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The semi-absolute anabelian geometry of geometrically pro-p arithmetic fundamental groups of associated low-dimensional configuration spaces

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THE SEMI-ABSOLUTE ANABELIAN GEOMETRY OF GEOMETRICALLY PRO-P ARITHMETIC FUNDAMENTAL GROUPS OF ASSOCIATED LOW-DIMENSIONAL CONFIGURATION SPACES

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Abstract. Let p be a prime number. In the present paper, we study geometrically pro-p arithmetic fundamental groups of low-dimensional configuration spaces associated to a given hyperbolic curve over an arithmetic field such as a number field or a p-adic local field. Our main results concern the group-theoretic reconstruction of the function field of certain tripods (i.e., copies of the projective line minus three points) that lie inside such a configuration space from the associated geometrically pro-p arithmetic fundamental group, equipped with the auxiliary data constituted by the collection of decomposition groups determined by the closed points of the associated compactified configuration space.

0. Introduction

Let $n \in \mathbb{Z}_{>1}$; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; p a prime number; k a number field or a p-adic local field; $X^{\log}$ a smooth log curve over k of type (g, r) (cf. Notation 1.3, (iv)). Write $\overline{\mathcal{M}}_{g,r}$ for the moduli stack (over k) of pointed stable curves of type (g, r) (with ordered marked points), and $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ for the open substack corresponding to the smooth curves (cf. Notation 1.3, (i)). In the present paper, we study the n-th log configuration space $X_n^{\log}$ associated to $X^{\log} \to \mathrm{Spec}(k)$ (cf. Definition 1.4). If $S^{\log}$ is a log scheme, then we shall write U_S for the interior of the log scheme $S^{\log}$ (cf. Notation 1.2, (vi)). The log scheme $X_n^{\log}$ may be thought of as a compactification of the usual n-th configuration space U_{X_n} associated to the smooth curve U_X. It is known that the function field of U_X may be reconstructed group-theoretically

- from its profinite arithmetic fundamental group whenever U_X is of strictly Belyi type (cf. [AbsTpIII], Theorem 1.9; [AbsTpIII], Corollary 1.10) or,
- from its geometrically pro-Σ arithmetic fundamental group, where Σ is a set of prime numbers of cardinality ≥ 2 that contains p, equipped with the auxiliary data constituted by the collection of decomposition groups associated to the closed points of U_X (cf. [AbsTpII], Corollary 2.9), regardless of whether or not U_X is of strictly Belyi type.

By contrast, in the present paper, we reconstruct the function field of certain tripods (i.e., copies of the projective line minus three points) that lie inside $X_n^{\log}$ group-theoretically from various geometrically pro-p arithmetic fundamental groups associated to U_{X_n}, equipped with the auxiliary data constituted by the collection of decomposition groups determined by the closed points of the underlying scheme X_n of $X_n^{\log}$.

Our main results are as follows:

Theorem 0.1. (Semi-absolute bi-anabelian formulation) *Let* $* \in \{\dagger, \ddagger\}$*;* ${}^*n \in \mathbb{Z}_{>1}$*;* $({}^*g, {}^*r)$ *a pair of nonnegative integers such that* $2({}^*g-1) + {}^*r > 0$*;* ${}^*\Box \in \{\mathrm{arb}, \mathrm{ord}\}$ *(cf. Notation 1.3, (iv));* $\Sigma_\Delta, \Sigma_{\mathrm{Gal}}$ *sets of prime numbers such that* $\Sigma_\Delta \subseteq \Sigma_{\mathrm{Gal}}$*, and* $\Sigma_\Delta, \Sigma_{\mathrm{Gal}}$ *are of cardinality* 1 *or equal to the set of prime numbers* $\mathfrak{Primes}$*;* $p \in \Sigma_\Delta$*;* ${}^*k$ *a generalized sub-p-adic local field (cf.* [Topics]*, Definition 4.11);* ${}^*\bar{k}$ *an algebraic closure of* ${}^*k$*;* ${}^*X^{\log}$ *a smooth log curve over* ${}^*k$ *of type* $({}^*g, {}^*r^{{}^*\Box})$ *(cf. Notation 1.3, (iv)). Write* ${}^*X^{\log}_{{}^*n}$ *for the* ${}^*n$*-th log configuration space associated to* ${}^*X^{\log} \to \mathrm{Spec}({}^*k)$ *(cf. Definition 1.4);* ${}^*K \subseteq {}^*\bar{k}$ *for the maximal pro-*Σ_{Gal} *subextension of* ${}^*\bar{k}/{}^*k$*;*

$$\Pi_{U_{{}^*X_{{}^*n}}} \stackrel{\mathrm{def}}{=} \begin{cases} \pi_1(U_{{}^*X_{{}^*n}})^{\Sigma_\Delta} & (\textit{if } \Sigma_\Delta = \Sigma_{\mathrm{Gal}}) \\ \pi_1(U_{{}^*X_{{}^*n}})^{[p]} & (\textit{if } \Sigma_\Delta \subsetneq \Sigma_{\mathrm{Gal}}), \end{cases}$$

where $\pi_1(U_{{}^*X_{{}^*n}})^{\Sigma_\Delta}$ *denotes the maximal pro-*Σ_Δ *quotient of* $\pi_1(U_{{}^*X_{{}^*n}})$*, and*

$$\pi_1(U_{{}^*X_{{}^*n}})^{[p]}$$

denotes the maximal geometrically pro-p quotient of $\pi_1(U_{{}^*X_{{}^*n}})$ *(cf. Notation 4.1);*

$$\Delta_{U_{{}^*X_{{}^*n}}} \stackrel{\mathrm{def}}{=} \pi_1(U_{{}^*X_{{}^*n}} \times_{{}^*k} {}^*\bar{k})^{\Sigma_\Delta}; \quad G^{\Sigma_{\mathrm{Gal}}}_{{}^*k} \stackrel{\mathrm{def}}{=} \mathrm{Gal}({}^*\bar{k}/{}^*k)^{\Sigma_{\mathrm{Gal}}};$$

$$\mathcal{D}_{{}^*X_{{}^*n}} \stackrel{\mathrm{def}}{=} \{D \subseteq \Pi_{U_{{}^*X_{{}^*n}}} \mid D \textit{ is a decomposition group associated to some } x \in {}^*X_{{}^*n}({}^*K)\}.$$

Suppose that the sequence

$$1 \longrightarrow \Delta_{U_{{}^*X_{{}^*n}}} \longrightarrow \Pi_{U_{{}^*X_{{}^*n}}} \longrightarrow G^{\Sigma_{\mathrm{Gal}}}_{{}^*k} \longrightarrow 1$$

is exact (cf. Notation 4.1; Remark 4.3), and that $({}^*X^{\log}, {}^*n)$ *is tripodally ample (cf. Definition 6.1). Thus,*

$$\mathscr{B}[{}^*X^{\log}_{{}^*n}] \stackrel{\mathrm{def}}{=} (\Pi_{U_{{}^*X_{{}^*n}}}, G^{\Sigma_{\mathrm{Gal}}}_{{}^*k}, \mathcal{D}_{{}^*X_{{}^*n}})$$

is a PGCS-collection of type $({}^*g, {}^*r^{{}^*\Box}, {}^*n, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ *(cf. Definition 4.2). Write*

$$\mathrm{Isom}(U_{{}^\dagger X_{{}^\dagger n}}, U_{{}^\ddagger X_{{}^\ddagger n}})$$

for the set of isomorphisms of schemes $U_{{}^\dagger X_{{}^\dagger n}} \xrightarrow{\sim} U_{{}^\ddagger X_{{}^\ddagger n}}$ *and*

$$\mathrm{Isom}^{\mathrm{Out}}(\mathscr{B}[{}^\dagger X^{\log}_{{}^\dagger n}], \mathscr{B}[{}^\ddagger X^{\log}_{{}^\ddagger n}])$$

for the set of equivalence classes of isomorphisms of PGCS-collections $\mathscr{B}[{}^\dagger X^{\log}_{{}^\dagger n}] \xrightarrow{\sim} \mathscr{B}[{}^\ddagger X^{\log}_{{}^\ddagger n}]$ *(cf. Definition 4.4) with respect to the equivalence relation given by composition with an inner automorphism arising from* $\Pi_{U_{{}^*X_{{}^*n}}}$*. Then the natural morphism*

$$\mathrm{Isom}(U_{{}^\dagger X_{{}^\dagger n}}, U_{{}^\ddagger X_{{}^\ddagger n}}) \to \mathrm{Isom}^{\mathrm{Out}}(\mathscr{B}[{}^\dagger X^{\log}_{{}^\dagger n}], \mathscr{B}[{}^\ddagger X^{\log}_{{}^\ddagger n}])$$

is bijective (cf. Theorem 6.4).

Theorem 0.2. (From PGCS-collections of type $(g, r^{\square}, n, \Sigma_{\Delta}, \Sigma_{\mathrm{Gal}})$ to certain function fields arising from tripods) *Let $n \in \mathbb{Z}_{>1}$; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; $\square \in \{\mathrm{arb}, \mathrm{ord}\}$; $\Sigma_{\Delta}, \Sigma_{\mathrm{Gal}}$ sets of prime numbers such that $\Sigma_{\Delta} \subseteq \Sigma_{\mathrm{Gal}}$, and $\Sigma_{\Delta}, \Sigma_{\mathrm{Gal}}$ are of cardinality 1 or equal to the set of prime numbers $\mathfrak{Primes}$. Let $\mathscr{B} = (\Pi_n, G, \mathcal{D}_n)$ be a PGCS-collection of type $(g, r^{\square}, n, \Sigma_{\Delta}, \Sigma_{\mathrm{Gal}})$ (cf. Definition 4.2). That is to say, Π_n is a profinite group; G is a quotient of Π_n; $\mathcal{D}_n$ is a set of subgroups of Π_n; there exist a prime number $p \in \Sigma_{\Delta}$, a generalized sub-p-adic local field k, an algebraic closure $\bar{k}$ of k, a smooth log curve $X^{\log}$ over k of type $(g, r^{\square})$, and an isomorphism*

$$\alpha \colon \Pi_n \xrightarrow{\sim} \Pi_{U_{X_n}} \overset{\mathrm{def}}{=} \begin{cases} \pi_1(U_{X_n})^{\Sigma_{\Delta}} & (\text{if } \Sigma_{\Delta} = \Sigma_{\mathrm{Gal}}) \\ \pi_1(U_{X_n})^{[p]} & (\text{if } \Sigma_{\Delta} \subsetneq \Sigma_{\mathrm{Gal}}) \end{cases}$$

such that, if we write $G_k \overset{\mathrm{def}}{=} \mathrm{Gal}(\bar{k}/k)$ and $K \subseteq \bar{k}$ for the maximal pro-Σ_{Gal} subextension of $\bar{k}/k$ (so $G_k^{\Sigma_{\mathrm{Gal}}} = \mathrm{Gal}(K/k)$), then the natural outer action $G_k \overset{\mathrm{out}}{\curvearrowright} \pi_1(U_{X_n} \times_k \bar{k})^{\Sigma_{\Delta}}$ (cf. Notation 4.1) factors through the natural surjection $G_k \twoheadrightarrow G_k^{\Sigma_{\mathrm{Gal}}}$, and α induces a commutative diagram

$$\begin{array}{ccc} \Pi_n & \xrightarrow[\alpha]{\sim} & \Pi_{U_{X_n}} \\ \downarrow & \circlearrowleft & \downarrow \\ G & \overset{\sim}{\underset{\alpha_G}{\dashrightarrow}} & G_k^{\Sigma_{\mathrm{Gal}}}, \end{array}$$

where the lower horizontal arrow α_G is an isomorphism, as well as a bijection

$$\mathcal{D}_n \xrightarrow{\sim} \mathcal{D}_{X_n} \overset{\mathrm{def}}{=} \{D \subseteq \Pi_{U_{X_n}} \mid D \text{ is a decomposition group associated to some } x \in X_n(K)\}.$$

Suppose that $(X^{\log}, n)$ is tripodally ample, and that k is a number field or a p-adic local field. Then:

(i) For any sufficiently small open normal subgroup H of G, one may construct a family (cf. the discussion of "choices" in the final portion of Remark 6.3) of a PGCS-collections $\{\mathscr{B}^{\mathrm{tpd}} = (\Pi_2^{\mathrm{tpd}}, H, \mathcal{D}_2^{\mathrm{tpd}})\}$ of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_{\Delta}, \Sigma_{\mathrm{Gal}})$ associated to the intrinsic structure of the PGCS-collection $\mathscr{B}$ (cf. Theorem 6.6, (i)).

(ii) Let $\beta_X \colon \mathscr{B} \xrightarrow{\sim} \mathscr{B}[X] \overset{\mathrm{def}}{=} (\Pi_{U_{X_n}}, G_k^{\Sigma_{\mathrm{Gal}}}, \mathcal{D}_{X_n})$ be an isomorphism of PGCS-collections and $\mathscr{B}^{\mathrm{tpd}} = (\Pi_2^{\mathrm{tpd}}, H, \mathcal{D}_2^{\mathrm{tpd}})$ a PGCS-collection of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_{\Delta}, \Sigma_{\mathrm{Gal}})$ associated to $\mathscr{B}$ (cf. (i)). Write $H[X] \overset{\mathrm{def}}{=} \mathrm{Ker}(G_k^{\Sigma_{\mathrm{Gal}}} \to G/H)$, where $G_k^{\Sigma_{\mathrm{Gal}}} \to G/H$ denotes the composite of the natural quotient $G \to G/H$ with the inverse of the isomorphism $(\beta_X)_G \colon G \xrightarrow{\sim} G_k^{\Sigma_{\mathrm{Gal}}}$ determined by β_X (cf. Definition 4.4). Let $Y^{\log}$ be a smooth log curve over k of type $(0, 3^{\mathrm{ord}})$; write

$$\Pi_{U_{Y_2}} \overset{\mathrm{def}}{=} \begin{cases} \pi_1(U_{Y_2})^{\Sigma_{\Delta}} & (\text{if } \Sigma_{\Delta} = \Sigma_{\mathrm{Gal}}) \\ \pi_1(U_{Y_2})^{[p]} & (\text{if } \Sigma_{\Delta} \subsetneq \Sigma_{\mathrm{Gal}}). \end{cases}$$

Then, for a suitable choice $\mathscr{B}^{\mathrm{tpd}}[X] = (\Pi_{U_{Y_2}}, H[X], \mathcal{D}_{Y_2})$ of PGCS-collection of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_{\Delta}, \Sigma_{\mathrm{Gal}})$ associated to $\mathscr{B}[X]$ (cf. (i); Remarks 6.2, 6.3),

β_X induces an isomorphism of PGCS-collections

$$\beta_Y^{\mathrm{tpd}} \colon \mathscr{B}^{\mathrm{tpd}} \xrightarrow{\sim} \mathscr{B}^{\mathrm{tpd}}[X] \overset{\mathrm{def}}{=} (\Pi_{U_{Y_2}}, H[X], \mathcal{D}_{Y_2})$$

(cf. Theorem 6.6, (ii)).

(iii) *One may construct a quotient group $\Pi_2^{\mathrm{tpd}} \twoheadrightarrow \Pi_{2\to 1}^{\mathrm{tpd}}[\mathscr{B}^{\mathrm{tpd}}]$ (cf. Definition 6.5) and a field $\mathrm{Frac}(R[\mathscr{B}^{\mathrm{tpd}}])$ (cf. Definition 6.5) equipped with an action by $\Pi_{2\to 1}^{\mathrm{tpd}}[\mathscr{B}^{\mathrm{tpd}}]$ associated to the intrinsic structure of the PGCS-collection $\mathscr{B}^{\mathrm{tpd}}$ (cf. Theorem 6.6, (iii)).*

(iv) *In the notation of (ii), (iii), write $\mathcal{E}_2[\mathscr{B}^{\mathrm{tpd}}] = \{E_1, \ldots, E_5\}$ for the set of generalized fiber subgroups $\subseteq \Pi_2^{\mathrm{tpd}}$ (cf. Definition 4.8, (ii));*

$$\Pi_{U_{Y_{2\to 1}}} \overset{\mathrm{def}}{=} \Pi_{U_{Y_2}} / \bigcap_{i=1}^{5} (\beta_Y^{\mathrm{tpd}})_{\Pi}(E_i),$$

where $(\beta_Y^{\mathrm{tpd}})_{\Pi} \colon \Pi_2^{\mathrm{tpd}} \xrightarrow{\sim} \Pi_{U_{Y_2}}$ denotes the isomorphism determined by β_Y^{tpd} (cf. Definition 4.4). Then the isomorphism $(\beta_Y^{\mathrm{tpd}})_{\Pi}$ induces a commutative diagram

$$\begin{array}{ccc} \Pi_2^{\mathrm{tpd}} & \xrightarrow[(\beta_Y^{\mathrm{tpd}})_{\Pi}]{\sim} & \Pi_{U_{Y_2}} \\ \downarrow & & \downarrow \\ \Pi_{2\to 1}^{\mathrm{tpd}}[\mathscr{B}^{\mathrm{tpd}}] & \xrightarrow{\sim} & \Pi_{U_{Y_{2\to 1}}}, \end{array}$$

where the vertical arrows are the natural projections, and $\Pi_{2\to 1}^{\mathrm{tpd}}[\mathscr{B}^{\mathrm{tpd}}] \xrightarrow{\sim} \Pi_{U_{Y_{2\to 1}}}$ denotes a uniquely determined isomorphism of profinite groups (cf. Theorem 6.6, (iv)).

