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Integrable Connections III: Frobenius-descent Data

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Abstract. — In the present paper, we prove that, for a smooth scheme over a field of characteristic $p > 0$, the natural morphism from the divided power stratification structure to the Frobenius-stratification structure is strictly p-integrable. In particular, as an application of the second fundamental correspondence, we obtain a natural bijection between the set of Frobenius-descent data and the set of p-dormant PD-connections on objects of weakly integrable categories fibered in groupoids over categories of schemes. This bijection may be regarded as a generalization of the well-known equivalence concerning dormant connections and Frobenius-descent data.

Contents

Introduction

I.0. — Let p be a prime number, S a scheme *over a field of characteristic* p, and X a scheme which is *smooth* and *separated* over S. Write $X^F \to S$ for the base-change of the structure morphism $X \to S$ of X by the absolute [p-th power] Frobenius endomorphism of S [cf. Definition 1.1],

$$\mathrm{Fr}_{X/S} \colon X \longrightarrow X^F$$

for the relative [p-th power] Frobenius morphism of X/S [cf. Definition 1.1], Sch_S for the category of schemes over S and morphisms of schemes over S [cf. [3], Definition 1.6, (i)], and

$$^{\mathrm{PD}}P^1 \subseteq X \times_S X$$

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for the closed subscheme of $X \times_S X$ defined by the quasi-coherent ideal of $\mathcal{O}_{X\times_S X}$ obtained by forming the square of the quasi-coherent ideal that defines the diagonal closed subscheme $X \subseteq X \times_S X$ of $X \times_S X$ [cf. [4], Definition 2.1, (ii); [2], Remark 4.2]. For each $i \in \{1, 2\}$, write, moreover,

$$^{\mathrm{PD}}\mathrm{pr}^1_i \colon {}^{\mathrm{PD}}P^1 \longrightarrow X$$

for the morphism over S obtained by forming the composite of the natural closed immersion ${}^{\mathrm{PD}}P^1 \hookrightarrow X \times_S X$ and the projection $X \times_S X \to X$ onto the i-th factor [cf. [3], Definition 2.3, (ii); [4], Lemma 2.2].

I.1. — In the remainder of the present Introduction, let

$$\mathcal{E}$$

be a locally free $\mathcal{O}_X$-module. Here, let us recall some discussions of [4], Introduction. Let us first recall that a *connection* on $\mathcal{E}$ [i.e., relative to X/S] is defined to be a homomorphism of $(X \to S)^{-1}\mathcal{O}_S$-modules

$$\nabla \colon \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

that satisfies the equality $\nabla(fe) = f\nabla(e) + e \otimes df$ — where f, e are local sections of $\mathcal{O}_X$, $\mathcal{E}$, respectively [cf. [4], Definition 1.6]. On the other hand, it is well-known [cf., e.g., [2], Proposition 2.9] that giving a connection on $\mathcal{E}$ [i.e., a homomorphism of $(X \to S)^{-1}\mathcal{O}_S$-modules as above] is *equivalent* to giving an *isomorphism of $\mathcal{O}_{{}^{\mathrm{PD}}P^1}$-modules*

$$({}^{\mathrm{PD}}\mathrm{pr}^1_2)^*\mathcal{E} \xrightarrow{\sim} ({}^{\mathrm{PD}}\mathrm{pr}^1_1)^*\mathcal{E}$$

that restricts, on the diagonal closed subscheme $X \subseteq {}^{\mathrm{PD}}P^1$, to the *identity automorphism* of $\mathcal{E}$ [cf. [4], Proposition 2.6, (ii)]. Moreover, such an isomorphism $({}^{\mathrm{PD}}\mathrm{pr}^1_2)^*\mathcal{E} \xrightarrow{\sim} ({}^{\mathrm{PD}}\mathrm{pr}^1_1)^*\mathcal{E}$ is naturally related to the notion of a *divided power stratification* [cf., e.g., [2], Definition 4.3] on $\mathcal{E}$.

Now let us observe that since the above "second" definition of the notion of a *connection* [i.e., an isomorphism $({}^{\mathrm{PD}}\mathrm{pr}^1_2)^*\mathcal{E} \xrightarrow{\sim} ({}^{\mathrm{PD}}\mathrm{pr}^1_1)^*\mathcal{E}$ as above] and the definition of the notion of a *divided power stratification* of [2], Definition 4.3, are "sufficiently abstract", one may apply these definitions [not only to locally free modules as above but also] to an object of a category fibered in groupoids over the category Sch_S. In the remainder of the present Introduction, let $\mathcal{F} \to \mathrm{Sch}_S$ be a category fibered in groupoids over Sch_S and ξ an object of $\mathcal{F}$ over X. Thus, one may define

- a PD-*connection* on ξ to be an isomorphism

$$({}^{\mathrm{PD}}\mathrm{pr}^1_2)^*\xi \xrightarrow{\sim} ({}^{\mathrm{PD}}\mathrm{pr}^1_1)^*\xi$$

in $\mathcal{F}$ over the identity automorphism of ${}^{\mathrm{PD}}P^1$ that restricts to the identity automorphism of $\xi = (({}^{\mathrm{PD}}\mathrm{pr}^1_2)^*\xi)|_X = (({}^{\mathrm{PD}}\mathrm{pr}^1_1)^*\xi)|_X$ [cf. [3], Definition 4.1, (iii); [4], Definition 2.5].

Moreover, one may also define

- a PD-*stratification* to be a collection of data similar to a divided power stratification defined in [2], Definition 4.3 [cf. [3], Definition 4.6; [4], Definition 2.5] and

• a PD-*integrable* PD-*connection* to be a PD-connection whose PD-curvature [cf. [3], Definition 4.7, (i); [4], Definition 2.5] is the identity automorphism [cf. [3], Definition 4.7, (ii); [4], Definition 2.5].

Write

$$^{\mathrm{PD}}\mathrm{Strt}(\xi), \qquad ^{\mathrm{PD}}\mathrm{IntCnn}(\xi)$$

for the sets of PD-stratifications, PD-integrable PD-connections on ξ, respectively [cf. [3], Definition 4.6; [3], Definition 4.7, (ii); [4], Definition 2.5].

Here, let us recall that one important result of [4] [cf. [4], Theorem A] asserts that

> if the category $\mathcal{F}$ fibered in groupoids over Sch_S is *weakly integrable* [cf. [3], Definition 1.8], then the natural map
>
> $$^{\mathrm{PD}}\mathrm{Strt}(\xi) \longrightarrow {}^{\mathrm{PD}}\mathrm{IntCnn}(\xi)$$
>
> [cf. [3], Lemma 4.8] is *bijective.*

Note that this bijection may be regarded as a *generalization* [cf. [4], Remark 4.3.1, (i)] of the well-known equivalence of the following two conditions on a given [classical] connection $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ on the locally free $\mathcal{O}_X$-module $\mathcal{E}$:

• The connection ∇ is *integrable* — i.e., the *curvature* of the connection ∇

$$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \xrightarrow{\nabla^1} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_{X/S}$$

[cf., e.g., the discussion preceding [2], Theorem 2.15] is *zero.*

• The connection ∇ extends to a *uniquely determined divided power stratification* on $\mathcal{E}$.

I.2. — We shall refer to a descent datum on the object ξ of the category $\mathcal{F}$ fibered in groupoids over the category Sch_S with respect to the relative Frobenius morphism $\mathrm{Fr}_{X/S}\colon X \to X^F$ of X/S — i.e., an isomorphism in $\mathcal{F}$, over the identity automorphism of $X \times_{X^F} X$, of the pull-back of ξ by the second projection $X \times_{X^F} X \to X$ with the pull-back of ξ by the first projection $X \times_{X^F} X \to X$ that satisfies a certain "cocycle condition" on $X \times_{X^F} X \times_{X^F} X$ — as a *Frobenius-descent datum* on ξ [cf. Definition 3.2, (iv)]. Write

$$\mathrm{FrDsc}(\xi)$$

for the set of Frobenius-descent data on ξ [cf. Definition 3.2, (iv)].

Now let us observe that one verifies easily that the natural closed immersion $^{\mathrm{PD}}P^1 \hookrightarrow X \times_S X$ *factors* through the closed immersion $X \times_{X^F} X \hookrightarrow X \times_S X$ determined by the morphism $X^F \to S$ [cf. Lemma 1.3]. Thus, by pulling back a Frobenius-descent datum on ξ by the resulting closed immersion $^{\mathrm{PD}}P^1 \hookrightarrow X \times_{X^F} X$, one obtains an isomorphism

$$(^{\mathrm{PD}}\mathrm{pr}^1_2)^*\xi \xrightarrow{\sim} (^{\mathrm{PD}}\mathrm{pr}^1_1)^*\xi$$

in $\mathcal{F}$ over the identity automorphism of ${}^{\mathrm{PD}}P^1$. Let us also observe that this isomorphism is in fact a PD-*connection* on ξ; moreover, this PD-connection naturally extends to a PD-*stratification* on ξ. In summary, one may construct a PD-stratification from a Frobenius-descent datum, i.e., obtain a map of sets

$$\mathrm{FrDsc}(\xi) \longrightarrow {}^{\mathrm{PD}}\mathrm{Strt}(\xi) \ (\longrightarrow {}^{\mathrm{PD}}\mathrm{IntCnn}(\xi))$$

[cf. Proposition 1.11; Proposition 3.3; [3], Lemma 4.8; [3], Lemma 4.12, (i)].

I.3. — Next, let us recall the notion of *p-curvature* [cf., e.g., [5], §5]. One may associate, to a divided power stratification [or, alternatively, an integrable connection] on the locally free $\mathcal{O}_X$-module $\mathcal{E}$, the *p-curvature*, i.e., a certain homomorphism of $\mathcal{O}_X$-modules

$$\mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathrm{Fr}^*_{X/S}\Omega^1_{X^F/S}.$$

The usual definition of the *p-curvature* of a divided power stratification *relies*, at least *a priori*, on the fact that $\mathcal{E}$ is an $\mathcal{O}_X$-module. Thus, the usual definition cannot be applied, at least in any immediate way, to a PD-stratification on an object of $\mathcal{F}$ discussed in §I.1. However, *S. Mochizuki* gave a "sufficiently abstract" definition of p-curvature [cf. [7], Proposition 1.7; [8], §2.3], which *does work* even if one works with the category $\mathcal{F}$ fibered in groupoids over Sch_S as in §I.1. We define

- the (π, p)-*curvature* of a PD-stratification — that is an automorphism of a certain object of $\mathcal{F}$ — by applying a similar definition to the definition by Mochizuki [cf. also Remark 3.9.1, (i)], i.e., [3], Definition 4.13, (i) [cf. Proposition 1.11; [3], Definition 4.13, (i)].

Moreover, we define

- a (π, p)-*dormant* PD-*connection* by applying [3], Definition 4.13, (iii), i.e., to be a [necessarily PD-integrable — cf. [3], Lemma 4.8] PD-connection that arises from a PD-stratification whose (π, p)-curvature is the identity automorphism [cf. Proposition 1.11; [3], Definition 4.13, (iii)].

Write

$${}^{\pi,p}\mathrm{DrmCnn}(\xi) \subseteq {}^{\mathrm{PD}}\mathrm{IntCnn}(\xi)$$

for the set of (π, p)-dormant PD-connections on ξ [cf. Proposition 1.11; [3], Definition 4.13, (iii)].

I.4. — Next, let us recall that it is well-known [cf., e.g., [5], Theorem 5.1] that, for a given integrable connection ∇ on the locally free $\mathcal{O}_X$-module $\mathcal{E}$, the following two conditions are *equivalent*:

(†) The integrable connection ∇ is *dormant* — i.e., the *p-curvature* of the integrable connection ∇

$$\mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathrm{Fr}^*_{X/S}\Omega^1_{X^F/S}$$

[cf. §I.3] is *zero*.

(‡) The integrable connection ∇ arises from a *uniquely determined Frobenius-descent datum* on $\mathcal{E}$ [cf. the final display of §I.2].

