degree ℓ. Thus, ψ_ℓ^{-1} induces an injection of $\mathbb{F}_\ell$-vector spaces

$$\psi_\ell^{-1,\text{ét}} : M^{\text{ét}}_{X_1^\bullet} \hookrightarrow M^{\text{ét}}_{X_2^\bullet}.$$

Moreover, since $\dim_{\mathbb{F}_\ell}(M^{\text{ét}}_{X_1^\bullet}) = 2g - r_{X_1} = 2g - r_{X_2} = \dim_{\mathbb{F}_\ell}(M^{\text{ét}}_{X_2^\bullet})$, we obtain that

$$\psi_\ell^{-1,\text{ét}} : M^{\text{ét}}_{X_1^\bullet} \xrightarrow{\sim} M^{\text{ét}}_{X_2^\bullet}$$

is an isomorphism. This completes the proof of the proposition. □

Proposition 4.5. *Write $M^{\text{top}}_{X_1^\bullet}$ and $M^{\text{top}}_{X_2^\bullet}$ for* $\mathrm{H}^1(\Gamma_{X_1^\bullet}, \mathbb{F}_\ell)$ *and* $\mathrm{H}^1(\Gamma_{X_2^\bullet}, \mathbb{F}_\ell)$*. Then the isomorphism $\psi_\ell^{\text{ét}} : M^{\text{ét}}_{X_2^\bullet} \xrightarrow{\sim} M^{\text{ét}}_{X_1^\bullet}$ induces an isomorphism*

$$\psi_\ell^{\text{top}} : M^{\text{top}}_{X_2^\bullet} \xrightarrow{\sim} M^{\text{top}}_{X_1^\bullet}$$

which fits into a commutative diagram as follows:

$$\begin{array}{ccc} M^{\text{ét}}_{X_2^\bullet} & \xrightarrow{\psi_\ell^{\text{ét}}} & M^{\text{ét}}_{X_1^\bullet} \\ \uparrow & & \uparrow \\ M^{\text{top}}_{X_2^\bullet} & \xrightarrow{\psi_\ell^{\text{top}}} & M^{\text{top}}_{X_1^\bullet}, \end{array}$$

where the vertical arrows are the natural injections.

Proof. For each line bundle $\beta \in M^{\text{ét}}_{X_2^\bullet}$ over $X_2^\bullet$, write $f^\bullet_{2,\beta} : Y^\bullet_{2,\beta} \to X_2^\bullet$ for the Galois étale covering corresponding to β. Then β induces a line bundle $\psi_\ell^{\text{ét}}(\beta) \in M^{\text{ét}}_{X_1^\bullet}$ over $X_1^\bullet$. Write $f^\bullet_{1,\psi_\ell^{\text{ét}}(\beta)} : Y^\bullet_{1,\psi_\ell^{\text{ét}}(\beta)} \to X_1^\bullet$ for the Galois étale covering corresponding to $\psi_\ell^{\text{ét}}(\beta)$.

Theorem 1.5 implies that

$$r_{Y_{1,\psi_\ell^{\text{ét}}(\beta)}} \leq r_{Y_{2,\beta}},$$

where $r_{Y_{1,\psi_\ell^{\text{ét}}(\beta)}}$ and $r_{Y_{2,\beta}}$ denote the Betti numbers of the dual semi-graphs of $Y^\bullet_{1,\psi_\ell^{\text{ét}}(\beta)}$ and $Y^\bullet_{2,\beta}$, respectively. Since $\#v^{\text{sp}}_{f^\bullet_{2,\beta}} = \#v(\Gamma_{X_2^\bullet}) = \#v(\Gamma_{X_1^\bullet})$, we have $\#v^{\text{sp}}_{f^\bullet_{1,\psi_\ell^{\text{ét}}(\beta)}} \leq \#v(\Gamma_{X_1^\bullet})$. Moreover, Lemma 1.7 implies that $\#v^{\text{sp}}_{f^\bullet_{1,\psi_\ell^{\text{ét}}(\beta)}} = \#v(\Gamma_{X_1^\bullet})$. Thus, we have

$$\psi_\ell^{\text{ét}}(\beta) \in M^{\text{top}}_{X_1^\bullet}.$$

Then $\psi_\ell^{\text{ét}}$ induces an injection

$$\psi_\ell^{\text{top}} : M^{\text{top}}_{X_2^\bullet} \hookrightarrow M^{\text{top}}_{X_1^\bullet}.$$

Moreover, since $r_{X_1} = r_{X_2}$, we have ψ_ℓ^{top} is an isomorphism. This completes the proof of the proposition. □

Lemma 4.6. *Let $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ be a Galois* **étale** *covering of degree ℓ over k_2 and $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ a Galois admissible covering of degree ℓ over k_1 induced by $f_2^\bullet$. Suppose that $\#v^{\text{ra}}_{f_2^\bullet} = 1$. Then we have $\#v^{\text{ra}}_{f_1^\bullet} = 1$.*

Proof. Theorem 1.5 and Proposition 4.4 imply that $r_{Y_1} \leq r_{Y_2}$. Then we have

$$\#e^{\text{cl}}(\Gamma_{X_1^\bullet}) - \ell(\#v(\Gamma_{X_1^\bullet}) - \#v^{\text{ra}}_{f_1^\bullet}) - \#v^{\text{ra}}_{f_1^\bullet} + 1 \leq \#e^{\text{cl}}(\Gamma_{X_2^\bullet}) - \ell(\#v(\Gamma_{X_2^\bullet}) - 1) - 1 + 1$$

Thus, we obtain that $\#v^{\text{ra}}_{f_1^\bullet} \leq 1$.

If $\#v^{\text{ra}}_{f_1^\bullet} = 0$, then the line bundle corresponding to $f_1^\bullet$ is contained in $M^{\text{top}}_{X_1^\bullet}$. Then Proposition 4.5 implies that the line bundle corresponding to $f_2^\bullet$ is contained in $M^{\text{top}}_{X_2^\bullet}$. This means that $\#v^{\text{ra}}_{f_2^\bullet} = 0$. This is a contradiction. Thus, $\#v^{\text{ra}}_{f_1^\bullet} = 1$. We completes the proof of the lemma. □

We reconstruct the sets of vertices and the sets of genera of irreducible components as follows.

Proposition 4.7. *For each $v \in v(\Gamma_{X_1^\bullet})$ (resp. $v \in v(\Gamma_{X_1^\bullet})$), we write $X_{1,v}$ (resp. $X_{2,v}$) for the irreducible component of $X_1^\bullet$ (resp. $X_2^\bullet$) corresponding to v and $g_{1,v}$ (resp. $g_{2,v}$) for the genus of the normalization $\widetilde{X_{1,v}}$ (resp. $\widetilde{X_{2,v}}$) of $X_{1,v}$ (resp. $X_{2,v}$). Then the isomorphism $\psi_\ell^{\text{ét}} : M_{X_2^\bullet}^{\text{ét}} \xrightarrow{\sim} M_{X_1^\bullet}^{\text{ét}}$ induces a bijection of the set of vertices*

$$\rho_\phi^{\text{vex}} : v(\Gamma_{X_2^\bullet}) \xrightarrow{\sim} v(\Gamma_{X_1^\bullet}),$$

where ρ_ϕ^{vex} does not depend on the choices of $\ell \in \mathfrak{Primes} \setminus \{p\}$. Moreover, we have

$$g_{2,v} = g_{1,\rho_\phi^{\text{vex}}(v)}$$

for each $v \in v(\Gamma_{X_2^\bullet})$.

Proof. Let $V_{\ell,X_1^\bullet} \subseteq V^*_{X_1^\bullet}$ and $V_{\ell,X_2^\bullet} \subseteq V^*_{X_2^\bullet}$ be the subsets of $M_{X_1^\bullet}^{\text{ét}}$ and $M_{X_2^\bullet}^{\text{ét}}$, respectively, defined in Section 2. By applying Lemma 4.2, we obtain that

$$\psi_\ell^{\text{ét}}(V^*_{X_2^\bullet}) = V^*_{X_1^\bullet}.$$

Moreover, Lemma 4.6 implies that

$$\psi_\ell^{\text{ét}}(V_{\ell,X_2^\bullet}) = V_{\ell,X_1^\bullet}.$$

Let $\alpha_1, \alpha_2 \in V_{\ell,X_2^\bullet}$ distinct from each other such that $\alpha_1 \sim \alpha_2$. It is easy to see that $a\alpha_1 + b\alpha_2 \in V_{\ell,X_2^\bullet}$ if and only $a\psi_\ell^{\text{ét}}(\alpha_1) + b\psi_\ell^{\text{ét}}(\alpha_2) \in V_{\ell,X_1^\bullet}$ for each $a, b \in \mathbb{F}_p^\times$. Thus, we obtain an injection of the set of vertices

$$V_{\ell,X_2^\bullet}/\sim \xrightarrow{\sim} V_{\ell,X_1^\bullet}/\sim.$$

Then the "non-moreover" part of the proposition follows from Remark 2.1.1.

Next, let us prove the "moreover" part of the proposition. For each $w_1 \in v(\Gamma_{X_1^\bullet})$ (resp. $w_2 \in v(\Gamma_{X_2^\bullet})$), we set

$$L_{X_1^\bullet}^{w_1,\ell} := \{\alpha \in M_{X_1^\bullet}^{\text{ét}} \mid \#v^{\text{ra}}_{f^\bullet_{1,\alpha}} = 1 \text{ and } (f^\bullet_{1,\alpha})^{-1}(X_{1,w_1}) \text{ is connected}\}$$

$$\text{(resp. } L_{X_2^\bullet}^{w_2,\ell} := \{\alpha \in M_{X_2^\bullet}^{\text{ét}} \mid \#v^{\text{ra}}_{f^\bullet_{2,\alpha}} = 1 \text{ and } (f^\bullet_{2,\alpha})^{-1}(X_{2,w_2}) \text{ is connected}\}).$$

Moreover, we denote by

$$[L_{X_1^\bullet}^{w_1,\ell}] \text{ (resp. } [L_{X_2^\bullet}^{w_2,\ell}])$$

for the image of $L_{X_1^\bullet}^{w_1,\ell}$ in $M_{X_1^\bullet}^{\text{ét}}/M_{X_1^\bullet}^{\text{top}}$ (resp. $L_{X_2^\bullet}^{w_2,\ell}$ in $M_{X_2^\bullet}^{\text{ét}}/M_{X_2^\bullet}^{\text{top}}$). Then we have

$$\#[L_{X_1^\bullet}^{w_1,\ell}] = \ell^{g_{1,w_1}} - 1 \text{ (resp. } \#[L_{X_2^\bullet}^{w_2,\ell}] = \ell^{g_{2,w_2}} - 1).$$

On the other hand, for each $v \in v(\Gamma_{X^\bullet})$, Proposition 4.5 and Lemma 4.6 imply that $\psi_\ell^{\text{ét}}$ induces an injection

$$[L_{X_2^\bullet}^{v,\ell}] \hookrightarrow [L_{X_1^\bullet}^{\rho_\phi^{\text{vex}}(v),\ell}].$$

Thus, we have

$$\ell^{g_{2,v}} - 1 = \#[L_{X_2^\bullet}^{v,\ell}] \leq \#[L_{X_1^\bullet}^{\rho_\phi^{\text{vex}}(v),\ell}] = \ell^{g_{1,\rho_\phi^{\text{vex}}(v)}} - 1.$$

This means that

$$g_{2,v} \leq g_{1,\rho_\phi^{\mathrm{vex}}(v)}$$

for each $v \in v(\Gamma_{X_2^\bullet})$. On the other hand, since

$$\sum_{w \in v(\Gamma_{X_1^\bullet})} g_{1,w} = g - r_{X_1} = g - r_{X_2} = \sum_{w \in v(\Gamma_{X_2^\bullet})} g_{2,w},$$

we have

$$g_{2,v} = g_{1,\rho_\phi^{\mathrm{vex}}(v)}$$

for each $v \in v(\Gamma_{X_2^\bullet})$. This completes the proof of the proposition. □

Next, let us reconstruct the sets of edges from surjections. In the remainder of the present section, we fix a triple

$$(\ell, d, f_1^\bullet : Y_1^\bullet \to X_1^\bullet)$$

associated to $X_1^\bullet$ (cf. Section 2 for definition). Then Lemma 4.2 implies that the triple induces a triple

$$(\ell, d, f_2^\bullet : Y_2^\bullet \to X_2^\bullet)$$

associated to $X_2^\bullet$. The surjection $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ induces a sujection of admissible fundamental groups

$$\phi_Y : \Pi_{Y_1^\bullet} \twoheadrightarrow \Pi_{Y_2^\bullet}$$

of $Y_1^\bullet$ and $Y_2^\bullet$. Moreover, the constructions of $Y_1^\bullet$ and $Y_2^\bullet$ imply that $Y_1^\bullet$ and $Y_2^\bullet$ satisfy Condition A and Condition B.

We write

$$M_{Y_1^\bullet}^{\text{ét}},\ M_{Y_1^\bullet},\ M_{Y_1^\bullet}^{\text{ra}},\ M_{Y_2^\bullet}^{\text{ét}},\ M_{Y_2^\bullet},\ \text{and}\ M_{Y_2^\bullet}^{\text{ra}}$$

for

$$\mathrm{H}^1_{\text{ét}}(Y_1^\bullet, \mathbb{F}_\ell),\ \mathrm{Hom}(\Pi_{Y_1^\bullet}, \mathbb{F}_\ell),\ M_{Y_1^\bullet}/M_{Y_1^\bullet}^{\text{ét}},\ \mathrm{H}^1_{\text{ét}}(Y_2^\bullet, \mathbb{F}_\ell),\ \mathrm{Hom}(\Pi_{Y_2^\bullet}, \mathbb{F}_\ell),\ \text{and}\ M_{Y_2^\bullet}/M_{Y_2^\bullet}^{\text{ét}},$$

respectively. Then, by Proposition 4.4, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{Y_2^\bullet}^{\text{ét}} & \longrightarrow & M_{Y_2^\bullet} & \longrightarrow & M_{Y_2^\bullet}^{\text{ra}} & \longrightarrow & 0 \\
 & & \big\downarrow & & {\scriptstyle \psi_{Y,\ell}}\big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & M_{Y_1^\bullet}^{\text{ét}} & \longrightarrow & M_{Y_1^\bullet} & \longrightarrow & M_{Y_1^\bullet}^{\text{ra}} & \longrightarrow & 0,
\end{array}$$

where all the vertical arrows are isomorphisms. Let $U_{\ell,Y_1^\bullet}^*$ and $U_{\ell,Y_2^\bullet}^*$ be the subsets of $M_{Y_1^\bullet}$ and $M_{Y_2^\bullet}$ (cf. Section 2), respectively. Since the actions of G_d on the exact sequences are compatible with the isomorphisms appeared in the commutative diagram above, we have

$$\psi_{Y,\ell}(U_{\ell,Y_1^\bullet}^*) = U_{\ell,Y_2^\bullet}^*.$$

Moreover, let

$$U^{\mathrm{nd}}_{\ell,Y_1^\bullet},\ U^{\mathrm{mp}}_{\ell,Y_1^\bullet},\ \{U^{\mathrm{nd}}_{\ell,Y_{1,e}^\bullet}\}_{e\in e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})},\ \{U^{\mathrm{mp}}_{\ell,Y_{1,e}^\bullet}\}_{e\in e^{\mathrm{op}}(\Gamma_{Y_1^\bullet})},\ \{U^{\mathrm{nd,sp=0}}_{\ell,Y_1^\bullet,e}\}_{e\in e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})},\{U^{\mathrm{mp,sp=0}}_{\ell,Y_1^\bullet,e}\}_{e\in e^{\mathrm{op}}(\Gamma_{Y_1^\bullet})},$$

$$U^{\mathrm{nd}}_{\ell,Y_2^\bullet},\ U^{\mathrm{mp}}_{\ell,Y_2^\bullet},\ U^{\mathrm{mp}}_{\ell,Y_2^\bullet},\ \{U^{\mathrm{nd}}_{\ell,Y_{2,e}^\bullet}\}_{e\in e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})},\ \{U^{\mathrm{mp}}_{\ell,Y_{2,e}^\bullet}\}_{e\in e^{\mathrm{op}}(\Gamma_{Y_2^\bullet})},\ \{U^{\mathrm{nd,sp=0}}_{\ell,Y_2^\bullet,e}\}_{e\in e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})},$$

and $\{U^{\mathrm{mp,sp=0}}_{\ell,Y_2^\bullet,e}\}_{e\in e^{\mathrm{op}}(\Gamma_{Y_2^\bullet})}$ be the subsets of $U^*_{\ell,Y_1^\bullet}$ and $U^*_{\ell,Y_2^\bullet}$ defined in Section 2, respectively. We have the following two lemmas.