(v) *In the notation of (iv), write $Z \to U_{Y_2}$ for the profinite étale covering corresponding to $(\Pi_{U_{Y_2}} \twoheadrightarrow)\ \Pi_{U_{Y_{2\to 1}}}$ and $\mathrm{Fnct}(Z)$ for the function field of Z. Then one may construct a field isomorphism*

$$\mathrm{Frac}(R[\mathscr{B}^{\mathrm{tpd}}]) \xrightarrow{\sim} \mathrm{Fnct}(Z)$$

associated to the intrinsic structure of the data $(\mathscr{B}^{\mathrm{tpd}}, \mathscr{B}^{\mathrm{tpd}}[X], \beta_Y^{\mathrm{tpd}} \colon \mathscr{B}^{\mathrm{tpd}} \xrightarrow{\sim} \mathscr{B}^{\mathrm{tpd}}[X])$, where the field isomorphism "$\xrightarrow{\sim}$" is equivariant with respect to the respective natural actions of the profinite groups $(\Pi_2^{\mathrm{tpd}} \twoheadrightarrow)\ \Pi_{2\to 1}^{\mathrm{tpd}}[\mathscr{B}]$, $(\Pi_{U_{Y_2}} \twoheadrightarrow)\ \Pi_{U_{Y_{2\to 1}}}$ (cf. the display of (iv); Theorem 6.6, (v)).

These main results are derived from the following results concerning tripods (i.e., the case where $(g, r^{\square}) = (0, 3^{\mathrm{ord}})$):

Theorem 0.3. (From PGCS-collections of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ to CFS-collections to base fields) *We maintain the following notation of Theorem 0.2: $(g, r^{\square}, n,\ \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$; $\mathscr{B} = (\Pi_n, G, \mathcal{D}_n)$; k; $\bar{k}$; $G_k^{\Sigma_{\mathrm{Gal}}}$; $X^{\log}$; K; $\alpha \colon \Pi_n \xrightarrow{\sim} \Pi_{U_{X_n}}$; $\alpha_G \colon G \xrightarrow{\sim} G_k^{\Sigma_{\mathrm{Gal}}}$. Suppose that $(g, r^{\square}, n) = (0, 3^{\mathrm{ord}}, 2)$. Let E be a generalized fiber subgroup of Π_2 (cf. Definition 4.8, (ii)). Such a $\mathscr{B}$ and E determine a collection of data*

$$\mathscr{A}[\mathscr{B}, E] \overset{\mathrm{def}}{=} (A[\mathscr{B}], B[\mathscr{B}, E], \partial B[\mathscr{B}, E], H[\mathscr{B}], M[\mathscr{B}, E])$$

(cf. Definition 4.8; Theorem 4.9, (ii)). Then:

(i) *Let $\mathscr{A} = (A, B, \partial B, H, M)$ be a CFS-collection (cf. Definition 3.2). That is to say, A, B are sets; $\partial B \subseteq B$ is a subset of cardinality 3; $H \subseteq \mathrm{Aut}(A)$ is a subgroup; M is a set of maps $A \to B$; there exist a field ${}^{\dagger}k$, a smooth log curve $Y^{\log}$ over ${}^{\dagger}k$ of type $(0, 3^{\mathrm{ord}})$, a bijection ${}^{\dagger}\alpha \colon A \overset{\sim}{\to} Y_2({}^{\dagger}k)$ (where Y_2 denotes the underlying scheme of the 2-nd log configuration space $Y_2^{\log}$), and a bijection ${}^{\dagger}\beta \colon B \overset{\sim}{\to} Y({}^{\dagger}k)$ such that*

(a) *${}^{\dagger}\beta$ induces a bijection $B \setminus \partial B \overset{\sim}{\to} U_Y({}^{\dagger}k)$;*

(b) *the isomorphism of groups $\mathrm{Aut}(A) \overset{\sim}{\to} \mathrm{Aut}(Y_2({}^{\dagger}k))$ determined by ${}^{\dagger}\alpha$ induces an isomorphism of groups $H \overset{\sim}{\to} \mathrm{Aut}_{{}^{\dagger}k}(U_{Y_2})$ $(\hookrightarrow \mathrm{Aut}(Y_2({}^{\dagger}k)))$;*

(c) *if we write M_Y for the set of maps $Y_2({}^{\dagger}k) \to Y({}^{\dagger}k)$ induced by the 30 natural morphisms $Y_2 \to Y$ (cf. Proposition 2.1; Definition 2.3, (ii); Proposition 2.6, (ii)), then there exists a bijection $M \overset{\sim}{\to} M_Y$ such that if $\lambda \mapsto q$ via this bijection, then*

$$\begin{array}{ccc} A & \xrightarrow[{}^{\dagger}\alpha]{\sim} & Y_2({}^{\dagger}k) \\ {\scriptstyle\lambda}\big\downarrow & \circlearrowright & \big\downarrow{\scriptstyle q} \\ B & \xrightarrow[{}^{\dagger}\beta]{\sim} & Y({}^{\dagger}k). \end{array}$$

Write S_5 for the symmetric group on 5 letters. Let $\phi \colon H \overset{\sim}{\to} S_5$ be an isomorphism. Such an isomorphism ϕ determines a subset $M_1[\phi] \subseteq M$ (cf. Definition 3.5). Let $\lambda \in M_1[\phi]$. Such an isomorphism ϕ and element $\lambda \in M_1[\phi]$ determine elements $0[\phi, \lambda]$, $1[\phi, \lambda]$, $\infty[\phi, \lambda] \in \partial B \subseteq B$ (cf. Definition 3.8). Then:

(1) *One may construct a field $F[\mathscr{A}, \phi, \lambda]$ associated to the intrinsic structure of the following collection of data: the CFS-collection $\mathscr{A}$, the isomorphism $\phi \colon H \overset{\sim}{\to} S_5$, and the element $\lambda \in M_1[\phi]$ (cf. Definition 3.12; Theorem 3.13, (i), (ii)).*

(2) *The bijection $B \overset{\sim}{\to} Y({}^{\dagger}k) \overset{\sim}{\to} {}^{\dagger}k \cup \{\infty\}$ given by the composite*

$$t_{{}^{\dagger}\beta(0[\phi,\lambda]), {}^{\dagger}\beta(1[\phi,\lambda]), {}^{\dagger}\beta(\infty[\phi,\lambda])} \circ {}^{\dagger}\beta$$

(cf. the notation of Proposition 2.8) determines a field isomorphism

$$F[\mathscr{A}, \phi, \lambda] \overset{\sim}{\to} {}^{\dagger}k$$

(cf. Theorem 3.13, (i), (ii)).

(ii) *The isomorphism $\alpha \colon \Pi_2 \overset{\sim}{\to} \Pi_{U_{X_2}}$ induces*

(a) *bijections (the latter two of which are compatible)*

$$A[\mathscr{B}] \overset{\sim}{\to} X_2(K), \quad B[\mathscr{B}, E] \overset{\sim}{\to} X(K), \quad \partial B[\mathscr{B}, E] \overset{\sim}{\to} X(K) \setminus U_X(K),$$

(b) *a group isomorphism $H[\mathscr{B}] \overset{\sim}{\to} \mathrm{Aut}_k(U_{X_2})$,*

(c) *a bijection*

$$M[\mathscr{B}, E] \overset{\sim}{\to} \{\text{the maps } X_2(K) \to X(K) \text{ induced by projection morphisms } U_{X_2} \twoheadrightarrow U_X\}$$

(cf. Theorem 4.9, (i)).

(iii) *The above collection of data $\mathscr{A}[\mathscr{B}, E]$ is a CFS-collection. In particular, one may construct a CFS-collection $\mathscr{A}[\mathscr{B}, E]$ associated to the intrinsic structure*

of the following collection of data: the PGCS-collection $\mathscr{B}$ *of type* $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ *and the generalized fiber subgroup* $E \subseteq \Pi_2$ *(cf. Theorem 4.9, (ii)).*

(iv) *Let* $\phi\colon H[\mathscr{B}] \xrightarrow{\sim} S_5$ *be an isomorphism and* $\lambda \in M[\mathscr{B}, E]_1[\phi] \subseteq M[\mathscr{B}, E]$ *(cf. (i), (iii)). Write* $\beta\colon B[\mathscr{B}, E] \xrightarrow{\sim} X(K)$ *for the second bijection of (ii), (a). Such an isomorphism* ϕ *and element* $\lambda \in M[\mathscr{B}, E]_1[\phi]$ *determine elements* $0[\phi, \lambda]$, $1[\phi, \lambda]$, $\infty[\phi, \lambda] \in \partial B[\mathscr{B}, E] \subseteq B[\mathscr{B}, E]$ *(cf. (i)). Then the bijection* $B[\mathscr{B}, E] \xrightarrow{\sim} X(K) \xrightarrow{\sim} K \cup \{\infty\}$ *given by the composite*

$$t_{\beta(0[\phi,\lambda]),\beta(1[\phi,\lambda]),\beta(\infty[\phi,\lambda])} \circ \beta$$

(cf. (ii), (a); Propositions 2.1, 2.8) determines a field isomorphism

$$F[\mathscr{A}[\mathscr{B}, E], \phi, \lambda] \xrightarrow{\sim} K$$

(cf. (i), (1), (2)) that is equivariant with respect to the respective natural actions of the profinite groups G, $G_k^{\Sigma_{\mathrm{Gal}}}$*, relative to the isomorphism* $\alpha_G\colon G \xrightarrow{\sim} G_k^{\Sigma_{\mathrm{Gal}}}$ *(cf. Definition 4.8, (iii); Theorem 4.9, (iii)).*

Theorem 0.4. (From PGCS-collections of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ **to function fields of tripods)** *We maintain the following notation of Theorem 0.2:* $(g, r^{\square}, n, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$; $\mathscr{B} = (\Pi_n, G, \mathcal{D}_n)$; $p \in \Sigma_\Delta$; k; $\bar{k}$; $G_k^{\Sigma_{\mathrm{Gal}}}$; $X^{\log}$; K; $\alpha\colon \Pi_n \xrightarrow{\sim} \Pi_{U_{X_n}}$; $\mathcal{D}_{X_n}$. *Let* Π_2^{prf} *be a profinite group which is isomorphic to the étale fundamental group* $\Pi_{U_{X_2}}^{\mathrm{prf}} \stackrel{\mathrm{def}}{=} \pi_1(U_{X_2})$ *(relative to a suitable choice of basepoint). Suppose that* $(g, r^{\square}, n) \stackrel{\mathrm{def}}{=} (0, 3^{\mathrm{ord}}, 2)$*. Then:*

(i) *Let* $E_{\mathscr{B}} \in \mathcal{E}_2[\mathscr{B}]$ *(cf. Definition 4.8, (ii)),* $\phi\colon H[\mathscr{B}] \xrightarrow{\sim} S_5$ *an isomorphism, and* $\lambda \in M[\mathscr{B}, E_{\mathscr{B}}]_1[\phi] \subseteq M[\mathscr{B}, E_{\mathscr{B}}]$*. Then one may construct from the PGCS-collection* $\mathscr{B}$ *a collection of isomorphisms between the fields* $F[\mathscr{A}[\mathscr{B}, E_{\mathscr{B}}], \phi, \lambda]$ *associated to any two choices of the data* $(E_{\mathscr{B}}, \phi, \lambda)$ *that is compatible with composition, i.e., satisfies the "cocycle condition" that arises when one considers three choices of the data* $(E_{\mathscr{B}}, \phi, \lambda)$*. In particular, one may construct*
   - *a field* $K[\mathscr{B}] \stackrel{\mathrm{def}}{=} F[\mathscr{A}[\mathscr{B}, E_{\mathscr{B}}], \phi, \lambda]$ *equipped with a natural action by* G *(cf. Theorem 0.3, (iv)),*
   - $k[\mathscr{B}] \stackrel{\mathrm{def}}{=} K[\mathscr{B}]^G$ *(cf. Notation 1.6)*

   associated to the intrinsic structure of the PGCS-collection $\mathscr{B}$*, i.e., which is independent of the choice of data* $(E_{\mathscr{B}}, \phi, \lambda)$ *(cf. Theorem 5.2, (i)).*

(ii) *Suppose that* k *is a number field or a* p*-adic local field. Then there exists an isomorphism of PGCS-collections* $\mathscr{B} \xrightarrow{\sim} \mathscr{B}[\Pi_2^{\mathrm{prf}}]$ *(cf. Theorem 5.1, (iv)). In particular, there exists an isomorphism*

$$\Pi_2 \xrightarrow{\sim} \Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}]$$

(cf. Theorem 5.1, (iv)). Let $E \in \mathcal{E}_2[\Pi_2^{\mathrm{prf}}]$ *(cf. Theorem 5.1, (v)) and* $\beta\colon \mathscr{B} \xrightarrow{\sim} \mathscr{B}[\Pi_2^{\mathrm{prf}}]$ *an isomorphism of PGCS-collections. Then the isomorphism* $\beta\colon \mathscr{B} \xrightarrow{\sim}$

$\mathscr{B}[\Pi_2^{\mathrm{prf}}]$ *induces a commutative diagram*

$$\begin{array}{ccccc}
\Pi_2^{\mathrm{prf}} & \twoheadrightarrow & \Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}] & \xleftarrow{\sim} & \Pi_2 \\
\downarrow & & & & \downarrow \\
\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] & & \twoheadrightarrow & & \Pi_1[\mathscr{B}, E|_{\Pi_2}] \\
\downarrow & & & & \downarrow \\
G[\Pi_2^{\mathrm{prf}}] & & \twoheadrightarrow & & G,
\end{array}$$

where $\Pi_2 \overset{\sim}{\to} \Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}]$ *denotes the isomorphism determined by* β*;* $\Pi_2^{\mathrm{prf}} \twoheadrightarrow \Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}]$ *denotes the natural surjection (cf. Theorem 5.1, (iv));* $E|_{\Pi_2} \subseteq \Pi_2$ *denotes the generalized fiber subgroup of* Π_2 *given by forming the image of* E *via the composite of arrows* $\Pi_2^{\mathrm{prf}} \twoheadrightarrow \Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}] \overset{\sim}{\leftarrow} \Pi_2$ *in the upper line of the diagram; the arrows* $\Pi_2^{\mathrm{prf}} \twoheadrightarrow \Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] \twoheadrightarrow G[\Pi_2^{\mathrm{prf}}]$ *denote the natural surjections (cf. Theorem 5.1, (i), (v)); the arrows* $\Pi_2 \twoheadrightarrow \Pi_1[\mathscr{B}, E|_{\Pi_2}] \twoheadrightarrow G$ *denote the natural surjections (cf. Definition 4.8, (i), (ii));* $\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] \twoheadrightarrow \Pi_1[\mathscr{B}, E|_{\Pi_2}]$*,* $G[\Pi_2^{\mathrm{prf}}] \twoheadrightarrow G$ *denote the unique surjections that render the diagram commutative. In particular, we obtain a field*

$$F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] \overset{\mathrm{def}}{=} F_1[\Pi_2^{\mathrm{prf}}, E]^{\mathrm{Ker}(\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] \twoheadrightarrow \Pi_1[\mathscr{B}, E|_{\Pi_2}])}$$

equipped with a natural action by $(\Pi_2 \twoheadrightarrow)\ \Pi_1[\mathscr{B}, E|_{\Pi_2}]$ *(cf. Theorems 5.1, (vi); 5.2, (ii)).*

(iii) In the notation of (ii), one may construct a field $F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta]$ *(cf. (ii)) equipped with an action by* Π_2 *associated to the intrinsic structure of the following collection of data:*
- *the PGCS-collection* $\mathscr{B}$*;*
- *a profinite group* Π_2^{prf} *isomorphic to* $\Pi_{U_{X_2}}^{\mathrm{prf}}$*;*
- $E \in \mathcal{E}_2[\Pi_2^{\mathrm{prf}}]$*;*
- *an isomorphism* $\beta\colon \mathscr{B} \overset{\sim}{\to} \mathscr{B}[\Pi_2^{\mathrm{prf}}]$*;*

such that if

$$\beta_X : \mathscr{B} \overset{\sim}{\to} \mathscr{B}[X] \overset{\mathrm{def}}{=} (\Pi_{U_{X_2}}, G_k^{\Sigma_{\mathrm{Gal}}}, \mathcal{D}_{X_2})$$

is an isomorphism of PGCS-collections of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$*, then one may construct a field isomorphism*

$$F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] \overset{\sim}{\to} \mathrm{Fnct}(W)$$

associated to the intrinsic structure of the data $(\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \beta_X)$*, where* W *denotes the pro-finite étale covering of* U_X *corresponding to* Π_{U_X} *(so* $\Pi_{U_X} = \mathrm{Gal}(W/U_X)$*);* $\mathrm{Fnct}(W)$ *denotes the function field of* W*; the isomorphism "*$\overset{\sim}{\to}$*" is equivariant with respect to the respective natural actions of the profinite groups* $(\Pi_2 \twoheadrightarrow)\ \Pi_1[\mathscr{B}, E|_{\Pi_2}]$*,* Π_{U_X} *(cf. Theorem 5.2, (iii)).*

(iv) In the notation of (i), (ii), (iii), suppose that $E_{\mathscr{B}} = E|_{\Pi_2}$*. Let* $\phi\colon H[\mathscr{B}] \overset{\sim}{\to} S_5$ *be an isomorphism,* $\lambda \in M[\mathscr{B}, E_{\mathscr{B}}]_1[\phi] \subseteq M[\mathscr{B}, E_{\mathscr{B}}]$*, and*

$$T \in F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta].$$

Then T induces, by restriction to decomposition groups (cf. also Proposition 4.7, (iv)), a map

$$T(-)\colon \mathcal{D}_1[\mathscr{B}, E_{\mathscr{B}}] \to K[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] \cup \{\infty\} \stackrel{\mathrm{def}}{=} \bar{k}[\Pi_2^{\mathrm{prf}}, E]^{\mathrm{Ker}(G[\Pi_2^{\mathrm{prf}}] \twoheadrightarrow G)} \cup \{\infty\}$$

(cf. (ii); Theorem 5.1, (vii)); there exists a unique element $T[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \phi, \lambda] \in F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta]^{\Pi_1[\mathscr{B}, E|_{\Pi_2}]}$ such that the zero divisor of $T[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \phi, \lambda]$ is of degree 1 (cf. [AbsTpIII], *Proposition 1.6, (iii)) and supported on $0[\phi, \lambda]$,*

$$T[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \phi, \lambda](1[\phi, \lambda]) = 1 \in K[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta],$$

the divisor of poles of $T[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \phi, \lambda]$ is of degree 1 (cf. [AbsTpIII], *Proposition 1.6, (iii)) and supported on $\infty[\phi, \lambda]$ (cf. Proposition 2.8). Moreover, the map*

$$T[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \phi, \lambda](-)\colon \mathcal{D}_1[\mathscr{B}, E_{\mathscr{B}}] \to K[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] \cup \{\infty\}$$

induces a field isomorphism

$$K[\mathscr{B}] \xrightarrow{\sim} K[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta],$$

where the isomorphism "$\xrightarrow{\sim}$" is equivariant with respect to the respective natural actions of G (cf. Theorem 5.2, (iv)).