Here, let us observe that the proof of the above relationship between dormant connections and Frobenius-descent data given in [5], §5, *relies*, at least *a priori*, on the fact that $\mathcal{E}$ is an $\mathcal{O}_X$-module [cf. the argument concerning the endomorphism "P" in the proof of [5], Theorem 5.1]. In particular, this proof cannot be applied, at least in any immediate way, in a situation in which we work with the category $\mathcal{F}$ fibered in groupoids over Sch_S as in §I.1.

In the present paper, we give a *"geometric proof"* of the above equivalence, which *does work* even if one works with a certain category $\mathcal{F}$ fibered in groupoids over Sch_S as in §I.1. One important result of the present paper — that may be regarded as an application of the *second fundamental correspondence* of [3], Definition 6.7 — is as follows [cf. Corollary 3.9].

THEOREM A. — *Let p be a prime number, S a scheme* **over a field of characteristic p**, *X a scheme which is* **smooth** *and* **separated** *over S,*

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a **weakly integrable** [*cf.* [3], *Definition* 1.8] *category fibered in groupoids over* Sch_S*, and ξ an object of $\mathcal{F}$ over X. Then the map*

$$\mathrm{FrDsc}(\xi) \longrightarrow {}^{\pi,p}\mathrm{DrmCnn}(\xi)$$

[*cf. Proposition* 1.11*; Definition* 3.2*,* (iv)*; Proposition* 3.3*;* [3]*, Definition* 4.13*,* (iii)*;* [3]*, Lemma* 4.14] *determined by the morphism $\pi\colon \mathrm{PD} \to \mathrm{Fr}$* [*cf. Proposition* 1.11] *is* **bijective**.

We shall say that the category $\mathcal{F}$ fibered in groupoids over Sch_S is *Frobenius-descent effective* if the relative [p-th power] Frobenius morphism of every smooth scheme over S is an effective descent morphism, i.e., relative to the category $\mathcal{F}$ fibered in groupoids over Sch_S [cf. Definition 3.2, (v)]. Since [one verifies easily that] the relative Frobenius morphism of a smooth scheme over S is a *finite flat universal homeomorphism*, one verifies immediately from elementary descent theory that both

- the category fibered in groupoids of *locally free modules* [cf. [4], Definition 1.1] and
- the category fibered in groupoids of *smooth schemes* [cf. [4], Definition 1.7]

are *Frobenius-descent effective* [cf. Remark 3.2.1].

As an application of Theorem A, we also obtain the following result [cf. Corollary 3.10].

THEOREM B. — *Let p be a prime number, S a scheme* **over a field of characteristic p**, *X a scheme which is* **smooth** *and* **separated** *over S, and*

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a **weakly integrable** [*cf.* [3]*, Definition* 1.8] *and* **Frobenius-descent effective** [*cf. Definition* 3.2*,* (v)] *category fibered in groupoids over* Sch_S*. Write $X^F \to S$ for the base-change of the structure morphism $X \to S$ of X by the absolute* [*p-th power*] *Frobenius endomorphism of S,*

$$\mathrm{Fr}_{X/S}\colon X \longrightarrow X^F$$

for the relative [p-th power] Frobenius morphism of X/S, and

$$\mathcal{F}|_X^{\mathrm{Drm}}$$

for the groupoid defined as follows:

- *An object of the category* $\mathcal{F}|_X^{\mathrm{Drm}}$ *is a pair* (ξ, ∇) *consisting of an object* ξ *of* $\mathcal{F}|_X$ *[cf. [3], Definition 1.6, (ii)] and a* $\boldsymbol{(\pi, p)}$**-dormant** PD*-connection* ∇ *on* ξ *[cf. Proposition 1.11; [3], Definition 4.1, (iii); [3], Definition 4.13, (iii); [4], Definition 2.5].*

- *If* (ξ_1, ∇_1) *and* (ξ_2, ∇_2) *are objects of the category* $\mathcal{F}|_X^{\mathrm{Drm}}$*, then a morphism* $(\xi_1, \nabla_1) \to (\xi_2, \nabla_2)$ *in the category* $\mathcal{F}|_X^{\mathrm{Drm}}$ *is defined to be an isomorphism* $\xi_1 \xrightarrow{\sim} \xi_2$ *in* $\mathcal{F}|_X$ *that is* **PD-horizontal** *[cf. [3], Definition 4.2; [4], Definition 2.5].*

Then the functor

$$\mathcal{F}|_{X^F} \longrightarrow \mathcal{F}|_X^{\mathrm{Drm}}$$

[cf. Corollary 3.9] determined by the morphism $\mathrm{Fr}_{X/S} \colon X \to X^F$ *is an* **equivalence** *of categories.*

Thus, we obtain *generalizations* of the equivalence of the two conditions (†) and (‡) in §I.4, i.e., Theorem A and Theorem B.

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1. FROBENIUS-STRATIFICATION STRUCTURES

In the present §1, we introduce and discuss the notion of a *Frobenius-stratification structure* on a smooth scheme over a field of positive characteristic [cf. Definition 1.8 below]. In the present §1, let p be a prime number, S a scheme *over a field of characteristic* p, and X a scheme which is *smooth* and *separated* over S. Thus, we have the *divided power stratification structure* on X/S

$$\mathrm{PD} = \big(({}^{\mathrm{PD}}P^n)_{n\geq 0},\ ({}^{\mathrm{PD}}\iota^n \colon {}^{\mathrm{PD}}P^n \to {}^{\mathrm{PD}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathrm{PD}}\sigma^n \colon {}^{\mathrm{PD}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathrm{PD}}\delta^{n_1,n_2} \colon {}^{\mathrm{PD}}P^{n_1,n_2} \to {}^{\mathrm{PD}}P^{n_1+n_2})_{n_1,\, n_2\geq 0} \big)$$

[cf. [4], Definition 2.5].

DEFINITION 1.1. — We shall write

$$\mathrm{Fr}_S \colon S \longrightarrow S$$

for the absolute [p-th power] Frobenius endomorphism of S,

$$\mathrm{Fr}_X \colon X \longrightarrow X$$

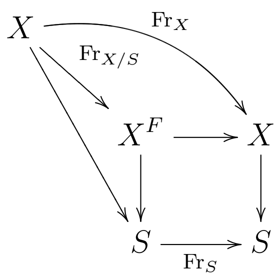
for the absolute [p-th power] Frobenius endomorphism of X,

$$X^F \longrightarrow S$$

for the base-change of the structure morphism $X \to S$ of X by Fr_S, and

$$\mathrm{Fr}_{X/S} \colon X \longrightarrow X^F$$

for the relative [p-th power] Frobenius morphism of X/S. Thus, we have a commutative diagram of schemes



— where the right-hand lower square is cartesian.

DEFINITION 1.2.

(i) We shall write

$${}^{\mathrm{Fr}}P \overset{\mathrm{def}}{=} X \times_{X_F} X$$

for the fiber product of two copies of $\mathrm{Fr}_{X/S} \colon X \to X^F$. Thus, the structure morphism $X^F \to S$ of X^F determines a closed immersion over S

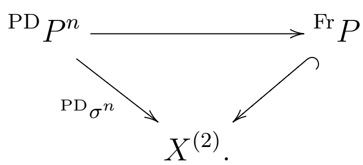
$${}^{\mathrm{Fr}}P \hookrightarrow X^{(2)}.$$

(ii) Let i be an element of $\{1, 2\}$. Then we shall write

$${}^{\mathrm{Fr}}\mathrm{pr}_i \colon {}^{\mathrm{Fr}}P \longrightarrow X$$

for the projection onto the i-th factor.

LEMMA 1.3. — *Let n be a nonnegative integer. Then the morphism ${}^{\mathrm{PD}}\sigma^n \colon {}^{\mathrm{PD}}P^n \to X^{(2)}$* **factors** *through the closed immersion ${}^{\mathrm{Fr}}P \hookrightarrow X^{(2)}$ of Definition* 1.2, (i)*:*



PROOF. — This assertion follows immediately from [4], Lemma 3.11 [cf. also [4], Lemma 3.2]. ☐

DEFINITION 1.4. — Let n be a nonnegative integer.

(i) We shall write

$$ {}^{\mathrm{Fr}}P^n \subseteq {}^{\mathrm{Fr}}P $$

for the closed subscheme of ${}^{\mathrm{Fr}}P$ obtained by forming the scheme-theoretic image of the morphism ${}^{\mathrm{PD}}P^n \to {}^{\mathrm{Fr}}P$ obtained in Lemma 1.3. Thus, we have a sequence of closed subschemes of ${}^{\mathrm{Fr}}P$

$$ {}^{\mathrm{Fr}}P \supseteq \ldots \supseteq {}^{\mathrm{Fr}}P^{n+1} \supseteq {}^{\mathrm{Fr}}P^n \supseteq \ldots \supseteq {}^{\mathrm{Fr}}P^2 \supseteq {}^{\mathrm{Fr}}P^1 \supseteq {}^{\mathrm{Fr}}P^0. $$

(ii) We shall write

$$ {}^{\mathrm{Fr}}\iota^n \colon {}^{\mathrm{Fr}}P^n \hookrightarrow {}^{\mathrm{Fr}}P^{n+1} $$

for the natural closed immersion over S.

(iii) We shall write

$$ {}^{\mathrm{Fr}}\sigma^n \colon {}^{\mathrm{Fr}}P^n \hookrightarrow X^{(2)} $$

for the closed immersion over S obtained by forming the composite of the natural closed immersion ${}^{\mathrm{Fr}}P^n \hookrightarrow {}^{\mathrm{Fr}}P$ and the closed immersion ${}^{\mathrm{Fr}}P \hookrightarrow X^{(2)}$ of Definition 1.2, (i).

(iv) We shall write

$$ \pi^n \colon {}^{\mathrm{PD}}P^n \longrightarrow {}^{\mathrm{Fr}}P^n $$

for the morphism over S obtained by the definition of ${}^{\mathrm{Fr}}P^n$. Thus, we have a commutative diagram of schemes over S

$$ \begin{array}{ccc} {}^{\mathrm{PD}}P^n & \xrightarrow{\quad \pi^n \quad} & {}^{\mathrm{Fr}}P^n \\ & {\scriptstyle {}^{\mathrm{PD}}\sigma^n} \searrow \qquad \swarrow {\scriptstyle {}^{\mathrm{Fr}}\sigma^n} & \\ & X^{(2)}. & \end{array} $$

LEMMA 1.5. — *The collection of data*

$$ \big(({}^{\mathrm{Fr}}P^n)_{n\geq 0},\ ({}^{\mathrm{Fr}}\iota^n \colon {}^{\mathrm{Fr}}P^n \to {}^{\mathrm{Fr}}P^{n+1})_{n\geq 0},\ ({}^{\mathrm{Fr}}\sigma^n \colon {}^{\mathrm{Fr}}P^n \to X^{(2)})_{n\geq 0} \big) $$

forms a **pre-stratification structure** [*cf.* [3]*, Definition* 2.2] *on* X/S*.*

PROOF. — It is immediate from the definitions of "${}^{\mathrm{Fr}}\iota^n$" and "${}^{\mathrm{Fr}}\sigma^n$" that the collection of data under consideration satisfies condition (1) of [3], Definition 2.2. Moreover, it follows from the commutative diagram of Definition 1.4, (iv), and [4], Lemma 2.2, that the collection of data under consideration satisfies condition (2) of [3], Definition 2.2. This completes the proof of Lemma 1.5. □

DEFINITION 1.6. — Let n_1, n_2 be nonnegative integers. Thus, we have the scheme ${}^{\mathrm{Fr}}P^{n_1,n_2}$ over S [cf. [3], Definition 2.4, (i), in the case where we take the "$\mathfrak{S}$" to be the pre-stratification structure of Lemma 1.5]. Then one verifies easily that the commutative diagram of schemes over S

$$\begin{array}{ccc} {}^{\mathrm{PD}}P^{n_1,n_2} & \xrightarrow{{}^{\mathrm{PD}}\delta^{n_1,n_2}} & {}^{\mathrm{PD}}P^{n_1+n_2} \\ {\scriptstyle {}^{\mathrm{PD}}\sigma^{n_1,n_2}}\downarrow & & \downarrow{\scriptstyle {}^{\mathrm{PD}}\sigma^{n_1+n_2}} \\ X^{(3)} & \xrightarrow[{}^{X}\mathrm{pr}^{(3)}_{\{1,3\}}]{} & X^{(2)} \end{array}$$

[cf. condition (1) of [3], Definition 2.5; [4], Proposition 2.4] determines a commutative diagram of schemes over S

$$\begin{array}{ccc} {}^{\mathrm{Fr}}P^{n_1,n_2} & \longrightarrow & {}^{\mathrm{Fr}}P^{n_1+n_2} \\ \downarrow & & \downarrow \\ X^{(3)} & \xrightarrow[{}^{X}\mathrm{pr}^{(3)}_{\{1,3\}}]{} & X^{(2)} \end{array}$$

— where the vertical arrows are the natural closed immersions. We shall write

$${}^{\mathrm{Fr}}\delta^{n_1,n_2} \colon {}^{\mathrm{Fr}}P^{n_1,n_2} \longrightarrow {}^{\mathrm{Fr}}P^{n_1+n_2}$$

for the upper horizontal arrow of this diagram.

