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On multiframings of 3-manifolds

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Abstract

We investigate multiframings of a closed oriented 3-manifold. We show that multiframings give a geometric realization of the tensor product of the homotopy set of framings and $\mathbb{Q}$. Then we give two applications of multiframings.

1 Introduction

A notion of multisections is a generalization of sections of bundles introduced by Fukaya and Ono in [2]. In this article we investigate multiframings of a closed oriented 3-manifold, namely multisections of the framed bundle of the tangent bundle of the 3-manifold.

The homotopy set of (usual) framings of a closed oriented 3-manifold M is isomorphic to $[M, SO(3)]$ the homotopy set of maps from M to $SO(3)$ and it is isomorphic to $H^3(M;\mathbb{Z}) \oplus$ torsion. The signature defect (Hirzebruch defect) is a homotopy invariant of framings and it gives an affine map from the homotopy set of framings to $\mathbb{Z}$. For many 3-manifolds M, this map may be, however, not surjective and not injective.

Multiframings give a geometric realization of the tensor product of the homotopy set of framings and $\mathbb{Q}$. In fact, we prove that there is an affine isomorphism between the homotopy set of multiframings and additive group $H^3(M;\mathbb{Q}) \cong \mathbb{Q}$.

We give two applications of multiframings.

The first application is that there exists a unique canonical multiframing up to homotopy on any closed oriented 3-manifold. A (usual) framing whose signature defect equal to 0 is called a canonical framing (see [1], [3] for more deitails). We generalize the notion of signature defect to multiframings. Then a canonical multiframing is defined to be a multiframing whose signature defect equal to 0. The generalized signature defect gives an affine isomorphism from the homotopy set of multiframings to $\mathbb{Q}$.

The second application is on surgeries of framed 3–manifolds. Let M be a framed 3–manifold and let $N \subset M$ be a submanifold. When we replace N to a new manifold N' such that $\partial N \cong \partial N'$, we need to extend a framing of $TM|_{\partial N}$ to N'. The primary obstruction of this extension is in $H^1(N', \partial N'; \mathbb{Z}/2)$ and in some cases, this obstruction is alive. For example, Lescop studied in [5],[6] a handle body replacement of 3–manifold to show that Kontsevich-Kuperberg-Thurston invariant (defined in [4]) is of finite. There are difficulties in the

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proofs of these surgery formulas occur from the primary obstruction to extend framings. We show that any framing of $TM|_{\partial N}$ is able to extend to N' as a multiframing.

The organization of this aritcle is as follows. In Section 2 we review the notion of multisections with a modification. In Section 3 we study the homotopy set of multiframings of a closed oriented 3-manifold. In Section 4 and 5 we give applications of multiframings.

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2 Review of multisections

Fukaya and Ono in [2] defined a notion of multisections on orbiflolds. In this section, we review the notion of multisections and partially reformulate it.

Let $E \to M$ be a F-bundle over a smooth manifold M where F is a smooth manifold. Let us denote by $\mathfrak{S}_k$ the k-th symmetric group. Let $SP_k(F) = F^k/\mathfrak{S}_k$ be the symmetric power of F and denote by $q : F^k \to SP_k(F)$ the projection. $SP_k(E) \to M$ is an $SP_k(F)$-bundle obtained by taking the symmetric power fiberwise. We denote by $q : E^k \to SP_k(E)$ the projection induced from the projection $F^k \to SP_k(F)$.

Definition 2.1 (k-fold section)**.** Let k be a positive integer. A section $s : M \to SP_k(E)$ is said to be a *k-fold section* if there exists a pair $(\{U^a\}_{a\in A}, \{s^a_i\}_{a\in A, i=1,\dots,k})$ such that the following conditions hold.

(1) $\{U^a\}_{a\in A}$ is an open covering of M and $s^a_1, \dots, s^a_k : U^a \to E|_{U^a}$ are smooth sections of the bundle $E|_{U^a}$.

(2) $s|_{U^a} = q \circ (s^a_1, \dots, s^a_k)$ for any $a \in A$.