(v) *In the notation of (i), (iii), (iv) (cf. also, Theorem 5.1, (vii)), the isomorphism $\beta_X\colon \mathscr{B} \xrightarrow{\sim} \mathscr{B}[X]$ induces a commutative diagram*

$$\begin{array}{ccccc} & & F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] & \xrightarrow{\sim} & \mathrm{Fnct}(W) \\ & & \cup & & \cup \\ K[\mathscr{B}] & \xrightarrow{\sim} & K[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] & \xrightarrow{\sim} & K \end{array}$$

associated to the intrinsic structure of the data $(\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \beta_X)$, where the horizontal arrows are the isomorphisms discussed so far in (iii), (iv), and Theorem 5.1, (vii); the $\cup$'s are the natural inclusions (cf. Theorem 5.2, (v)).

This paper is organized as follows: In §1, we explain some notations. In §2, we describe the field structure of a field k using the projections $\mathcal{M}_{0,5}(k) \to \mathcal{M}_{0,4}(k)$ (determined by forgetting a marked point), together with certain elements $\tau_{\mathrm{rf}}, \tau_{\mathrm{ra}}, \tau_{\mathrm{cr}} \in S_5$ (cf. Definition 2.9) of the symmetric group on 5 letters S_5, which we regard as acting on $\mathcal{M}_{0,5}$, by permuting the 5 marked points (cf. Proposition 2.2, (i)). In §3, we define the notion of a CFS-collection and construct a field associated to the intrinsic structure of a CFS-collection — i.e.,

$$\text{CFS-collection} \rightsquigarrow \text{field}$$

(cf. Theorem 0.3, (i)). In §4, we define the notion of a PGCS-collection and construct a CFS-collection (hence also a (base) field) associated to the intrinsic structure of a PGCS-collection — i.e.,

$$\text{PGCS-collection} \rightsquigarrow \text{CFS-collection} \rightsquigarrow \text{(base) field}$$

(cf. Theorem 0.3, (ii), (iii), (iv)). In §5, §6, we construct certain function fields associated to the intrinsic structure of a PGCS-collection — i.e.,

$$\text{PGCS-collection} \rightsquigarrow \text{certain function fields}$$

— first in the case of PGCS-collections of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_{\mathrm{Gal}}, \Sigma_{\Delta})$ (cf. Theorem 0.4, which is proven in §5), then in the case of PGCS-collections of type $(g, r^{\square}, n, \Sigma_{\mathrm{Gal}}, \Sigma_{\Delta})$ (cf. Theorems 0.1, 0.2, which are proven in §6).

1. Notations

Notation 1.1. Let S be a scheme and X a scheme over S, whose structure morphism $X \to S$ we denote by f.

(i) Write $\mathrm{Aut}(X)$ for the group of automorphisms of the scheme X.
(ii) Write $\mathrm{Aut}(X \to S) \subseteq \mathrm{Aut}(X) \times \mathrm{Aut}(S)$ for the subgroup of elements (α_X, α_S) such that $f \circ \alpha_X = \alpha_S \circ f$.
(iii) Write $\mathrm{Aut}_S(X) \subseteq \mathrm{Aut}(X \to S)$ for the subgroup of elements (α_X, α_S) such that α_S is the identity automorphism of S. When $S = \mathrm{Spec}(A)$, where A is a commutative ring with unity, we shall write $\mathrm{Aut}_A(X) \stackrel{\mathrm{def}}{=} \mathrm{Aut}_S(X)$.

Notation 1.2. Let $S^{\log}$ be an fs log scheme (cf. [Nky], Definition 1.7).

(i) Write S for the underlying scheme of $S^{\log}$.
(ii) Write $\mathcal{M}_S$ for the sheaf of monoids that defines the log structure of $S^{\log}$.
(iii) Let $\overline{s}$ be a geometric point of S. Then we shall denote by $I(\overline{s}, \mathcal{M}_S)$ the ideal of $\mathcal{O}_{S,\overline{s}}$ generated by the image of $\mathcal{M}_{S,\overline{s}} \setminus \mathcal{O}^{\times}_{S,\overline{s}}$ via the homomorphism of monoids $\mathcal{M}_{S,\overline{s}} \to \mathcal{O}_{S,\overline{s}}$ induced by the morphism $\mathcal{M}_S \to \mathcal{O}_S$ which defines the log structure of $S^{\log}$.
(iv) Let $s \in S$ and $\overline{s}$ a geometric point of S which lies over s. Write $(\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}})^{\mathrm{gp}}$ for the groupification of $\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}}$. Then we shall refer to the rank of the finitely generated free abelian group $(\mathcal{M}_{S,\overline{s}}/\mathcal{O}^{\times}_{S,\overline{s}})^{\mathrm{gp}}$ as the *log rank* at s. Note that one verifies easily that this rank is independent of the choice of $\overline{s}$, i.e., depends only on s.
(v) Let $m \in \mathbb{Z}$. Then we shall write

$$S^{\log \leq m} \stackrel{\mathrm{def}}{=} \{s \in S \mid \text{the log rank at } s \text{ is } \leq m\}.$$

Note that since $S^{\log \leq m}$ is open in S (cf. [MzTa], Proposition 5.2, (i)), we shall also regard (by abuse of notation) $S^{\log \leq m}$ as an open subscheme of S.
(vi) We shall write $U_S \stackrel{\mathrm{def}}{=} S^{\log \leq 0}$ and refer to U_S as the *interior* of $S^{\log}$. When $U_S = S$, we shall often use the notation S to denote the log scheme $S^{\log}$.

Notation 1.3. Let (g, r) be a pair of nonnegative integers such that $2g - 2 + r > 0$ and k a field.

(i) Write $\overline{\mathcal{M}}_{g,r}$ for the moduli stack (over k) of pointed stable curves of type (g, r), and $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ for the open substack corresponding to the smooth curves (cf. [Knu]). Here, we assume the marked points to be ordered.
(ii) Write

$$\overline{\mathcal{C}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$$

for the tautological curve over $\overline{\mathcal{M}}_{g,r}$; $\overline{\mathcal{D}}_{g,r} \stackrel{\mathrm{def}}{=} \overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r}$ for the divisor at infinity.
(iii) Write $\overline{\mathcal{M}}^{\log}_{g,r}$ for the log stack obtained by equipping the moduli stack $\overline{\mathcal{M}}_{g,r}$ with the log structure determined by the divisors with normal crossings $\overline{\mathcal{D}}_{g,r}$.

(iv) The divisor of $\overline{\mathcal{C}}_{g,r}$ given by the union of $\overline{\mathcal{C}}_{g,r} \times_{\overline{\mathcal{M}}_{g,r}} \overline{\mathcal{D}}_{g,r}$ with the divisor of $\overline{\mathcal{C}}_{g,r}$ determined by the marked points determines a log structure on $\overline{\mathcal{C}}_{g,r}$; we denote the resulting log stack by $\overline{\mathcal{C}}^{\log}_{g,r}$. Thus, we obtain a morphism of log stacks

$$\overline{\mathcal{C}}^{\log}_{g,r} \to \overline{\mathcal{M}}^{\log}_{g,r},$$

which we refer to as the *tautological log curve* over $\overline{\mathcal{M}}^{\log}_{g,r}$. If $S^{\log}$ is an arbitrary log scheme, then we shall refer to a morphism

$$C^{\log} \to S^{\log}$$

whose pull-back to some finite étale covering $T \to S$ is isomorphic to the pull-back of the tautological log curve via some morphism $T^{\log} \stackrel{\text{def}}{=} S^{\log} \times_S T \to \overline{\mathcal{M}}^{\log}_{g,r}$ as a *stable log curve* (of type (g,r)). If $C \to S$ is smooth, i.e., every geometric fiber of $C \to S$ is free of nodes, then we shall refer to $C^{\log} \to S^{\log}$ as a *smooth log curve* (of type (g,r)). If $C \to S$ is smooth, and the marked points of $X^{\log}$ are equipped with an ordering, then we shall refer to $C^{\log} \to S^{\log}$ as a *smooth log curve of type* (g, r^{ord}). When it is necessary to distinguish "g, r" from "g, r^{ord}", we shall occasionally write "g, r^{arb}" for "g, r".

Definition 1.4. Let k be a field; $\square \in \{\mathrm{arb}, \mathrm{ord}\}$; $S \stackrel{\text{def}}{=} \mathrm{Spec}(k)$; (g,r) a pair of nonnegative integers such that $2g-2+r>0$;

$$X^{\log} \to S$$

(cf. Notation 1.2, (vi)) a smooth log curve of type $(g, r^{\square})$; $n \in \mathbb{Z}_{>0}$. Suppose first that $\square = \mathrm{ord}$. Then the smooth log curve $X^{\log}$ over S determines a *classifying morphism* $S \to \overline{\mathcal{M}}^{\log}_{g,r}$. Thus, by pulling back via this morphism $S \to \overline{\mathcal{M}}^{\log}_{g,r}$ the morphism $\overline{\mathcal{M}}^{\log}_{g,r+n} \to \overline{\mathcal{M}}^{\log}_{g,r}$ given by forgetting the last n marked points, we obtain a morphism of log schemes

$$X^{\log}_n \to S.$$

Observe that since the above construction is manifestly functorial with respect to permutations of the marked points, we conclude, by an easy étale descent argument, that one may, in fact, define $X^{\log}_n$ even if $\square = \mathrm{arb}$. We shall refer to $X^{\log}_n$ as the *n-th log configuration space associated to* $X^{\log} \to S$. Note that $X^{\log}_1 = X^{\log}$. Write $X^{\log}_0 \stackrel{\text{def}}{=} S$.

Definition 1.5. Let $n \in \mathbb{Z}_{>0}$; $\square \in \{\mathrm{arb}, \mathrm{ord}\}$; (g,r) a pair of nonnegative integers such that $2g-2+r>0$; Σ a nonempty set of prime numbers; k a field of characteristic $\notin \Sigma$; $X^{\log}$ a smooth log curve over k of type $(g, r^{\square})$; P a point of X_n; $\overline{P}$ a geometric point of X_n which lies over P.

(i) $\overline{P}$ parametrizes a pointed stable curve of type $(g, r+n)$ over some separably closed field (cf. Notation 1.3, (iv)). Thus, $\overline{P}$ determines a semi-graph of anabelioids of pro-Σ PSC-type (cf. [CmbGC], Definition 1.1, (i)), which is in fact easily verified to be independent, up to (a non-unique!) isomorphism, of the choice of the geometric point $\overline{P}$ lying over P. We shall write $\mathcal{G}_{\overline{P}}$ for this semi-graph of anabelioids of pro-Σ PSC-type.

(ii) Suppose that $\square = \mathrm{ord}$. Let us fix an ordered set

$$\mathcal{C}_{r,n} \stackrel{\text{def}}{=} \{c_1, \ldots, c_{r+n}\}.$$

Thus, by definition, we have a natural bijection $\mathcal{C}_{r,n} \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}_{\overline{P}})$ that determines a bijection between the subset $\{c_1, \ldots, c_r\}$ and the set of cusps of $X^{\log}$ (cf. [Hgsh], Definition 2.2, (v)). In the following, let us identify the set $\mathrm{Cusp}(\mathcal{G}_{\overline{P}})$ with $\mathcal{C}_{r,n}$.

(iii) We shall refer to an irreducible divisor of X_n contained in the complement $X_n \setminus U_{X_n}$ of the interior U_{X_n} of X_n as a *log divisor* of $X_n^{\log}$. That is to say, a log divisor of $X_n^{\log}$ is an irreducible divisor of X_n whose generic point parametrizes a pointed stable curve with precisely two irreducible components (cf. [Hgsh], Definition 2.2, (vi)).

(iv) Let V be a log divisor of $X_n^{\log}$. Then we shall write $\mathcal{G}_{\overline{V}}$ for "$\mathcal{G}_{\overline{P}}$" in the case where we take "P" to be the generic point of V, and $\overline{V}$ to be a geometric point that lies over the generic point of V.

(v) Suppose that $\square = \mathrm{ord}$. Let $m \in \mathbb{Z}_{>1}$; $y_1, \ldots, y_m \in \mathcal{C}_{r,n}$ distinct elements such that $\sharp(\{y_1, \ldots, y_m\} \cap \{c_1, \ldots, c_r\}) \leq 1$. Then one verifies immediately — by considering *clutching morphisms* (cf. [Knu], Definition 3.8) — that there exists a unique log divisor V of $X_n^{\log}$, which we shall denote by $V(y_1, \ldots, y_m)$, that satisfies the following condition: the semi-graph of anabelioids $\mathcal{G}_{\overline{V}}$ (for some geometric point $\overline{V}$ that lies over V) has precisely two vertices v_1, v_2 such that v_1 is of type $(0, m+1)$, v_2 is of type $(g, n+r-m+1)$, and $y_1, \ldots, y_m$ are cusps of $\mathcal{G}_{\overline{V}}|_{v_1}$ (cf. [CbTpI], Definition 2.1, (iii)).

Notation 1.6. Let K be a field and G a group that acts on K. Then we write K^G for the subfield of G-invariants of K.

Notation 1.7. Write $\mathfrak{Primes}$ for the set of prime numbers. Let G be a profinite group and $\Sigma \subseteq \mathfrak{Primes}$. Then we shall write G^{Σ} for the maximal pro-Σ quotient of G.

Notation 1.8. Let G be a profinite group and H a closed normal subgroup of G. Then we shall write $\mathrm{Aut}(G)$ for the group of automorphisms of G, $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the subgroup of inner automorphisms of G arising from elements of G, $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$,

$$\mathrm{Aut}_{G/H}(G) \stackrel{\mathrm{def}}{=} \{\sigma \in \mathrm{Aut}(G) \mid \sigma(H) = H, \text{ and } \sigma \text{ lies over} \\ \text{the identity automorphism of } G/H\},$$

and $\mathrm{Inn}_H(G) \subseteq \mathrm{Aut}_{G/H}(G)$ for the subgroup of inner automorphisms of G arising from elements of H. Note that it follows immediately from the various definitions involved that $\mathrm{Inn}_H(G)$ is a normal subgroup of $\mathrm{Aut}_{G/H}(G)$. Write $\mathrm{Out}_{G/H}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}_{G/H}(G)/\mathrm{Inn}_H(G)$.

Notation 1.9. Let G be a profinite group and H a closed normal subgroup of G such that H is center-free. Then the conjugation action of G on H induces a natural outer action $G/H \stackrel{\mathrm{out}}{\curvearrowright} H$ of G/H on H. Since H is center-free, this outer action $G/H \stackrel{\mathrm{out}}{\curvearrowright} H$, in turn, induces a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H & \longrightarrow & \mathrm{Aut}(H) & \longrightarrow & \mathrm{Out}(H) & \longrightarrow & 1 \end{array}$$

in which the rows are exact, hence also a natural isomorphism

$$G \xrightarrow{\sim} \mathrm{Aut}(H) \times_{\mathrm{Out}(H)} G/H.$$

In particular, one may reconstruct the group G from the natural outer action $G/H \overset{\mathrm{out}}{\curvearrowright} H$.

Notation 1.10. Let G be a profinite group and H a subgroup of G. Then we shall write

$$C_G(H) \overset{\mathrm{def}}{=} \{g \in G \mid (gHg^{-1}) \cap H \text{ has finite index in } H,\ gHg^{-1}\}$$

for the commensurator of H in G.

Notation 1.11. Let ${}^\dagger\Pi, {}^\ddagger\Pi, G$ be profinite groups, ${}^\dagger\epsilon\colon {}^\dagger\Pi \twoheadrightarrow G$, ${}^\ddagger\epsilon\colon {}^\ddagger\Pi \twoheadrightarrow G$ sujections. Then we shall write ${}^\dagger\Delta \overset{\mathrm{def}}{=} \mathrm{Ker}({}^\dagger\epsilon)$, ${}^\ddagger\Delta \overset{\mathrm{def}}{=} \mathrm{Ker}({}^\ddagger\epsilon)$,

$$\mathrm{Isom}({}^\dagger\Pi, {}^\ddagger\Pi) \overset{\mathrm{def}}{=} \{\sigma\colon {}^\dagger\Pi \xrightarrow{\sim} {}^\dagger\Pi\colon \text{isomorphism}\},$$

$$\mathrm{Isom}_G({}^\dagger\Pi, {}^\ddagger\Pi) \overset{\mathrm{def}}{=} \{\sigma \in \mathrm{Isom}({}^\dagger\Pi, {}^\ddagger\Pi) \mid \sigma({}^\dagger\Delta) = {}^\ddagger\Delta \text{ and } \sigma \text{ lies over the identity automorphism of } G\},$$

and

$$\mathrm{Isom}_G^{\mathrm{Out}}({}^\dagger\Pi, {}^\ddagger\Pi)$$

for the set of equivalence classes of $\sigma \in \mathrm{Isom}_G({}^\dagger\Pi, {}^\ddagger\Pi)$ with respect to the equivalence relation given by composition with an inner automorphism arising from ${}^\ddagger\Delta$.