PROPOSITION 1.7. — *The collection of data*

$$\mathrm{Fr} = \Big(({}^{\mathrm{Fr}}P^n)_{n\geq 0},\ ({}^{\mathrm{Fr}}\iota^n \colon {}^{\mathrm{Fr}}P^n \to {}^{\mathrm{Fr}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathrm{Fr}}\sigma^n \colon {}^{\mathrm{Fr}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathrm{Fr}}\delta^{n_1,n_2} \colon {}^{\mathrm{Fr}}P^{n_1,n_2} \to {}^{\mathrm{Fr}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\Big)$$

forms a **stratification structure** [*cf.* [3], *Definition* 2.5] *on* X/S.

PROOF. — This assertion follows immediately from Lemma 1.5 and [4], Proposition 2.4, together with the definition of "${}^{\mathrm{Fr}}\delta^{n_1,n_2}$". □

DEFINITION 1.8. — We shall refer to the stratification structure of Proposition 1.7

$$\mathrm{Fr} = \Big(({}^{\mathrm{Fr}}P^n)_{n\geq 0},\ ({}^{\mathrm{Fr}}\iota^n \colon {}^{\mathrm{Fr}}P^n \to {}^{\mathrm{Fr}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathrm{Fr}}\sigma^n \colon {}^{\mathrm{Fr}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathrm{Fr}}\delta^{n_1,n_2} \colon {}^{\mathrm{Fr}}P^{n_1,n_2} \to {}^{\mathrm{Fr}}P^{n_1+n_2})_{n_1,\, n_2\geq 0}\Big)$$

as the *Frobenius-stratification structure* on X/S.

LEMMA 1.9. — *Let* n *be a nonnegative integer. Then the closed immersion* ${}^{\mathrm{Fr}}\iota^n \colon {}^{\mathrm{Fr}}P^n \hookrightarrow {}^{\mathrm{Fr}}P^{n+1}$ *is* **square-nilpotent** [*cf.* [3], *Definition* 1.2, (ii)]. *In particular, the stratification structure* Fr *is* **ι-quasi-nil-retraction-like** [*cf.* [3], *Definition* 3.1, (i)].

PROOF. — This assertion follows from [4], Lemma 2.7 [cf. also [3], Remark 1.2.1]. □

LEMMA 1.10. — *Let r be a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Write $n \stackrel{\text{def}}{=} \sum_{i=1}^{r} n_i$. Then the morphisms in the diagram of schemes over S*

$$\begin{array}{ccc} {}^{\mathrm{Fr}}P^{\times n} & \xrightarrow{{}^{\mathrm{Fr}}\delta^{\times n}_{n_1,\ldots,n_r}} & {}^{\mathrm{Fr}}P^{n_1,\ldots,n_r} \\ & {\scriptstyle {}^{\mathrm{Fr}}\delta^{\times n}} \searrow \quad \swarrow {\scriptstyle {}^{\mathrm{Fr}}\delta^{n_1,\ldots,n_r}} & \\ & {}^{\mathrm{Fr}}P^{n} & \end{array}$$

of [3], *Definition* 2.6 [*i.e., in the case where we take the "$\mathfrak{S}$" to be the Frobenius-stratification structure* Fr], *are* **quasi-nil-retraction-like** [*cf.* [3], *Definition* 1.2, (iii)].

PROOF. — This assertion follows from Lemma 1.9 and [3], Remark 3.1.1, (iii). $\square$

PROPOSITION 1.11. — *The collection of data*

$$\pi \stackrel{\text{def}}{=} (\pi^n \colon {}^{\mathrm{PD}}P^n \to {}^{\mathrm{Fr}}P^n)_{n \geq 0}$$

forms a **morphism of stratification structures** [*cf.* [3], *Definition* 2.8]

$$\mathrm{PD} \longrightarrow \mathrm{Fr}.$$

PROOF. — This assertion follows immediately from the various definitions involved. $\square$

LEMMA 1.12. — *Let r be a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Then the morphism $\pi^{n_1,\ldots,n_r} \colon {}^{\mathrm{PD}}P^{n_1,\ldots,n_r} \to {}^{\mathrm{Fr}}P^{n_1,\ldots,n_r}$* [*cf.* [3], *Definition* 2.9, *i.e., in the case where we take the "Φ" to be the morphism π of Proposition* 1.11] *is* **quasi-nil-retraction-like**.

PROOF. — This assertion follows — in light of [3], Remark 3.7.1 — from Lemma 1.9 and [4], Lemma 2.7. $\square$

2. DIVIDED POWER POLYNOMIAL ALGEBRAS IN POSITIVE CHARACTERISTIC

In the present §2, we discuss *divided power polynomial algebras in positive characteristic*. The results obtained in the present §2 will be applied in §3 to prove that the Frobenius-stratification structures have some good properties.

In the present §2, let R be a ring and A an R-algebra. Suppose that there exist a nonnegative integer d and d elements $a_1, \ldots, a_d \in A$ of A such that the a_i's determine an *étale* morphism $R[s_1, \ldots, s_d] \to A$ over R — where $s_1, \ldots, s_d$ are indeterminates. Thus, we are in the situation of [4], §3. In particular, we are given the ring $A^{(2)} \stackrel{\text{def}}{=} A \otimes_R A$, the kernel $I \subseteq A^{(2)}$ of the multiplication $A^{(2)} \twoheadrightarrow A$, and the homomorphisms

$${}^{X}_{\sharp}\mathrm{pr}_1^{(2)},\ {}^{X}_{\sharp}\mathrm{pr}_2^{(2)} \colon A \longrightarrow A^{(2)}$$

given by mapping $a \in A$ to $a \otimes 1$, $1 \otimes a \in A^{(2)}$, respectively. In the present §2, let p be a prime number. Moreover, suppose that the ring R is a ring *over a field of characteristic* p.

DEFINITION 2.1.

(i) We shall write

$$J_p \subseteq A^{(2)}$$

for the ideal of $A^{(2)}$ generated by

$$\,^{X}_{\sharp}\mathrm{pr}_2^{(2)}(a_1)^p - \,^{X}_{\sharp}\mathrm{pr}_1^{(2)}(a_1)^p, \ldots, \,^{X}_{\sharp}\mathrm{pr}_2^{(2)}(a_d)^p - \,^{X}_{\sharp}\mathrm{pr}_1^{(2)}(a_d)^p$$

[cf. also [4], Lemma 3.11] and

$$^{\mathrm{Fr}}A \stackrel{\mathrm{def}}{=} A^{(2)}/J_p$$

for the quotient of $A^{(2)}$ by the ideal $J_p \subseteq A^{(2)}$ of $A^{(2)}$.

(ii) Let i be an element of $\{1, 2\}$. Then we shall write

$$\,^{\mathrm{Fr}}_{\sharp}\mathrm{pr}_i \colon A \longrightarrow {}^{\mathrm{Fr}}A$$

for the homomorphism obtained by forming the composite of the homomorphism $\,^{X}_{\sharp}\mathrm{pr}_i^{(2)} \colon A \to A^{(2)}$ and the natural surjective homomorphism $A^{(2)} \twoheadrightarrow {}^{\mathrm{Fr}}A$.

(iii) Let n be a nonnegative integer. Then we shall write

$$^{\mathrm{Fr}}A^n \subseteq {}^{\mathrm{PD}}A^n$$

[cf. [4], Definition 3.1, (ii)] for the subring of ${}^{\mathrm{PD}}A^n$ obtained by forming the image of the homomorphism $({}^{\mathrm{Fr}}A^n \twoheadrightarrow)\ A^{(2)}/(I^{n+1} + J) \to {}^{\mathrm{PD}}A^n$ obtained in [4], Lemma 3.11.

(iv) Let n be a nonnegative integer. Then we shall write

$$\,^{\mathrm{Fr}}_{\sharp}\iota^n \colon {}^{\mathrm{Fr}}A^{n+1} \longrightarrow\!\!\!\!\to {}^{\mathrm{Fr}}A^n$$

for the natural surjective homomorphism.

(v) Let n be a nonnegative integer. Then we shall write

$$\,^{\mathrm{Fr}}_{\sharp}\sigma^n \colon A^{(2)} \longrightarrow {}^{\mathrm{Fr}}A^n$$

for the natural surjective homomorphism.

(vi) Let n be a nonnegative integer. Then we shall write

$$_{\sharp}\pi^n \colon {}^{\mathrm{Fr}}A^n \hookrightarrow {}^{\mathrm{PD}}A^n$$

for the natural injective homomorphism.

LEMMA 2.2. — *Write*

$$X \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A) \longrightarrow S \stackrel{\mathrm{def}}{=} \mathrm{Spec}(R).$$

In particular, we are in the situation of §1*, which thus implies that we are given the morphisms*

$$^{\mathrm{Fr}}\mathrm{pr}_1, {}^{\mathrm{Fr}}\mathrm{pr}_2 \colon {}^{\mathrm{Fr}}P \longrightarrow X, \qquad \pi^n \colon {}^{\mathrm{PD}}P^n \longrightarrow {}^{\mathrm{Fr}}P^n$$

— *where* n *is a nonnegative integer — and the pre-stratification structure on* X/S *of Lemma* 1.5

$$\big(({}^{\mathrm{Fr}}P^n)_{n\geq 0},\ ({}^{\mathrm{Fr}}\iota^n\colon {}^{\mathrm{Fr}}P^n \to {}^{\mathrm{Fr}}P^{n+1})_{n\geq 0},\ ({}^{\mathrm{Fr}}\sigma^n\colon {}^{\mathrm{Fr}}P^n \to X^{(2)})_{n\geq 0}\big).$$

Then there exist **isomorphisms** *of schemes*

$${}^{\mathrm{Fr}}P \xrightarrow{\sim} \mathrm{Spec}({}^{\mathrm{Fr}}A), \qquad {}^{\mathrm{Fr}}P^n \xrightarrow{\sim} \mathrm{Spec}({}^{\mathrm{Fr}}A^n)$$

such that the morphisms of schemes

$${}^{\mathrm{Fr}}\mathrm{pr}_1, \qquad {}^{\mathrm{Fr}}\mathrm{pr}_2, \qquad \pi^n, \qquad {}^{\mathrm{Fr}}\iota^n, \qquad {}^{\mathrm{Fr}}\sigma^n$$

coincide *— relative to the isomorphisms of schemes under consideration and the isomorphisms of schemes of* [4], *Lemma* 3.2 *— with the morphisms of schemes induced by the homomorphisms*

$${}^{\mathrm{Fr}}_{\sharp}\mathrm{pr}_1, \qquad {}^{\mathrm{Fr}}_{\sharp}\mathrm{pr}_2, \qquad {}_{\sharp}\pi^n, \qquad {}^{\mathrm{Fr}}_{\sharp}\iota^n, \qquad {}^{\mathrm{Fr}}_{\sharp}\sigma^n$$

defined in Definition 2.1, *respectively.*

PROOF. — This assertion follows immediately from [4], Lemma 3.2, together with the various definitions involved. □

DEFINITION 2.3. — Let r be a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Write $n \overset{\mathrm{def}}{=} \sum_{i=1}^{r} n_i$.