(3) For each $a \neq b \in A$ satisfying $U^a \cap U^b \neq \emptyset$, there exists a permutation $\sigma^{ab} \in \mathfrak{S}_k$ such that

$$s^a_{\sigma^{ab}(i)}|_{U^a\cap U^b} = s^b_i|_{U^a\cap U^b} \quad (i = 1, \dots, k).$$

Note that the notion of 1-fold section coincides with the notion of section. We call the pair $(\{U^a\}_{a\in A}, \{s^a_i\}_{a,i})$ or the triple $(\{U^a\}_{a\in A}, \{s^a_i\}_{a,i}, \{\sigma^{ab}\}_{a,b})$ *a locally liftable neighborhood system* of s. Sometimes we drop $\{s^a_i\}_{a,i}$ from the notation. For each $x \in M$, we call the neighborhood U^a satisfying $U^a \ni x$ a *locally liftable neighborhood of x*.

We remark that it is not required the cocycle condition: $\sigma^{xy} \circ \sigma^{yz} = \sigma^{xz}$ in the condition (3) on the above definition. For a k-fold section s and a natural number l, let us denote by $s^{(l)}$ the kl-fold section $s^{(l)} = q \circ (s, \ldots, s)$.

Definition 2.2 (multisection)**.** A *multisection* is a equivalent class or a representative element of the quotient set

$$\{(s,k) | k \text{ is a natural number}, s \text{ is a } k\text{-fold section}\} / \sim .$$

Here the equivalent relation $\sim$ is defined as:

$$(s,k) \sim (t,l) \Longleftrightarrow s^{(l)} = t^{(k)}.$$

Sometimes a multisection is denoted by $s : M \to SP_\infty(E)$.

A multimap, a multiform, and a multiconnection are defined as a factional sections of certain fiber bundles.

Definition 2.3 (pull-back)**.** Let s be a multisection of a F-bundle $E \to M$ and let $f : N \to M$ be a smooth map between smooth manifolds. Let $\{U^a\}$ be a local liftable neighborhood system of s with $s|_{U^a} = q \circ (s_1^a, \ldots, s_k^a)$. Then the multisection f^*s of f^*E is locally defined by the following equation:

$$f^*s|_{f^{-1}(U^a)} = q \circ (f^*s_1^a, \ldots, f^*s_k^a).$$

We call *f^*s the pull-back of s along f.*

We can define a pull-back of a form along a multimap in a similar way. We next define a homotopy between multisections.

Definition 2.4 (homotopy)**.**

(1) Let $s_0, s_1 : M \to SP_k(E)$ be k-fold sections. Let $p : M \times [0,1] \to M$ denote the projection. A k-fold section $S : M \times [0,1] \to SP_k(p^*E)$ of the bundle p^*E is said to be a *homotopy between s_0 and s_1* if $S|_{M \times \{i\}} = s_i (i = 0,1)$. We denote $s_0 \simeq_k s_1$.

(2) Let s_0 and s_1 are multisections. s_0 and s_1 are *multisection homotopic* if there exists a natural number k such that $s_0 \simeq_k s_1$. We write $s_0 \simeq s_1$.

We next consider fiber bundles equipped with some structures.

Let E be a vector bundle.

Definition 2.5 (average)**.** Let s be a multisection which is locally written by $s|_{U^a} = q(s_1^a, \ldots, s_k^a)$ on U^a, where $\{U^a\}$ is a locally liftable neighborhood system of s. The *average of s* is the multisection of E defined by the following equation:

$$\operatorname{av}(s)|_{U^a} = \frac{1}{k}(s_1^a + \ldots + s_k^a).$$

Let $f : M \to N$ be a multimap between smooth manifolds and let $\omega \in A^n(N)$ be a closed n-form on N. By definition, the n-form $\operatorname{av}(f^*\omega)$ is closed too.

Let F be a Lie group and the structure group of the bundle $E \to M$ be $Aut(F)$. We define two kinds of products of multisections.

Definition 2.6. Let s_0, s_1 be multisections of the bundle E. Let s_0 be represented by $s_0 : M \to SP_k(E)$ for some integer k and let s_1 be represented by $s_1 : M \to SP_l(E)$ for some integer l. Let $s_0(x) = q(x_1^0, \ldots, x_k^0), s_1(x) = q(x_1^1, \ldots, x_l^1)$ for each $x \in M$. The multisection $s_0 \cdot s_1$ is defined by

$$s_0 \cdot s_1(x) = q(\{x_i^0 \cdot x_j^1\}_{i,j}),$$

for $x \in M$.