Notation 1.12. Let E_1, E_2 be sets. Then we shall write

$$\mathrm{Maps}(E_1, E_2)$$

for the set of maps $E_1 \to E_2$. Let G be a topological group and

$$Q \overset{\mathrm{def}}{=} \{p_i; E_1 \twoheadrightarrow Q_i\}_{i \in I}$$

a collection of quotients of E_1 indexed by a nonempty set I. Suppose further that each of the sets E_1 and E_2 is equipped with a topology and a continuous action by G, and that the topology and continuous action of G on E_1 induce a topology and continuous action of G on each of the quotients Q_i, for $i \in I$. For $i \in I$, we shall refer to a subset $F \subseteq Q_i$ of Q_i as *G-cofinite* if, for some open subgroup $H \subseteq G$, the subset $F \subseteq Q_i$ is stabilized by H, and, moreover, the set F/H of H-orbits of F is finite. We shall say that a subset $F \subseteq E_1$ is *pre-(G,Q)-cofinite* if, for some $i \in I$, the image $p_i(F)$ of F in Q_i is G-cofinite. We shall say that a subset $F \subseteq E_1$ is *(G,Q)-cofinite* if it is a finite union of pre-(G,Q)-cofinite subsets of E_1. Let us assume that E_1 is *not (G,Q)-cofinite*. Observe that if ${}^\dagger F \subseteq {}^\ddagger F \subseteq E_1$ are (G,Q)-cofinite subsets, then the inclusion $E_1 \setminus {}^\ddagger F \subseteq E_1 \setminus {}^\dagger F$ induces a natural map

$$\mathrm{Maps}(E_1 \setminus {}^\dagger F, E_2) \to \mathrm{Maps}(E_1 \setminus {}^\ddagger F, E_2).$$

We shall write

$$\mathrm{RatMaps}(E_1, E_2) \overset{\mathrm{def}}{=} \varinjlim_{F \subseteq E_1} \mathrm{Maps}(E_1 \setminus F, E_2),$$

where F ranges over the (G,Q)-cofinite subsets of E_1. Observe that, if $F \subseteq E_1$ is a (G,Q)-cofinite subset, then any $\sigma \in G$ induces a natural bijection

$$\mathrm{Maps}(E_1 \setminus F, E_2) \xrightarrow{\sim} \mathrm{Maps}(E_1 \setminus \sigma^{-1}(F), E_2)$$

given by taking, for $e_1 \in E_1 \setminus F$, $(f^\sigma)(e_1) \stackrel{\text{def}}{=} (f(\sigma^{-1}(e_1)))^\sigma$. These natural bijections induce natural actions of G on $\mathrm{Maps}(E_1, E_2)$ and $\mathrm{RatMaps}(E_1, E_2)$.

2. **Geometric description of the structure of a field**

Let $n \in \mathbb{Z}_{>1}$. Write S_n for the symmetric group on n letters. In the present §2, we describe the field structure of a field k using the projections $\mathcal{M}_{0,5}(k) \to \mathcal{M}_{0,4}(k)$, together with certain elements $\tau_{\mathrm{rf}}, \tau_{\mathrm{ra}}, \tau_{\mathrm{cr}} \in S_5$ (cf. Definition 2.9 below).

Proposition 2.1. *Let $n \in \mathbb{Z}_{>0}$, k a field, and $X^{\log}$ a smooth log curve over k of type $(0, 3^{\mathrm{ord}})$. Then there exist natural isomorphisms*

$$U_{X_n} \xrightarrow{\sim} \mathcal{M}_{0,3+n}, \quad X_n^{\log} \xrightarrow{\sim} \overline{\mathcal{M}}_{0,3+n}^{\log}$$

arising from the well-known modular interpretation of the moduli stacks in the codomains of these isomorphisms.

Proof. This follows immediately from the definitions. □

Proposition 2.2. *Let k be a field. Then the following hold:*

(i) The homomorphism

$$S_5 \to \mathrm{Aut}_k(\mathcal{M}_{0,5})$$

obtained by considering the permutations of the labels ($\in \{1, 2, 3, 4, 5\}$) on the five marked points is an isomorphism. Let us identify S_5 with $\mathrm{Aut}_k(\mathcal{M}_{0,5})$ by means of this isomorphism. Thus, S_5 acts on $\overline{\mathcal{M}}_{0,5}$ and $\overline{\mathcal{M}}_{0,5}(k)$.

(ii) The homomorphism

$$S_3 \to \mathrm{Aut}_k(\mathcal{M}_{0,4})$$

obtained by considering the permutations of the labels ($\in \{1, 2, 3\}$) on the first three marked points is an isomorphism. Let us identify S_3 with $\mathrm{Aut}_k(\mathcal{M}_{0,4})$ by means of this isomorphism. Thus, S_3 acts on $\overline{\mathcal{M}}_{0,4}$ and $\overline{\mathcal{M}}_{0,4}(k)$.

(iii) By considering the permutations of the labels ($\in \{1, 2, 3, 4\}$) on the four marked points, we obtain a homomorphism

$$S_4 \to \mathrm{Aut}_k(\mathcal{M}_{0,4}).$$

Let $a, b, c, d \in \{1, 2, 3, 4\}$ be distinct elements such that $a, b \in \{1, 2, 3\}$. Then the action of the transposition $(a, b) \in S_4$ on $\overline{\mathcal{M}}_{0,4}(k)$, the action of the transposition $(a, b) \in S_3$ on $\overline{\mathcal{M}}_{0,4}(k)$, and the action of the transposition $(c, d) \in S_4$ on $\overline{\mathcal{M}}_{0,4}(k)$ coincide.

Proof. Assertions (i), (ii) follow immediately from [NaTa], Theorem D (cf. also [NaTa], Theorem 4.4; [Nkm], Theorem A). Assertion (iii) follows immediately from the definitions. □

Definition 2.3. Let k be a field.

(i) Let $i \in \{1, 2, 3, 4, 5\}$. Write $p_i^{\mathrm{tpd}} \colon \overline{\mathcal{M}}_{0,5}(k) \twoheadrightarrow \overline{\mathcal{M}}_{0,4}(k)$ for the projection given by forgetting the i-th marked point $\mathcal{M}_{0,5} \to \mathcal{M}_{0,4}$.

(ii) By considering the composites of the projections of (i) with the automorphisms arising from the action of S_3 on $\overline{\mathcal{M}}_{0,4}(k)$, we obtain a set of surjective maps $\overline{\mathcal{M}}_{0,5}(k) \twoheadrightarrow \overline{\mathcal{M}}_{0,4}(k)$. We shall write $M \stackrel{\text{def}}{=} \{\overline{\mathcal{M}}_{0,5}(k) \twoheadrightarrow \overline{\mathcal{M}}_{0,4}(k)\}$ for this set of morphisms. Note that the action of S_5 on $\overline{\mathcal{M}}_{0,5}(k)$ induces an action of S_5

on M from the right; the action of S_3 on $\overline{\mathcal{M}}_{0,4}(k)$ induces a *free* action of S_3 on M from the left.

(iii) We define an equivalence relation on M as follows: For ${}^{\dagger}q, {}^{\ddagger}q \in M$,

$$ {}^{\dagger}q \sim {}^{\ddagger}q \overset{\text{def}}{\Longleftrightarrow} \{{}^{\dagger}q^{-1}(z)\}_{z\in\overline{\mathcal{M}}_{0,4}(k)\setminus\mathcal{M}_{0,4}(k)} = \{{}^{\ddagger}q^{-1}(z)\}_{z\in\overline{\mathcal{M}}_{0,4}(k)\setminus\mathcal{M}_{0,4}(k)}. $$

Note that the action of S_5 on M (cf. (ii)) induces an action of S_5 on the set $M_{\sim}$ of equivalence classes with respect to this equivalence relation, while the action of S_3 on M induces the trivial action of S_3 on $M_{\sim}$.

(iv) Let $i \in \{1,2,3,4,5\}$. Then we shall write $M_i \in M_{\sim}$ for the equivalence class (cf. (iii)) that contains p_i^{tpd}.

Proposition 2.4. *Let k be a field. Then the following hold:*

(i) It holds that

$$ \overline{\mathcal{M}}_{0,4} \setminus \mathcal{M}_{0,4} = \bigsqcup_{V:\ a\ log\ divisor\ of\ \overline{\mathcal{M}}_{0,4}^{\log}} V $$

(cf. Definition 1.5, (iii), and Proposition 2.1).

(ii) It holds that

$$ \{\text{log divisors of } \overline{\mathcal{M}}_{0,4}^{\log}\} = \{V(c_1,c_4), V(c_2,c_4), V(c_3,c_4)\} $$

where $c_1, c_2, c_3, c_4 \in \mathcal{C}_{3,1} = \{c_1,c_2,c_3,c_4\}$ (cf. Definition 1.5, (ii), (v), and Proposition 2.1).

(iii) It holds that

$$ \sharp(\overline{\mathcal{M}}_{0,4}(k) \setminus \mathcal{M}_{0,4}(k)) = 3. $$

(iv) Let $z \in \overline{\mathcal{M}}_{0,4}(k) \setminus \mathcal{M}_{0,4}(k)$ be an element. Then there exists a unique log divisor V of $\overline{\mathcal{M}}_{0,4}^{\log}$ such that $\{z\} = V(k) \subseteq \overline{\mathcal{M}}_{0,4}(k)$.

We shall regard, by a slight abuse of notation, log divisors of $\overline{\mathcal{M}}_{0,4}^{\log}$ as elements of $\overline{\mathcal{M}}_{0,4}(k) \setminus \mathcal{M}_{0,4}(k)$ (cf. (iv)) and write $0 \overset{\text{def}}{=} V(c_1,c_4)$, $1 \overset{\text{def}}{=} V(c_2,c_4)$, $\infty \overset{\text{def}}{=} V(c_3,c_4) \in \overline{\mathcal{M}}_{0,4}(k) \setminus \mathcal{M}_{0,4}(k)$.

Proof. Assertion (i) follows from Definition 1.5, (iii), and Proposition 2.1. Assertion (ii) follows immediately (cf. Definition 1.5, (v)). Assertion (iii) follows from Proposition 2.1. Assertion (iv) follows from assertions (i), (ii), (iii). □

Proposition 2.5. *Let k be a field. Then the following hold:*

(i) It holds that

$$ \overline{\mathcal{M}}_{0,5} \setminus \mathcal{M}_{0,5} = \bigcup_{V:\ a\ log\ divisor\ of\ \overline{\mathcal{M}}_{0,5}^{\log}} V $$

(cf. Definition 1.5, (iii), and Proposition 2.1).

(ii) It holds that

$$ \begin{aligned} &\sharp\{\text{log divisors of } \overline{\mathcal{M}}_{0,5}^{\log}\} \\ =&\sharp\{V(c_4,c_5),\ V(c_i,c_4,c_5),\ V(c_i,c_j) \mid i \in \{1,2,3\},\ j \in \{4,5\}\} = 10. \end{aligned} $$

(iii) It holds that

$$(p_5^{\mathrm{tpd}})^{-1}(0) = V(c_1, c_4) \cup V(c_1, c_4, c_5), \quad (p_5^{\mathrm{tpd}})^{-1}(1) = V(c_2, c_4) \cup V(c_2, c_4, c_5),$$
$$(p_5^{\mathrm{tpd}})^{-1}(\infty) = V(c_3, c_4) \cup V(c_3, c_4, c_5),$$

where $c_1, c_2, c_3, c_4, c_5 \in \mathcal{C}_{3,2} = \{c_1, \ldots, c_5\}$ *(cf. Definition 1.5, (ii), (v); Proposition 2.1). In particular,*

$$p_5^{\mathrm{tpd}}(V(c_i, c_4)) = p_5^{\mathrm{tpd}}(V(c_i, c_4, c_5)) = V(c_i, c_4),$$

where $i \in \{1, 2, 3\}$*.*

(iv) Let $i \in \{1, 2, 3, 4, 5\}$*.*

- *If* $i \in \{1, 2, 3\}$*, write* $\{i', i''\} = \{1, 2, 3\} \setminus \{i\}$*; then*

$$\begin{aligned}&\{(p_i^{\mathrm{tpd}})^{-1}(z)\}_{z \in \overline{\mathcal{M}}_{0,4}(k) \setminus \mathcal{M}_{0,4}(k)} \\ =&\{V(c_{i'}, c_4) \cup V(c_{i''}, c_5), V(c_{i''}, c_4) \cup V(c_{i'}, c_5), V(c_i, c_4, c_5) \cup V(c_4, c_5)\}.\end{aligned}$$

- *If* $i \in \{4, 5\}$*, write* $\{i'''\} = \{4, 5\} \setminus \{i\}$*; then*

$$\{(p_i^{\mathrm{tpd}})^{-1}(z)\}_{z \in \overline{\mathcal{M}}_{0,4}(k) \setminus \mathcal{M}_{0,4}(k)} = \{V(c_j, c_{i'''}) \cup V(c_j, c_4, c_5) \mid j \in \{1, 2, 3\}\}.$$

Here, $c_1, c_2, c_3, c_4, c_5 \in \mathcal{C}_{3,2} = \{c_1, \ldots, c_5\}$ *(cf. Definition 1.5, (ii)).*

(v) It holds that

$$\overline{\mathcal{M}}_{0,5}(k) \setminus \mathcal{M}_{0,5}(k) = \bigcup_{z \in \overline{\mathcal{M}}_{0,4}(k) \setminus \mathcal{M}_{0,4}(k), i \in \{1, \ldots, 5\}} (p_i^{\mathrm{tpd}})^{-1}(z).$$

Proof. Assertion (i) follows from Definition 1.5, (iii), and Proposition 2.1. Assertion (ii) follows from Definition 1.5, (v), and Proposition 2.1. Assertions (iii), (iv), (v) follow immediately from the well-known modular interpretation of the moduli stacks involved. □

Proposition 2.6. *Let* k *be a field. Then the following hold:*

(i) For each $i \in \{2, 3, 4, 5\}$*, it holds that* $p_i^{\mathrm{tpd}} = p_{i-1}^{\mathrm{tpd}} \circ (i-1, i)$*, where* $(i-1, i) \in S_5 \xrightarrow{\sim} \mathrm{Aut}_k(\mathcal{M}_{0,5})$ *denotes the permutation that maps* $i - 1 \mapsto i$*,* $i \mapsto i - 1$*.*

(ii) The assignment $\{1, 2, 3, 4, 5\} \ni i \mapsto M_i \in M_\sim$ *determines a bijection*

$$\{1, 2, 3, 4, 5\} \xrightarrow{\sim} M_\sim$$

(cf. Definition 2.3, (iii), (iv)). In particular, the fibers of the natural projection $M \twoheadrightarrow M_\sim$ *are* S_3*-torsors (relative to the action of* S_3 *from the left — cf. Definition 2.3, (ii)); the set* M *is of cardinality* 30*.*

(iii) Let $a, b, c \in \{1, 2, 3, 4, 5\}$ *be distinct elements. Then* $M_a = M_a \circ (b, c)$ *(cf. Definition 2.3, (iv)), where* $(b, c) \in S_5 \xrightarrow{\sim} \mathrm{Aut}_k(\mathcal{M}_{0,5})$ *denotes the permutation that maps* $b \mapsto c$*,* $c \mapsto b$*.*

(iv) Note that the action of S_5 *on* $\overline{\mathcal{M}}_{0,5}$ *induces an action of* S_5 *on the set of log divisors of* $\overline{\mathcal{M}}_{0,5}^{\log}$ *(cf. Definition 1.5, (iii), and Proposition 2.1). Let* $i, j \in \{1, 2, 3, 4, 5\}$ *distinct elements such that* $\{i, j\} \not\subseteq \{1, 2, 3\}$ *and* $\sigma \in S_5$*. Then*

$$\sigma(V(c_i, c_j)) = \begin{cases} V(c_{\sigma(i)}, c_{\sigma(j)}) & (\text{if } \{\sigma(i), \sigma(j)\} \not\subseteq \{1, 2, 3\}) \\ V(c_l, c_4, c_5) & (\text{if } \{l\} \cup \{\sigma(i), \sigma(j)\} = \{1, 2, 3\}), \end{cases}$$

where $c_i, c_j, c_{\sigma(i)}, c_{\sigma(j)}, c_l \in \mathcal{C}_{3,2} = \{c_1, c_2, c_3, c_4, c_5\}$ *(cf. Definition 1.5, (ii)).*

(v) Let $x, y \in \overline{\mathcal{M}}_{0,4}(k)$ be distinct elements. Then there exists a unique element $z \in \overline{\mathcal{M}}_{0,5}(k)$ such that $p_5^{\mathrm{tpd}}(z) = x$, $p_4^{\mathrm{tpd}}(z) = y$. We shall write $(x, y) \in \overline{\mathcal{M}}_{0,5}(k)$ for this unique element.

