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Reconstruction of Invariants of Configuration Spaces of Hyperbolic Curves from Associated Lie Algebras

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RECONSTRUCTION OF INVARIANTS OF CONFIGURATION SPACES OF HYPERBOLIC CURVES FROM ASSOCIATED LIE ALGEBRAS

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Abstract. In the present paper, we study the Lie algebra associated to (the fundamental group of) the configuration space of a hyperbolic curve over an algebraically closed field of characteristic zero and give explicit algorithms for reconstructing various objects from the Lie algebra. Among others, we reconstruct the set of generalized fiber ideals. As an application, we give an algorithm for reconstructing the set of generalized fiber subgroups of the fundamental group of the configuration space. Moreover, as another application, we give a Grothendieck conjecture-type result for a configuration space and a hyperbolic polycurve.

Introduction

Let K be a field of characteristic zero, X a hyperbolic curve of type (g, r) over K, n a positive integer, l a prime number, and Σ a set of prime numbers which coincides with either $\{l\}$ or the set of all prime numbers. Write X_n for the n-th configuration space of X over K (see Definition 1.4(ii)), i.e., the complement of the union of various weak diagonals in the product of n copies of X.

If K is algebraically closed, then write Π_n^Σ for the maximal pro-Σ quotient of the étale fundamental group $\pi_1(X_n)$. In [HMM], certain explicit group-theoretic algorithms for reconstructing some objects from Π_n^Σ are given. For instance, n and the set of generalized fiber subgroups (of given co-length) of Π_n^Σ (see Definition 1.7(i)) can be reconstructed from Π_n^Σ, and, moreover, if $n \geq 2$, then (g, r) can be reconstructed from Π_n^Σ (cf. [HMM] Theorem 2.5). In the present paper, at first we consider the Lie algebra analogues of these reconstruction algorithms:

Theorem A (cf. Theorems 3.10, 4.3, 4.7)**.** *Write* $\mathrm{Gr}(\Pi_n^l)$ *for the* $\mathbb{Z}_{>0}$*-graded Lie algebra over* $\mathbb{Z}_l$ *determined by the weight filtration of* $\Pi_n^{\{l\}} (=: \Pi_n^l)$ *(see Definition 1.5). Then there exist algorithms for reconstructing* n *and the set of generalized fiber ideals of given co-length of* $\mathrm{Gr}(\Pi_n^l)$ *(see Definition 1.7(iii)) from (the Lie algebra obtained by forgetting the grading of) the graded Lie algebra* $\mathrm{Gr}(\Pi_n^l)$ *over* $\mathbb{Z}_l$*. Moreover, if* $n \geq 2$*, then there exist algorithms for reconstructing* (g, r) *and* $\mathrm{Gr}(\Pi_n^l)(m)$ *for* $m \geq 1$ *(see Definition 1.5(ii)) from (the Lie algebra obtained by forgetting the grading of) the graded Lie algebra* $\mathrm{Gr}(\Pi_n^l)$ *over* $\mathbb{Z}_l$*.*

The following Theorem B plays a central role in the reconstruction of the set of generalized fiber ideals of $\mathrm{Gr}(\Pi_n^l)$. This is proved by the classification of surjective

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homomorphisms (as abstract Lie algebras over $\mathbb{Z}_l$) from $\mathrm{Gr}(\Pi_n^l)$ to a surface algebra (see Definition 1.5(iv)), which is given in Theorem 3.1.

Theorem B (cf. Theorem 3.5(ii))**.** *Let* $\mathfrak{i} \subset \mathrm{Gr}(\Pi_n^l)$ *be a (not necessarily homogeneous) Lie ideal of* $\mathrm{Gr}(\Pi_n^l)$ *over* $\mathbb{Z}_l$*. Then* $\mathfrak{i}$ *is a generalized fiber ideal of co-length one if and only if* $\mathrm{Gr}(\Pi_n^l)/\mathfrak{i}$ *is isomorphic to* $\mathrm{Gr}(\Pi_1^l)$ *as abstract Lie algebras over* $\mathbb{Z}_l$*.*

As an application of the consideration of Lie algebras, we obtain the following group-theoretic analogue of Theorem B:

Theorem C (cf. Theorem 5.13)**.** *Let* $N \subset \Pi_n^\Sigma$ *be a normal closed subgroup of* Π_n^Σ*. Then* N *is a generalized fiber subgroup of co-length one (see Definition 1.7(i)) if and only if* Π_n^Σ/N *is isomorphic to* Π_1^Σ *as profinite groups.*

Theorem C gives a group-theoretic algorithm for reconstructing the set of generalized fiber subgroups of co-length one in a unified way (i.e., without depending on invariants, especially on (g, r)). Let us observe that Theorem C asserts that a normal closed subgroup N of Π_n^Σ such that the quotient Π_n^Σ/N is isomorphic to the surface group Π_1^Σ must be "geometric" in some sense. By considering normal closed subgroups of Π_n^Σ such that the quotient is isomorphic to a surface group more closely, we obtain the following Grothendieck conjecture-type result for a configuration space and a hyperbolic polycurve:

Theorem D (cf. Theorem 7.14)**.** *Suppose that* K *is generalized sub-*l*-adic (see Definition 7.6). Let* $(Z, Z = Z_{(n)} \to Z_{(n-1)} \to \cdots \to Z_{(1)} \to \operatorname{Spec} K = Z_{(0)})$ *be a parametrized hyperbolic polycurve of dimension* n *over* K *(see Definition 7.1(ii)) and* $\varphi : \pi_1(X_n) \xrightarrow{\sim} \pi_1(Z)$ *an isomorphism from* $\pi_1(X_n)$ *to* $\pi_1(Z)$ *over the absolute Galois group* G_K *of* K*. If* $(g, r) = (0, 3)$*, then suppose that* $X \cong \mathbb{P}_K^1 \setminus \{0, 1, \infty\}$*. Moreover, suppose that at least one of the following holds:*

- $g \geq 2$*.*
- *For each integer* m *such that* $1 \leq m \leq n$*, the hyperbolic curve* $Z_{(m)}/Z_{(m-1)}$ *is not of type* $(0, 3), (1, 1)$*.*

Then there exist generalized projection morphisms $p_m : X_m \to X_{m-1}$ $(1 \leq m \leq n)$ *(see Definition 1.6) such that* φ *arises from a unique isomorphism* $(X_n, X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} \operatorname{Spec} K = X_0) \xrightarrow{\sim} (Z, Z = Z_{(n)} \to Z_{(n-1)} \to \cdots \to \operatorname{Spec} K = Z_{(0)})$ *of parametrized hyperbolic polycurves over* K*. In particular, the natural map*

$$\mathrm{Isom}_K(X_n, Z) \to \mathrm{Isom}_{G_K}(\pi_1(X_n), \pi_1(Z))/\operatorname{Inn}(\pi_1(Z \times_K \overline{K}))$$

is bijective.

0. Notations and conventions

Definition 0.1. Let R be a commutative ring (with unit). A *Lie algebra* over R is an R-module $\mathfrak{g}$ with an R-bilinear map $[-, -] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for any $a, b, c \in \mathfrak{g}$, it holds that $[a, a] = 0, [[a, b], c] + [[b, c], a] + [[c, a], b] = 0$. We shall refer to the R-bilinear map $[-, -]$ as the *Lie bracket* of the Lie algebra $\mathfrak{g}$. Note that for any $a, b \in \mathfrak{g}$, $0 = [a + b, a + b] = [a, b] + [b, a]$, i.e, $[b, a] = -[a, b]$. A Lie algebra $\mathfrak{g}$ is (not necessarily commutative or associative) ring without unit whose product is the Lie bracket.

Definition 0.2. Let R be a commutative ring and $\mathfrak{g}$ a Lie algebra over R.

(i) We shall say that $\mathfrak{g}$ is *abelian* if $[a,b]=0$ for any $a,b\in\mathfrak{g}$.
(ii) We shall say that a subset $\mathfrak{s}\subset\mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{g}$ over R if $\mathfrak{s}$ is an R-submodule of $\mathfrak{g}$ and closed under multiplication.
(iii) We shall say that a subset $\mathfrak{i}\subset\mathfrak{g}$ is a Lie ideal of $\mathfrak{g}$ over R if $\mathfrak{i}$ is an R-submodule, and, moreover, $\mathfrak{i}$ is an ideal of $\mathfrak{g}$ with respect to the ring structure on $\mathfrak{g}$, i.e., for any $a\in\mathfrak{i}$ and $b\in\mathfrak{g}$, it holds that $[a,b],[b,a]\in\mathfrak{i}$.
(iv) Let $S\subset\mathfrak{g}$ be a subset of $\mathfrak{g}$. Then we shall write $\langle S\rangle_R$ for the Lie subalgebra of $\mathfrak{g}$ over R generated by S, and $\langle S\rangle_{R,\mathfrak{g}}$ for the Lie ideal of $\mathfrak{g}$ over R generated by S.
(v) Let $\mathfrak{s},\mathfrak{t}\subset\mathfrak{g}$ be R-submodules (resp. Lie ideals over R) of $\mathfrak{g}$. Then we shall write $[\mathfrak{s},\mathfrak{t}]$ for the submodule of $\mathfrak{g}$ generated by elements of the form $[s,t]$ ($s\in\mathfrak{s},t\in\mathfrak{t}$) (note that $[\mathfrak{s},\mathfrak{t}]$ is necessarily an R-submodule (resp. Lie ideal over R)).
(vi) Let m be a positive integer. Then we shall define $\mathfrak{g}[m]\subset\mathfrak{g}$ as follows:

$$\mathfrak{g}[1]:=\mathfrak{g},\ \mathfrak{g}[m]:=[\mathfrak{g},\mathfrak{g}[m-1]]\ (m\geq 2).$$

Then it is clear that $\mathfrak{g}[m]$ is a Lie ideal of $\mathfrak{g}$ over R.
(vii) Let m be a positive integer. Then we shall write $(\mathfrak{L}_m)_R$ for the free Lie algebra over R of rank m.

Definition 0.3. Let G be a group.

(i) Let $S\subset G$ be a subset of G. Then we shall write $\langle S\rangle$ for the subgroup of G generated by S.
(ii) Let $A,B\subset G$ be subgroups of G. Then we shall write $[A,B]:=\langle\{[a,b]\mid a\in A,\ b\in B\}\rangle$, where $[a,b]=aba^{-1}b^{-1}$.
(iii) Let m be a positive integer. Then we shall write F_m for the free group of rank m.

Definition 0.4. Let G be a profinite group.

(i) A *G-module* A is a discrete abelian group A together with a continuous action of G on A.
(ii) Let A be a G-module and d a nonnegative integer. Then we shall write $H^d(G,A)$ for the d-th cohomology group of G with coefficients in A.

Definition 0.5. Let X be a connected noetherian scheme X. Then we shall write $\pi_1(X)$ for the étale fundamental group of X (for some choice of basepoint). If X is a variety over $\mathbb{C}$, then we shall write $\pi_1^{\mathrm{top}}(X):=\pi_1^{\mathrm{top}}(X^{\mathrm{an}})$ for the topological fundamental group of (the complex analytic space associated to) X (for some choice of ($\mathbb{C}$-rational) base point).

Definition 0.6. We shall write $\mathfrak{Primes}$ for the set of all prime numbers.

1. Generalities on the Configuration Spaces of Hyperbolic Curves

In the present §1, we study generalities on the configuration spaces of hyperbolic curves.

Definition 1.1. Let G be a group and Σ a set of prime numbers. Then we shall write

$$G^{\Sigma}:=\varprojlim_{N} G/N,$$

where N runs over all normal subgroups of G of finite index such that every prime factor of $[G : N]$ is contained in Σ. We shall refer to G^{Σ} as the *pro-Σ completion* of G. For a prime number l, we shall write simply

$$G^{l}$$

instead of $G^{\{l\}}$. Moreover, we shall write simply

$$G^{\wedge}$$

instead of $G^{\mathfrak{Primes}}$.

Remark 1.1.1. If G is a topologically finitely generated profinite group, then, since every homomorphism from G to any finite group is continuous (cf. [NS] Theorem 1.1), G^{Σ} coincides with the maximal pro-Σ quotient of G.

Definition 1.2. Let S be a scheme and X a scheme over S. Then we shall say that X is a *hyperbolic curve* (*of type*(g, r)) over S if there exist

- a pair of nonnegative integers (g, r);
- a scheme X^{cpt} which is smooth, proper, geometrically connected, and of relative dimension one over S;
- a (possibly empty) closed subscheme $D \subset X^{\mathrm{cpt}}$ of X^{cpt} which is finite and étale over S

such that

- $2g - 2 + r > 0$;
- any geometric fiber of $X^{\mathrm{cpt}} \to S$ is (a necessarily smooth proper connected curve) of genus g;
- the finite étale covering $D \hookrightarrow X^{\mathrm{cpt}} \to S$ is of degree r;
- X is isomorphic to $X^{\mathrm{cpt}} \setminus D$ over S.

We shall refer to the above integer g as the genus of X over S.

Definition 1.3. Let Σ be a nonempty set of prime numbers and (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$. Then we shall write

$$\Pi_{g,r} := \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_r \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_r = 1 \rangle.$$

We shall refer to a group isomorphic to $\Pi_{g,r}$ as a discrete surface group. Moreover, we shall refer to a profinite group isomorphic to $\Pi_{g,r}^{\Sigma}$ as a (pro-Σ) surface group (cf. [MT] Definition 1.2).

Remark 1.3.1.

(i) Let X be a hyperbolic curve of type (g, r) over an algebraically closed field of characteristic zero. Then $\pi_1(X)^{\Sigma}$ is isomorphic to $\Pi_{g,r}^{\Sigma}$. Moreover, if X is a hyperbolic curve over $\mathbb{C}$, then $\pi_1^{\mathrm{top}}(X)$ is isomorphic to $\Pi_{g,r}$.

(ii) An open subgroup (resp. a subgroup of finite index) of a pro-Σ (resp. discrete) surface group is a pro-Σ (resp. discrete) surface group.

Definition 1.4.

(i) Let S be a scheme, X a hyperbolic curve (of type (g, r)) over S, and n a positive integer. Then we shall write $P_n = P_n(X/S) := \overbrace{X \times_S \cdots \times_S X}^{n}$.

(ii) Let (S, X, n) be as in (i). For (i, j) a pair of integers such that $1 \leq i < j \leq n$. Then we shall write $\pi_{i,j} : P_n \twoheadrightarrow P_2 = X \times_S X$ for the projection to the i-th and j-th factors. Then we shall write $X_n := P_n \setminus (\bigcup_{i,j} \pi_{i,j}^{-1}(\Delta))$, where $\Delta \subset \Pi_2$ is the diagonal of $P_2 = X \times_S X$. We shall refer to X_n as the *n-th configuration space* of X over S (cf. [MT] Definition 2.1). (For convenience, we set $X_0 := S$.)

(iii) Let K be an algebraically closed field of characteristic zero, X a hyperbolic curve over K, n a positive integer, and Σ a set of prime numbers such that $\sharp\Sigma = 1$ or $\Sigma = \mathfrak{Primes}$. Then we shall write

$$\Pi_n^{\Sigma} = \Pi_{n,g,r}^{\Sigma} = \Pi_n^{\Sigma}(X) := \pi_1(X_n)^{\Sigma}.$$

We shall refer to a profinite group isomorphic to Π_n^{Σ} as a (pro-Σ) configuration space group (cf. [HMM] §0). For a prime number l, we shall write simply

$$\Pi_n^{l} = \Pi_{n,g,r}^{l} = \Pi_n^{l}(X)$$

instead of $\Pi_n^{\{l\}}$. Moreover, we shall write simply

$$\Pi_n^{\wedge} = \Pi_{n,g,r}^{\wedge} = \Pi_n^{\wedge}(X)$$

instead of $\Pi_n^{\mathfrak{Primes}}$.

(iv) Let X be a hyperbolic curve over $\mathbb{C}$ and n a positive integer. Then we shall write $\Pi_n = \Pi_{n,g,r} = \Pi_n(X) := \pi_1^{\mathrm{top}}(X_n)$. We shall refer to a group isomorphic to Π_n as a discrete configuration space group.

(v) Let G be a configuration space group. Then we shall refer to a collection of data

$$(K, X, n, \Sigma, \Pi_n^{\Sigma}(X) \xrightarrow{\sim} G)$$

consisting of an algebraically closed field K of characteristic zero, a hyperbolic curve X over K, a positive integer n, a set of prime numbers Σ such that $\sharp\Sigma = 1$ or $\Sigma = \mathfrak{Primes}$, and an isomorphism of profinite groups $\Pi_n^{\Sigma}(X) \xrightarrow{\sim} G$ as a *CS-envelope* for G. For a discrete configuration space group G, we shall define a CS-envelope $(X, n, \Pi_n(X) \xrightarrow{\sim} G)$ for G similarly (where $\Pi_n(X) \xrightarrow{\sim} G$ is an isomorphism of abstract groups).

Remark 1.4.1. It may seem that there are two definitions of "Π_n^{Σ}". However, if we take a subfield K' of K such that K' is an algebraic closure of a finitely generated field over $\mathbb{Q}$ and that X has a model X' over K', and if we fix an inclusion $K' \hookrightarrow \mathbb{C}$, then $\Pi_n^{\Sigma}(X)$ is isomorphic to the pro-Σ completion of $\Pi_n(X' \times_{K'} \mathbb{C})$ via the fixed inclusion $K' \hookrightarrow \mathbb{C}$.

Definition 1.5. Let K be an algebraically closed field of characteristic zero, X a hyperbolic curve of type (g, r) over K, and n a positive integer.

(i) Let l be a prime number. Then we shall write

$$\Pi_n^{l}(1) := \Pi_n^{l},$$

$$\Pi_n^{l}(2) := \ker(\Pi_n^{l} \to (\pi_1(\overbrace{X^{\mathrm{cpt}} \times_K \cdots \times_K X^{\mathrm{cpt}}}^{n})^{l})^{\mathrm{ab}}),$$

$$\Pi_n^{l}(m) := \overline{\langle [\Pi_n^{l}(m_1), \Pi_n^{l}(m_2)] \mid m_1 + m_2 = m,\ m_1 \geq 1,\ m_2 \geq 1 \rangle}\ (m \geq 3),$$

where $\Pi_n^l \to (\pi_1(\overbrace{X^{\mathrm{cpt}} \times_K \cdots \times_K X^{\mathrm{cpt}}}^{n})^l)^{\mathrm{ab}}$ is the homomorphism induced by the open immersion $X_n \hookrightarrow \overbrace{X^{\mathrm{cpt}} \times_K \cdots \times_K X^{\mathrm{cpt}}}^{n}$ (cf. [Ho1] Definitions 1.1, 1.4).

(ii) In the notation of (i), we shall write

$$\mathrm{Gr}^m(\Pi_n^l) := \Pi_n^l(m)/\Pi_n^l(m+1) \ (m \geq 1),$$
$$\mathrm{Gr}(\Pi_n^l)(d) := \bigoplus_{m \geq d} \mathrm{Gr}^m(\Pi_n^l) \ (d \geq 1),$$
$$\mathrm{Gr}(\Pi_n^l) := \mathrm{Gr}(\Pi_n^l)(1).$$

We define the Lie bracket $[A, B] \in \mathrm{Gr}^{m_1+m_2}(\Pi_n^l)$ of $A \in \mathrm{Gr}^{m_1}(\Pi_n^l)$ and $b \in \mathrm{Gr}^{m_2}(\Pi_n^l)$ by $[A, B] = aba^{-1}b^{-1} \mod \Pi_n^l(m_1+m_2+1)$, where $a \in \Pi_n^l(m_1)$ and $b \in \Pi_n^l(m_2)$ are representatives of A, B, respectively. Then (the Lie bracket is well-defined and) $\mathrm{Gr}(\Pi_n^l)$ is a $\mathbb{Z}_{>0}$-graded Lie algebra over $\mathbb{Z}_l$. Moreover, for each positive integer d, $\mathrm{Gr}(\Pi_n^l)(d)$ is a Lie ideal of $\mathrm{Gr}(\Pi_n^l)$ over $\mathbb{Z}_l$.

(iii) If $K = \mathbb{C}$, then we shall write

$$\Pi_n(1) := \Pi_n,$$
$$\Pi_n(2) := \ker(\Pi_n \to \pi_1^{\mathrm{top}}(\overbrace{X^{\mathrm{cpt}} \times_{\mathbb{C}} \cdots \times_{\mathbb{C}} X^{\mathrm{cpt}}}^{n})^{\mathrm{ab}}),$$
$$\Pi_n(m) := \langle [\Pi_n(m_1), \Pi_n(m_2)] \mid m_1 + m_2 = m, \ m_1 \geq 1, \ m_2 \geq 1 \rangle \ (m \geq 3),$$

where $\Pi_n \to \pi_1^{\mathrm{top}}(\overbrace{X^{\mathrm{cpt}} \times_{\mathbb{C}} \cdots \times_{\mathbb{C}} X^{\mathrm{cpt}}}^{n})^{\mathrm{ab}}$ is the homomorphism induced by the open immersion $X_n \hookrightarrow \overbrace{X^{\mathrm{cpt}} \times_{\mathbb{C}} \cdots \times_{\mathbb{C}} X^{\mathrm{cpt}}}^{n}$.

(iv) In the notation of (iii), we shall write

$$\mathrm{Gr}^m(\Pi_n) := \Pi_n(m)/\Pi_n(m+1) \ (m \geq 1),$$
$$\mathrm{Gr}(\Pi_n)(d) := \bigoplus_{m \geq d} \mathrm{Gr}^m(\Pi_n) \ (d \geq 1),$$
$$\mathrm{Gr}(\Pi_n) := \mathrm{Gr}(\Pi_n)(1).$$

Then, as in (ii), $\mathrm{Gr}(\Pi_n)$ is a $\mathbb{Z}_{>0}$-graded Lie algebra over $\mathbb{Z}$ and for each positive integer d, $\mathrm{Gr}(\Pi_n)(d)$ is a Lie ideal of $\mathrm{Gr}(\Pi_n)$. Moreover, for an integral domain R, we shall write

$$\mathrm{Gr}_R^m(\Pi_n) := \mathrm{Gr}^m(\Pi_n) \otimes_{\mathbb{Z}} R \ (m \geq 1),$$
$$\mathrm{Gr}_R(\Pi_n)(d) := \mathrm{Gr}(\Pi_n)(d) \otimes_{\mathbb{Z}} R \ (d \geq 1),$$
$$\mathrm{Gr}_R(\Pi_n) := \mathrm{Gr}_R(\Pi_n)(1)(= \mathrm{Gr}(\Pi_n) \otimes_{\mathbb{Z}} R).$$

We shall refer to a Lie algebra over R isomorphic to $\mathrm{Gr}_R(\Pi_n)$ (as an abstract Lie algebra over R) as a *configuration space algebra* over R. We shall also refer to a Lie algebra over R isomorphic to $\mathrm{Gr}_R(\Pi_1)$ (as an abstract Lie algebra over R) as a *surface algebra* over R. (Note that, though $\mathrm{Gr}_R(\Pi_n)$ has a natural grading structure, we do not treat configuration space algebras and surface algebras as graded Lie algebras unless otherwise specified).

(v) Let R be an integral domain and $\mathfrak{g}$ a configuration space algebra over R. Then we shall refer to a collection of data

$$(X, n, \mathrm{Gr}_R(\Pi_n) \xrightarrow{\sim} \mathfrak{g})$$

consisting of a hyperbolic curve X over $\mathbb{C}$, a positive integer n, and an isomorphism of abstract Lie algebras $\mathrm{Gr}_R(\Pi_n(X)) \xrightarrow{\sim} \mathfrak{g}$ over R as a *CS-envelope* for $\mathfrak{g}$.

Remark 1.5.1 (cf. [NT](2.7.3)). The isomorphism appearing in Remark 1.4.1(i) determines an isomorphism $\mathrm{Gr}(\Pi_n^l) \xrightarrow{\sim} \mathrm{Gr}_{\mathbb{Z}_l}(\Pi_n)$. In particular, $\mathrm{Gr}(\Pi_n^l)$ is a configuration space algebra over $\mathbb{Z}_l$.

Definition 1.6. Let K be a field and X a hyperbolic curve of type (g, r) over K. Write $\varepsilon := r$ if $X \cong \mathbb{P}^1_K \setminus \{0, 1, \infty\}$ (hence $(g, r) = (0, 3)$) or $(g, r) = (1, 1)$, and write $\varepsilon := 0$ if otherwise. For each subset $I' \subset I$ such that $0 \leq \sharp I' \leq n$, let us define the *generalized projection morphism* $p_{I'} : X_n \to X_{n-\sharp I'}$ as follows:

(a) If $X \cong \mathbb{P}^1_K \setminus \{0, 1, \infty\}$ (e.g., $(g, r) = (0, 3)$ and K is algebraically closed), then there is a natural isomorphism $X_n \xrightarrow{\sim} (\mathcal{M}_{0,n+3})_K$, $(x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n, 0, 1, \infty]$ (where $\mathcal{M}_{0,n+3}$ is the moduli space of ordered $(n+3)$-pointed curves of genus zero). We shall define $p_{I'}$ as

$$p_{I'} : X_n \xrightarrow{\sim} (\mathcal{M}_{0,n+3})_K \to (\mathcal{M}_{0,n-\sharp I'+3})_K \xrightarrow{\sim} X_{n-\sharp I'},$$

where $(\mathcal{M}_{0,n+3})_K \to (\mathcal{M}_{0,n-\sharp I'+3})_K$ is the morphism obtained by forgetting the marked points corresponding to the elements of I'.

(b) If $(g, r) = (1, 1)$, then, since the compactification E of X is an elliptic curve over K, E has an addition whose identity element is O, where $\{O\} = E \setminus X$. In this case, there is a natural isomorphism $X_n \xrightarrow{\sim} E_{n+1}/E$, $(x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n, O]$, where E_{n+1} is the "$(n+1)$-st configuration space" of E, and the action of E on E_{n+1} is the diagonal translation determined by the addition of E. We shall define $p_{I'}$ as

$$p_{I'} : X_n \xrightarrow{\sim} E_{n+1}/E \to E_{n-\sharp I'+1}/E \xrightarrow{\sim} X_{n-\sharp I'},$$

where $E_{n+1}/E \to E_{n-\sharp I'+1}/E$ is the morphism obtained by forgetting the factors corresponding to I'.

(c) If $X \not\cong \mathbb{P}^1_K \setminus \{0, 1, \infty\}$ and $(g, r) \neq (1, 1)$, then we shall define $p_{I'}$ as the projection obtained by forgetting the factors corresponding to I'.

(In particular, $p_\emptyset = \mathrm{id}_{X_n}$. Moreover, if $\sharp I' = n$, then $p_{I'}$ is the structure morphism $X_n \to X_0 = \mathrm{Spec}\, K$.) If $p_{I'}$ coincides with a projection from X_n to $X_{n-\sharp I'}$ obtained by forgetting some $\sharp I'$ factors (i.e., $I' \subset \{1, \ldots, n\}$ or $\sharp I' = n$), then we shall also refer to $p_{I'}$ as a *projection morphism*. We shall refer to $\sharp I'$ (resp. $n - \sharp I'$) as the *length* (resp. *co-length*) of $p_{I'}$.

Definition 1.7. Let K be an algebraically closed field of characteristic zero, X a hyperbolic curve of type (g, r) over K, and n a positive integer.

(i) Let Σ be a set of prime numbers such that $\sharp\Sigma = 1$ or $\Sigma = \mathfrak{Primes}$. Then we shall refer to the kernel $N \subset \Pi_n^\Sigma$ of the natural (outer) surjection $\Pi_n^\Sigma \twoheadrightarrow \Pi_m^\Sigma$ (cf. e.g., [NT] (2.4.2)) induced by a generalized projection morphism p as a *generalized fiber subgroup* of Π_n^Σ. We shall refer to the length (resp. co-length) of p as the *length* (resp. *co-length*) of $N \subset \Pi_n^\Sigma$. If p is a projection morphism, then we shall also refer to N as a *fiber subgroup* of Π_n^Σ. If $K = \mathbb{C}$, then we shall define (generalized) fiber subgroups of Π_n similarly.

(ii) Let m be a nonnegative integer and Σ a set of prime numbers such that $\sharp\Sigma = 1$ or $\Sigma = \mathfrak{Primes}$. Then we shall write $\mathrm{GFS}_m(\Pi_n^\Sigma)$ for the set of generalized fiber subgroups of co-length m of Π_n^Σ and $\mathrm{FS}_m(\Pi_n^\Sigma)$ for the set of fiber subgroups of co-length m of Π_n^Σ. (If $m > n$, then, since there is no generalized projection of co-length m, these sets are empty.) If $K = \mathbb{C}$, then we shall define $\mathrm{GFS}_m(\Pi_n)$ and $\mathrm{FS}_m(\Pi_n)$ similarly.

(iii) Let l be a prime number. Then we shall refer to the kernel $\mathfrak{i} \subset \mathrm{Gr}(\Pi_n^l(X))$ of the natural surjection $\mathrm{Gr}(\Pi_n^l(X)) \twoheadrightarrow \mathrm{Gr}(\Pi_m^l(X))$ induced by a generalized projection morphism p as a *generalized fiber ideal* of $\mathrm{Gr}(\Pi_n^l(X))$. We shall refer to the length (resp. co-length) of p as the *length* (resp. *co-length*) of $\mathfrak{i} \subset \mathrm{Gr}(\Pi_n^l(X))$. If p is a projection morphism, then we shall also refer to $\mathfrak{i}$ as a *fiber ideal* of $\mathrm{Gr}(\Pi_n^l(X))$. If $K = \mathbb{C}$, then, for an integral domain R, we shall define (generalized) fiber ideals of $\mathrm{Gr}_R(\Pi_n)$ similarly.

(iv) Let l be a prime number and m a nonnegative integer. Then we shall write $\mathrm{GFI}_m(\mathrm{Gr}(\Pi_n^l))$ for the set of generalized fiber ideals of co-length m of $\mathrm{Gr}(\Pi_n^l)$ and $\mathrm{FI}_m(\mathrm{Gr}(\Pi_n^l))$ for the set of fiber ideals of co-length m of $\mathrm{Gr}(\Pi_n^l)$. If $K = \mathbb{C}$, then, for an integral domain R, we shall define $\mathrm{GFI}_m(\mathrm{Gr}_R(\Pi_n))$ and $\mathrm{FI}_m(\mathrm{Gr}_R(\Pi_n))$ similarly.

Remark 1.7.1.

(i) The notions of a "generalized projection morphism" (if K is an algebraically closed field of characteristic zero) coincides with that of [HMM] Definition 2.1(i). Moreover, the notion of a "generalized fiber subgroup" coincides with that of [HMM] Definition 2.1(ii) and [Hi] Definition 9.1 (cf. [NT] Theorem D).

(ii) The isomorphism appearing in Remark 1.4.1(i) determines bijections $\mathrm{GFS}_m(\Pi_n^\Sigma) \stackrel{1:1}{\to} \mathrm{GFS}_m(\Pi_n)$, $\mathrm{FS}_m(\Pi_n^\Sigma) \stackrel{1:1}{\to} \mathrm{FS}_m(\Pi_n)$. Moreover, the isomorphism of Remark 1.5.1 determines bijections $\mathrm{GFI}_m(\mathrm{Gr}(\Pi_n^l)) \stackrel{1:1}{\to} \mathrm{GFI}_m(\mathrm{Gr}_{\mathbb{Z}_l}(\Pi_n))$, $\mathrm{FI}_m(\mathrm{Gr}(\Pi_n^l)) \stackrel{1:1}{\to} \mathrm{FI}_m(\mathrm{Gr}_{\mathbb{Z}_l}(\Pi_n))$.