Definition 2.7. Let s be a multisection represented by $s : M \to SP_k(E)$ for some integer k. Set $s(x) = q(x_1, \ldots, x_k)$ for each $x \in M$. For any $n \in \mathbb{Z}$, the multisection $\psi^n(s)$ is defined by the following equation:

$$\psi^n(s)(x) = q(x_1^n, \ldots, x_k^n).$$

We will use the notation s^{-1} instead of $\psi^{-1}(s)$.

3 Multiframings of a 3-manifold

In this section, we study multiframings of a 3-manifold, namely multisections of the frame bundle of the tangent bundle of the 3-manifold.

Let M be a closed oriented 3-dimensional manifold. Let $V = GL^+(\mathbb{R}^n)$ ($n \geq 3, n \neq 4$).

Definition 3.1 (homotopy set of multimaps)**.** We define $[M, V]_\infty = \{f : M \to SP_\infty(V), \text{multimap}\}/\simeq$ to be the homotopy set of multimaps.

In this section we show that the homotopy set $[M, V]_\infty$ has a natural group structure and give the isomorphism between $[M, V]_\infty$ and the additive group $\mathbb{Q}$:

$$[M, V]_\infty \cong \mathbb{Q}.$$

Remark 3.2. If we fix a framing $TM \cong M \times \mathbb{R}^3$, the homotopy set of multiframings of M is naturally identified with to $[M, GL_+(\mathbb{R}^3)]_\infty$.

We first define a map $\mu : [M, V]_\infty \to \mathbb{R}$.

Since $GL^+(\mathbb{R}^n)$ is homotopy equivalent to the maximal compact subgroup $SO(n)$, $[M, GL^+(\mathbb{R}^n)]_\infty \cong [M, SO(n)]_\infty$. Suppose $\iota : SO(3) \to SO(n)$ is the embedding induced by the natural embedding $\mathbb{R}^3 \hookrightarrow \mathbb{R}^3 \oplus 0 \subset \mathbb{R}^n$. Let $\omega_n \in A^3(SO(n))$ be a closed 3-form satisfying $\int_{SO(3)} \iota^* \omega_n = 1$. Since $H^3(SO(n); \mathbb{R}) = \mathbb{R}$ for $n \geq 3, n \neq 4$, $[\omega_n]$ generates $H^3(SO(n); \mathbb{R})$. Recall that $\text{av}(f^*\omega_n) \in A^3(M)$ is a closed form for any multimap $f : M \to SP_\infty(SO(n))$.

Lemma 3.3. *The value $\int_M \text{av}(f^*\omega_n) \in \mathbb{R}$ is independent of the choice of ω_n and the multimap homotopy of f.*

Proof. Let ω'_n be the alternative choice. Since $\iota^* : H^3(SO(n)) \to H^3(SO(3))$ is injective, there exists a 2-form $\eta \in A^2(SO(n))$ satisfying $\omega'_n = \omega_n + d\eta$. Because

$\mathrm{av}(f^*d\eta) = d\mathrm{av}(f^*\eta) \in A^3(M)$, $\int_M \mathrm{av}(f^*\omega_n') = \int_M \mathrm{av}(f^*\omega_n) + \int_M d\mathrm{av}(f^*\eta) = \int_M \mathrm{av}(f^*\omega_n)$. From this, $\int_M \mathrm{av}(f^*\omega_n)$ is independent of the choice of ω.

Set $\mu_n(f) = \int_M \mathrm{av}(f^*\omega_n)$. We assume that f is represented by a k-fold map. For any $l \in \mathbb{N}$, $\mu_n(f^{(l)}) = \frac{1}{l}(\mu_n(f) + \ldots + \mu_n(f))$. Hence $\mu_n(f) = \mu_n(f^{(l)})$. According to Stokes' theorem, $f \simeq_\infty g$ implies that $\mu_n(f) = \mu_n(g)$. □

Definition 3.4. $\mu_n(f) = \int_M \mathrm{av}(f^*\omega_n) \in \mathbb{R}$.

By abuse of notation, we write $\mu(f)$ instead of $\mu_n(f)$.