Proof. Assertion (i) follows from Proposition 2.2, (i), and Definition 2.3, (i). Next, we consider assertion (ii). Let $q \in M$ be an element. By Definition 2.3, (ii), there exist $\tau \in S_3$ and $i \in \{1, 2, 3, 4, 5\}$ such that $q = \tau \circ p_i^{\mathrm{tpd}}$. In particular, $q \sim p_i^{\mathrm{tpd}}$ (cf. Definition 2.3, (iii)), so the map $\{1, 2, 3, 4, 5\} \to M_\sim$ is surjective. The injectivity of the map $\{1, 2, 3, 4, 5\} \to M_\sim$ follows from Proposition 2.5, (ii), (iv). This completes the proof of assertion (ii). Next, we consider assertion (iii). By conjugating by S_5, we may suppose that $a = 5$. Then it follows immediately that $M_a \circ (b, c) = (b, c) \circ M_a = M_a$ (cf. Definition 2.3, (iii)). Assertions (iv), (v) follow from the well-known modular interpretation of the moduli stacks involved. □

Proposition 2.7. *Let k be a field, $x \in \overline{\mathcal{M}}_{0,4}(k) \setminus \mathcal{M}_{0,4}(k)$, and $y \in \overline{\mathcal{M}}_{0,4}(k) \setminus \{x\}$. Then we obtain an element*

$$p_5^{\mathrm{tpd}}((i, j)(x, y)) = (i, j)(p_5^{\mathrm{tpd}}(x, y)) \in \{0, 1, \infty\},$$

where $i, j \in \{1, 2, 3\}$ are distinct elements and, by a slight abuse of notation, we write (i, j) for the corresponding transpositions $\in S_5 \xrightarrow{\sim} \mathrm{Aut}_k(\mathcal{M}_{0,5})$, $\in S_3 \xrightarrow{\sim} \mathrm{Aut}_k(\mathcal{M}_{0,4})$ (cf. Proposition 2.2, (i), (ii)). Then the following hold:

(i) Let ${}^\dagger y, {}^\ddagger y \in \overline{\mathcal{M}}_{0,4}(k) \setminus \{x\}$. Then

$$p_5^{\mathrm{tpd}}((i, j)(x, {}^\dagger y)) = p_5^{\mathrm{tpd}}((i, j)(x, {}^\ddagger y)).$$

(ii) It holds that

$$x = 0 \Longleftrightarrow x = p_5^{\mathrm{tpd}}((2, 3)(x, y)).$$

(iii) It holds that

$$x = 1 \Longleftrightarrow x = p_5^{\mathrm{tpd}}((1, 3)(x, y)).$$

(iv) It holds that

$$x = \infty \Longleftrightarrow x = p_5^{\mathrm{tpd}}((1, 2)(x, y)).$$

Proof. Assertions (i), (ii), (iii), (iv) follow immediately from the various definitions involved. □

Proposition 2.8. *Let k be a field. For every three distinct elements $z_1, z_2, z_3 \in \overline{\mathcal{M}}_{0,4}(k) \setminus \mathcal{M}_{0,4}(k)$, there exists a unique regular function $t_{z_1,z_2,z_3} \in \Gamma(\mathcal{M}_{0,4}, \mathcal{O}_{\mathcal{M}_{0,4}})$ (which may be regarded as a rational function on $\overline{\mathcal{M}}_{0,4}$) such that*

- *t_{z_1,z_2,z_3} induces a bijection*

$$t_{z_1,z_2,z_3} \colon \overline{\mathcal{M}}_{0,4}(k) \xrightarrow{\sim} k \cup \{\infty\};$$

- *the zero divisor of t_{z_1,z_2,z_3} is of degree 1 and supported on z_1;*
- *$t_{z_1,z_2,z_3}(z_2) = 1$;*
- *the divisor of poles of t_{z_1,z_2,z_3} is of degree 1 and supported on z_3.*

Proof. This follows immediately from the well-known geometry of the projective line (i.e., $\overline{\mathcal{M}}_{0,4}$). □

In the remainder of the present §2, we suppose that $(z_1, z_2, z_3) = (0, 1, \infty)$ (cf. the final portion of Proposition 2.4) and consider the bijection

$$t_{z_1,z_2,z_3} \colon \overline{\mathcal{M}}_{0,4}(k) \xrightarrow{\sim} k \cup \{\infty\}$$

of Proposition 2.8. In the following, we shall think of k as a subset of $\overline{\mathcal{M}}_{0,4}(k)$ by means of this bijection. Our goal will be to describe the field structure of k using the projections $p_i^{\mathrm{tpd}} \colon \mathcal{M}_{0,5}(k) \to \mathcal{M}_{0,4}(k)$ $(i \in \{1,2,3,4,5\})$ (cf. Definition 2.3, (i)) and $\tau_{\mathrm{rf}}, \tau_{\mathrm{ra}}, \tau_{\mathrm{cr}} \in S_5$ (cf. Definition 2.9 below).

Definition 2.9. Let k be a field and $x, y \in \mathcal{M}_{0,4}(k)$ distinct elements. From a computational point of view, it is often useful to recall that $(x, y) \in \mathcal{M}_{0,5}(k)$ corresponds to the genus 0 curve with 5 ordered marked points given by $(0, 1, \infty, x, y)$.

(i) **(Reflection)** We write

$$\tau_{\mathrm{rf}} \overset{\mathrm{def}}{=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix} \in S_5,$$

i.e.,

$$\tau_{\mathrm{rf}} \colon \mathcal{M}_{0,5}(k) \xrightarrow{\sim} \mathcal{M}_{0,5}(k) \colon (x, y) \mapsto \left(1 - x, \tfrac{y(x-1)}{x-y}\right).$$

(ii) **(Ratio)** We write

$$\tau_{\mathrm{ra}} \overset{\mathrm{def}}{=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix} \in S_5,$$

i.e.,

$$\tau_{\mathrm{ra}} \colon \mathcal{M}_{0,5}(k) \simeq \mathcal{M}_{0,5}(k) \colon (x, y) \mapsto \left(\tfrac{1}{x}, \tfrac{y}{x}\right).$$

(iii) **(Cross ratio)** We write

$$\tau_{\mathrm{cr}} \overset{\mathrm{def}}{=} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix} \in S_5,$$

i.e.,

$$\tau_{\mathrm{cr}} \colon \mathcal{M}_{0,5}(k) \simeq \mathcal{M}_{0,5}(k) \colon (x, y) \mapsto \left(\tfrac{y-x}{y}, \tfrac{y-x}{y-1}\right).$$

Proposition 2.10. *Let k be a field, $\tau \in S_5$, and $x, y \in \mathcal{M}_{0,4}(k)$ distinct elements. Then the following hold:*

(i) $\tau = \tau_{\mathrm{rf}} \iff$

$$M_4 \circ \tau = M_3,\ M_3 \circ \tau = M_4,\ M_i \circ \tau = M_i\ (i \in \{1, 2, 5\}).$$

(ii) $\tau = \tau_{\mathrm{ra}} \iff$

$$M_2 \circ \tau = M_4,\ M_4 \circ \tau = M_2,\ M_j \circ \tau = M_j\ (j \in \{1, 3, 5\}).$$

(iii) $\tau = \tau_{\mathrm{cr}} \iff$

$$M_4 \circ \tau = M_1,\ M_5 \circ \tau = M_2,\ M_1 \circ \tau = M_3,$$
$$M_2 \circ \tau = M_4,\ M_3 \circ \tau = M_5.$$

(iv)

$$p_5^{\mathrm{tpd}}(\tau_{\mathrm{rf}}(x, y)) = 1 - x.$$

(v)

$$p_5^{\mathrm{tpd}}(\tau_{\mathrm{ra}}(x, y)) = \tfrac{1}{x}.$$

(vi)
$$p_4^{\text{tpd}}(\tau_{\text{ra}}(x,y)) = \tfrac{y}{x}.$$
(vii)
$$p_5^{\text{tpd}}(\tau_{\text{cr}}(x,y)) = \tfrac{y-x}{y}.$$

Proof. Assertions (i), (ii), (iii) follow from Proposition 2.6, (ii), together with the various definitions involved. Assertions (iv), (v), (vi), (vii) follow from Definition 2.9, (i), (ii), (iii). □

Proposition 2.11. *Let k be a field and $x, y \in \mathcal{M}_{0,4}(k)$ distinct elements. Then the following hold:*

(i) $\frac{1}{x} = p_5^{\text{tpd}}(\tau_{\text{ra}}(x,y))$.
(ii) *If* $y \neq \frac{1}{x}$, *then* $x \cdot y = p_4^{\text{tpd}}(\tau_{\text{ra}}(\frac{1}{x}, y))$.

Proof. Assertions (i), (ii) follow immediately from Proposition 2.10, (v), (vi). □

Proposition 2.12. *Let k be a field such that $\sharp k \neq 3$. Then it holds that k is a field of characteristic $\neq 2 \iff$ there exists an element $x \in \mathcal{M}_{0,4}(k)$ such that $\frac{1}{x} = x$.*

Proof. Assertion follows immediately from the various definitions involved. □

Proposition 2.13. *Let k be a field and $x, y \in \mathcal{M}_{0,4}(k)$ distinct elements. We suppose that k is a field of characteristic 2. Then the following hold:*

(i) $x + 1 = p_5^{\text{tpd}}(\tau_{\text{rf}}(x,y))$.
(ii) $x + y = y(x \cdot \frac{1}{y} + 1)$.

Proof. Assertion (i) follows immediately from Proposition 2.10, (iv). Assertion (ii) follows immediately from the various definitions involved. □

Proposition 2.14. *Let k be a field and $x, y \in \mathcal{M}_{0,4}(k)$ distinct elements. We suppose that k is a field of characteristic $\neq 2$. Then the following hold:*

(i) $x = -1 \iff \frac{1}{x} = x$.
(ii) *If* $x \neq -1$, *then* $1 + 1 = p_5^{\text{tpd}}(\tau_{\text{rf}}(-1, x))$.
(iii) *If* $x \neq -1$, *then* $x + 1 = p_5^{\text{tpd}}(\tau_{\text{cr}}(x, -1))$.
(iv) $x + y = y(x \cdot \frac{1}{y} + 1)$.

Proof. Assertions (i), (iv) follow immediately from the various definitions involved. Assertions (ii), (iii) follow immediately from Proposition 2.10, (iv), (vii). □

3. **Construction of a field associated to a CFS-collection**

In the present §3, we introduce the notion of a CFS-collection (cf. Definition 3.2 below) and construct a field associated to the intrinsic structure of a CFS-collection (cf. Theorem 3.13 below).

Definition 3.1. Let $A, B, \partial B, H, M$ be sets. We shall refer to $\mathscr{A} = (A, B, \partial B, H, M)$ as a *model CFS-collection* ("model configuration-theoretic field structure collection") if there exist a field k and a smooth log curve $X^{\log}$ over k of type $(0, 3^{\text{ord}})$ such that $A = X_2(k)$; $B = X(k)$; $\partial B = X(k) \setminus U_X(k)$; H is the set of automorphisms of $X_2(k)$ induced by automorphisms of U_{X_2} over k (cf. Proposition 2.2, (i)); M is the set of maps $X_2(k) \to X(k)$ induced by the 30 natural morphisms $X_2 \to X$ (cf. Proposition 2.1; Definition 2.3, (ii); Proposition 2.6, (ii)).

Definition 3.2. Let A, B be sets; $\partial B \subseteq B$ a subset of cardinality 3; $H \subseteq \mathrm{Aut}(A)$ a subgroup; M a set of maps $A \to B$. Then we shall say that $\mathscr{A} = (A, B, \partial B, H, M)$ is a *CFS-collection* if it satisfies the following condition: There exist a field k, a smooth log curve $X^{\log}$ over k of type $(0, 3^{\mathrm{ord}})$, a bijection $\alpha \colon A \xrightarrow{\sim} X_2(k)$, and a bijection $\beta \colon B \xrightarrow{\sim} X(k)$, such that

(i) β induces a bijection $B \setminus \partial B \xrightarrow{\sim} U_X(k)$;

(ii) the isomorphism of groups $\mathrm{Aut}(A) \xrightarrow{\sim} \mathrm{Aut}(X_2(k))$ determined by α induces an isomorphism of groups $H \xrightarrow{\sim} \mathrm{Aut}_k(U_{X_2})$ ($\hookrightarrow \mathrm{Aut}(X_2(k))$) (so $H \cong S_5$ (cf. Propositions 2.1 and 2.2, (i)));

(iii) if we write M_X for the set of maps $X_2(k) \to X(k)$ induced by the 30 natural morphisms $X_2 \to X$ (cf. Proposition 2.1; Definition 2.3, (ii); Proposition 2.6, (ii)), then there exists a bijection $M \xrightarrow{\sim} M_X$ such that if $\lambda \mapsto q$ via this bijection, then

$$\begin{array}{ccc} A & \xrightarrow[\alpha]{\sim} & X_2(k) \\ {\scriptstyle\lambda}\downarrow & \circlearrowright & \downarrow{\scriptstyle q} \\ B & \xrightarrow[\beta]{\sim} & X(k). \end{array}$$

Remark 3.3. It is immediate that any model CFS-collection is a CFS-collection. Moreover, relative to the terminology introduced in Definition 3.9 below, the data (α, β) that appears in Definition 3.2 may be regarded as an isomorphism of CFS-collections between the CFS-collection under consideration in Definition 3.2 and some model CFS-collection.

Definition 3.4. Let $(A, B, \partial B, H, M)$ be a CFS-collection. Let ${}^{\dagger}\lambda, {}^{\ddagger}\lambda \in M$. We define an equivalence relation

$${}^{\dagger}\lambda \sim {}^{\ddagger}\lambda \overset{\mathrm{def}}{\iff} \{{}^{\dagger}\lambda^{-1}(b)\}_{b \in \partial B} = \{{}^{\ddagger}\lambda^{-1}(b)\}_{b \in \partial B}.$$

The set of equivalence classes of M is of cardinality 5 (cf. Remark 3.3; Proposition 2.6, (ii)).

Definition 3.5. Let $(A, B, \partial B, H, M)$ be a CFS-collection and $\phi \colon H \xrightarrow{\sim} S_5$ an isomorphism. Here, we remark that H acts naturally on M (cf. Remark 3.3; Definition 2.3, (ii)). Also, we recall the well-known elementary fact that every automorphism of S_5 is inner; thus, ϕ is *unique* up to composition with an inner automorphism of S_5. Let $a \in \{1, 2, 3, 4, 5\}$ be an element. Write $M_a[\phi]$ for the unique equivalence class as in Definiiton 3.4 such that $M_a[\phi] = M_a[\phi] \circ (\phi^{-1}(b, c))$, for all transpositions $(b, c) \in S_5$ such that $a \notin \{b, c\}$ (cf, Remark 3.3; Proposition 2.6, (ii), (iii)). Thus, $M = M_1[\phi] \sqcup \cdots \sqcup M_5[\phi]$. Let $\lambda \in M_1[\phi]$. Write $p_1[\phi, \lambda] \overset{\mathrm{def}}{=} \lambda$, $p_i[\phi, \lambda] \overset{\mathrm{def}}{=} p_{i-1}[\phi, \lambda] \circ (\phi^{-1}(i-1, i))$, where $i \in \{2, 3, 4, 5\}$ and $(i-1, i) \in S_5$.

Definition 3.6. Let $(A, B, \partial B, H, M)$ be a CFS-collection, $\phi \colon H \xrightarrow{\sim} S_5$ an isomorphism, and $\lambda \in M_1[\phi]$. We define

$$\begin{aligned} H_B[\phi, \lambda] \overset{\mathrm{def}}{=} \{\gamma \in \mathrm{Aut}(B) \mid & \text{ there exists an element } \sigma \in S_5 \\ & \text{such that } \sigma(1) = 1 \text{ and } \gamma \circ \lambda = \lambda \circ \phi^{-1}(\sigma)\}. \end{aligned}$$

Let ${}^{\dagger}\phi \colon H \xrightarrow{\sim} S_5$ be an isomorphism, and ${}^{\dagger}\lambda \in M_1[{}^{\dagger}\phi]$. Then one verifies immediately (cf. Definition 2.2, (i), (ii); Remark 3.3) that $H_B[{}^{\dagger}\phi, {}^{\dagger}\lambda] = H_B[\phi, \lambda]$. Write

$H_B \stackrel{\text{def}}{=} H_B[\phi, \lambda] \subseteq \mathrm{Aut}(B)$. Finally, one verifies immediately (cf. Definition 2.2, (i), (ii); Remark 3.3) that the assignment $\gamma \mapsto \gamma|_{\partial B}$ determines an isomorphism of groups $H_B \xrightarrow{\sim} \mathrm{Aut}(\partial B)$. Here, we recall that $\mathrm{Aut}(\partial B)$ is isomorphic to S_3.

Definition 3.7. Let $(A, B, \partial B, H, M)$ be a CFS-collection, $\phi\colon H \xrightarrow{\sim} S_5$ an isomorphism, and $\lambda \in M_1[\phi]$. Let $x, y \in B$ be distinct elements. Then there exists an unique element $z \in A$ such that $p_5[\phi, \lambda](z) = x$, $p_4[\phi, \lambda](z) = y$ (cf. Remark 3.3; Proposition 2.6, (v)). Write $(x, y)[\phi, \lambda] \stackrel{\text{def}}{=} z$.

Definition 3.8. Let $(A, B, \partial B, H, M)$ be a CFS-collection, $\phi\colon H \xrightarrow{\sim} S_5$ an isomorphism, and $\lambda \in M_1[\phi]$. Then (cf. Remark 3.3; Proposition 2.7, (ii), (iii), (iv)):

- We shall write $0[\phi, \lambda]$ for the unique element $x \in \partial B$ such that for every $y \in B \setminus \{x\}$, it holds that $x = p_5[\phi, \lambda]((\phi^{-1}(2, 3))(x, y)[\phi, \lambda])$.
- We shall write $1[\phi, \lambda]$ for the unique element $x \in \partial B$ such that for every $y \in B \setminus \{x\}$, it holds that $x = p_5[\phi, \lambda]((\phi^{-1}(1, 3))(x, y)[\phi, \lambda])$.
- We shall write $\infty[\phi, \lambda]$ for the unique element $x \in \partial B$ such that for every $y \in B \setminus \{x\}$, it holds that $x = p_5[\phi, \lambda]((\phi^{-1}(1, 2))(x, y)[\phi, \lambda])$.

Thus, $\{0[\phi, \lambda], 1[\phi, \lambda], \infty[\phi, \lambda]\} = \partial B$.

Definition 3.9. Let

$$^{\dagger}\mathscr{A} = (^{\dagger}A, {}^{\dagger}B, {}^{\dagger}\partial B, {}^{\dagger}H, {}^{\dagger}M), \ {}^{\ddagger}\mathscr{A} = (^{\ddagger}A, {}^{\ddagger}B, {}^{\ddagger}\partial B, {}^{\ddagger}H, {}^{\ddagger}M)$$

be CFS-collections. We shall refer to $(\alpha, \beta)\colon {}^{\dagger}\mathscr{A} \xrightarrow{\sim} {}^{\ddagger}\mathscr{A}$ as an *isomorphism of CFS-collections* if $\alpha\colon {}^{\dagger}A \xrightarrow{\sim} {}^{\ddagger}A$, $\beta\colon {}^{\dagger}B \xrightarrow{\sim} {}^{\ddagger}B$ are bijections of sets such that $\beta({}^{\dagger}\partial B) = {}^{\ddagger}\partial B$, $\alpha \circ {}^{\dagger}H \circ \alpha^{-1} = {}^{\ddagger}H$, $\beta \circ {}^{\dagger}M \circ \alpha^{-1} = {}^{\ddagger}M$.