Lemma 4.8. *We have*

$$\psi^{-1}_{Y,\ell}(\bigcup_{e\in e^{\mathrm{op}}(\Gamma_{Y_1^\bullet})} U^{\mathrm{mp,sp=0}}_{\ell,Y_1^\bullet,e}) \subseteq \bigcup_{e\in e^{\mathrm{op}}(\Gamma_{Y_2^\bullet})} U^{\mathrm{mp,sp=0}}_{\ell,Y_2^\bullet,e}.$$

Moreover, we have

$$\psi^{-1}_{Y,\ell}(U^{\mathrm{mp}}_{\ell,Y_1^\bullet}) = U^{\mathrm{mp}}_{\ell,Y_2^\bullet}.$$

Proof. Let $e\in e^{\mathrm{op}}(\Gamma_{Y_1^\bullet})$ and $\alpha\in U^{\mathrm{mp,sp=0}}_{\ell,Y_1^\bullet,e}$. Then the admissible covering

$$g^\bullet_{1,\alpha}: Y^\bullet_{1,\alpha}\to Y^\bullet_1$$

corresponding to α induces a Galois admissible covering

$$g^\bullet_{2,\beta}: Y^\bullet_{2,\beta}\to Y^\bullet_2$$

over k_2 of degree ℓ. Write $\beta := \psi^{-1}_{Y,\ell}(\alpha)$ for the line bundle corresponding to $g^\bullet_{2,\beta}$. We have

$$\beta\in U^*_{\ell,Y_2^\bullet}.$$

Write $g_{Y_{1,\alpha}}$ and $g_{Y_{2,\beta}}$ for the genera of $Y^\bullet_{1,\alpha}$ and $Y^\bullet_{2,\beta}$, $r_{Y_{1,\alpha}}$ and $r_{Y_{2,\beta}}$ for the Betti numbers of the dual semi-graphs $\Gamma_{Y^\bullet_{1,\alpha}}$ and $\Gamma_{Y^\bullet_{2,\beta}}$, respectively. Then the Riemann-Hurwitz formula implies that

$$g_{Y_{1,\alpha}} - g_{Y_{2,\beta}} = \frac{1}{2}(d - \#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}})(\ell-1).$$

On the other hand, we have

$$r_{Y_{1,\alpha}} = \ell\#e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet}) - \#v(\Gamma_{Y_1^\bullet}) + 1$$

and

$$r_{Y_{2,\alpha}} = \ell\#e^{\mathrm{cl,\acute{e}t}}_{g^\bullet_{2,\beta}} + \#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} - \ell\#v^{\mathrm{cl,sp}}_{g^\bullet_{2,\beta}} - \#v^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} + 1.$$

Then Theorem 1.5 implies that

$$g_{Y_{1,\alpha}} - g_{Y_{2,\beta}} \geq r_{Y_{1,\alpha}} - r_{Y_{2,\alpha}}.$$

Thus, we have

$$\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} + \#v^{\mathrm{sp}}_{g^\bullet_{2,\beta}} + \frac{1}{2}\#\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} \leq \frac{d}{2}.$$

If $\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} = \#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} = 0$, then $g^\bullet_{2,\beta}$ is an étale covering. By replacing $X_1^\bullet$ and $X_2^\bullet$ by $Y_1^\bullet$ and $Y_2^\bullet$, respectively, Proposition 4.4 implies that $g^\bullet_{1,\alpha}$ is an étale covering. This is a contradiction. Thus, either $\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} \neq 0$ or $\#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}} \neq 0$ holds. If $\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} \neq 0$, then $\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} \geq d$. This is a contradiction. Thus, we have $\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} = 0$.

If $\#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}} \neq 0$, then we have $\#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}} = d$ and $\#v^{\mathrm{sp}}_{g^\bullet_{2,\beta}} = 0$. This means that

$$\beta \in \bigcup_{e \in e^{\mathrm{op}}(\Gamma_{Y_2^\bullet})} U^{\mathrm{mp,sp}=0}_{\ell, Y_2^\bullet, e}.$$

Then we have

$$\psi^{-1}_{Y,\ell}\Big(\bigcup_{e \in e^{\mathrm{op}}(\Gamma_{Y_1^\bullet})} U^{\mathrm{mp,sp}=0}_{\ell, Y_1^\bullet, e}\Big) \subseteq \bigcup_{e \in e^{\mathrm{op}}(\Gamma_{Y_2^\bullet})} U^{\mathrm{mp,sp}=0}_{\ell, Y_2^\bullet, e}.$$

Moreover, for each $\gamma \in U^{\mathrm{mp}}_{\ell, Y_1^\bullet, e}$, γ is a linear combination of the elements of $U^{\mathrm{mp,sp}=0}_{\ell, Y_1^\bullet, e}$. Then we have

$$\psi^{-1}_{Y,\ell}(U^{\mathrm{mp}}_{\ell, Y_1^\bullet}) \subseteq U^{\mathrm{mp}}_{\ell, Y_2^\bullet}.$$

On the other hand, since $g_{Y_1} = g_{Y_2}$, Proposition 2.3 implies that $\#U^{\mathrm{mp}}_{\ell, Y_1^\bullet} = \#U^{\mathrm{mp}}_{\ell, Y_2^\bullet}$. Thus, we obtain

$$\psi^{-1}_{Y,\ell}(U^{\mathrm{mp}}_{\ell, Y_1^\bullet}) = U^{\mathrm{mp}}_{\ell, Y_2^\bullet}.$$

This completes the proof of the lemma. □

Lemma 4.9. *We have*

$$\psi^{-1}_{Y,\ell}\Big(\bigcup_{e \in e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})} U^{\mathrm{nd,sp}=0}_{\ell, Y_1^\bullet, e}\Big) \subseteq \bigcup_{e \in e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})} U^{\mathrm{nd,sp}=0}_{\ell, Y_2^\bullet, e}.$$

Moreover, we have

$$\psi^{-1}_{Y,\ell}(U^{\mathrm{nd}}_{\ell, Y_1^\bullet}) = U^{\mathrm{nd}}_{\ell, Y_2^\bullet}.$$

Proof. Let $e \in e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})$ and $\alpha \in U^{\mathrm{nd,sp}=0}_{Y_1^\bullet, e}$. Then the admissible covering

$$g^\bullet_{1,\alpha} : Y^\bullet_{1,\alpha} \to Y^\bullet_1$$

corresponding to α induces a Galois admissible covering

$$g^\bullet_{2,\beta} : Y^\bullet_{2,\beta} \to Y^\bullet_2$$

over k_2 of degree ℓ. Write β for the line bundle corresponding to $g^\bullet_{2,\beta}$. We have

$$\beta \in U^*_{\ell, Y_2^\bullet}.$$

Write $g_{Y_{1,\alpha}}$ and $g_{Y_{2,\beta}}$ for the genera of $Y^\bullet_{1,\alpha}$ and $Y^\bullet_{2,\beta}$, $r_{Y_{1,\alpha}}$ and $r_{Y_{2,\beta}}$ for the Betti numbers of the dual semi-graphs $\Gamma_{Y^\bullet_{1,\alpha}}$ and $\Gamma_{Y^\bullet_{2,\beta}}$, respectively. Then the Riemann-Hurwitz formula implies that

$$g_{Y_{1,\alpha}} - g_{Y_{2,\beta}} = -\frac{1}{2}(\#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}})(\ell - 1).$$

On the other hand, we have

$$r_{Y_{1,\alpha}} = \ell(\#e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet}) - d) + d - \#v(\Gamma_{Y_1^\bullet}) + 1$$

and

$$r_{Y_{2,\beta}} = \ell\#e^{\mathrm{cl,ét}}_{g^\bullet_{2,\beta}} + \#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} - \ell\#v^{\mathrm{cl,sp}}_{g^\bullet_{2,\beta}} - \#v^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} + 1.$$

Then Theorem 1.5 implies that

$$g_{Y_{1,\alpha}} - g_{Y_{2,\beta}} \geq r_{Y_{1,\alpha}} - r_{Y_{2,\beta}}.$$

Thus, we have

$$\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} + \#v^{\mathrm{sp}}_{g^\bullet_{2,\beta}} + \frac{1}{2}\#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}} \leq d.$$

If $\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} = \#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}} = 0$, then $g^\bullet_{2,\beta}$ is an étale covering. By replacing $X_1^\bullet$ and $X_2^\bullet$ by $Y_1^\bullet$ and $Y_2^\bullet$, respectively, Proposition 4.4 implies that $g^\bullet_{1,\alpha}$ is an étale covering. This is a contradiction. Thus, either $\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} \neq 0$ or $\#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}} \neq 0$ holds.

If $\#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}} \neq 0$, then we have $\#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}} \leq 2d$ and $\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} = 0$. Then the "moreover" part of Lemma 4.8 implies that $g^\bullet_{1,\alpha} \in U^{\mathrm{mp}}_{\ell,Y_1^\bullet}$. This is a contradiction. Then we obtain $\#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}} = 0$.

If $\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} \neq 0$, then we have $\#e^{\mathrm{cl,ra}}_{g^\bullet_{2,\beta}} = d$ and $\#v^{\mathrm{sp}}_{g^\bullet_{2,\beta}} = \#e^{\mathrm{op,ra}}_{g^\bullet_{2,\beta}} = 0$. This means that

$$\beta \in \bigcup_{e \in e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})} U^{\mathrm{nd,sp=0}}_{\ell,Y_2^\bullet,e}.$$

Thus, we have

$$\psi^{-1}_{Y,\ell}\Big(\bigcup_{e \in e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})} U^{\mathrm{nd,sp=0}}_{\ell,Y_1^\bullet,e}\Big) \subseteq \bigcup_{e \in e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})} U^{\mathrm{nd,sp=0}}_{\ell,Y_2^\bullet,e}.$$

Moreover, for each $\gamma \in U^{\mathrm{nd}}_{\ell,Y_1^\bullet,e}$, γ is a linear combination of the elements of $U^{\mathrm{nd,sp=0}}_{\ell,Y_1^\bullet,e}$. Then we have

$$\psi^{-1}_{Y,\ell}(U^{\mathrm{nd}}_{\ell,Y_1^\bullet}) \subseteq U^{\mathrm{nd}}_{\ell,Y_2^\bullet}.$$

On the other hand, since $g_{Y_1} = g_{Y_2}$, Proposition 2.3 implies that $\#U^{\mathrm{nd}}_{\ell,Y_1^\bullet} = \#U^{\mathrm{nd}}_{\ell,Y_2^\bullet}$. Thus, we obtain

$$\psi^{-1}_{Y,\ell}(U^{\mathrm{nd}}_{\ell,Y_1^\bullet}) = U^{\mathrm{nd}}_{\ell,Y_2^\bullet}.$$

This completes the proof of the lemma. □

By Theorem 2.2, Remark 2.2.1, Lemma 4.8, and Lemma 4.9, we have the following theorem.

Theorem 4.10. *The isomorphism $\psi_{Y,\ell} : M_{Y_2^\bullet} \xrightarrow{\sim} M_{Y_1^\bullet}$ induces a bijection of the set of closed edges (resp. open edges)*

$$\rho^{\mathrm{cl,edge}}_{\phi} : e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) \xrightarrow{\sim} e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})$$

$$(\text{resp. } \rho^{\mathrm{op,edge}}_{\phi} : e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) \xrightarrow{\sim} e^{\mathrm{op}}(\Gamma_{X_1^\bullet})).$$

Moreover, $\rho^{\mathrm{cl,edge}}_{\phi}$ (resp. $\rho^{\mathrm{op,edge}}_{\phi}$) does not depend on the choices of ℓ, d, and the étale covering $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$.

5 Reconstruction of sets of p-rank via surjections

Let $k_i, i \in \{1,2\}$, be an algebraically closed field of characteristic $p > 0$. Let $X_i^\bullet$, $i \in \{1,2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over k_i, $\Pi_{X_i^\bullet}$ the admissible fundamental groups of $X_i^\bullet$, $\Gamma_{X_i^\bullet}$ the dual semi-graphs of $X_i^\bullet$, and r_{X_i} the Betti number of $\Gamma_{X_i^\bullet}$. In this section, we suppose that $X_1^\bullet$ and $X_2^\bullet$ **satisfy Condition A and Condition B**, and that

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

is an open continuous surjective homomorphism of the admissible fundamental groups of $X_1^\bullet$ and $X_2^\bullet$. Moreover, we denote by

$$(g, n) := (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}).$$

The surjection ϕ induced a surjection of the maximal pro-p quotients

$$\phi^p : \Pi^p_{X_1^\bullet} \twoheadrightarrow \Pi^p_{X_2^\bullet}.$$

Then each Galois admissible covering

$$f_2^\bullet : Y_2^\bullet \to X_2^\bullet$$

of degree p over k_2 induces a Galois admissible covering

$$f_1^\bullet : Y_1^\bullet \to X_1^\bullet$$

of degree p over k_1. Note that, by the definition of admissible coverings, $f_1^\bullet$ and $f_2^\bullet$ are étale coverings. Moreover, ϕ^p induces an injection

$$\psi_p : \mathrm{H}^1_{\text{ét}}(X_2^\bullet, \mathbb{F}_p) \hookrightarrow \mathrm{H}^1_{\text{ét}}(X_1^\bullet, \mathbb{F}_p).$$

For $i \in \{1,2\}$, write g_{Y_i}, for the genus of $Y_i^\bullet$, $\Gamma_{Y_i^\bullet}$ for the dual semi-graph of $Y_i^\bullet$, and r_{Y_i} for the Betti number of $\Gamma_{Y_i^\bullet}$.

Lemma 5.1. *Let $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ be a Galois étale covering of degree p over k_1 which is induced by a Galois étale covering $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ of degree p over k_2. Suppose that $\#v^{\mathrm{ra}}_{f_1^\bullet} = 0$. Then $f_2^\bullet$ is an étale covering, and $\#v^{\mathrm{ra}}_{f_2^\bullet} = 0$. In particular, ψ_p induces an isomorphism*

$$\mathrm{H}^1(\Gamma_{X_2^\bullet}, \mathbb{F}_p) \xrightarrow{\sim} \mathrm{H}^1(\Gamma_{X_1^\bullet}, \mathbb{F}_p).$$

Proof. Since $f_1^\bullet$ and $f_2^\bullet$ are étale coverings, the Riemann-Hurwitz formula implies that

$$g_{Y_1} = g_{Y_2}.$$

Thus, similar arguments to the arguments given in the proofs of Proposition 4.5 imply that

$$\#v^{\mathrm{ra}}_{f_2^\bullet} = 0.$$

This completes the proof of the lemma. □

Lemma 5.2. *Let $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ be a Galois étale covering of degree p over k_1 which is induced by a Galois étale covering $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ of degree p over k_2. Suppose that $\#v_{f_1^\bullet}^{\mathrm{ra}} = 1$. Then $f_2^\bullet$ is an étale covering, and $\#v_{f_2^\bullet}^{\mathrm{ra}} = 1$.*

Proof. Similar arguments to the arguments given in the proofs of Lemma 4.6 imply that

$$\#v_{f_2^\bullet}^{\mathrm{ra}} \leq 1.$$

If $\#v_{f_2^\bullet}^{\mathrm{ra}} = 0$, then Lemma 5.1 implies that the line bundle corresponding to $\#v_{f_1^\bullet}^{\mathrm{ra}} = 0$. This is a contradiction. Then we obtain that

$$\#v_{f_2^\bullet}^{\mathrm{ra}} = 1.$$

□

The main theorem of the present section is as follows.

Theorem 5.3. *For each $v \in v(\Gamma_{X_1^\bullet})$ (resp. $v \in v(\Gamma_{X_1^\bullet})$), we write $X_{1,v}$ (resp. $X_{2,v}$) for the irreducible component of $X_1^\bullet$ (resp. $X_2^\bullet$) corresponding to v and $\sigma_{1,v}$ (resp. $\sigma_{2,v}$) for the p-rank of the normalization $\widetilde{X_{1,v}}$ (resp. $\widetilde{X_{2,v}}$) of $X_{1,v}$ (resp. $X_{2,v}$). Then the injection $\psi_p : \mathrm{H}^1_{\text{ét}}(X_2^\bullet, \mathbb{F}_p) \hookrightarrow \mathrm{H}^1_{\text{ét}}(X_1^\bullet, \mathbb{F}_p)$ induces an injection of the set of vertices*

$$\rho_\phi^{\mathrm{vex},p} : v(\Gamma_{X_2^\bullet})^{>0,p} \hookrightarrow v(\Gamma_{X_1^\bullet})^{>0,p}.$$

Moreover, we have

$$\sigma_{2,v} \leq \sigma_{1,\rho_\phi^{\mathrm{vex},p}(v)}$$

for each $v \in v(\Gamma_{X_2^\bullet})^{>0,p}$.

Proof. For the prime number p, write $V_{p,X_1^\bullet}$ and $V_{p,X_1^\bullet}$ for the sets of line bundles defined in Section 2. Lemma 5.2 implies that

$$\psi_p(V_{p,X_2^\bullet}) \subseteq V_{p,X_1^\bullet}.$$

Let $\alpha_1, \alpha_2 \in V_{p,X_2^\bullet}$ distinct from each other such that $\alpha_1 \sim \alpha_2$. It is easy to see that $a\alpha_1 + b\alpha_2 \in V_{p,X_2^\bullet}$ if and only $a\psi_p(\alpha_1) + b\psi_p(\alpha_2) \in V_{p,X_1^\bullet}$ for each $a, b \in \mathbb{F}_p^\times$. Thus, we obtain an injection of the set of vertices

$$\rho_\phi^{\mathrm{vex},p} : v(\Gamma_{X_2^\bullet})^{>0,p} \hookrightarrow v(\Gamma_{X_1^\bullet})^{>0,p}.$$

For each $w_1 \in v(\Gamma_{X_1^\bullet})$ (resp. $w_2 \in v(\Gamma_{X_2^\bullet})$), write

$$L_{X_1^\bullet}^{w_1,p} := \{\alpha \in \mathrm{H}^1_{\text{ét}}(X_1^\bullet, \mathbb{F}_p) \mid \#v_{f_{1,\alpha}^\bullet}^{\mathrm{ra}} = 1 \text{ and } (f_{1,\alpha}^\bullet)^{-1}(X_{1,w_1}) \text{ is connected}\}$$

$$(\text{resp. } L_{X_2^\bullet}^{w_2,p} := \{\alpha \in \mathrm{H}^1_{\text{ét}}(X_2^\bullet, \mathbb{F}_p) \mid \#v_{f_{2,\alpha}^\bullet}^{\mathrm{ra}} = 1 \text{ and } (f_{2,\alpha}^\bullet)^{-1}(X_{2,w_2}) \text{ is connected}\}),$$

where $f_{1,\alpha}^\bullet$ (resp. $f_{2,\alpha}^\bullet$) denotes the Galois étale covering corresponding to α. Moreover, we denote by

$$[L_{X_1^\bullet}^{w_1,p}] \text{ (resp. } [L_{X_2^\bullet}^{w_2,p}])$$

for the image of $L^{w_1,p}_{X_1^\bullet}$ in $\mathrm{H}^1_{\text{ét}}(X_1^\bullet, \mathbb{F}_p)/\mathrm{H}^1(\Gamma_{X_1^\bullet}, \mathbb{F}_p)$ (resp. $L^{w_2,\ell}_{X_2^\bullet}$ in $\mathrm{H}^1_{\text{ét}}(X_2^\bullet, \mathbb{F}_p)/\mathrm{H}^1(\Gamma_{X_2^\bullet}, \mathbb{F}_p)$). Then we have

$$\#[L^{w_1,p}_{X_1^\bullet}] = p^{\sigma_{1,w_1}} - 1 \text{ (resp. } \#[L^{w_2,p}_{X_2^\bullet}] = p^{\sigma_{2,w_2}} - 1).$$