Now we obtain the map

$$\mu_n : [M, GL^+(\mathbb{R}^n)]_\infty \to \mathbb{R}.$$

Example 3.5. (1) For any 3-manifold M, there exists a map $f : M \to SO(3)$ satisfying $\mu(\phi) = 2$. Let $B^3 \subset M$ be a small ball in M. We take a map $f : (M, M \setminus B^3) \to (S^3, 1)$ such that $\deg f = 1$. Suppose $\pi : S^3 \to SO(3)$ is the double covering. Then $\mu(\pi \circ f) = 2\deg(\pi \circ f) = 2$. Similarly, for $n \geq 5$ we can take a map $f : M \to SO(n)$ satisfying $\mu(\phi) = 2$ by the composition of $\pi \circ f$ and $\iota : SO(3) \to SO(n)$.

Note that $\iota^*(1) = 2$ for the map $\iota^* : H^3(SO(n); \mathbb{R}) \to H^3(SO(3); \mathbb{R})$ for $n \geq 5$. Then $[\omega_n]$ does not belong to $H^3(SO(n); \mathbb{Z})$.

(2) For the identity map $\mathrm{id} : \mathbb{R}P^3 \to \mathbb{R}P^3 \cong SO(3) (\subset GL^+(\mathbb{R}^3))$, we have $\mu_3(\mathrm{id}) = 1$. On the other hand, there is no map $f : M \to GL^+(\mathbb{R}^3)$ satisfying $\mu_3(f) = 1$ for any integral homology 3-sphere M. In fact any $f : M \to SO(3)$ can be lifted to $S^3 = Spin(3)$.

The following two lemmas play a fundamental role when we deform an element of $[M, V]_\infty$ by homotopy.

Lemma 3.6. *Let $B^3 \subset M$ be a small ball in M. Let $\phi : (M, M \setminus B^3) \to (V, 1)$ be a smooth map. Then $\mu(\phi^k) = k\mu(\phi)$ for any integer $k \in \mathbb{Z}$.*

Proof. We first assume that $k \geq 0$. Let $B_1^3, \ldots, B_k^3 \subset B^3$ be disjoint balls in B^3. There exists a map $\phi_i : M \to V$ homotopic to ϕ for each i such that $\phi_i|_{M \setminus B_i^3} \equiv 1$, since $M \setminus B_i^3$ is a deformation retract of $M \setminus B^3$. Since $\phi^k \simeq_1 \phi_1 \cdot \ldots \cdot \phi_k$, we have

$$\mu(\phi^k) = \mu(\phi_1 \cdot \ldots \cdot \phi_k) = k\mu(q \circ (\phi_1 \cdot \ldots \cdot \phi_k, 1, \ldots, 1)) = k\mu(\phi_1, \ldots, \phi_k) = k\mu(\phi).$$

We next set $k < 0$. The equation $\phi^k \cdot \phi^{-k} \simeq_\infty 1$ implies that $\mu(\phi^{-k}) = -\mu(\phi^k) = -k\mu(\phi)$. □

In the next lemma we may choose $Spin(n)$ as V.

Lemma 3.7. *Let B^3 be a ball in M. For any smooth maps $f_1, \ldots, f_k : (M, \partial B^3) \to (V, 1)$, there exist smooth maps $g_1, \ldots, g_k : (M, \partial B^3) \to (V, 1)$ satisfying the following conditions.*

(1) $q \circ (g_1, \ldots, g_k) \simeq_k q \circ (f_1, \ldots, f_k)$.

(2) *For* $i = 2, 3, \ldots, k$, $g_i|_{B^3} \equiv 1$.

(3) *For* $i = 1, 2, \ldots, k$, $f_i|_{M\setminus B^3} = g_i|_{M\setminus B^3}$.

Proof. Let $B^3_1, \ldots, B^3_k \subset B^3$ be disjoint balls in B^3. As the proof of the lemma 3.6, we get a map $f'_i : M \to V$ for each i, such that $f'_i|_{B^3\setminus B^3_i} \equiv 1$ and there is a homotopy relative to $M\setminus B^3$ between f'_i and f_i. Let $g_1|_{B^3} = f'_1 \cdot f'_2 \cdot \ldots \cdot f'_k|_{B^3}$, $g_2|_{B^3} = \ldots = g_k|_{B^3} :\equiv 1$, $g_i|_{M\setminus B^3} = f_i|_{M\setminus B^3}$ for $i = 1, \ldots, k$. Since the balls $B^3_1, \ldots, B^3_k$ are disjoint, $q \circ (f_1, \ldots, f_k) \simeq_k q \circ (g_1, \ldots, g_k)$. □

We will use the following lemma when we study the structure of the group $[M, V]_\infty$.