Proposition 1.8. *Let (m, n) be a pair of positive integers such that $m < n$, X a hyperbolic curve of type (g, r) over an algebraically closed field K of characteristic zero, $p : X_n \to X_m$ a generalized projection of co-length m, and $\overline{x} \to X_m$ a geometric point. Then the geometric fiber $X_n \times_{X_m} \overline{x}$ is the $(n-m)$-th configuration space of the hyperbolic curve $X_{m+1} \times_{X_m} \overline{x}$ of type $(g, r+m)$. Moreover, if we write $N \subset \Pi_n^\Sigma(X)$ (resp. $\mathfrak{i} \subset \mathrm{Gr}(\Pi_n^l(X))$) for the generalized fiber subgroup (resp. generalized fiber ideal) corresponding to p, then $N \cong \Pi_{n-m}^\Sigma(X_{m+1} \times_{X_m} \overline{x})$ (resp. $\mathfrak{i} \cong \mathrm{Gr}(\Pi_{n-m}^l(X_{m+1} \times_{X_m} \overline{x}))$ as graded Lie algebras over $\mathbb{Z}_l$). If $K = \mathbb{C}$ and $\overline{x} \to X_m$ is a $\mathbb{C}$-valued point, then similar holds for $N \subset \Pi_n(X)$ (resp. $\mathfrak{i} \subset \mathrm{Gr}_R(\Pi_n(X))$ for an integral domain R).*

Proof. See e.g. [NT] (2.4.1), (2.6.1), [Ho2] Proposition 2.4(i), [MT] Proposition 2.2. □

Remark 1.8.1. For $N \in \mathrm{GFS}_m(\Pi_n^\Sigma(X))$ (resp. $\mathfrak{i} \in \mathrm{GFI}_m(\mathrm{Gr}(\Pi_n^l(X)))$), in light of the above identification, we can define $\mathrm{GFS}_{m'}(N) = \mathrm{GFS}_{m'}(\Pi_{n-m}^\Sigma(X_{m+1} \times_{X_m} \overline{x}))$ (resp. $\mathrm{GFI}_{m'}(\mathfrak{i}) = \mathrm{GFI}_{m'}(\mathrm{Gr}(\Pi_{n-m}^l(X_{m+1} \times_{X_m} \overline{x})))$). If $N' \in \mathrm{GFS}_{m'}(N)$ (resp. $\mathfrak{i}' \in \mathrm{GFI}_{m'}(\mathfrak{i})$), then $N' \in \mathrm{GFS}_{m+m'}(\Pi_n^\Sigma(X))$ (resp. $\mathfrak{i}' \in \mathrm{GFI}_{m+m'}(\mathrm{Gr}(\Pi_n^l(X)))$).

In particular, N' (resp. $\mathfrak{i}'$) is a normal subgroup of $\Pi_n^{\Sigma}(X)$ (resp. a Lie ideal of $\mathrm{Gr}(\Pi_n^l(X))$ over $\mathbb{Z}_l$). If $K = \mathbb{C}$, then similar holds for $N \in \mathrm{GFS}_m(\Pi_n(X))$ (resp. $\mathfrak{i} \in \mathrm{GFI}_m(\mathrm{Gr}_R(\Pi_n(X)))$ for an integral domain R).

Corollary 1.9. *A configuration space group (resp. discrete configuration space group) is topologically finitely generated (resp. finitely generated).*

Proof. This follows from Proposition 1.8 and Remark 1.3.1(i). □

Definition 1.10. Let K be an algebraically closed field of characteristic zero, X a hyperbolic curve of type (g, r) over K, and n a positive integer.

(i) Suppose that $g = 0$ (resp. $g = 1$). Then there is a hyperbolic curve Y of type $(0, 3)$ (resp. (1,1)) such that X is an open subscheme of Y. We shall say that a morphism $p : X_n \to Y$ is an *exceptional morphism* if p is a composite of an immersion $X_n \hookrightarrow Y_n$ determined by the open immersion $X \hookrightarrow Y$ and a generalized projection morphism $Y_n \to Y$ of co-length one which is not a projection morphism.

(ii) Let Σ be a set of prime numbers such that $\sharp\Sigma = 1$ or $\Sigma = \mathfrak{Primes}$. Then we shall refer to the kernel $N \subset \Pi_n^{\Sigma} = \Pi_n^{\Sigma}(X)$ of the (outer) surjection $\Pi_n^{\Sigma}(X) \twoheadrightarrow \Pi_1^{\Sigma}(Y) \cong F_2^{\Sigma}$ (cf. [Ho2] Lemma 1.2) induced by an exceptional morphism $p : X_n \to Y$ as an *exceptional subgroup* of Π_n^{Σ}. we shall write $\mathrm{ES}(\Pi_n^{\Sigma})$ for the set of exceptional subgroups of Π_n^{Σ}. If $K = \mathbb{C}$, then we shall define an exceptional subgroup of Π_n similarly, and write $\mathrm{ES}(\Pi_n)$ for the set of exceptional subgroups of Π_n.

(iii) Let l be a prime number. Then we shall refer to the kernel $\mathfrak{i} \subset \mathrm{Gr}(\Pi_n^l(X))$ of the natural surjection $\mathrm{Gr}(\Pi_n^l(X)) \twoheadrightarrow \mathrm{Gr}(\Pi_1^l(Y)) \cong (\mathfrak{L}_2)_{\mathbb{Z}_l}$ induced by an exceptional morphism $p : X_n \to Y$ as an *exceptional ideal* of $\mathrm{Gr}(\Pi_n^l(X))$. We shall write $\mathrm{EI}(\mathrm{Gr}(\Pi_n^l))$ for the set of exceptional ideals of $\mathrm{Gr}(\Pi_n^l)$. If $K = \mathbb{C}$, then, for an integral domain R, we shall define an exceptional ideal of $\mathrm{Gr}_R(\Pi_n)$ similarly, and write $\mathrm{EI}(\mathrm{Gr}_R(\Pi_n))$ for the set of exceptional subgroups of $\mathrm{Gr}_R(\Pi_n)$.

Remark 1.10.1.

(i) By definition, $\mathrm{ES}(\Pi_n^l)$ and $\mathrm{EI}(\mathrm{Gr}(\Pi_n^l))$ are empty if $g \geq 2$. If $(g, r) \in \{(0, 3), (1, 1)\}$, then exceptional morphisms are nothing but generalized projection morphisms of co-length one which are not projection morphisms.

(ii) The isomorphism appearing in Remark 1.4.1(i) determines a bijection $\mathrm{ES}(\Pi_n^{\Sigma}) \overset{1:1}{\to} \mathrm{ES}(\Pi_n)$. Moreover, the isomorphism of Remark 1.5.1 determines a bijection $\mathrm{EI}(\mathrm{Gr}(\Pi_n^l)) \overset{1:1}{\to} \mathrm{EI}(\mathrm{Gr}_{\mathbb{Z}_l}(\Pi_n))$.

Throughout this paper, i (resp. j, k, h) is an integer such that $1 \leq i \leq g$ (resp. $1 \leq j \leq r$, $1 \leq k \leq n$, $1 \leq h \leq n$) unless otherwise specified. Moreover, we apply the same rule to i', i_1, and so on.

Proposition 1.11. *The graded Lie algebra* $\mathrm{Gr}_R(\Pi_{n,g,r})$ *has the following presentation:*

generators: $X_i^{(k)}$, $Y_i^{(k)}$, $Z_j^{(k)}$, $W_h^{(k)}$.

relations:

$$\sum_{i=1}^{g}[X_i^{(k)}, Y_i^{(k)}] + \sum_{j=1}^{r} Z_j^{(k)} + \sum_{h=1}^{n} W_h^{(k)} = 0, \tag{R1}$$

$$W_k^{(k)} = 0, \tag{R2}$$

$$W_h^{(k)} = W_k^{(h)}, \tag{R3}$$

$$[X_i^{(k)}, X_{i'}^{(k')}] = [Y_i^{(k)}, Y_{i'}^{(k')}] = 0 \ (k \neq k'), \tag{R4}$$

$$[X_i^{(k)}, Y_{i'}^{(k')}] = 0 \ (i \neq i', k \neq k'), \tag{R5}$$

$$[X_i^{(k)}, Y_i^{(k')}] = W_k^{(k')} \ (k \neq k'), \tag{R6}$$

$$[X_i^{(k)}, Z_j^{(k')}] = [Y_i^{(k)}, Z_j^{(k')}] = 0 \ (k \neq k'), \tag{R7}$$

$$[Z_j^{(k)}, Z_{j'}^{(k')}] = 0 \ (j \neq j', k \neq k'), \tag{R8}$$

$$[X_i^{(k)}, W_h^{(k')}] = [Y_i^{(k)}, W_h^{(k')}] = [Z_j^{(k)}, W_h^{(k')}] = 0 \ (k \notin \{k', h\}), \tag{R9}$$

$$[W_h^{(k)}, W_{h'}^{(k')}] = 0 \ (\{k, h\} \cap \{k', h'\} = \emptyset). \tag{R10}$$

Here, $X_i^{(k)}, Y_i^{(k)} \in \mathrm{Gr}^1(\Pi_{n,g,r})$*,* $Z_j^{(k)}, W_h^{(k)} \in \mathrm{Gr}^2(\Pi_{n,g,r})$*. Moreover, (by a suitable choice of generators, we may assume that,) if we write* $\Phi_k := \langle X_i^{(k')}, Y_i^{(k')}, Z_j^{(k')}, W_h^{(k')} \mid k' \neq k\rangle_R$*, then* Φ_k *is the fiber ideal of co-length one determined by the projection morphism* $X_n \to X_1$ *corresponding to* $\{1, \ldots, k-1, k+1, \ldots, n\}$ *in the notation of Definition 1.6. In particular, if* $r > 0$*, then* $\mathrm{Gr}_R(\Pi_1) = \mathrm{Gr}_R(\Pi_{1,g,r})$ *is isomorphic to* $(\mathfrak{L}_{2g+r-1})_R$ *as an abstract Lie algebra over R. Moreover, if* $r = 0$*, then* $\mathrm{Gr}_R(\Pi_1)$ *is isomorphic to the Lie algebra over R with* $2g$ *generators* $X_1, \ldots, X_g, Y_1, \ldots, Y_g$ *and one relation* $\sum_{i=1}^{g}[X_i, Y_i] = 0$ *as an abstract Lie algebra over R.*

Proof. See [NT] (3.2). (Note that $(g, n, r, X_i^{(k)}, Y_i^{(k)}, Z_j^{(k)}, W_h^{(k)})$ corresponds to "$(g, r, n, X_i^{(k)}, X_{g+i}^{(k)}, Z_j^{(k)}, Z_{h+r}^{(k)})$" in [NT] (3.2).) □

Hereinafter, we fix generators as in Proposition 1.11. We shall write $\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n)) := \{X_i^{(k)},\ Y_i^{(k)},\ Z_j^{(k)},\ W_h^{(k)} \mid k \neq h\} \subset \mathrm{Gr}_R(\Pi_n)$.

Proposition 1.12. *The following equalities hold:*

$$[Z_j^{(k)}, Z_j^{(h)} + W_k^{(h)}] = 0, \tag{R11}$$

$$[W_h^{(k)}, W_h^{(k')} + W_k^{(k')}] = 0. \tag{R12}$$

Proof. First, we verify equality (R11). If $h = k$, then (R11) follows from (R2). If $h \neq k$, then by (R1), we obtain that

$$[Z_j^{(k)}, Z_j^{(h)} + W_k^{(h)}] = -\sum_{i=1}^{g}[Z_j^{(k)}, [X_i^{(h)}, Y_i^{(h)}]] - \sum_{j' \neq j}[Z_j^{(k)}, Z_{j'}^{(h)}] - \sum_{k' \neq k}[Z_j^{(k)}, W_{k'}^{(h)}].$$

Moreover, it follows from (R7)–(R9) and (R2) that

$$\sum_{i=1}^{g}[Z_j^{(k)},[X_i^{(h)},Y_i^{(h)}]]=\sum_{i=1}^{g}(-[X_i^{(h)},[Y_i^{(h)},Z_j^{(k)}]]-[Y_i^{(h)},[Z_j^{(k)},X_i^{(h)}]])=0,$$
$$\sum_{j'\neq j}[Z_j^{(k)},Z_{j'}^{(h)}]=0,$$
$$\sum_{k'\neq k}[Z_j^{(k)},W_{k'}^{(h)}]=0.$$

Thus, we conclude that $[Z_j^{(k)},Z_j^{(h)}+W_k^{(h)}]=0$. This completes the proof of equality (R11).

Next, we verify equality (R12). If $h=k$, then (R12) follows from (R2). If $h\neq k$ and $k'\in\{h,k\}$, then (R12) follows from (R2) and (R3). If $h\neq k$ and $k'\notin\{h,k\}$, then by (R1), we obtain that

$$[W_h^{(k)},W_h^{(k')}+W_k^{(k')}]=-\sum_{i=1}^{g}[W_h^{(k)},[X_i^{(k')},Y_i^{(k')}]]-\sum_{j=1}^{r}[W_h^{(k)},Z_j^{(k')}]-\sum_{h'\notin\{h,k\}}[W_h^{(k)},W_{h'}^{(k')}].$$

Thus, it follows from an argument similar to the above argument, together with (R9) and (R10), that $[W_h^{(k)},W_h^{(k')}+W_k^{(k')}]=0$. This completes the proof of equality (R12), hence also of Proposition 1.12. □

Remark 1.12.1. By using (R3), equality (R11) also follows directly from equality (A6) of [NT].

Remark 1.12.2. If $(g,r)=(0,3)$, then, by writing $W_h^{(k)}=W_k^{(h)}$ $(1\leq k\leq h\leq n+3,$ $h>n)$ as follows, we obtain an $(n+3)$-symmetric presentation of $\mathrm{Gr}_R(\Pi_n)$, whose relations are (R1)–(R3), (R10) for $1\leq h,k\leq n+3$ (cf. [Ih] Proposition 3.2.1):

- If $h=k$, then $W_h^{(k)}=W_k^{(h)}:=0$.
- If $k\leq n$, then $W_h^{(k)}=W_k^{(h)}:=Z_{h-n}^{(k)}$.
- If $n<k<h$, then let $s\in\{1,2,3\}$ be such that $\{k-n,h-n,s\}=\{1,2,3\}$. Write $W_h^{(k)}=W_k^{(h)}:=\sum_{t=1}^{n}Z_s^{(t)}+\sum_{1\leq u<v\leq n}W_u^{(v)}$.

If $(g,r)=(1,1)$, then, by writing $X_1^{(n+1)}:=-\sum_{m=1}^{n}X_1^{(m)}$, $Y_1^{(n+1)}:=-\sum_{m=1}^{n}Y_1^{(m)}$, $W_h^{(n+1)}=W_{n+1}^{(h)}:=\begin{cases}Z_1^{(h)} & (1\leq h\leq n)\\ 0 & (h=n+1)\end{cases}$, we obtain an $(n+1)$-symmetric presentation of $\mathrm{Gr}_R(\Pi_n)$, whose relations are $\sum_{k=1}^{n+1}X_1^{(k)}=\sum_{k=1}^{n+1}Y_1^{(k)}=0$ and (R1)–(R10) for $1\leq k,h\leq n+1$.

Lemma 1.13. *In the notation of Definition 1.6, the generalized projections correspond to distinct sets $I,I'\subset\{1,\dots,n+\varepsilon\}$ such that $\sharp I,\sharp I'\leq n-1$ determine distinct generalized fiber ideals.*

Proof. In light of Proposition 1.11, Remark 1.12.2, and [NT] Theorem D, by applying Proposition 1.8 inductively, it holds that the generalized fiber ideal corresponding to $I\subset\{1,\dots,n+\varepsilon\}$ coincides with $\langle X_i^{(k')},Y_i^{(k')},Z_j^{(k')},W_h^{(k')}\mid 1\leq h,k'\leq n+\varepsilon,\ k'\notin I\rangle_R$. Lemma 1.13 immediately follows from this fact. □

2. Surface algebras

In the present §2, we study surface algebras. Let R be an integral domain. Write F for the field of fractions of R.

Lemma 2.1. *Let $\mathfrak{g}$ be a finitely generated $\mathbb{Z}_{>0}$-graded Lie algebra over R. Then $\mathfrak{g}$ is hopfian as an abstract Lie algebra over R, i.e., any surjective endomorphism $\mathfrak{g} \twoheadrightarrow \mathfrak{g}$ (not necessarily graded) over R is an isomorphism.*

Proof. Let $\varphi : \mathfrak{g} \twoheadrightarrow \mathfrak{g}$ be a surjective endomorphism over R. Since $\mathfrak{g}$ is finitely generated, for each positive integer m, the R-module $\mathfrak{g}/\mathfrak{g}[m]$ is finitely generated. For each positive integer m, φ determines a surjective endomorphism of R-module $\mathfrak{g}/\mathfrak{g}[m] \twoheadrightarrow \mathfrak{g}/\mathfrak{g}[m]$, which is necessarily an isomorphism (cf. [Mat] Theorem 2.4). Thus, $\ker\varphi \subset \bigcap_{m\geq 1}\mathfrak{g}[m]$. On the other hand, since $\mathfrak{g}$ is $\mathbb{Z}_{>0}$-graded, it holds that $\bigcap_{m\geq 1}\mathfrak{g}[m] = \{0\}$, which implies that φ is an isomorphism. This completes the proof of Lemma 2.1. □

Theorem 2.2 (cf. [Sh1] (see also [Sh2]))**.** *A Lie subalgebra of a free Lie algebra over a field is a free Lie algebra.*

Lemma 2.3. *Let A be a principal ideal domain and m an integer such that $m \geq 2$. Write $\mathfrak{L}$ for the free Lie algebra over A which is freely generated by $\{X_1, \ldots, X_m\}$. We give $\mathfrak{L}$ the $\mathbb{Z}_{>0}$-grading induced by setting $\deg X_i = 1$ for each $1 \leq i \leq m$. Let $Z \in \langle X_2, \ldots, X_m\rangle_A \subset \mathfrak{L}$ be a homogeneous element of degree 2. Write $\mathfrak{g} := \mathfrak{L}/\langle [X_1, X_2] - Z\rangle_{A,\mathfrak{L}}$, and for each integer i such that $1 \leq i \leq m$, write x_i for the image of $X_i \in \mathfrak{L}$ by the natural surjection $\mathfrak{L} \twoheadrightarrow \mathfrak{g}$. Moreover, write $\mathfrak{i} := \langle x_2, \ldots, x_m\rangle_{A,\mathfrak{g}}$.*

For each pair of integers (i, d) such that $i \geq 2$ and $d \geq 1$, let us define $x_{i,d} \in \mathfrak{i}$ as follows:

$$x_{i,1} := x_i,\ \ x_{i,d} := [x_1, x_{i,d-1}]\ (d \geq 2).$$

Then $\mathfrak{g}$ is a free A-module. Moreover, $\mathfrak{i}$ is a free Lie algebra over A which is freely generated by $\{x_{i,d} \mid i \geq 3, d \geq 1 \text{ or } (i, d) = (2, 1)\}$.

Proof. For each positive integer d, write $\mathfrak{g}^d$ for the set of all homogeneous elements of degree d in $\mathfrak{g}$ (whose grading is induced by that of $\mathfrak{L}$), and $\mathfrak{i}^d := \mathfrak{i} \cap \mathfrak{g}^d$. Then it is clear that $\mathfrak{i}^1 = x_2A + \cdots + x_mA \subset \mathfrak{g}^1 = x_1A + \cdots + x_mA$, $\mathfrak{i}^d = \mathfrak{g}^d$ $(d \geq 2)$.

First, we verify that, if we write $\mathfrak{j} := \langle x_{i,d} \mid i \geq 3, d \geq 1 \text{ or } (i, d) = (2, 1)\rangle_A \subset \mathfrak{g}$, then it holds that $\mathfrak{i} = \mathfrak{j}$. Since $x_i = x_{i,1} \in \mathfrak{j}$ $(i \geq 2)$, it suffices to verify that, if $y_1 \in \mathfrak{g}$, $y_2 \in \mathfrak{j}$, then $[y_1, y_2] \in \mathfrak{j}$. We may assume that y_1 is a Lie monomial with respect to $x_1, x_2, \ldots, x_m$ and y_2 is a Lie monomial with respect to $x_{i,d}$ $(i \geq 3, d \geq 1$ or $(i, d) = (2, 1))$. We verify $[y_1, y_2] \in \mathfrak{j}$ by induction on $\deg y_1$. If $\deg y_1 = 1$, then y_1 is a scalar multiple of one of $x_1, x_2, \ldots, x_m$. If y_1 is not a scalar multiple of x_1, then it is clear that $[y_1, y_2] \in \mathfrak{j}$. Let us consider the case y_1 is a scalar multiple of x_1. Write z for the image of $Z \in \langle X_2, \ldots, X_m\rangle_A \subset \mathfrak{L}$ by the natural surjection $\mathfrak{L} \twoheadrightarrow \mathfrak{g}$. Then, since $Z \in \langle X_2, \ldots, X_m\rangle_A$, it holds that $z \in \langle x_2, \ldots, x_m\rangle_A (\subset \mathfrak{j})$. Moreover, $[x_1, x_2] = z$. Thus, since $x_{i,d} = [x_1, x_{i,d-1}]$, for any $y \in \{x_{i,d} \mid i \geq 3, d \geq 1 \text{ or } (i, d) = (2, 1)\}$, $[x_1, y] \in \mathfrak{j}$, which implies that $[y_1, y_2] \in \mathfrak{j}$.

Now suppose that $\deg y_1 \geq 2$, and that the induction hypothesis is in force. Then there exist Lie monomials $y_3, y_4 \in \mathfrak{g}$ (with respect to $x_1, x_2, \ldots, x_m$) such that $y_1 = [y_3, y_4]$. It holds that $[y_1, y_2] = [[y_3, y_4], y_2] = -[[y_2, y_3], y_4] - [[y_4, y_2], y_3]$. Now it follows from the induction hypothesis that $[y_2, y_3], [y_4, y_2] \in \mathfrak{j}$. By applying the induction hypothesis once again, $[[y_2, y_3], y_4], [[y_4, y_2], y_3] \in \mathfrak{j}$. Thus, $[y_1, y_2] \in \mathfrak{j}$. This completes the proof of the equality $\mathfrak{i} = \mathfrak{j}$.

Let $a_{i,d}$ ($i \geq 3, d \geq 1$ or $(i, d) = (2, 1)$) be variables. Write $\mathfrak{F}$ for the free Lie algebra over A freely generated by $\{a_{i,d} \mid i \geq 3, d \geq 1 \text{ or } (i, d) = (2, 1)\}$. We give $\mathfrak{F}$ the $\mathbb{Z}_{>0}$-grading induced by setting $\deg a_{i,d} = d$. Then there is a surjective homomorphism $\mathfrak{F} \twoheadrightarrow \mathfrak{i}$ of graded Lie algebra over A, which sends $a_{i,d}$ to $x_{i,d}$. We verify that this surjection is an isomorphism.

For each positive integer d, write $\mathfrak{F}^d$ for the set of all homogeneous elements of degree d in $\mathfrak{F}$, and $f_d := \operatorname{rank}_A \mathfrak{F}^d$. Then, by substituting $x_1 = \cdots = x_{m-1} = t, x_m = \cdots = x_{2m-3} = t^2, x_{2m-2} = \cdots = x_{3m-5} = t^3, \ldots$ to the equation appearing in [Wi] P.156, we obtain that $\prod_{d\geq 1}(1-t^d)^{f_d} = 1-(m-1)t-\sum_{d\geq 2}(m-2)t^d \in \mathbb{Z}[[t]]$. (Note that only finite variables appear in the equation of [Wi] P.156, but since $\mathfrak{F}^d$ is finitely generated, to calculate $\prod_{d\geq 1}(1 - t^d)^{f_d}$, first we consider $\langle \bigoplus_{d\leq t} \mathfrak{F}^d\rangle_A$ for $t \geq 1$ and then take the limit.)

On the other hand, it holds that $[X_1, X_2] - Y \notin \mathfrak{m}\mathfrak{L}$ for each maximal ideal $\mathfrak{m}$ of A. Thus, since A is a principal ideal domain, it follows from (the proof of) [La] Proposition 4 that $\mathfrak{g}$ is a free A-module, and if we write $e_d := \operatorname{rank}_A \mathfrak{g}^d$, then it holds that $\prod_{d\geq 1}(1 - t^d)^{e_d} = 1 - mt + t^2 \in \mathbb{Z}[[t]]$. Since $(1 - t)(1 - (m - 1)t - \sum_{d\geq 2}(m - 2)t^d) = 1 - mt + t^2$, we obtain that $f_1 = e_1 - 1$, $f_d = e_d$ $(d \geq 2)$, which implies that $\operatorname{rank}_A \mathfrak{F}^d = \operatorname{rank}_A \mathfrak{i}^d$ for each positive integer d. Thus, it follows from [Mat] Theorem 2.4 that, for each positive integer t, the surjective homomorphism $\bigoplus_{d\leq t} \mathfrak{F}^d \twoheadrightarrow \bigoplus_{d\leq t} \mathfrak{i}^d$ induced by $\mathfrak{F} \twoheadrightarrow \mathfrak{i}$ is an isomorphism. This implies that the surjection $\mathfrak{F} \twoheadrightarrow \mathfrak{i}$ is an isomorphism. Since $\mathfrak{F}$ is freely generated by $\{a_{i,d} \mid i \geq 3, d \geq 1 \text{ or } (i, d) = (2, 1)\}$, $\mathfrak{i}$ is freely generated by $\{x_{i,d} \mid i \geq 3, d \geq 1 \text{ or } (i, d) = (2, 1)\}$. This completes the proof of Lemma 2.3. □

Theorem 2.4. *Let X be a hyperbolic curve over $\mathbb{C}$. Then a proper graded Lie subalgebra of a graded Lie algebra $\mathrm{Gr}_F(\Pi_1(X))$ over F is a free Lie algebra.*

Proof. Write (g, r) for the type of $X/\mathbb{C}$. Let $\mathfrak{h}$ be a proper graded Lie subalgebra of $\mathrm{Gr}_F(\Pi_1)$. If $r \geq 1$, then, since $\mathrm{Gr}_F(\Pi_1)$ is a free Lie algebra over F, it follows from Theorem 2.2 that $\mathfrak{h}$ is free. Thus, we may assume that $r = 0$. Then, since $\mathrm{Gr}_F(\Pi_1)$ is generated by $\mathrm{Gr}^1_F(\Pi_1)$, it holds that $\mathfrak{h} \cap \mathrm{Gr}^1_F(\Pi_1) \subsetneq \mathrm{Gr}^1_F(\Pi_1)$.

Now we consider the symplectic form ω on $\mathrm{Gr}^1_F(\Pi_1)$ determined by

$$\omega(X^{(1)}_i, X^{(1)}_{i'}) = \omega(Y^{(1)}_i, Y^{(1)}_{i'}) = 0,$$
$$\omega(X^{(1)}_i, Y^{(1)}_{i'}) = \begin{cases} 1 & (i = i') \\ 0 & (i \neq i'). \end{cases}$$

Since $\mathfrak{h} \cap \mathrm{Gr}^1_F(\Pi_1) \subsetneq \mathrm{Gr}^1_F(\Pi_1)$, there exists a nonzero element v in the orthogonal complement of $\mathfrak{h} \cap \mathrm{Gr}^1_F(\Pi_1)$ in $\mathrm{Gr}^1_F(\Pi_1)$ with respect to the symplectic form ω. If we write $J := \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}$ and $\mathrm{GSp}(2g, F) := \{M \in \mathrm{M}(2g, F) \mid {}^tMJM = \lambda J \text{ for some } \lambda \in F^\times\}$ (where $\mathrm{M}(2g, F)$ is the set of all $2g \times 2g$ matrices with entries in F), then $\mathrm{GSp}(2g, F)$ acts naturally on $\mathrm{Gr}_F(\Pi_1)$ and preserves ω up to scalar multiple. Moreover, the action of $\mathrm{GSp}(2g, F)$ on $\mathrm{Gr}^1_F(\Pi_1) \setminus \{0\}$ is transitive (cf. [Di] Proposition 1). Thus, we may assume that $v = X^{(1)}_1$. Then $\mathfrak{h} \subset \langle X^{(1)}_2, \ldots, X^{(1)}_g, Y^{(1)}_1, \ldots, Y^{(1)}_g\rangle_{F,\mathrm{Gr}_F(\Pi_1)}$.

Now write $\mathfrak{L}$ for the free Lie algebra over F which is free generated by $2g$ elements $\{X_1, \ldots, X_g, Y_1, \ldots, Y_g\}$, and write $Z = -\sum_{i=2}^{g}[X_i, Y_i]$. Then it follows from Proposition 1.11 that the surjective homomorphism $\mathfrak{L} \twoheadrightarrow \mathrm{Gr}_F^1(\Pi_1)$, $X_i \mapsto X_i^{(1)}$, $Y_i \mapsto Y_i^{(1)}$ induces an isomorphism $\mathfrak{g} := \mathfrak{L}/\langle [X_1, Y_1] - Z\rangle_{F,\mathfrak{L}} \xrightarrow{\sim} \mathrm{Gr}_F^1(\Pi_1)$. Thus, by applying Lemma 2.3, where we take the data "$(A, \{X_1, \ldots, X_m\}, Z)$" to be $(F, \{X_1, Y_1, X_2, \ldots, Y_g\}, Z = -\sum_{i=2}^{g}[X_i, Y_i])$, we conclude that $\langle X_2^{(1)}, \ldots, X_g^{(1)}, Y_1^{(1)}, \ldots, Y_g^{(1)}\rangle_{F,\mathrm{Gr}_F(\Pi_1)}$ is a free Lie algebra over F. Thus, it follows from Theorem 2.2 that $\mathfrak{h}$ is free. This completes the proof of Theorem 2.4. □

Lemma 2.5. *Let $\mathfrak{g}$ be a surface algebra over R.*

(i) *$\mathfrak{g}$ is a free R-module.*
(ii) *$\mathfrak{g}$ is hopfian as an abstract Lie algebra over R.*
(iii) *Suppose that $\mathfrak{g} \not\cong (\mathfrak{L}_2)_R$. Let $a, b \in \mathfrak{g}$. Then it holds that $\langle a, b\rangle_R \subsetneq \mathfrak{g}$.*
(iv) *There exists a surjective homomorphism $\mathfrak{g} \twoheadrightarrow (\mathfrak{L}_2)_R$ over R.*
(v) *Let $a, b \in \mathfrak{g}$. Then $[a, b] = 0$ if and only if a and b are linearly dependent over R (i.e., there exists $(s, t) \in R^2 \setminus \{(0, 0)\}$ such that $sa = tb$), or, equivalently, linearly dependent over F.*

Proof. Assertion (i) follows from the fact that $\mathrm{Gr}(\Pi_{1,g,r})$ is a free $\mathbb{Z}$-module (cf. Lemma 2.3). Assertion (ii) follows from Lemma 2.1. Assertion (iii) follows from the easily verified fact that $\mathrm{rank}_R(\mathfrak{g}/\mathfrak{g}[2]) \geq 3$ if $\mathfrak{g} \not\cong (\mathfrak{L}_2)_R$. Next, we verify assertion (iv). If $\mathfrak{g} \cong \mathrm{Gr}_R(\Pi_{1,g,r})$ and $r > 0$, then, since $\mathfrak{g} \cong (\mathfrak{L}_{2g+r-1})_R$, assertion (iv) is immediate. If $\mathfrak{g} \cong \mathrm{Gr}_R(\Pi_{1,g,0})$ $(g \geq 2)$, then we obtain a surjective homomorphism $\mathfrak{g} \cong \mathrm{Gr}_R(\Pi_{1,g,0}) \twoheadrightarrow (\mathfrak{L}_2)_R$ by $X_1^{(1)} \mapsto \alpha$, $X_2^{(1)} \mapsto \beta$, $X_i^{(1)} \mapsto 0$ $(i \geq 3)$, $Y_i^{(1)} \mapsto 0$, where $(\mathfrak{L}_2)_R$ is freely generated by α and β. This completes the proof of assertion (iv). Finally, we verify assertion (v). We may assume that $R = F$. Let us choose a CS-envelope for $\mathfrak{g}$ and we treat $\mathfrak{g}$ as a graded Lie algebra over F whose grading is induced by the grading of the fixed CS-envelope.

If a and b are homogeneous, then it follows from (iii) and Theorem 2.4 that $\langle a, b\rangle_F \subset \mathfrak{g}$ is free, which implies that a and b are linearly dependent. In the general case, write

$$a = \sum_{d \geq 1} a_d, \ \ b = \sum_{d \geq 1} b_d,$$

where a_d and b_d are homogeneous elements of degree d. Since the case $a = 0$ or $b = 0$ is clear, we assume that $a, b \neq 0$, and write d_1 (resp. d_2) for the minimum positive integer d such that $a_d \neq 0$ (resp. $b_d \neq 0$). Then it holds that $[a_{d_1}, b_{d_2}] = 0$. Thus, it follows from assertion (iv) in the case a and b are homogeneous, a_{d_1} and b_{d_2} are linearly dependent. In particular, $d_1 = d_2$.