On the other hand, for each $v \in v(\Gamma_{X^\bullet})$, Lemma 5.1 implies that ψ_p induces an injection

$$[L^{v,p}_{X_2^\bullet}] \hookrightarrow [L^{\rho^{\text{vex}}_\phi(v),p}_{X_1^\bullet}].$$

Thus, we have

$$p^{\sigma_{2,v}} - 1 = \#[L^{v,p}_{X_2^\bullet}] \leq \#[L^{\rho^{\text{vex}}_\phi(v),p}_{X_1^\bullet}] = p^{\sigma_{1,\rho^{\text{vex}}_\phi(v)}} - 1.$$

This means that

$$\sigma_{2,v} \leq \sigma_{1,\rho^{\text{vex}}_\phi(v)}$$

for each $v \in v(\Gamma_{X_2^\bullet})$. On the other hand, since

$$\sum_{w \in v(\Gamma_{X_1^\bullet})} \sigma_{1,w} = \sigma(X_1^\bullet) - r_{X_1} \geq \sigma(X_2^\bullet) - r_{X_2} = \sum_{w \in v(\Gamma_{X_2^\bullet})} \sigma_{2,w},$$

we have

$$\sigma_{2,v} \leq \sigma_{1,\rho^{\text{vex}}_\phi(v)}$$

for each $v \in v(\Gamma_{X_2^\bullet})$. This completes the proof of the theorem. $\square$

6 Reconstruction of dual semi-graphs via surjections

Let $k_i, i \in \{1, 2\}$, be an algebraically closed field of characteristic $p > 0$ and $\ell \neq p$ a prime number. Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over k_i, $\Pi_{X_i^\bullet}$ the admissible fundamental groups of $X_i^\bullet$, $\Gamma_{X_i^\bullet}$ the dual semi-graphs of $X_i^\bullet$, and r_{X_i} the Betti number of $\Gamma_{X_i^\bullet}$. In this section, we suppose that $X_1^\bullet$ and $X_2^\bullet$ **satisfy Condition A and Condition B**, and that

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

is an open continuous surjective homomorphism of the admissible fundamental groups of $X_1^\bullet$ and $X_2^\bullet$. Moreover, we denote by

$$(g, n) := (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}).$$

Let H_2 be an open normal subgroup of $\Pi^\bullet_{X_2}$ and $H_1 := \phi^{-1}(H_2)$ the open normal subgroup of $\Pi_{X_1^\bullet}$. Write ϕ_{H_1} for the surjection $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$,

$$f^\bullet_{X_{H_1}} : X^\bullet_{H_1} \to X_1^\bullet$$

and

$$f^\bullet_{X_{H_2}} : X^\bullet_{H_1} \to X_2^\bullet$$

for the Galois admissible coverings over k_1 and k_2, $(g_{X_{H_1}}, n_{X_{H_1}})$ and $(g_{X_{H_2}}, n_{X_{H_2}})$ for the types of $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$, $\Gamma_{X^\bullet_{H_1}}$ and $\Gamma_{X^\bullet_{H_1}}$ for the dual semi-graphs of $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$, respectively. Note that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A. Furthermore, we suppose that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ **satisfy Condition B**.

Let

$$(\ell, d, f_2^\bullet : Y_2^\bullet \to X_2^\bullet)$$

be a triple associated to $X_2^\bullet$ such that $(\ell, \#(\Pi_{X_2^\bullet}/H_2)) = 1$ and $(d, \#(\Pi_{X_2^\bullet}/H_2)) = 1$. By Lemma 4.2, we obtain a triple

$$(\ell, d, f_1^\bullet : Y_1^\bullet \to X_1^\bullet)$$

associated to $X_1^\bullet$ induced by ϕ and $(\ell, d, f_2^\bullet : Y_2^\bullet \to X_2^\bullet)$. On the other hand, we have a triple

$$(\ell, d, h_2^\bullet : W_2^\bullet := Y_2^\bullet \times_{X_2^\bullet} X^\bullet_{H_2} \to X^\bullet_{H_2})$$

associated to $X^\bullet_{H_2}$. By Lemma 4.2 again, we obtain a triple

$$(\ell, d, h_1^\bullet : W_1^\bullet := Y_1^\bullet \times_{X_1^\bullet} X^\bullet_{H_1} \to X^\bullet_{H_1})$$

associated to $X^\bullet_{H_1}$ induced by ϕ_{H_1} and $(\ell, d, h_2^\bullet : W_2^\bullet \to X^\bullet_{H_2})$.

Then the morphisms $g_1^\bullet$ and $g_2^\bullet$ induce respectively the natural morphisms

$$\gamma_{f_1^\bullet} : \Gamma_{X^\bullet_{H_1}} \to \Gamma_{X_1^\bullet} \text{ and } \gamma_{f_2^\bullet} : \Gamma_{X^\bullet_{H_1}} \to \Gamma_{X_2^\bullet},$$

$$\gamma^{\mathrm{vex}}_{f_1^\bullet} : v(\Gamma_{X^\bullet_{H_1}}) \to v(\Gamma_{X_1^\bullet}) \text{ and } \gamma^{\mathrm{vex}}_{f_2^\bullet} : v(\Gamma_{X^\bullet_{H_2}}) \to v(\Gamma_{X_2^\bullet}),$$

$$\gamma^{\mathrm{cl,edge}}_{f_1^\bullet} : e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_1}}) \to e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) \text{ and } \gamma^{\mathrm{cl,edge}}_{f_2^\bullet} : e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_2}}) \to e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}),$$

$$\gamma^{\mathrm{op,edge}}_{f_1^\bullet} : e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_1}}) \to e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \text{ and } \gamma^{\mathrm{op,edge}}_{f_2^\bullet} : e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_2}}) \to e^{\mathrm{op}}(\Gamma_{X_2^\bullet}).$$

Write ϕ_{H_1} for $\phi|_{H_1} : H_1 \twoheadrightarrow H_2$. By Theorem 3.7 and Theorem 3.10, ϕ_{H_1} induces

$$\rho^{\mathrm{vex}}_{\phi_{H_1}} : v(\Gamma_{X^\bullet_{H_2}}) \to v(\Gamma_{X^\bullet_{H_1}}),\ \rho^{\mathrm{cl,edge}}_{\phi_{H_1}} : e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_2}}) \to e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_1}}), \text{ and } \rho^{\mathrm{op,edge}}_{\phi_{H_1}} : e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_2}}) \to e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_1}}).$$

Then we have the following lemma.

Lemma 6.1. *The natural diagrams*

$$\begin{tikzcd}
v(\Gamma_{X^\bullet_{H_2}}) \arrow[r, "\rho^{\mathrm{vex}}_{\phi_{H_1}}"] \arrow[d, "\gamma^{\mathrm{vex}}_{f_2^\bullet}"'] & v(\Gamma_{X^\bullet_{H_1}}) \arrow[d, "\gamma^{\mathrm{vex}}_{f_1^\bullet}"'] \\
v(\Gamma_{X_2^\bullet}) \arrow[r, "\rho^{\mathrm{vex}}_{\phi}"] & v(\Gamma_{X_1^\bullet}),
\end{tikzcd}$$

$$\begin{tikzcd}
e^{\mathrm{cl}}(\Gamma_{X^\bullet_{H_2}}) \arrow[r, "\rho^{\mathrm{cl,edge}}_{\phi_{H_1}}"] \arrow[d, "\gamma^{\mathrm{cl,edge}}_{f_2^\bullet}"'] & v(\Gamma_{X^\bullet_{H_1}}) \arrow[d, "\gamma^{\mathrm{cl,edge}}_{f_1^\bullet}"'] \\
e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) \arrow[r, "\rho^{\mathrm{cl,edge}}_{\phi}"] & e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}),
\end{tikzcd}$$

and

$$\begin{array}{ccc} e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_2}}) & \xrightarrow{\rho^{\mathrm{op,edge}}_{\phi_{H_1}}} & v(\Gamma_{X^\bullet_{H_1}}) \\ {\scriptstyle \gamma^{\mathrm{op,edge}}_{f_2^\bullet}}\downarrow & & {\scriptstyle \gamma^{\mathrm{op,edge}}_{f_1^\bullet}}\downarrow \\ e^{\mathrm{op}}(\Gamma_{X_2^\bullet}) & \xrightarrow{\rho^{\mathrm{op,edge}}_{\phi}} & e^{\mathrm{op}}(\Gamma_{X_1^\bullet}) \end{array}$$

are commutative. Moreover, all the commutative diagrams above are compatible with the natural actions of $\Pi_{X_2^\bullet}/H_2 = \Pi_{X_1^\bullet}/H_1$.

Proof. We only treat the third diagram. Let

$$e_{X_{H_2}} \in e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_2}}),\ e_{X_{H_1}} := \rho^{\mathrm{op,edge}}_{\phi_{H_1}}(e_{X_{H_2}}) \in e^{\mathrm{op}}(\Gamma_{X^\bullet_{H_1}}),\ e_2 := \gamma^{\mathrm{op,edge}}_{f_2^\bullet}(e_{X_{H_2}}) \in e^{\mathrm{op}}(\Gamma_{X_2^\bullet}),$$

$$e_1 := (\gamma^{\mathrm{op,edge}}_{f_1^\bullet} \circ \rho^{\mathrm{op,edge}}_{\phi_{H_1}})(e_{X_{H_2}}) \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}),\ \text{and } e_1' := \rho^{\mathrm{op,edge}}_{\phi}(e_2) \in e^{\mathrm{op}}(\Gamma_{X_1^\bullet}).$$

Let us prove that $e_1 = e_1'$.

Write $S_{X_{H_1}}$ and $S_{X_{H_2}}$ for the sets $(\gamma^{\mathrm{op,edge}}_{f_2^\bullet})^{-1}(e_1')$ and $(\gamma^{\mathrm{op,edge}}_{f_2^\bullet})^{-1}(e_2)$, respectively. Note that $e_{X_{H_2}} \in S_{X_{H_2}}$. To verify $e_1 = e_1'$, it is sufficient to prove that $e_{X_{H_1}} \in S_{X_{H_1}}$

Let $\alpha_2 \in U^{\mathrm{mp}}_{\ell, Y_2^\bullet, e_2}$ (cf. Section 2). Then the proof of Lemma 4.8 implies that α_2 induces an element

$$\alpha_1 \in U^{\mathrm{mp}}_{\ell, Y_1^\bullet, e_1'}.$$

Write $Y^\bullet_{\alpha_1}$ and $Y^\bullet_{\alpha_2}$ for the pointed stable curves over k_1 and k_2 corresponding to α_1 and α_2, respectively. Consider the connected admissible Galois covering

$$Y^\bullet_{\alpha_2} \times_{X_2^\bullet} X^\bullet_{H_2} \to W_2^\bullet$$

of degree ℓ over k_2, and write β_2 for the element of $U^*_{W_2^\bullet}$ (cf. Section 2 for definition) corresponding to this connected admissible Galois covering. Then we have

$$\beta_2 = \sum_{c_2 \in S_{X_{H_2}}} t_{c_2}\beta_{c_2},$$

where $t_{c_2} \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ and $\beta_{c_2} \in U^{\mathrm{mp}}_{\ell, W_2^\bullet, c_2}$. Note that $t_{e_2} \neq 0$. On the other hand, the proof of Lemma 4.8 implies that β_{c_2} induced an element $\beta_{\rho^{\mathrm{op,edge}}_{\phi_{H_1}}(c_2)} \in U^{\mathrm{mp}}_{\ell, W_1^\bullet, \rho^{\mathrm{op,edge}}_{\phi_{H_1}}(c_2)}$. Then β_2 induces an element

$$\beta_1 := \sum_{c_2 \in S_{X_{H_2}} \setminus \{e_{X_{H_2}}\}} t_{c_2}\beta_{\rho^{\mathrm{op,edge}}_{\phi_{H_1}}(c_2)} + t_{e_{X_{H_2}}}\rho^{\mathrm{op,edge}}_{\phi_{H_1}}(e_{X_{H_2}}) \in U^*_{\ell, W_1^\bullet}.$$

Note that since β_1 corresponds to the connected admissible Galois covering $Y^\bullet_{\alpha_1} \times_{X_1^\bullet} X^\bullet_{H_1} \to W_1^\bullet$, we have the composition of the connected admissible Galois covering $Y^\bullet_{\alpha_1} \times_{X_1^\bullet} X^\bullet_{H_1} \to W_1^\bullet$ and the étale Galois covering $h_1^\bullet : W_1^\bullet \to X^\bullet_{H_1}$ is ramified over $S_{X_{H_2}}$. This means that $e_{X_{H_1}} = \rho^{\mathrm{op,edge}}_{\phi_{H_1}}(e_{X_{H_2}})$ is contained in $S_{X_{H_1}}$.

Similar arguments to the arguments given above imply the first and the second diagrams are commutative. It is easy to check the "moreover" part of the lemma. This completes the proof of the lemma. □

Lemma 6.2. *Write $\Gamma_{Y_1^\bullet}$ and $\Gamma_{Y_2^\bullet}$ for the dual semi-graphs of $Y_1^\bullet$ and $Y_2^\bullet$, respectively, and*

$$\psi^{\text{ét}}_{Y,\ell}: \mathrm{H}^1_{\text{ét}}(Y_2^\bullet, \mathbb{F}_\ell) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(Y_1^\bullet, \mathbb{F}_\ell)$$

the natural isomorphism induced by ϕ. Let $e_2 \in e^{\text{cl}}(\Gamma_{X_2^\bullet})$ and $e_1 := \rho_\phi^{\text{cl,edge}}(e_2) \in e^{\text{cl}}(\Gamma_{X_1^\bullet})$ (resp. $e_2 \in e^{\text{op}}(\Gamma_{X_2^\bullet})$ and $e_1 := \rho_\phi^{\text{op,edge}}(e_2) \in e^{\text{op}}(\Gamma_{X_1^\bullet})$). Let $m \in \mathbb{Z}_{\geq 0}$ such that $U^{\text{nd,sp}=m}_{\ell, Y_1^\bullet, e_1} \neq \emptyset$ and $U^{\text{nd,sp}=m}_{\ell, Y_2^\bullet, e_2} \neq \emptyset$ (resp. $U^{\text{mp,sp}=m}_{\ell, Y_1^\bullet, e_1} \neq \emptyset$ and $U^{\text{mp,sp}=m}_{\ell, Y_2^\bullet, e_2} \neq \emptyset$), $\alpha_2 \in U^{\text{nd}}_{\ell, Y_2^\bullet, e_2}$ (resp, $\alpha_2 \in U^{\text{mp}}_{\ell, Y_2^\bullet, e_2}$), and $\alpha_1 := \psi^{\text{ét}}_{Y,\ell}(\alpha_2) \in U^{\text{nd}}_{\ell, Y_1^\bullet, e_1}$ (resp. $\alpha_1 := \psi^{\text{ét}}_{Y,\ell}(\alpha_2) \in U^{\text{mp}}_{\ell, Y_1^\bullet, e_1}$) the line bundle induced by α_2. Then we have

$$\alpha_1 \in U^{\text{nd,sp}=m}_{\ell, Y_1^\bullet, e_1} \text{ if and only if } \alpha_2 \in U^{\text{nd,sp}=m}_{\ell, Y_2^\bullet, e_2}$$

$$(\text{resp. } \alpha_1 \in U^{\text{mp,sp}=m}_{\ell, Y_1^\bullet, e_1} \text{ if and only if } \alpha_2 \in U^{\text{mp,sp}=m}_{\ell, Y_2^\bullet, e_2}).$$

Proof. We only treat the case where e_1 and e_2 are closed edges. Write

$$f^\bullet_{1,\alpha_1}: Y^\bullet_{1,\alpha_1} \to X_1^\bullet \text{ and } f^\bullet_{2,\alpha_2}: Y^\bullet_{2,\alpha_2} \to X_2^\bullet$$

for the Galois admissible coverings over k_1 and k_2 of degree ℓ corresponding to α_1 and α_2, respectively. Write $g_{Y_{1,\alpha_1}}$ and $g_{Y_{2,\alpha_2}}$ for the genera of $Y^\bullet_{1,\alpha_1}$ and $Y^\bullet_{2,\alpha_2}$, $r_{Y_{1,\alpha_1}}$ and $r_{Y_{2,\alpha_2}}$ for the Betti numbers of the dual semi-graphs $\Gamma_{Y^\bullet_{1,\alpha_1}}$ and $\Gamma_{Y^\bullet_{2,\alpha_2}}$, respectively. Note that we have $g_{Y_{1,\alpha_1}} = g_{Y_{2,\alpha_2}}$.

First, we prove the "if" part of the lemma. We have

$$r_{Y_{2,\alpha_2}} = r_{Y_{2,\alpha_2}} = \ell(\#e^{\text{cl}}(\Gamma_{Y_2^\bullet}) - d) + d - \ell m - (\#v(\Gamma_{Y_2^\bullet}) - m) + 1$$

$$= \ell(\#e^{\text{cl}}(\Gamma_{Y_2^\bullet}) - d) + d - \ell(\#v(\Gamma_{Y_2^\bullet}) - (\#v(\Gamma_{Y_2^\bullet}) - m)) - (\#v(\Gamma_{Y_2^\bullet}) - m) + 1$$

and

$$r_{Y_{1,\alpha_1}} = \ell\#e^{\text{cl,ét}}_{f^\bullet_{1,\alpha_1}} + \#e^{\text{cl,ra}}_{f^\bullet_{1,\alpha_1}} - \ell\#v^{\text{cl,sp}}_{f^\bullet_{1,\alpha_1}} - \#v^{\text{cl,ra}}_{f^\bullet_{1,\alpha_1}} + 1.$$

Then Theorem 1.5 implies that

$$0 = g_{Y_{1,\alpha_1}} - g_{Y_{2,\alpha_2}} = r_{Y_{1,\alpha_1}} - r_{Y_{2,\alpha_2}}.$$

Thus, we have

$$\#e^{\text{cl,ra}}_{f^\bullet_{1,\alpha_1}} + \#v^{\text{ra}}_{f^\bullet_{1,\alpha_1}} = d - (\#v(\Gamma_{Y_2^\bullet}) - m).$$

Since $\#e^{\text{cl,ra}}_{f^\bullet_{1,\alpha_1}} = d$, we obtain that

$$\#v^{\text{ra}}_{f^\bullet_{1,\alpha_1}} = \#v(\Gamma_{Y_2^\bullet}) - m.$$

This means that $\alpha_1 \in U^{\text{nd,sp}=m}_{\ell, Y_1^\bullet, e_1}$.