(i) Let $m_1, \ldots, m_r$ be nonnegative integers such that $m_i \leq n_i$ for each $i \in \{1, \ldots, r\}$; j an element of $\{1, \ldots, r+1\}$; j' an element of $\{1, \ldots, r\}$. Then it follows from Lemma 2.2 that we obtain rings

$${}^{\mathrm{Fr}}A^{n_1,\ldots,n_r}, \qquad {}^{\mathrm{Fr}}A^{\times r}$$

that "correspond" — relative to the isomorphisms of schemes of Lemma 2.2 — to the schemes

$${}^{\mathrm{Fr}}P^{n_1,\ldots,n_r}, \qquad {}^{\mathrm{Fr}}P^{\times r}$$

defined in [3], Definition 2.4, (i), (vi) [i.e., in the case where we take the "$\mathfrak{S}$" to be the pre-stratification structure of Lemma 1.5], respectively, and homomorphisms

$${}^{\mathrm{Fr}}_{\sharp}\iota^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}\colon {}^{\mathrm{Fr}}A^{n_1,\ldots,n_r} \longrightarrow {}^{\mathrm{Fr}}A^{m_1,\ldots,m_r},$$

$${}^{\mathrm{Fr}}_{\sharp}\mathrm{pr}^{n_1,\ldots,n_r}_j\colon A \longrightarrow {}^{\mathrm{Fr}}A^{n_1,\ldots,n_r}, \qquad {}^{\mathrm{Fr}}_{\sharp}\mathrm{pr}^{n_1,\ldots,n_r}_{\{j',j'+1\}}\colon {}^{\mathrm{Fr}}A^{n_{j'}} \longrightarrow {}^{\mathrm{Fr}}A^{n_1,\ldots,n_r},$$

$${}^{\mathrm{Fr}}_{\sharp}\mathrm{pr}^{\times r}_j\colon A \longrightarrow {}^{\mathrm{Fr}}A^{\times r}, \qquad {}^{\mathrm{Fr}}_{\sharp}\mathrm{pr}^{\times r}_{\{j',j'+1\}}\colon {}^{\mathrm{Fr}}A^{1} \longrightarrow {}^{\mathrm{Fr}}A^{\times r}$$

that "correspond" — relative to the isomorphisms of schemes of Lemma 2.2 — to the morphisms of schemes

$${}^{\mathrm{Fr}}\iota^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}, \qquad {}^{\mathrm{Fr}}\mathrm{pr}^{n_1,\ldots,n_r}_j, \qquad {}^{\mathrm{Fr}}\mathrm{pr}^{n_1,\ldots,n_r}_{\{j',j'+1\}}, \qquad {}^{\mathrm{Fr}}\mathrm{pr}^{\times r}_j, \qquad {}^{\mathrm{Fr}}\mathrm{pr}^{\times r}_{\{j',j'+1\}}$$

defined in [3], Definition 2.4, (ii), (iv), (v), (vi) [i.e., in the case where we take the "$\mathfrak{S}$" to be the pre-stratification structure of Lemma 1.5], respectively.

(ii) It follows from Lemma 2.2 that we obtain homomorphisms

$$ {}^{\mathrm{Fr}}_{\sharp}\delta^{\times n}_{n_1,\dots,n_r} \colon {}^{\mathrm{Fr}}A^{n_1,\dots,n_r} \longrightarrow {}^{\mathrm{Fr}}A^{\times n}, \qquad {}^{\mathrm{Fr}}_{\sharp}\delta^{\times n} \colon {}^{\mathrm{Fr}}A^{n} \longrightarrow {}^{\mathrm{Fr}}A^{\times n}, $$

$$ {}^{\mathrm{Fr}}_{\sharp}\delta^{n_1,\dots,n_r} \colon {}^{\mathrm{Fr}}A^{n} \longrightarrow {}^{\mathrm{Fr}}A^{n_1,\dots,n_r} $$

that "correspond" — relative to the isomorphisms of schemes of Lemma 2.2 — to the morphisms of schemes

$$ {}^{\mathrm{Fr}}\delta^{\times n}_{n_1,\dots,n_r}, \qquad {}^{\mathrm{Fr}}\delta^{\times n}, \qquad {}^{\mathrm{Fr}}\delta^{n_1,\dots,n_r} $$

defined in [3], Definition 2.6 [i.e., in the case where we take the "$\mathfrak{S}$" to be the stratification structure of Definition 1.8], respectively.

(iii) Let i be an element of $\{1, \dots, r+1\}$. Suppose that $n \geq 1$. Then it follows from Lemma 2.2 that we obtain rings

$$ {}^{\mathrm{Fr}}\underline{A}^{n_1,\dots,n_r}, \qquad {}^{\mathrm{Fr}}\underline{A}^{\times r} $$

that "correspond" — relative to the isomorphisms of schemes of Lemma 2.2 — to the schemes

$$ {}^{\mathrm{Fr}}\underline{P}^{n_1,\dots,n_r}, \qquad {}^{\mathrm{Fr}}\underline{P}^{\times r} $$

defined in [3], Definition 2.7, (i), (v) [i.e., in the case where we take the "$\mathfrak{S}$" to be the stratification structure of Definition 1.8], respectively, and homomorphisms

$$ {}^{\mathrm{Fr}}_{\sharp}\underline{\iota}^{n_1,\dots,n_r} \colon {}^{\mathrm{Fr}}A^{n_1,\dots,n_r} \longrightarrow {}^{\mathrm{Fr}}\underline{A}^{n_1,\dots,n_r}, \qquad {}^{\mathrm{Fr}}_{\sharp}\underline{\delta}^{n_1,\dots,n_r} \colon {}^{\mathrm{Fr}}A^{n-1} \longrightarrow {}^{\mathrm{Fr}}\underline{A}^{n_1,\dots,n_r}, $$

$$ {}^{\mathrm{Fr}}_{\sharp}\underline{\mathrm{pr}}^{n_1,\dots,n_r}_{i} \colon A \longrightarrow {}^{\mathrm{Fr}}\underline{A}^{n_1,\dots,n_r}, \qquad {}^{\mathrm{Fr}}_{\sharp}\underline{\iota}^{0}_{n_1,\dots,n_r} \colon {}^{\mathrm{Fr}}\underline{A}^{n_1,\dots,n_r} \longrightarrow A, $$

$$ {}^{\mathrm{Fr}}_{\sharp}\underline{\iota}^{\times r} \colon {}^{\mathrm{Fr}}A^{\times r} \longrightarrow {}^{\mathrm{Fr}}\underline{A}^{\times r}, \qquad {}^{\mathrm{Fr}}_{\sharp}\underline{\delta}^{\times r} \colon {}^{\mathrm{Fr}}A^{r-1} \longrightarrow {}^{\mathrm{Fr}}\underline{A}^{\times r}, $$

$$ {}^{\mathrm{Fr}}_{\sharp}\underline{\mathrm{pr}}^{\times r}_{i} \colon A \longrightarrow {}^{\mathrm{Fr}}\underline{A}^{\times r}, \qquad {}^{\mathrm{Fr}}_{\sharp}\underline{\iota}^{0}_{\times r} \colon {}^{\mathrm{Fr}}\underline{A}^{\times r} \longrightarrow A $$

that "correspond" — relative to the isomorphisms of schemes of Lemma 2.2 — to the morphisms of schemes

$$ {}^{\mathrm{Fr}}\underline{\iota}^{n_1,\dots,n_r}, \qquad {}^{\mathrm{Fr}}\underline{\delta}^{n_1,\dots,n_r}, \qquad {}^{\mathrm{Fr}}\underline{\mathrm{pr}}^{n_1,\dots,n_r}_{i}, \qquad {}^{\mathrm{Fr}}\underline{\iota}^{0}_{n_1,\dots,n_r}, $$

$$ {}^{\mathrm{Fr}}\underline{\iota}^{\times r}, \qquad {}^{\mathrm{Fr}}\underline{\delta}^{\times r}, \qquad {}^{\mathrm{Fr}}\underline{\mathrm{pr}}^{\times r}_{i}, \qquad {}^{\mathrm{Fr}}\underline{\iota}^{0}_{\times r} $$

defined in [3], Definition 2.7, (ii), (iii), (iv), (v) [i.e., in the case where we take the "$\mathfrak{S}$" to be the stratification structure of Definition 1.8], respectively.

(iv) Let $m' \leq m \leq n$ be nonnegative integers and i an element of $\{1, 2\}$. Then it follows from Lemma 2.2 that we obtain a ring

$$ {}^{\pi}\underline{A}^{n|m} $$

that "corresponds" — relative to the isomorphisms of schemes of Lemma 2.2 and [4], Lemma 3.2, and — to the scheme

$$ {}^{\pi}\underline{P}^{n|m} $$

defined in [3], Definition 2.10, (i) [i.e., in the case where we take the "Φ" to be the morphism of Proposition 1.11], and homomorphisms

$$ {}_{\sharp}\underline{\pi}^{n|m} \colon {}^{\mathrm{Fr}}A^{m} \longrightarrow {}^{\pi}\underline{A}^{n|m}, \qquad {}^{\pi}_{\sharp}\underline{\iota}^{n|m'}_{n|m} \colon {}^{\pi}\underline{A}^{n|m} \longrightarrow {}^{\pi}\underline{A}^{n|m'}, $$

$$ {}^{\pi}_{\sharp}\underline{\mathrm{pr}}^{n|m}_{i} \colon A \longrightarrow {}^{\pi}\underline{A}^{n|m}, \qquad {}^{\pi}_{\sharp}\underline{\iota}^{0}_{n|m} \colon {}^{\pi}\underline{A}^{n|m} \longrightarrow A $$

that "correspond" — relative to the isomorphisms of schemes of Lemma 2.2 and [4], Lemma 3.2 — to the morphisms of schemes

$$ \underline{\pi}^{n|m}, \qquad {}^{\pi}\underline{\iota}^{n|m'}_{n|m}, \qquad {}^{\pi}\underline{\mathrm{pr}}^{n|m}_{i}, \qquad {}^{\pi}\underline{\iota}^{0}_{n|m} $$

defined in [3], Definition 2.10, (ii), (iii), (iv), (v) [i.e., in the case where we take the "Φ" to be the morphism of Proposition 1.11], respectively.

DEFINITION 2.4. — Let n be a positive integer. Then we shall write

$$ [n,d]^{<p} \subseteq [n,d] $$

for the subset of the set $[n,d]$ of [4], Definition 3.4, (i), consisting of maps $\{1,\dots,n\} \to \{0,\dots,d\}$ such that $\sharp\mathfrak{m}^{-1}(\{i\}) < p$ whenever $i \neq 0$. Thus, the equivalence relation "$\sim$" on the set $[n,d]$ of [4], Definition 3.4, (ii), determines an equivalence relation on the subset $[n,d]^{<p}$.