Let $(s, t) \in F^2 \setminus \{(0, 0)\}$ be such that $sa_{d_1} + tb_{d_1} = 0$. Suppose that $sa + tb \neq 0$. Then, since $[sa + tb, b] = s[a, b] + t[b, b] = 0$, it follows from the above argument that the homogeneous component of degree d_1 of $sa + tb$ is nonzero. However, since the homogeneous component of degree d_1 of $sa + tb$ is $sa_{d_1} + tb_{d_1} = 0$, we obtain a contradiction. This completes the proof of assertion (v), hence also of Lemma 2.5. □

Corollary 2.6. *Let X be a proper hyperbolic curve over $\mathbb{C}$. Then $\mathrm{Gr}_R(\Pi_1(X))$ is not a free Lie algebra over R.*

Proof. Write g for the genus of X. It follows from Proposition 1.11 that $\mathrm{rank}_R(\mathfrak{g}/\mathfrak{g}[2]) = 2g$. Thus, if $\mathrm{Gr}_R(\Pi_1(X))$ is a free Lie algebra, then $\mathrm{Gr}_R(\Pi_1(X)) \cong (\mathfrak{L}_{2g})_R$. However, since it follows from Proposition 1.11 that there exists a surjective homomorphism $(\mathfrak{L}_{2g})_R \twoheadrightarrow \mathrm{Gr}_R(\Pi_1(X))$ which is not injective, we obtain a contradiction (cf. Lemma 2.5(ii)). This completes the proof of Corollary 2.6. □

Now, as an application of Lemma 2.5(v), we prove the following Corollary 2.7, which is a result stronger than [St] Proposition 205.

Corollary 2.7. *Let $\mathfrak{g}$ be a surface algebra over R and $\mathfrak{A}$ a nontrivial Lie subalgebra of $\mathfrak{g}$. Suppose that $\mathfrak{A}$ is finitely generated as an R-module. Then it holds that* $\dim_F(\mathfrak{A} \otimes_R F) = 1$.

Proof. For $a = \sum_{d \geq 1} a_d \in \mathfrak{A} \backslash \{0\}$ ($a_d \in \mathrm{Gr}^d_R(\Pi_1)$), write $d_a := \min\{d \geq 1 \mid a_d \neq 0\}$. Since $\mathfrak{A}$ is finitely generated as an R-module, $d := \max_{a \in \mathfrak{A} \backslash \{0\}} d_a < \infty$. Let us choose $a \in \mathfrak{A} \setminus \{0\}$ such that $d_a = d$. Note that, for each $b \in \mathfrak{A}$, if $[a, b] \neq 0$, then $d_{[a,b]} > d_a$. Thus, by our choice of a, it holds that $[a, b] = 0$. It follows from Lemma 2.5 that a and b are linearly dependent over R. Since $b \in \mathfrak{A}$ is arbitrary, we conclude that $\mathfrak{A} \subset aF$. This completes the proof of Corollary 2.7. □

3. Reconstruction of generalized fiber ideals

In the present §3, we give an algorithm for reconstructing the set of generalized fiber ideals of given co-length. Let X be a hyperbolic curve of type (g, r) over $\mathbb{C}$, n a positive integer, and R an integral domain unless otherwise specified. Write F for the field of fractions of R.

Theorem 3.1. *Let $\mathfrak{h}$ be a Lie algebra over R, $\varphi : \mathrm{Gr}_R(\Pi_n) \twoheadrightarrow \mathfrak{h}$ a surjective homomorphism of Lie algebras over R. Suppose that the following conditions are satisfied:*

- *$\mathfrak{h}$ is a free R-module, and* $\mathrm{rank}_R\, \mathfrak{h} \geq \begin{cases} 2 & (g = 0) \\ 3 & (g \geq 1). \end{cases}$
- *For $a, b \in \mathfrak{h}$, if $[a, b] = 0$, then a and b are linearly dependent over R.*
- *For each $\mathfrak{i} \in \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n))$, $\mathfrak{i} \not\subset \ker \varphi$.*

Then one of the following holds:

(1) *It holds that $g = 1$. Moreover, there exist distinct integers k_1, k_2 and elements $\alpha, \beta \in \mathfrak{h}$ such that $\langle \alpha, \beta \rangle_R = \mathfrak{h}$ and that for $A \in \mathrm{Gen}(\mathrm{Gr}_R(\Pi_n))$,*

$$\varphi(A) = \begin{cases} \alpha & (A = X_1^{(k_1)}) \\ -\alpha & (A = X_1^{(k_2)}) \\ -\beta & (A = Y_1^{(k_1)}) \\ \beta & (A = Y_1^{(k_2)}) \\ [\alpha, \beta] & (A = W_{k_1}^{(k_2)}) \\ 0 & (\textit{otherwise}). \end{cases}$$

(2) *It holds that $g = 0$. Moreover, there exist distinct integers j_1, j_2, distinct integers k_1, k_2, and elements $\alpha, \beta \in \mathfrak{h}$ such that $\langle \alpha, \beta \rangle_R = \mathfrak{h}$ and that for*

$A \in \mathrm{Gen}(\mathrm{Gr}_R(\Pi_n))$,

$$\varphi(A) = \begin{cases} \alpha & (A = Z_{j_1}^{(k_1)}, Z_{j_2}^{(k_2)}) \\ \beta & (A = Z_{j_1}^{(k_2)}, Z_{j_2}^{(k_1)}) \\ -\alpha-\beta & (A = W_{k_1}^{(k_2)}) \\ 0 & (\textit{otherwise}). \end{cases}$$

(3) *It holds that* $g = 0$. *Moreover, there exist an integer* j_1*, distinct integers* k_1, k_2, k_3*, and elements* $\alpha, \beta \in \mathfrak{h}$ *such that* $\langle \alpha, \beta \rangle_R = \mathfrak{h}$ *and that for* $A \in \mathrm{Gen}(\mathrm{Gr}_R(\Pi_n))$,

$$\varphi(A) = \begin{cases} \alpha & (A = Z_{j_1}^{(k_1)}, W_{k_2}^{(k_3)}) \\ \beta & (A = Z_{j_1}^{(k_2)}, W_{k_3}^{(k_1)}) \\ -\alpha-\beta & (A = Z_{j_1}^{(k_3)}, W_{k_1}^{(k_2)}) \\ 0 & (\textit{otherwise}). \end{cases}$$

(4) *It holds that* $g = 0$. *Moreover, there exist distinct integers* k_1, k_2, k_3, k_4 *and elements* $\alpha, \beta \in \mathfrak{h}$ *such that* $\langle \alpha, \beta \rangle_R = \mathfrak{h}$ *and that for* $A \in \mathrm{Gen}(\mathrm{Gr}_R(\Pi_n))$,

$$\varphi(A) = \begin{cases} \alpha & (A = W_{k_1}^{(k_2)}, W_{k_3}^{(k_4)}) \\ \beta & (A = W_{k_1}^{(k_3)}, W_{k_2}^{(k_4)}) \\ -\alpha-\beta & (A = W_{k_1}^{(k_4)}, W_{k_2}^{(k_3)}) \\ 0 & (\textit{otherwise}). \end{cases}$$

Proof. First, let us consider the case $g \geq 1$. In this case, relation (R6) implies that there exists $A \in \mathrm{Gen}(\mathrm{Gr}_R(\Pi_n)) \setminus \{W_h^{(k)} \mid k \neq h\}$ such that $\varphi(A) \neq 0$. Write $\alpha := \varphi(A)$. We divide the situation into five cases:

(i) There exist integers i_1, k_1 such that $A = X_{i_1}^{(k_1)}$. Moreover, there exists an integer k_2 such that $k_2 \neq k_1$ and that $\varphi(Y_{i_1}^{(k_2)}) \notin \alpha F (\subset \mathfrak{h} \otimes_R F)$.

(i)′ There exist integers i_1, k_1 such that $A = Y_{i_1}^{(k_1)}$. Moreover, there exists an integer k_2 such that $k_2 \neq k_1$ and that $\varphi(X_{i_1}^{(k_2)}) \notin \alpha F$.

(ii) There exist integers i_1, k_1 such that $A = X_{i_1}^{(k_1)}$. Moreover, for any integer k such that $k \neq k_1$, it holds that $\varphi(Y_{i_1}^{(k)}) \in \alpha F$.

(ii)′ There exist integers i_1, k_1 such that $A = Y_{i_1}^{(k_1)}$. Moreover, for any integer k such that $k \neq k_1$, it holds that $\varphi(X_{i_1}^{(k)}) \in \alpha F$.

(iii) There exist integers j_1, k_1 such that $A = Z_{j_1}^{(k_1)}$.

We verify that (1) holds in the cases (i) or (i)′ and that $\varphi(\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n)) \cap \Phi_{k_1}) \subset \alpha F$ holds in other cases. Note that, by considering the automorphism of $\mathrm{Gr}_R(\Pi_n)$ determined by $X_i^{(k)} \mapsto Y_i^{(k)}$, $Y_i^{(k)} \mapsto -X_i^{(k)}$, $Z_j^{(k)} \mapsto Z_j^{(k)}$, $W_h^{(k)} \mapsto W_h^{(k)}$, cases (i)′, (ii)′ are reduced to (i), (ii), respectively.

First, we consider the case (i). We may assume that $i_1 = k_1 = 1, k_2 = 2$. Write $\beta := \varphi(Y_1^{(2)})$. Then (R4) implies that, for any i and any $k \neq 1$, $[\varphi(X_i^{(k)}), \alpha] = 0$. Thus, by an assumption on $\mathfrak{h}$,

$$\varphi(X_i^{(k)}) \in \alpha F \ (k \neq 1). \tag{i-1}$$

Similarly,

(i-2) $$\varphi(Y_i^{(k)}) \in \beta F \ (k \neq 2).$$

Next, (R5) implies that

(i-3) $$\varphi(X_i^{(k)}) \in \beta F \ (i \neq 1,\ k \neq 2),$$

(i-4) $$\varphi(Y_i^{(k)}) \in \alpha F \ (i, k \neq 1).$$

In particular, by (i-1)–(i-4), for any $i \neq 1$ and any $k \notin \{1, 2\}$, it holds that $\varphi(X_i^{(k)})$, $\varphi(Y_i^{(k)}) \in \alpha F \cap \beta F = \{0\}$, i.e.,

(i-5) $$\varphi(X_i^{(k)}) = \varphi(Y_i^{(k)}) = 0 \ (i \neq 1,\ k \notin \{1, 2\}).$$

Next, (R7) implies that

(i-6) $$\varphi(Z_j^{(k)}) \in \alpha F \ (k \neq 1),$$

(i-7) $$\varphi(Z_j^{(k)}) \in \beta F \ (k \neq 2).$$

In particular,

(i-8) $$\varphi(Z_j^{(k)}) = 0 \ (k \notin \{1, 2\}).$$

Next, (R9) implies that

(i-9) $$\varphi(W_h^{(k)}) \in \alpha F \ (h, k \neq 1),$$

(i-10) $$\varphi(W_h^{(k)}) \in \beta F \ (h, k \neq 2).$$

In particular,

(i-11) $$\varphi(W_h^{(k)}) = 0 \ (h, k \notin \{1, 2\}).$$

Next, (R3) and (R6) imply that

(i-12) $$\varphi(W_1^{(2)}) = \varphi(W_2^{(1)}) = [\alpha, \beta].$$

By (i-1)–(i-12), we obtain that $\varphi(\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n))) \subset \alpha F \cup \beta F \cup \{[\alpha, \beta]\}$. If $[\alpha, \beta] \in \alpha F + \beta F$, then, since φ is surjective and $\mathrm{Gr}_R(\Pi_n)$ is generated by $\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n))$ as a Lie algebra over R, we conclude that $\mathfrak{h} \subset \alpha F + \beta F$, which contradicts our assumption that $\mathrm{rank}_R\, \mathfrak{h} \geq 3$. Thus, it holds that $[\alpha, \beta] \notin \alpha F + \beta F$.

Now it follows from (R1), (R2), together with (i-2), (i-3), (i-7), (i-10), and (i-12), that $[\alpha, \varphi(Y_1^{(1)})] + [\alpha, \beta] \in \beta F$. Since $[\alpha, \beta] \notin \beta F$ and $\varphi(Y_1^{(1)}) \in \beta F$ (cf. (i-2)), we obtain that

(i-13) $$\varphi(Y_1^{(1)}) = -\beta.$$

Similarly, we obtain that

(i-14) $$\varphi(X_1^{(2)}) = -\alpha.$$

By applying the above argument, it follows from (i-14) that

(i-15) $$\varphi(X_i^{(1)}) \in \alpha F.$$

By (i-3) and (i-15), we obtain that

(i-16) $$\varphi(X_i^{(1)}) = 0 \ (i \neq 1).$$

Similarly, we obtain that

$$\varphi(X_i^{(2)}) = \varphi(Y_i^{(1)}) = \varphi(Y_i^{(2)}) = 0 \ (i \neq 1), \tag{i-17}$$

$$\varphi(Z_j^{(1)}) = \varphi(Z_j^{(2)}) = 0, \tag{i-18}$$

$$\varphi(W_k^{(1)}) = \varphi(W_k^{(2)}) = \varphi(W_1^{(k)}) = \varphi(W_2^{(k)}) = 0 \ (k \notin \{1,2\}). \tag{i-19}$$

Next, (R6) and (i-19) imply that $0 = \varphi(W_k^{(2)}) = [\varphi(X_1^{(k)}), \varphi(Y_1^{(2)})]$, which implies that

$$\varphi(X_1^{(k)}) \in \beta F \ \ (k \notin \{1,2\}). \tag{i-20}$$

By (i-1) and (i-20), we obtain that

$$\varphi(X_1^{(k)}) = 0 \ (k \notin \{1,2\}). \tag{i-21}$$

Similarly, we obtain that

$$\varphi(Y_1^{(k)}) = 0 \ (k \notin \{1,2\}). \tag{i-22}$$

Finally, (R6) implies that $0 \neq [\alpha, \beta] = \varphi(W_1^{(2)}) = [\varphi(X_i^{(1)}), \varphi(Y_i^{(2)})]$ for any i, which implies, in light of (i-17), that $g = 1$. Now (i-8), (i-11), (i-12)–(i-14), (i-18), (i-19), (i-21), (i-22), and the surjectivity of φ imply that (1) holds.

Next, we consider the case (ii). We may assume that $i_1 = k_1 = 1$. It follows from our assumption that $\varphi(Y_1^{(k)}) \in \alpha F$ for $k \neq 1$, together with (R4) and (R5), that

$$\varphi(X_i^{(k)}), \varphi(Y_i^{(k)}) \in \alpha F \ \ (k \neq 1). \tag{ii-1}$$

Moreover, (R7) implies that

$$\varphi(Z_j^{(k)}) \in \alpha F \ \ (k \neq 1). \tag{ii-2}$$

Finally, (R6) and (ii-1) imply that

$$\varphi(W_h^{(k)}) = [\varphi(X_1^{(h)}), \varphi(Y_1^{(k)})] = 0 \ (k \neq 1, \ h \neq k). \tag{ii-3}$$

Thus, it holds that $\varphi(\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n)) \cap \Phi_1) \subset \alpha F$.

Next, we consider the case (iii). We may assume that $j_1 = k_1 = 1$. It follows from (R7) that

$$\varphi(X_i^{(k)}), \varphi(Y_i^{(k)}) \in \alpha F \ \ (k \neq 1). \tag{iii-1}$$

Moreover, (R8) implies that

$$\varphi(Z_j^{(k)}) \in \alpha F \ \ (j, k \neq 1). \tag{iii-2}$$

Next, (R6) and (iii-1) imply that

$$\varphi(W_h^{(k)}) = [\varphi(X_1^{(h)}), \varphi(Y_1^{(k)})] = 0 \ (h, k \neq 1, \ h \neq k). \tag{iii-3}$$

In light of (R1), to verify $\varphi(\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n)) \cap \Phi_1) \subset \alpha F$, it suffices to show that for any $k \neq 1$, $\varphi(Z_1^{(k)}) \in \alpha F$ holds. Suppose that there exists $k \neq 1$ such that

$\varphi(Z_1^{(k)}) \notin \alpha F$. We may assume that $k = 2$. Write $\beta := \varphi(Z_1^{(2)})$. Then, by applying the above argument, we obtain that

$$\text{(iii-4)} \qquad \varphi(X_i^{(1)}), \varphi(Y_i^{(1)}) \in \beta F,$$

$$\text{(iii-5)} \qquad \varphi(Z_j^{(1)}) \in \beta F \ (j \neq 1),$$

$$\text{(iii-6)} \qquad \varphi(W_k^{(1)}) = \varphi(W_1^{(k)}) = 0 \ (k \notin \{1, 2\}).$$

Moreover, it follows from (R1)–(R3), together with (iii-1)–(iii-3) and (iii-6), that

$$\text{(iii-7)} \qquad \varphi(Z_1^{(k)}) \in \alpha F \ (k \notin \{1, 2\}),$$

$$\text{(iii-8)} \qquad \varphi(W_1^{(2)}) + \beta = \varphi(W_2^{(1)}) + \beta \in \alpha F.$$

Now (iii-8) implies that $\varphi(W_1^{(2)}) \in (\alpha F + \beta F) \setminus \{0\}$. This implies, in light of (R6), (iii-1) and (iii-4), that $\varphi(W_1^{(2)}) \in [\alpha, \beta]F \cap (\alpha F + \beta F) \setminus \{0\}$. Thus, we conclude that $[\alpha, \beta] \in \alpha F + \beta F$. On the other hand, it follows from (iii-1)–(iii-8) that $\varphi(\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n))) \subset \alpha F + \beta F$. Since $[\alpha, \beta] \in \alpha F + \beta F$, φ is surjective, and $\mathrm{Gr}_R(\Pi_n)$ is generated by $\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n))$ as a Lie algebra over R, we obtain that $\mathfrak{h} \subset \alpha F + \beta F$, which contradicts our assumption that $\mathrm{rank}_R\, \mathfrak{h} \geq 3$. This completes the proof of $\varphi(\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n)) \cap \Phi_{k_1}) \subset \alpha F$ in the case (iii).

Now we verify that (1) holds if $g \geq 1$. Suppose that (1) does not hold. Then there exists an integer k_1 such that, if we write $T := \{X_1^{(k_1)}, \ldots, X_g^{(k_1)}, Y_1^{(k_1)}, \ldots, Y_g^{(k_1)}, Z_1^{(k_1)}, \ldots, Z_r^{(k_1)}\}$, then $A \in T$ and $\varphi(\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n)) \cap \Phi_{k_1}) \subset \alpha F$. Since (we have assumed that) $\mathrm{rank}_R\, \mathfrak{h} \geq 3$, there exists $B \in T$ such that α and $\beta := \varphi(B)$ are linearly independent over R. This implies that $\varphi(\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n)) \cap \Phi_{k_1}) \subset \alpha F \cap \beta F = \{0\}$. However, since Φ_{k_1} is generated by $\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n)) \cap \Phi_{k_1}$ as a Lie algebra over R, we obtain that $\Phi_{k_1} \subset \ker \varphi$, which contradicts our assumption that for each $\mathfrak{i} \in \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n))$, $\mathfrak{i} \not\subset \ker \varphi$. This completes the proof of Theorem 3.1 in the case $g \geq 1$.

In the rest of the proof of Theorem 3.1, we consider the case $g = 0$. We divide the situation into three cases:

- (iv) There exist integers j_1, k_1 such that $\varphi(Z_{j_1}^{(k_1)}) \neq 0$. Moreover, there exists an integer k_2 such that $k_2 \neq k_1$, and that $\varphi(Z_{j_1}^{(k_2)}) \notin \varphi(Z_{j_1}^{(k_1)})F$.
- (v) There exist integers j_1, k_1 such that $\varphi(Z_{j_1}^{(k_1)}) \neq 0$. Moreover, for any integer k such that $k \neq k_1$, it holds that $\varphi(Z_{j_1}^{(k)}) \in \varphi(Z_{j_1}^{(k_1)})F$.
- (vi) For any integers j, k, it holds that $\varphi(Z_j^{(k)}) = 0$.

We verify that (2) or (3) holds in the case (iv), that (4) holds in the case (vi), and that the case (v) does not occur.

First, we consider the case (iv). We may assume that $j_1 = k_1 = 1, k_2 = 2$. Write $\alpha := \varphi(Z_1^{(1)})$, $\beta := \varphi(Z_1^{(2)})$. Then (R8) implies that

$$\text{(iv-1)} \qquad \varphi(Z_j^{(k)}) \in \alpha F \ (j, k \neq 1),$$

$$\text{(iv-2)} \qquad \varphi(Z_j^{(k)}) \in \beta F \ (j \neq 1, \ k \neq 2).$$

In particular,

$$\text{(iv-3)} \qquad \varphi(Z_j^{(k)}) = 0 \ (j \neq 1, \ k \notin \{1, 2\}).$$

Next, (R9) implies that

(iv-4) $$\varphi(W_h^{(k)}) \in \alpha F \ (h, k \neq 1),$$

(iv-5) $$\varphi(W_h^{(k)}) \in \beta F \ (h, k \neq 2).$$

In particular,

(iv-6) $$\varphi(W_h^{(k)}) = 0 \ (h, k \notin \{1, 2\}).$$

Now it follows from (R1), (R2), together with (iv-2), (iv-5), that $\varphi(W_2^{(1)}) + \alpha \in \beta F$. Similarly, we obtain that $\varphi(W_1^{(2)}) + \beta \in \alpha F$. By (R3), we conclude that

(iv-7) $$\varphi(W_1^{(2)}) = \varphi(W_2^{(1)}) = -\alpha - \beta.$$

By (iv-1) and (iv-2), for each $j \neq 1$, there exist $a_j, b_j \in F$ such that $\varphi(Z_j^{(2)}) = \alpha a_j$, $\varphi(Z_j^{(1)}) = \beta b_j$. Moreover, by (iv-4) and (iv-5), for each $h \notin \{1, 2\}$, there exist $a_{r+h}, b_{r+h} \in F$ such that $\varphi(W_h^{(2)}) = \alpha a_{r+h}$, $\varphi(W_h^{(1)}) = \beta b_{r+h}$. Now it follows from (R1), (R2) that

(iv-8) $$\sum_{m \in \{2, \ldots, r, r+3, \ldots, r+n\}} a_m = \sum_{m \in \{2, \ldots, r, r+3, \ldots, r+n\}} b_m = 1.$$

Moreover, by (R8)–(R10), we obtain that

(iv-9) $$a_m b_{m'} = 0 \ (m, m' \in \{2, \ldots, r, r+3, \ldots, r+n\}, \ m \neq m').$$

Thus, there exists $m_1 \in \{2, \ldots, r, r+3, \ldots, r+n\}$ such that $a_{m_1} = b_{m_1} = 1$, $a_m = b_m = 0$ $(m \neq m_1)$. In light of (R1)–(R3), it is clear that (2) (resp. (3)) holds if $m_1 \leq r$ (resp. $m_1 \geq r+3$).

Next, we consider the case (v). We may assume that $j_1 = k_1 = 1$. Write $\alpha := \varphi(Z_1^{(1)})$. Then (R8), together with our assumption, that

(v-1) $$\varphi(Z_j^{(k)}) \in \alpha F \ (k \neq 1).$$

Next, (R9) implies that

(v-2) $$\varphi(W_h^{(k)}) \in \alpha F \ (h, k \neq 1).$$

It follows from (R1) and (R3), together with (v-1) and (v-2), that

(v-3) $$\varphi(W_1^{(k)}) = \varphi(W_k^{(1)}) \in \alpha F \ (k \neq 1).$$

Since φ is surjective and $\mathrm{Gr}_R(\Pi_n)$ is generated by $\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n))$ as a Lie algebra over R, it follows from (v-1)–(v-3), together with our assumption that $\mathrm{rank}_R \, \mathfrak{h} \geq 2$, that there exists an integer $j_2 \neq 1$ such that $\beta := \varphi(Z_{j_2}^{(1)}) \notin \alpha F$. Moreover, by (R1), (R2), and (v-3), there exists an integer $j_3 \notin \{1, j_2\}$ such that $\gamma := \varphi(Z_{j_3}^{(1)}) \notin \alpha F$.

Now (R8) implies that

(v-4) $$\varphi(Z_j^{(k)}) \in \beta F \ (j \neq j_2, k \neq 1),$$

(v-5) $$\varphi(Z_j^{(k)}) \in \gamma F \ (j \neq j_3, k \neq 1).$$

By (v-1), (v-4), and (v-5), we obtain that

(v-6) $$\varphi(Z_j^{(k)}) = 0 \ (k \neq 1).$$

Moreover, (R9) implies that

$$\varphi(W_h^{(k)}) \in \beta F \ (h, k \neq 1). \tag{v-7}$$

By (v-2) and (v-7), we obtain that

$$\varphi(W_h^{(k)}) = 0 \ (h, k \neq 1). \tag{v-8}$$

Finally, it follows from (R1), together with (v-6) and (v-8), that

$$\varphi(W_1^{(k)}) = 0 \ (k \neq 1). \tag{v-9}$$

However, since Φ_1 is generated by $\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n)) \cap \Phi_1$ as a Lie algebra over R, (v-6), (v-8), (v-9) implies that $\Phi_1 \subset \ker\varphi$, which contradicts our assumption that for each $\mathfrak{i} \in \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n))$, $\mathfrak{i} \not\subset \ker\varphi$. Thus, the case (v) does not occur.

Finally, we consider the case (vi). In this case, in light of (R2), there exist integers h_1, k_1 such that $h_1 \neq k_1$, $\varphi(W_{h_1}^{(k_1)}) \neq 0$. We may assume that $h_1 = 1, k_1 = 2$. Write $\alpha := \varphi(W_1^{(2)})$. Then (R10) implies that

$$\varphi(W_h^{(k)}) \in \alpha F \ (h, k \notin \{1, 2\}). \tag{vi-1}$$

Since φ is surjective and $\mathrm{Gr}_R(\Pi_n)$ is generated by $\mathrm{Gen}(\mathrm{Gr}_R(\Pi_n))$ as a Lie algebra over R, it follows from our assumption that $\mathrm{rank}_R\, \mathfrak{h} \geq 2$, that there exist integers h_2, k_2 such that $\beta := \varphi(W_{h_2}^{(k_2)}) \notin \alpha F$. In light of (R2), (R3), and (vi-1), we may assume that $h_2 = 1, k_2 = 3$. By applying the above argument, we obtain that

$$\varphi(W_h^{(k)}) \in \beta F \ (h, k \notin \{1, 3\}). \tag{vi-2}$$

By (vi-1) and (vi-2), we obtain that

$$\varphi(W_h^{(k)}) = 0 \ (h, k \notin \{1, 2, 3\}). \tag{vi-3}$$

Now it follows from (R1), (R2), together with (vi-2), that $\varphi(W_3^{(2)}) + \alpha \in \beta F$. Similarly, we obtain that $\varphi(W_2^{(3)}) + \beta \in \alpha F$. By (R3), we conclude that

$$\varphi(W_2^{(3)}) = \varphi(W_3^{(2)}) = -\alpha - \beta. \tag{vi-4}$$

By applying the above argument, we obtain that

$$\varphi(W_1^{(k)}) = \varphi(W_k^{(1)}) \in (-\alpha - \beta)F \ (k \notin \{1, 2, 3\}). \tag{vi-5}$$

By (vi-1), (vi-2), and (vi-5), that for each $k \notin \{1, 2, 3\}$, there exist $a_j, b_j, c_j \in F$ such that $\varphi(W_k^{(3)}) = \alpha a_k$, $\varphi(W_k^{(2)}) = \beta b_k$, $\varphi(W_k^{(1)}) = (-\alpha - \beta)c_k$. Then it follows from (R1), (R2) that

$$\sum_{k=4}^{n} a_k = \sum_{k=4}^{n} b_k = \sum_{k=4}^{n} c_k = 1. \tag{vi-6}$$

Moreover, (R10) implies that

$$a_k b_{k'} = b_k c_{k'} = c_k a_{k'} = 0 \ (k \neq k',\ k, k' \notin \{1, 2, 3\}). \tag{vi-7}$$

Thus, there exists $k_3 \notin \{1, 2, 3\}$ such that $a_{k_3} = b_{k_3} = c_{k_3} = 1$, $a_k = b_k = c_k = 0$ $(k \notin \{1, 2, 3, k_3\})$. In light of (R3), we conclude that (4) holds. This completes the proof of Theorem 3.1. □

Definition 3.2.

(i) If $g = 1$, then, for each pair of integers (k_1, k_2) such that $k_1 < k_2$, we shall write

$$I_1(k_1,k_2) := \langle\{\{X_1^{(k)} \mid k \neq k_1,k_2\} \cup \{Y_1^{(k)} \mid k \neq k_1,k_2\} \cup \{Z_j^{(k)}\} \cup \{W_h^{(k)} \mid \{h,k\} \neq \{k_1,k_2\}\} \cup \{X_1^{(k_1)} + X_1^{(k_2)}, Y_1^{(k_1)} + Y_1^{(k_2)}\}\}\rangle_{R,\mathrm{Gr}_R(\Pi_n)}.$$

(ii) If $g = 0$, then;

(ii-a) for each quadruplet of integers $(j_1, j_2; k_1, k_2)$ such that $j_1 < j_2$, $k_1 < k_2$, we shall write

$$I_2(j_1,j_2;k_1,k_2) := \langle\{\{Z_j^{(k)} \mid j \neq j_1,j_2 \text{ or } k \neq k_1,k_2\} \cup \{W_h^{(k)} \mid \{h,k\} \neq \{k_1,k_2\}\} \cup \{Z_{j_1}^{(k_1)} - Z_{j_2}^{(k_2)}, Z_{j_2}^{(k_1)} - Z_{j_1}^{(k_2)}\}\}\rangle_{R,\mathrm{Gr}_R(\Pi_n)}.$$

(ii-b) for each quadruplet of integers $(j_1; k_1, k_2, k_3)$ such that $k_1 < k_2 < k_3$, we shall write

$$I_3(j_1;k_1,k_2,k_3) := \langle\{\{Z_j^{(k)} \mid j \neq j_1 \text{ or } k \neq k_1,k_2,k_3\} \cup \{W_h^{(k)} \mid \{h,k\} \not\subset \{k_1,k_2,k_3\}\} \cup \{Z_{j_1}^{(k_1)} - W_{k_2}^{(k_3)}, Z_{j_1}^{(k_2)} - W_{k_3}^{(k_1)}, Z_{j_1}^{(k_3)} - W_{k_1}^{(k_2)}\}\}\rangle_{R,\mathrm{Gr}_R(\Pi_n)}.$$

(ii-c) for each quadruplet of integers (k_1, k_2, k_3, k_4) such that $k_1 < k_2 < k_3 < k_4$, we shall write

$$I_4(k_1,k_2,k_3,k_4) := \langle\{\{Z_j^{(k)}\} \cup \{W_h^{(k)} \mid \{h,k\} \not\subset \{k_1,k_2,k_3,k_4\}\} \cup \{W_{k_1}^{(k_2)} - W_{k_3}^{(k_4)}, W_{k_1}^{(k_3)} - W_{k_2}^{(k_4)}, W_{k_1}^{(k_4)} - W_{k_2}^{(k_3)}\}\}\rangle_{R,\mathrm{Gr}_R(\Pi_n)}.$$

(iii) Write

$$\widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n)) := \begin{cases} \emptyset & (g \geq 2) \\ \{I_1(k_1,k_2) \mid k_1 < k_2\} & (g = 1) \\ \{I_2(j_1,j_2;k_1,k_2) \mid j_1 < j_2,\ k_1 < k_2\} & \\ \ \cup\{I_3(j_1;k_1,k_2,k_3) \mid k_1 < k_2 < k_3\} & (g = 0). \\ \ \cup\{I_4(k_1,k_2,k_3,k_4) \mid k_1 < k_2 < k_3 < k_4\} & \end{cases}$$

Corollary 3.3. *Let $\mathfrak{h}$ be a Lie algebra over R, $\varphi : \mathrm{Gr}_R(\Pi_n) \twoheadrightarrow \mathfrak{h}$ a surjective homomorphism of Lie algebras over R. Suppose that the following conditions are satisfied:*

- *$\mathfrak{h}$ is a free R-module, and* $\mathrm{rank}_R\,\mathfrak{h} \geq \begin{cases} 2 & (g = 0) \\ 3 & (g \geq 1) \end{cases}$.
- *For $a, b \in \mathfrak{h}$, if $[a, b] = 0$, then a and b are linearly dependent over R.*

Then there exists an ideal $\mathfrak{i} \in \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n)) \cup \widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n))$ such that $\ker\varphi \supset \mathfrak{i}$. In particular, if further suppose that $\langle a, b\rangle_R \subsetneq \mathfrak{h}$ for any $a, b \in \mathfrak{h}$, then there exists $\mathfrak{i} \in \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n))$ such that $\ker\varphi \supset \mathfrak{i}$.