Similar arguments to the arguments given in above imply the "only if" part and the "resp" part of the lemma. This completes the proof of the lemma. □

Lemma 6.3. *We maintain the notations introduced in Lemma 6.2. Then we have*

(i) $e_1 \in e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})$ *is a closed edge that corresponds to a node of* $Y_1^\bullet$ *which is contained in a unique irreducible component of* $Y_1^\bullet$ *if and only if* $e_2 \in e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})$ *is a closed edge that corresponds to a node of* $Y_2^\bullet$ *which is contained in a unique irreducible component of* $Y_2^\bullet$*.*

(ii) $e_1 \in e^{\mathrm{cl}}(\Gamma_{Y_1^\bullet})$ *is a closed edge that corresponds to a node of* $Y_1^\bullet$ *which is contained in two different irreducible components of* $Y_1^\bullet$ *if and only if* $e_2 \in e^{\mathrm{cl}}(\Gamma_{Y_2^\bullet})$ *is a closed edge that corresponds to a node of* $Y_2^\bullet$ *which is contained in two different irreducible components of* $Y_2^\bullet$*;*

Proof. Since (ii) can be deduced from (i), we only prove (i). Let us prove the "if" part of (i) of the lemma. Let $m := \#v(\Gamma_{Y_1^\bullet}) - 1 = \#v(\Gamma_{Y_2^\bullet}) - 1$. Then we have $U^{\mathrm{nd},\mathrm{sp}=m}_{\ell, Y_2^\bullet, e_2} \neq \emptyset$. Let

$$\alpha_2 \in U^{\mathrm{nd},\mathrm{sp}=m}_{\ell, Y_2^\bullet, e_2}.$$

Lemma 6.2 implies that

$$\alpha_1 := \psi^{\text{ét}}_{Y,\ell}(\alpha_2) \in U^{\mathrm{nd},\mathrm{sp}=m}_{\ell, Y_1^\bullet, e_1}.$$

Thus, $U^{\mathrm{nd},\mathrm{sp}=m}_{\ell, Y_1^\bullet, e_1} \neq \emptyset$. On the other hand, if e_1 is a closed edge that corresponds to a node of $Y_1^\bullet$ which is contained in two different irreducible components of $Y_1^\bullet$, then we have

$$U^{\mathrm{nd},\mathrm{sp}=m}_{\ell, Y_1^\bullet, e_1} = \emptyset.$$

Thus, e_1 is a closed edge that corresponds to a node of $Y_1^\bullet$ which is contained in a unique irreducible component of $Y_1^\bullet$.

Similar arguments to the arguments given in above imply the "only if" part of (i) of the lemma. $\square$

Next, we reconstruct the dual semi-graphs. The main theorem of the present section is as follows.

Theorem 6.4. *The surjections* $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$ *and* $\phi_{H_1} : \Pi_{X_{H_1}^\bullet} \twoheadrightarrow \Pi_{X_{H_1}^\bullet}$ *induce isomorphisms of dual semi-graphs*

$$\theta_\phi : \Gamma_{X_1^\bullet} \xrightarrow{\sim} \Gamma_{X_2^\bullet}$$

and

$$\theta_{\phi_{H_1}} : \Gamma_{X_{H_1}^\bullet} \xrightarrow{\sim} \Gamma_{X_{H_1}^\bullet}$$

such that

$$\theta_\phi|_{v(\Gamma_{X_1^\bullet})} = (\rho_\phi^{\mathrm{vex}})^{-1}, \ \theta_\phi|_{e^{\mathrm{cl}}(\Gamma_{X_1^\bullet})} = (\rho_\phi^{\mathrm{cl,edge}})^{-1}, \ \theta_\phi|_{e^{\mathrm{op}}(\Gamma_{X_1^\bullet})} = (\rho_\phi^{\mathrm{op,edge}})^{-1},$$

$$\theta_{\phi_{H_1}}|_{v(\Gamma_{X_{H_1}^\bullet})} = (\rho_{\phi_{H_1}}^{\mathrm{vex}})^{-1}, \ \theta_{\phi_{H_1}}|_{e^{\mathrm{cl}}(\Gamma_{X_{H_1}^\bullet})} = (\rho_{\phi_{H_1}}^{\mathrm{cl,edge}})^{-1}, \ \text{and } \theta_{\phi_{H_1}}|_{e^{\mathrm{op}}(\Gamma_{X_{H_1}^\bullet})} = (\rho_{\phi_{H_1}}^{\mathrm{op,edge}})^{-1}.$$

Moreover, the natural digram

$$\begin{array}{ccc} \Gamma_{X_{H_1}^\bullet} & \xrightarrow{\theta_{\phi_{H_1}}} & \Gamma_{X_{H_2}^\bullet} \\ {\scriptstyle \gamma_{f_1^\bullet}} \downarrow & & {\scriptstyle \gamma_{f_2^\bullet}} \downarrow \\ \Gamma_{X_1^\bullet} & \xrightarrow{\theta_\phi} & \Gamma_{X_2^\bullet} \end{array}$$

is commutative, and the commutative diagram is compatible with the natural actions of $\Pi_{X_2^\bullet}/H_2 = \Pi_{X_1^\bullet}/H_1$*.*

Proof. Let us construct θ_ϕ. We only need to prove that, for each $e_1 = \{b_{e_1,1}, b_{e_1,2}\} \in e(\Gamma_{X_1^\bullet})$, we have

$$\{\zeta_{e_1}^{\Gamma_{X_1^\bullet}}(b_{e_1,1}), \zeta_{e_1}^{\Gamma_{X_1^\bullet}}(b_{e_1,2})\} = \{(\rho_\phi^{\mathrm{vex}})^{-1}(\zeta_{e_2}^{\Gamma_{X_2^\bullet}}(b_{e_2,1})), (\rho_\phi^{\mathrm{vex}})^{-1}(\zeta_{e_2}^{\Gamma_{X_2^\bullet}}(b_{e_2,2}))\},$$

where

$$e_2 = (b_{e_2,1}, b_{e_2,2}) := (\rho_\phi^{\mathrm{cl,edge}})^{-1}(e_1)$$

when e_1 is a closed edge of $\Gamma_{X_1^\bullet}$,

$$e_2 = (b_{e_2,1}, b_{e_2,2}) := (\rho_\phi^{\mathrm{op,edge}})^{-1}(e_1)$$

when e_1 is an open edge of $\Gamma_{X_1^\bullet}$, and $b_{e_i,j},\ i,j \in \{1,2\}$, denotes the branches of e_i (cf. [M5, Section 1]).

First, let us treat the case where e_1 is a closed edge of $\Gamma_{X_1^\bullet}$ and corresponds to a node of $X_1^\bullet$ which is contained in two different irreducible components of $X_1^\bullet$. We maintains the notations introduced in Lemma 6.2. Then the construction of $Y_2^\bullet$ and Lemma 6.3 imply that e_2 corresponds to a node of $X_2^\bullet$ which is contained in two different irreducible components of $X_2^\bullet$. We maintain the notations introduced in Lemma 6.2. Let $m = \#v(\Gamma_{Y_1^\bullet}) - 2 = \#v(\Gamma_{Y_2^\bullet}) - 2$ and $\alpha_2 \in U_{\ell, Y_2^\bullet, e_2}^{\mathrm{nd,sp}=m}$. Then Lemma 6.2 implies that

$$\alpha_1 := \psi_{Y,\ell}^{\text{ét}}(\alpha_2) \in U_{\ell, Y_1^\bullet, e_1}^{\mathrm{nd,sp}=m}.$$

Thus, Lemma 6.1 implies the following commutative diagrams:

$$\begin{array}{ccc}
v(\Gamma_{Y_{2,\alpha_2}^\bullet}) & \xrightarrow{\rho^{\mathrm{vex}}_{\phi_{Y,\alpha}}} & v(\Gamma_{Y_{1,\alpha_1}^\bullet}) \\
{\scriptstyle \gamma_{f_{2,\alpha_2}^\bullet}}\downarrow & & {\scriptstyle \gamma_{f_{1,\alpha_1}^\bullet}}\downarrow \\
v(\Gamma_{Y_2^\bullet}) & \xrightarrow{\rho^{\mathrm{vex}}_{\phi_Y}} & v(\Gamma_{Y_1^\bullet}) \\
{\scriptstyle \gamma_{f_2^\bullet}}\downarrow & & {\scriptstyle \gamma_{f_1^\bullet}}\downarrow \\
v(\Gamma_{X_2^\bullet}) & \xrightarrow{\rho^{\mathrm{vex}}_{\phi}} & v(\Gamma_{X_1^\bullet}),
\end{array}$$

where ϕ_Y denotes the surjections between the admissible fundamental groups of $Y_1^\bullet$ and $Y_2^\bullet$ induced by ϕ, and $\phi_{Y,\alpha}$ denotes the surjection between the admissible fundamental groups of $Y_{1,\alpha_1}^\bullet$ and $Y_{2,\alpha_2}^\bullet$ induced ϕ. Write

$$\{v_{e_1,1}, v_{e_1,2}\}$$

for $\{\zeta_{e_1}^{\Gamma_{X_1^\bullet}}(b_{e_1,1}), \zeta_{e_1}^{\Gamma_{X_1^\bullet}}(b_{e_1,2})\}$ and

$$\{v_{e_2,1}, v_{e_2,2}\}$$

for $\{\zeta_{e_2}^{\Gamma_{X_2^\bullet}}(b_{e_2,1}), \zeta_{e_2}^{\Gamma_{X_2^\bullet}}(b_{e_2,2})\}$. Moreover, we have that

$$\{v_{e_1,1}, v_{e_1,2}\} = \{v \in v(\Gamma_{X_1^\bullet}) \mid \#(\gamma_{f_1^\bullet} \circ \gamma_{f_{1,\alpha_1}^\bullet})^{-1}(v) = 1\}$$

and

$$\{v_{e_2,1}, v_{e_2,2}\} = \{v \in v(\Gamma_{X_2^\bullet}) \mid \#(\gamma_{f_2^\bullet} \circ \gamma_{f_{2,\alpha_2}^\bullet})^{-1}(v) = 1\}.$$

Then the commutative diagram above implies that

$$\{v_{e_1,1}, v_{e_1,2}\} = \{(\rho_\phi^{\mathrm{vex}})^{-1}(v_{e_2,1}), (\rho_\phi^{\mathrm{vex}})^{-1}(v_{e_2,2})\}.$$

By applying similar arguments to the arguments given in above imply that

$$\{\zeta_{e_1}^{\Gamma_{X_1^\bullet}}(b_{e_1,1}), \zeta_{e_1}^{\Gamma_{X_1^\bullet}}(b_{e_1,2})\} = \{(\rho_\phi^{\mathrm{vex}})^{-1}(\zeta_{e_2}^{\Gamma_{X_2^\bullet}}(b_{e_2,1})), (\rho_\phi^{\mathrm{vex}})^{-1}(\zeta_{e_2}^{\Gamma_{X_2^\bullet}}(b_{e_2,2}))\},$$

holds when e_1 is a closed edge of $\Gamma_{X_1^\bullet}$ and corresponds to a node of $X_1^\bullet$ which is contained in a unique irreducible component of $X_1^\bullet$ (resp. e_1 is an open edge of $\Gamma_{X_1^\bullet}$).

On the other hand, by applying Lemma 6.1, it is easy to check that the diagram

$$\begin{array}{ccc} \Gamma_{X_{H_1}^\bullet} & \xrightarrow{\theta_{\phi_{H_1}}} & \Gamma_{X_{H_2}^\bullet} \\ \gamma_{f_1^\bullet}\Big\downarrow & & \gamma_{f_2^\bullet}\Big\downarrow \\ \Gamma_{X_1^\bullet} & \xrightarrow{\theta_\phi} & \Gamma_{X_2^\bullet} \end{array}$$

is commutative, and the commutative diagram is compatible with the natural actions of $\Pi_{X_2^\bullet}/H_2 = \Pi_{X_1^\bullet}/H_1$. This completes the proof of the theorem. $\square$

7 Mono-anabelian reconstruction algorithm for dual semi-graphs via surjections

Let k_i, $i \in \{1, 2\}$, be an algebraically closed field of characteristic $p > 0$. Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over k_i, $\Pi_{X_i^\bullet}$ the admissible fundamental groups of $X_i^\bullet$, $\Gamma_{X_i^\bullet}$ the dual semi-graphs of $X_i^\bullet$, and r_{X_i} the Betti number of $\Gamma_{X_i^\bullet}$. In this section, we suppose that $X_1^\bullet$ and $X_2^\bullet$ **satisfy Condition A and Condition B**, and that

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

is an open continuous surjective homomorphism of the admissible fundamental groups of $X_1^\bullet$ and $X_2^\bullet$. Moreover, we denote by

$$(g, n) := (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}).$$

In this section, we prove the mono-anabelian versions of Theorem 4.7, Theorem 4.10, Theorem 5.3, and Theorem 6.4. First, the mono-anabelian version of Theorem 4.7 is as follows:

Theorem 7.1. *We maintain the notations and conditions introduced in Theorem 4.7. Then the bijection of the set of vertices*

$$\rho_\phi^{\mathrm{vex}} : v(\Gamma_{X_2^\bullet}) \xrightarrow{\sim} v(\Gamma_{X_1^\bullet})$$

can be mono-anabelian reconstructed from the surjection $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$.

Proof. Since Lemma 3.2 implies that p can be mono-anabelian reconstructed from the surjection $\Pi_{X_1^\bullet}$ or $\Pi_{X_2^\bullet}$, we may choice a prime number $\ell \neq p$. Then Lemma 3.4 implies that $\mathrm{Hom}(\Pi_{X_1^\bullet}^{\text{ét,ab}}, \mathbb{F}_\ell)$, $\mathrm{Hom}(\Pi_{X_2^\bullet}^{\text{ét,ab}}, \mathbb{F}_\ell)$, and

$$\mathrm{Hom}(\Pi_{X_2^\bullet}^{\text{ét,ab}}, \mathbb{F}_\ell) \xrightarrow{\sim} \mathrm{Hom}(\Pi_{X_1^\bullet}^{\text{ét,ab}}, \mathbb{F}_\ell)$$

can be mono-anabelian reconstructed from $\Pi_{X_1^\bullet}$, $\Pi_{X_2^\bullet}$, and ϕ, respectively. Moreover, Theorem 3.5 implies that $V_{\ell, X_1^\bullet}$ and $V_{\ell, X_2^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$, respectively. Thus, the proof of Proposition 4.7 implies that

$$\rho_\phi^{\text{vex}} : v(\Gamma_{X_2^\bullet}) \xrightarrow{\sim} v(\Gamma_{X_1^\bullet})$$

can be mono-anabelian reconstructed from the surjection $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$. $\square$

The mono-anabelian version of Theorem 4.10 is as follows:

Theorem 7.2. *We maintain the notations and conditions introduced in Theorem 4.10. Then the bijections of the set of closed edges (resp. open edges)*

$$\rho_\phi^{\text{cl,edge}} : e^{\text{cl}}(\Gamma_{X_2^\bullet}) \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X_1^\bullet})$$

$$(\text{resp. } \rho_\phi^{\text{op,edge}} : e^{\text{op}}(\Gamma_{X_2^\bullet}) \xrightarrow{\sim} e^{\text{op}}(\Gamma_{X_1^\bullet}))$$

can be mono-anabelian reconstructed from the surjection $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$.

Proof. We only treat the case of $\rho_\phi^{\text{cl,edge}}$. Since Lemma 3.2 implies that p can be mono-anabelian reconstructed from $\Pi_{X_1^\bullet}$ or $\Pi_{X_2^\bullet}$, we may choice a triple

$$(\ell, d, f_2^\bullet : Y_2^\bullet \to X_2^\bullet)$$

associated to $X_2^\bullet$. Then the proof of Theorem 4.10 implies that the surjection ϕ induces group-theoretically a triple

$$(\ell, d, f_1^\bullet : Y_1^\bullet \to X_1^\bullet)$$

associated to $X_1^\bullet$. Write $\Pi_{Y_1^\bullet}$ and $\Pi_{Y_2^\bullet}$ for the admissible fundamental groups corresponding to $Y_1^\bullet$ and $Y_2^\bullet$, respectively. Then Lemma 3.4 implies that $\mathrm{Hom}(\Pi_{Y_1^\bullet}^{\text{ab}}, \mathbb{F}_\ell)$, $\mathrm{Hom}(\Pi_{Y_2^\bullet}^{\text{ab}}, \mathbb{F}_\ell)$, and

$$\mathrm{Hom}(\Pi_{Y_2^\bullet}^{\text{ab}}, \mathbb{F}_\ell) \xrightarrow{\sim} \mathrm{Hom}(\Pi_{Y_1^\bullet}^{\text{ab}}, \mathbb{F}_\ell)$$

can be mono-anabelian reconstructed from $\Pi_{X_1^\bullet}$, $\Pi_{X_2^\bullet}$, and ϕ, respectively. Moreover, Theorem 3.6 implies that $U_{\ell, Y_1^\bullet}^{\text{nd}}$ and $U_{\ell, Y_2^\bullet}^{\text{nd}}$ can be mono-anabelian reconstructed from $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$, respectively. Thus, Theorem 3.6 and the proof of Theorem 4.10 imply that $\rho_\phi^{\text{cl,edge}} : e^{\text{cl}}(\Gamma_{X_2^\bullet}) \xrightarrow{\sim} e^{\text{cl}}(\Gamma_{X_1^\bullet})$ can be mono-anabelian reconstructed from the surjection $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$. $\square$

The mono-anabelian version of Theorem 5.3 is as follows:

Theorem 7.3. *We maintain the notations and conditions introduced in Theorem 5.3. Then the injection of the set of vertices*

$$\rho_\phi^{\mathrm{vex},p} : v(\Gamma_{X_2^\bullet})^{>0,p} \hookrightarrow v(\Gamma_{X_1^\bullet})^{>0,p}$$

can be mono-anabelian reconstructed from the surjection $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$.