LEMMA 2.5. — *Let n be a positive integer. Then the following hold:*

(i) *Let $\overline{\mathfrak{m}}$ be an element of $[n,d]^{<p}/\sim$ ($\subseteq [n,d]/\sim$). Then the element $x^{[\overline{\mathfrak{m}}]} \in {}^{\mathrm{PD}}A^{n}$ defined in* [4], *Definition* 3.4, (iii), *is* **contained** *in the subring ${}^{\mathrm{Fr}}A^{n} \subseteq {}^{\mathrm{PD}}A^{n}$ of ${}^{\mathrm{PD}}A^{n}$.*

(ii) *If one regards ${}^{\mathrm{Fr}}A^{n}$ as an A-module by the homomorphism ${}^{\mathrm{Fr}}_{\sharp}\mathrm{pr}^{n}_{1}$ (respectively, ${}^{\mathrm{Fr}}_{\sharp}\mathrm{pr}^{n}_{2}$), then the A-module ${}^{\mathrm{Fr}}A^{n}$ is* **free**, *and the subset $\{x^{[\overline{\mathfrak{m}}]}\}_{\overline{\mathfrak{m}}\in[n,d]^{<p}/\sim} \subseteq {}^{\mathrm{Fr}}A^{n}$ of ${}^{\mathrm{Fr}}A^{n}$* [*cf.* (i)] *forms a* **basis** *of the free A-module ${}^{\mathrm{Fr}}A^{n}$.*

PROOF. — Since [it is immediate that] $(p-1)!$ is *invertible* in R, these assertions follow immediately from condition (4) of [2], Definition 3.1, and [4], Lemma 3.5, (i). □

LEMMA 2.6. — *Let n be a nonnegative integer. Then the following hold:*

(i) *It holds that the injective homomorphism ${}_{\sharp}\pi^{n} \colon {}^{\mathrm{Fr}}A^{n} \hookrightarrow {}^{\mathrm{PD}}A^{n}$ is an* **isomorphism** *if and only if the inequality $n \leq p-1$ holds.*

In the remainder of the present paper, if $n \leq p-1$, then let us identify ${}^{\mathrm{Fr}}A^{n}$ with ${}^{\mathrm{PD}}A^{n}$ by means of this isomorphism ${}_{\sharp}\pi^{n} \colon {}^{\mathrm{Fr}}A^{n} \xrightarrow{\sim} {}^{\mathrm{PD}}A^{n}$:

$$ {}^{\mathrm{Fr}}A^{n} = {}^{\mathrm{PD}}A^{n}. $$

(ii) *It holds that the natural surjective homomorphism ${}^{\mathrm{Fr}}A \twoheadrightarrow {}^{\mathrm{Fr}}A^{n}$ is an* **isomorphism** *if and only if the inequality $d(p-1) \leq n$ holds.*

PROOF. — These assertions follow immediately from Lemma 2.5, (ii), and [4], Lemma 3.5, (i). □

LEMMA 2.7. — *Let r be a positive integer; $n_1, \ldots, n_r$ nonnegative integers. Write $n \stackrel{\mathrm{def}}{=} \sum_{i=1}^{r} n_i$. Let us recall the commutative diagram of rings*

$$\begin{array}{ccccc} & & {}^{\mathrm{Fr}}A^{n} & & \\ & {}^{\mathrm{Fr}}_{\sharp}\delta^{n_1,\ldots,n_r} \swarrow & & \searrow {}^{\mathrm{Fr}}_{\sharp}\delta^{\times n} & \\ {}^{\mathrm{Fr}}A^{n_1,\ldots,n_r} & & \xrightarrow[{}^{\mathrm{Fr}}_{\sharp}\delta^{\times n}_{n_1,\ldots,n_r}]{} & & {}^{\mathrm{Fr}}A^{\times n} \end{array}$$

[cf. [3], Definition 2.6, i.e., in the case where we take the "$\mathfrak{S}$" to be the stratification structure of Definition 1.8]. Then the homomorphisms in this diagram are **injective***.*

PROOF. — This assertion follows from [4], Lemma 3.8. □

LEMMA 2.8. — *Suppose that $n \geq 3$. Let us recall the commutative diagram of* **injective** *[cf. Lemma 2.7] homomorphisms of rings*

$$\begin{array}{ccc} {}^{\mathrm{Fr}}A^{n} & \xhookrightarrow{{}^{\mathrm{Fr}}_{\sharp}\delta^{1,n-1}} & {}^{\mathrm{Fr}}A^{1,n-1} \\ {\scriptstyle {}^{\mathrm{Fr}}_{\sharp}\delta^{n-1,1}} \big\downarrow & \searrow {\scriptstyle {}^{\mathrm{Fr}}_{\sharp}\delta^{\times n}} & \big\downarrow {\scriptstyle {}^{\mathrm{Fr}}_{\sharp}\delta^{\times n}_{1,n-1}} \\ {}^{\mathrm{Fr}}A^{n-1,1} & \xhookrightarrow[{}^{\mathrm{Fr}}_{\sharp}\delta^{\times n}_{n-1,1}]{} & {}^{\mathrm{Fr}}A^{\times n}. \end{array}$$

Then the following two conditions are equivalent:

(1) *Either $n \neq p$ or $d = 0$.*

(2) *The intersection of the image of ${}^{\mathrm{Fr}}_{\sharp}\delta^{\times n}_{1,n-1}$ and the image of ${}^{\mathrm{Fr}}_{\sharp}\delta^{\times n}_{n-1,1}$* **coincides** *with the image of ${}^{\mathrm{Fr}}_{\sharp}\delta^{\times n}$.*

PROOF. — First, we verify the implication (1) $\Rightarrow$ (2). If $d = 0$, then it is immediate that condition (2) is satisfied. If $n \leq p - 1$, then it follows from Lemma 2.6, (i), and [4], Lemma 3.9, that condition (2) is satisfied. Suppose that $n \geq p + 1$ and $d \neq 0$.

Let z be an element of ${}^{\mathrm{PD}}A^{\times n} = {}^{\mathrm{Fr}}A^{\times n}$ [cf. Lemma 2.6, (i); [4], Definition 3.3, (i)] contained in the intersection under consideration. Thus, it follows from [4], Lemma 3.9, that z is *contained* in the image of ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n}$ [cf. [4], Definition 3.3, (ii)]. For each $\mathfrak{m} \in [n, d]$, write $a_{\mathfrak{m}} \in A$ for the uniquely determined element of A such that if one regards ${}^{\mathrm{PD}}A^{\times n} = {}^{\mathrm{Fr}}A^{\times n}$ as an A-module by the homomorphism ${}^{\mathrm{PD}}_{\sharp}\mathrm{pr}_1^{\times n} = {}^{\mathrm{Fr}}_{\sharp}\mathrm{pr}_1^{\times n}$ [cf. Lemma 2.6, (i); [4], Definition 3.3, (i)], then the equality $z = \sum_{\mathfrak{m} \in [n,d]} a_{\mathfrak{m}} x_{\otimes \mathfrak{m}}$ holds [cf. [4], Definition 3.4, (iv); [4], Lemma 3.5, (ii)].

Next, let us observe that it follows — in light of [4], Lemma 3.7, (i) — from Lemma 2.5, (ii), and [4], Lemma 3.5, (i), that, to verify the implication (1) $\Rightarrow$ (2), it suffices to verify the following claim:

Claim 2.8.A: If $\mathfrak{m} \in [n,d] \setminus [n,d]^{<p}$, then $a_{\mathfrak{m}} = 0$.

To this end, let $\mathfrak{m}$ be an element of $[n,d] \setminus [n,d]^{<p}$. Now let us observe that since z is *contained* in the image of ${}^{\mathrm{PD}}_{\sharp}\delta^{\times n}$, it follows from [4], Lemma 3.7, (i), that, to verify the equality $a_{\mathfrak{m}} = 0$, we may assume without loss of generality, by replacing $\mathfrak{m}$ by a suitable element of $[n,d]$ equivalent [i.e., with respect to the equivalence relation of [4], Definition 3.4, (ii) — cf. [4], Remark 3.6.1] to $\mathfrak{m}$, that the map $\mathfrak{m}\colon \{1,\ldots,n\} \to \{0,\ldots,d\}$ sends every element of the subset $\{1,\ldots,p\} \subseteq \{1,\ldots,n\}$ [cf. our assumption that $n \geq p+1$] to $1 \in \{0,\ldots,d\}$. Then since z is *contained* in the image of ${}^{\mathrm{Fr}}_{\sharp}\delta^{\times n}_{n-1,1}$, and $n-1 \geq p$, it follows immediately from Lemma 2.5, (ii), and [4], Lemma 3.7, (i), that $a_{\mathfrak{m}} = 0$, as desired. This completes the proof of Claim 2.8.A, hence also of the implication (1) $\Rightarrow$ (2).

Next, to verify the implication (2) $\Rightarrow$ (1), suppose that $n = p$ and $d \neq 0$. Write $\mathfrak{m} \in [n,d]$ for the map $\{1,\ldots,n\} \to \{0,\ldots,d\}$ given by mapping every $i \in \{1,\ldots,n\}$ to $1 \in \{0,\ldots,d\}$. Then one verifies easily from Lemma 2.5, (i), and [4], Lemma 3.7, (i), that the element $x_{\otimes\overline{\mathfrak{m}}} \in {}^{\mathrm{PD}}A^{\times n} = {}^{\mathrm{Fr}}A^{\times n}$ [cf. [4], Definition 3.4, (v)] is *contained* in the intersection under consideration. On the other hand, one verifies easily from Lemma 2.5, (ii), and [4], Lemma 3.7, (i), that the element $x_{\otimes\overline{\mathfrak{m}}} \in {}^{\mathrm{PD}}A^{\times n}$ is *not contained* in the image of ${}^{\mathrm{Fr}}_{\sharp}\delta^{\times n}$. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Lemma 2.8. □

LEMMA 2.9. — *Let r be a positive integer; $n_1,\ldots,n_r$ nonnegative integers. Write $n \stackrel{\mathrm{def}}{=} \sum_{i=1}^{r} n_i$. Suppose that $n \geq 1$. Then the kernel of the surjective homomorphism ${}^{\mathrm{Fr}}_{\sharp}\underline{\iota}^{n_1,\ldots,n_r}\colon {}^{\mathrm{Fr}}A^{n_1,\ldots,n_r} \twoheadrightarrow {}^{\mathrm{Fr}}\underline{A}^{n_1,\ldots,n_r}$ — i.e., the ideal of ${}^{\mathrm{Fr}}A^{n_1,\ldots,n_r}$ generated by the image of ${}^{\mathrm{Fr}}A^{n} \cap ({}^{\mathrm{PD}}I^{[n]}/{}^{\mathrm{PD}}I^{[n+1]}) \subseteq {}^{\mathrm{Fr}}A^{n}$ [cf. [4], Definition 3.1, (i)] by ${}^{\mathrm{Fr}}_{\sharp}\delta^{n_1,\ldots,n_r}$ — is* **annihilated** *by the kernel of the surjective homomorphism ${}^{\mathrm{Fr}}_{\sharp}\iota^{0,\ldots,0}_{n_1,\ldots,n_r}\colon {}^{\mathrm{Fr}}A^{n_1,\ldots,n_r} \twoheadrightarrow {}^{\mathrm{Fr}}A^{0,\ldots,0} = A$.*

PROOF. — This assertion follows immediately from [4], Lemma 3.10. □

LEMMA 2.10. — *Let $n \leq p$ be a nonnegative integer. If one regards ${}^{\mathrm{PD}}A^{p}$ as an A-module by the homomorphism ${}^{\mathrm{PD}}_{\sharp}\mathrm{pr}^{p}_{1}$, then the kernel of the surjective homomorphism ${}^{\pi}_{\sharp}\underline{\iota}^{p|n}_{p|p}\colon {}^{\mathrm{PD}}A^{p} = {}^{\pi}\underline{A}^{p|p} \twoheadrightarrow {}^{\pi}\underline{A}^{p|n}$* **coincides** *with the A-submodule of ${}^{\mathrm{PD}}A^{p}$* **freely** *[cf. [4], Lemma 3.5, (i)]* **generated** *by the $x^{[\overline{\mathfrak{m}}]}$'s, where $\overline{\mathfrak{m}}$ ranges over the elements of $[p,d]^{<p}/\sim$ $(\subseteq [p,d]/\sim)$ such that every $\mathfrak{m} \in \overline{\mathfrak{m}}$ $(\subseteq [p,d]^{<p})$ satisfies the inequality $\sharp\mathfrak{m}^{-1}(\{1,\ldots,d\}) \geq n+1$.*