Definition 3.10. Let

$$^{\dagger}\mathscr{A} = (^{\dagger}A, {}^{\dagger}B, {}^{\dagger}\partial B, {}^{\dagger}H, {}^{\dagger}M), \ {}^{\ddagger}\mathscr{A} = (^{\ddagger}A, {}^{\ddagger}B, {}^{\ddagger}\partial B, {}^{\ddagger}H, {}^{\ddagger}M)$$

be CFS-collections, $(\alpha, \beta)\colon {}^{\dagger}\mathscr{A} \xrightarrow{\sim} {}^{\ddagger}\mathscr{A}$ an isomorphism of CFS-collections, ${}^{\dagger}\phi\colon {}^{\dagger}H \xrightarrow{\sim} S_5$ an isomorphism, and ${}^{\dagger}\lambda \in {}^{\dagger}M_1[{}^{\dagger}\phi]$ (cf. Definition 3.5). Write ${}^{\ddagger}\lambda \stackrel{\text{def}}{=} \beta \circ {}^{\dagger}\lambda \circ \alpha^{-1}$; ${}^{\ddagger}\phi\colon {}^{\ddagger}H \xrightarrow{\sim} S_5$ for the isomorphism obtained by composing ${}^{\dagger}\phi$ with the isomorphism ${}^{\ddagger}H \xrightarrow{\sim} {}^{\dagger}H$ obtained by conjugating by α^{-1}. In this situation, we shall write

$$(\alpha, \beta)({}^{\dagger}\phi) \stackrel{\text{def}}{=} {}^{\ddagger}\phi, \quad (\alpha, \beta)({}^{\dagger}\lambda) \stackrel{\text{def}}{=} {}^{\ddagger}\lambda.$$

Then we have a commutative diagram

$$\begin{array}{ccc} {}^{\dagger}A & \xrightarrow{\alpha} & {}^{\ddagger}A \\ {\scriptstyle p_i[{}^{\dagger}\phi, {}^{\dagger}\lambda]}\downarrow & \circlearrowright & \downarrow{\scriptstyle p_i[{}^{\ddagger}\phi, {}^{\ddagger}\lambda]} \\ {}^{\dagger}B & \xrightarrow[\beta]{} & {}^{\ddagger}B, \end{array}$$

where $i \in \{1, 2, 3, 4, 5\}$ (cf. Definition 3.5).

Definition 3.11. Let $(A, B, \partial B, H, M)$ be a CFS-collection and $\phi\colon H \xrightarrow{\sim} S_5$ an isomorphism. Let $M_i[\phi]$ be as in Definition 3.5. Then we shall write $\tau_{\mathrm{rf}}[\phi], \tau_{\mathrm{ra}}[\phi], \tau_{\mathrm{cr}}[\phi] \in H$ for the unique elements of H such that

$$M_4[\phi] \circ \tau_{\mathrm{rf}}[\phi] = M_3[\phi], \ M_3[\phi] \circ \tau_{\mathrm{rf}}[\phi] = M_4[\phi], \ M_i[\phi] \circ \tau_{\mathrm{rf}}[\phi] = M_i[\phi],$$
$$M_2[\phi] \circ \tau_{\mathrm{ra}}[\phi] = M_4[\phi], \ M_4[\phi] \circ \tau_{\mathrm{ra}}[\phi] = M_2[\phi], \ M_j[\phi] \circ \tau_{\mathrm{ra}}[\phi] = M_j[\phi],$$
$$M_4[\phi] \circ \tau_{\mathrm{cr}}[\phi] = M_1[\phi], \ M_5[\phi] \circ \tau_{\mathrm{cr}}[\phi] = M_2[\phi], \ M_1[\phi] \circ \tau_{\mathrm{cr}}[\phi] = M_3[\phi],$$

$$M_2[\phi] \circ \tau_{\mathrm{cr}}[\phi] = M_4[\phi],\ M_3[\phi] \circ \tau_{\mathrm{cr}}[\phi] = M_5[\phi],$$

where $i \in \{1, 2, 5\}$ and $j \in \{1, 3, 5\}$ (cf. Remark 3.3; Proposition 2.10, (i), (ii), (iii)).

Definition 3.12. Let $\mathscr{A} = (A, B, \partial B, H, M)$ be a CFS-collection, $\phi\colon H \xrightarrow{\sim} S_5$ an isomorphism, and $\lambda \in M_1[\phi]$ an element. We shall say that a collection of maps

$$\boxplus, \boxtimes\colon (B \setminus \{\infty[\phi, \lambda]\}) \times (B \setminus \{\infty[\phi, \lambda]\}) \to (B \setminus \{\infty[\phi, \lambda]\}),$$

$$\boxminus\colon (B \setminus \{\infty[\phi, \lambda]\}) \to (B \setminus \{\infty[\phi, \lambda]\}),$$

$$\boxslash\colon (B \setminus \{0[\phi, \lambda], \infty[\phi, \lambda]\}) \to (B \setminus \{0[\phi, \lambda], \infty[\phi, \lambda]\})$$

is *CFS-admissible* if the following conditions are satisfied:

(1) First, we consider general properties (cf. Proposition 2.11):
  (a) $\boxminus(0[\phi, \lambda]) = 0[\phi, \lambda]$, $\boxslash(1[\phi, \lambda]) = 1[\phi, \lambda]$.
  (b) For $x, y \in B \setminus \{\infty[\phi, \lambda]\}$, $\boxplus(x, y) = \boxplus(y, x)$, $\boxtimes(x, y) = \boxtimes(y, x)$.
  (c) For $x \in B\setminus\{\infty[\phi, \lambda]\}$, $\boxplus(0[\phi, \lambda], x) = x$, $\boxtimes(0[\phi, \lambda], x) = 0[\phi, \lambda]$, $\boxplus(x, \boxminus(x)) = 0[\phi, \lambda]$.
  (d) For $x \in B \setminus \{0[\phi, \lambda], \infty[\phi, \lambda]\}$, $\boxtimes(1[\phi, \lambda], x) = x$, $\boxtimes(x, \boxslash(x)) = 1[\phi, \lambda]$.
  (e) Let $x, y \in B \setminus \partial B$ such that $x \neq y$. Then $\boxslash(x) = p_5[\phi, \lambda](\tau_{\mathrm{ra}}[\phi, \lambda](x, y))$.
  (f) Let $x, y \in B \setminus \partial B$ such that $y \neq \boxslash(x)$. Then

$$\boxtimes(x, y) = p_4[\phi, \lambda](\tau_{\mathrm{ra}}[\phi, \lambda](\boxslash(x), y)).$$

(2) Suppose that $\sharp B = 4$. Then we define the maps $\boxplus, \boxtimes, \boxminus, \boxslash$ for $B \setminus \{\infty[\phi, \lambda]\}$ as follows: write $\{a\} = B \setminus \{0[\phi, \lambda], 1[\phi, \lambda], \infty[\phi, \lambda]\}$; then

$\boxplus$	$0[\phi, \lambda]$	$1[\phi, \lambda]$	a
$0[\phi, \lambda]$	$0[\phi, \lambda]$	$1[\phi, \lambda]$	a
$1[\phi, \lambda]$	$1[\phi, \lambda]$	a	$0[\phi, \lambda]$
a	a	$0[\phi, \lambda]$	$1[\phi, \lambda]$

$\boxtimes$	$0[\phi, \lambda]$	$1[\phi, \lambda]$	a
$0[\phi, \lambda]$	$0[\phi, \lambda]$	$0[\phi, \lambda]$	$0[\phi, \lambda]$
$1[\phi, \lambda]$	$0[\phi, \lambda]$	$1[\phi, \lambda]$	a
a	$0[\phi, \lambda]$	a	$1[\phi, \lambda]$

$$\boxminus(0[\phi, \lambda]) = 0[\phi, \lambda], \quad \boxminus(1[\phi, \lambda]) = a, \quad \boxminus(a) = 1[\phi, \lambda],$$

$$\boxslash(1[\phi, \lambda]) = 1[\phi, \lambda], \quad \boxslash(a) = a.$$

(3) Suppose that there does not exist $x \in B \setminus \partial B$ such that

$$\boxslash(x) = x$$

(cf. Proposition 2.12). Then (cf. Proposition 2.13):
  (a) Let $x \in B \setminus \{\infty[\phi, \lambda]\}$. Then $\boxminus(x) = x$ and $\boxplus(x, x) = 0[\phi, \lambda]$.
  (b) Let $x, y \in B \setminus \partial B$ such that $x \neq y$. Then

$$\boxplus(x, 1[\phi, \lambda]) = p_5[\phi, \lambda](\tau_{\mathrm{rf}}[\phi, \lambda](x, y)).$$

  (c) Let $x, y \in B \setminus \{\infty[\phi, \lambda]\}$ such that $y \neq 0[\phi, \lambda]$. Then

$$\boxplus(x, y) = \boxtimes(y, \boxplus(\boxtimes(x, \boxslash(y)), 1[\phi, \lambda])).$$

(4) Suppose that $\sharp B \neq 4$, and that there exists an element $x \in B \setminus \partial B$ such that

$$\boxslash(x) = x$$

(cf. Proposition 2.12). Then (cf. Proposition 2.14):
  (a) Let $x, y \in B \setminus \partial B$ such that $\boxslash(x) = x$. Then $\boxminus(1[\phi, \lambda]) = x \in B$, and $\boxminus(y) = \boxtimes(x, y)$.
  (b) Let $x \in B \setminus (\partial B \sqcup \{\boxminus(1[\phi, \lambda])\})$. Then

$$\boxplus(1[\phi, \lambda], 1[\phi, \lambda]) = p_5[\phi, \lambda](\tau_{\mathrm{rf}}[\phi, \lambda](\boxminus(1[\phi, \lambda]), x)).$$

(c) Let $x \in B \setminus (\partial B \sqcup \{\boxminus(1[\phi,\lambda])\})$. Then

$$\boxplus(x, 1[\phi,\lambda]) = p_5[\phi,\lambda](\tau_{\mathrm{cr}}[\phi,\lambda](x, \boxminus(1[\phi,\lambda]))).$$

(d) Let $x, y \in B \setminus \{\infty[\phi,\lambda]\}$ such that $y \neq 0[\phi,\lambda]$. Then

$$\boxplus(x, y) = \boxtimes(y, \boxplus(\boxtimes(x, \boxslash(y)), 1[\phi,\lambda])).$$

Observe that it follows formally from the above conditions (1), (2), (3), (4) that if one fixes the data $(\mathscr{A}, \phi, \lambda)$, then any CFS-admissible collection of maps is *unique*. Thus, if the data $(\mathscr{A}, \phi, \lambda)$ admits a CFS-admissible collection of maps, then we shall write $F[\mathscr{A}, \phi, \lambda] \stackrel{\mathrm{def}}{=} (B \setminus \{\infty[\phi,\lambda]\}, \boxplus, \boxtimes, \boxminus, \boxslash)$ for the set $B \setminus \{\infty[\phi,\lambda]\}$, equipped with the maps $\boxplus$, $\boxtimes$, $\boxminus$, $\boxslash$.

Theorem 3.13. (From CFS-collections to fields) *Let $\mathscr{A} = (A, B, \partial B, H, M)$ be a CFS-collection (cf. Definition 3.2); $\phi\colon H \xrightarrow{\sim} S_5$ an isomorphism; $i \in \{1,2,3,4,5\}$; $M_i[\phi]$ as in Definition 3.5; $\lambda \in M_1[\phi]$; $p_i[\phi,\lambda] \in M_i[\phi]$ as in Definition 3.5; $0[\phi,\lambda], 1[\phi,\lambda], \infty[\phi,\lambda] \in \partial B$ as in Definition 3.8; $\tau_{\mathrm{rf}}[\phi], \tau_{\mathrm{ra}}[\phi], \tau_{\mathrm{cr}}[\phi] \in H$ as in Definition 3.11. Then:*

(i) Suppose, further, that the following conditions hold: $\mathscr{A} = (A, B, \partial B, H, M)$ is a model CFS-collection; k and $X^{\log}$ are as in Definition 3.1; $\phi\colon H \xrightarrow{\sim} S_5$ is the composite of the natural isomorphisms

$$H \xrightarrow{\sim} \mathrm{Aut}_k(U_{X_2}) \xrightarrow{\sim} \mathrm{Aut}_k(\mathcal{M}_{0,5}) \xleftarrow{\sim} S_5$$

(cf. Propositions 2.1; 2.2, (i)); $\lambda = p_1^{\mathrm{tpd}} \in M_1[\phi]$. Then the bijection

$$B = X(k) \xrightarrow{\sim} \overline{\mathcal{M}}_{0,4}(k) \xrightarrow{\sim} k \cup \{\infty\}$$

induced by $t_{0,1,\infty}$ (cf. Proposition 2.1, the final portion of Proposition 2.4, and Proposition 2.8), together with

> *the operations of addition, multiplication, additive inversion, and multiplicative inversion arising from the field structure on k,*

determines a CFS-admissible collection of maps for $(\mathscr{A}, \phi, \lambda)$ (cf. Definition 3.12). In particular, the resulting object $F[\mathscr{A}, \phi, \lambda]$ of Definition 3.12 may be regarded as a field structure on the set $B \setminus \{\infty[\phi,\lambda]\}$.

(ii) Let

$$^{\dagger}\mathscr{A} = (^{\dagger}A, {}^{\dagger}B, {}^{\dagger}\partial B, {}^{\dagger}H, {}^{\dagger}M), \quad {}^{\ddagger}\mathscr{A} = (^{\ddagger}A, {}^{\ddagger}B, {}^{\ddagger}\partial B, {}^{\ddagger}H, {}^{\ddagger}M)$$

be CFS-collections; ${}^{\dagger}\phi\colon {}^{\dagger}H \xrightarrow{\sim} S_5$ an isomorphism; ${}^{\ddagger}\phi\colon {}^{\ddagger}H \xrightarrow{\sim} S_5$ an isomorphism; ${}^{\dagger}\lambda \in {}^{\dagger}M_1[{}^{\dagger}\phi]$; ${}^{\ddagger}\lambda \in {}^{\ddagger}M_1[{}^{\ddagger}\phi]$; $(\alpha,\beta)\colon {}^{\dagger}\mathscr{A} \xrightarrow{\sim} {}^{\ddagger}\mathscr{A}$ an isomorphism of CFS-collections. Suppose that

$$(\alpha,\beta)({}^{\dagger}\phi) = {}^{\ddagger}\phi, \quad (\alpha,\beta)({}^{\dagger}\lambda) = {}^{\ddagger}\lambda$$

(cf. Definition 3.10), and that $({}^{\dagger}\mathscr{A}, {}^{\dagger}\phi, {}^{\dagger}\lambda)$ admits a CFS-admissible collection of maps. Then $({}^{\ddagger}\mathscr{A}, {}^{\ddagger}\phi, {}^{\ddagger}\lambda)$ admits a CFS-admissible collection of maps. Moreover, $F[{}^{\dagger}\mathscr{A}, {}^{\dagger}\phi, {}^{\dagger}\lambda]$, $F[{}^{\ddagger}\mathscr{A}, {}^{\ddagger}\phi, {}^{\ddagger}\lambda]$ may be regarded, respectively, as field structures on the sets ${}^{\dagger}B \setminus \{\infty[{}^{\dagger}\phi, {}^{\dagger}\lambda]\}$, ${}^{\ddagger}B \setminus \{\infty[{}^{\ddagger}\phi, {}^{\ddagger}\lambda]\}$ (cf. (i)), with respect to which β induces a field isomorphism $F[{}^{\dagger}\mathscr{A}, {}^{\dagger}\phi, {}^{\dagger}\lambda] \xrightarrow{\sim} F[{}^{\ddagger}\mathscr{A}, {}^{\ddagger}\phi, {}^{\ddagger}\lambda]$.

(iii) Let ${}^{\dagger}\phi, {}^{\ddagger}\phi \colon H \xrightarrow{\sim} S_5$ *be isomorphisms and* ${}^{\dagger}\lambda \in M_1[{}^{\dagger}\phi], {}^{\ddagger}\lambda \in M_1[{}^{\ddagger}\phi]$*. Then there exists a unique element* $\gamma \in \mathrm{Aut}(\partial B)$ *such that* $\gamma(z[{}^{\dagger}\phi, {}^{\dagger}\lambda]) = z[{}^{\ddagger}\phi, {}^{\ddagger}\lambda]$ *and* $\gamma \in \mathrm{Aut}(\partial B) \xleftarrow{\sim} H_B \subseteq \mathrm{Aut}(B)$ *(cf. Definition 3.6) determines a field isomorphism*

$$F[\mathscr{A}, {}^{\dagger}\phi, {}^{\dagger}\lambda] \xrightarrow{\sim} F[\mathscr{A}, {}^{\ddagger}\phi, {}^{\ddagger}\lambda]$$

relative to the field structures discussed in (i), (ii).

Proof. First, we consider assertion (i). Since we suppose that $\phi \colon H(= \mathrm{Aut}_k(U_{X_2})) \xrightarrow{\sim} S_5$ is the composite of the natural isomorphisms

$$\mathrm{Aut}_k(U_{X_2}) \xrightarrow{\sim} \mathrm{Aut}_k(\mathcal{M}_{0,5}) \xleftarrow{\sim} S_5,$$

by Proposition 2.6, (ii), (iii), and Definitions 2.3, (iv); 3.5, it holds that $M_i[\phi] = S_3 \circ p_i^{\mathrm{tpd}}$ $(i \in \{1, 2, 3, 4, 5\})$. Recall that $\lambda = p_1^{\mathrm{tpd}} \in M_1[\phi]$. Thus, by Definition 3.5, and Proposition 2.6, (i),

$$p_1[\phi, \lambda] \stackrel{\mathrm{def}}{=} \lambda = p_1^{\mathrm{tpd}},$$

$$p_i[\phi, \lambda] \stackrel{\mathrm{def}}{=} p_{i-1}[\phi, \lambda] \circ (\phi^{-1}(i-1, i)) = p_{i-1}^{\mathrm{tpd}} \circ (i-1, i) = p_i^{\mathrm{tpd}},$$

where $i \in \{2, 3, 4, 5\}$.