Proof. The first assertion immediately follows from Theorem 3.1. The second assertion follows from the easily verified fact that, for any $\mathfrak{i} \in \widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n))$, $\mathrm{Gr}_R(\Pi_n)/\mathfrak{i}$ is generated by two elements as a Lie algebra over R. □

Definition 3.4. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras over R. Then we shall write $I_R(\mathfrak{g}, \mathfrak{h})$ for the set of all Lie ideals $\mathfrak{i}$ of $\mathfrak{g}$ over R such that $\mathfrak{g}/\mathfrak{i} \cong \mathfrak{h}$ as Lie algebras over R. It follows directly from the definition that $I_R(\mathfrak{g}, \mathfrak{h}) \neq \emptyset$ if and only if there exists a surjective homomorphism $\mathfrak{g} \twoheadrightarrow \mathfrak{h}$ of Lie algebras over R.

Theorem 3.5.

(i) *It holds that* $\mathrm{EI}(\mathrm{Gr}_R(\Pi_n)) = \widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n))$.

(ii) *It holds that* $\mathrm{GFI}_1(\mathrm{Gr}_R(\Pi_n)) = I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_1))$.

(iii) *It holds that* $\mathrm{EI}(\mathrm{Gr}_R(\Pi_n)) = \{\mathfrak{i} \in I_R(\mathrm{Gr}_R(\Pi_n), (\mathfrak{L}_2)_R) \mid \mathfrak{j} \not\subset \mathfrak{i}$ *for any* $\mathfrak{j} \in \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n))\}$.

Proof. First, let us suppose that $(g, r) \in \{(0, 3), (1, 1)\}$. In this case, assertion (iii) immediately follows from assertion (ii). To verify assertions (i) and (ii), we claim that $\mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n)) \cup \widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n)) \supset I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_1))$. Indeed, let $\mathfrak{i} \in I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_1))$. Then it follows from Corollary 3.3, Lemma 2.5(i),(iii), that there exists $\mathfrak{j} \in \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n)) \cup \widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n))$ such that $\mathfrak{i} \supset \mathfrak{j}$. Since (we can easily verify that) $\mathrm{Gr}_R(\Pi_n)/\mathfrak{j}$ is generated by two elements as a Lie algebra over R, it follows from Lemma 2.5(ii) that the natural surjective homomorphism $\mathrm{Gr}_R(\Pi_n)/\mathfrak{j} \twoheadrightarrow \mathrm{Gr}_R(\Pi_n)/\mathfrak{i} \cong (\mathfrak{L}_2)_R$ is injective, which implies that $\mathfrak{i} = \mathfrak{j} \in \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n)) \cup \widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n))$.

Moreover, it is clear that $I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_1)) \supset \mathrm{GFI}_1(\mathrm{Gr}_R(\Pi_n))$. Now it follows from Lemma 1.13 that

$$\sharp\, \mathrm{GFI}_1(\mathrm{Gr}_R(\Pi_n)) = \begin{cases} \binom{n+1}{2} & ((g,r) = (1,1)) \\ \binom{n+3}{4} & ((g,r) = (0,3)) \end{cases}$$

(cf. Definition 1.6). On the other hand, it follows from Definition 3.2 that

$$\sharp\widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n)) = \begin{cases} \binom{n}{2} & ((g,r) = (1,1)) \\ \binom{3}{2} \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} + \binom{n}{4} & ((g,r) = (0,3)). \end{cases}$$

Thus, we can easily check that

$$\begin{aligned} \sharp\, \mathrm{GFI}_1(\mathrm{Gr}_R(\Pi_n)) &= n + \sharp\widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n)) \\ &= \sharp\, \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n)) + \sharp\widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n)) \end{aligned}$$

(note that this number coincides with the coefficient of x^4 in the expansion of $(1+x)^{n+3} = (1+x)^3 \cdot (1+x)^n$). Thus, we conclude that $\mathrm{GFI}_1(\mathrm{Gr}_R(\Pi_n)) = I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_1)) = \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n)) \cup \widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n))$. Since $\mathrm{GFI}_1(\mathrm{Gr}_R(\Pi_n)) = \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n)) \cup \mathrm{EI}(\mathrm{Gr}_R(\Pi_n))$, we can easily verify that $\mathrm{EI}(\mathrm{Gr}_R(\Pi_n)) = \widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n))$. This completes the proof of Theorem 3.5 in the case $(g, r) \in \{(0, 3), (1, 1)\}$.

Next, let us suppose that $(g, r) \notin \{(0, 3), (1, 1)\}$. In this case, assertion (i) follows from "(i)" in the case $(g, r) \in \{(0, 3), (1, 1)\}$ (which has already been verified) and the definitions of $\mathrm{EI}(\mathrm{Gr}_R(\Pi_n))$ and $\widetilde{\mathrm{EI}}(\mathrm{Gr}_R(\Pi_n))$. Next, we verify assertion (ii). It is clear that $\mathrm{GFI}_1(\mathrm{Gr}_R(\Pi_n)) \subset I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_1))$. Let $\mathfrak{i} \in I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_1))$. Since $(g, r) \notin \{(0, 3), (1, 1)\}$, it holds that $\mathrm{Gr}_R(\Pi_1) \not\cong (\mathfrak{L}_2)_R$. Thus, it follows from Corollary 3.3, Lemma 2.5(i),(iii),(v) that there exists $\mathfrak{j} \in \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n))$ such that $\mathfrak{i} \supset \mathfrak{j}$. By an argument similar to the above argument, we obtain that $\mathfrak{i} = \mathfrak{j} \in \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n)) (= \mathrm{GFI}_1(\mathrm{Gr}_R(\Pi_n)))$. This completes the proof of assertion (ii) in the case $(g, r) \notin \{(0, 3), (1, 1)\}$. By an argument similar to the above argument, assertion (iii) follows from Corollary 3.3, Lemma 2.5(i),(ii),(v). This completes the proof of Theorem 3.5. □

Proposition 3.6. *There exists a unique surface algebra* $\mathfrak{s}$ *over* R *(up to isomorphism as Lie algebras over* R*) which satisfies the following conditions:*

(1) $I_R(\mathrm{Gr}_R(\Pi_n), \mathfrak{s}) \neq \emptyset$.

(2) *For any surface algebra* $\mathfrak{h}$ *over* R*, if* $I_R(\mathrm{Gr}_R(\Pi_n), \mathfrak{h}) \neq \emptyset$*, then* $I_R(\mathfrak{s}, \mathfrak{h}) \neq \emptyset$*.*

Moreover, this unique surface algebra $\mathfrak{s}$ *is isomorphic to* $\mathrm{Gr}_R(\Pi_1)$ *as Lie algebras over* R*.*

Proof. First, we verify that the surface algebra $\mathrm{Gr}_R(\Pi_1)$ satisfies conditions (1), (2). Condition (1) is clear. To verify that $\mathrm{Gr}_R(\Pi_1)$ satisfies condition (2), let $\mathfrak{h}$ be a surface algebra over R such that $I_R(\mathrm{Gr}_R(\Pi_n), \mathfrak{h}) \neq \emptyset$. If $\mathfrak{h} \cong (\mathfrak{L}_2)_R$, then it follows from Lemma 2.5(iv) that $I_R(\mathrm{Gr}_R(\Pi_1), \mathfrak{h}) \neq \emptyset$. Thus, we may assume that $\mathfrak{h} \ncong (\mathfrak{L}_2)_R$. Let $\mathfrak{j} \in I_R(\mathrm{Gr}_R(\Pi_n), \mathfrak{h})$. Then it follows from Lemma 2.5(i),(iii),(v) and Corollary 3.3 that there exists an ideal $\mathfrak{i} \in \mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n))$ such that $\mathfrak{i} \subset \mathfrak{j}$. Since $\mathrm{Gr}_R(\Pi_n)/\mathfrak{i} \cong \mathrm{Gr}_R(\Pi_1)$, we conclude that $I_R(\mathrm{Gr}_R(\Pi_1), \mathfrak{h}) \neq \emptyset$.

Next, we verify that $\mathrm{Gr}_R(\Pi_1)$ is the unique surface algebra over R which satisfies conditions (1), (2). Let $\mathfrak{s}$ be a surface algebra over R which satisfies conditions (1), (2). Then it holds that $I_R(\mathfrak{s}, \mathrm{Gr}_R(\Pi_1)) \neq \emptyset$, $I_R(\mathrm{Gr}_R(\Pi_1), \mathfrak{s}) \neq \emptyset$. Thus, there exist surjective homomorphisms $\varphi : \mathfrak{s} \twoheadrightarrow \mathrm{Gr}_R(\Pi_1), \psi : \mathrm{Gr}_R(\Pi_1) \twoheadrightarrow \mathfrak{s}$ of Lie algebras over R. Now it follows from Lemma 2.5(ii) that the composite $\psi \circ \varphi : \mathfrak{s} \twoheadrightarrow \mathfrak{s}$ is an isomorphism. Thus, it holds that $\varphi : \mathfrak{s} \twoheadrightarrow \mathrm{Gr}_R(\Pi_1)$ is an isomorphism. This completes the proof of Proposition 3.6. □

Definition 3.7. Let $\mathfrak{g}$ be a configuration space algebra over R. Then it follows from Proposition 3.6 that there exists a unique surface algebra $\mathfrak{s}$ over R (up to isomorphism as Lie algebras over R) which satisfies the following conditions:

(1) $I_R(\mathfrak{g}, \mathfrak{s}) \neq \emptyset$.

(2) For any surface algebra $\mathfrak{h}$ over R, if $I_R(\mathfrak{g}, \mathfrak{h}) \neq \emptyset$, then $I_R(\mathfrak{s}, \mathfrak{h}) \neq \emptyset$.

We shall write $\mathrm{G}_1(\mathfrak{g})$ for this unique surface algebra over R. Moreover, we shall write $\mathrm{GF}_1(\mathfrak{g}) := I_R(\mathfrak{g}, \mathrm{G}_1(\mathfrak{g}))$, $\mathrm{E}(\mathfrak{g}) := \{\mathfrak{i} \in I_R(\mathfrak{g}, (\mathfrak{L}_2)_R) \mid \mathfrak{j} \not\subset \mathfrak{i} \text{ for any } \mathfrak{j} \in \mathrm{GF}_1(\mathfrak{g})\}$.

Theorem 3.8. *Let* $\mathfrak{g}$ *be a configuration space algebra over* R *and* $(X, n, \varphi : \mathrm{Gr}_R(\Pi_n) \xrightarrow{\sim} \mathfrak{g})$ *a CS-envelope for* $\mathfrak{g}$*. Write* (g, r) *for the type of the hyperbolic curve* $X/\mathbb{C}$*. Then the following hold:*

(i) $\mathrm{Gr}_R(\Pi_1) \cong \mathrm{G}_1(\mathfrak{g})$ *as Lie algebras over* R*.*

(ii) *The isomorphism* φ *induces a bijection* $\mathrm{GFI}_1(\mathrm{Gr}_R(\Pi_n)) \overset{1:1}{\to} \mathrm{GF}_1(\mathfrak{g})$, $\mathfrak{i} \mapsto \varphi(\mathfrak{i})$.

(iii) *Suppose that* $(g, r) \notin \{(0, 3), (1, 1)\}$ *(which is equivalent to the condition* $\mathrm{G}_1(\mathfrak{g}) \ncong (\mathfrak{L}_2)_R$*). Then the isomorphism* φ *induces a bijection* $\mathrm{EI}(\mathrm{Gr}_R(\Pi_n)) \overset{1:1}{\to} \mathrm{E}(\mathfrak{g})$, $\mathfrak{i} \mapsto \varphi(\mathfrak{i})$.

(iv) *Suppose that* $(g, r) \in \{(0, 3), (1, 1)\}$*. Then it holds that* $\mathrm{E}(\mathfrak{g}) = \emptyset$*.*

Proof. Assertion (i) follows from Proposition 3.6. For any Lie ideal $\mathfrak{i}$ over R of $\mathrm{Gr}_R(\Pi_n)$, assertion (i) implies that $\varphi(\mathfrak{i}) \in \mathrm{GF}_1(\mathfrak{g}) = I_R(\mathfrak{g}, \mathrm{G}_1(\mathfrak{g}))$ if and only if $\mathfrak{i} \in I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_1))$. Thus, (ii) follows from Theorem 3.5. In light of assertions (i), (ii) and Theorem 3.5, assertion (iii) follows from an argument similar to the above argument. Assertion (iv) follows from the fact that $\mathrm{G}_1(\mathfrak{g}) \cong (\mathfrak{L}_2)_R$ if $(g, r) \in \{(0, 3), (1, 1)\}$. □

Remark 3.8.1. If $(g, r) \in \{(0, 3), (1, 1)\}$, then, since there exists an automorphism of $\mathrm{Gr}_R(\Pi_n)$ which preserves neither $\mathrm{FI}_1(\mathrm{Gr}_R(\Pi_n))$ nor $\mathrm{EI}(\mathrm{Gr}_R(\Pi_n))$, we cannot reconstruct these sets from $\mathrm{Gr}_R(\Pi_n)$.

Definition 3.9. Let m be a nonnegative integer and $\mathfrak{g}$ a configuration space algebra over R. We shall define a set $\mathrm{GF}_m(\mathfrak{g})$ of Lie subalgebras of $\mathfrak{g}$ over R as follows:

- $\mathrm{GF}_0(\mathfrak{g}) := \{\mathfrak{g}\}$, $\mathrm{GF}_1(\mathfrak{g}) := I_R(\mathfrak{g}, \mathrm{G}_1(\mathfrak{g}))$ (cf. Definition 3.7). Moreover, for convenience, we shall define $\mathrm{GF}_1(\{0\}) := \emptyset$.
- If $m \geq 2$, then $\mathrm{GF}_m(\mathfrak{g}) := \bigcup_{\mathfrak{i} \in \mathrm{GF}_{m-1}(\mathfrak{g})} \mathrm{GF}_1(\mathfrak{i})$. Here, let us observe, by applying Theorem 3.8(ii) and Proposition 1.8 inductively, that for each $\mathfrak{i} \in \mathrm{GF}_1(\mathfrak{g})$, if $\mathfrak{i} \neq \{0\}$, then $\mathfrak{i}$ is also a configuration space algebra over R.

Theorem 3.10. *Let m be a nonnegative integer, $\mathfrak{g}$ a configuration space algebra over R, and $(X, n, \varphi : \mathrm{Gr}_R(\Pi_n) \xrightarrow{\sim} \mathfrak{g})$ a CS-envelope for $\mathfrak{g}$. Then the isomorphism φ determines a bijection $\mathrm{GFI}_m(\mathrm{Gr}_R(\Pi_n)) \overset{1:1}{\to} \mathrm{GF}_m(\mathfrak{g})$, $\mathfrak{i} \mapsto \varphi(\mathfrak{i})$.*

Proof. We verify Theorem 3.10 by induction on m. If $m = 0$, then Theorem 3.10 is clear. If $m = 1$, then Theorem 3.10 is nothing but Theorem 3.8(ii). Now suppose that $m \geq 2$, and that the induction hypothesis is in force. Let $\mathfrak{i} \in \mathrm{GFI}_m(\mathrm{Gr}_R(\Pi_n))$. Then it follows from Proposition 1.8 that there exists $\mathfrak{j} \in \mathrm{GFI}_{m-1}(\mathrm{Gr}_R(\Pi_n))$ such that $\mathfrak{i} \subset \mathfrak{j}$, and, moreover, $\mathfrak{i}$ is a (generalized) fiber ideal of co-length one of $\mathfrak{j}$ with respect to the natural structure of configuration space algebra on $\mathfrak{j}$ (cf. Remark 1.8.1). Thus, Theorem 3.10 follows from Theorem 3.8(ii) and the induction hypothesis. This completes the proof of Theorem 3.10. $\square$

Remark 3.10.1. The proof of [NT] Theorem D gives a "bi-anabelian" reconstruction (in the notation of [Mo3] Remark 1.9.8) of the set of generalized fiber ideals of length one of the graded Lie algebra $\mathrm{Gr}_R(\Pi_n)$ (under the assumption that R is a field of characteristic zero), that is to say, any graded automorphism of $\mathrm{Gr}_R(\Pi_n)$ preserves the set of generalized fiber ideals of length one(, hence also the set of generalized fiber ideals of arbitrary given (co-)length).

Proposition 3.11. *Let m be a positive integer. Then it holds that $\mathrm{GFI}_m(\mathrm{Gr}_R(\Pi_n)) = I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_m))$.*

Proof. It is clear that $\mathrm{GFI}_m(\mathrm{Gr}_R(\Pi_n)) \subset I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_m))$. We verify $\mathrm{GFI}_m(\mathrm{Gr}_R(\Pi_n)) \supset I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_m))$ by induction on m. If $m = 1$, then this follows from Theorem 3.5(ii). Now suppose that $m \geq 2$, and that the induction hypothesis is in force. Let us fix a surjection $\varphi : \mathrm{Gr}_R(\Pi_m) \twoheadrightarrow \mathrm{Gr}_R(\Pi_{m-1})$ determined by a fiber ideal of length one of $\mathrm{Gr}_R(\Pi_m)$. Then it follows from Proposition 1.8 that $\ker \varphi$ is a surface algebra over R arising from a hyperbolic curve of type $(g, r + m - 1)$. Let $\mathfrak{i} \in I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_m))$. Write $\mathfrak{j}$ for the kernel of the composite of φ and the natural surjection $\mathrm{Gr}_R(\Pi_n) \twoheadrightarrow \mathrm{Gr}_R(\Pi_m)$ determined by $\mathfrak{i}$. Then the sequence $0 \to \mathfrak{i} \to \mathfrak{j} \to \ker \varphi \to 0$ is exact. Moreover, it follows from the induction hypothesis that $\mathfrak{j} \in I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_{m-1})) = \mathrm{GFI}_{m-1}(\mathrm{Gr}_R(\Pi_n))$. Thus, by Proposition 1.8, $\mathfrak{j}$ is a configuration space algebra arising from an $(n - m + 1)$-th configuration space of a hyperbolic curve of type $(g, r + m - 1)$. In particular, it follows from Theorem 3.5(ii) that $\mathfrak{i} \in I_R(\mathfrak{j}, \ker \varphi)$ is a (generalized) fiber ideal of co-length one of $\mathfrak{j}$, which implies that $\mathfrak{i} \in \mathrm{GFI}_m(\mathrm{Gr}_R(\Pi_n))$ (cf. Remark 1.8.1). This completes the proof of $\mathrm{GFI}_m(\mathrm{Gr}_R(\Pi_n)) \supset I_R(\mathrm{Gr}_R(\Pi_n), \mathrm{Gr}_R(\Pi_m))$, hence also of Proposition 3.11. $\square$

4. Reconstruction of some invariants

In the present §4, we give algorithms for reconstructing some invariants of a configuration space. Let X be a hyperbolic curve of type (g, r) over $\mathbb{C}$, n a positive integer, and R an integral domain unless otherwise specified. Write F for the field of fractions of R.

Lemma 4.1. *Let s be a positive integer. Then the following conditions are equivalent:*

(1) $s \leq n$.
(2) *There exists an abelian Lie subalgebra $\mathfrak{a}$ of $\mathrm{Gr}_R(\Pi_n)$ over R such that $\mathfrak{a} \cong R^{\oplus s}$ as R-modules.*

Proof. First, we verify the implication (1) $\Rightarrow$ (2). Suppose that $s \leq n$. Since at least one of $g > 0$ or $r > 0$ holds, it suffices to find an abelian Lie subalgebra $\mathfrak{a}$ over R such that $\mathfrak{a} \cong R^{\oplus s}$ in each case. First, suppose that $g > 0$. Write $\mathfrak{a} := \langle \{X_1^{(k)} \mid 1 \leq k \leq s\} \rangle_R$. Let us verify that $\mathfrak{a} \cong R^{\oplus s}$. It follows from (R4) that $\mathfrak{a}$ is abelian. Thus, it suffices to show that $X_1^{(1)}, \ldots, X_1^{(s)}$ are linearly independent over R. Let $a_1, \ldots, a_s \in R$ be such that $\sum_{k=1}^{s} a_k X_1^{(k)} = 0$. By induction on s, it suffices to show that $a_s = 0$. Now it holds that $X_1^{(1)}, \ldots, X_1^{(s-1)} \in \Phi_s$ and $X_1^{(s)} \notin \Phi_s$. Since $\mathrm{Gr}_R(\Pi_n)/\Phi_s \cong \mathrm{Gr}_R(\Pi_1)$ is a free R-module (cf. Lemma 2.5(i)), the image of $X_1^{(s)}$ by the natural surjection $\mathrm{Gr}_R(\Pi_n) \twoheadrightarrow \mathrm{Gr}_R(\Pi_n)/\Phi_s$ is not an R-torsion element. Thus, we conclude that $a_s = 0$. This completes the proof of the existence of an abelian Lie subalgebra $\mathfrak{a}$ such that $\mathfrak{a} \cong R^{\oplus s}$ in the case $g > 0$.

Next, suppose that $r > 0$. Write $A^{(k)} := Z_1^{(k)} + \sum_{h=1}^{k-1} W_h^{(k)}$ and $\mathfrak{a} := \langle \{A^{(k)} \mid 1 \leq k \leq s\} \rangle_R$. Let us verify that $\mathfrak{a} \cong R^{\oplus s}$. For each pair of integers (k_1, k_2) such that $1 \leq k_1 < k_2 \leq s$, it follows from (R9) and (R10) that

$$[A^{(k_1)}, A^{(k_2)}] = [Z_1^{(k_1)}, Z_1^{(k_2)} + W_{k_1}^{(k_2)}] + \sum_{h=1}^{k_1-1} [W_h^{(k_1)}, W_h^{(k_2)} + W_{k_1}^{(k_2)}],$$

which implies, in light of (R11) and (R12), that $[A^{(k_1)}, A^{(k_2)}] = 0$. Thus, it suffices to show that $A^{(1)}, \ldots, A^{(s)}$ are linearly independent over R. Let $a_1, \ldots, a_s \in R$ be such that $\sum_{k=1}^{s} a_k A^{(k)} = 0$. By induction on s, it suffices to show that $a_s = 0$. Now it holds that $Z_1^{(1)}, \ldots, Z_1^{(s-1)} \in \Phi_s$ and $Z_1^{(s)} \notin \Phi_s$. Moreover, it follows from (R3) that $W_h^{(k)} = W_k^{(h)} \in \Phi_s$ $(1 \leq h < k \leq s)$. Thus, $A^{(1)}, \ldots, A^{(s-1)} \in \Phi_s$ and $A^{(s)} \notin \Phi_s$. Since $\mathrm{Gr}_R(\Pi_n)/\Phi_s \cong \mathrm{Gr}_R(\Pi_1)$ is a free R-module (cf. Lemma 2.5(i)), the image of $A^{(s)}$ by the natural surjection $\mathrm{Gr}_R(\Pi_n) \twoheadrightarrow \mathrm{Gr}_R(\Pi_n)/\Phi_s$ is not an R-torsion element. Thus, we conclude that $a_s = 0$. This completes the proof of the implication (1) $\Rightarrow$ (2).

Next, we verify the implication (2) $\Rightarrow$ (1) by induction on n. Let $\mathfrak{a} \subset \mathrm{Gr}_R(\Pi_n)$ be as in (2). By considering $\mathfrak{a} \otimes_R F \subset \mathrm{Gr}_F(\Pi_n) = \mathrm{Gr}_R(\Pi_n) \otimes_R F$ if necessary, we may assume that $R = F$. If $n = 1$, then it follows from Lemma 2.5(v) that $s \leq 1 = n$. Now suppose that $n \geq 2$, and that the induction hypothesis is in force. Let $\mathfrak{i} \in \mathrm{FI}_1(\mathrm{Gr}_F(\Pi_n))$. Write $\varphi : \mathrm{Gr}_F(\Pi_n) \twoheadrightarrow \mathrm{Gr}_F(\Pi_n)/\mathfrak{i}$ for the natural surjection. Then it follows from Proposition 1.8, together with the induction hypothesis, that $\dim_F(\mathfrak{a} \cap \mathfrak{i}) \leq n - 1$. Moreover, it follows from the case $n = 1$ that $\dim_F \varphi(\mathfrak{a}) \leq 1$. Thus, we conclude that $s = \dim_F \mathfrak{a} \leq n$. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Lemma 4.1. □

Definition 4.2. Let $\mathfrak{g}$ be a configuration space algebra over R. Then it follows from Lemma 4.1 that the set $\{s \in \mathbb{Z}_{>0} \mid \text{there exists an abelian Lie subalgebra } \mathfrak{a} \subset \mathfrak{g} \text{ such that } \mathfrak{a} \cong R^{\oplus s}\}$ is finite and nonempty. We shall write $n(\mathfrak{g})$ for the largest integer in this set.

Theorem 4.3. *Let $\mathfrak{g}$ be a configuration space algebra over R and $(X, n, \mathrm{Gr}_R(\Pi_n) \overset{\sim}{\to} \mathfrak{g})$ a CS-envelope for $\mathfrak{g}$. Then it holds that $n = n(\mathfrak{g})$.*

Proof. This follows from Lemma 4.1. □

Remark 4.3.1. The algorithm of Definition 4.2 is a Lie algebra analogue of the group-theoretic reconstruction algorithm appearing in [HMM] (cf. [HMM] Theorem 1.6). Since we have already reconstructed the set of generalized fiber ideals of given co-length, we can give other algorithms for reconstructing n. For example, n coincides with the unique nonnegative integer m such that $\mathrm{GF}_m(\mathfrak{g}) = \{\{0\}\}$. Moreover, if we fix $\mathfrak{i} \in \mathrm{GF}_1(\mathfrak{g})$, then it holds that $n = \sharp\,\mathrm{GF}_1(\mathfrak{i}) + 1 = \sharp\{\mathfrak{j} \in \mathrm{GF}_2(\mathfrak{g}) \mid \mathfrak{j} \subset \mathfrak{i}\} + 1$.

Lemma 4.4. *Suppose that $n \geq 2$ and $g \geq 1$. Let $\mathfrak{i}, \mathfrak{j}$ be distinct elements of $\mathrm{GFI}_{n-1}(\mathrm{Gr}_R(\Pi_n))$. Write $\mathfrak{k} := \{a \in \mathfrak{i} \mid [a, b] \in \mathrm{Gr}_R(\Pi_n)[3]$ for any $b \in \mathfrak{j}\}$. Then it holds that $\mathfrak{k} = \mathfrak{i} \cap \mathrm{Gr}_R(\Pi_n)(2)$. In particular, $\mathrm{rank}_R(\mathfrak{i}/\mathfrak{k}) = 2g$.*

Proof. The second assertion follows immediately from the first assertion and Proposition 1.8. We verify the first assertion. By considering the graded automorphism of $\mathrm{Gr}_R(\Pi_n)$ induced by an automorphism of X_n determined by a permutation of labeling if necessary, we may assume that $\mathfrak{i} = \langle X_i^{(1)}, Y_i^{(1)}, Z_j^{(1)}, W_k^{(1)} \mid 1 \leq i \leq g, 1 \leq j \leq r, 1 \leq k \leq n\rangle_R$, $\mathfrak{j} = \langle X_i^{(2)}, Y_i^{(2)}, Z_j^{(2)}, W_k^{(2)} \mid 1 \leq i \leq g, 1 \leq j \leq r, 1 \leq k \leq n\rangle_R$.

First, we verify $\mathfrak{i} \cap \mathrm{Gr}_R(\Pi_n)(2) \subset \mathfrak{k}$. One verifies easily that $\mathfrak{k}$ is a Lie ideal of $\mathrm{Gr}_R(\Pi_n)$ over R, and, moreover, $\mathfrak{i} \cap \mathrm{Gr}_R(\Pi_n)[2] \subset \mathfrak{k}$. In particular, it follows from (R2), (R6), together with our assumtion that $g \geq 1$, that $W_h^{(1)} \in \mathfrak{i} \cap \mathrm{Gr}_R(\Pi_n)[2] \subset \mathfrak{k}$ for any h. Thus, to verify $\mathfrak{i} \cap \mathrm{Gr}_R(\Pi_n)(2) \subset \mathfrak{k}$, it suffices to show that $Z_j^{(1)} \in \mathfrak{k}$ for any j, i.e., it suffices to show that $[Z_j^{(1)}, b] \in \mathrm{Gr}_R(\Pi_n)[3]$ for any $b \in \mathfrak{j}$. If $b \in \mathfrak{j} \cap \mathrm{Gr}_R(\Pi_n)[2]$, then it is clear that $[Z_j^{(1)}, b] \in \mathrm{Gr}_R(\Pi_n)[3]$. Thus, it suffices to show that $[Z_j^{(1)}, X_i^{(2)}]$, $[Z_j^{(1)}, Y_i^{(2)}]$, $[Z_j^{(1)}, Z_{j'}^{(2)}]$, $[Z_j^{(1)}, W_h^{(2)}] \in \mathrm{Gr}_R(\Pi_n)[3]$ for any i, j', h. Now it follows from an argument similar to the above argument that $W_h^{(2)} \in \mathrm{Gr}_R(\Pi_n)[2]$, which implies that $[Z_j^{(1)}, W_h^{(2)}] \in \mathrm{Gr}_R(\Pi_n)[3]$. On the other hand, it follows from (R7), (R8) that for any i and any $j' \neq j$, it holds that $[Z_j^{(1)}, X_i^{(2)}] = [Z_j^{(1)}, Y_i^{(2)}] = [Z_j^{(1)}, Z_{j'}^{(2)}] = 0$. Moreover, it follows from (R11) and (R6) that $[Z_j^{(1)}, Z_j^{(2)}] = -[Z_j^{(1)}, W_1^{(2)}] = -[Z_j^{(1)}, [X_1^{(1)}, Y_1^{(2)}]] \in \mathrm{Gr}_R(\Pi_n)[3]$. This completes the proof of $\mathfrak{i} \cap \mathrm{Gr}_R(\Pi_n)(2) \subset \mathfrak{k}$.

Next, we verify $\mathfrak{i} \cap \mathrm{Gr}_R(\Pi_n)(2) \supset \mathfrak{k}$. Since $\mathfrak{k}$ is a Lie ideal of $\mathrm{Gr}_R(\Pi_n)$ over R, it follows from $\mathfrak{i} \cap \mathrm{Gr}_R(\Pi_n)(2) \subset \mathfrak{k}$ (which has already been verified) that, it suffices to show that if $a_i, b_i \in R$ $(1 \leq i \leq g)$ and $\sum_{i=1}^{g}(a_i X_i^{(1)} + b_i Y_i^{(1)}) \in \mathfrak{k}$, then $a_i = b_i = 0$ for any i. Write $A := \sum_{i=1}^{g}(a_i X_i^{(1)} + b_i Y_i^{(1)})$. Then (R4)–(R6) imply that $[A, Y_i^{(2)}] = a_i W_1^{(2)}$. Since $W_1^{(2)}$ is a nonzero homogeneous element of degree 2, $W_1^{(2)} \notin \mathrm{Gr}_R(\Pi_n)[3]$. Thus, we obtain that $a_i = 0$. Similarly, $[A, X_i^{(2)}] = -b_i W_2^{(1)}$ implies that $b_i = 0$. This completes the proof of $\mathfrak{i} \cap \mathrm{Gr}_R(\Pi_n)(2) \supset \mathfrak{k}$, hence also of Lemma 4.4. □

Lemma 4.5. *Let $\mathfrak{g}$ be a configuration space algebra over R and $(X, n, \mathrm{Gr}_R(\Pi_n) \overset{\sim}{\to} \mathfrak{g})$ a CS-envelope for $\mathfrak{g}$. Write (g, r) for the type of the hyperbolic curve $X/\mathbb{C}$. Then the following hold:*

(i) $\sharp\,\mathrm{GF}_1(\mathfrak{g}) = \begin{cases} \binom{n+3}{4} & ((g,r)=(0,3)) \\ \binom{n+1}{2} & ((g,r)=(1,1)) \\ n & ((g,r)\notin\{(0,3),(1,1)\}). \end{cases}$

(ii) $\sharp\,\mathrm{E}(\mathfrak{g}) = \begin{cases} 0 & (g \geq 2 \textit{ or } (g,r)\in\{(0,3),(1,1)\}) \\ \binom{n}{2} & (g=1 \textit{ and } r\neq 1) \\ \binom{r}{2}\cdot\binom{n}{2} + r\cdot\binom{n}{3} + \binom{n}{4} & (g=0 \textit{ and } r\neq 3). \end{cases}$

(iii) *$r > 0$ if and only if $\mathrm{G}_1(\mathfrak{g})$ is a free Lie algebra over R. Moreover, if $r > 0$, then $\mathrm{G}_1(\mathfrak{g}) \cong (\mathfrak{L}_{2g+r-1})_R$.*

Proof. Assertions (i), (ii) follows from Theorem 3.5(i), Theorem 3.8(ii), (iii), (iv), Definition 3.2. Assertion (iii) follows from Proposition 1.11, Corollary 2.6, Theorem 3.8(i). ∎

Remark 4.5.1. In the definition of exceptional ideals (cf. Definition 1.10), we can choose arbitrarily an open immersion $X \hookrightarrow Y$, where $g = 1$ (resp. $g = 0$) and Y is of type $(1,1)$ (resp. $(0,3)$), that is to say, we can arbitrarily choose one point (resp. three points) from $X^{\mathrm{cpt}} \setminus X$. Thus, if $g = 1$, then there are r different choices of a point, and for each choice, we can define $\binom{n}{2}$ exceptional ideals. However, Lemma 4.5(ii) implies that there exist only $\binom{n}{2}$ exceptional ideals in all. This implies that in fact the set of exceptional ideals arising from each choice of the point does not depend on the choice of the point. Let us recall that the ideal $I_1(k_1,k_2) \subset \mathrm{Gr}_R(\Pi_n)$ (cf. Definition 3.2) contains all $Z_j^{(k)}$s.