Proof. By Lemma 3.2, the prime number p can be mono-anabelian reconstructed from $\Pi_{X_1^\bullet}$ or $\Pi_{X_2^\bullet}$. Then Lemma 3.4 implies that $\mathrm{Hom}(\Pi_{X_1^\bullet}^{\text{ét,ab}}, \mathbb{F}_p)$, $\mathrm{Hom}(\Pi_{X_2^\bullet}^{\text{ét,ab}}, \mathbb{F}_p)$, and

$$\mathrm{Hom}(\Pi_{X_2^\bullet}^{\text{ét,ab}}, \mathbb{F}_p) \xrightarrow{\sim} \mathrm{Hom}(\Pi_{X_1^\bullet}^{\text{ét,ab}}, \mathbb{F}_p)$$

can be mono-anabelian reconstructed from $\Pi_{X_1^\bullet}$, $\Pi_{X_2^\bullet}$, and ϕ, respectively. Moreover, Theorem 3.5 implies that $V_{p,X_1^\bullet}$ and $V_{p,X_2^\bullet}$ can be mono-anabelian reconstructed from $\Pi_{X_1^\bullet}$ and $\Pi_{X_2^\bullet}$, respectively. Thus, the proof of Proposition 5.3 implies that

$$\rho_\phi^{\mathrm{vex}} : v(\Gamma_{X_2^\bullet})^{>0,p} \hookrightarrow v(\Gamma_{X_1^\bullet})^{>0,p}$$

can be mono-anabelian reconstructed from the surjection $\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$. □

The mono-version of Theorem 6.4 is as follows:

Theorem 7.4. *We maintain the notations and conditions introduced in Theorem 6.4. Then commutative diagram*

$$\begin{array}{ccc}
\Gamma_{X_{H_1}^\bullet} & \xrightarrow{\theta_{\phi_{H_1}}} & \Gamma_{X_{H_2}^\bullet} \\
{\scriptstyle\gamma_{f_1^\bullet}}\Big\downarrow & & {\scriptstyle\gamma_{f_2^\bullet}}\Big\downarrow \\
\Gamma_{X_1^\bullet} & \xrightarrow{\theta_\phi} & \Gamma_{X_2^\bullet}
\end{array}$$

can be mono-anabelian reconstructed from the natural commutative diagram of profinite groups

$$\begin{array}{ccc}
H_1 & \xrightarrow{\phi_{H_1}} & H_2 \\
\Big\downarrow & & \Big\downarrow \\
\Pi_{X_1^\bullet} & \xrightarrow{\phi} & \Pi_{X_2^\bullet},
\end{array}$$

where the vertical arrows of the commutative diagram above are natural injections. Moreover, the commutative diagram is compatible with the natural actions of $\Pi_{X_2^\bullet}/H_2 = \Pi_{X_1^\bullet}/H_1$.

Proof. Theorem 3.8, Lemma 6.1, and Theorem 7.1 imply that the natural commutative natural diagrams

$$\begin{array}{ccc}
v(\Gamma_{X_{H_2}^\bullet}) & \xrightarrow{\rho_{\phi_{H_1}}^{\mathrm{vex}}} & v(\Gamma_{X_{H_1}^\bullet}) \\
{\scriptstyle\gamma_{f_2^\bullet}^{\mathrm{vex}}}\Big\downarrow & & {\scriptstyle\gamma_{f_1^\bullet}^{\mathrm{vex}}}\Big\downarrow \\
v(\Gamma_{X_2^\bullet}) & \xrightarrow{\rho_\phi^{\mathrm{vex}}} & v(\Gamma_{X_1^\bullet}),
\end{array}$$

$$
\begin{array}{ccc}
e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_{H_2}}) & \xrightarrow{\rho^{\mathrm{cl,edge}}_{\phi_{H_1}}} & v(\Gamma_{X^{\bullet}_{H_1}}) \\
\gamma^{\mathrm{cl,edge}}_{f^{\bullet}_2}\Big\downarrow & & \gamma^{\mathrm{cl,edge}}_{f^{\bullet}_1}\Big\downarrow \\
e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_2}) & \xrightarrow{\rho^{\mathrm{cl,edge}}_{\phi}} & e^{\mathrm{cl}}(\Gamma_{X^{\bullet}_1}),
\end{array}
$$

and

$$
\begin{array}{ccc}
e^{\mathrm{op}}(\Gamma_{X^{\bullet}_{H_2}}) & \xrightarrow{\rho^{\mathrm{op,edge}}_{\phi_{H_1}}} & v(\Gamma_{X^{\bullet}_{H_1}}) \\
\gamma^{\mathrm{op,edge}}_{f^{\bullet}_2}\Big\downarrow & & \gamma^{\mathrm{op,edge}}_{f^{\bullet}_1}\Big\downarrow \\
e^{\mathrm{op}}(\Gamma_{X^{\bullet}_2}) & \xrightarrow{\rho^{\mathrm{op,edge}}_{\phi}} & e^{\mathrm{op}}(\Gamma_{X^{\bullet}_1})
\end{array}
$$

can be mono-anabelian reconstructed from the natural commutative diagram of profinite groups

$$
\begin{array}{ccc}
H_1 & \xrightarrow{\phi_{H_1}} & H_2 \\
\Big\downarrow & & \Big\downarrow \\
\Pi_{X^{\bullet}_1} & \xrightarrow{\phi} & \Pi_{X^{\bullet}_2}.
\end{array}
$$

On the other hand, Theorem 3.8 implies that the natural morphisms of dual semi-graphs

$$\gamma_{f^{\bullet}_1} : \Gamma_{X^{\bullet}_{H_1}} \to \Gamma_{X^{\bullet}_1} \text{ and } \gamma_{f^{\bullet}_2} : \Gamma_{X^{\bullet}_{H_2}} \to \Gamma_{X^{\bullet}_2}$$

can be mono-anabelian reconstructed from the natural injections

$$H_1 \hookrightarrow \Pi_{X^{\bullet}_1} \text{ and } H_2 \hookrightarrow \Pi_{X^{\bullet}_2},$$

respectively. Thus, to verify the first part of the theorem, we only need to check the morphisms of the sets of vertices, the sets of closed edges, and the sets of open edges obtained above induce a commutative diagram of dual semi-graphs. Then the first part of the theorem follows from Theorem 6.4, and the "moreover" part of the theorem follows immediately from Theorem 3.8 and Theorem 6.4. $\square$

8 Condition A and Condition B

Let $k_i,\ i \in \{1,2\}$, be an algebraically closed field of characteristic $p > 0$, and let $X^{\bullet}_i, i \in \{1,2\}$, be a pointed stable curve of type (g_{X_i}, n_{X_i}) over k_i, $\Pi_{X^{\bullet}_i}$ the admissible fundamental groups of $X^{\bullet}_i$, $\Gamma_{X^{\bullet}_i}$ the dual semi-graphs of $X^{\bullet}_i$, and r_{X_i} the Betti number of $\Gamma_{X^{\bullet}_i}$. In this section, we suppose that $X^{\bullet}_1$ and $X^{\bullet}_2$ **satisfy Condition A and Condition B**, and that

$$\phi : \Pi_{X^{\bullet}_1} \twoheadrightarrow \Pi_{X^{\bullet}_2}$$

is an open continuous surjective homomorphism of the admissible fundamental groups of $X^{\bullet}_1$ and $X^{\bullet}_2$. Moreover, we denote by

$$(g, n) := (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}).$$

Let H_2 be an open normal subgroup of $\Pi^{\bullet}_{X_2}$ and $H_1 := \phi^{-1}(H_2)$ the open normal subgroup $\Pi_{X_1^\bullet}$. Write ϕ_{H_1} for the surjection $\phi|_{H_1} : H_1 \to H_2$ induced by ϕ,

$$f^{\bullet}_{X_{H_1}} : X^{\bullet}_{H_1} \to X^{\bullet}_1$$

and

$$f^{\bullet}_{X_{H_2}} : X^{\bullet}_{H_2} \to X^{\bullet}_2$$

for the Galois admissible coverings over k_1 and k_2, $(g_{X_{H_1}}, n_{X_{H_1}})$ and $(g_{X_{H_2}}, n_{X_{H_2}})$ for the types of $X^{\bullet}_{H_1}$ and $X^{\bullet}_{H_2}$, $\Gamma_{X^{\bullet}_{H_1}}$ and $\Gamma_{X^{\bullet}_{H_2}}$ for the dual semi-graphs of $X^{\bullet}_{H_1}$ and $X^{\bullet}_{H_2}$, and $r_{X_{H_1}}$ and $r_{X_{H_2}}$ for the Betti numbers of $\Gamma_{X^{\bullet}_{H_1}}$ and $\Gamma_{X^{\bullet}_{H_2}}$, respectively. The main goal of the present section is proving that there exists an open subgroup $H'_2 \subseteq H_2$ such that the pointed stable curves corresponding to $H'_1 := \phi^{-1}(H'_2) \subseteq H_1$ and H'_2 over k_1 and k_2, respectively, satisfy Condition A and Condition B.

Proposition 8.1. *Suppose that the order of $G := \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$ is prime to p. Then $X^{\bullet}_{H_1}$ and $X^{\bullet}_{H_2}$ satisfy Condition A and Condition B.*

Proof. It is easy to see that $X^{\bullet}_{H_1}$ and $X^{\bullet}_{H_2}$ satisfy Condition A. We only need to prove that $X^{\bullet}_{H_1}$ and $X^{\bullet}_{H_2}$ satisfy Condition B. To verify the proposition, by Theorem 5.4, it is sufficient to prove that there exists a characteristic subgroup $H^*_2 \subseteq H_2$ such that the pointed stable curves corresponding to $H^*_1 := \phi^{-1}(H^*_2)$ and H^*_2 over k_1 and k_2, respectively, satisfy Condition B. Moreover, without loss of generality, we may assume that the image of H_2 in $\Pi^{p'}_{X_2^\bullet}$ is a characteristic subgroup of $\Pi^{p'}_{X_2^\bullet}$. Since ϕ induces an isomorphism

$$\Pi^{p'}_{X_1^\bullet} \xrightarrow{\sim} \Pi^{p'}_{X_2^\bullet},$$

the image of H_1 in $\Pi^{p'}_{X_1^\bullet}$ is also a characteristic subgroup of $\Pi^{p'}_{X_1^\bullet}$. Thus, the lemma follows immediately from Proposition 4.7 and Theorem 6.4. $\square$

Proposition 8.2. *Suppose that $G := \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$ is a finite p-group. Then $X^{\bullet}_{H_1}$ and $X^{\bullet}_{H_2}$ satisfy Condition A and Condition B.*

Proof. It is easy to see that $X^{\bullet}_{H_1}$ and $X^{\bullet}_{H_2}$ satisfy Condition A. We only need to prove that $X^{\bullet}_{H_1}$ and $X^{\bullet}_{H_2}$ satisfy Condition B. To verify the proposition, without loss the generality, it is sufficient to treat the case where $G \cong \mathbb{Z}/p\mathbb{Z}$. For each $w_1 \in v(\Gamma_{X_1^\bullet})$ (resp. $w_2 \in v(\Gamma_{X_1^\bullet})$), write

$$L^{w_1,p}_{X_1^\bullet} := \{\alpha \in \mathrm{H}^1_{\text{ét}}(X^{\bullet}_1, \mathbb{F}_p) \mid \#v^{\mathrm{ra}}_{f^{\bullet}_{1,\alpha}} = 1 \text{ and } (f^{\bullet}_{1,\alpha})^{-1}(X_{1,w_1}) \text{ is connected}\}$$

$$(\text{resp. } L^{w_2,p}_{X_2^\bullet} := \{\alpha \in \mathrm{H}^1_{\text{ét}}(X^{\bullet}_2, \mathbb{F}_p) \mid \#v^{\mathrm{ra}}_{f^{\bullet}_{2,\alpha}} = 1 \text{ and } (f^{\bullet}_{2,\alpha})^{-1}(X_{2,w_2}) \text{ is connected}\}),$$

where $f^{\bullet}_{1,\alpha}$ (resp. $f^{\bullet}_{2,\alpha}$) denotes the Galois étale covering corresponding to α. Since $\mathrm{H}^1_{\text{ét}}(X^{\bullet}_1, \mathbb{F}_p)$ (resp. $\mathrm{H}^1_{\text{ét}}(X^{\bullet}_2, \mathbb{F}_p)$) is generated by $L^{w_1,p}_{X_1^\bullet}$ (resp. $L^{w_2,p}_{X_2^\bullet}$), the proposition follows immediately from Lemma 4.2. $\square$

Lemma 8.3. *Suppose that $G := \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$ is a finite* **simple group**. *Then there exists an open subgroup $Q_2 \subseteq \Pi_{X_2^\bullet}$ satisfies the following conditions:*

(i) let $Q_1 := \phi^{-1}(Q_2)$; then we have

$$(\#\Pi_{X_1^\bullet}/Q_1, p) = 1,$$

$$\#(\Pi_{X_1}/(Q_1 \cap H_1)) = \#(\Pi_{X_1}/Q_1)\#(\Pi_{X_1}/H_1)$$

and

$$\#(\Pi_{X_2}/(Q_2 \cap H_2)) = \#(\Pi_{X_2}/Q_2)\#(\Pi_{X_2}/H_2),$$

where $\#(-)$ denotes the cardinality of the quotient set $(-)$;

(ii) write $X_{Q_1}^\bullet$ and $X_{Q_2}^\bullet$ for the pointed stable curves over k_1 and k_2 corresponding to Q_1 and Q_2, respectively; then $X_{Q_1}^\bullet$ and $X_{Q_2}^\bullet$ satisfy Condition A and Condition B;

*(iii) let $P_2 := Q_2 \cap H_2$ and $X_{P_2}^\bullet$ the pointed stable curve over k_2 corresponding to P_2; then the Galois admissible covering $h_{P_2}^\bullet : X_{P_2}^\bullet \to X_{Q_2}^\bullet$ induced by $P_2 \subseteq Q_2$ is a connected G-**étale** covering over k_2;*

*(iv) let $P_1 := Q_1 \cap H_1 = \phi^{-1}(P_2)$ and $X_{P_1}^\bullet$ the pointed stable curve over k_1 corresponding to P_1; then the Galois admissible covering $h_{P_1}^\bullet : X_{P_1}^\bullet \to X_{Q_1}^\bullet$ induced by $P_1 \subseteq Q_1$ is a connected G-**étale** covering over k_1;*

(v) write $\Gamma_{X_{Q_1}^\bullet}$ and $\Gamma_{X_{Q_2}^\bullet}$ for the dual semi-graphs of $X_{Q_1}^\bullet$ and $X_{Q_2}^\bullet$, respectively; then we have $\#v(\Gamma_{X_{Q_1}^\bullet}) = \#v(\Gamma_{X_1^\bullet})$ and $\#v(\Gamma_{X_{Q_2}^\bullet}) = \#v(\Gamma_{X_2^\bullet})$.

Proof. Let G_p be a Sylow-p-subgroup of G and q the index of G_p. Let d be a prime number distinct from p such that $(d, q) = 1$. Let $f_2^\bullet : Y_2^\bullet \to X_2^\bullet$ be a Galois étale covering of degree d over k_2 such that $\#v_{f_2^\bullet}^{\mathrm{sp}} = 0$ and $f_1^\bullet : Y_1^\bullet \to X_1^\bullet$ the Galois admissible covering of degree d over k_1 induced by ϕ and $f_2^\bullet$. Then Lemma 4.2 implies that $f_1^\bullet$ is étale and $\#v_{f_1^\bullet}^{\mathrm{sp}} = 0$. Note that $Y_1^\bullet$ and $Y_2^\bullet$ satisfy Condition A and Condition B.

Next, we consider a $\mathbb{Z}/q\mathbb{Z}$-Galois admissible covering

$$f_2^{*,\bullet} : Y_2^{*,\bullet} \to Y_2^\bullet$$

over k_2 such that $f_2^{*,\bullet}$ is totally ramified over all the nodes and all the marked points of $Y_2^\bullet$. Write

$$f_1^{*,\bullet} : Y_1^{*,\bullet} \to Y_1^\bullet$$

for the $\mathbb{Z}/q\mathbb{Z}$-Galois admissible covering over k_1 induced by $f_2^{*,\bullet}$, $\Gamma_{Y_2^{*,\bullet}}$ and $\Gamma_{Y_1^{*,\bullet}}$ for the dual semi-graphs of $Y_2^{*,\bullet}$ and $Y_1^{*,\bullet}$, respectively. Since $f_2^{*,\bullet}$ induces an isomorphism between $\Gamma_{Y_2^{*,\bullet}}$ and the dual semi-graph $\Gamma_{Y_2^\bullet}$ of $Y_2^\bullet$, Lemma 1.7 implies that $f_1^{*,\bullet}$ induces an isomorphism between $\Gamma_{Y_1^{*,\bullet}}$ and the dual semi-graph $\Gamma_{Y_1^\bullet}$ of $Y_1^\bullet$. Thus, $f_1^{*,\bullet}$ is totally ramified over all the nodes and all the marked points of $Y_1^\bullet$.

We take $Q_2 := \Pi_{Y_2^{*,\bullet}} \subseteq \Pi_{X_2}$, where $\Pi_{Y_2^{*,\bullet}}$ denotes the admissible fundamental group of $Y_2^{*,\bullet}$. Then $Q_1 := \phi^{-1}(Q_2)$ is the admissible fundamental group $\Pi_{Y_1^{*,\bullet}}$ of $Y_1^{*,\bullet}$. Thus, we obtain that

$$X_{P_1}^\bullet = X_{H_1}^\bullet \times_{X_1^\bullet} Y_1^{*,\bullet}$$

and

$$X_{P_2}^\bullet = X_{H_2}^\bullet \times_{X_2^\bullet} Y_2^{*,\bullet}.$$

If G is a commutative simple group, it is easy to see that we may choose suitable $\mathbb{Z}/q\mathbb{Z}$-Galois admissible coverings $f_1^{*,\bullet}$ and $f_2^{*,\bullet}$ such that $X_{P_1}^\bullet$ and $X_{P_2}^\bullet$ are connected. Moreover, if G is a non-commutative simple group, Since G is a non-commutative simple group, the constructions of $X_{P_1}^\bullet$ and $X_{P_2}^\bullet$ imply that $X_{P_1}^\bullet$ and $X_{P_2}^\bullet$ are connected. Then the condition (i) holds. This means that $X_{P_1}^\bullet$ and $X_{P_2}^\bullet$ are connected. By the constructions of $Y_1^{*,\bullet}$ and $Y_2^{*,\bullet}$, $Y_1^{*,\bullet}$ and $Y_2^{*,\bullet}$ satisfy Condition A and Condition B. Then condition (ii) holds. Moreover, Abhyankar's lemma implies that (iii) and (iv) hold. The constructions of $X_{Q_1^\bullet}$ and $X_{Q_2^\bullet}$ implies that (v) holds. This completes the proof of the lemma. □

Next, we prove a generalized version of Proposition 4.5.