Lemma 3.8. *Let* $V = GL^+(\mathbb{R}^n)$ $(n \geq 3, n \neq 4)$. *Suppose* $1 \in V$ *is the unit element of* V. *Let* $B^3 \subset M$ *be a small ball. Let* $\phi : (M, M \setminus B^3) \to (V, 1)$ *be a smooth map satisfying* $\mu(\phi) = 2$ $(n = 3)$ *or* $\mu(\phi) = 1$ $(n \geq 5)$. *For any multimap* $f : M \to SP_\infty(V)$, *there exists integers* $l, m \in \mathbb{Z}$ *such that*

$$f \simeq_\infty q \circ (\phi^l, 1, \ldots, 1).$$

(*The number of letter "1" is* m.)

Remark 3.9. The map ϕ satisfying $\mu(\phi) = 2$ $(n = 3)$ or $\mu(\phi) = 1$ $(n \geq 5)$ and $\phi|_{M\setminus B^3} \equiv 1$ is unique up to homotopy. It follows from the obstruction theory.

Proof. Suppose the multimap f is represented by a k-fold map. We will show that we can take $m = 2k - 1$.

Let $\pi : Spin(n) \to SO(n)$ be the double covering. Suppose $(\{U^a\}, \{\sigma^{ab}_0\}, \{f^a_i\})$ be a local liftable neighborhood system (i.e. $f|_{U^a} = q \circ (f^a_1, \ldots, f^a_k)$). Taking a subdivision of $\{U^a\}$ if necessary, we may assume that $\{U^a\}$ is a good cover, namely any $\cap_{a_1,\ldots,a_k} U^{a_k}$ is contractible or empty set.

Let $f^a_{i,+}, f^a_{i,-}$ be different lifts of $f^a_i : U^a \to SO(n)$ along π for $i = 1, \ldots, k$. Let $\sigma^{ab}_1 \in \mathfrak{S}_{2k} = Aut(\{1, \ldots, k\} \times \{+, -\})$ be permutations satisfying the following conditions;

For a point $x \in U^a \cap U^b$,

(1) There is a transposition $\tau^{ab}_i : \{\pm\} \to \{\pm\}$ satisfying $\sigma^{ab}_1(i, +)$
$= (\sigma^{ab}_0(i), \tau^{ab}_i(+))$.

(2) $f^a_{i,+}(x) = f^b_{\sigma^{ab}_1(i,+)}(x)$.

We claim that σ^{ab}_1 is independent of the choice of x. Indeed, σ^{ab}_1 is uniquely determined for each x and $U^a \cap U^b$ is arcwise connected. Hence the $2k$-fold map $\tilde{f} : M \to SP_{2k}(Spin(n))$ which has $(\{U^a\}, \{\sigma^{ab}_1\})$ as a locally liftable neighborhood system is defined by $\tilde{f}|_{U^a} = q \circ (f^a_{1,+}, f^a_{1,-}, \ldots, f^a_{k,+}, f^a_{k,-})$. By the construction, $\pi \circ \tilde{f} = q \circ (f, f)$.

We next deform $\tilde{f}$ by multimap homotopy.

Let $|M|$ be a cell decomposition of M satisfying the following condition.

- For each cell σ of $|M|$, there is a subscript a such that $\sigma \subset U^a$.

Write $\tilde{f}|_\sigma = q \circ (\tilde{f}_1^\sigma, \ldots, \tilde{f}_{2k}^\sigma)$.

We deform $\tilde{f}$ by a multimap homotopy in the following steps. Suppose $\varepsilon > 0$.

Step 0 For each 0-cell σ^0, we deform $\tilde{f}_i^{\sigma^0}$ on the ε-neighborhood of σ^0 by a homotopy to satisfy $\tilde{f}|_{\sigma^0} = 1^{(k)}$. It is possible to carry out this deformation because $\pi_0(Spin(n)) = 0$. (This operation defines a homotopy of $\tilde{f}$ because of the definition of multimap. The same thing works on the following steps.)

Step 1 For each 1-cell σ^1, we deform $\tilde{f}_i^{\sigma^1}$ on the $\varepsilon/2$-neighborhood of σ^1 by a homotopy to satisfy$\tilde{f}|_{\sigma^1} = 1^{(k)}$. It is possible to carry out this deformation because $\pi_1(Spin(n)) = 0$.