In the case $g = 1$, it follows from the direct calculation of coordinates that an exceptional morphism $X_n \to Y = E \setminus \{O\}$ factors as $X_n \hookrightarrow E_n \twoheadrightarrow E_n/E \twoheadrightarrow E_2/E \xrightarrow{\sim} E \setminus \{O\}$, where $E_n \twoheadrightarrow E_n/E$ is the natural surjective morphism and $E_n/E \twoheadrightarrow E_2/E$ is a projection morphism obtained by forgetting $(n-2)$ factors. Thus, we can also define exceptional ideals as the kernel of the homomorphism $\mathrm{Gr}_R(\Pi_n) \to (\mathfrak{L}_2)_R$ arising from the composite $X_n \hookrightarrow E_n \twoheadrightarrow E_n/E \twoheadrightarrow E_2/E$, where $E_n/E \twoheadrightarrow E_2/E$ is a projection morphism.

The case $g = 0$ is more complicated than the case $g = 1$. We can check that $I_2(j_1,j_2;k_1,k_2)$ (resp. $I_3(j_1;k_1,k_2,k_3)$, $I_4(k_1,k_2,k_3,k_4)$) arises from $(r-2)$ (resp. $\binom{r-1}{2}$, $\binom{r}{3}$) different choices of three points.

Definition 4.6. Let $\mathfrak{g}$ be a configuration space algebra over R such that $n(\mathfrak{g}) \geq 2$.

(i) Let $\mathfrak{i}, \mathfrak{j}$ be distinct elements of $\mathrm{GF}_{n(\mathfrak{g})-1}(\mathfrak{g})$. Then we shall write $\mathfrak{k}(\mathfrak{g};\mathfrak{i},\mathfrak{j}) := \{a \in \mathfrak{i} \mid [a,b] \in \mathfrak{g}[3] \text{ for any } b \in \mathfrak{j}\}$.

(ii) Let us define a nonnegative integer $g(\mathfrak{g})$ as follows:
- If $\sharp\,\mathrm{GF}_1(\mathfrak{g}) = \binom{n(\mathfrak{g})+3}{4}$ or $\sharp\,\mathrm{E}(\mathfrak{g}) > \binom{n(\mathfrak{g})}{2}$, then $g(\mathfrak{g}) := 0$.
- If $\sharp\,\mathrm{GF}_1(\mathfrak{g}) \neq \binom{n(\mathfrak{g})+3}{4}$ and $\sharp\,\mathrm{E}(\mathfrak{g}) \leq \binom{n(\mathfrak{g})}{2}$, then let us choose distinct elements $\mathfrak{i}, \mathfrak{j}$ of $\mathrm{GF}_{n(\mathfrak{g})-1}(\mathfrak{g})$ and write $g(\mathfrak{g}) := \mathrm{rank}_R(\mathfrak{i}/\mathfrak{k}(\mathfrak{g};\mathfrak{i},\mathfrak{j}))/2$. (Note that it follows from Theorems 3.10, 4.3 and Lemma 4.4 that $\sharp\,\mathrm{GF}_{n(\mathfrak{g})-1}(\mathfrak{g}) \geq 2$, $\mathfrak{k}(\mathfrak{g};\mathfrak{i},\mathfrak{j})$ is a Lie ideal of $\mathfrak{i}$ over R, and $\mathfrak{i}/\mathfrak{k}(\mathfrak{g};\mathfrak{i},\mathfrak{j})$ is a free R-module of finite rank.)

(iii) Let us define a nonnegative integer $r(\mathfrak{g})$ as follows:
- If $\mathrm{G}_1(\mathfrak{g})$ is a free Lie algebra over R, then $r(\mathfrak{g}) := \mathrm{rank}_R(\mathrm{G}_1(\mathfrak{g})/\mathrm{G}_1(\mathfrak{g})[2]) + 1 - 2g(\mathfrak{g})$.
- If $\mathrm{G}_1(\mathfrak{g})$ is not a free Lie algebra over R, then $r(\mathfrak{g}) := 0$.

(iv) Let m be a positive integer. Then let us define $\mathfrak{g}(m) \subset \mathfrak{g}$ as follows:

- $\mathfrak{g}(1) := \mathfrak{g}$.
- If $g(\mathfrak{g}) = 0$, then $\mathfrak{g}(2) := \mathfrak{g}$.
- If $g(\mathfrak{g}) \neq 0$, then $\mathfrak{g}(2) := \langle \bigcup_{\mathfrak{i},\mathfrak{j}} \mathfrak{k}(\mathfrak{g}; \mathfrak{i}, \mathfrak{j}) \rangle_R$, where $(\mathfrak{i}, \mathfrak{j})$ runs over all pairs of distinct elements of $\mathrm{GF}_{n(\mathfrak{g})-1}(\mathfrak{g})$.
- If $m \geq 3$, then $\mathfrak{g}(m) := \langle [\mathfrak{g}(m_1), \mathfrak{g}(m_2)] \mid m_1 + m_2 = m,\, m_1, m_2 \geq 1 \rangle_R$.

Theorem 4.7. *Let $\mathfrak{g}$ be a configuration space algebra over R such that $n(\mathfrak{g}) \geq 2$ and $(X, n, \varphi : \mathrm{Gr}_R(\Pi_n) \xrightarrow{\sim} \mathfrak{g})$ a CS-envelope for $\mathfrak{g}$. Then the following hold:*

(i) *$g(\mathfrak{g})$ is well-defined (i.e., does not depend on the choice of $\mathfrak{i}, \mathfrak{j}$ in Definition 4.6(ii)) and $X/\mathbb{C}$ is of type $(g(\mathfrak{g}), r(\mathfrak{g}))$.*

(ii) *For any positive integer m, it holds that $\varphi(\mathrm{Gr}_R(\Pi_n)(m)) = \mathfrak{g}(m)$ (here, $\mathrm{Gr}_R(\Pi_n)(m)$ appearing in the left hand side is to be interpreted in the sense of Definition 1.5(iv)).*

Proof. This follows from Proposition 1.11, Theorems 3.10, 4.3, Lemmas 4.4, 4.5, together with the (easily verified) fact that $\binom{n+3}{4} > \binom{n+1}{2} > n$ if $n \geq 2$. $\square$

Remark 4.7.1. $(g(\mathfrak{g}), r(\mathfrak{g})) = (1, 1)$ if and only if $\sharp\, \mathrm{GF}_1(\mathfrak{g}) = \binom{n(\mathfrak{g})+1}{2}$. $g = 1$ and $r \neq 1$ if and only if $\sharp\, \mathrm{E}(\mathfrak{g}) = \binom{n(\mathfrak{g})}{2}$.

5. Generalized fiber subgroups

In the present §5, as an application of §3, we give a group-theoretic algorithm for reconstructing the set of generalized fiber subgroups of given co-length (in a unified way). Let K be an algebraically closed field of characteristic zero, X a hyperbolic curve of type (g, r) over K, n a positive integer, l a prime number, and Σ a set of prime numbers such that $\Sigma = \{l\}$ or $\Sigma = \mathfrak{Primes}$ unless otherwise specified (in particular, $l \in \Sigma$ in either case).

Definition 5.1. Let G be a profinite group.

(i) We shall say that G is *hopfian* (as a profinite group) if any surjective (continuous) endomorphism $G \twoheadrightarrow G$ is an isomorphism.

(ii) (cf. [Mo2] Definition 1.1(ii)) We shall say that G is *elastic* if any topologically finitely generated closed subgroup that is normal in an open subgroup of G is either trivial or open in G.

(iii) (cf. [MT] Definition 6.1) We shall say that G is *nearly abelian* if G has a normal closed subgroup $N \subset G$ such that N is topologically normally genrated by a single element in G and that G/N has an abelian open subgroup.

(iv) (cf. [MT] Definition 1.1(iii)) We shall say that G is *strongly torsion-free* if G is topologically finitely generated, and, moreover, U^{ab} is torsion-free for any open subgroup $U \subset G$.

(v) (cf. [MT] §0) We shall say that G is *slim* if every open subgroup of G is center-free.

(vi) (cf. [MT] Definition 3.1) We shall say that G is *indecomposable* if, for any pair of profinite groups G_1, G_2, if $G \cong G_1 \times G_2$, then either G_1 or G_2 is trivial. We shall say that G is *strongly indecomposable* if every open subgroup of G is indecomposable.

Lemma 5.2 (cf. [RZ] Proposition 2.5.2)**.** *A topologically finitely generated profinite group is hopfian.*

Lemma 5.3. *Let (g, r) be a nonnegative integer such that $2g - 2 + r > 0$. Then the following hold:*

(i) $\Pi^{\Sigma}_{g,r}$ *is elastic, strongly torsion-free, slim, and strongly indecomposable.*

(ii) $(g, r) \in \{(0, 3), (1, 1)\}$ *if and only if* $\Pi^{\Sigma}_{g,r}$ *is nearly abelian.*

Proof. Assertion (i) follows from [MT] Theorem 1.5, [MT] Remark 1.2.2, [MT] Propositions 1.4, 3.2. Assertion (ii) follows from [MT] Proposition 6.2. □

Remark 5.3.1. Lemma 5.3 holds for arbitrary nonempty set of prime numbers Σ.

Theorem 5.4. *Let $N \subset \Pi^{\Sigma}_n$ be a normal closed subgroup of Π^{Σ}_n. Suppose that Π^{Σ}_n/N is elastic and not nearly abelian. Then there exists $N' \in \mathrm{FS}_1(\Pi_n)$ such that $N' \subset N$.*

Proof. Theorem 5.4 in the case Π^{Σ}_n/N is a (not nearly abelian) pro-Σ surface group is proved in [HMM] Proposition 2.3. The proof of [HMM] Proposition 2.3 is also valid in the case Π^{Σ}_n/N is elastic and not nearly abelian. □

Definition 5.5. Let G be a profinite group. Then, for each positive integer m, we shall define a closed subgroup $\gamma_m(G)$ of G as follows: $\gamma_1(G) := G$, $\gamma_{m+1}(G) := \overline{[G, \gamma_m(G)]}$ $(m \geq 1)$. Moreover, we shall write $\mathrm{Gr}^{\mathrm{lcs}}(G) := \bigoplus_{m\geq 1} \gamma_m(G)/\gamma_{m+1}(G)$. It follows from Lemma 5.6 below that $\mathrm{Gr}^{\mathrm{lcs}}(G)$ is a $\mathbb{Z}_{>0}$-graded Lie algebra over $\mathbb{Z}$ (whose Lie bracket is defined in a manner similar to Definition 1.5(ii)). If G is a pro-l group, then $\mathrm{Gr}^{\mathrm{lcs}}(G)$ can be regarded as a $\mathbb{Z}_{>0}$-graded Lie algebra over $\mathbb{Z}_l$ in a natural way.

Lemma 5.6 (cf. [MKS] Theorem 5.3(4))**.** *Let m_1, m_2 be positive integers and G a profinite group. Then it holds that $\overline{[\gamma_{m_1}(G), \gamma_{m_2}(G)]} \subset \gamma_{m_1+m_2}(G)$.*

Lemma 5.7. *Let m be a positive integer. If $r \leq 1$, then $\Pi^l_n(m) = \gamma_m(\Pi^l_n)$. If $g = 0$, then $\Pi^l_n(2m-1) = \Pi^l_n(2m) = \gamma_m(\Pi^l_n)$. In particular, if $g = 0$ or $r \leq 1$, then $\mathrm{Gr}(\Pi^l_n) \cong \mathrm{Gr}^{\mathrm{lcs}}(\Pi^l_n)$ as abstract Lie algebras over $\mathbb{Z}_l$.*

Proof. The third assertion follows from the first and second assertions. First, we verify the equality $\Pi^l_n(m) = \gamma_m(\Pi^l_n)$ under the assumption that $r \leq 1$ by induction on m. $\Pi^l_n(1) = \gamma_1(\Pi^l_n)$ is clear. To verify $\Pi^l_n(2) = \gamma_2(\Pi^l_n)$, let us observe that it follows from the explicit description of $\Pi^{\wedge}_1$ and $\pi_1(X^{\mathrm{cpt}})$ that the surjective homomorphism $(\Pi^l_1)^{\mathrm{ab}} \twoheadrightarrow (\pi_1(X^{\mathrm{cpt}})^l)^{\mathrm{ab}}$ induced by the open immersion $X \hookrightarrow X^{\mathrm{cpt}}$ is an isomorphism. On the other hand, it follows from [HMM] Proposition 2.2(ii) that the surjective homomorphism $(\Pi^l_n)^{\mathrm{ab}} \twoheadrightarrow \overbrace{(\Pi^l_1)^{\mathrm{ab}} \times \cdots \times (\Pi^l_1)^{\mathrm{ab}}}^{n}$ induced by the open immersion $X_n \hookrightarrow \overbrace{X \times_k \cdots \times_k X}^{n}$ is an isomorphism. Thus, we conclude that $\Pi^l_n(2) = \ker(\Pi^l_n \twoheadrightarrow (\Pi^l_n)^{\mathrm{ab}}) = \gamma_2(\Pi^l_n)$. Now suppose that $m \geq 3$, and that the induction hypothesis is in force. Then it follows from the induction hypothesis that $\Pi^l_n(m) \supset \overline{[\Pi^l_n(1), \Pi^l_n(m-1)]} = \overline{[\Pi^l_n, \gamma_{m-1}(\Pi^l_n)]} = \gamma_m(\Pi^l_n)$. Moreover, it follows from the induction hypothesis and Lemma 5.6 that, for each pair of positive integers (m_1, m_2) such that $m_1 + m_2 = n$, it holds that $\overline{[\Pi^l_n(m_1), \Pi^l_n(m_2)]} = \overline{[\gamma_{m_1}(\Pi^l_n), \gamma_{m_2}(\Pi^l_n)]} \subset \gamma_m(\Pi^l_n)$, which implies that $\Pi^l_n(m) \subset \gamma_m(\Pi^l_n)$. This completes the proof of $\Pi^l_n(m) = \gamma_m(\Pi^l_n)$ under the assumption that $r \geq 1$.

Finally, we verify the equalities $\Pi^l_n(2m-1) = \Pi^l_n(2m) = \gamma_m(\Pi^l_n)$ under the assumption that $g = 0$ by induction on m. Since $\pi_1(X^{\mathrm{cpt}}) = \{1\}$, $\Pi^l_n(1) = \Pi^l_n(2) =$

$\Pi_n^l = \gamma_1(\Pi_n^l)$ is clear. Now suppose that $m \geq 2$, and that the induction hypothesis is in force. Then it follows from the induction hypothesis that $\Pi_n^l(2m-1) \supset \Pi_n^l(2m) \supset \overline{[\Pi_n^l(2), \Pi_n^l(2m-2)]} = \overline{[\Pi_n^l, \gamma_{m-1}(\Pi_n^l)]} = \gamma_m(\Pi_n^l)$. Thus, it suffices to show that $\Pi_n^l(2m-1) \subset \gamma_m(\Pi_n^l)$, i.e., it suffices to show that, for each pair of positive integers (m_1, m_2) such that $m_1+m_2 = 2m-1$, it holds that $\overline{[\Pi_n^l(m_1), \Pi_n^l(m_2)]} \subset \gamma_m(\Pi_n^l)$. Now since $m_1 + m_2$ is odd, one of m_1 and m_2 is odd. We may assume that m_1 is odd. Then it follows from the induction hypothesis and Lemma 5.6 that $\overline{[\Pi_n^l(m_1), \Pi_n^l(m_2)]} = \overline{[\gamma_{(m_1+1)/2}(\Pi_n^l), \gamma_{m_2/2}(\Pi_n^l)]} \subset \gamma_{(m_1+m_2+1)/2}(\Pi_n^l) = \gamma_m(\Pi_n^l)$. This completes the proof of $\Pi_n^l(2m-1) = \Pi_n^l(2m) = \gamma_m(\Pi_n^l)$ under the assumption that $g = 0$, hence also of Lemma 5.7. □

Definition 5.8. Let $H \subset \Pi_n^l$ be a closed subgroup of Π_n^l. Then, for each positive integer m, we shall write $H(m) := H \cap \Pi_n^l(m)$, $\mathrm{Gr}^m_{\Pi_n^l}(H) := H(m)/H(m+1) \subset \mathrm{Gr}^m(\Pi_n^l)$. Moreover, we shall write $\mathrm{Gr}_{\Pi_n^l}(H) := \bigoplus_{m\geq 1} \mathrm{Gr}^m_{\Pi_n^l}(H) \subset \mathrm{Gr}(\Pi_n^l)$. $\mathrm{Gr}_{\Pi_n^l}(H)$ is a Lie subalgebra of $\mathrm{Gr}(\Pi_n^l)$ over $\mathbb{Z}_l$. Moreover, if H is normal in Π_n^l, then $\mathrm{Gr}_{\Pi_n^l}(H)$ is a Lie ideal of $\mathrm{Gr}(\Pi_n^l)$ over $\mathbb{Z}_l$.

Remark 5.8.1. If H is normal in Π_n^l, then we obtain a natural exact sequence

$$0 \to \mathrm{Gr}_{\Pi_n^l}(H) \to \mathrm{Gr}(\Pi_n^l) \to \bigoplus_{m=1}^{\infty} (\Pi_n^l(m)/H(m))/(\Pi_n^l(m+1)/H(m+1)) \to 0.$$

In particular, if $N \in \mathrm{GFS}_m(\Pi_n^l)$ (resp. $N \in \mathrm{ES}(\Pi_n^l)$), then $\mathrm{Gr}_{\Pi_n^l}(N) \in \mathrm{GFI}_m(\mathrm{Gr}(\Pi_n^l))$ (resp. $\mathrm{Gr}_{\Pi_n^l}(N) \in \mathrm{EI}(\mathrm{Gr}(\Pi_n^l))$), and, moreover, any element of $\mathrm{GFI}_m(\mathrm{Gr}(\Pi_n^l)), \mathrm{EI}(\mathrm{Gr}(\Pi_n^l))$ is of the form above.

Lemma 5.9. *In the notation of Definition 1.6, the generalized projections correspond to distinct sets $I, I' \subset \{1, \ldots, n+\varepsilon\}$ such that $\sharp I, \sharp I' \leq n-1$ determine distinct generalized fiber subgroups.*

Proof. This follows from Lemma 1.13 and Remark 5.8.1. □

Lemma 5.10. *Let $N \in \mathrm{ES}(\Pi_n^\Sigma)$. Then N is topologically finitely generated.*

Proof. We may assume that $g \leq 1$. Let $p : X_n \to Y$ be an exceptional morphism, where Y is a hyperbolic curve of type $(0,3)$ (resp. $(1,1)$) if $g = 0$ (resp. $g = 1$). By [Ho2] Lemma 2.11, it suffices to show that p is surjective. Suppose that p is not surjective. Then there exists a proper open subscheme $Z \subsetneq Y$ such that p factors through $Z \hookrightarrow Y$. Since the resulting morphism $X_n \to Z$ is dominant and locally of finite type, it follows from (the proof of) [Ho2] Lemma 1.3 that the outer homomorphism $\Pi_n^\wedge \to \pi_1(Z)$ induced by $X_n \to Z$ is open. Since $Z \subsetneq Y$, $\pi_1(Z)$, hence also $\mathrm{Im}(\Pi_n^\wedge \to \pi_1(Z))$, is a surface group which is not isomorphic to $F_2^\wedge$. Thus, it follows from Lemma 5.3, Theorem 5.4 that there exists $N' \in \mathrm{FS}_1(\Pi_n^\wedge)$ such that $\ker(\Pi_n^\wedge \to \pi_1(Z)) \supset N'$. Thus, it holds that $N = \ker(\Pi_n^\wedge \to \pi_1(Y)) \supset N'$.

Write $p' : X_n \to X$ for the projection morphism which corresponds to N' and $\widetilde{p} : Y_n \to Y$ (resp. $\widetilde{p}' : Y_n \to Y$) for the generalized projection morphism which is not a projection morphism (resp. the projection morphism) such that the composite of $X_n \hookrightarrow Y_n$ and $\widetilde{p}$ (resp. $\widetilde{p}'$) coincides with the exceptional morphism p (resp. the composite of p' and $X \hookrightarrow Y$). Moreover, write $\widetilde{N}, \widetilde{N}'$ for the generalized fiber subgroups of $\Pi_n^\wedge(Y)$ corresponding to $\widetilde{p}, \widetilde{p}'$, respectively. Let $x \in X \subset Y$. Then it is clear that $X_n \times_X x \hookrightarrow Y_n \times_Y x$ is an open immersion (where we take $X_n \to X, Y_n \to$

Y to be $p', \widetilde{p}'$, respectively). Thus, it follows from Proposition 1.8 and [Ho2] Lemma 1.2 that the outer homomorphism $N' \to \widetilde{N}'$ induced by $\Pi_n^{\wedge} \twoheadrightarrow \Pi_n^{\wedge}(Y)$ is surjective. Thus, $N \supset N'$ implies that $\widetilde{N} \supset \widetilde{N}'$. Since $\Pi_n^{\wedge}(Y)/\widetilde{N} \cong \Pi_n^{\wedge}(Y)/\widetilde{N}' \cong \Pi_1^{\wedge}(Y)$, it follows from Lemma 5.2 that $\widetilde{N} = \widetilde{N}'$. However, since $\widetilde{p}$ is not a projection morphism and $\widetilde{p}'$ is a projection morphism, it follows from Lemma 5.9 that $\widetilde{N} \neq \widetilde{N}'$. Thus, we obtain a contradiction. This completes the proof of the surjectivity of p, hence also of Lemma 5.10. □

Remark 5.10.1. Actually, we prove a stronger result by a different method later (cf. Lemma 6.17 below). Note that the proof of Lemma 6.17 does not resort to Theorem 5.4.

Theorem 5.11. *Let $N \subset \Pi_n^{\Sigma}$ be a normal closed subgroup of Π_n^{Σ}. Suppose that the following conditions are satisfied:*

- $g = 0$ *or* $r \leq 1$.
- $\mathrm{Gr}^{\mathrm{lcs}}((\Pi_n^{\Sigma}/N)^l)$ *is a free* $\mathbb{Z}_l$*-module, and* $\mathrm{rank}_{\mathbb{Z}_l} \mathrm{Gr}^{\mathrm{lcs}}((\Pi_n^{\Sigma}/N)^l) \geq \begin{cases} 2 & (g = 0) \\ 3 & (g \geq 1) \end{cases}$.
- *For* $a, b \in \mathrm{Gr}^{\mathrm{lcs}}((\Pi_n^{\Sigma}/N)^l)$, *if* $[a, b] = 0$, *then* a *and* b *are linearly dependent over* $\mathbb{Z}_l$.
- Π_n^{Σ}/N *and* $(\Pi_n^{\Sigma}/N)^l$ *are elastic.*

Then there exists $N' \in \mathrm{FS}_1(\Pi_n^{\Sigma}) \cup \mathrm{ES}(\Pi_n^{\Sigma})$ *such that* $N' \subset N$. *Moreover, if further suppose that* $\langle a, b\rangle_{\mathbb{Z}_l} \subsetneq \mathrm{Gr}^{\mathrm{lcs}}((\Pi_n^{\Sigma}/N)^l)$ *for any* $a, b \in \mathrm{Gr}^{\mathrm{lcs}}((\Pi_n^{\Sigma}/N)^l)$, *then there exists* $N' \in \mathrm{FS}_1(\Pi_n^{\Sigma})$ *such that* $N' \subset N$.

Proof. First, suppose that $\Sigma = \{l\}$. Then it follows from Lemma 5.7 that the exact sequence

$$1 \to N \to \Pi_n^l \to \Pi_n^l/N \to 1$$

determines an exact sequence

$$0 \to \mathrm{Gr}_{\Pi_n^l}(N) \to \mathrm{Gr}(\Pi_n^l) \to \mathrm{Gr}^{\mathrm{lcs}}(\Pi_n^l/N) \to 0.$$

Thus, it follows from Corollary 3.3 and Theorem 3.5(i) that there exists $\mathfrak{i} \in \mathrm{FI}_1(\mathrm{Gr}(\Pi_n^l)) \cup \mathrm{EI}(\mathrm{Gr}(\Pi_n^l))$ such that $\mathrm{Gr}_{\Pi_n^l}(N) \supset \mathfrak{i}$. Now it follows from Remark 5.8.1 that there exists $N' \in \mathrm{FS}_1(\Pi_n^l) \cup \mathrm{ES}(\Pi_n^l)$ such that $\mathrm{Gr}_{\Pi_n^l}(N') = \mathfrak{i}$. Note that if $\langle a, b\rangle_{\mathbb{Z}_l} \subsetneq \mathrm{Gr}^{\mathrm{lcs}}(\Pi_n^l/N)$ for any $a, b \in \mathrm{Gr}^{\mathrm{lcs}}(\Pi_n^l/N)$, then it follows from Corollary 3.3 that $N' \in \mathrm{FS}_1(\Pi_n^l)$. Since $\mathrm{Gr}_{\Pi_n^l}(N') \subset \mathrm{Gr}_{\Pi_n^l}(N)$, it follows from Lemma 5.7 that $N'/N' \cap \gamma_2(\Pi_n^l) \subset N/N \cap \gamma_2(\Pi_n^l) (\subset \Pi_n^l/\gamma_2(\Pi_n^l))$. In other words, it holds that $\ker((\Pi_n^l)^{\mathrm{ab}} \twoheadrightarrow (\Pi_n^l/N')^{\mathrm{ab}}) \subset \ker((\Pi_n^l)^{\mathrm{ab}} \twoheadrightarrow (\Pi_n^l/N)^{\mathrm{ab}})$.

Write $p_1 : \Pi_n^l \twoheadrightarrow \Pi_n^l/N$ for the natural surjection. Then, since N' is topologically finitely generated (cf. Lemma 5.10, Corollary 1.9), it holds that $p_1(N') \subset \Pi_n^l/N$ is a topologically finitely generated normal closed subgroup of Π_n^l/N. Moreover, since $\ker((\Pi_n^l)^{\mathrm{ab}} \twoheadrightarrow (\Pi_n^l/N')^{\mathrm{ab}}) \subset \ker((\Pi_n^l)^{\mathrm{ab}} \twoheadrightarrow (\Pi_n^l/N)^{\mathrm{ab}})$, it follows that the image of $p_1(N') \subset \Pi_n^l/N$ by the natural surjection $\Pi_n^l/N \twoheadrightarrow (\Pi_n^l/N)^{\mathrm{ab}}$ is trivial. On the other hand, since $\mathrm{Gr}^{\mathrm{lcs}}(\Pi_n^l/N)$ is nontrivial, $\Pi_n^l/N \supsetneq \gamma_2(\Pi_n^l/N)$. Moreover, since $\mathrm{Gr}^{\mathrm{lcs}}(\Pi_n^l/N)$ is a free $\mathbb{Z}_l$-module, $(\Pi_n^l/N)^{\mathrm{ab}} = (\Pi_n^l/N)/\gamma_2(\Pi_n^l/N) \subset \mathrm{Gr}^{\mathrm{lcs}}(\Pi_n^l/N)$ is infinite, which implies that $p_1(N') \subset \Pi_n^l/N$ is not open. Thus, since Π_n^l/N is elastic, we conclude that $p_1(N')$ is trivial, i.e., $N' \subset N$. This completes the proof of Theorem 5.11 in the case $\Sigma = \{l\}$.

Next, suppose that $\Sigma = \mathfrak{Primes}$. Then it follows from the case $\Sigma = \{l\}$ that there exists $N' \in \mathrm{FS}_1(\Pi_n^\wedge) \cup \mathrm{ES}(\Pi_n^\wedge)$ such that $\mathrm{Im}(N' \hookrightarrow \Pi_n^\wedge \twoheadrightarrow \Pi_n^l) \subset \mathrm{Im}(N \hookrightarrow \Pi_n^\wedge \twoheadrightarrow \Pi_n^l)$. Write $p_2 : \Pi_n^\wedge \twoheadrightarrow \Pi_n^\wedge/N$ for the natural surjection. Then, since N' is topologically finitely generated (cf. Lemma 5.10, Corollary 1.9), it holds that $p_2(N') \subset \Pi_n^\wedge/N$ is a topologically finitely generated normal closed subgroup of $\Pi_n^\wedge/N$. Moreover, since $\mathrm{Im}(N' \hookrightarrow \Pi_n^\wedge \twoheadrightarrow \Pi_n^l) \subset \mathrm{Im}(N \hookrightarrow \Pi_n^\wedge \twoheadrightarrow \Pi_n^l)$, it follows that the image of $p_2(N') \subset \Pi_n^\wedge/N$ by the natural surjection $\Pi_n^\wedge/N \twoheadrightarrow (\Pi_n^\wedge/N)^l$ is trivial. On the other hand, since $\mathrm{Gr}^{\mathrm{lcs}}((\Pi_n^\wedge/N)^l)$ is a nontrivial free $\mathbb{Z}_l$-module, it follows from an argument similar to the above argument that $((\Pi_n^\wedge/N)^l)^{\mathrm{ab}}$ is infinite. In particular, $(\Pi_n^\wedge/N)^l$ is infinite, which implies that $p_2(N') \subset \Pi_n^\wedge/N$ is not open. Thus, since $\Pi_n^\wedge/N$ is elastic, we conclude that $p_2(N')$ is trivial, i.e., $N' \subset N$. This completes the proof of Theorem 5.11. □

Definition 5.12. Let G, H be profinite groups. Then we shall write $N(G, H)$ for the set of all normal closed subgroups N of G such that $G/N \cong H$. It follows directly from the definition that $N(G, H) \neq \emptyset$ if and only if there exists a surjective homomorphism $G \twoheadrightarrow H$ of profinite groups.

Theorem 5.13. *It holds that* $\mathrm{GFS}_1(\Pi_n^\Sigma) = N(\Pi_n^\Sigma, \Pi_1^\Sigma)$.

Proof. $\mathrm{GFS}_1(\Pi_n^\Sigma) \subset N(\Pi_n^\Sigma, \Pi_1^\Sigma)$ is clear. We verify $\mathrm{GFS}_1(\Pi_n^\Sigma) \supset N(\Pi_n^\Sigma, \Pi_1^\Sigma)$. Let $N \in N(\Pi_n^\Sigma, \Pi_1^\Sigma)$. First, suppose that $g = 0$ or $r \leq 1$. Then it follows from Lemma 2.5(i),(v), Lemma 5.3(i), Theorem 5.11 that there exists $N' \in \mathrm{FS}_1(\Pi_n^\Sigma) \cup \mathrm{ES}(\Pi_n^\Sigma)$ such that $N' \subset N$. If $N' \in \mathrm{FS}_1(\Pi_n^\Sigma)$, then, since $\Pi_n^\Sigma/N' \cong \Pi_n^\Sigma/N \cong \Pi_1^\Sigma$, it follows from Lemma 5.2 that the natural surjection $\Pi_n^\Sigma/N' \twoheadrightarrow \Pi_n^\Sigma/N$ is an isomorphism. Thus, we conclude that $N = N' \in \mathrm{FS}_1(\Pi_n^\Sigma) \subset \mathrm{GFS}_1(\Pi_n^\Sigma)$. If $N' \in \mathrm{ES}(\Pi_n^\Sigma)$, then the inclusion $N' \hookrightarrow N$ determines a natural surjection $\Pi_n^\Sigma/N' \twoheadrightarrow \Pi_n^\Sigma/N \cong \Pi_1^\Sigma$. Since Π_n^Σ/N' is isomorphic to F_2^Σ, it holds that Π_1^Σ, hence also $(\Pi_1^\Sigma)^{\mathrm{ab}}$, is topologically generated by two elements. Now it follows from the explicit description of $\Pi_1^\wedge$ that $(g, r) \in \{(0, 3), (1, 1)\}$. In particular, $\Pi_1^\Sigma \cong F_2^\Sigma$. Thus, it follows from Lemma 5.2 that the natural surjection $\Pi_n^\Sigma/N' \twoheadrightarrow \Pi_n^\Sigma/N$ is an isomorphism. This implies that $N = N' \in \mathrm{ES}(\Pi_n^\Sigma) \subset \mathrm{GFS}_1(\Pi_n^\Sigma)$.