Next, observe that, relative to the identifications induced by the natural isomorphisms $X(k) \xrightarrow{\sim} \mathcal{M}_{0,4}(k)$, $X_2(k) \xrightarrow{\sim} \mathcal{M}_{0,5}(k)$ (cf. Definition 2.1), the following hold:

- Let $x, y \in B$ be distinct elements. Then by Proposition 2.6, (v), and Definition 3.7, it holds that $(x, y)[\phi, \lambda] = (x, y)$.
- By the final portion of Proposition 2.4, Proposition 2.7, (ii), (iii), (iv), and Definition 3.8, it holds that $z[\phi, \lambda] = z$ for $z \in \{0, 1, \infty\}$.
- By Definitions 2.9, 3.11, and Proposition 2.10, (i), (ii), (iii), it holds that $\tau_{\mathrm{rf}}[\phi] = \tau_{\mathrm{rf}}$, $\tau_{\mathrm{ra}}[\phi] = \tau_{\mathrm{ra}}$, $\tau_{\mathrm{cr}}[\phi] = \tau_{\mathrm{cr}}$.

Thus, assertion (i) follows from Definition 3.12 and Propositions 2.11, 2.12, 2.13, 2.14.

Assertion (ii) follows formally from assertion (i); Definitions 3.2, 3.5, 3.6, 3.12 (cf., especially, the *uniqueness* of a CFS-collection of maps associated to a given "$(\mathscr{A}, \phi, \lambda)$"); Remark 3.3; Proposition 2.2, (i), (ii); Proposition 2.6, (ii). (More details may be found in the (essentially similar) argument given in the final portion of the proof of assertion (iii).)

Next, we consider assertion (iii). Since $\mathrm{Aut}(\partial B)$ is isomorphic to S_3, it follows immediately from the various definitions involved that there exists a *unique* element $\gamma \in \mathrm{Aut}(\partial B)$ such that $\gamma(z[{}^{\dagger}\phi, {}^{\dagger}\lambda]) = z[{}^{\ddagger}\phi, {}^{\ddagger}\lambda]$ for $z \in \{0, 1, \infty\}$. By Definition 3.2 and Remark 3.3, we may assume without loss of generality that $\mathscr{A}$ is a model CFS-collection. Moreover, by Definitions 3.5, 3.6; Proposition 2.2, (i), (ii); Proposition 2.6, (ii), we may assume without loss of generality that ${}^{\dagger}\phi$, ${}^{\dagger}\lambda$ are, respectively, the "ϕ", "λ" of assertion (i). (More details may be found in the (essentially similar) argument given in the following paragraph.)

Next, observe that there exists a unique element $\alpha \in H \subseteq \mathrm{Aut}(A)$ such that ${}^{\ddagger}\phi \colon H \xrightarrow{\sim} S_5$ is the isomorphism obtained by composing ${}^{\dagger}\phi$ with the isomorphism $H \xrightarrow{\sim} H$ obtained by conjugating by α^{-1} (cf. Definition 3.5). Write $\beta \in H_B \subseteq \mathrm{Aut}(B)$ for the unique element such that ${}^{\ddagger}\lambda = \beta \circ {}^{\dagger}\lambda \circ \alpha^{-1}$ (cf. Proposition 2.6, (ii); Definition 3.6). Thus, the pair (α, β) may be regarded as an isomorphism of collections of data $(\mathscr{A}, {}^{\dagger}\phi, {}^{\dagger}\lambda) \xrightarrow{\sim} (\mathscr{A}, {}^{\ddagger}\phi, {}^{\ddagger}\lambda)$ (cf. Definition 3.10). In particular,

$\beta|_{\partial B} = \gamma$ (cf. Definitions 3.6, 3.8). By assertion (ii), we may regard the field structure on $F[\mathscr{A}, {}^{\ddagger}\phi, {}^{\ddagger}\lambda]$ as the result of transporting the field structure on $F[\mathscr{A}, {}^{\dagger}\phi, {}^{\dagger}\lambda]$ via β. In particular, β determines a field isomorphism

$$F[\mathscr{A}, {}^{\dagger}\phi, {}^{\dagger}\lambda] \xrightarrow{\sim} F[\mathscr{A}, {}^{\ddagger}\phi, {}^{\ddagger}\lambda],$$

as desired. □

4. Construction of a CFS-collection associated to a PGCS-collection

In the present §4, we introduce the notion of a PGCS-collection (cf. Definition 4.2 below) and construct a CFS-collection (cf. §3) associated to the intrinsic structure of a PGCS-collection (cf. Theorem 4.9 below).

Notation 4.1. Let $n \in \mathbb{Z}_{>0}$; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; p a prime number; k a generalized sub-p-adic local field (cf. [Topics], Definition 4.11); $\bar{k}$ an algebraic closure of k; $X^{\log}$ a smooth log curve over k of type (g, r). Write $G_k = \mathrm{Gal}(\bar{k}/k)$. In the following, we shall consider the commutative diagram of étale fundamental groups "$\pi_1(-)$" (relative to suitable choices of basepoints) and their quotients:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(U_{X_n} \times_k \bar{k}) & \longrightarrow & \pi_1(U_{X_n}) & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \pi_1(U_{X_n} \times_k \bar{k})^{(p)} & \longrightarrow & \pi_1(U_{X_n})^{[p]} & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
 & & \pi_1(U_{X_n} \times_k \bar{k})^{(p)} & \longrightarrow & \pi_1(U_{X_n})^{(p)} & \longrightarrow & G_k^{(p)} & \longrightarrow & 1,
\end{array}$$

where we append the superscript (p) to a profinite group to denote its maximal pro-p quotient, and we write

$$\pi_1(U_{X_n})^{[p]} \stackrel{\text{def}}{=} \pi_1(U_{X_n})/\mathrm{Ker}(\pi_1(U_{X_n} \times_k \bar{k}) \to \pi_1(U_{X_n} \times_k \bar{k})^{(p)}).$$

Definition 4.2. Let $n \in \mathbb{Z}_{>1}$; (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$; $\Box \in \{\mathrm{arb}, \mathrm{ord}\}$; $\Sigma_\Delta, \Sigma_{\mathrm{Gal}}$ sets of prime numbers such that $\Sigma_\Delta \subseteq \Sigma_{\mathrm{Gal}}$, and $\Sigma_\Delta, \Sigma_{\mathrm{Gal}}$ are of cardinality 1 or equal to $\mathfrak{Primes}$; Π_n a profinite group; G a quotient of Π_n; $\mathcal{D}_n$ a set of subgroups of Π_n. We shall refer to $(\Pi_n, G, \mathcal{D}_n)$ as a *PGCS-collection* ("point-theoretic Galois configuration space collection") *of type* $(g, r^{\Box}, n, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ if there exists a collection of data as follows:

- a prime number $p \in \Sigma_\Delta$; a generalized sub-p-adic local field k; an algebraic closure $\bar{k}$ of k; a smooth log curve $X^{\log}$ over k of type $(g, r^{\Box})$;
- an isomorphism

$$\alpha\colon \Pi_n \xrightarrow{\sim} \Pi_{U_{X_n}} \stackrel{\text{def}}{=} \begin{cases} \pi_1(U_{X_n})^{\Sigma_\Delta} & (\text{if } \Sigma_\Delta = \Sigma_{\mathrm{Gal}}) \\ \pi_1(U_{X_n})^{[p]} & (\text{if } \Sigma_\Delta \subsetneq \Sigma_{\mathrm{Gal}}) \end{cases}$$

such that, if we write G_k for the Galois group $\mathrm{Gal}(\bar{k}/k)$ of k and $K \subseteq \bar{k}$ for the maximal pro-Σ_{Gal} subextension of $\bar{k}/k$ (so $G_k^{\Sigma_{\mathrm{Gal}}} = \mathrm{Gal}(K/k)$), then the natural outer action $G_k \overset{\text{out}}{\curvearrowright} \Delta_{U_{X_n}} \overset{\text{def}}{=} \pi_1(U_{X_n} \times_k \bar{k})^{\Sigma_\Delta}$ factors through the natural surjection $G_k \twoheadrightarrow G_k^{\Sigma_{\text{Gal}}}$, and α induces a commutative diagram

$$\begin{array}{ccc} \Pi_n & \xrightarrow[\alpha]{\sim} & \Pi_{U_{X_n}} \\ \downarrow & \circlearrowleft & \downarrow \\ G & \overset{\sim}{\underset{\alpha_G}{\dashrightarrow}} & G_k^{\Sigma_{\text{Gal}}}, \end{array}$$

where the lower horizontal arrow α_G is an isomorphism, as well as a bijection

$$\mathcal{D}_n \xrightarrow{\sim} \mathcal{D}_{X_n} \overset{\text{def}}{=} \{D \subseteq \Pi_{U_{X_n}} \mid D \text{ is a decomposition group associated to some } x \in X_n(K)\}.$$

Remark 4.3. Let p be a prime number. Then, in the situation of Definition 4.2, if one takes $(g, r^{\square}, n, \Sigma_\Delta, \Sigma_{\text{Gal}})$ to be $(0, r^{\text{ord}}, n, \{p\}, \mathfrak{Primes})$, then one verifies immediately (cf. [MzTa], Proposition 2.2, (i)) that the natural outer action $G_k \overset{\text{out}}{\curvearrowright} \pi_1(U_{X_n} \times_k \bar{k})^{\Sigma_\Delta}$ factors through the natural surjection $G_k \twoheadrightarrow G_k^{(p)}$ if and only if k contains a primitive p-th root of unity.

Definition 4.4. Let ${}^\dagger\mathscr{B} = ({}^\dagger\Pi_n, {}^\dagger G, {}^\dagger\mathcal{D}_n)$, ${}^\ddagger\mathscr{B} = ({}^\ddagger\Pi_n, {}^\ddagger G, {}^\ddagger\mathcal{D}_n)$ be PGCS-collections of type $(g, r^{\square}, n, \Sigma_\Delta, \Sigma_{\text{Gal}})$. We shall refer to $\beta = (\beta_\Pi, \beta_G, \beta_{\mathcal{D}})\colon {}^\dagger\mathscr{B} \xrightarrow{\sim} {}^\ddagger\mathscr{B}$ as an *isomorphism of PGCS-collections* if β_Π is an isomorphism of profinite groups ${}^\dagger\Pi_n \xrightarrow{\sim} {}^\ddagger\Pi_n$ such that β_Π induces a commutative diagram of homomorphisms of profinite groups

$$\begin{array}{ccc} {}^\dagger\Pi_n & \xrightarrow[\beta_\Pi]{\sim} & {}^\ddagger\Pi_n \\ \downarrow & \circlearrowleft & \downarrow \\ {}^\dagger G & \overset{\sim}{\underset{\beta_G}{\dashrightarrow}} & {}^\ddagger G, \end{array}$$

where ${}^\dagger\Pi_n \to {}^\dagger G$, ${}^\ddagger\Pi_n \to {}^\ddagger G$ are the natural quotient homomorphisms, and $\beta_G\colon {}^\dagger G \xrightarrow{\sim} {}^\ddagger G$ is an isomorphism, as well as a bijection

$${}^\dagger\mathcal{D}_n \overset{\sim}{\underset{\beta_{\mathcal{D}}}{\dashrightarrow}} {}^\ddagger\mathcal{D}_n.$$

Definition 4.5. Let $\mathscr{B} = (\Pi_n, G, \mathcal{D}_n)$ be a PGCS-collection of type $(g, r^{\square}, n, \Sigma_\Delta, \Sigma_{\text{Gal}})$. Then we shall write $\text{Aut}(\mathscr{B})$ for the group of automorphisms of $\mathscr{B}$, $\text{Aut}_G(\mathscr{B}) \subseteq \text{Aut}(\mathscr{B})$ for the subgroup of automorphisms of $\mathscr{B}$ lying over the identity automorphism of G,

$$\text{Out}(\mathscr{B})$$

for the group of equivalence classes of automorphisms of the PGCS-collection $\mathscr{B}$ with respect to the equivalence relation given by composition with an inner automorphism arising from Π_n, and

$$\text{Out}_G(\mathscr{B})$$

for the quotient of $\text{Aut}_G(\mathscr{B})$ by the normal subgroup of inner automorphisms arising from $\text{Ker}(\Pi_n \to G)$.

Definition 4.6. Let ${}^\dagger\mathscr{B} = ({}^\dagger\Pi_n, G, {}^\dagger\mathcal{D}_n)$, ${}^\ddagger\mathscr{B} = ({}^\ddagger\Pi_n, G, {}^\ddagger\mathcal{D}_n)$ be PGCS-collections of type $(g, r^{\square}, n, \Sigma_\Delta, \Sigma_{\text{Gal}})$. Then we shall write

$$\text{Isom}({}^\dagger\mathscr{B}, {}^\ddagger\mathscr{B}) \subseteq \text{Isom}({}^\dagger\Pi_n, {}^\ddagger\Pi_n)$$

for the set of isomorphisms ${}^{\dagger}\mathscr{B} \xrightarrow{\sim} {}^{\ddagger}\mathscr{B}$ of PGCS-collections,

$$\mathrm{Isom}_G({}^{\dagger}\mathscr{B}, {}^{\ddagger}\mathscr{B}) \stackrel{\mathrm{def}}{=} \mathrm{Isom}_G({}^{\dagger}\Pi_n, {}^{\ddagger}\Pi_n) \cap \mathrm{Isom}({}^{\dagger}\mathscr{B}, {}^{\ddagger}\mathscr{B}),$$

and

$$\mathrm{Isom}_G^{\mathrm{Out}}({}^{\dagger}\mathscr{B}, {}^{\ddagger}\mathscr{B})$$

for the image of $\mathrm{Isom}_G({}^{\dagger}\mathscr{B}, {}^{\ddagger}\mathscr{B})$ via the natural surjection

$$\mathrm{Isom}_G({}^{\dagger}\Pi_n, {}^{\ddagger}\Pi_n) \twoheadrightarrow \mathrm{Isom}_G^{\mathrm{Out}}({}^{\dagger}\Pi_n, {}^{\ddagger}\Pi_n)$$

(cf. Notation 1.11).

Proposition 4.7. *Let* $n \in \{1, 2\}$*;* Σ_Δ, Σ_{Gal} *sets of prime numbers such that* $\Sigma_\Delta \subseteq \Sigma_{\mathrm{Gal}}$*, and* Σ_Δ, Σ_{Gal} *are of cardinality* 1 *or equal to* $\mathfrak{Primes}$*;* $p \in \Sigma_\Delta$*;* k *a generalized sub-p-adic local field;* $\bar{k}$ *an algebraic closure of* k*;* $X^{\log}$ *a smooth log curve of type* $(0, 3^{\mathrm{ord}})$*. Write* $K \subseteq \bar{k}$ *for the maximal pro-*Σ_{Gal} *subextension of* $\bar{k}/k$ *and*

$$\Delta_{U_{X_n}} \stackrel{\mathrm{def}}{=} \pi_1(U_{X_n} \times_k \bar{k})^{\Sigma_\Delta}, \quad \Pi_{U_{X_n}} \stackrel{\mathrm{def}}{=} \begin{cases} \pi_1(U_{X_n})^{\Sigma_\Delta} & (\textit{if } \Sigma_\Delta = \Sigma_{\mathrm{Gal}}) \\ \pi_1(U_{X_n})^{[p]} & (\textit{if } \Sigma_\Delta \subsetneq \Sigma_{\mathrm{Gal}}). \end{cases}$$

For $x \in X_n(K)$*, let* $D_x \subseteq \Pi_{U_{X_n}}$ *be a decomposition group of* $\Pi_{U_{X_n}}$ *at* x*. Write* $[D_x]$ *for the* $\Delta_{U_{X_n}}$*-conjugacy class of* D_x*. If* $\Sigma_{\mathrm{Gal}} = \{p\}$*, suppose that* k *contains a primitive p-th root of unity. Then the following hold:*

(i) The natural morphism $U_{X_2} \times_k \bar{k} \to U_{X_2} \to \mathrm{Spec}(k)$ *induces an isomorphism*

$$\Delta_{U_{X_2}} \xrightarrow{\sim} \mathrm{Ker}(\Pi_{U_{X_2}} \twoheadrightarrow G_k^{\Sigma_{\mathrm{Gal}}}).$$

(ii) We shall refer to a subgroup of $\Delta_{U_{X_2}}$ *as a* **generalized fiber subgroup** *if it coincides with the subgroup* $\mathrm{Ker}(\Delta_{U_{X_2}} \to \Delta_{U_X})$ *associated to one of the* 30 *projection morphisms*

$$(U_{X_2} \xrightarrow{\sim})\ \mathcal{M}_{0,5} \to \mathcal{M}_{0,4}\ (\xrightarrow{\sim} U_X)$$

(cf. Proposition 2.1) given, up to pre-/post-composition with automorphisms of $\mathcal{M}_{0,5}$*,* $\mathcal{M}_{0,4}$ *(cf. Proposition 2.2, (i), (ii)), by forgetting one of the marked points (cf.* [HMM]*, Definition 2.1, (ii)). Then there exists a group-theoretic characterization of the set* $\mathcal{E}_{X_2}$ *of generalized fiber subgroups (cf.* [HMM]*, Theorem 2.5, (iv)).*

(iii) Let E *be a generalized fiber subgroup of* $\Delta_{U_{X_2}}$ *(cf. (ii)). Then there exists a projection morphism* $p_{2/1}^X \colon U_{X_2} \to U_X$ *as in (ii) that induces isomorphisms*

$$(\Delta_{U_{X_2}} \twoheadrightarrow)\ \Delta_{U_{X_2}}/E \xrightarrow{\sim} \Delta_{U_X}, \quad (\Pi_{U_{X_2}} \twoheadrightarrow)\ \Pi_{U_{X_2}}/E \xrightarrow{\sim} \Pi_{U_X}.$$

(iv) Write

$$\mathcal{D}_{X_n} \stackrel{\mathrm{def}}{=} \{D \subseteq \Pi_{U_{X_n}} \mid D \textit{ is a decomposition group associated to some } x \in X_n(K)\}$$

and $[\mathcal{D}_{X_n}]$ *for the set of* $\Delta_{U_{X_n}}$*-conjugacy classes of subgroups* $\in \mathcal{D}_{X_n}$*. Then the map*

$$X_n(K) \to [\mathcal{D}_{X_n}] \colon x \mapsto [D_x]$$

is bijective.