Step 2 For each 2-cell σ^2, we deform $\tilde{f}_i^{\sigma^2}$ on the $\varepsilon/4$-neighborhood of σ^2 by a homotopy to satisfy$\tilde{f}|_{\sigma^2} = 1^{(k)}$. It is possible to carry out this deformation because $\pi_2(Spin(n)) = 0$.

Step 3 Now, $\tilde{f}$ is constant on the 2-skeleton $M^{(2)}$: $\tilde{f}|_{M^{(2)}} = 1$. Then for each 3-cell σ^3, $\tilde{f}|_{\sigma^3} = q \circ (\tilde{f}_1^{\sigma^3}, \ldots, \tilde{f}_{2k}^{\sigma^3})$, $\tilde{f}_i^{\sigma^3} : (\sigma_3, \partial\sigma_3) \to (Spin(n), 1)$. We apply Lemma 3.7 for each 3-cell. Then we obtain a multimap $\tilde{f}$ satisfying $\tilde{f}|_{\sigma^3} = q \circ (g_0^{\sigma^3}, 1, \ldots, 1)$ where $g_0^{\sigma^3} : (\sigma^3, \partial\sigma^3) \to (Spin(n), 1)$ is an appropriate smooth map. According to $g_0^{\sigma^3}|_{\partial\sigma^3} \equiv 1$, there exists a smooth map $g : M \to Spin(n)$ on M satisfying $g|_{\sigma^3} = g_0^{\sigma^3}$ for each 3-cell σ^3. By the construction of g, we have $\tilde{f} \simeq_{2k} q \circ (g_0, 1, \ldots, 1)$. Since g is constant on the 2-skeleton $M^{(2)}$, there is an integer $l \in \mathbb{Z}$ such that $g_0 \simeq_1 \varphi_n^l$, where $\varphi_n : (M, M \setminus B^3) \to (Spin(n), 1)$ is a smooth map with $(\varphi_n)_* : \pi_3(M, M \setminus B^3) \cong \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \cong \pi_3(Spin(n))$.

We have $\tilde{f} \simeq_{2k} q \circ (\varphi_n^l, 1, \ldots, 1)$. Then $f \sim q \circ (f, f) = \pi \circ \tilde{f} \simeq_{2k} \pi \circ q \circ (\varphi_n^l, 1, \ldots, 1) = q \circ ((\pi \circ \varphi_n)^l, 1, \ldots, 1)$.

Finally, we calculate $\mu(\pi \circ \varphi_n : (M, M \setminus B^3) \to (SO(n), 1))$.

When $n = 3$, $\mu(\pi \circ \varphi_3) = \varphi_3^* \pi^* 1 = \varphi_3^* 2 = 2$. We now turn to the case $n \geq 5$. Let $\iota : SO(3) \hookrightarrow SO(n)$ be the map induced by the embedding $\mathbb{R}^3 \xrightarrow{\cong} \mathbb{R}^3 \oplus 0 \subset \mathbb{R}^n$. The pull-back of the double covering $\pi : Spin(n) \to SO(n)$ along ι is $\pi : Spin(3) \to SO(3)$. Suppose $\tilde{\iota} : Spin(3) \to Spin(n)$ is the map induced by the bundle map ι. Then $\mu(\pi \circ \tilde{\iota} \circ \varphi_3) = \mu(\iota \circ \pi \circ \varphi_3) = \int_M \varphi_3^* \omega_3 = 2$. Since $\tilde{\iota}_*(1) = 2$ for $\tilde{\iota}_* : H_3(Spin(3); \mathbb{R}) \to H_3(Spin(n); \mathbb{R})$, we have $\mu(\pi \circ \varphi_n) = 1$.

Take $\phi = \pi \circ \varphi_n$. We finish the proof. □

Proposition 3.10. *Let $V = GL^+(\mathbb{R}^n)$ $(n \geq 3, n \neq 4)$. The homotopy set $[M, V]_\infty$ is a group with $\cdot$ as a product, namely, the followings hold.*

(1) *If $f_1 \simeq_\infty f_2$ and $g_1 \simeq_\infty g_2$, then $f_1 \cdot g_1 \simeq_\infty f_2 \cdot g_2$. Hence it is possible to define $[f] \cdot [g] = [f \cdot g]$.*

(2) $[f] \cdot [f^{-1}] = [c_1]$.