Next, suppose that $g \geq 1$ and $r \geq 2$. Then $\Pi_1^\Sigma \ncong F_2^\Sigma$. Thus, it follows from Lemma 5.3, Theorem 5.4 that there exists $N' \in \mathrm{FS}_1(\Pi_n)$ such that $N' \subset N$. Thus, by applying the above argument, we obtain that $N = N' \in \mathrm{FS}_1(\Pi_n^\Sigma) = \mathrm{GFS}_1(\Pi_n^\Sigma)$. This completes the proof of the inclusion $\mathrm{GFS}_1(\Pi_n^\Sigma) \supset N(\Pi_n^\Sigma, \Pi_1^\Sigma)$, hence also of Theorem 5.13. □

Remark 5.13.1. If $g \geq 1$ and $r \geq 2$, then, since we cannot use Lemma 5.7, we cannot apply the method using Lie algebras. If $(g, r) \in \{(0, 3), (1, 1)\}$, then, since Π_1^Σ is nearly abelian (cf. Lemma 5.3(ii)), we cannot apply Theorem 5.4. Note that if $g \geq 2$, then we can show $\mathrm{GFS}_1(\Pi_n) = N(\Pi_n, \Pi_1)$ by using Theorem 7.11 below. However, we cannot use Theorem 7.11 if $g \leq 1$. Thus, to prove Theorem 5.13, we need both Theorem 5.4 and the method using Lie algebras (so far as is known).

Proposition 5.14. *There exists a unique surface group* Π *(up to isomorphism as profinite groups) which satisfies the following conditions:*

(1) $N(\Pi_n^\Sigma, \Pi) \neq \emptyset$.
(2) *For any surface group* Π'*, if* $N(\Pi_n^\Sigma, \Pi') \neq \emptyset$*, then* $N(\Pi, \Pi') \neq \emptyset$.

Moreover, this unique surface group Π *is isomorphic to* $\Pi \cong \Pi_1^\Sigma$ *as profinite groups.*

Proof. In light of Lemma 5.2, Remark 5.3.1, and Theorem 5.4, this follows from an argument similar to the argument of Proposition 3.6. (Note that, if Π' is a pro-Σ' surface group and $N(\Pi_n^\Sigma, \Pi') \neq \emptyset$, then $\Sigma \supset \Sigma'$.) □

Definition 5.15. Let G be a configuration space group. Then it follows from Proposition 5.14 that there exists a unique surface group Π (up to isomorphism as profinite groups) which satisfies the following conditions:

(1) $N(G, \Pi) \neq \emptyset$.
(2) For any surface group Π', if $N(G, \Pi') \neq \emptyset$, then $N(\Pi, \Pi') \neq \emptyset$.

We shall write $\Pi_1(G)$ for this unique surface group. Moreover, we shall write $\mathrm{GF}_1(G) := N(G, \Pi_1(G))$.

Theorem 5.16. *Let G be a configuration space group and $(K, X, n, \Sigma, \varphi : \Pi_n^\Sigma \xrightarrow{\sim} G)$ a CS-envelope for G. Then it holds that $\Pi_1^\Sigma \cong \Pi_1(G)$. Moreover, the isomorphism φ induces a bijection $\mathrm{GFS}_1(\Pi_n^\Sigma) \xrightarrow{1:1} \mathrm{GF}_1(G)$, $N \mapsto \varphi(N)$.*

Proof. This follows from Proposition 5.14, Theorem 5.13. □

Definition 5.17. Let m be a nonnegative integer and G a configuration space group. We shall define a set $\mathrm{GF}_m(G)$ of closed subgroups of G as follows:

- $\mathrm{GF}_0(G) := \{G\}$, $\mathrm{GF}_1(G) := N(G, \Pi_1(G))$ (cf. Definition 5.15). Moreover, for convenience, we shall define $\mathrm{GF}_1(\{1\}) := \emptyset$.
- If $m \geq 2$, then $\mathrm{GF}_m(G) := \bigcup_{N \in \mathrm{GF}_{m-1}(G)} \mathrm{GF}_1(N)$. Here, let us observe, by applying Theorem 5.16 and Proposition 1.8 inductively, that for each $N \in \mathrm{GF}_{m-1}(G)$, if $N \neq \{1\}$, then N is also a configuration space group.

Theorem 5.18. *Let m be a nonnegative integer, G a configuration space group, and $(K, X, n, \Sigma, \varphi : \Pi_n^\Sigma \xrightarrow{\sim} G)$ a CS-envelope for G. Then the isomorphism φ determines a bijection $\mathrm{GFS}_m(\Pi_n^\Sigma) \xrightarrow{1:1} \mathrm{GF}_m(G)$, $N \mapsto \varphi(N)$.*

Proof. In light of Theorem 5.16, this follows from an argument similar to the argument of Theorem 3.10. □

Remark 5.18.1. Another algorithm for reconstructing $\mathrm{GFS}_m(\Pi_n^\Sigma)$ is given in [HMM] Theorem 2.5. Note that the method for reconstructing $\mathrm{GFS}_1(\Pi_n^\Sigma)$ in [HMM] Theorem 2.5 in the case $(g, r) \in \{(0, 3), (1, 1)\}$ is quite different from that in the case $(g, r) \notin \{(0, 3), (1, 1)\}$.

Proposition 5.19. *Let m be a positive integer. Then it holds that $\mathrm{GFS}_m(\Pi_n^\Sigma) = N(\Pi_n^\Sigma, \Pi_m^\Sigma)$.*

Proof. In light of Theorem 5.13, this follows from an argument similar to the argument of Proposition 3.11. □

6. Exceptional subgroups

In the present §6, we discuss some properties of exceptional subgroups. Let K be an algebraically closed field of characteristic zero, X a hyperbolic curve of type (g, r) over K, n a positive integer, l a prime number, and Σ a set of prime numbers such that $\Sigma = \{l\}$ or $\Sigma = \mathfrak{Primes}$ unless otherwise specified.

Now we give a group-theoretic algorithm for reconstructing the set of exceptional subgroups (except for the case $(g, r) \in \{(0, 3), (1, 1)\}$). For this aim, to begin with, we reconstruct some invariants of X_n from Π_n^Σ.

Theorem 6.1 ([HMM] Theorem 1.6)**.** *It holds that* $n = \max\{s \in \mathbb{Z}_{>0} \mid$ *there exists a closed subgroup* $N \subset \Pi_n^l$ *such that* $N \cong \mathbb{Z}_l^{\oplus s}\}$.

Lemma 6.2 (cf. [HM] Lemma 1.3(iv))**.** *Suppose that* $r > 0$. *Let* $N \in \mathrm{FS}_1(\Pi_2^l)$. *Then the kernel of the natural action* $\Pi_1^l \cong \Pi_2^l/N \to \mathrm{Aut}(N^{\mathrm{ab}})$ *determined by the outer representation* $\Pi_1^l \to \mathrm{Out}(N)$ *coincides with the kernel of the surjection* $\Pi_2^l/N \twoheadrightarrow (\pi_1(X^{\mathrm{cpt}})^l)^{\mathrm{ab}}$.

Definition 6.3. Let G be a configuration space group.

(i) We shall write $\Sigma(G)$ for the set of all prime numbers l such that $G^l \neq \{1\}$.

(ii) Let us fix $l \in \Sigma(G)$. Then it follows from Theorem 6.1, together with Theorem 6.4(i) below, that the set $\{s \in \mathbb{Z}_{>0} \mid$ there exists a closed subgroup $N \subset G^l$ such that $N \cong \mathbb{Z}_l^{\oplus s}\}$ is finite and nonempty. We shall write $n(G)$ for the largest integer in this set.

(iii) Suppose that $n(G) \geq 2$, $\sharp\Sigma(G) = 1$, and $\Pi_1(G)$ is a free pro-$\Sigma(G)$ group. Let $H \in \mathrm{GF}_1(G)$ and $N \in \mathrm{GF}_1(H)$. (Note that it follows from Theorem 5.16, together with Theorem 6.4(ii) below, that H is a configuration space group and $\mathrm{GF}_1(H) \neq \emptyset$.) Then it follows from Theorem 5.16 and Remark 1.8.1 that N is a normal closed subgroup of G. We shall write $J(G; H, N)$ for the kernel of the composite of the natural surjection $G \twoheadrightarrow G/H$ and the natural action $G/H \to \mathrm{Aut}((H/N)^{\mathrm{ab}})$ determined by the outer representation $G/H \to \mathrm{Out}(H/N)$.

(iv) Suppose that $n(G) \geq 2$ and $\sharp\Sigma(G) = 1$. Let m be a positive integer. Then let us define a closed subgroup $G(m) \subset G$ as follows:
- $G(1) := G$.
- If $\Pi_1(G)$ is not free, then $G(2) := \overline{[G, G]}$.
- If $\Pi_1(G)$ is free, then $G(2) := \bigcap_{H \in \mathrm{GF}_1(G)} \bigcap_{N \in \mathrm{GF}_1(H)} J(G; H, N)$.
- If $m \geq 3$, then $G(m) := \overline{\langle [G(m_1), G(m_2)] \mid m_1 + m_2 = m,\ m_1, m_2 \geq 1 \rangle}$.

Theorem 6.4. *Let* G *be a configuration space group and* $(K, X, n, \Sigma, \varphi : \Pi_n^\Sigma \xrightarrow{\sim} G)$ *a CS-envelope for* G. *Then the following holds:*

(i) *It holds that* $\Sigma = \Sigma(G)$.

(ii) $n(G)$ *is well-defined (i.e., does not depend on the choice of* $l \in \Sigma(G)$*), and it holds that* $n = n(G)$.

(iii) *Suppose that* $n = n(G) \geq 2$ *and* $\sharp\Sigma = \sharp\Sigma(G) = 1$. *Let* m *be a positive integer. Then it holds that* $\varphi(\Pi_n^\Sigma(m)) = G(m)$ *(here,* $\mathrm{Gr}_R(\Pi_n)(m)$ *appearing in the left hand side is to be interpreted in the sense of Definition 1.5(i)).*

Proof. Assertion (i) is clear. Assertion (ii) follows from assertion (i) and Theorem 6.1. Assertion (iii) follows from Lemma 5.7, Theorem 5.16, Lemma 6.2, together with the fact that for any $H \in \mathrm{GFS}_1(\Pi_n^l)$, there is an automorphism φ of Π_n^l such that $\varphi(H) \in \mathrm{FS}_1(\Pi_n^l)$ and that φ preserves the filtration $(\Pi_n^l(m))_{m\geq 1}$. □

Remark 6.4.1. By using $\mathrm{GF}_m(G)$, we can also reconstruct n as in Remark 4.3.1. Moreover, yet another algorithm for reconstructing n is given in [S1] Theorem 2.15.

Lemma 6.5. *Suppose that* $n \geq 2$. *Let* $N \subset \Pi_n^\Sigma$ *be a normal closed subgroup of* Π_n^Σ. *Write* $\widetilde{N}$ *for the image of* N *by the natural surjection* $\Pi_n^\Sigma \twoheadrightarrow \Pi_n^l$. *Then the following conditions are equivalent:*

(1) N *is topologically finitely generated,* Π_n^Σ/N *is elastic, and* $\mathrm{Gr}_{\Pi_n^l}(\widetilde{N}) \in \mathrm{EI}(\mathrm{Gr}(\Pi_n^l))$.

(2) $N \in \mathrm{ES}(\Pi_n^\Sigma)$.

Proof. The implication (2) $\Rightarrow$ (1) follows from Lemma 5.3(i), Lemma 5.10, and Remark 5.8.1. We verify the implication (1) $\Rightarrow$ (2). Suppose that N is topologically finitely generated, that Π_n^Σ/N is elastic, and that $\mathrm{Gr}_{\Pi_n^l}(\widetilde{N}) \in \mathrm{EI}(\mathrm{Gr}(\Pi_n^l))$. Then it follows from Remark 5.8.1 that there exists $N' \in \mathrm{ES}(\Pi_n^\Sigma)$ such that, if we write $\widetilde{N}'$ for the image of N' by the natural surjection $\Pi_n^\Sigma \twoheadrightarrow \Pi_n^l$, then it holds that $\mathrm{Gr}_{\Pi_n^l}(\widetilde{N}') = \mathrm{Gr}_{\Pi_n^l}(\widetilde{N})$. Write $m = \begin{cases} 1 & (g > 0) \\ 2 & (g = 0) \end{cases}$. Then, in light of Lemma 5.7, $\Pi_n^l(m+1)$ is a normal closed subgroup of Π_n^l and it holds that $\Pi_n^l/\widetilde{N}' \cdot \Pi_n^l(m+1) \cong \mathrm{Gr}^m(\Pi_n^l)/\mathrm{Gr}^m_{\Pi_n^l}(\widetilde{N}')$. It follows from the explicit description of exceptional ideals (cf. Definition 3.2, Theorem 3.5(i)) that $\mathrm{Gr}^m(\Pi_n^l)/\mathrm{Gr}^m_{\Pi_n^l}(\widetilde{N}')$ is infinite. Moreover, since $\mathrm{Gr}_{\Pi_n^l}(\widetilde{N}') = \mathrm{Gr}_{\Pi_n^l}(\widetilde{N})$, it holds that $\widetilde{N}'/\widetilde{N}' \cap \Pi_n^l(m+1) = \widetilde{N}/\widetilde{N} \cap \Pi_n^l(m+1)$ in $\Pi_n^l/\Pi_n^l(m+1) = \mathrm{Gr}^m(\Pi_n^l)$.

Write $p : \Pi_n^\Sigma \twoheadrightarrow \Pi_n^\Sigma/N' (\cong F_2^\Sigma)$ for the natural surjection. Then $p(N) \subset \Pi_n^\Sigma/N'$ is a topologically finitely generated normal closed subgroup of Π_n^Σ/N'. Moreover, since $\widetilde{N}'/\widetilde{N}' \cap \Pi_n^l(m+1) = \widetilde{N}/\widetilde{N} \cap \Pi_n^l(m+1)$, the image of $p(N)$ by the natural surjection $\Pi_n^\Sigma/N' \twoheadrightarrow \Pi_n^l/\widetilde{N}' \cdot \Pi_n^l(m+1)$ is trivial. Since $\Pi_n^l/\widetilde{N}' \cdot \Pi_n^l(m+1)$ is infinite, $p(N) \subset \Pi_n^\Sigma/N'$ is not open. Thus, it follows from Lemma 5.3(i) that $p(N)$ is trivial, i.e., $N \subset N'$. Now it follows from Lemma 5.10 that $N'/N \subset \Pi_n^\Sigma/N$ is a topologically finitely generated normal closed subgroup of Π_n^Σ/N. Moreover, since Π_n^Σ/N' is infinite, N'/N is not open in Π_n^Σ/N. Thus, since (we have assumed that) Π_n^Σ/N is elastic, we conclude that N'/N is trivial, i.e., $N = N' \in \mathrm{ES}(\Pi_n^\Sigma)$. This completes the proof of Lemma 6.5. □

Definition 6.6. Let G be a configuration space group. If $n(G) = 1$, then we shall write $\mathrm{E}(G) := \emptyset$. If $n(G) \geq 2$, then we define $\mathrm{E}(G)$ as follows: Let $l \in \Sigma(G)$. Then it follows from Theorem 6.4 that $\mathfrak{g} := \bigoplus_{m \geq 1} G^l(m)/G^l(m+1)$ is a configuration space algebra over $\mathbb{Z}_l$. We shall write $\mathrm{E}(G)$ for the set of all topologically finitely generated normal closed subgroups N of G such that G/N is elastic and that $\bigoplus_{m \geq 1} \widetilde{N} \cap G^l(m)/\widetilde{N} \cap G^l(m+1) \in \mathrm{E}(\mathfrak{g})$, where $\widetilde{N}$ is the image of N by the natural surjection $G \twoheadrightarrow G^l$.

Theorem 6.7. *Let G be a configuration space group and $(K, X, n, \Sigma, \varphi : \Pi_n^\Sigma \xrightarrow{\sim} G)$ a CS-envelope for G. Then $\mathrm{E}(G)$ is well-defined (i.e., does not depend on the choice of $l \in \Sigma(G)$). Moreover, if $\Pi_1(G) \not\cong F_2^{\Sigma(G)}$, then the isomorphism φ determines a bijection $\mathrm{ES}(\Pi_n^\Sigma) \overset{1:1}{\to} \mathrm{E}(G)$, $N \mapsto \varphi(N)$.*

Proof. This follows from Theorem 3.8(iii), Theorems 5.16, 6.4, and Lemma 6.5. □

Remark 6.7.1. If $g = 0$, then, by Theorem 5.11, we obtain that $\mathrm{ES}(\Pi_n^\Sigma) = \{N \in N(\Pi_n^\Sigma, F_2^\Sigma) \mid N' \not\subset N \text{ for any } N' \in \mathrm{FS}_1(\Pi_n^\Sigma)\}$. If $g = 0$ and $r \geq 4$, this gives another group-theoretic algorithm for reconstructing $\mathrm{ES}(\Pi_n^\Sigma)$.

Next, we prove that there are some exceptional subgroups which is not of SESG-type (cf. Definition 6.9).

Definition 6.8 (cf. [S1] Definition 2.6)**.** A *successive extension of surface groups* is data $(G, (G_m)_{0 \leq m \leq n}, (\Sigma_m)_{1 \leq m \leq n})$ consisting of

- a profinite group G;
- a sequence of profinite groups $(G_m)_{0\leq m\leq n}$;
- a sequence of nonempty sets of prime numbers $(\Sigma_m)_{1\leq m\leq n}$

such that

- $G_0 = G$, $G_n = \{1\}$;
- for any integer m such that $1 \leq m \leq n$, G_m is a normal closed subgroup of G_{m-1}, and, moreover, G_{m-1}/G_m is a pro-Σ_m surface group.

Definition 6.9 (cf. [S3] Definition 1.12)**.** Let G be a profinite group and $(G_m)_{0\leq m\leq n}$ a sequence of subgroups of G. Then we shall say that $(G_m)_{0\leq m\leq n}$ is an *SESG-filtration* on G if there exists a sequence of nonempty sets of prime numbers $(\Sigma_m)_{1\leq m\leq n}$ such that $(G, (G_m)_{0\leq m\leq n}, (\Sigma_m)_{1\leq m\leq n})$ is a successive extension of surface groups. We shall say that a profinite group G is of *SESG-type* if G has an SESG-filtration.

Remark 6.9.1 (cf. [S1] Remark 2.6.2(ii)). An open subgroup of a profinite group of SESG-type is of SESG-type.

Lemma 6.10 (cf. [S1] Lemma 2.10)**.** *Let d be a nonnegative integer, G a profinite group of SESG-type, and A a finite G-module. Then $H^d(G, A)$ is finite.*

Definition 6.11 (cf. [S1] Definition 1.3)**.** Let G be a profinite group.

(i) Let A be a G-module. For each nonnegative integer d, we shall write

$$h^d(G, A) := \log(\sharp H^d(G, A)).$$

(ii) Let A be a G-module. Suppose that $h^d(G, A) < \infty$ for any integer d, and that $h^d(G, A) = 0$ for all but finitely many integers d. Then we shall write

$$\chi(G, A) := \sum_{d=0}^{\infty} (-1)^d h^d(G, A).$$

In this case, we shall say that "$\chi(G, A)$ is defined".

(iii) Let Σ be a nonempty set of prime numbers. Suppose that there exists a (unique) constant $b \in \mathbb{R}$ such that, for any finite Σ-torsion G-module A (i.e., for any $a \in A$, there exists a positive integer n such that $na = 0$ and that every prime factor of n is contained in Σ), it holds that $\chi(G, A)$ is defined, and that $\chi(G, A) = b \log(\sharp A)$. Then we shall write

$$\chi_\Sigma(G) := b.$$

In this case, we shall say that "$\chi_\Sigma(G)$ is defined".

Lemma 6.12 (cf. [S1] Lemma 1.4(ii))**.** *Let Σ be a nonempty set of prime numbers, $1 \to G_1 \to G_2 \to G_3 \to 1$ a short exact sequence of profinite groups, and A a finite Σ-torsion G_2-module. Suppose that $\chi_\Sigma(G_3)$ is defined. Then, if $\chi(G_1, A)$ is defined, then $\chi(G_2, A)$ is also defined, and it holds that $\chi(G_2, A) = \chi(G_1, A) \cdot \chi_\Sigma(G_3)$. In particular, if $\chi_\Sigma(G_1)$ is defined, then $\chi_\Sigma(G_2)$ is also defined, and it holds that $\chi_\Sigma(G_2) = \chi_\Sigma(G_1) \cdot \chi_\Sigma(G_3)$.*

Lemma 6.13 (cf. [S1] Proposition 2.7(i),(iv))**.** *Let Σ be a nonempty set of prime numbers, (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$, and $l \in \Sigma$. Then the following hold:*

(i) $\mathrm{cd}_l(\Pi_{g,r}^\Sigma) = \begin{cases} 1 & (r > 0) \\ 2 & (r = 0) \end{cases}$.

(ii) $\chi_\Sigma(\Pi_{g,r}^\Sigma)$ *is defined, and it holds that* $\chi_\Sigma(\Pi_{g,r}^\Sigma) = 2 - 2g - r$.

Lemma 6.14 (cf. [S1] Corollary 2.11)**.** *Let l be a prime number and $(G, (G_j)_{0 \le j \le n}, (\Sigma_j)_{1 \le j \le n})$ a successive extension of surface groups. Then it holds that* $\mathrm{cd}_l\, G = \sum_{j=1}^n \mathrm{cd}_l(G_{j-1}/G_j)$.

Corollary 6.15. *The following hold:*

(i) $\mathrm{cd}_l(\Pi_n^\Sigma) = \begin{cases} n & (r > 0) \\ n+1 & (r = 0) \end{cases}$.

(ii) $\chi_\Sigma(\Pi_n^\Sigma)$ *is defined, and it holds that* $\chi_\Sigma(\Pi_n^\Sigma) = \prod_{m=0}^{n-1}(2 - 2g - r - m)$.

Proof. This follows from Lemmas 6.12, 6.13, 6.14, and Proposition 1.8. □

Definition 6.16. Let X, Y be connected noetherian schemes and $f : X \to Y$ a morphism.

(i) We shall write

$$\Delta_f^{(\Sigma)} = \Delta_{X/Y}^{(\Sigma)} \subset \pi_1(X)^\Sigma$$

for the kernel of the outer homomorphism $\pi_1(X)^\Sigma \to \pi_1(Y)^\Sigma$ induced by f. We shall write simply

$$\Delta_{X/Y} := \Delta_{X/Y}^{(\mathfrak{Primes})}.$$

If $Y = \mathrm{Spec}\, A$, then by abuse of notation we often write

$$\Delta_{X/A},\ \Delta_{X/A}^{(\Sigma)}$$

instead of $\Delta_{X/Y}$, $\Delta_{X/Y}^{(\Sigma)}$, respectively.

(ii) We shall write

$$\Pi_{X/Y}^\Sigma$$

for the quotient of $\pi_1(X)$ by the kernel of the natural surjection from $\Delta_{X/Y}$ to its maximal pro-Σ quotient (which is a characteristic subgroup of $\Delta_{X/Y}$). For a prime number l, we shall write simply

$$\Pi_{X/Y}^l$$

instead of $\Pi_{X/Y}^{\{l\}}$. If $Y = \mathrm{Spec}\, A$, then by abuse of notation we often write

$$\Pi_{X/A}^\Sigma$$

instead of $\Pi_{X/S}^\Sigma$.

Lemma 6.17. *Let $N \in \mathrm{ES}(\Pi_n^\Sigma)$. Then N is topologically generated by $n(2g+r) + \frac{(n+1)(n-4)}{2}$ elements.*

Proof. Let $p : X_n \to Y$ be an exceptional morphism such that $\Delta_p^{(\Sigma)} = N$ and factor p into $X_n \hookrightarrow Y_n \to Y$ as in Definition 1.10(i). Then there exists a generalized projection morphism $p' : Y_n \to Y_{n-1}$ such that p' is not a projection morphism and that p' factors through $Y_n \to Y$. Write $\iota : X_n \hookrightarrow Y_n$. First, let us observe that, to prove Lemma 6.17, it suffices to show that the composite $p' \circ \iota : X_n \to Y_{n-1}$ is surjective, and, moreover, there exists a open subscheme $U \subset Y_{n-1}$ of Y_{n-1} such that $(p' \circ \iota)^{-1}(U) \subset X_n$ is a hyperbolic curve of type $(g, n(2g - 2 + r) -$

$2g+2$) over U. Indeed, if so, then by applying [Ho2] Proposition 1.10, where we take the data "$(Y, X, S, Y \to X, X \to S, \overline{s} \to S)$" to be $(X_n, X_n, Y_{n-1}, \mathrm{id}_{X_n}, p' \circ \iota, \overline{\eta} \to Y_{n-1})$ (where $\overline{\eta}$ is a generic geometric point of Y_{n-1}), we obtain a surjective homomorphism $\Pi^{\Sigma}_{X_n \times_{Y_{n-1}} \overline{\eta}} \twoheadrightarrow \Delta^{(\Sigma)}_{p' \circ \iota}$. Since $X_n \times_{Y_{n-1}} \overline{\eta} \cong (p' \circ \iota)^{-1}(U) \times_U \overline{\eta}$ is a hyperbolic curve of type $(g, n(2g-2+r)-2g+2)$ over $\overline{\eta}$, it holds that $\Pi^{\Sigma}_{X_n \times_{Y_{n-1}} \overline{\eta}} \cong \Pi^{\Sigma}_{g, n(2g-2+r)-2g+2} \cong F^{\Sigma}_{n(2g-2+r)+1}$. On the other hand, it follows from Proposition 1.8 that, for each integer m such that $2 \le m \le n-1$, it holds that $\Delta^{(\Sigma)}_{Y_m/Y_{m-1}} \cong F^{\Sigma}_{m+1}$. Since $p' \circ \iota$ and p determine an exact sequence $1 \to \Delta^{(\Sigma)}_{p' \circ \iota} \to \Delta^{(\Sigma)}_{p} \to \Delta^{(\Sigma)}_{Y_{n-1}/Y} \to 1$, we conclude that $N = \Delta^{(\Sigma)}_p$ is generated by $n(2g-2+r)+1+\sum_{m=2}^{n-1}(m+1) = n(2g+r) + \frac{(n+1)(n-4)}{2}$ elements.

Now we consider the morphism $p' \circ \iota$. First, suppose that $g = 1$. In this case, p' corresponds to $\{n+1\} \subset \{1, \dots, n+1\}$ in the notation of Definition 1.6(b). Let us identify $Y = E \setminus \{O\}$ as in Definition 1.6(b) and write $\{b_1, \dots, b_{r-1}\} = Y \setminus X$. Moreover, write φ_n for the isomorphism $Y_n \xrightarrow{\sim} E_{n+1}/E$, $(y_1, \dots, y_n) \mapsto [y_1, \dots, y_n, O]$ (cf. Definition 1.6(b)) and ψ for the automorphism of E_{n+1}/E determined by $[y_1, \dots, y_{n-1}, y_n, O] \mapsto [y_1, \dots, y_{n-1}, O, y_n] (= [y_1 - y_n, \dots, y_{n-1} - y_n, -y_n, O])$. Furthermore, write $\widetilde{X_n} := X_n \times_{Y_n} Y_n$, where $Y_n \to Y_n$ is the automorphism $\varphi_n^{-1} \circ \psi \circ \varphi_n$. Then the first projection morphism $p_1 : \widetilde{X_n} \to X_n$ is an isomorphism. Write $f := p' \circ \iota \circ p_1$. Then it holds that $f(y_1, \dots, y_{n-1}, y_n) = (y_1, \dots, y_{n-1})$. Thus, since $\varphi_n^{-1} \circ \psi \circ \varphi_n(y_1, \dots, y_n) = (y_1 - y_n, \dots, y_{n-1} - y_n, -y_n)$, for $a = (a_1, \dots, a_{n-1}) \in Y_{n-1}$, the fiber $f^{-1}(a)$ is

$$\{(y_1, \dots, y_{n-1}, y_n) \in Y_n \mid y_1 - y_n, \dots, y_{n-1} - y_n, -y_n \in X,\ (y_1, \dots, y_{n-1}) = a\},$$

i.e., $f^{-1}(a) = \{a\} \times (E \setminus D_a) \subset Y_{n-1} \times E$, where

$$\begin{aligned} D_a :=& \{O\} \cup \{a_s \mid 1 \le s \le n-1\} \cup \\ & \{a_s - b_t \mid 1 \le s \le n-1,\ 1 \le t \le r-1\} \cup \{-b_t \mid 1 \le t \le r-1\}. \end{aligned}$$

This implies that, if we write $D \subset Y_{n-1} \times E$ for the divisor determined by nr equations

$$y_n = O,\ y_n = y_s,\ y_n = -b_t,\ y_n = y_s - b_t\ (1 \le s \le n-1,\ 1 \le t \le r-1),$$

then $D_a \subset E$ is a fiber of the composite of the immersion $D \hookrightarrow Y_{n-1} \times E$ and the first projection $Y_{n-1} \times E \to Y_{n-1}$ over $a \in Y_{n-1}$, and, moreover, $\widetilde{X_n} \hookrightarrow Y_n$ determines an isomorphism $\widetilde{X_n} \xrightarrow{\sim} (Y_{n-1} \times E) \setminus D$. Since $D_a \subset E$ is finite, the fiber $f^{-1}(a)$ is nonempty for any $a \in Y_{n-1}$, i.e., f is surjective. Moreover, if we write $U \subset Y_{n-1}$ for the complement of the divisors of Y_{n-1} determined by equations

$$\begin{gathered} y_s = b_t,\ y_s = -b_t\ (1 \le s \le n-1,\ 1 \le t \le r-1), \\ y_s = y_{s'} - b_t\ (1 \le s, s' \le n-1,\ 1 \le t, \le r-1,\ s \ne s'), \\ -b_t = y_s - b_{t'}\ (1 \le s \le n-1,\ 1 \le t, t' \le r-1,\ t \ne t'), \\ y_s - b_t = y_{s'} - b_{t'}\ (1 \le s, s' \le n-1,\ 1 \le t, t' \le r-1,\ s \ne s',\ t \ne t'), \end{gathered}$$

then $\sharp D_a = nr$ for all $a \in U$. Thus, $f^{-1}(U) \subset \widetilde{X_n}$, hence also $(p' \circ \iota)^{-1}(U) \subset X_n$, is a hyperbolic curve of type $(1, nr)$ over U.