PROOF. — Let us first observe that [it is immediate that] $(p-1)!$ is *invertible* in R. Thus, Lemma 2.10 follows immediately — in light of condition (4) of [2], Definition 3.1 — from Lemma 2.5, (ii), and [4], Lemma 3.5, (i). This completes the proof of Lemma 2.10. □

LEMMA 2.11. — *Let $n \leq p$ be a positive integer. Then the kernel of the surjective homomorphism ${}^{\pi}_{\sharp}\underline{\iota}^{p|n-1}_{p|n}\colon {}^{\pi}\underline{A}^{p|n} \twoheadrightarrow {}^{\pi}\underline{A}^{p|n-1}$ is* **annihilated** *by the kernel of the natural surjective homomorphism ${}^{\pi}_{\sharp}\underline{\iota}^{0}_{p|n}\colon {}^{\pi}\underline{A}^{p|n} \twoheadrightarrow A$.*

PROOF. — This assertion follows immediately from Lemma 2.10, together with condition (4) of [2], Definition 3.1. □

3. An Application of the Second Fundamental Correspondence

In the present §3, we prove [cf. Theorem 3.8 below] that, for a smooth scheme *over a field of characteristic* $p > 0$, the morphism $\pi\colon \mathrm{PD} \to \mathrm{Fr}$ of Proposition 1.11 is *strictly* p*-integrable* [cf. [3], Definition 3.9]. Moreover, we discuss an application of the second fundamental correspondence of [3], Definition 6.7 [cf. Corollary 3.9 below]. In the present §3, let p be a prime number, S a scheme *over a field of characteristic* p, X a scheme which is *smooth* and *separated* over S,

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a category fibered in groupoids over Sch_S, and

$$\xi$$

an object of $\mathcal{F}$ over X.

Lemma 3.1. — *Let n be a nonnegative integer. Then the following hold:*

(i) *It holds that the morphism* $\pi^n\colon {}^{\mathrm{PD}}P^n \to {}^{\mathrm{Fr}}P^n$ *is an* **isomorphism** *if and only if the inequality* $n \leq p-1$ *holds.*

(ii) *It holds that the natural closed immersion* ${}^{\mathrm{Fr}}P^n \hookrightarrow {}^{\mathrm{Fr}}P$ *is an* **isomorphism** *if and only if the inequality* $d(p-1) \leq n$ *holds.*

Proof. — Assertion (i) (respectively, (ii)) follows from Lemma 2.6, (i) (respectively, (ii)). $\square$

Definition 3.2.

(i) We shall write

$${}^{\mathrm{Fr}}T \stackrel{\mathrm{def}}{=} X \times_{X_F} X \times_{X_F} X$$

for the fiber product of three copies of $\mathrm{Fr}_{X/S}\colon X \to X^F$.

(ii) Let $I \subseteq \{1,2,3\}$ be a subset of $\{1,2,3\}$ of cardinality two. Then we shall write

$${}^{\mathrm{Fr}}\mathrm{pr}^T_I\colon {}^{\mathrm{Fr}}T \longrightarrow {}^{\mathrm{Fr}}P$$

for the projection onto the factors labeled by the elements of I, i.e., the morphism given by "$(x_1, x_2, x_3) \mapsto (x_{i_1}, x_{i_2})$", where $I = \{i_1, i_2\}$ and $i_1 < i_2$.

(iii) Let i be an element of $\{1,2,3\}$. Then we shall write

$${}^{\mathrm{Fr}}\mathrm{pr}^T_i\colon {}^{\mathrm{Fr}}T \longrightarrow X$$

for the projection onto the i-th factor.

(iv) We shall refer to a descent datum on ξ with respect to the morphism $\mathrm{Fr}_{X/S}\colon X \to X^F$ — i.e., an isomorphism in $\mathcal{F}|_{{}^{\mathrm{Fr}}P}$

$$\theta\colon {}^{\mathrm{Fr}}\mathrm{pr}_2^*\xi \xrightarrow{\sim} {}^{\mathrm{Fr}}\mathrm{pr}_1^*\xi$$

that satisfies the following "cocycle condition" as a *Frobenius-descent datum* on ξ: The composite

$$\begin{aligned}
& ({}^{\mathrm{Fr}}\mathrm{pr}_3^T)^*\xi = ({}^{\mathrm{Fr}}\mathrm{pr}_{\{2,3\}}^T)^{*\mathrm{Fr}}\mathrm{pr}_2^*\xi \xrightarrow[\sim]{({}^{\mathrm{Fr}}\mathrm{pr}_{\{2,3\}}^T)^*\theta} ({}^{\mathrm{Fr}}\mathrm{pr}_{\{2,3\}}^T)^{*\mathrm{Fr}}\mathrm{pr}_1^*\xi \\
= \quad & ({}^{\mathrm{Fr}}\mathrm{pr}_2^T)^*\xi = ({}^{\mathrm{Fr}}\mathrm{pr}_{\{1,2\}}^T)^{*\mathrm{Fr}}\mathrm{pr}_2^*\xi \xrightarrow[\sim]{({}^{\mathrm{Fr}}\mathrm{pr}_{\{1,2\}}^T)^*\theta} ({}^{\mathrm{Fr}}\mathrm{pr}_{\{1,2\}}^T)^{*\mathrm{Fr}}\mathrm{pr}_1^*\xi \\
= \quad & ({}^{\mathrm{Fr}}\mathrm{pr}_1^T)^*\xi = ({}^{\mathrm{Fr}}\mathrm{pr}_{\{1,3\}}^T)^{*\mathrm{Fr}}\mathrm{pr}_1^*\xi \xrightarrow[\sim]{({}^{\mathrm{Fr}}\mathrm{pr}_{\{1,3\}}^T)^*(\theta^{-1})} ({}^{\mathrm{Fr}}\mathrm{pr}_{\{1,3\}}^T)^{*\mathrm{Fr}}\mathrm{pr}_2^*\xi \\
= \quad & ({}^{\mathrm{Fr}}\mathrm{pr}_3^T)^*\xi
\end{aligned}$$

— where we write θ^{-1} for the inverse of the isomorphism θ — is the identity automorphism of $({}^{\mathrm{Fr}}\mathrm{pr}_3^T)^*\xi$. We shall write

$$\mathrm{FrDsc}(\xi)$$

for the set of Frobenius-descent data on ξ.

(v) We shall say that the category $\mathcal{F}$ fibered in groupoids over Sch_S is *Frobenius-descent effective* if the relative [p-th power] Frobenius morphism of every smooth scheme over S is an effective descent morphism, i.e., relative to the category $\mathcal{F}$ fibered in groupoids over Sch_S.

REMARK 3.2.1. — Since [one verifies easily that] the relative Frobenius morphism of a smooth scheme over S is a *finite flat universal homeomorphism*, one verifies immediately from elementary descent theory that both

- the category $\mathrm{LcFr} \to \mathrm{Sch}_S$ fibered in groupoids of [4], Definition 1.1, and
- the category $\mathrm{SmSch} \to \mathrm{Sch}_S$ fibered in groupoids of [4], Definition 1.7,

are *Frobenius-descent effective*.

PROPOSITION 3.3. — *The notion of a* **Frobenius-descent datum** *on ξ of Definition* 3.2, (iv)*, is the* **same** *as the notion of an* **Fr-stratification** *on ξ in the sense of Definition* 1.8 *and* [3]*, Definition* 4.6*:*

$$\mathrm{FrDsc}(\xi) = {}^{\mathrm{Fr}}\mathrm{Strt}(\xi)$$

[*cf. Definition* 1.8*;* [3]*, Definition* 4.6]*.*

PROOF. — This assertion follows immediately from Lemma 3.1, (ii), together with the various definitions involved [cf. also [3], Remark 4.6.2]. ☐

LEMMA 3.4. — *The following hold:*

(i) *The stratification structure* Fr *satisfies condition* (1) *of* [3], *Definition* 3.2.

(ii) *Suppose that X is* **of positive relative dimension** *over S. Then the stratification structure* Fr *is* **δ-strictly cocartesian of level $\geq p+1$** [*cf.* [3], *Definition* 3.1, (iv)] **but not δ-strictly cocartesian of level $\geq p$**.

PROOF. — First, we verify assertion (i). Let us first recall from Lemma 1.9 that Fr is *ι-quasi-nil-retraction-like*. Moreover, it follows from Lemma 1.10 and Lemma 2.7 that Fr is *δ-nil-retraction-like* [cf. [3], Definition 3.1, (iii)]. In particular, the stratification structure Fr satisfies condition (1) of [3], Definition 3.2, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us recall from the first paragraph of the present proof of Lemma 3.4 that Fr is *δ-nil-retraction-like*. Thus, it follows immediately from Lemma 2.8 that the stratification structure Fr is *δ-strictly cocartesian of level $\geq p+1$ but not δ-strictly cocartesian of level $\geq p$*, as desired. This completes the proof of assertion (ii), hence also of Lemma 3.4. ☐

LEMMA 3.5. — *The following hold:*

(i) *The stratification structure* Fr *satisfies condition* (1) *of* [3], *Definition* 3.6.

(ii) *The stratification structure* Fr *satisfies condition* (2) *of* [3], *Definition* 3.6.

(iii) *Suppose that $p \neq 2$ (respectively, $p = 2$). Then the surjective homomorphism of $\mathcal{O}_{T^1}$-modules from $({}^{\mathrm{Fr}}\iota_T^0)_*(\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \Omega^1_{X/S})$ [i.e., the conormal sheaf of the closed immersion ${}^{\mathrm{Fr}}\iota^T_{\times 2}\colon T^1 \hookrightarrow {}^{\mathrm{Fr}}P^{\times 2}$ — cf.* (i); [3], *Lemma* 3.5, (i), (iv)] *to the conormal sheaf of the square-nilpotent closed immersion $T^1 \hookrightarrow {}^{\mathrm{Fr}}\underline{P}^{\times 2}$* [*cf.* (ii)] *determined by the closed immersion ${}^{\mathrm{Fr}}\underline{\iota}^{\times 2}\colon {}^{\mathrm{Fr}}\underline{P}^{\times 2} \hookrightarrow {}^{\mathrm{Fr}}P^{\times 2}$* [*cf.* (i)] *determines an* **isomorphism** *of*

$$({}^{\mathrm{Fr}}\iota_T^0)_*\Omega^2_{X/S} \qquad \textit{(respectively, } ({}^{\mathrm{Fr}}\iota_T^0)_*(\mathbb{S}^2\Omega^1_{X/S})$$

— where we write $\mathbb{S}^2\Omega^1_{X/S}$ for the symmetric product of $\Omega^1_{X/S}$ of degree two) with the conormal sheaf of the square-nilpotent closed immersion $T^1 \hookrightarrow {}^{\mathrm{Fr}}\underline{P}^{\times 2}$.

PROOF. — Let us first recall from Lemma 3.1, (i), that ${}^{\mathrm{Fr}}P^n$ may be identified with ${}^{\mathrm{PD}}P^n$ whenever $n \leq p-1$. Thus, since $1 \leq p-1$, assertion (i) follows from the fact that PD satisfies condition (1) of [3], Definition 3.6 [cf. [4], Theorem 4.1]. Next, assertions (ii), (iii) follow immediately from Lemma 2.5, (ii), and [4], Lemma 3.7, (i). This completes the proof of Lemma 3.5. ☐

THEOREM 3.6. — *Let p be a prime number, S a scheme* **over a field of characteristic p**, *and X a scheme which is* **smooth** *and* **separated** *over S* **of relative dimension d**. *Then the following hold:*

(i) *The following two conditions are equivalent:*

(i-1) *Either $p \neq 2$ or $d = 0$.*

(i-2) *The Frobenius-stratification structure* Fr [*cf. Definition* 1.8] *is* **of standard type** [*cf.* [3], *Definition* 3.6].