Proposition 8.4. *Let $G := \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$. Suppose that $f_{H_2}^\bullet$ is an* **étale** *covering over k_2, and that $\#v^{\rm sp}_{f^\bullet_{H_2}} = \#v(\Gamma_{X_2^\bullet})$. Then $f_{H_1}^\bullet$ is an* **étale** *covering over k_1, and that $\#v^{\rm sp}_{f^\bullet_{H_1}} = \#v(\Gamma_{X_1^\bullet})$. In particular, $X_{H_1}^\bullet$ and $X_{H_2}^\bullet$ satisfy Condition A and Condition B; moreover, ϕ induces an isomorphism*

$$\Pi^{\rm top}_{X_1^\bullet} \overset{\sim}{\to} \Pi^{\rm top}_{X_2^\bullet}.$$

Proof. First, let us prove that, if $f_{H_1}^\bullet$ is an étale covering over k_1, then $\#v^{\rm sp}_{f^\bullet_{H_1}} = \#v(\Gamma_{X_1^\bullet})$. Write $\gamma_{f^\bullet_{H_1}} : \Gamma_{X^\bullet_{H_1}} \to \Gamma_{X_2^\bullet}$ for the morphism of dual semi-graphs induced by $f^\bullet_{H_1}$. Then we have

$$r_{X_{H_2}} = \#G\#e^{\rm cl}(\Gamma_{X_2^\bullet}) - \#G\#v(\Gamma_{X_2^\bullet}) + 1$$

and

$$r_{X_{H_1}} = \#G\#e^{\rm cl}(\Gamma_{X_1^\bullet}) - \#G\#v^{\rm sp}_{f^\bullet_{H_1}} - (\sum_{v\in v(\Gamma_{X_1^\bullet})\setminus v^{\rm sp}_{f^\bullet_{X_{H_1}}}} (\#G/\#G_v)) + 1,$$

where $\#G_v$ denotes the order of the stabilizer of a vertices of $\gamma^{-1}_{f^\bullet_{H_1}}(v)$ under the natural action of G on $\Gamma_{X^\bullet_{H_1}}$ (note that $\#G_v$ does not depend on the choices of $v \in \gamma^{-1}_{f^\bullet_{H_1}}(v)$). Since $f^\bullet_{H_1}$ and $f^\bullet_{H_2}$ are étale, we have $g_{X_{H_1}} = g_{X_{H_2}}$. Then Theorem 1.5 implies that

$$r_{X_{H_2}} = r_{X_{H_1}}.$$

We obtain

$$\#G\#(v(\Gamma_{X_1^\bullet}) \setminus v^{\rm sp}_{f^\bullet_{X_{H_1}}}) = \sum_{v\in v(\Gamma_{X_1^\bullet})\setminus v_{f^\bullet_{X_{H_1}}}} (\#G/\#G_v).$$

Thus, we obtain that

$$\#G_v = 1.$$

This means that $\#v^{\rm sp}_{f^\bullet_{H_1}} = \#v(\Gamma_{X_1^\bullet})$.

We only need to prove that $f^\bullet_{H_1}$ is étale. By applying Proposition 8.1 and Proposition 8.2, we may assume that G is a non-commutative simple group such that $(\#G, p) \neq 1$. Then Lemma 8.3 implies that there exists an open subgroup $Q_2 \subseteq \Pi_{X_2^\bullet}$ satisfying the conditions (i) (ii) (iii) (iv) (v) in Lemma 8.3. We maintain the notations introduced in Lemma 8.3. Then $h^\bullet_{P_2} : X^\bullet_{P_2} \to X^\bullet_{H_2}$ and $h^\bullet_{P_1} : X^\bullet_{P_1} \to X^\bullet_{H_1}$ are étale, and $\#v^{\rm sp}_{h^\bullet_{P_2}} =$

$\#v(\Gamma_{X^\bullet_{H_2}})$. Moreover, the proof above implies that $\#v^{\mathrm{sp}}_{h^\bullet_{P_1}} = \#v(\Gamma_{X^\bullet_{H_1}})$. Thus, $X^\bullet_{P_1}$ and $X^\bullet_{P_2}$ satisfy Condition A and Condition B. Write ϕ_{P_1} for the surjection $\phi|_{P_1} : P_1 \twoheadrightarrow P_2$ induced by ϕ and ϕ_Y for the surjection between the admissible fundamental groups of $Y^\bullet_1$ and $Y^\bullet_2$ induced by ϕ. Moreover, Theorem 6.4 implies the following commutative diagram

$$\begin{array}{ccc} \Gamma_{X^\bullet_{P_1}} & \xrightarrow{\theta_{\phi_{P_1}}} & \Gamma_{X^\bullet_{P_2}} \\ \downarrow & & \downarrow \\ \Gamma_{Y^\bullet_1} & \xrightarrow{\theta_{\phi_Y}} & \Gamma_{Y^\bullet_2}, \end{array}$$

and the commutative diagram above compatible the natural actions of

$$G \times \mathbb{Z}/q\mathbb{Z},$$

where q denotes the index of a Sylow-p-subgroup of G. On the other hand, we note that each ramification groups associated to the marked points of $X^\bullet_{P_2}$ (i.e., the stabilizers of each open edge of $\Gamma_{X^\bullet_{P_2}}$ under the action of $G \times \mathbb{Z}/q\mathbb{Z}$) are contained in $\mathbb{Z}/q\mathbb{Z}$. This implies that each ramification groups associated to the marked points of $X^\bullet_{P_1}$ (i.e., the stabilizers of each open edge of $\Gamma_{X^\bullet_{P_1}}$ under the action of $G \times \mathbb{Z}/q\mathbb{Z}$) are contained in $\mathbb{Z}/q\mathbb{Z}$. This means that

$$Y^\bullet_1 \times_{X^\bullet_1} X^\bullet_{H_1} \to Y^\bullet_1$$

is étale. Since $Y^\bullet_1 \to X^\bullet_1$ is étale, we have $f^\bullet_{H_1} : X^\bullet_{H_1} \to X^\bullet_1$ is étale. This completes the proof of the proposition. □

Proposition 8.5. *Let $G := \Pi_{X^\bullet_1}/H_1 = \Pi_{X^\bullet_2}/H_2$. Suppose that $f^\bullet_{H_1}$ and $f^\bullet_{H_2}$ are* **étale***, and that $\#v^{\mathrm{sp}}_{f^\bullet_{H_2}} = \#v(\Gamma_{X^\bullet_2}) - 1$ and $\#v^{\mathrm{ra}}_{f^\bullet_{H_2}} = 1$. Then we have*

$$\#v^{\mathrm{sp}}_{f^\bullet_{H_1}} = \#v(\Gamma_{X^\bullet_1}) - 1 \textit{ and } \#v^{\mathrm{ra}}_{f^\bullet_{H_1}} = 1.$$

In particular, $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A and Condition B.

Proof. Write $\gamma_{f^\bullet_{H_1}} : \Gamma_{X^\bullet_{H_1}} \to \Gamma_{X^\bullet_1}$ for the morphism of dual semi-graphs induced by $f^\bullet_{H_1}$. We have

$$r_{X_{H_2}} = \#G\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_2}) - \#G(\#v(\Gamma_{X^\bullet_2}) - 1) - 1 + 1$$

and

$$r_{X_{H_1}} = \#G\#e^{\mathrm{cl}}(\Gamma_{X^\bullet_1}) - (\sum_{v \in v(\Gamma_{X^\bullet_1})} (\#G/\#G_v)) + 1,$$

where $\#G_v$ denotes the order of the stabilizer of a vertices of $\gamma^{-1}_{f^\bullet_{H_1}}(v)$ under the natural action of G on $\Gamma_{X^\bullet_{H_1}}$ (note that $\#G_v$ does not depend on the choices of $v \in \gamma^{-1}_{f^\bullet_{H_1}}(v)$). Since $f^\bullet_{H_1}$ and $f^\bullet_{H_2}$ are étale, we have $g_{X_{H_1}} = g_{X_{H_2}}$. Moreover, Theorem 1.5 implies that

$$r_{X_{H_1}} = r_{X_{H_2}}.$$

Then we obtain
$$\#G\#v(\Gamma_{X_1^\bullet}) - (\sum_{v\in v(\Gamma_{X_1^\bullet})}(\#G/\#G_v)) = \#G-1.$$
This implies that
$$(\sum_{v\in v(\Gamma_{X_1^\bullet})}(\frac{\#G_v-1}{\#G_v}))(\frac{\#G}{\#G-1}) = 1.$$

Since $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition A and Condition B, Theorem 5.4 implies ϕ induces a morphism of dual semi-graphs
$$\theta_\phi : \Gamma_{X_1^\bullet} \to \Gamma_{X_2^\bullet}$$
of $X_1^\bullet$ and $X_2^\bullet$. Let ℓ be any prime number such that $(\ell, \#G)=1$. Moreover, let $v^{\mathrm{ra}}_{f^\bullet_{H_2}} := \{v_2\}$, and let
$$f_2^\bullet : Y_2^\bullet \to X_2^\bullet$$
be a Galois étale covering over k_2 of degree ℓ such that $v^{\mathrm{ra}}_{f_2^\bullet} := \{v_2\}$, and, by Proposition 4.4,
$$f_1^\bullet : Y_1^\bullet \to X_1^\bullet$$
the Galois étale covering over k_1 of degree ℓ induced by ϕ such that
$$v^{\mathrm{ra}}_{f_1^\bullet} = \{v_1 := \theta_\phi^{-1}(v_2)\}.$$
Consider the Galois étale covering
$$g_2^\bullet : Z_2^\bullet := X^\bullet_{H_2} \times_{X_2^\bullet} Y_2^\bullet \to X_2^\bullet$$
over k_2 whose Galois group is isomorphic to $H := G \times \mathbb{Z}/\ell\mathbb{Z}$. Then ϕ induces a Galois étale covering
$$g_1^\bullet : Z_1^\bullet := X^\bullet_{H_1} \times_{X_1^\bullet} Y_1^\bullet \to X_1^\bullet$$
over k_1 whose Galois group is isomorphic to H. Note that, by the construction above, we have $v^{\mathrm{ra}}_{g_2^\bullet} = \{v_2\}$. Write $\gamma_{g_1^\bullet} : \Gamma_{Z_1^\bullet} \to \Gamma_{X_1^\bullet}$ for the morphism of dual semi-graphs induced by $g_1^\bullet$. Moreover, for each $v \in v(\Gamma_{X_1^\bullet})$, $\#H_v$ denotes the order of the stabilizer of a vertices of $\gamma^{-1}_{g_1^\bullet}(v)$ under the natural action of H on $\Gamma_{Z_1^\bullet}$ (note that $\#G_v$ does not depend on the choices of $v \in \gamma^{-1}_{g_1^\bullet}(v)$). Then, for each $v \in v(\Gamma_{X_1^\bullet})$, we have
$$\#H_v = \#G_v$$
if $v \neq v_1$ and
$$\#H_{v_1} = \ell\#G_{v_1}.$$
On the other hand, we have
$$r_{Z_1} = \#G\#e^{\mathrm{cl}}(\Gamma_{X_1^\bullet}) - (\sum_{v\in v(\Gamma_{X_1^\bullet})}(\#H/\#H_v)) + 1$$

and

$$r_{Z_2} = \#H\#e^{\mathrm{cl}}(\Gamma_{X_2^\bullet}) - \#H(\#v(\Gamma_{X_2^\bullet}) - 1) - 1 + 1,$$

where r_{Z_1} denotes the Betti number of $\Gamma_{Z_1^\bullet}$, and r_{Z_2} denotes the Betti number of $\Gamma_{Z_2^\bullet}$. Since $g_1^\bullet$ and $g_2^\bullet$ are étale, we have $g_{Z_1} = g_{Z_2}$. Moreover, Theorem 1.5 implies that

$$r_{Z_1} = r_{Z_2}.$$

Then we obtain

$$\#H\#v(\Gamma_{X_1^\bullet}) - (\sum_{v\in v(\Gamma_{X_1^\bullet})} (\#H/\#H_v)) = \#H - 1.$$

This implies that

$$(\sum_{v\in v(\Gamma_{X_1^\bullet})} (\frac{\#H_v - 1}{\#H_v}))(\frac{\#H}{\#H-1}) = (\sum_{v\in v(\Gamma_{X_1^\bullet})\setminus\{v_1\}} (\frac{\#G_v - 1}{\#G_v}) + \frac{\ell\#G_{v_1} - 1}{\ell\#G_{v_1}})(\frac{\ell\#G}{\ell\#G-1}) = 1$$

holds for any ℓ if $(\#G, \ell) = 1$. This implies that

$$\#G_v = 1$$

if $v \neq v_1$ and

$$\#G_{v_1} = \#G.$$

This means that

$$\#v^{\mathrm{ra}}_{f^\bullet_{H_1}} = 1 \text{ and } \#v^{\mathrm{sp}}_{f^\bullet_{H_1}} = \#v(\Gamma_{X_1^\bullet}) - 1.$$

This completes the proof of the proposition. $\square$

Now, let us prove the main theorem of the present section.

Theorem 8.6. *There exists open subgroups $H_2' \subseteq H_2$ and $H_1' := \phi^{-1}(H_2') \subseteq H_1$ such that the pointed stable curves $X^\bullet_{H_1'}$ and $X^\bullet_{H_2'}$ over k_1 and k_2 corresponding to H_1' and H_2', respectively, satisfy Condition A and Condition B.*

Proof. It is easy to see that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition A. We only need to prove that $X^\bullet_{H_1}$ and $X^\bullet_{H_2}$ satisfy Condition B. To verify the theorem, without loss of generality, we may assume that $G := \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$ is a **simple group**.

If G is commutative, then the theorem follows from Proposition 8.1 and Proposition 8.2. Then we may assume that $G := \Pi_{X_1^\bullet}/H_1 = \Pi_{X_2^\bullet}/H_2$ is a finite **non-commutative simple group**. Furthermore, by applying Lemma 8.3 and Proposition 8.4, we may assume that $f^\bullet_{H_1}$ and $f^\bullet_{H_2}$ are **étale**, and that all the irreducible components of $X_1^\bullet$ and $X_2^\bullet$ are **smooth** over k_1 and k_2, respectively.

For each $v_2 \in \Gamma_{X_2^\bullet}$, write X_{2,v_2} for the irreducible component of the underlying curve X_2 corresponding to v_2. We define a smooth pointed stable curve

$$X^\bullet_{2,v_2} := (X_{2,v_2}, D_{X_{2,v_2}} := (D_{X_2} \cup \mathrm{Nod}(X_2^\bullet)) \cap X_{2,v_2})$$

over k_2, where $\mathrm{Nod}(X_2^\bullet)$ denotes the set of nodes of $X_2^\bullet$. Write $\Pi_{X_{2,v_2}^\bullet}$ for the admissible fundamental group of $X_{2,v_2}^\bullet$. Then we have an outer injective homomorphism

$$\Pi_{X_{2,v_2}^\bullet} \hookrightarrow \Pi_{X_2^\bullet},$$

and, moreover, we fix an injection. Let $H_{2,v_2} := \Pi_{X_{2,v_2}^\bullet} \cap H_2$. It is easy to construct a Galois étale covering

$$f_{2,v_2}^\bullet : X_{P_{2,v_2}}^\bullet \to X_2^\bullet$$

over k_2 which corresponds to an open normal subgroup $P_{2,v_2} \subseteq \Pi_{X_2^\bullet}$ satisfying the following conditions:

(i) $P_{2,v_2} \cap \Pi_{X_2^\bullet,v_2} = H_{2,v_2}$;

(ii) $v_{f_{2,v_2}^\bullet}^{\mathrm{ra}} = \{v_2\}$ and $\# v_{f_{2,v_2}^\bullet}^{\mathrm{sp}} = \# v(\Gamma_{X_2^\bullet}) - 1$.

Since $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition A and Condition B, Theorem 6.4 implies ϕ induces a morphism of dual semi-graphs

$$\theta_\phi : \Gamma_{X_1^\bullet} \to \Gamma_{X_2^\bullet}$$

of $X_1^\bullet$ and $X_2^\bullet$. Write v_1 for $\theta_\phi^{-1}(v_2)$. Then $f_{2,v_2}^\bullet$ and ϕ induces a Galois admissible covering

$$f_{1,v_1}^\bullet : X_{P_{1,v_1}}^\bullet \to X_1^\bullet$$

over k_1 which corresponds to an open normal subgroup $P_{1,v_1} := \phi^{-1}(P_{2,v_2}) \subseteq \Pi_{X_1^\bullet}$.