Next, suppose that $g = 0$. We may assume that p' corresponds to $\{n+1\} \subset \{1, \dots, n+3\}$ in the notation of Definition 1.6(a). Let us identify $Y = \mathbb{P}^1_K \setminus$

$\{0,1,\infty\}$ as in Definition 1.6(a) and write $\{b_1,\dots,b_{r-3}\} = Y \setminus X$. Moreover, write φ_n for the isomorphism $Y_n \xrightarrow{\sim} (\mathcal{M}_{0,n+3})_K$, $(y_1,\dots,y_n) \mapsto [y_1,\dots,y_n,0,1,\infty]$ (cf. Definition 1.6(a)) and ψ for the automorphism of $(\mathcal{M}_{0,n+3})_K$ determined by $[y_1,\dots,y_{n-1},y_n,0,1,\infty] \mapsto [y_1,\dots,y_{n-1},0,y_n,1,\infty](= [\frac{y_n-y_1}{y_n-1},\dots,\frac{y_n-y_{n-1}}{y_n-1},\frac{y_n}{y_n-1},0,1,\infty])$. Furthermore, write $\widetilde{X_n} := X_n \times_{Y_n} Y_n$, where $Y_n \to Y_n$ is the automorphism $\varphi_n^{-1} \circ \psi \circ \varphi_n$. Then $p_1 : \widetilde{X_n} \to X_n$ is an isomorphism. Write $f := p' \circ \iota \circ p_1$. Then it holds that $f(y_1,\dots,y_{n-1},y_n) = (y_1,\dots,y_{n-1})$. Thus, since $\varphi_n^{-1} \circ \psi \circ \varphi_n(y_1,\dots,y_n) = (\frac{y_n-y_1}{y_n-1},\dots,\frac{y_n-y_{n-1}}{y_n-1},\frac{y_n}{y_n-1})$, for $a = (a_1,\dots,a_{n-1}) \in Y_{n-1}$, the fiber $f^{-1}(a)$ is

$$\left\{(y_1,\dots,y_{n-1},y_n) \in Y_n \,\middle|\, \frac{y_n-y_1}{y_n-1},\dots,\frac{y_n-y_{n-1}}{y_n-1},\frac{y_n}{y_n-1} \in X,\ (y_1,\dots,y_{n-1}) = a\right\},$$

i.e., $f^{-1}(a) = \{a\} \times (\mathbb{P}^1_K \setminus D_a) \subset Y_{n-1} \times \mathbb{P}^1_K$, where

$$\begin{aligned} D_a :=& \{0,1,\infty\} \cup \{a_s \mid 1 \le s \le n-1\} \cup \\ & \left\{\frac{b_t-a_s}{b_t-1} \,\middle|\, 1 \le s \le n-1,\ 1 \le t \le r-3\right\} \cup \left\{\frac{b_t}{b_t-1} \,\middle|\, 1 \le t \le r-3\right\}. \end{aligned}$$

This implies that, if we write $D \subset Y_{n-1} \times \mathbb{P}^1_K$ for the divisor determined by $n(r-2)+2$ equations

$$y_n = 0,\ y_n = 1,\ y_n = \infty,\ y_n = y_s,\ y_n = \frac{b_t}{b_t-1},\ y_n = \frac{b_t-y_s}{b_t-1}\ (1 \le s \le n-1,\ 1 \le t \le r-3),$$

then $D_a \subset \mathbb{P}^1_K$ is a fiber of the composite of the immersion $D \hookrightarrow Y_{n-1} \times \mathbb{P}^1_K$ and the first projection $Y_{n-1} \times \mathbb{P}^1_K \to Y_{n-1}$ over $a \in Y_{n-1}$, and, moreover, $\widetilde{X_n} \hookrightarrow Y_n$ determines an isomorphism $\widetilde{X_n} \xrightarrow{\sim} (Y_{n-1} \times \mathbb{P}^1_K) \setminus D$. Since $D_a \subset \mathbb{P}^1_K$ is finite, the fiber $f^{-1}(a)$ is nonempty for any $a \in Y_{n-1}$, i.e., f is surjective. Moreover, if we write $U \subset Y_{n-1}$ for the complement of the divisors of Y_{n-1} determined by equations

$$\begin{aligned} y_s = b_t,\ y_s = \frac{b_t}{b_t-1}\ & (1 \le s \le n-1,\ 1 \le t \le r-3), \\ y_s = \frac{b_t-y_{s'}}{b_t-1}\ & (1 \le s,s' \le n-1,\ 1 \le t \le r-3,\ s \ne s'), \\ \frac{b_t}{b_t-1} = \frac{b_{t'}-y_s}{b_{t'}-1}\ & (1 \le s \le n-1,\ 1 \le t,t' \le r-3,\ t \ne t'), \\ \frac{b_t-y_s}{b_t-1} = \frac{b_{t'}-y_{s'}}{b_{t'}-1}\ & (1 \le s,s' \le n-1,\ 1 \le t,t' \le r-3,\ s \ne s',\ t \ne t'), \end{aligned}$$

then $\sharp D_a = n(r-2)+2$ for all $a \in U$. Thus, $f^{-1}(U) \subset \widetilde{X_n}$, hence also $(p' \circ \iota)^{-1}(U) \subset X_n$, is a hyperbolic curve of type $(0, n(r-2)+2)$ over U. This completes the proof of Lemma 6.17. □

Definition 6.18. Suppose that $g = 0$. We divide the set $\mathrm{ES}(\Pi_n^\Sigma)$ into three sets $\mathrm{ES}^m(\Pi_n^\Sigma)$ $(m \in \{1,2,3\})$ as follows: Let $N \in \mathrm{ES}(\Pi_n^\Sigma)$. Let $p : X_n \to Y$ be an exceptional morphism such that $\Delta_p^{(\Sigma)} = N$. Factor p into $X_n \hookrightarrow Y_n \to Y$ as in Definition 1.10(i), and write $I' \subset \{1,\dots,n+3\}$ for the subset of $\{1,\dots,n+3\}$ of cardinality $n-1$ such that the generalized projection morphism $Y_n \twoheadrightarrow Y$ coincides with $p_{I'}$ in the notation of Definition 1.6(a). Then $N \in \mathrm{ES}^m(\Pi_n^\Sigma)$ if $\sharp(I' \cap \{n+1,n+2,n+3\}) = m$. If $K = \mathbb{C}$, then we shall define $\mathrm{ES}^m(\Pi_n)$ similarly.

Theorem 6.19. *Let $N \in \mathrm{ES}(\Pi_n^{\Sigma})$ and A a finite nonzero Σ-torsion Π_n^{Σ}-module. Suppose that one of the following hold:*

- *$g = 1$ and $r \geq 2$.*
- *$g = 0$, $r \geq 4$, and $N \in \mathrm{ES}^1(\Pi_n^{\Sigma})$.*

Then $H^n(N, A)$ is infinite. In particular, N is not of SESG-type.

Proof. The second assertion follows from the first assertion and Lemma 6.10. We verify the first assertion by induction on n. Let $p : X_n \to Y$ be an exceptional morphism such that $\Delta_p^{(\Sigma)} = N$ and factor p into $X_n \hookrightarrow Y_n \to Y$ as in Definition 1.10(i). First, let us suppose that $n = 2$. Moreover, suppose that $H^2(N, A)$ is finite. It follows from Lemma 6.17 that there exists a surjective homomorphism $F_{4g+2r-3}^{\Sigma} \twoheadrightarrow N$, which implies that $h^0(N, A) = h^0(F_{4g+2r-3}^{\Sigma}, A)$, $h^1(N, A) \leq h^1(F_{4g+2r-3}^{\Sigma}, A) (< \infty)$. Moreover, it follows from Corollary 6.15(i) that $\mathrm{cd}_{l'}(N) \leq \mathrm{cd}_{l'}(\Pi_2^{\Sigma}) = 2$ for all $l' \in \Sigma$. Thus, since (we have assumed that) $H^2(N, A)$ is finite, $\chi(N, A)$ is defined and it holds that

$$\begin{aligned}\chi(N, A) &\geq h^0(N, A) - h^1(N, A) \\ &\geq h^0(F_{4g+2r-3}^{\Sigma}, A) - h^1(F_{4g+2r-3}^{\Sigma}, A) \\ &= \chi(F_{4g+2r-3}^{\Sigma}, A) = -(4g + 2r - 4)\log(\sharp A)\end{aligned}$$

(cf. Lemma 6.13).

Now it follows from Lemma 6.12 and Lemma 6.13(ii) that $\chi(\Pi_2^{\Sigma}, A) = \chi(N, A) \cdot \chi_{\Sigma}(\Pi_1^{\Sigma}(Y)) = -\chi(N, A) \leq (4g+2r-4)\log(\sharp A)$. On the other hand, it follows from Corollary 6.15 that $\chi(\Pi_2^{\Sigma}, A) = \chi_{\Sigma}(\Pi_2^{\Sigma})\log(\sharp A) = (2g + r - 1)(2g + r - 2)\log(\sharp A)$. Thus, we conclude that $(2g + r - 1)(2g + r - 2) \leq 4g + 2r - 4$. However, since $(2g + r - 1)(2g + r - 2) - (4g + 2r - 4) = (2g + r - 2)(2g + r - 3) > 0$, we obtain a contradiction. Thus, we conclude that $H^2(N, A)$ is infinite.

Now suppose that $n \geq 3$, and that the induction hypothesis is in force. Then it follows from our assumption that we can factor the exceptional morphism $p : X_n \to Y$ into $p_2 \circ p_1$, where $p_1 : X_n \to X_{n-1}$ is a projection morphism and $p_2 : X_{n-1} \to Y$ is an exceptional morphism. Now we obtain an exact sequence

$$1 \to \Delta_{p_1}^{(\Sigma)} \to \Delta_p^{(\Sigma)} \to \Delta_{p_2}^{(\Sigma)} \to 1.$$

Note that $\Delta_{p_1}^{(\Sigma)} \cong \Pi_{g,r+n-1}^{\Sigma}$, $\Delta_p^{(\Sigma)} = N$, and $\Delta_{p_2}^{(\Sigma)} \in \mathrm{ES}(\Pi_{n-1}^{\Sigma})$. Let us consider the Hochschild-Serre spectral sequence

$$E_2^{st} = H^s(\Delta_{p_2}^{(\Sigma)}, H^t(\Delta_{p_1}^{(\Sigma)}, A)) \Rightarrow H^{s+t}(N, A)$$

(cf. e.g., [NSW] Theorem (2.4.1)). Since A is a Σ-torsion Π_n^{Σ}-module, $H^1(\Delta_{p_1}^{(\Sigma)}, A)$ is a Σ-torsion Π_{n-1}^{Σ}-module. Moreover, it follows from Lemma 6.13 that $H^1(\Delta_{p_1}^{(\Sigma)}, A)$ is finite and nonzero. Thus, it follows from the induction hypothesis that $E_2^{n-1,1} = H^{n-1}(\Delta_{p_2}^{(\Sigma)}, H^1(\Delta_{p_1}^{(\Sigma)}, A))$ is infinite. On the other hand, it follows from Corollary 6.15 that $\mathrm{cd}_{l'}(\Delta_{p_2}^{(\Sigma)}) \leq \mathrm{cd}_{l'}(\Pi_{n-1}^{\Sigma}) = n-1$ and $\mathrm{cd}_{l'}(\Delta_{p_1}^{(\Sigma)}) = 1$ for any prime number $l' \in \Sigma$. Thus, we obtain that $H^n(N, A) \cong E_2^{n-1,1}$, which implies that $H^n(N, A)$ is infinite. This completes the proof of Theorem 6.19. □

Remark 6.19.1. In the notation of the proof of Lemma 6.17, by an argument similar to the argument in the proof of Theorem 6.19, we can show that if $(g, r) \notin \{(0, 3), (1, 1)\}$ and $\mathrm{cd}_l(\Delta_{p' \circ \iota}^{(\Sigma)}) \leq 2$, then, for any finite nonzero Σ-torsion Π_n^{Σ}-module

A, it holds that $H^2(\Delta^{(\Sigma)}_{p'\circ\iota}, A)$ is infinite. On the other hand, [S1] Theorem 2.15 implies that, if $\Delta^{(\Sigma)}_{p'\circ\iota}$ is of SESG-type, then $\Delta^{(\Sigma)}_{p'\circ\iota}$ must be a surface group. Thus, it follows from Lemma 6.13 that $\Delta^{(\Sigma)}_{p'\circ\iota}$ is not of SESG-type. However, the author does not know at the time of writing whether $N \in \mathrm{ES}^2(\Pi_n^\Sigma) \cup \mathrm{ES}^3(\Pi_n^\Sigma)$ is of SESG-type or not. The author conjectures that, for any finite nonzero Σ-torsion Π_n^Σ-module A, it holds that $H^n(N, A)$ is infinite.

Note that we can reconstruct $\mathrm{ES}^m(\Pi_n^\Sigma)$ if $g = 0$ and $r \geq 4$. More specifically, the following Proposition 6.20 gives a group-theoretic algorithm for reconstructing $\mathrm{ES}^m(\Pi_n^\Sigma)$.

Proposition 6.20. *Let $N \in \mathrm{ES}(\Pi_n^\Sigma)$ and $m \in \{1, 2, 3\}$. Then the following conditions are equivalent:*

1. $N \in \mathrm{ES}^m(\Pi_n^\Sigma)$.
2. *There exists $N' \in \mathrm{FS}_{m+1}(\Pi_n^\Sigma)$ such that $N' \subset N$. Moreover, for any $N' \in \mathrm{FS}_m(\Pi_n^\Sigma)$, it holds that $N' \not\subset N$.*
3. $\sharp\{N' \in \mathrm{FS}_{n-1}(\Pi_n^\Sigma) \mid N' \subset N\} = n - m - 1$.

Proof. The equivalence (2) $\Leftrightarrow$ (3) follows from [MT] Proposition 2.4(iv). We consider the equivalence (1) $\Leftrightarrow$ (2). To prove this equivalence, it suffices to show the implication (1) $\Rightarrow$ (2). Let I' be as in Definition 6.1. Suppose that $N \in \mathrm{ES}^m(\Pi_n^\Sigma)$. Write $\overline{I}' := I' \cap \{1, \ldots, n\}$. Then $p_{I'} : Y_n \to Y$ factors through the projection morphism $p_{\overline{I}'} : Y_n \to Y_{m+1}$. Moreover, it is clear that the composite $X_n \hookrightarrow Y_n \overset{p_{\overline{I}'}}{\to} Y_{m+1}$ factors through the projection morphism $X_n \to X_{m+1}$ obtained by forgetting the factors corresponding to $\overline{I}'$. Thus, if we write $N' \in \mathrm{FS}_{m+1}(\Pi_n^\Sigma)$ for the fiber subgroup corresponding to the projection morphism $X_n \to X_{m+1}$, then it holds that $N' \subset N$.

Now let us verify that for any $N'' \in \mathrm{FS}_m(\Pi_n^\Sigma)$, it holds that $N'' \not\subset N$. In light of [MT] Proposition 2.4(iv), we may assume that $N' \subset N''$, where $N' \in \mathrm{FS}_{m+1}(\Pi_n^\Sigma)$ as above. Write $\overline{I}'' \subset \{1, \ldots, n\}$ for the subset of $\{1, \ldots, n\}$ such that N'' corresponds to $\overline{I}''$. Then it follows from an argument similar to the argument appearing in the proof of Lemma 5.10 that the images of N', N'' by the surjection $\Pi_n^\Sigma \twoheadrightarrow \Pi_n^\Sigma(Y)$ is the fiber subgroup of $\Pi_n^\Sigma(Y)$ corresponding to $\overline{I}', \overline{I}''$, respectively. Thus, $\Pi_n^\Sigma(Y) \twoheadrightarrow \Pi_{m+1}^\Sigma(Y) \twoheadrightarrow \Pi_1^\Sigma(Y)$ factors through $\Pi_{m+1}^\Sigma(Y) \twoheadrightarrow \Pi_m^\Sigma(Y)$ induced by the projection morphism forgetting the factor corresponding to $\overline{I}'' \setminus \overline{I}'$. Now it follows from Theorem 5.13 that the kernel of the resulting surjection $\Pi_m^\Sigma(Y) \twoheadrightarrow \Pi_1^\Sigma(Y)$ is in $\mathrm{GFS}_1(\Pi_m^\Sigma(Y))$. Thus, we conclude that there exists $I'' \subset \{1, \ldots, n+3\}$ (of cardinality $(n-1)$) such that $\overline{I}'' \subset I''$ and that the generalized fiber subgroup of $\Pi_n^\Sigma(Y)$ corresponding to I'' coincides with that of I'. This implies, in light of Lemma 5.9, that $I' = I''$. However, since $I' \cap \{1, \ldots, n\} = \overline{I}' \subsetneq \overline{I}'' \subset I'' \cap \{1, \ldots, n\}$, it holds that $I' \neq I''$. Thus, we obtain a contradiction. This completes the proof of the equivalence (1) $\Leftrightarrow$ (2), hence also of Proposition 6.20. $\square$

7. Grothendieck conjecture-type result between a configuration space and a hyperbolic polycurve

In the present §7, we consider Grothendieck's anabelian conjecture for (a certain étale covering of) a configuration space and a hyperbolic polycurve (cf. Definition

7.1). Let K be a field of characteristic zero (not necessarily algebraically closed), $\overline{K}$ an algebraic closure of K, X a hyperbolic curve of type (g, r) over K, n a positive integer, l a prime number, and Σ a set of prime numbers such that $\Sigma = \{l\}$ or $\Sigma = \mathfrak{Primes}$ unless otherwise specified. Write $G_K := \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group of K.

Definition 7.1 (cf. [S3] Definitions 2.2, 2.3)**.** Let S be a scheme.

(i) We shall say that Y is a *hyperbolic polycurve (of relative dimension n)* over S if there exist a positive integer n and a (not necessarily unique) factorization of the structure morphism $Y \to S$
$$Y = Y_{(n)} \to Y_{(n-1)} \to \cdots \to Y_{(2)} \to Y_{(1)} \to S = Y_{(0)}$$
such that, for each integer k such that $1 \leq m \leq n$, $Y_{(m)} \to Y_{(m-1)}$ is a hyperbolic curve. We shall refer to the above factorization of $Y \to S$ as a *sequence of parametrizing morphisms*.

(ii) A *parametrized hyperbolic polycurve (of relative dimension n)* is a pair $\mathfrak{Y} = (Y, Y = Y_{(n)} \to Y_{(n-1)} \to \cdots \to Y_{(1)} \to S = Y_{(0)})$ consisting of a hyperbolic polycurve Y (of relative dimension n) over S and a sequence of parametrizing morphisms $Y = Y_{(n)} \to Y_{(n-1)} \to \cdots \to Y_{(1)} \to S = Y_{(0)}$ of Y/S. We shall refer to Y (over S) as an *underlying hyperbolic polycurve* of $\mathfrak{Y}$.

(iii) Let $\mathfrak{Y}$ be a parametrized hyperbolic polycurve over S whose underlying hyperbolic polycurve is Y. Then we shall write $\Delta_{\mathfrak{Y}/S} := \Delta_{Y/S}$, $\Pi^{\Sigma}_{\mathfrak{Y}/S} := \Pi^{\Sigma}_{Y/S}$.

(iv) Let T be a scheme and $\mathfrak{Y} = (Y, Y = Y_{(n)} \to Y_{(n-1)} \to \cdots \to Y_{(1)} \to S = Y_{(0)})$, $\mathfrak{Z} = (Z, Z = Z_{(n)} \to Z_{(n-1)} \to \cdots \to Z_{(1)} \to T = Z_{(0)})$ parametrized hyperbolic polycurves of relative dimension n. An *isomorphism* from $\mathfrak{Z}$ to $\mathfrak{Y}$ is defined to be a collection of isomorphisms of schemes $\{Z_{(m)} \xrightarrow{\sim} Y_{(m)}\}_{0 \leq m \leq n}$ such that, for each integer m such that $1 \leq m \leq n$, the diagram
$$\begin{array}{ccc} Z_{(m)} & \xrightarrow{\sim} & Y_{(m)} \\ \downarrow & & \downarrow \\ Z_{(m-1)} & \xrightarrow{\sim} & Y_{(m-1)} \end{array}$$
commutes. If $S = T$, then an *S-isomorphism* $\mathfrak{Z} \xrightarrow{\sim} \mathfrak{Y}$ is defined to be an isomorphism $\{Z_{(m)} \xrightarrow{\sim} Y_{(m)}\}_{0 \leq m \leq n}$ from $\mathfrak{Z}$ to $\mathfrak{Y}$ such that $T = Z_{(0)} \xrightarrow{\sim} Y_{(0)} = S$ is the identity morphism (i.e., each $Z_{(m)} \xrightarrow{\sim} Y_{(m)}$ is an S-isomorphism).

Proposition 7.2 (cf. [Ho2] Proposition 2.3)**.** *Let $\mathfrak{Y} = (Y, Y = Y_{(n)} \to Y_{(n-1)} \to \cdots \to Y_{(1)} \to S = Y_{(0)})$ be a parametrized hyperbolic polycurve over S and $Z \to Y$ a connected finite étale covering of Y. For each integer m such that $0 \leq m \leq n$, write $Z_{(m)}$ for the normalization of $Y_{(m)}$ in the finite extension of the function field of $Y_{(m)}$ obtained by forming the algebraic closure of the function field of $Y_{(m)}$ in the function field of Z. Then $\mathfrak{Z} := (Z, Z = Z_{(n)} \to Z_{(n-1)} \to \cdots \to Z_{(1)} \to Z_{(0)})$ is a parametrized hyperbolic polycurve.*

Definition 7.3 (cf. [S3] Definition 2.4)**.** Let n be a positive integer, S a connected noetherian separated normal scheme over K, and $\mathfrak{Y} = (Y, Y = Y_{(n)} \to Y_{(n-1)} \to \cdots \to Y_{(1)} \to S = Y_{(0)})$ a parametrized hyperbolic polycurve of relative dimension

n over S. Then we shall say that $\mathfrak{Y}/S$ satisfies condition $(*)_l$ if for any triplet of integers (i, j, k) such that $0 \leq i < j < k \leq n$, the sequence of profinite groups

$$1 \to \Delta^l_{Y_{(k)}/Y_{(j)}} \to \Delta^l_{Y_{(k)}/Y_{(i)}} \to \Delta^l_{Y_{(j)}/Y_{(i)}} \to 1$$

is exact.

Example 7.4. For each $1 \leq m \leq n$, let $p_m : X_m \to X_{m-1}$ be a generalized projection morphism. Then $(X_n, X_n \stackrel{p_n}{\to} X_{n-1} \stackrel{p_{n-1}}{\to} \cdots \stackrel{p_2}{\to} X_1 \stackrel{p_1}{\to} \operatorname{Spec} K = X_0)$ is a parametrized hyperbolic polycurve over K satisfying condition $(*)_l$ (cf. Proposition 1.8).

Remark 7.4.1. Let S be a connected noetherian separated normal scheme over K and $\mathfrak{Y} = (Y, Y = Y_{(n)} \to Y_{(n-1)} \to \cdots \to Y_{(1)} \to S = Y_{(0)})$ a parametrized hyperbolic polycurve over S. If $\Sigma = \{l\}$, then suppose that $\mathfrak{Y}$ satisfies condition $(*)_l$. Then the data $(\Delta^{\Sigma}_{Y/S}, (\Delta^{\Sigma}_{Y/Y_{(m)}})_{0 \leq m \leq n}, (\Sigma)_{1 \leq m \leq n})$ is a successive extension of surface groups (cf. [S2] Remark 3.8).

Proposition 7.5 (cf. [S2] Lemma 3.22(i))**.** *Let* $\mathfrak{Y} = (Y, Y = Y_{(n)} \to Y_{(n-1)} \to \cdots \to Y_{(1)} \to S = Y_{(0)})$ *be a parametrized hyperbolic polycurve over* S *and* $U \subset \Pi^l_{\mathfrak{Y}/S}$ *an open subgroup of* $\Pi^l_{\mathfrak{Y}/S}$*. Write* $Z \to Y$ *for the connected finite étale covering of* Y *corresponding to the inverse image of* $U \subset \Pi^l_{\mathfrak{Y}/S}$ *by the natural surjection* $\pi_1(Y) \twoheadrightarrow \Pi^l_{\mathfrak{Y}/S}$*, and* $\mathfrak{Z}$ *for the parametrized hyperbolic polycurve determined by* $Z \to Y$ *and* $\mathfrak{Y}$ *in the notation of Proposition 7.2. Suppose that* $\mathfrak{Y}$ *satisfies condition* $(*)_l$*. Then* $\mathfrak{Z}$ *satisfies* $(*)_l$*.*

Definition 7.6 (cf. [Mo1] Definition 4.11)**.** We shall say that K is *generalized sub-l-adic* if K is isomorphic to a subfield of a finitely generated extension of the quotient field of $W(\overline{\mathbb{F}_l})$ (the ring of Witt vectors with coefficients in $\overline{\mathbb{F}_l}$).

Theorem 7.7 (cf. [S3] Theorem 2.6)**.** *Suppose that* K *is generalized sub-l-adic. Let* $\mathfrak{Y} = (Y, Y = Y_{(n)} \to Y_{(n-1)} \to \cdots \to Y_{(1)} \to \operatorname{Spec} K = Y_{(0)})$*,* $\mathfrak{Z} = (Z, Z = Z_{(n)} \to Z_{(n-1)} \to \cdots \to Z_{(1)} \to \operatorname{Spec} K = Z_{(0)})$ *be parametrized hyperbolic polycurves of dimension* n *over* K *and* $\varphi : \Pi^{\Sigma}_{\mathfrak{Y}/K} \xrightarrow{\sim} \Pi^{\Sigma}_{\mathfrak{Z}/K}$ *an isomorphism from* $\Pi^{\Sigma}_{\mathfrak{Y}/K}$ *to* $\Pi^{\Sigma}_{\mathfrak{Z}/K}$ *over* G_K*. If* $\Sigma = \{l\}$*, then suppose that both* $\mathfrak{Y}$ *and* $\mathfrak{Z}$ *satisfy condition* $(*)_l$*. Moreover, suppose that for each integer* m *such that* $0 \leq m \leq n$*, it holds that* $\varphi(\Delta^{\Sigma}_{Y/Y_{(m)}}) = \Delta^{\Sigma}_{Z/Z_{(m)}}$*. Then* φ *arises from a unique isomorphism* $\mathfrak{Y} \xrightarrow{\sim} \mathfrak{Z}$ *over* K*.*

Now, to give the Grothendieck conjecture-type result mentioned at the beginning of this section, we study the generalized fiber subgroups of $\Delta_{X_n/K} \cong \Pi_n(X \times_K \overline{K})$ in more detail. Let us assume that $K = \overline{K}$ until Corollary 7.13 below.

Definition 7.8 (cf. [MT] Definition 2.3(ii))**.** Let $H \subset \Pi^{\Sigma}_n$ be a closed subgroup of Π^{Σ}_n. Then we shall say that H is *product-theoretic* if H is the inverse image of a closed subgroup of $\pi_1(P_n)^{\Sigma}$ by the natural surjection $\Pi^{\Sigma}_n \twoheadrightarrow \pi_1(P_n)^{\Sigma}$ (cf. Definition 1.4).

Theorem 7.9. *Let* $H \subset \Pi^{\Sigma}_n$ *be a product-theoretic open subgroup of* Π^{Σ}_n*,* G *a strongly torsion-free profinite group, and* $H \to G$ *a homomorphism of profinite groups. Suppose that* $g \geq 2$*. Moreover, suppose that* $r = 0$ *or* G *is pro-solvable. Then* $\ker(H \to G) \subset \Pi^{\Sigma}_n$ *is product-theoretic.*

Proof. If $\Sigma = \mathfrak{Primes}$, then Theorem 7.9 follows from [MT] Theorems 4.7, 5.6(ii). Suppose that $\Sigma = \{l\}$. Let us observe that the inverse image of $\ker(\Pi_n^l \twoheadrightarrow \pi_1(P_n)^l) \subset \Pi_n^l$ by the natural surjection $\Pi_n^\wedge \twoheadrightarrow \Pi_n^l$ contains $\ker(\Pi_n^\wedge \twoheadrightarrow \pi_1(P_n)) \subset \Pi_n^\wedge$, and, moreover, since the operation of taking the maximal pro-l quotient of a profinite group is right exact, the resulting homomorphism $\ker(\Pi_n^\wedge \twoheadrightarrow \pi_1(P_n)) \to \ker(\Pi_n^l \twoheadrightarrow \pi_1(P_n)^l)$ is surjective. Write $\widetilde{H} \subset \Pi_n^\wedge$ for the inverse image of $H \subset \Pi_n^l$ by the natural surjection $\Pi_n^\wedge \twoheadrightarrow \Pi_n^l$. Then it follows from the observation above that $\widetilde{H}$ contains $\ker(\Pi_n^\wedge \twoheadrightarrow \pi_1(P_n))$, i.e., $\widetilde{H} \subset \Pi_n^\wedge$ is product-theoretic. Thus, it follows from the case $\Sigma = \mathfrak{Primes}$ that $\ker(\widetilde{H} \twoheadrightarrow H \to G) \subset \Pi_n^\wedge$ is product-theoretic. Thus, it follows from the observation above that $\ker(H \to G) \subset \Pi_n^l$ contains $\ker(\Pi_n^l \twoheadrightarrow \pi_1(P_n)^l)$, i.e., $\ker(H \to G) \subset \Pi_n^l$ is product-theoretic. This completes the proof of Theorem 7.9. □

Proposition 7.10 (cf. [MT] Proposition 3.3)**.** *Let $G_1, \ldots, G_n; Q$ be profinite groups and $\phi : \prod_{m=1}^n G_m \twoheadrightarrow Q$ a surjective homomorphism of profinite groups. Then there exist normal closed subgroups $H_m \subset G_m$ $(1 \le m \le n)$, $N \subset Q$ such that N is contained in the center of Q and that ϕ and the natural surjection $Q \twoheadrightarrow Q/N$ induce an isomorphism $\prod_{m=1}^n G_m/H_m \xrightarrow{\sim} Q/N$. In particular, if Q is center-free and indecomposable, then there exists $m \in \{1, \ldots, n\}$ such that ϕ induces an isomorphism $G_m/H_m \xrightarrow{\sim} Q$.*

Theorem 7.11. *Let $H \subset \Pi_n^\Sigma$ be a product-theoretic open subgroup of Π_n^Σ and $N \subset H$ a normal closed subgroup of H. Suppose that $g \ge 2$ and that H/N is elastic. Moreover, suppose that there exists a profinite group Q such that the following hold:*

- *There exists a surjective homomorphism $H/N \twoheadrightarrow Q$ of profinite groups.*
- *Q is infinite, slim, strongly indecomposable, and strongly torsion-free.*
- *$r = 0$ or Q is pro-solvable.*

Then there exists $N' \in \mathrm{FS}_1(\Pi_n^\Sigma)$ such that $N' \cap H \subset N$.

Proof. The following proof is essentially due to the proof of [MT] Corollary 4.8. Let us fix a profinite group Q and a surjective homomorphism $H/N \twoheadrightarrow Q$ as in the statement. Then it follows from Theorem 7.9 that $\ker(H \twoheadrightarrow H/N \twoheadrightarrow Q)$ is a product-theoretic subgroup of Π_n^Σ. Thus, the composite $H \twoheadrightarrow H/N \twoheadrightarrow Q$ factors through $H \to \mathrm{Im}(H \hookrightarrow \Pi_n^\Sigma \twoheadrightarrow \pi_1(P_n)^\Sigma)$. Write $\varphi : \mathrm{Im}(H \to \pi_1(P_n)^\Sigma) \to Q$ for the resulting (surjective) homomorphism.

Now let us observe that $\pi_1(P_n)^\Sigma$ has a decomposition $\pi_1(P_n)^\Sigma = V_1 \times \cdots \times V_n$, where $V_m \cong \Pi_1^\Sigma$ corresponds to the m-th factor of $P_n = \overbrace{X \times_k \cdots \times_k X}^{n}$. Since $H \subset \Pi_n^\Sigma$ is open, $\mathrm{Im}(H \to \pi_1(P_n)^\Sigma) \subset \pi_1(P_n)^\Sigma$ is open. Thus, there exist open subgroups $U_m \subset V_m$ $(1 \le m \le n)$ such that, if we write $U := U_1 \times \cdots \times U_n \subset V_1 \times \cdots \times V_m$, then $U \subset \mathrm{Im}(H \to \pi_1(P_n)^\Sigma) \subset \pi_1(P_n)^\Sigma$. Since Q is slim and strongly indecomposable, the open subgroup $\varphi(U) \subset Q$ is center-free and indecomposable. Thus, it follows from Proposition 7.10 that there exists an integer $m \in \{1, \ldots, n\}$ such that $U_1 \times \cdots \times U_{m-1} \times U_{m+1} \times \cdots \times U_n \subset \ker \varphi|_U$. This implies that, if we write $N' \in \mathrm{FS}_1(\Pi_n^\Sigma)$ for the fiber subgroup determined by the projection morphism corresponding to $\{1, \ldots, m-1, m+1, \ldots, n\}$ in the notation of Definition 1.6(c), then it holds that $\mathrm{Im}(N' \hookrightarrow \Pi_n^\Sigma \twoheadrightarrow \pi_1(P_n)^\Sigma) \cap U \subset \ker \varphi$. Thus, since $U \subset \pi_1(P_n)^\Sigma$ is open, $\mathrm{Im}(N' \cap H \hookrightarrow H \twoheadrightarrow H/N \twoheadrightarrow Q) \subset Q$ is finite. This implies, in light of the infiniteness of Q, that $\mathrm{Im}(N' \cap H \hookrightarrow H \twoheadrightarrow H/N) \subset H/N$ is of infinite index in H/N. Moreover, since N', hence also $N' \cap H$, is topologically finitely generated (cf. Corollary 1.9), it holds that $\mathrm{Im}(N' \cap H \hookrightarrow H \twoheadrightarrow H/N) \subset H/N$ is a topologically finitely generated normal closed subgroup of H/N. Thus, it follows from the elasticity of H/N that $\mathrm{Im}(N' \cap H \hookrightarrow H \twoheadrightarrow H/N)$ is trivial, i.e., $N' \cap H \subset N$. This completes the proof of Theorem 7.11. □

Theorem 7.12. *Let $H \subset \Pi_n^\Sigma$ be a product-theoretic open subgroup of Π_n^Σ and $N \subset H$ a normal closed subgroup of H. Suppose that H/N is a surface group. Moreover, suppose that at least one of the following holds:*

- *$n = 1$ or $g \geq 2$.*
- *$H = \Pi_n^\Sigma$ and $g = 0$.*
- *$H = \Pi_n^\Sigma$ and $r \leq 1$.*
- *$H = \Pi_n^\Sigma$ and H/N is not nearly abelian.*

Then there exists $N' \in \mathrm{FS}_1(\Pi_n^\Sigma) \cup \mathrm{ES}(\Pi_n^\Sigma)$ such that $N \supset N' \cap H$. Moreover, if further suppose that N is topologically finitely generated, then it holds that $N = N' \cap H$.