(ii) *The following three conditions are equivalent:*

(ii-1) *Either* $p = 2$ *or* $d = 0$.

(ii-2) *The Frobenius-stratification structure* Fr *is* **integrable** [*cf.* [3], *Definition* 3.2].

(ii-3) *The Frobenius-stratification structure* Fr *is* **strictly integrable** [*cf.* [3], *Definition* 3.3].

PROOF. — Assertion (i) follows immediately from Lemma 3.5, (i), (ii), (iii). Next, we verify assertion (ii). Let us first verify the following assertion:

> Claim 3.6.A: The Frobenius-stratification structure Fr satisfies conditions (1), (3), and (4) of [3], Definition 3.2.

To this end, let us first recall from Lemma 3.4, (i), that Fr satisfies condition (1) of [3], Definition 3.2. Moreover, it follows from Lemma 1.9 that Fr satisfies condition (3) of [3], Definition 3.2. Finally, one verifies immediately from a similar argument to the argument applied in the fifth paragraph of the proof of [4], Theorem 4.1, together with Lemma 2.9, that Fr satisfies condition (4) of [3], Definition 3.2. This completes the proof of Claim 3.6.A.

Next, let us verify the following assertion:

> Claim 3.6.B: The Frobenius-stratification structure Fr satisfies conditions (2), (3) of [3], Definition 3.3.

To this end, let us first observe that it follows from Lemma 2.5, (ii), that Fr satisfies condition (2) of [3], Definition 3.3. Moreover, it follows immediately from Lemma 3.5, (iii), together with a similar argument to the argument applied in [3], Remark 3.6.2, that Fr satisfies condition (3) of [3], Definition 3.3. This completes the proof of Claim 3.6.B.

Now it follows from Claim 3.6.A and Claim 3.6.B, together with Lemma 3.4, (ii), that assertion (ii) holds. This completes the proof of assertion (ii), hence also of Theorem 3.6. □

REMARK 3.6.1. — Suppose that $p = 2$. Then, as discussed in Lemma 3.5, (iii), the conormal sheaf of the square-nilpotent closed immersion $T^1 \hookrightarrow {}^{\mathrm{Fr}}\underline{P}^{\times 2}$ [cf. Lemma 3.5, (ii)] is *isomorphic* to the $\mathcal{O}_{T^1}$-module $({}^{\mathrm{Fr}}\iota^0_T)_*(\mathbb{S}^2\Omega^1_{X/S})$. Now let us observe that [since $p = 2$] the $\mathcal{O}_X$-module $\mathbb{S}^2\Omega^1_{X/S}$ fits into the following exact sequence of $\mathcal{O}_{T^1}$-modules

$$0 \longrightarrow \mathrm{Fr}^*_{X/S}\Omega^1_{X^F/S} \longrightarrow \mathbb{S}^2\Omega^1_{X/S} \longrightarrow \Omega^2_{X/S} \longrightarrow 0.$$

Moreover, one verifies immediately that

- the quotient module $\Omega^2_{X/S}$ is *closely related* to the conormal sheaf of the square-nilpotent closed immersion ${}^{\mathrm{PD}}\underline{\iota}^T_{\times 2} \colon T^1 \hookrightarrow {}^{\mathrm{PD}}\underline{P}^{\times 2}$ [cf. [4], Theorem 4.1], i.e., *closely related* to the notion of the PD-*curvature* [cf. also [4], Remark 4.3.1, (i), (ii)], and

• the submodule $\mathrm{Fr}^*_{X/S}\Omega^1_{X^F/S}$ is *isomorphic* to the conormal sheaf of the square-nilpotent closed immersion ${}^{\pi}\underline{\iota}^{0}_{2|0}\colon X \hookrightarrow {}^{\pi}\underline{P}^{2|0}$, i.e., *closely related* to the notion of the (π, p)-*curvature* [cf. Remark 3.9.1, (i), (ii), below].

COROLLARY 3.7. — *Let S be a scheme* **over a field of characteristic two**, *X a scheme which is* **smooth** *and* **separated** *over S,*

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a **weakly integrable** [*cf.* [3], *Definition* 1.8] *category fibered in groupoids over* Sch_S, *and ξ an object of $\mathcal{F}$ over X. Then the natural map*

$$\mathrm{FrDsc}(\xi) \longrightarrow {}^{\mathrm{Fr}}\mathrm{IntCnn}(\xi)$$

[*cf. Definition* 1.8*; Definition* 3.2*,* (iv)*; Proposition* 3.3*;* [3]*, Definition* 4.7*,* (ii)] *is* **bijective**.

PROOF. — This assertion follows from Theorem 3.6, (ii), and [3], Theorem 5.6. □

The main result of the present paper is as follows.

THEOREM 3.8. — *Let p be a prime number, S a scheme* **over a field of characteristic p**, *and X a scheme which is* **smooth** *and* **separated** *over S. Then the morphism* [*cf.* [3]*, Definition* 2.8] *from the divided power stratification structure* [*cf.* [4]*, Definition* 2.5] *to the Frobenius-stratification structure* [*cf. Definition* 1.8]

$$\pi\colon \mathrm{PD} = \big(({}^{\mathrm{PD}}P^n)_{n\geq 0},\ ({}^{\mathrm{PD}}\iota^n\colon {}^{\mathrm{PD}}P^n \to {}^{\mathrm{PD}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathrm{PD}}\sigma^n\colon {}^{\mathrm{PD}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathrm{PD}}\delta^{n_1,n_2}\colon {}^{\mathrm{PD}}P^{n_1,n_2} \to {}^{\mathrm{PD}}P^{n_1+n_2})_{n_1,\,n_2\geq 0}\big)$$

$$\longrightarrow \mathrm{Fr} = \big(({}^{\mathrm{Fr}}P^n)_{n\geq 0},\ ({}^{\mathrm{Fr}}\iota^n\colon {}^{\mathrm{Fr}}P^n \to {}^{\mathrm{Fr}}P^{n+1})_{n\geq 0},$$

$$({}^{\mathrm{Fr}}\sigma^n\colon {}^{\mathrm{Fr}}P^n \to X^{(2)})_{n\geq 0},\ ({}^{\mathrm{Fr}}\delta^{n_1,n_2}\colon {}^{\mathrm{Fr}}P^{n_1,n_2} \to {}^{\mathrm{Fr}}P^{n_1+n_2})_{n_1,\,n_2\geq 0}\big)$$

[*cf. Proposition* 1.11] *is* **strictly p-integrable** [*cf.* [3]*, Definition* 3.9].

PROOF. — Let us first observe that it follows from Lemma 2.5, (ii), and [4], Lemma 3.5, (i), together with Lemma 1.12, that π is *nil-retraction-like* [cf. [3], Definition 3.7, (i)]. In particular, the morphism π satisfies condition (1) of [3], Definition 3.8.

Next, it follows from [4], Theorem 4.1 (respectively, Lemma 3.4, (i), (ii); Lemma 1.9) that condition (2) (respectively, (3); (4)) of [3], Definition 3.8, is satisfied.

Next, let us verify that π satisfies condition (5) of [3], Definition 3.8. Let $n \leq p$ be a positive integer. Let us first observe that it follows immediately from Lemma 2.5, (ii), and [4], Lemma 3.5, (i), together with Lemma 2.10, that the morphism $\underline{\pi}^{p|n}\colon {}^{\pi}\underline{P}^{p|n} \to {}^{\mathrm{Fr}}P^n$ is *nil-retraction-like* [cf. [3], Definition 1.2, (iv)], which thus implies [cf. [3], Remark 1.3.1] that the morphism $\underline{\pi}^{p|n}\colon {}^{\pi}\underline{P}^{p|n} \to {}^{\mathrm{Fr}}P^n$ satisfies conditions (1), (2) of [3], Definition 1.3. Now let us take the "$\overline{X}$" (respectively, "$\overline{Y}$") of [3], Definition 1.3, to be X (respectively, X) and the closed immersion "i_X" (respectively, "i_Y") of [3], Definition 1.3, to be the closed immersion ${}^{\pi}\underline{\iota}^{0}_{p|n}\colon X \hookrightarrow {}^{\pi}\underline{P}^{p|n}$ (respectively, ${}^{\mathrm{Fr}}\iota^{0}_{n}\colon X = {}^{\mathrm{Fr}}P^{0} \hookrightarrow {}^{\mathrm{Fr}}P^{n}$). Then it follows from Lemma 2.11 that condition (3) of [3], Definition 1.3, is satisfied. Moreover, condition (4) of [3], Definition 1.3, is immediate. This completes the proof of the assertion that π satisfies condition (5) of [3], Definition 3.8. In particular, the morphism π satisfies condition (1) of [3], Definition 3.9.

Next, it follows from immediately from Lemma 2.5, (ii), and [4], Lemma 3.5, (i), together with Lemma 2.10, that π is p-$\underline{\mathrm{pr}}$-*finite flat* [cf. [3], Definition 3.7, (ii)]. In particular, the morphism π satisfies condition (2) of [3], Definition 3.9.

Finally, it follows from Lemma 2.5, (ii), that Fr is pr-*finite flat* [cf. [3], Definition 3.1, (ii)]. In particular, condition (3) of [3], Definition 3.9, is satisfied. Thus, we conclude that the morphism π is *strictly p-integrable*. This completes the proof of Theorem 3.8. □

One main application of the second fundamental correspondence of [3], Definition 6.7, is as follows.

COROLLARY 3.9. — *Let p be a prime number, S a scheme* **over a field of characteristic p**, *X a scheme which is* **smooth** *and* **separated** *over S,*

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a **weakly integrable** *[cf. [3], Definition 1.8] category fibered in groupoids over Sch_S, and ξ an object of $\mathcal{F}$ over X. Then the map*

$$\mathrm{FrDsc}(\xi) \longrightarrow {}^{\pi,p}\mathrm{DrmCnn}(\xi)$$

[cf. Proposition 1.11; Definition 3.2, (iv); Proposition 3.3; [3], Definition 4.13, (iii); [3], Lemma 4.14] determined by the morphism $\pi\colon \mathrm{PD} \to \mathrm{Fr}$ [cf. Proposition 1.11] is **bijective**.

PROOF. — This assertion follows from Theorem 3.8 and [3], Theorem 6.6. □

REMARK 3.9.1. — Suppose that we are in the situation of Corollary 3.9.

(i) Suppose that one takes the "$\mathcal{F} \to \mathrm{Sch}_S$" of Corollary 3.9 to be the category $\mathrm{LcFr} \to \mathrm{Sch}_S$ fibered in groupoids of [4], Definition 1.1 [cf. also [4], Proposition 1.5], which thus implies that the object ξ corresponds to a *locally free $\mathcal{O}_X$-module $\mathcal{E}$*.

Let ∇ be a PD-*integrable* [cf. [3], Definition 4.7, (ii); [4], Definition 2.5] PD-connection [cf. [3], Definition 4.1, (iii); [4], Definition 2.5] on ξ. Then it follows from [4], Proposition 2.6, (ii), that the PD-connection ∇ corresponds to a *classical connection* on $\mathcal{E}$, i.e., a certain homomorphism of $(X \to S)^{-1}\mathcal{O}_S$-modules [cf. [4], Definition 1.6]

$$\mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}.$$

Moreover, it follows from [4], Corollary 4.3 [or the equivalence between (i) and (iii) of [2], Theorem 4.8 — cf. [4], Remark 4.3.1, (i)], that the PD-*integrable* PD-connection ∇ on ξ extends to a *uniquely determined* PD-*stratification* [cf. [3], Definition 4.6; [4], Definition 2.5] $\widetilde{\nabla}$ on ξ.