First, we suppose that $F := \Pi_{X_1^\bullet}/P_{1,v_1} = \Pi_{X_2^\bullet}/P_{2,v_2}$ is a simple group. By applying Lemma 8.3, there exists an open subgroup $Q_2^* \subseteq \Pi_{X_1^\bullet}$ such that the admissible coverings

$$g_{2,v_2}^{*,\bullet} : X_{P_{2,v_2}\cap Q_2^*}^\bullet \to X_{Q_2^*}^\bullet$$

and

$$g_{1,v_1}^{*,\bullet} : X_{P_{1,v_1}\cap Q_1^*}^\bullet \to X_{Q_1^*}^\bullet$$

are étale, where $Q_1^* := \phi^{-1}(Q_2^*)$. Moreover, by the construction of Q_2^*, we have

(i) $P_{2,v_2} \cap Q_2^* \cap \Pi_{X_2^\bullet,v_2} = H_{2,v_2} \cap Q_2^*$;

(ii) write $\gamma_{Q_2^*} : \Gamma_{X_{Q_2^*}^\bullet} \to \Gamma_{X_2^\bullet}$ for the morphism of dual semi-graphs of $X_{Q_2^*}^\bullet$ and $X_2^\bullet$ induced by the morphism $X_{Q_2^*}^\bullet \to X_2^\bullet$; then we have

$$v_{g_{2,v_2}^{*,\bullet}}^{\mathrm{ra}} = \gamma_{Q_2^*}^{-1}(v_2) \text{ and } \# v_{g_{2,v_2}^{*,\bullet}}^{\mathrm{sp}} = \# v(\Gamma_{X_{Q_2^*}^\bullet}) - 1;$$

note that $\#\gamma_{Q_2^*}^{-1}(v_2) = 1$.

If $F := \Pi_{X_1^\bullet}/P_{1,v_1} = \Pi_{X_2^\bullet}/P_{2,v_2}$ is not a simple group, then we have a sequence of subgroups

$$\{1\} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n-1} \subseteq F_n = F$$

such that F_i/F_{i-1}, $i \in \{1, \ldots, n\}$, is a simple group. For each $i \in \{0, \ldots, n\}$, write $P_{2,v_2,i} \subseteq \Pi_{X_2^\bullet}$ for the inverse images of F_i of the natural morphism $\Pi_{X_2^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}/P_{2,v_2}$ and

$P_{1,v_1,i} := \phi^{-1}(P_{2,v_2,i})$. By applying similar proof of the case where F is a simple group to $P_{1,v_1,i}$ and $P_{2,v_2,i}$, $i \in \{1, \ldots, n\}$, we may obtain that there exists an open subgroup $Q_2 \subseteq \Pi_{X_1^\bullet}$ such that the admissible coverings

$$g^\bullet_{2,v_2} : X^\bullet_{P_{2,v_2} \cap Q_2} \to X^\bullet_{Q_2}$$

and

$$g^\bullet_{1,v_1} : X^\bullet_{P_{1,v_1} \cap Q_1} \to X^\bullet_{Q_1}$$

are étale, where $Q_1 := \phi^{-1}(Q_2)$. Moreover, by the construction of Q_2, we have

(i) $P_{2,v_2} \cap Q_2 \cap \Pi_{X_2^\bullet, v_2} = H_{2,v_2} \cap Q_2$;

(ii) write $\gamma_{Q_2} : \Gamma_{X^\bullet_{Q_2}} \to \Gamma_{X_2^\bullet}$ for the morphism of dual semi-graphs of $X^\bullet_{Q_2}$ and $X_2^\bullet$ induced by $g^\bullet_{2,v_2}$; then we have

$$v^{\mathrm{ra}}_{g^\bullet_{2,v_2}} = \gamma^{-1}_{Q_2}(v_2) \text{ and } \#v^{\mathrm{sp}}_{g^\bullet_{2,v_2}} = \#v(\Gamma_{X^\bullet_{Q_2}}) - 1;$$

note that $\#\gamma^{-1}_{Q_2}(v_2) = 1$.

Then by replacing

$$X_1^\bullet,\ X_2^\bullet,\ X^\bullet_{H_1},\ X^\bullet_{H_2},\ X^\bullet_{P_{1,v_1}},\ \text{and } X^\bullet_{P_{2,v_2}}$$

by

$$X^\bullet_{Q_1},\ X^\bullet_{Q_2},\ X^\bullet_{Q_1 \cap H_1},\ X^\bullet_{Q_2 \cap H_2},\ X^\bullet_{P_{1,v_1} \cap Q_1},\ \text{and } X^\bullet_{P_{2,v_2} \cap Q_2},$$

respectively, to verify the theorem, we may assume that

$$f^\bullet_{1,v_1} : X^\bullet_{P_{1,v_1}} \to X_1^\bullet$$

is also a Galois étale covering over k_1. Moreover, Proposition 8.5 implies that $v^{\mathrm{ra}}_{f^\bullet_{1,v_1}} = \{v_1\}$ and $\#v^{\mathrm{sp}}_{f^\bullet_{1,v_1}} = \#v(\Gamma_{X_1^\bullet}) - 1$. Note that $X^\bullet_{P_{1,v_1}}$ and $X^\bullet_{P_{2,v_2}}$ satisfy Condition A and Condition B.

We consider the fiber product

$$Z_1^\bullet := \bigtimes_{X_1^\bullet, v_1 \in v(\Gamma_{X_1^\bullet})} X^\bullet_{P_{1,v_1}}$$

of curves $X_{P_{1,v_1}}, v_1 \in v(\Gamma_{X_1^\bullet})$, over $X_1^\bullet$ and the fiber product

$$Z_2^\bullet := \bigtimes_{X_2^\bullet, v_2 \in v(\Gamma_{X_2^\bullet})} X^\bullet_{P_{2,v_2}}$$

of curves $X_{P_{2,v_2}}, v_2 \in v(\Gamma_{X_2^\bullet})$, over $X_2^\bullet$. Note that $Z_1^\bullet$ and $Z_2^\bullet$ are connected which corresponding to the open normal subgroups

$$\bigcap_{v_1 \in v(\Gamma_{X_1^\bullet})} P_{1,v_1} \subseteq \Pi_{X_1^\bullet} \text{ and } \bigcap_{v_2 \in v(\Gamma_{X_2^\bullet})} P_{2,v_2} \subseteq \Pi_{X_2^\bullet},$$

respectively. Moreover, we have that $Z_1^\bullet$ and $Z_2^\bullet$ satisfy Condition A and Condition B. Write $\Gamma_{Z_1^\bullet}$ and $\Gamma_{Z_2^\bullet}$ for the dual semi-graphs of $Z_1^\bullet$ and $Z_2^\bullet$, respectively.

Next, we consider

$$h_1^\bullet : W_1^\bullet \hookrightarrow Z_1^\bullet \times_{X_1^\bullet} X_{H_1}^\bullet \to Z_1^\bullet$$

and

$$h_2^\bullet : W_2^\bullet \hookrightarrow Z_2^\bullet \times_{X_2^\bullet} X_{H_2}^\bullet \to Z_2^\bullet,$$

where $W_1^\bullet$ and $W_2^\bullet$ denotes the pointed stable curves over k_1 and k_2 corresponding to the open normal subgroups

$$H_1 \cap (\bigcap_{v_1 \in v(\Gamma_{X_1^\bullet})} P_{1,v_1}) \subseteq H_1 \subseteq \Pi_{X_1^\bullet} \text{ and } H_2 \cap (\bigcap_{v_2 \in v(\Gamma_{X_2^\bullet})} P_{2,v_2}) \subseteq H_2 \subseteq \Pi_{X_2^\bullet}.$$

By the construction of $Z_2^\bullet$, we obtain that

$$\#v_{h_2^\bullet}^{\mathrm{sp}} = \#v(\Gamma_{Z_2^\bullet}).$$

Then Proposition 8.4 implies that

$$\#v_{h_1^\bullet}^{\mathrm{sp}} = \#v(\Gamma_{Z_1^\bullet}).$$

Thus, $W_1^\bullet$ and $W_2^\bullet$ satisfy Condition A and Condition B. Then we may take

$$H_1' := H_1 \cap (\bigcap_{v_1 \in v(\Gamma_{X_1^\bullet})} P_{1,v_1}) \text{ and } H_2' := H_2 \cap (\bigcap_{v_2 \in v(\Gamma_{X_2^\bullet})} P_{2,v_2}).$$

This completes the proof of the theorem. □

9 Mono-anabelian versions of combinatorial Grothendieck conjecture

In this section, we prove mono-anabelian versions of combinatorial Grothendieck conjecture for semi-graphs of anabelioids of PSC-type associated to pointed stable curves over algebraically closed fields of characteristic $p > 0$. Let k be an algebraically closed field of characteristic $p > 0$, and let

$$X^\bullet$$

be a pointed stable curves of type (g_X, n_X) over k, $\Pi_{X^\bullet}$ the admissible fundamental group of $X^\bullet$, and $\Gamma_{X^\bullet}$ the dual semi-graph of $X^\bullet$. We write

$$\mathcal{G}_{X^\bullet}$$

for the semi-graph of anabelioids of PSC-type associated to $X^\bullet$ and write $\Gamma_{\mathcal{G}_{X^\bullet}}$ for the underlying semi-graph of $\mathcal{G}_{X^\bullet}$ (cf. [M4], [M5], [M6] for the general theories of anabelioids, semi-graphs of anabelioids, and semi-graphs of anabelioids of PSC-type, respectively).

Then we have $\Gamma_{\mathcal{G}_{X^\bullet}} = \Gamma_{X^\bullet}$. We choose a base point $\beta_{\mathcal{G}_{X^\bullet}}$ of $\mathcal{G}_{X^\bullet}$. Then we obtain the fundamental group

$$\Pi_{\mathcal{G}_{X^\bullet}} := \pi_1(\mathcal{G}_{X^\bullet}, \beta_{\mathcal{G}_{X^\bullet}})$$

of $\mathcal{G}_{X^\bullet}$. By the definition of semi-graph of anabelioids of PSC-type, we have

$$\Pi_{\mathcal{G}_{X^\bullet}} \cong \Pi_{X^\bullet}.$$

Moreover, for suitable choices of base points of $X^\bullet$ and $\beta_{\mathcal{G}_{X^\bullet}}$, we may assume that $\Pi_{\mathcal{G}_{X^\bullet}} = \Pi_{X^\bullet}$. We have the following theorem, which is the first main theorem of the present paper.

Theorem 9.1. *The semi-graphs of anabelioids of PSC-type $\mathcal{G}_{X^\bullet}$ associated to $X^\bullet$ can be mono-anabelian reconstructed from its fundamental group $\Pi_{\mathcal{G}_{X^\bullet}}$.*

Proof. Note that since $\Pi_{\mathcal{G}_{X^\bullet}}$ is topologically finitely generated, there exists a set of open normal subgroups $\{H_i\}_{i\in\mathbb{N}}$ (e.g. characteristic subgroups) of $\Pi_{\mathcal{G}_{X^\bullet}}$ such that

(i) $H_i \supseteq H_{i+1}$, for each $i \in \mathbb{N}$;

(ii) $\varprojlim_{i\in\mathbb{N}} \Pi_{\mathcal{G}_{X^\bullet}}/H_i = \Pi_{\mathcal{G}_{X^\bullet}}$.

Then to verify that $\mathcal{G}_{X^\bullet}$ can be mono-anabelian reconstructed from its fundamental group $\Pi_{\mathcal{G}_{X^\bullet}}$, it is sufficient to prove that

(i) for each i, the dual semi-graph $\Gamma_{X^\bullet_{H_i}}$ of the curve $X^\bullet_{H_i}$ corresponding to H_i can be mono-anabelian reconstructed from H_i;

(ii) for each i, the natural map of dual semi-graphs

$$\Gamma_{X^\bullet_{H_i}} \to \Gamma_{X^\bullet} = \Gamma_{\mathcal{G}_{X^\bullet}}$$

can be mono-anabelian reconstructed from the natural injection $H_i \hookrightarrow \Pi_{\mathcal{G}_{X^\bullet}}$, and the natural action of $\Pi_{\mathcal{G}_{X^\bullet}}/H_i$ on $\Gamma_{X^\bullet_{H_i}}$ induced by the natural action of $\Pi_{\mathcal{G}_{X^\bullet}}/H_i$ on $X^\bullet_{H_i}$ can be mono-anabelian reconstructed from the natural injection $H_i \hookrightarrow \Pi_{\mathcal{G}_{X^\bullet}}$.

By [Y1, Lemma 5.3], we may assume that $H_0 = \Pi_{\mathcal{G}_{X^\bullet}}$, and that the curve $X^\bullet_{H_1}$ corresponding to H_1 satisfies Condition A. Then Theorem 3.7 implies that, for each $i \leq 1$, $\Gamma_{X^\bullet_{H_i}}$ admits a natural action of $\Pi_{\mathcal{G}_{X^\bullet}}/H_i$. For each $i, j \leq 1$ such that $j > i$, by applying Theorem 3.7 again, we may identify naturally $\Gamma_{X^\bullet_{H_j}}/(H_i/H_j)$ with $\Gamma_{X^\bullet_{H_i}}$. Moreover, we may identify naturally $\Gamma_{X^\bullet_{H_j}}/H_j$ with $\Gamma_{X^\bullet_{H_i}}/H_i$. Thus, we can define

$$\Gamma_{\mathcal{G}_{X^\bullet}} = \Gamma_{X^\bullet} := \Gamma_{X^\bullet_{H_1}}/H_1.$$

Then the theorem follows from Theorem 3.7. □

Remark 9.1.1. Let $v \in v(\Gamma_{X^\bullet})$, X_v the irreducible component of X corresponding to v, and $\mathrm{nl}_v : \widetilde{X}_v \to X_v$ the normalization morphism of X_v. We define

$$\widetilde{X}^\bullet_v := (\widetilde{X}_v, \mathrm{nl}_v^{-1}((\mathrm{Node}(X) \cup D_X) \cap X_v))$$

to be a smooth pointed stable curve of type (g_v, n_v) over k, where $\mathrm{Node}(X)$ denotes the sets of nodes of X. For suitable of choices of base point of $\widetilde{X}_v^\bullet$, we obtain the admissible fundamental group

$$\Pi_{\widetilde{X}_v^\bullet}$$

of $\widetilde{X}_v^\bullet$. Moreover, we have an outer injective homomorphism of fundamental groups

$$\Pi_{\widetilde{X}_v^\bullet} \hookrightarrow \Pi_{X^\bullet}.$$

Then Theorem 9.1 means that, there exists a group-theoretic algorithm whose input datum is $\Pi_{X^\bullet}$, and whose output data are as follows:

- (g_X, n_X) and $\Gamma_{X^\bullet}$;
- the conjugacy class of the inertia group of every marked point of $X^\bullet$ in $\Pi_{X^\bullet}$;
- the conjugacy class of the inertia group of every node of $X^\bullet$ in $\Pi_{X^\bullet}$;
- (g_v, n_v) and the conjugacy class of $\Pi_{\widetilde{X}_v^\bullet}$ for each $v \in v(\Gamma_{X^\bullet})$.

Remark 9.1.2. If $X^\bullet$ are smooth over k, then Theorem 9.1 has been obtained by Tamagawa (cf. [T2, Theorem 0.5 and Theorem 5.2]).

Remark 9.1.3. This theorem is a mono-anabelian version of [Y1, Theorem 0.2].

In the remainder of the present section, let k_i, $i \in \{1, 2\}$, be an algebraically closed field of characteristic $p > 0$, and let

$$X_i^\bullet$$

be a pointed stable curve of type (g_{X_i}, n_{X_i}) over k_i, $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Moreover, for each $i \in \{1, 2\}$, we write

$$\mathcal{G}_{X_i^\bullet}$$

for the semi-graph of anabelioids of PSC-type associated to $X_i^\bullet$, $\Gamma_{\mathcal{G}_{X_i^\bullet}}$ for the underlying semi-graph of $\mathcal{G}_{X_i^\bullet}$. Let

$$\Phi : \mathcal{G}_{X_1^\bullet} \to \mathcal{G}_{X_2^\bullet}$$

be a morphism of semi-graphs of anabelioids (cf. [M5, Remark 2.4.2]). We choose a base point $\beta_{\mathcal{G}_{X_1^\bullet}}$ of $\mathcal{G}_{X_1^\bullet}$ and denote by $\beta_{\mathcal{G}_{X_2^\bullet}}$ the resulting base point of $\mathcal{G}_{X_2^\bullet}$ induced by Φ and $\beta_{\mathcal{G}_{X_1^\bullet}}$. Then we obtain a morphism of fundamental groups

$$\Pi_{\mathcal{G}_{X_1^\bullet}} := \pi_1(\mathcal{G}_{X_1^\bullet}, \beta_{\mathcal{G}_{X_1^\bullet}}) \to \Pi_{\mathcal{G}_{X_2^\bullet}} := \pi_1(\mathcal{G}_{X_2^\bullet}, \beta_{\mathcal{G}_{X_2^\bullet}})$$

of $\mathcal{G}_{X_1^\bullet}$ and $\mathcal{G}_{X_2^\bullet}$. Note that we have two isomorphisms as follows:

$$\Pi_{\mathcal{G}_{X_1^\bullet}} \cong \Pi_{X_1^\bullet} \text{ and } \Pi_{\mathcal{G}_{X_2^\bullet}} \cong \Pi_{X_2^\bullet}.$$

Moreover, by choosing suitable base points of $X_1^\bullet$ and $X_2^\bullet$, respectively, we may assume that

$$\Pi_{\mathcal{G}_{X_1^\bullet}} = \Pi_{X_1^\bullet} \text{ and } \Pi_{\mathcal{G}_{X_2^\bullet}} = \Pi_{X_2^\bullet}.$$

Definition 9.2. We shall call that a morphism $\Phi : \mathcal{G}_{X_1^\bullet} \to \mathcal{G}_{X_2^\bullet}$ of semi-graphs of anabelioids of PSC-type is a **unramified π_1-epimorphism** if the following conditions are satisfied:

(i) $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$;

(ii) Φ induces an isomorphism on underlying semi-graphs;

(iii) each of the induced morphisms between the respective constituent anabelioids (cf. [M5, Definition 2.1]) is a π_1-epimorphism (i.e., induces an open continuous surjective outer homomorphism on associated fundamental groups).

We denote by

$$\mathrm{Hom}^{\mathrm{unep}}(\mathcal{G}_{X_1^\bullet}, \mathcal{G}_{X_2^\bullet})$$

the set of unramified π_1-epimorphisms between $\mathcal{G}_{X_1^\bullet}$ and $\mathcal{G}_{X_2^\bullet}$ (possibly empty).