Proof. First, we verify the first assertion. Let us observe that for a prime number l', it follows from the well-known fact that every finite l'-group is solvable, together with Lemma 5.3(i), that pro-l' surface group is infinite, slim, strongly indecomposable, strongly torsion-free, and pro-solvable. Thus, if H/N is a pro-Σ' surface group, then the first assertion in the case $g \geq 2$ follows from Theorem 7.11, where we take "$H/N \twoheadrightarrow Q$" for the natural surjection $H/N \twoheadrightarrow (H/N)^{l'}$ for a fixed prime number $l' \in \Sigma'$. Moreover, the first assertion in other three cases follow from Lemma 2.5(i),(v), Lemma 5.3(i), Theorems 5.4, 5.11.

Next, we verify the second assertion. Suppose that N is topologically finitely generated. Then $N/N' \cap H = \ker(H/N' \cap H \twoheadrightarrow H/N) \subset H/N' \cap H$ is a topologically finitely generated normal closed subgroup of $H/N' \cap H$. Moreover, since H/N is a surface group (hence H/N is infinite), $N/N' \cap H$ is not open in $H/N' \cap H$. On the other hand, since $H/N' \cap H$ is an open subgroup of a surface group Π_n^Σ/N', it follows from Lemma 5.3(i) that $H/N' \cap H$ is elastic. Thus, we conclude that $N/N' \cap H = \{1\}$, i.e., $N = N' \cap H$. This completes the proof of Theorem 7.12. □

Corollary 7.13. *Let $H \subset \Pi_n^\Sigma$ be a product-theoretic open subgroup of Π_n^Σ and $N \subset H$ a normal closed subgroup of H. Suppose that at least one of the following holds:*

- *$n = 1$ or $g \geq 2$.*
- *$H = \Pi_n^\Sigma$ and $(g, n) = (0, 2)$.*
- *$H = \Pi_n^\Sigma$ and $(g, r) \in \{(0, 3), (1, 1)\}$.*
- *$H = \Pi_n^\Sigma$ and H/N is not nearly abelian.*

Then the following conditions are equivalent:

1. *N is trivial or of SESG-type, and H/N is a surface group.*
2. *There exists $N' \in \mathrm{GFS}_1(\Pi_n^\Sigma)$ such that $N = N' \cap H$.*

Proof. The implication (1) ⇒ (2) follows from Theorems 7.12, 6.19, together with the easily verified fact that a profinite group of SESG-type is topologically finitely generated (note that if $n = 2$ and $g = 0$, then $\mathrm{ES}(\Pi_n^\Sigma) = \mathrm{ES}^1(\Pi_n^\Sigma)$). The implication (2) ⇒ (1) follows from Remarks 1.3.1, 6.9.1. □

Now we are ready to give a Grothendieck conjecture-type result as an application of Theorem 7.7 and Corollary 7.13.

Theorem 7.14. *Suppose that K is generalized sub-l-adic. Let $H \subset \Pi^{\Sigma}_{X_n/K}$ be an open subgroup of $\Pi^{\Sigma}_{X_n/K}$ such that $H \cap \Delta^{\Sigma}_{X_n/K}$ is a product-theoretic (open) subgroup of $\Delta^{\Sigma}_{X_n/K} \cong \Pi^{\Sigma}_n(X \times_K \overline{K})$. Write $Y \to X_n$ for the connected finite étale covering of X_n corresponding to the inverse image of the open subgroup $H \subset \Pi^{\Sigma}_{X_n/K}$ by the natural surjection $\pi_1(X_n) \twoheadrightarrow \Pi^{\Sigma}_{X_n/K}$, and L for the finite extension field of K corresponding to the open subgroup $\mathrm{Im}(H \hookrightarrow \Pi^{\Sigma}_{X_n/K} \twoheadrightarrow G_K)$ of G_K. Let n' be a positive integer, $\mathfrak{Z} = (Z, Z = Z_{(n')} \to Z_{(n'-1)} \to \cdots \to Z_{(1)} \to \mathrm{Spec}\, L = Z_{(0)})$ a parametrized hyperbolic polycurve of dimension n' over L, and $\varphi : H \xrightarrow{\sim} \Pi^{\Sigma}_{\mathfrak{Z}/L}$ an isomorphism from H to $\Pi^{\Sigma}_{\mathfrak{Z}/L}$ over G_L. If $\Sigma = \{l\}$, then suppose that $\mathfrak{Z}$ satisfies condition $(*)_l$. Moreover, suppose that at least one of the following holds:*

- *$n = 1$ or $g \geq 2$.*
- *$H \supset \Delta^{\Sigma}_{X_n/K}$ and $(g, n) = (0, 2)$.*
- *$H \supset \Delta^{\Sigma}_{X_n/K}$ and $(g, r, n) \in \{(0, 3, 3), (1, 1, 2)\}$.*
- *$H \supset \Delta^{\Sigma}_{X_n/K}$ and for each integer m such that $1 \leq m \leq n'$, the hyperbolic curve $Z_{(m)}/Z_{(m-1)}$ is not of type $(0, 3), (1, 1)$.*
- *$H \supset \Delta^{\Sigma}_{X_n/K}$ and for each integer m such that $1 \leq m \leq n'$, it holds that $Z_{(m)} = (Z_{(1)})_m$ and $Z_{(m)} \to Z_{(m-1)}$ is a generalized projection morphism.*

Then the following hold:

(i) *It holds that $n = n'$.*

(ii) *The natural map*

$$\mathrm{Isom}_L(Y, Z) \to \mathrm{Isom}_{G_L}(H, \Pi^{\Sigma}_{Z/L}) / \mathrm{Inn}(\Delta^{\Sigma}_{Z/L})$$

is bijective.

(iii) *If $(g, r) = (0, 3)$, then suppose that $X \cong \mathbb{P}^1_K \setminus \{0, 1, \infty\}$. Then there exist generalized projection morphisms $p_m : X_m \to X_{m-1}$ $(1 \leq m \leq n)$ such that, if we write $\mathfrak{Y}$ for the parametrized hyperbolic polycurve determined by the connected finite étale covering $Y \to X_n$ with respect to $(X_n, X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} X_1 \xrightarrow{p_1} \mathrm{Spec}\, K = X_0)$ in the notation of Proposition 7.2, then φ arises from a unique isomorphism $\mathfrak{Y} \xrightarrow{\sim} \mathfrak{Z}$ over L.*

Proof. Assertion (i) follows from [S1] Theorem 2.15. First, we verify that assertion (ii) follows from assertion (iii). It follows from [Ho2] Proposition 3.2(i), [S2] Proposition 4.2(i) that the map $\mathrm{Isom}_L(Y, Z) \to \mathrm{Isom}_{G_L}(H, \Pi^{\Sigma}_{Z/L}) / \mathrm{Inn}(\Delta^{\Sigma}_{Z/L})$ is injective. Thus, it suffices to show that this map is surjective. If $(g, r) \neq (0, 3)$, then the surjectivity immediately follows from assertion (iii). Now suppose that $(g, r) = (0, 3)$. Then there exists a finite field extension M of L such that $X \times_K M \cong \mathbb{P}^1_M \setminus \{0, 1, \infty\}$. Let us observe that the inverse image of $G_M \subset G_L (\subset G_K)$ by $\Pi^{\Sigma}_{X_n/K} \twoheadrightarrow G_K$ (resp. $H \twoheadrightarrow G_L$, $\Pi^{\Sigma}_{Z/L} \twoheadrightarrow G_L$) is naturally identified with $\Pi^{\Sigma}_{(X \times_K M)_n/M}$ (resp. $\Pi^{\Sigma}_{Y \times_L M/M}$, $\Pi_{Z \times_L M/M}$).

Let $\psi \in \mathrm{Isom}_{G_L}(H, \Pi^{\Sigma}_{Z/L})$. Then the restriction of ψ to $\Pi^{\Sigma}_{Y \times_L M/M} \subset H$ determines an isomorphism $\Pi^{\Sigma}_{Y \times_L M/M} \xrightarrow{\sim} \Pi^{\Sigma}_{Z \times_L M/M}$ over G_M. Now it follows from assertion (iii) that the isomorphism $\Pi^{\Sigma}_{Y\times_L M/M} \overset{\sim}{\to} \Pi^{\Sigma}_{Z\times_L M/M}$ arises from an isomorphism $Y \times_L M \overset{\sim}{\to} Z \times_L M$. Moreover, it follows from [Ho2] Proposition 3.2(i), [S2] Proposition 4.2(i) that the isomorphism $Y \times_L M \overset{\sim}{\to} Z \times_L M$ is compatible with the natural actions of $\mathrm{Gal}(M/L)$. Thus, by descending the morphism, we obtain an isomorphism $Y \overset{\sim}{\to} Z$ over L. Since $\Delta^{\Sigma}_{Z/L}$ is slim (cf. [Ho2] Proposition 2.4(iii), [S2] Proposition 3.16(iii)), it follows from [S2] Lemma 2.20 that the isomorphism $H = \Pi^{\Sigma}_{Y/L} \overset{\sim}{\to} \Pi^{\Sigma}_{Z/L}$ arising from $Y \overset{\sim}{\to} Z$ belongs to the $\Delta^{\Sigma}_{Z/L}$-conjugacy class determined by ψ, which implies that ψ arises from an isomorphism $Y \overset{\sim}{\to} Z$.

Thus, to verify Theorem 7.14, it suffices to verify assertion (iii). First, we show that there exists $N^{(1)} \in \mathrm{GFS}_1(\Delta^{\Sigma}_{X_n/K})$ such that $\varphi(N^{(1)} \cap H) = \Delta^{\Sigma}_{Z/Z_{(1)}}$. In the fifth case, this follows from Theorem 5.16. In other case, it holds that $\Delta^{\Sigma}_{Z/Z_{(1)}} \subset \Delta^{\Sigma}_{Z/L}$ is trivial or of SESG-type, and $\Delta^{\Sigma}_{Z/L}/\Delta^{\Sigma}_{Z/Z_{(1)}} \cong \Delta^{\Sigma}_{Z_{(1)}/L}$ is a surface group. Moreover, if $Z_{(1)}/L$ is not of type $(0,3),(1,1)$, then it follows from Lemma 5.3(ii) that $\Delta^{\Sigma}_{Z_{(1)}/L}$ is not nearly abelian. Thus, it follows from Corollary 7.13 that there exists $N^{(1)} \in \mathrm{GFS}_1(\Delta^{\Sigma}_{X_n/K})$ such that $\varphi^{-1}(\Delta^{\Sigma}_{Z/L}) = N^{(1)} \cap H$, i.e., $\varphi(N^{(1)} \cap H) = \Delta^{\Sigma}_{Z/Z_{(1)}}$.

Now it follows from Proposition 1.8 that $N^{(1)}$ corresponds to the $(n-1)$-st configuration space of a hyperbolic curve of type $(g, r+1)$. Moreover, if $g \geq 2$ (hence $\mathrm{GFS}_1(\Delta^{\Sigma}_{X_n/K}) = \mathrm{FS}_1(\Delta^{\Sigma}_{X_n/K})$), then it is clear that $N^{(1)} \cap H \subset N^{(1)}$ is product-theoretic. Thus, by applying the above argument to $N^{(1)}$ (note that if $(g,r,n) = (0,3,3)$, then $(g,n-1) = (0,2)$), there exists $N^{(2)} \in \mathrm{GFS}_1(N^{(1)})$ (with respect to the identification of Proposition 1.8) such that $\varphi(N^{(2)} \cap H) = \Delta^{\Sigma}_{Z/Z_{(2)}}$. By applying the above argument repeatedly, we obtain a sequence of subgroups $N^{(n)} \subset N^{(n-1)} \subset \cdots \subset N^{(2)} \subset N^{(1)} \subset \Delta^{\Sigma}_{X_n/K} =: N^{(0)}$ such that, for each integer m such that $1 \leq m \leq n$, it holds that $N^{(m)} \in \mathrm{GFS}_1(N^{(m-1)})$ and $\varphi(N^{(m)} \cap H) = \Delta^{\Sigma}_{Z/Z_{(m)}}$. Thus, if we write $p_m : X_m \to X_{m-1}$ for the generalized projection morphism corresponding to $N^{(m)} \subset N^{(m-1)}$ (note that, to ensure the existence of the generalized projection morphism, we have to assume that $X \cong \mathbb{P}^1_K \setminus \{0,1,\infty\}$ if $(g,r) = (0,3)$) and write $\mathfrak{Y}$ for the parametrized hyperbolic polycurve determined by $Y \to X_n$ with respect to $(X_n, X_n \overset{p_n}{\to} X_{n-1} \overset{p_{n-1}}{\to} \cdots \overset{p_2}{\to} X_1 \overset{p_1}{\to} \mathrm{Spec}\, K = X_0)$ in the notation of Proposition 7.2, then it follows from Theorem 7.7 that φ arises from a unique isomorphism $\mathfrak{Y} \overset{\sim}{\to} \mathfrak{Z}$ over L (note that, if $\Sigma = \{l\}$, then it follows from Example 7.4, Proposition 7.5 that $\mathfrak{Y}$ satisfies condition $(*)_l$). This completes the proof of assertion (iii), hence also of Theorem 7.14. □

Remark 7.14.1. If $n \leq 4$ or $n' \leq 4$, then the bijectivity of the map

$$\mathrm{Isom}_L(Y,Z) \to \mathrm{Isom}_{G_L}(H, \Pi^{\Sigma}_{Z/L})/\,\mathrm{Inn}(\Delta^{\Sigma}_{Z/L})$$

appearing in Theorem 7.14 follows from [Ho2] Corollary 3.18 and [S2] Corollary 4.22.

8. Discrete analogues

In the present §8, we prove results on discrete configuration space groups that are analogous to some results appearing in §5-7. Let X be a hyperbolic curve of type (g,r) over $\mathbb{C}$, and n a positive integer unless otherwise specified.

Definition 8.1. Let G be a group.

(i) We shall say that G is *hopfian* (as an abstract group) if any surjective endomorphism $G \twoheadrightarrow G$ is an isomorphism.
(ii) We shall say that G is *residually finite* if the intersection of all subgroups of G of finite index is trivial, or, equivalently, the natural homomorphism $G \to G^\wedge$ is injective.

Proposition 8.2 (cf. [Mal] (See also [MKS] §6.5))**.** *A finitely generated residually finite group is hopfian.*

Lemma 8.3. *A discrete configuration space group is residually finite. In particular, a discrete configuration space group is hopfian.*

Proof. The first assertion follows from [MT] Proposition 7.1(ii). The second assertion follows from the first assertion, together with Corollary 1.9 and Proposition 8.2. □

Lemma 8.4. *Let G be a residually finite group and $N \subset G$ a normal subgroup. Write $\overline{N} \subset G^\wedge$ for the closure of N in $G^\wedge$. Then the sequence*

$$1 \to \overline{N} \to G^\wedge \to (G/N)^\wedge \to 1$$

is exact.

Proof. This follows from [RZ] Lemma 3.2.4, Proposition 3.2.5. □

Lemma 8.5. *Let G be a residually finite group and $N \subset G$ a normal subgroup. Suppose that G/N is residually finite. Write $\overline{N} \subset G^\wedge$ for the closure of N in $G^\wedge$. Then it holds that $N = \overline{N} \cap G (\subset G^\wedge)$.*

Proof. Write $H := \overline{N} \cap G$. It is clear that $N \subset H$. Moreover, we can easily verify that the closure of H in $G^\wedge$ coincides with $\overline{N} \subset G^\wedge$. Thus, it follows from Lemma 8.4 that the natural surjection $G/N \twoheadrightarrow G/H$ determines an isomorphism $(G/N)^\wedge \xrightarrow{\sim} (G/H)^\wedge$. Since (we have assumed that) G/N is residually finite, it follows from Lemma 8.4 that the closure of $\ker(G/N \twoheadrightarrow G/H)$ in $(G/N)^\wedge$ coincides with $\ker((G/N)^\wedge \xrightarrow{\sim} (G/H)^\wedge) = \{1\}$, which implies that $\ker(G/N \twoheadrightarrow G/H) = \{1\}$. Thus, we conclude that $N = H = \overline{N} \cap G$. □

Definition 8.6. Let $H \subset \Pi_n$ be a subgroup of Π_n. Then we shall say that H is *product-theoretic* if H is the inverse image of a subgroup of $\pi_1^{\mathrm{top}}(P_n)$ by the natural surjection $\Pi_n \twoheadrightarrow \pi_1^{\mathrm{top}}(P_n)$ (cf. Definition 1.4).

Theorem 8.7. *Let $H \subset \Pi_n$ be a product-theoretic subgroup of Π_n of finite index and $N \subset H$ a normal subgroup of H. Suppose that H/N is a discrete surface group. Moreover, suppose that at least one of the following holds:*

- $n = 1$ *or* $g \geq 2$.
- $H = \Pi_n$ *and* $g = 0$.
- $H = \Pi_n$ *and* $r \leq 1$.
- $H = \Pi_n$ *and* $H/N \not\cong F_2$.

Then there exists $N' \in \mathrm{FS}_1(\Pi_n) \cup \mathrm{ES}(\Pi_n)$ such that $N \supset N' \cap H$. Moreover, if further suppose that N is finitely generated, then it holds that $N = N' \cap H$.

Proof. Since $H \subset \Pi_n$ is of finite index, we may regard $H^\wedge$ as an open subgroup of $\Pi_n^\wedge$. Moreover, since the natural homomorphism $(\pi_1^{\mathrm{top}}(P_n))^\wedge \to \pi_1(P_n)$ is isomorphic, it is clear that $H^\wedge \subset \Pi_n^\wedge$ is product-theoretic. Write $\overline{N} \subset H^\wedge$ for the closure of N in $H^\wedge$ (hence in $\Pi_n^\wedge$). Then it follows from Lemma 8.4 that $H^\wedge/\overline{N}$ is a profinite surface group. Moreover, if $H/N \not\cong F_2$, then $H^\wedge/\overline{N} \not\cong F_2^\wedge$, which implies that $H^\wedge/\overline{N}$ is not nearly abelian (cf. Lemma 5.3(ii)). Thus, it follows from Theorem 7.12 that there exists $N' \in \mathrm{FS}_1(\Pi_n) \cup \mathrm{ES}(\Pi_n))$ such that, if we write $\overline{N}'$ for the closure of N' in $\Pi_n^\wedge$, then it holds that $\overline{N} \supset \overline{N}' \cap H^\wedge$. Note that if N is finitely generated, then, since $\overline{N}$ is topologically finitely generated, it holds that $\overline{N} = \overline{N}' \cap H^\wedge$. Now it follows from Lemma 8.3 that $H \subset \Pi_n$ and H/N are residually finite, which implies, in light of Lemma 8.5, that $N = \overline{N} \cap H$. On the other hand, since Π_n and Π_n/N' are residually finite, it follows from Lemma 8.5 that $N' = \overline{N}' \cap \Pi_n$, which implies that $\overline{N}' \cap H = N' \cap H$. Thus, we conclude that $N = \overline{N} \cap H \supset (\overline{N}' \cap H^\wedge) \cap H = N' \cap H$, and, moreover, if N is finitely generated, then it holds that $N = N' \cap H$. This completes the proof of Theorem 8.7. $\square$

Remark 8.7.1. In the case $g \geq 2$ and $H = \Pi_n$, (the first assertion of) Theorem 8.7 is proved in [Ch2] Lemma 2.5 (see also [Ch1] §3).

Definition 8.8. Let G, H be groups. Then we shall write $N^{\mathrm{abst}}(G, H)$ for the set of all normal subgroups N of G such that $G/N \cong H$. It follows directly from the definition that $N^{\mathrm{abst}}(G, H) \neq \emptyset$ if and only if there exists a surjective homomorphism $G \twoheadrightarrow H$ of abstract groups.

Theorem 8.9. *It holds that* $\mathrm{GFS}_1(\Pi_n) = N^{\mathrm{abst}}(\Pi_n, \Pi_1)$.

Proof. $\mathrm{GFS}_1(\Pi_n) \subset N^{\mathrm{abst}}(\Pi_n, \Pi_1)$ is clear. We verify $\mathrm{GFS}_1(\Pi_n) \supset N^{\mathrm{abst}}(\Pi_n, \Pi_1)$. Let $N \in N^{\mathrm{abst}}(\Pi_n, \Pi_1)$. Then it follows from Theorem 8.7 that there exists $N' \in \mathrm{FS}_1(\Pi_n) \cup \mathrm{ES}(\Pi_n) = \mathrm{GFS}_1(\Pi_n)$ such that $N \supset N'$. If $(g, r) \in \{(0, 3), (1, 1)\}$, then it follows from Lemma 8.3 that $\Pi_n/N' \cong \Pi_n/N \cong \Pi_1$ is hopfian. Thus, the natural surjection $\Pi_n/N' \twoheadrightarrow \Pi_n/N$ is an isomorphism, which implies that $N = N' \in \mathrm{FS}_1(\Pi_n) \cup \mathrm{ES}(\Pi_n) = \mathrm{GFS}_1(\Pi_n)$. If $(g, r) \notin \{(0, 3), (1, 1)\}$, then, by considering the natural surjection $\Pi_n/N' \twoheadrightarrow \Pi_n/N \cong \Pi_1$, it holds that $\Pi_n/N' \not\cong F_2$, which implies that $N' \notin \mathrm{ES}(\Pi_n)$. Thus, $N' \in \mathrm{FS}_1(\Pi_n)$. By the above argument, we conclude that $N = N' \in \mathrm{FS}_1(\Pi_n) = \mathrm{GFS}_1(\Pi_n)$. This completes the proof of Theorem 8.9. $\square$

Proposition 8.10. *There exists a unique discrete surface group* Π *(up to isomorphism) which satisfies the following conditions:*

(1) $N^{\mathrm{abst}}(\Pi_n, \Pi) \neq \emptyset$.
(2) *For any discrete surface group* Π'*, if* $N^{\mathrm{abst}}(\Pi_n, \Pi') \neq \emptyset$*, then* $N^{\mathrm{abst}}(\Pi, \Pi') \neq \emptyset$.

Moreover, this unique discrete surface group Π *is isomorphic to* $\Pi \cong \Pi_1$.

Proof. In light of Lemma 8.3 and Theorem 8.7, this follows from an argument similar to the argument of Proposition 3.6. $\square$

Definition 8.11. Let G be a discrete configuration space group. Then it follows from Proposition 8.10 that there exists a unique discrete surface group Π (up to isomorphism) which satisfies the following conditions:

(1) $N^{\mathrm{abst}}(G, \Pi) \neq \emptyset$.

(2) For any discrete surface group Π', if $N^{\text{abst}}(G, \Pi') \neq \emptyset$, then $N^{\text{abst}}(\Pi, \Pi') \neq \emptyset$.

We shall write $\Pi_1(G)$ for this unique discrete surface group. Moreover, we shall write $\text{GF}_1(G) := N^{\text{abst}}(G, \Pi_1(G))$.

Theorem 8.12. *Let G be a discrete configuration space group and $(X, n, \varphi : \Pi_n \overset{\sim}{\to} G)$ a CS-envelope for G. Then it holds that $\Pi_1 \cong \Pi_1(G)$. Moreover, the isomorphism φ induces a bijection $\text{GFS}_1(\Pi_n) \overset{1:1}{\to} \text{GF}_1(G)$, $N \mapsto \varphi(N)$.*

Proof. This follows from Proposition 8.10, Theorem 8.9. □

Definition 8.13. Let m be a nonnegative integer and G a discrete configuration space group. We shall define a set $\text{GF}_m(G)$ of subgroups of G as follows:

- $\text{GF}_0(G) := \{G\}$, $\text{GF}_1(G) := N^{\text{abst}}(G, \Pi_1(G))$ (cf. Definition 8.11). Moreover, for convenience, we shall define $\text{GF}_1(\{1\}) := \emptyset$.
- If $m \geq 2$, then $\text{GF}_m(G) := \bigcup_{N \in \text{GF}_{m-1}(G)} \text{GF}_1(N)$. Here, let us observe, by applying Theorem 8.12 and Proposition 1.8 inductively, that for each $N \in \text{GF}_{m-1}(G)$, if $N \neq \{1\}$, then N is also a discrete configuration space group.

Theorem 8.14. *Let m be a nonnegative integer, G a discrete configuration space group, and $(X, n, \varphi : \Pi_n \overset{\sim}{\to} G)$ a CS-envelope for G. Then the isomorphism φ determines a bijection $\text{GFS}_m(\Pi_n) \overset{1:1}{\to} \text{GF}_m(G)$, $N \mapsto \varphi(N)$.*

Proof. In light of Theorem 8.12, this follows from an argument similar to the argument of Theorem 3.10. □

Proposition 8.15. *Let m be a positive integer. Then it holds that $\text{GFS}_m(\Pi_n) = N^{\text{abst}}(\Pi_n, \Pi_m)$.*

Proof. In light of Theorem 8.9, this follows from an argument similar to the argument of Proposition 3.11. □

Proposition 8.16. *Let $N \subset \Pi_n$ be a normal subgroup of Π_n. Write $\overline{N}$ for the closure of N in $\Pi_n^\wedge$. Suppose that Π_n/N is residually finite, and that $\overline{N} \in \text{ES}(\Pi_n^\wedge)$ (cf. Remark 1.4.1). Then it holds that $N \in \text{ES}(\Pi_n)$.*

Proof. There exists $N' \in \text{ES}(\Pi_n)$ that, if we write $\overline{N}'$ for the closure of N' in $\Pi_n^\wedge$, then $\overline{N} = \overline{N}'$. Since Π_n, Π_n/N, and $\Pi_n/N' \cong F_2$ are residually finite (cf. Lemma 8.3), it follows from Lemma 8.5 that $N = \overline{N} \cap \Pi_n = \overline{N}' \cap \Pi_n = N' \in \text{ES}(\Pi_n)$. This completes the proof of Proposition 8.16. □

Definition 8.17. Let G be a discrete configuration space group. If $n(G^\wedge) = 1$ (cf. Remark 1.4.1, Definition 6.3(ii)), then we shall write $\text{E}(G) := \emptyset$. If $n(G^\wedge) \geq 2$, then we shall write $\text{E}(G)$ for the set of normal subgroups $N \subset G$ such that G/N is residually finite and the closure of N in $G^\wedge$ is an element of $\text{E}(G^\wedge)$ (cf. Definition 6.6).

Theorem 8.18. *Let G be a discrete configuration space group and $(X, n, \varphi : \Pi_n \overset{\sim}{\to} G)$ a CS-envelope for G. Suppose that $\Pi_1(G) \not\cong F_2$. Then the isomorphism φ determines a bijection $\text{ES}(\Pi_n) \overset{1:1}{\to} \text{E}(G)$, $N \mapsto \varphi(N)$.*

Proof. This follows from Theorem 6.7, Lemma 8.3, Proposition 8.16. □

Lemma 8.19. *Let $N \in \mathrm{ES}(\Pi_n)$. Then N is topologically generated by $n(2g+r)+\frac{(n+1)(n-4)}{2}$ elements.*

Proof. Suppose that $g=0$ (the case $g=1$ follows from a similar argument). Write $f: X_n \to Y_{n-1}$ and $U \subset Y_{n-1}$ be as in the proof of Lemma 6.17 (under the assumption $K=\mathbb{C}$). Since $f^{-1}(U)$ is a hyperbolic curve of type $(0, n(r-2)+2)$ over U, $\ker(\pi_1^{\mathrm{top}}(f^{-1}(U)) \twoheadrightarrow \pi_1^{\mathrm{top}}(U))$ is a free group of rank $n(r-2)+1$. By an argument similar to the argument appearing in the former part of the proof of Lemma 6.17, it suffices to show that the homomorphism $\ker(\pi_1^{\mathrm{top}}(f^{-1}(U)) \twoheadrightarrow \pi_1^{\mathrm{top}}(U)) \to \ker(\pi_1^{\mathrm{top}}(\widetilde{X_n}) \twoheadrightarrow \pi_1^{\mathrm{top}}(Y_{n-1}))$ is surjective. Let us recall that $\widetilde{X_n}$ is identified with $(Y_{n-1} \times \mathbb{P}^1_{\mathbb{C}}) \setminus D$, where $D \subset Y_{n-1} \times \mathbb{P}^1_{\mathbb{C}}$ for the divisor determined by $n(r-2)+2$ equations

$$y_n = 0,\ y_n = 1,\ y_n = \infty,\ y_n = y_s,\ y_n = \frac{b_t}{b_t - 1},\ y_n = \frac{b_t - y_s}{b_t - 1}\ (1 \le s \le n-1,\ 1 \le t \le r-3),$$

$f(y_1, \dots, y_n) = (y_1, \dots, y_{n-1})$ (under the above identification), and $U = Y_{n-1} \setminus (\bigcup_m D'_m)$, where $\{D'_m\}$ is the set of prime divisors of Y_{n-1} determined by equations

$$y_s = b_t,\ y_s = \frac{b_t}{b_t - 1}\ (1 \le s \le n-1,\ 1 \le t \le r-3),$$
$$y_s = \frac{b_t - y_{s'}}{b_t - 1}\ (1 \le s, s' \le n-1,\ 1 \le t \le r-3,\ s \ne s'),$$
$$\frac{b_t}{b_t - 1} = \frac{b_{t'} - y_s}{b_{t'} - 1}\ (1 \le s \le n-1,\ 1 \le t, t' \le r-3,\ t \ne t'),$$
$$\frac{b_t - y_s}{b_t - 1} = \frac{b_{t'} - y_{s'}}{b_{t'} - 1}\ (1 \le s, s' \le n-1,\ 1 \le t, t' \le r-3,\ s \ne s',\ t \ne t').$$

Now, by considering suitable tubular neighborhoods of prime divisors D'_m and applying the Seifert-van Kampen theorem inductively that the homomorphism $\pi_1^{\mathrm{top}}(U) \to \pi_1^{\mathrm{top}}(Y_{n-1})$ is surjective, and, moreover, the kernel of this homomorphism is normally generated by the homotopy classes γ_m of simple loops around D'_m. Write $\widetilde{\gamma}_m$ for the homotopy class of a simple loop around $f^{-1}(D'_m)$. Then the image of $\widetilde{\gamma}_m$ by the surjective homomorphism $\pi_1^{\mathrm{top}}(f^{-1}(U)) \twoheadrightarrow \pi_1^{\mathrm{top}}(U)$ is γ_m. Moreover, $\widetilde{\gamma}_m$ is contained in the kernel of the (surjective) homomorphism $\pi_1^{\mathrm{top}}(f^{-1}(U)) \twoheadrightarrow \pi_1^{\mathrm{top}}(\widetilde{X_n})$. Thus, the homomorphism $\ker(\pi_1^{\mathrm{top}}(f^{-1}(U)) \twoheadrightarrow \pi_1^{\mathrm{top}}(\widetilde{X_n})) \to \ker(\pi_1^{\mathrm{top}}(U) \twoheadrightarrow \pi_1^{\mathrm{top}}(Y_{n-1}))$ is surjective, which implies that $\ker(\pi_1^{\mathrm{top}}(f^{-1}(U)) \twoheadrightarrow \pi_1^{\mathrm{top}}(U)) \to \ker(\pi_1^{\mathrm{top}}(\widetilde{X_n}) \twoheadrightarrow \pi_1^{\mathrm{top}}(Y_{n-1}))$ is surjective. This completes the proof of Lemma 8.19. □

Theorem 8.20. *Let $N \in \mathrm{ES}(\Pi_n)$ and A a finite nonzero Π_n-module. Suppose that one of the following hold:*

- *$g = 1$ and $r \ge 2$.*
- *$g = 0$, $r \ge 4$, and $N \in \mathrm{ES}^1(\Pi_n)$.*

Then the n-th cohomology group $H^n(N, A)$ of N (as an abstract group) with coefficient A is infinite.

Proof. It follows from the proof of [S1] Proposition 2.7 that the discrete version of Lemma 6.13 holds. Thus, in light of Lemma 8.19, Theorem 8.20 follows from an argument similar to the argument of Theorem 6.19. □

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