(3) $[f]\cdot[c_1] = [c_1]\cdot[f] = [f]$.

Proof. (1) Let F be a multimap homotopy between f_1 and f_2, and G be a multimap homotopy between g_1 and g_2. Then $F\cdot G$ is a multimap homotopy between $f_1\cdot g_1$ and $f_2\cdot g_2$.

(2) (3)According to Lemma 3.8, we only need to consider the case $f = q\circ(\phi^l, 1, \ldots, 1)$. The rest of the proof is straightforward. □

Finally, we obtain the following theorem about the structure of the group $([M,V]_\infty, \cdot)$.

Theorem 3.11. *Let $V = GL^+(\mathbb{R}^n)$ $(n \geq 3, n \neq 4)$. The map $\mu : ([M,V]_\infty, \cdot) \cong (\mathbb{Q}, +)$ is a group isomorphism.*

Proof. Let $B^3 \subset M$ be a small ball in M and let $\phi : (M, M\setminus B^3) \to (V, 1)$ be a map such that $\mu(\phi) = 2$. According to Lemma 3.8, the natural quotient map $\{q\circ(\phi^l, 1, \ldots, 1)\}_{(l,k);\text{co-prime}} \to [M,V]_\infty$ is surjective. Since $\mu(q\circ(\phi^l, 1, \ldots, 1)) = \frac{2l}{k} \in \mathbb{Q}$, the map $\mu : \{q\circ(\phi^l, 1, \ldots, 1)\} \to H^3(M;\mathbb{Q})$ is surjective.

We next prove that μ is injective.

Let f be a k-fold map and let g be a l-fold map satisfying $\mu(f) = \mu(g)$ as multimap. Thanks to Lemma 3.8, there are integers $m, n \in \mathbb{Z}$ such that

$$f \simeq_\infty q\circ(\phi^m, 1, \ldots, 1) \text{ (the number of 1 is } 2k),$$

$$g \simeq_{2l} q\circ(\phi^n, 1, \ldots, 1) \text{ (that of 1 is } 2l).$$

Then

$$f^{(l)} \simeq_\infty q\circ(\phi^{lm}, 1, \ldots, 1) \text{ (the number of 1 is } 2kl),$$

$$g^{(k)} \simeq_\infty q\circ(\phi^{kn}, 1, \ldots, 1) \text{ (that of 1 is } 2kl).$$

Then we have $\mu(f) = \frac{2lm}{2kl}, \mu(g) = \frac{2kn}{2kl}$. Since $\mu(f) = \mu(g)$, we have $lm = kn$. Then

$$q\circ(\phi^{lm}, 1, \ldots, 1) = q\circ(\phi^{kn}, 1, \ldots, 1).$$

Then $f \simeq_\infty g$. It shows that μ is an injection. □

4 Applications of multiframings: (1) Canonical framing

Atiyah in [1] showed that, for any closed oriented 3–manifold, there is a canonical 2–framing, namely a framing of the double of the tangent bundle such that its signature defect equal to 0. Note that there may be no canonical (1–)framing or there are not unique homotopy classes of canonical framings. In this section, we generalize the notion of the signature defect to multiframings and then we show that there exists a unique canonical multiframing up to multiframing homotopy.

Let τ be a multiframing of a 3-dimensional manifold M. We define a rational number $\delta(\tau)$ for τ using the Chern-Weil theory. This is a generalization of the notion of the signature defect of usual framings.

Let W be a 4-dimensional manifold with $\partial W = M$. Let $E \to W$ be a real vector-bundle of rank 4. Let ∇ be a multi affine connection on E.

Suppose ∇ is locally written by $\nabla|_U = q \circ (\nabla_1, \dots, \nabla_k)$ on each appropriate open set $U \subset W$ where $\nabla_1, \dots, \nabla_k$ are connections on U. Let R_i is the curvature form of ∇_i for $i = 1, \dots, k$. For any invariant polynomial f, $f(R_i)$ is a closed form on U for each i. Let $f(\nabla)$ be the multi form locally given as $q \circ (f(R_1), \dots, f(R_k))$.

Let $p_1(\nabla)$ be the 1st Pontrjagin form of ∇.