Remark 9.2.1. Suppose that $\Phi : \mathcal{G}_{X_1^\bullet} \to \mathcal{G}_{X_2^\bullet}$ is a unramified π_1-epimorphism. Then Φ induces an open continuous surjective homomorphism of fundamental groups

$$\phi : \Pi_{\mathcal{G}_{X_1^\bullet}} \twoheadrightarrow \Pi_{\mathcal{G}_{X_2^\bullet}}$$

of $\mathcal{G}_{X_1^\bullet}$ and $\mathcal{G}_{X_2^\bullet}$. Let Σ be a set of prime numbers such that $p \notin \Sigma$. Write $\mathcal{G}^{\Sigma}_{X_1^\bullet}$ and $\mathcal{G}^{\Sigma}_{X_2^\bullet}$ for the semi-graphs of anabelioids of pro-Σ PSC-type associated to $X_1^\bullet$ and $X_2^\bullet$, respectively (cf. [M6, Definition 1.1 (i)]). Then Φ induces an isomorphism

$$\Phi^{\Sigma} : \mathcal{G}^{\Sigma}_{X_1^\bullet} \xrightarrow{\sim} \mathcal{G}^{\Sigma}_{X_2^\bullet}.$$

We denote by

$$\mathrm{Hom}^{\mathrm{open}}(\Pi_{\mathcal{G}_{X_1^\bullet}}, \Pi_{\mathcal{G}_{X_2^\bullet}})$$

the set of open continuous homomorphisms between $\Pi_{\mathcal{G}_{X_1^\bullet}}$ and $\Pi_{\mathcal{G}_{X_2^\bullet}}$. Then we obtain the following natural map

$$\pi_1^{\mathrm{unep}} : \mathrm{Hom}^{\mathrm{unep}}(\mathcal{G}_{X_1^\bullet}, \mathcal{G}_{X_2^\bullet}) \to \mathrm{Hom}^{\mathrm{open}}(\Pi_{\mathcal{G}_{X_1^\bullet}}, \Pi_{\mathcal{G}_{X_2^\bullet}})/\mathrm{Inn}(\Pi_{\mathcal{G}_{X_2^\bullet}}).$$

The second main theorem of the present paper is as follows:

Theorem 9.3. *Suppose that $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition A and Condition B. Then the natural map*

$$\pi_1^{\mathrm{unep}} : \mathrm{Hom}^{\mathrm{unep}}(\mathcal{G}_{X_1^\bullet}, \mathcal{G}_{X_2^\bullet}) \to \mathrm{Hom}^{\mathrm{open}}(\Pi_{\mathcal{G}_{X_1^\bullet}}, \Pi_{\mathcal{G}_{X_2^\bullet}})/\mathrm{Inn}(\Pi_{\mathcal{G}_{X_2^\bullet}})$$

is a bijection. Moreover, let $\phi \in \mathrm{Hom}^{\mathrm{open}}(\Pi_{\mathcal{G}_{X_1^\bullet}}, \Pi_{\mathcal{G}_{X_2^\bullet}})$. Then the unramified π_1-epimorphism

$$\Phi := (\pi_1^{\mathrm{unep}})^{-1}([\phi]) : \mathcal{G}_{X_1^\bullet} \to \mathcal{G}_{X_2^\bullet}$$

can be mono-anabelian reconstructed from $\phi : \Pi_{\mathcal{G}_{X_1^\bullet}} \to \Pi_{\mathcal{G}_{X_2^\bullet}}$.

Proof. The injectivity of π_1^{unep} follows immediately from the definitions of semi-graphs of anabelioids, and the surjectivity of π_1^{unep} follows from the "moreover" part of the theorem. Then, to verify the theorem, we only need to prove the "moreover" part of the theorem.

Let

$$\phi \in \mathrm{Hom}^{\mathrm{open}}(\Pi_{\mathcal{G}_{X_1^\bullet}}, \Pi_{\mathcal{G}_{X_2^\bullet}}).$$

Then we have ϕ is a surjection. Since $\Pi_{\mathcal{G}_{X_2^\bullet}}$ is topologically finitely generated, Theorem 8.6 implies that there exists a set of open normal subgroups $\{H_{2,i}\}_{i\in\mathbb{N}}$ of $\Pi_{\mathcal{G}_{X_2^\bullet}}$ such that

(i) $H_{2,i} \supseteq H_{2,i+1}$, for each $i \in \mathbb{N}$;

(ii) $\varprojlim_{i\in\mathbb{N}} \Pi_{\mathcal{G}_{X_2^\bullet}}/H_{2,i} = \Pi_{\mathcal{G}_{X_2^\bullet}}$;

(iii) write $\{H_{1,i}\}_{i\in\mathbb{N}}$ for the set of open normal subgroups $\{\phi^{-1}(H_{1,i})\}_{i\in\mathbb{N}}$ of $\Pi_{\mathcal{G}_{X_1^\bullet}}$; then, for each i, the pointed stable curve $X^\bullet_{H_{1,i}}$ and $X^\bullet_{H_{2,i}}$ corresponding to $H_{1,i}$ and $H_{2,i}$, respectively, satisfy Condition A and Condition B.

Moreover, by Theorem 9.1, the sets $\{H_{1,i}\}_{i\in\mathbb{N}}$, $\{H_{2,i}\}_{i\in\mathbb{N}}$, and $\{\phi|_{H_{1,i}} : H_{1,i} \twoheadrightarrow H_{2,i}\}_{i\in\mathbb{N}}$ can be mono-anabelian reconstructed from $\Pi_{\mathcal{G}_{X_1^\bullet}}$, $\Pi_{\mathcal{G}_{X_2^\bullet}}$, and $\phi : \Pi_{\mathcal{G}_{X_1^\bullet}} \to \Pi_{\mathcal{G}_{X_2^\bullet}}$. Then, to verify the "moreover" part of the theorem, it is sufficient to prove that

for each i, there exists a group-theoretic algorithm whose input datum is the natural commutative digram profinite groups

$$\begin{array}{ccc} H_{1,i} & \xrightarrow{\phi|_{H_{1,i}}} & H_{2,i} \\ \downarrow & & \downarrow \\ \Pi_{\mathcal{G}_{X_1^\bullet}} & \xrightarrow{\phi} & \Pi_{\mathcal{G}_{X_2^\bullet}}, \end{array}$$

and whose output datum is a commutative dual semi-graphs

$$\begin{array}{ccc} \Gamma_{X^\bullet_{H_{1,i}}} & \xrightarrow{\theta_{\phi|_{H_{1,i}}}} & \Gamma_{X^\bullet_{H_{2,i}}} \\ \downarrow & & \downarrow \\ \Gamma_{\mathcal{G}_{X_1^\bullet}} = \Gamma_{X_1^\bullet} & \xrightarrow{\theta_\phi} & \Gamma_{X_2^\bullet} = \Gamma_{\mathcal{G}_{X_2^\bullet}} \end{array}$$

which induced by the commutative digram of profinite groups above, where $\Gamma_{(-)}$ denotes the dual semi-graph of the curve $(-)$. Moreover, the commutative diagram is compatible with the natural action of $\Pi_{\mathcal{G}_{X_1^\bullet}}/H_{1,i} = \Pi_{\mathcal{G}_{X_2^\bullet}}/H_{2,i}$.

Thus, the theorem follows immediately from Theorem 7.4. $\square$

The Theorem 9.3 implies the following corollaries.

Corollary 9.4. *Let*

$$\phi : \Pi_{\mathcal{G}_{X_1^\bullet}} \twoheadrightarrow \Pi_{\mathcal{G}_{X_2^\bullet}}$$

be an open continuous surjective homomorphism, $H_2 \subseteq \Pi_{\mathcal{G}_{X_2^\bullet}}$ an open subgroup, and $H_1 := \phi^{-1}(H_2)$. Write $X_{H_1}^\bullet$, $i \in \{1,2\}$, for the curve corresponding to H_i. If we suppose that $X_{H_i}^\bullet$, $i \in \{1,2\}$, satisfies Condition A and Condition B, then there exists a unramified π_1-epimorphism

$$\Phi : \mathcal{G}_{X_1^\bullet} \to \mathcal{G}_{X_2^\bullet}$$

such that $\pi_1^{\mathrm{unep}}(\Phi) = [\phi]$, where $[\phi]$ denotes the image of ϕ in $\mathrm{Hom}^{\mathrm{open}}(\Pi_{\mathcal{G}_{X_1^\bullet}}, \Pi_{\mathcal{G}_{X_2^\bullet}})/\mathrm{Inn}(\Pi_{\mathcal{G}_{X_2^\bullet}})$. In particular, if there exists an isomorphism of dual semi-graphs

$$\rho : \Gamma_{X_1^\bullet} \xrightarrow{\sim} \Gamma_{X_2^\bullet}$$

such that, for each $v_1 \in v(\Gamma_{X_1^\bullet})$, the genus of the normalization of the irreducible component of $X_1^\bullet$ corresponding to v_1 is equal to the genus of the normalization of the irreducible component of $X_2^\bullet$ corresponding to $\rho(v_1)$, then there exists a unramified π_1-epimorphism

$$\Phi : \mathcal{G}_{X_1^\bullet} \to \mathcal{G}_{X_2^\bullet}$$

such that $\pi_1^{\mathrm{unep}}(\Phi) = [\phi]$, where $[\phi]$ denotes the image of ϕ in $\mathrm{Hom}^{\mathrm{open}}(\Pi_{\mathcal{G}_{X_1^\bullet}}, \Pi_{\mathcal{G}_{X_2^\bullet}})/\mathrm{Inn}(\Pi_{\mathcal{G}_{X_2^\bullet}})$.

Proof. The first part of the corollary follows immediately from Theorem 9.3. Let us prove the "in particular" part of the corollary. Write $\Pi_{X_i^\bullet}^{p'}$, $i \in \{1,2\}$, for the maximal prime-to-p quotient of $\Pi_{X_i^\bullet}$. The assumptions implies $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Then ϕ induces an isomorphism

$$\phi^{p'} : \Pi_{X_1^\bullet}^{p'} \xrightarrow{\sim} \Pi_{X_2^\bullet}^{p'}.$$

By [Y1, Lemma 5.3] and the assumptions, there exists a characteristic subgroup $H_2^{p'} \subseteq \Pi_{X^\bullet}^{p'}$ such that the curve corresponding to the curves corresponding to $H_1^{p'}$ and $H_2^{p'}$ satisfy Condition A and Condition B. Thus, the "in particular" part of the corollary follows from the first part of the corollary. □

Remark 9.4.1. Corollary 9.4 generalizes [Y2, Theorem 3.4] to the case of arbitrary pointed stable curves.

Corollary 9.5. *Let*

$$\phi : \Pi_{X_1^\bullet} \twoheadrightarrow \Pi_{X_2^\bullet}$$

be an open continuous surjective homomorphism, H_2 an arbitrary open subgroup, and $H_1 := \phi^{-1}(H_2)$. Write $X_{H_1}^\bullet$ and $X_{H_2}^\bullet$ for the pointed stable curves corresponding to H_1 and H_2 over k_1 and k_2, respectively. Suppose that $X_1^\bullet$ and $X_2^\bullet$ satisfy Condition A and Condition B. Then we have

$$\mathrm{Arv}_p(H_1) = \mathrm{Arv}_p(H_2).$$

Proof. The corollary follows immediately from Theorem 1.5 and Theorem 9.3. □

10 Applications to the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$

In this section, we apply the results obtained in Section 9 to the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, and let

$$\overline{\mathcal{M}}_{g,n}$$

be the moduli stack over $\overline{\mathbb{F}}_p$ parameterizing pointed stable curves of type (g, n). We denote by

$$\overline{M}_{g,n}$$

the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$. Let k be an algebraically closed field, and let $X^\bullet$ be a pointed stable curve of type (g, n) over k. Then $X^\bullet \to \operatorname{Spec} k$ determines a classifying morphism

$$c_{X^\bullet} : \operatorname{Spec} k \to \overline{\mathcal{M}}_{g,n} \to M_{g,n},$$

and we denote by $q_X \in M_{g,n}$ the image of $c_{X^\bullet}$. Write $k(q_X)$ for the residue field of q_X and $\overline{k(q_X)}$ for an algebraic closure of $k(q_X)$. We denote by

$$X^\bullet_{q_X} := \overline{\mathcal{M}}_{g,n+1} \times_{\overline{\mathcal{M}}_{g,n}} \operatorname{Spec} \overline{k(q_X)}$$

the pointed curve of type (g, n) over $\overline{k(q_X)}$ induced by the natural morphism

$$\operatorname{Spec} \overline{k(q_X)} \to \operatorname{Spec} k(q_X) \to \overline{M}_{g,n}.$$

We shall call that $X^\bullet_{q_X}$ is a **minimal model** of $X^\bullet$ (cf. [T2, Definition 1.30 and Lemma 1.31] for the case of smooth pointed stable curves). Note that the admissible fundamental group of $X^\bullet$ is naturally isomorphic to the admissible fundamental group of $X^\bullet_{q_X}$.

We fix some notations. Suppose that $\overline{\mathbb{F}}_p \subseteq k_i$, $i \in \{1, 2\}$, and let

$$X_i^\bullet := (X_i, D_{X_i})$$

be a pointed stable curve of type (g_{X_i}, n_{X_i}) over k_i and

$$\Pi_{X_i^\bullet}$$

the admissible fundamental group of $X_i^\bullet$. For each $i \in \{1, 2\}$ and each $v \in v(\Gamma_{X_i^\bullet})$, write $X_{i,v}$ for the irreducible component of X_i corresponding to v and $\mathrm{nl}_{i,v} : \widetilde{X}_{i,v} \to X_{i,v}$ for the normalization morphism of $X_{i,v}$; we define

$$\widetilde{X}^\bullet_{i,v} := (\widetilde{X}_{i,v}, \mathrm{nl}_{i,v}^{-1}((\mathrm{Node}(X_i) \cup D_{X_i}) \cap X_{i,v})),$$

and

$$X^\bullet_{i,v} := (X_{i,v}, D_{X_i} \cap X_{i,v}),$$

to be a smooth pointed stable curve of type $(g_{i,v}, n_{i,v})$ and an irreducible pointed stable curve of type $(g_{X_{i,v}}, n_{X_{i,v}})$ over k_i, respectively, where $\mathrm{Node}(X_i)$ denotes the set of nodes of X_i. We shall call $X^\bullet_{i,v}$ a **pointed irreducible component** of $X^\bullet_i$. For suitable of choices of base points of $\widetilde{X}^\bullet_{i,v}$ and $X^\bullet_{i,v}$, we obtain the admissible fundamental group

$$\Pi_{\widetilde{X}^\bullet_{i,v}} \text{ and } \Pi_{X^\bullet_{i,v}}$$

of $\widetilde{X}^\bullet_{i,v}$ and $X^\bullet_{i,v}$, respectively; moreover, we have outer injective homomorphisms of admissible fundamental groups

$$\Pi_{\widetilde{X}^\bullet_{i,v}} \hookrightarrow \Pi_{X^\bullet_{i,v}} \hookrightarrow \Pi_{X^\bullet_i}.$$

Then the third main theorem of the present paper is as follows:

Theorem 10.1. *(a) Suppose that, for each $i \in \{1, 2\}$ and each $v \in v(\Gamma_{X^\bullet_i})$, $(g_{i,v}, n_{i,v})$ is equal to either $(0, n_{i,v})$ or $(1, 1)$. Moreover, suppose that $p \neq 2$ when there exits $v \in v(\Gamma_{X^\bullet_i})$ such that $(g_{i,v}, n_{i,v}) = (1, 1)$.*

(a-i) Suppose that $k_1 = k_2 = \overline{\mathbb{F}}_p$, and that $X^\bullet_1$ is an irreducible pointed stable curve over $\overline{\mathbb{F}}_p$. Then we can detect whether or not $X^\bullet_1$ is isomorphic to a pointed irreducible component of $X^\bullet_2$ as schemes group-theoretically from $\Pi_{X^\bullet_1}$ and $\Pi_{X^\bullet_2}$.

(a-ii) Suppose that $k_1 = \overline{\mathbb{F}}_p$, that $(g, n) = (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$, that

$$\phi : \Pi_{X^\bullet_1} \twoheadrightarrow \Pi_{X^\bullet_2}$$

an open continuous surjective homomorphism, and that there exists an isomorphism of dual semi-graphs

$$\rho : \Gamma_{X^\bullet_1} \xrightarrow{\sim} \Gamma_{X^\bullet_2}$$

such that, for each $v \in v(\Gamma_{X^\bullet_1})$, $(g_{1,v}, n_{1,v}) = (g_{2,\rho(v)}, n_{2,\rho(v)})$. Let $X^\bullet_{q_{X_2}}$ be a minimal model $X^\bullet_{q_{X_2}}$ of $X^\bullet_2$. Then $X^\bullet_{q_{X_2}}$ is a pointed stable curve over $\overline{\mathbb{F}}_p$; moreover, if we suppose that $X^\bullet_{q_{X_2}} = X^\bullet_2$ (i.e., $k_2 = \overline{\mathbb{F}}_p$), then, for each $v \in v(\Gamma_{X^\bullet_1})$, $X^\bullet_{1,v}$ is isomorphic to $X^\bullet_{2,\rho(v)}$ as schemes. In particular, if $X^\bullet_i$, $i \in \{1, 2\}$, is irreducible, then $X^\bullet_1$ is isomorphic to $X^\bullet_{q_{X_2}}$ as schemes if and only if

$$\mathrm{Hom}^{\mathrm{open}}(\Pi_{X^\bullet_1}, \Pi_{X^\bullet_2}) \neq \emptyset,$$

where $\mathrm{Hom}^{\mathrm{open}}(-, -)$ denotes the set of open continuous homomorphisms of profinite groups.

(b) Suppose that $k_1 = \overline{\mathbb{F}}_p$. Then there are at most finitely many $\overline{\mathbb{F}}_p$-isomorphism classes of irreducible pointed stable curves over $\overline{\mathbb{F}}_p$ whose admissible fundamental groups are isomorphic to the admissible fundamental group of a pointed irreducible component of $X^\bullet_1$.

Proof. The part (a) of the theorem follows immediately from [Y2, Theorem 4.3 and Remark 4.3.3], Theorem 9.1, and Corollary 9.4. The part (b) of the theorem follows immediately from Theorem 9.1 and [T3, Theorem 0.1]. □

Remark 10.1.1. Theorem 10.1 generalizes [Y1, Theorem 6.6 (ii) and Theorem 6.9] and [Y2, Theorem 4.3 and Remark 4.3.3].

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