Now let us observe that one verifies easily from Lemma 2.10 that the closed immersion ${}^{\pi}\underline{\iota}^{0}_{p|0}\colon X \hookrightarrow {}^{\pi}\underline{P}^{p|0}$ is *square-nilpotent*. Moreover, it is well-known that the conormal sheaf of this square-nilpotent closed immersion ${}^{\pi}\underline{\iota}^{0}_{p|0}\colon X \hookrightarrow {}^{\pi}\underline{P}^{p|0}$ is *isomorphism* to the $\mathcal{O}_X$-module $\mathrm{Fr}^*_{X/S}\Omega^1_{X^F/S}$ [cf., e.g., [7], Proposition 1.6]. Thus, it follows from [4], Lemma 1.2, (ii), that the subgroup

$$\mathrm{Lift}_{{}^{\pi}\underline{\iota}^{0}_{p|0}}\big((\underline{\pi}^{p|0})^*\xi, (\underline{\pi}^{p|0})^*\xi; \mathrm{id}_\xi\big) \subseteq \mathrm{Aut}_{\mathcal{F}|_{{}^{\pi}\underline{P}^{p|0}}}\big((\underline{\pi}^{p|0})^*\xi\big)$$

may be naturally identified with the module

$$\Gamma\big(X, \mathrm{Fr}^*_{X/S}\Omega^1_{X^F/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\big).$$

In particular, we conclude that the (π,p)-*curvature* [cf. Proposition 1.11; [3], Definition 4.13, (i)] of the PD-stratification $\widetilde{\nabla}$ may be naturally identified with a global section of

$$\mathrm{Fr}^*_{X/S}\Omega^1_{X^F/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}).$$

Moreover, in this case, one verifies easily from a straightforward calculation [cf., e.g., [7], Proposition 1.7] that this global section of $\mathrm{Fr}^*_{X/S}\Omega^1_{X^F/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\xi)$ *coincides*, up to sign, with the *p-curvature* of the corresponding classical connection on $\mathcal{E}$

$$\mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

in the usual sense [cf., e.g., the discussion preceding [5], Theorem 5.1].

In particular, we conclude from Remark 3.2.1 that the *bijection* of Corollary 3.9 may be regarded as a *generalization* of the equivalence between

- quasi-coherent $\mathcal{O}_{X^F}$-modules and
- quasi-coherent $\mathcal{O}_X$-modules equipped with dormant connections [i.e., integrable connections of p-curvature zero]

by *Cartier* [cf., e.g., [5], Theorem 5.1] [i.e., in the case where the modules under consideration are locally free].

(ii) Suppose that one takes "$\mathcal{F} \to \mathrm{Sch}_S$" of Corollary 3.9 to be the category SmSch $\to$ Sch_S fibered in groupoids of [4], Definition 1.7 [cf. also [4], Prposition 1.11], which thus implies that the object ξ is a smooth morphism $Z \to X$ of schemes over S. Let ∇ be a quasi-p-PD-connection [cf. [3], Definition 4.1, (i); [4], Definition 2.5] on ξ. Then it follows immediately from a similar argument to the argument of (i), together with [4], Lemma 1.8, (ii), that the (π,p)-*curvature* of the quasi-p-PD-connection ∇ may be naturally identified with a global section of

$$\mathrm{Fr}^*_{X/S}\Omega^1_{X^F/S} \otimes_{\mathcal{O}_X} \xi_*\mathcal{T}_{Z/X}.$$

COROLLARY 3.10. — *Let p be a prime number, S a scheme* **over a field of characteristic p**, *X a scheme which is* **smooth** *and* **separated** *over S, and*

$$\mathcal{F} \longrightarrow \mathrm{Sch}_S$$

a **weakly integrable** *[cf. [3], Definition 1.8] and* **Frobenius-descent effective** *[cf. Definition 3.2, (v)] category fibered in groupoids over* Sch_S*. Write $X^F \to S$ for the base-change of the structure morphism $X \to S$ of X by the absolute [p-th power] Frobenius* *endomorphism of* S,

$$\mathrm{Fr}_{X/S} \colon X \longrightarrow X^F$$

for the relative [*p-th power*] *Frobenius morphism of* X/S, *and*

$$\mathcal{F}|_X^{\mathrm{Drm}}$$

for the groupoid defined as follows:

• *An object of the category* $\mathcal{F}|_X^{\mathrm{Drm}}$ *is a pair* (ξ, ∇) *consisting of an object* ξ *of* $\mathcal{F}|_X$ [*cf.* [3], *Definition* 1.6, (ii)] *and a* $(\boldsymbol{\pi}, \boldsymbol{p})$**-dormant** PD-*connection* ∇ *on* ξ [*cf. Proposition* 1.11*;* [3], *Definition* 4.1, (iii)*;* [3], *Definition* 4.13, (iii)*;* [4], *Definition* 2.5].

• *If* (ξ_1, ∇_1) *and* (ξ_2, ∇_2) *are objects of the category* $\mathcal{F}|_X^{\mathrm{Drm}}$, *then a morphism* $(\xi_1, \nabla_1) \to (\xi_2, \nabla_2)$ *in the category* $\mathcal{F}|_X^{\mathrm{Drm}}$ *is defined to be an isomorphism* $\xi_1 \xrightarrow{\sim} \xi_2$ *in* $\mathcal{F}|_X$ *that is* **PD-horizontal** [*cf.* [3], *Definition* 4.2*;* [4], *Definition* 2.5].

Then the functor

$$\mathcal{F}|_{X^F} \longrightarrow \mathcal{F}|_X^{\mathrm{Drm}}$$

[*cf. Corollary* 3.9] *determined by the morphism* $\mathrm{Fr}_{X/S} \colon X \to X^F$ *is an* **equivalence** *of categories.*

PROOF. — This assertion follows from Corollary 3.9. □

Corollary 3.9 and [4], Corollary 4.3, give an *alternative proof* [of a slight generalization] of [1], Theorem B.0.1, and [6], Lemma 3.5 [cf. also [1], Remark B.0.2], i.e., the following assertion.

COROLLARY 3.11. — *Let* p *be a prime number;* S *a scheme* **over a field of characteristic** $\boldsymbol{p}$*;* X, Z **smooth** *schemes over* S*;* $\xi \colon Z \to X$ *a* **smooth** *morphism over* S. *Write* $X^F \to S$ *for the base-change of the structure morphism* $X \to S$ *of* X *by the absolute* [*p-th power*] *Frobenius endomorphism of* S *and*

$$\mathrm{Fr}_{X/S} \colon X \longrightarrow X^F$$

for the relative [*p-th power*] *Frobenius morphism of* X/S. *Suppose that the* $\mathcal{O}_X$*-module* $\xi_* \mathcal{T}_{Z/X}$ *is* **zero**. *Then the following two conditions are equivalent:*

(1) *The image of the* **Kodaira-Spencer homomorphism** $\mathcal{T}_{X/S} \to \mathbb{R}^1 \xi_* \mathcal{T}_{Z/X}$ *associated to* Z/X *is* **zero**.

(2) *There exists a* **unique**, *up to isomorphism, smooth morphism* $Y \to X^F$ *of schemes over* S *that fits into the following* **cartesian** *diagram of schemes over* S*:*

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ {\scriptstyle \xi}\downarrow & & \downarrow \\ X & \xrightarrow[\mathrm{Fr}_{X/S}]{} & X^F. \end{array}$$

PROOF. — Let us first observe that, to verify Corollary 3.11, we may assume [cf. Remark 3.2.1] without loss of generality, by replacing S, X by affine open subschemes of S, X, respectively, that both S and X are *affine* [which thus implies that X is *separated* over S].

Next, let us observe that since X is *affine*, one verifies easily that condition (1) is *equivalent* to the condition that the *Kodaira-Spencer class* $\in H^1(Z, \xi^*\Omega^1_{X/S} \otimes_{\mathcal{O}_Z} \mathcal{T}_{Z/X})$ is *trivial*. Thus, since [we have assumed that] the $\mathcal{O}_X$-module $\xi_*\mathcal{T}_{Z/X}$ is *zero*, it follows from Proposition 3.12, (i), (ii), below that condition (1) is *equivalent* to the following condition:

(1′) The smooth morphism $\xi \colon Z \to X$ [that is an object over X of the category SmSch fibered in groupoids over Sch_S of [4], Definition 1.7] admits a *unique* PD-*connection*.

Next, let us observe that, again by our assumption that the $\mathcal{O}_X$-module $\xi_*\mathcal{T}_{Z/X}$ is *zero*, it follows from [4], Remark 4.3.1, (ii), that every PD-connection on $\xi \colon Z \to X$ is PD-*integrable*. Thus, it follows from [4], Prposition 1.11, and [4], Corollary 4.3, that every PD-connection on $\xi \colon Z \to X$ extends to a *uniquely determined* PD-*stratification* on $\xi \colon Z \to X$. On the other hand, again by our assumption that the $\mathcal{O}_X$-module $\xi_*\mathcal{T}_{Z/X}$ is *zero*, it follows from Remark 3.9.1, (ii), that every PD-stratification on $\xi \colon Z \to X$ is (π, p)-*dormant*. Thus, in summary, we conclude from Corollary 3.9 and [4], Prposition 1.11, that condition (1) is *equivalent* to the following condition:

(1″) The smooth morphism $\xi \colon Z \to X$ [that is an object over X of the category SmSch fibered in groupoids over Sch_S of [4], Definition 1.7] admits a *unique Frobenius-descent datum*.

On the other hand, the equivalence (1″) $\Leftrightarrow$ (2) follows from Remark 3.2.1. This completes the proof of Corollary 3.11. □

PROPOSITION 3.12. — *Let Z be a scheme over S and $\xi \colon Z \to X$ a* **smooth** *morphism over S, which thus determines an object over X of the category* SmSch *fibered in groupoids over* Sch_S *of* [4]*, Definition* 1.7*. Then the following hold:*

(i) *The following two conditions are equivalent:*

(1) *The object ξ of* SmSch *over X admits a* **PD-connection** [*cf.* [3]*, Definition* 4.1*,* (iii)*;* [4]*, Definition* 2.5]*, i.e., the set* ${}^{\mathrm{PD}}\mathrm{Cnn}^1(\xi)$ [*cf.* [3]*, Remark* 4.1.1*;* [3]*, Definition* 4.4*;* [4]*, Definition* 2.5] *is* **nonempty**.

(2) *The* **Kodaira-Spencer class** $\in H^1(Z, \xi^*\Omega^1_{X/S} \otimes_{\mathcal{O}_Z} \mathcal{T}_{Z/X})$ *associated to Z/X is* **trivial**.

(ii) *If the set* ${}^{\mathrm{PD}}\mathrm{Cnn}^1(\xi)$ *is* **nonempty***, then the set* ${}^{\mathrm{PD}}\mathrm{Cnn}^1(\xi)$ *has a* **natural structure of torsor** *under the module*

$$\Gamma(X, \Omega^1_{X/S} \otimes_{\mathcal{O}_Z} \xi_*\mathcal{T}_{Z/X}).$$

PROOF. — These assertions follow immediately from a straightforward calculation concerning [4], Lemma 1.8, (i), (ii), in a specific situation in which we take the "$(X, \underline{X}, \underline{X} \hookrightarrow$

$X, \xi_1, \xi_2, \phi)$" of [4], Lemma 1.8, to be

$$\big({}^{\mathrm{PD}}P^1,\ X = {}^{\mathrm{PD}}P^0,\ {}^{\mathrm{PD}}\iota^0,\ ({}^{\mathrm{PD}}\mathrm{pr}_2^1)^*Z \to {}^{\mathrm{PD}}P^1,\ ({}^{\mathrm{PD}}\mathrm{pr}_1^1)^*Z \to {}^{\mathrm{PD}}P^1,\ \mathrm{id}_Z\big)$$

[cf. [4], Lemma 2.7], which thus implies [cf. [2], Remark 4.2] that the "$\mathcal{O}_{\underline{X}}$-module $\mathcal{I}_X$" of [4], Lemma 1.8, is *isomorphic* to the $\mathcal{O}_X$-module $\Omega^1_{X/S}$ [cf. also the second paragraph of the proof of [6], Lemma 3.5]. □

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