Lemma 4.1. *Suppose W is a closed manifold. Let ∇ be a multi connection and let ∇' be a usual connection. Then $\int_W \mathrm{av}(p_1(\nabla)) = \int_W p_1(\nabla')$. In particular, $\frac{1}{4\pi^2}\int_W \mathrm{av}(p_1(\nabla)) = 3\mathrm{Sign}(W)$.*

Proof. Suppose ∇ is locally written by $\nabla|_{U^a} = q \circ (\nabla^a_1, \cdots, \nabla^a_k)$ where $\{U^a\}_{a \in A}$ is a locally liftable neighborhood system. We denote by $\pi : W \times [0,1] \to W$ the projection. Let $\tilde{\nabla}$ be the multi connection of $\pi^* E$ defined as:

$$\tilde{\nabla}|_{U^a \times \{t\}} = q \circ (t\nabla^a_1 + (1-t)\nabla', \cdots, t\nabla^a_k + (1-t)\nabla').$$

Then the 4-form $\mathrm{av}(p_1(\tilde{\nabla})) \in A^4(W \times [0,1])$ satisfies $\mathrm{av}(p_1(\tilde{\nabla}))|_{W\times\{0\}} = \mathrm{av}(p_1(\nabla'))$, $\mathrm{av}(p_1(\tilde{\nabla}))|_{W\times\{1\}} = \mathrm{av}(p_1(\nabla))$. Thanks to Stokes' theorem, we have $\int_W \mathrm{av}(p_1(\nabla)) = \int_W p_1(\nabla')$. □

Take a 4-dimensional manifold W such that $\partial W = M$. The multi connection on $TW|_{\partial W}$ is given by the outward vector of ∂W and the multiframing τ. Let ∇_τ be a multi connection on W which is one of the extensions of the above multi connection. Then we have the 1st Pontrjagin form $p_1(\nabla_\tau)$.

Proposition 4.2. *The real number $\delta(\tau) = \frac{1}{4\pi^2}\int_W \mathrm{av}(p_1(\nabla)) - 3sign(W) \in \mathbb{Q}$ is independent of the choice of W and ∇_τ.*

Proof. This follows by the above lemma and Hirzebruch's signature theorem. □

Definition 4.3 (signature defect)**.** We call $\delta(\tau)$ *the signature defect of* τ.

When τ is a usual framing, $\delta(\tau)$ coincides with the usual signature defect by the definition.

Definition 4.4. A multiframing satisfying $\delta(\tau) = 0$ is said to be *canonical multiframing.*

Thanks to Theorem 3.11, for any closed oriented 3-manifold, there exists a canonical multiframing up to multiframing homotopy:

Proposition 4.5. *For any closed oriented 3-manifold M, there exist canonical multiframings of M. Moreover, if both τ_0 and τ_1 are canonical multiframings of M then $\tau_0 \simeq \tau_1$.*

5 Applications of multiframings: (2) Surgery

By using the multiframings, we can clarify arguments about surgery of framed 3–manifold. Let $N_0 \subset M$ be a 3-dimensional submanifold of closed framed 3-manifold M. For example, N_0 is solid torus. When we replace N_0 to other 3-manifold N, we often enable to extend given framing of $M \setminus N$ to new manifold. We can, however, always extend as a multiframing.

Let N be an oriented 3-dimensional smooth manifold with boundary. Let $f : \partial N \to SP_k(SO(3))$ be a multimap.

Lemma 5.1. *There exists a multimap $\tilde{f} : N \to SP_\infty(SO(3))$ such that $\tilde{f}|_{\partial N} = f$.*

Proof. By a similar argument of the proof of Theorem 3.8 on the boundary ∂N, we have $f \simeq_\infty 1$. We denote by $F = \{f_t\}_{t\in I}$ the homotopy between $f = f_0$ and $1 = f_1$. Suppose $\partial N \times [0,1] \subset N$ is a collar neighborhood of ∂N. The following $\tilde{f}$ is a desired multimap.

$$\tilde{f} = \begin{cases} f_t & (x \in \partial N \times \{t\}) \\ f_1 = 1 & (\text{otherwise}) \end{cases}$$

□

Corollary 5.2. *Any multiframing τ_0 of $TN|_{\partial N}$ can extend to N.*

Proof. TN is trivial, then arguments of multiframings reduce to that of multimaps. □

References

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