(v) Write $p_{2/1}^{\Pi} \colon \Pi_{U_{X_2}} \twoheadrightarrow \Pi_{U_X}$ *for the surjection of (iii). Then it holds that*

$$\mathcal{D}_X = \{C_{\Pi_{U_X}}(p_{2/1}^{\Pi}(D)) \mid D \in \mathcal{D}_{X_2}\}$$

(cf. Notation 1.10).

(vi) For each $x \in X(K)$, it holds that

$$x \in U_X(K) \iff D_x \cap \Delta_{U_X} = \{1\}.$$

(vii) The natural morphism

$$(S_5 \cong)\ \mathrm{Aut}_k(U_{X_2}) \to \mathrm{Out}_{G_k^{\Sigma_{\mathrm{Gal}}}}(\Pi_{U_{X_2}})$$

(cf. Notation 1.8; Proposition 2.2, (i)) is bijective.

(viii) Let $p^X_{2/1} \colon U_{X_2} \to U_X$ be a projection morphism as in (iii). Then the set of composite morphisms $p^X_{2/1} \circ \mathrm{Aut}_k(U_{X_2})$ coincides with the set of projection morphisms $U_{X_2} \to U_X$ as in (ii).

Proof. Assertion (i) follows from [MzTa], Proposition 2.2, (i). Assertions (ii), (iii) follow from [HMM], Theorem 2.5, (iv).

Next, we consider assertion (iv). Since the surjectivity of the map under consideration follows immediately from the various definitions involved, it suffices to verify injectivity. First, we consider the case where $\Sigma_{\mathrm{Gal}} = \mathfrak{Primes}$. Then assertion (iv) follows from [Topics], Theorem 4.12 (i.e., via the same argument as the argument applied in the proof of [LocAn], Theorem 19.1, to derive [LocAn], Theorem 19.1, from [LocAn], Theorem A — cf. also the Remark following the statement of [Topics], Theorem 4.12). Next, we consider the case where $\Sigma_{\mathrm{Gal}} = \{p\}$. Let $x \in X_n(K)$ and $D^{[p]}_x \subseteq \pi_1(U_{X_n})^{[p]}, D^{(p)}_x \subseteq \Pi_{U_{X_2}} = \pi_1(U_{X_n})^{(p)}$ decomposition groups associated to x such that $(\pi_1(U_{X_n})^{[p]} \to \pi_1(U_{X_n})^{(p)})(D^{[p]}_x) = D^{(p)}_x$. Then one verifies easily that the composite morphism $D^{[p]}_x \subseteq \pi_1(U_{X_n})^{[p]} \to G_k$ is injective, and that the image $\mathrm{Im}(D^{[p]}_x)$ of this composite morphism is $G_{\kappa(x)} \subseteq G_k$, where $\kappa(x)$ denotes the residue field of $x \in X_n$. Since $K \supseteq \kappa(x)$, and

$$\mathrm{Ker}(\mathrm{Im}(D^{[p]}_x) \twoheadrightarrow \mathrm{Im}(D^{(p)}_x)) = \mathrm{Im}(D^{[p]}_x) \cap \mathrm{Ker}(G_k \twoheadrightarrow G^{(p)}_k),$$

where $\mathrm{Im}(D^{[p]}_x)$ denotes the image of the composite morphism $D^{(p)}_x \subseteq \pi_1(U_{X_n})^{(p)} \to G^{(p)}_k$, it holds that

$$\mathrm{Im}(D^{[p]}_x) \cap \mathrm{Ker}(G_k \twoheadrightarrow G^{(p)}_k) = G_{\kappa(x)} \cap G_K = G_K = \mathrm{Ker}(G_k \twoheadrightarrow G^{(p)}_k).$$

Thus, the composite morphism $D^{(p)}_x \subseteq \pi_1(U_{X_n})^{(p)} \to G^{(p)}_k$ is injective, and we have a natural isomorphism, together with equalities of subgroups, as follows:

$$\begin{aligned} \mathrm{Ker}(\pi_1(U_{X_n})^{[p]} \twoheadrightarrow \pi_1(U_{X_n})^{(p)}) &= \mathrm{Ker}(D^{[p]}_x \twoheadrightarrow D^{(p)}_x) \\ \xrightarrow{\sim} \mathrm{Ker}(\mathrm{Im}(D^{[p]}_x) \twoheadrightarrow \mathrm{Im}(D^{(p)}_x)) &= \mathrm{Ker}(G_k \twoheadrightarrow G^{(p)}_k) = G_K. \end{aligned}$$

Now let $x, y \in X_n(K)$ be distinct elements. If it holds that $D^{(p)}_x = D^{(p)}_y$ (where we use similar notation for "y" to the notation already introduced for "x"), then it holds that

$$\begin{aligned} D^{[p]}_y &\subseteq D^{[p]}_x \cdot \mathrm{Ker}(\pi_1(U_{X_n})^{[p]} \to \pi_1(U_{X_n})^{(p)}) \\ &= D^{[p]}_x \cdot \mathrm{Ker}(D^{[p]}_x \to D^{(p)}_x) = D^{[p]}_x, \end{aligned}$$

and hence, by symmetry, that $D^{[p]}_x = D^{[p]}_y$. Thus, we conclude that $x = y$ by applying assertion (v) in the case where "$\Sigma_{\mathrm{Gal}} = \mathfrak{Primes}$" (which has already been verified). This completes the proof of assertion (iv). Assertion (v) follows immediately from assertion (iv). Assertion (vi) follows immediately from the various definitions involved (cf. also, e.g., [CmbGC], Remark 1.1.3). Assertion (vii) follows immediately, in light of assertion (ii), from [Topics], Theorem 4.12 (applied successively to the second and first arrows of the composite morphisms $U_{X_2} \to U_X \to \mathrm{Spec}(k)$ arising from the natural projections). Assertion (viii) follows immediately from the various definitions involved. $\square$

Definition 4.8. Let Σ_Δ, Σ_{Gal} be sets of prime numbers such that $\Sigma_\Delta \subseteq \Sigma_{\mathrm{Gal}}$, and Σ_Δ, Σ_{Gal} are of cardinality 1 or equal to $\mathfrak{Primes}$; $\mathscr{B} = (\Pi_2, G, \mathcal{D}_2)$ a PGCS-collection of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$.

(i) Write $\Pi_2 \to G$ for the natural quotient homomorphism (cf. Definition 4.2), $\Delta_2[\mathscr{B}] \overset{\mathrm{def}}{=} \mathrm{Ker}(\Pi_2 \to G)$, and
$$A[\mathscr{B}]$$
for the set of $\Delta_2[\mathscr{B}]$-conjugacy classes of subgroups $\in \mathcal{D}_2$.

(ii) There exists a group-theoretic characterization of the set $\mathcal{E}_2[\mathscr{B}]$ of generalized fiber subgroups $\subseteq \Pi_2$ (cf. Proposition 4.7, (ii)). Let E be a generalized fiber subgroup of Π_2. Write $\Delta_1[\mathscr{B}, E] \overset{\mathrm{def}}{=} \Delta_2[\mathscr{B}]/E$, $\Pi_1[\mathscr{B}, E] \overset{\mathrm{def}}{=} \Pi_2/E$, and $p^{\Pi}_{2/1}[\mathscr{B}, E] \colon \Pi_2 \to \Pi_1[\mathscr{B}, E]$ for the natural quotient homomorphism.

(iii) Write $\mathcal{D}_1[\mathscr{B}, E] \overset{\mathrm{def}}{=} \{C_{\Pi_1[\mathscr{B},E]}(p^{\Pi}_{2/1}[\mathscr{B}, E](D)) \mid D \in \mathcal{D}_2\}$ and
$$B[\mathscr{B}, E]$$
for the set of $\Delta_1[\mathscr{B}, E]$-conjugacy classes of subgroups $\in \mathcal{D}_1[\mathscr{B}, E]$. Thus, $B[\mathscr{B}, E]$ is equipped with a natural action by G $(= \Pi_1[\mathscr{B}, E]/\Delta_1[\mathscr{B}, E])$.

(iv) Write
$$\partial B[\mathscr{B}, E] \overset{\mathrm{def}}{=} \{[D] \in B[\mathscr{B}, E] \mid D \cap \Delta_1[\mathscr{B}, E] \neq \{1\}\} \subseteq B[\mathscr{B}, E],$$
where $D \in \mathcal{D}_1[\mathscr{B}, E]$, and $[D]$ denotes the $\Delta_1[\mathscr{B}, E]$-conjugacy class of $D \in \mathcal{D}_1[\mathscr{B}, E]$.

(v) Write
$$H[\mathscr{B}] \subseteq \mathrm{Aut}(A[\mathscr{B}])$$
for the group of bijections $A[\mathscr{B}] \xrightarrow{\sim} A[\mathscr{B}]$ induced by the group of $\Delta_2[\mathscr{B}]$-outer automorphisms (i.e., equivalence classes of automorphisms, relative to the equivalence relation given by composition with inner automorphisms arising from elements of $\Delta_2[\mathscr{B}]$) of the profinite group Π_2 lying over the identity automorphism of G.

(vi) Write $p^{A/B}[\mathscr{B}, E] \colon A[\mathscr{B}] \to B[\mathscr{B}, E]$ for the map induced by the quotient homomorphism $p^{\Pi}_{2/1}[\mathscr{B}, E] \colon \Pi_2 \to \Pi_1[\mathscr{B}, E]$ and
$$M[\mathscr{B}, E]$$
for the $H[\mathscr{B}]$-orbit of $p^{A/B}[\mathscr{B}, E]$, relative to the tautological action of $H[\mathscr{B}]$ on $A[\mathscr{B}]$ (i.e., the domain of $p^{A/B}[\mathscr{B}, E]$).

Theorem 4.9. (From PGCS-collections of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ to CFS-collections to base fields) *We maintain the following notation of Definition 4.2: $(g, r^{\square}, n, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$; $(\Pi_n, G, \mathcal{D}_n)$; k; $\bar{k}$; $G_k^{\Sigma_{\mathrm{Gal}}}$; $X^{\log}$; K; $\alpha \colon \Pi_n \xrightarrow{\sim} \Pi_{U_{X_n}}$; $\alpha_G \colon G \xrightarrow{\sim} G_k^{\Sigma_{\mathrm{Gal}}}$. Write $\mathscr{B} \overset{\mathrm{def}}{=} (\Pi_n, G, \mathcal{D}_n)$. Suppose that $(g, r^{\square}, n) = (0, 3^{\mathrm{ord}}, 2)$. Let E be a generalized fiber subgroup of Π_2 (cf. Definition 4.8, (ii)). Then:*

(i) The isomorphism $\alpha \colon \Pi_2 \xrightarrow{\sim} \Pi_{U_{X_2}}$ induces

(a) bijections (the latter two of which are compatible)

$$A[\mathscr{B}] \xrightarrow{\sim} X_2(K), \quad B[\mathscr{B}, E] \xrightarrow{\sim} X(K), \quad \partial B[\mathscr{B}, E] \xrightarrow{\sim} X(K) \setminus U_X(K),$$

(b) a group isomorphism $H[\mathscr{B}] \xrightarrow{\sim} \mathrm{Aut}_k(U_{X_2})$,

(c) a bijection

$$M[\mathscr{B}, E] \xrightarrow{\sim} \{\textit{the maps } X_2(K) \to X(K) \textit{ induced by projection morphisms } U_{X_2} \twoheadrightarrow U_X \textit{ as in Proposition 4.7, (ii)}\}.$$

(ii) The above collection of data

$$\mathscr{A}[\mathscr{B}, E] \stackrel{\mathrm{def}}{=} (A[\mathscr{B}], B[\mathscr{B}, E], \partial B[\mathscr{B}, E], H[\mathscr{B}], M[\mathscr{B}, E])$$

is a CFS-collection. In particular, one may construct a CFS-collection $\mathscr{A}[\mathscr{B}, E]$ *associated to the intrinsic structure of the following collection of data: the PGCS-collection* $\mathscr{B}$ *of type* $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ *and the generalized fiber subgroup* $E \subseteq \Pi_2$.

(iii) Let $\phi\colon H[\mathscr{B}] \xrightarrow{\sim} S_5$ *be an isomorphism and* $\lambda \in M[\mathscr{B}, E]_1[\phi] \subseteq M[\mathscr{B}, E]$ *(cf. (ii); Definition 3.5). Write* $\beta\colon B[\mathscr{B}, E] \xrightarrow{\sim} X(K)$ *for the second bijection of (i), (a). Such an isomorphism* ϕ *and element* $\lambda \in M[\mathscr{B}, E]_1[\phi]$ *determine elements* $0[\phi, \lambda]$, $1[\phi, \lambda]$, $\infty[\phi, \lambda] \in \partial B[\mathscr{B}, E] \subseteq B[\mathscr{B}, E]$ *(cf. Definition 3.8). Then the bijection* $B[\mathscr{B}, E] \xrightarrow{\sim} X(K) \xrightarrow{\sim} K \cup \{\infty\}$ *given by the composite*

$$t_{\beta(0[\phi,\lambda]),\beta(1[\phi,\lambda]),\beta(\infty[\phi,\lambda])} \circ \beta$$

(cf. (i), (a); Propositions 2.1, 2.8) determines a field isomorphism

$$F[\mathscr{A}[\mathscr{B}, E], \phi, \lambda] \xrightarrow{\sim} K$$

(cf. Definition 3.12; Theorem 3.13, (i)) that is equivariant with respect to the respective natural actions of the profinite groups G, $G_k^{\Sigma_{\mathrm{Gal}}}$, *relative to the isomorphism* $\alpha_G : G \xrightarrow{\sim} G_k^{\Sigma_{\mathrm{Gal}}}$ *(cf. Definition 4.8, (iii)).*

Proof. Assertion (i) follows immediately from Proposition 4.7, (iv), (v), (vi), (vii), (viii), and Definition 4.8, (i), (iii), (iv), (v), (vi) (cf. also, in the case of (b), Proposition 2.6, (ii)). Assertion (ii) follows immediately from assertion (i) and Definition 3.2. Assertion (iii) follows from Theorem 3.13, (i), (ii). □

5. Construction of a function field for a tripod associated to a PGCS-collection

In the present §5, we construct a certain function field associated to the intrinsic structure of a PGCS-collection of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ (cf. Theorem 5.2 below).

Theorem 5.1. (Review of known results) *Let* $\Sigma_\Delta, \Sigma_{\mathrm{Gal}}$ *be sets of prime numbers such that* $\Sigma_\Delta \subseteq \Sigma_{\mathrm{Gal}}$, *and* $\Sigma_\Delta, \Sigma_{\mathrm{Gal}}$ *are of cardinality* 1 *or equal to* $\mathfrak{Primes}$; $p \in \Sigma_\Delta$; k *a number field or a p-adic local field;* $\bar{k}$ *an algebraic closure of* k; $X^{\log}$ *a smooth log curve over* k *of type* $(0, 3^{\mathrm{ord}})$; Π_2^{prf} *a profinite group which is isomorphic to the étale fundamental group* $\pi_1(U_{X_2})$ *(relative to a suitable choice of basepoint). Write* $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\bar{k}/k)$. *Then:*

(i) *One may construct a surjection* $\Pi_2^{\mathrm{prf}} \twoheadrightarrow G[\Pi_2^{\mathrm{prf}}]$ *associated to the intrinsic structure of the profinite group* Π_2^{prf} *such that the following property is satisfied: Any isomorphism* $\Pi_2^{\mathrm{prf}} \overset{\sim}{\to} \pi_1(U_{X_2})$ *of profinite groups induces a commutative diagram*

$$\begin{array}{ccc} \Pi_2^{\mathrm{prf}} & \xrightarrow{\sim} & \pi_1(U_{X_2}) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ G[\Pi_2^{\mathrm{prf}}] & \xrightarrow{\sim} & G_k, \end{array}$$

where $\pi_1(U_{X_2}) \twoheadrightarrow G_k$ *denotes the natural surjection;* $\Pi_2^{\mathrm{prf}} \overset{\sim}{\to} \pi_1(U_{X_2})$ *denotes the given isomorphism;* $G[\Pi_2^{\mathrm{prf}}] \overset{\sim}{\to} G_k$ *denotes a uniquely determined isomorphism.*

(ii) *One may construct a set of subgroups* $\mathcal{D}_2[\Pi_2^{\mathrm{prf}}]$ *of* Π_2^{prf} *associated to the intrinsic structure of the profinite group* Π_2^{prf} *such that the following property is satisfied: Any isomorphism* $\Pi_2^{\mathrm{prf}} \overset{\sim}{\to} \pi_1(U_{X_2})$ *of profinite groups induces a bijection*

$$\mathcal{D}_2[\Pi_2^{\mathrm{prf}}] \overset{\sim}{\to} \mathcal{D}_{X_2}^{\mathrm{prf}} \overset{\mathrm{def}}{=} \{D \subseteq \pi_1(U_{X_2}) \mid D \textit{ is a decomposition group associated to some } x \in X_2(\bar{k})\}.$$

(iii) *One may construct a PGCS-collection*

$$(\Pi_2^{\mathrm{prf}}, G[\Pi_2^{\mathrm{prf}}], \mathcal{D}_2[\Pi_2^{\mathrm{prf}}])$$

of type $(0, 3^{\mathrm{ord}}, 2, \mathfrak{Primes}, \mathfrak{Primes})$ *associated to the intrinsic structure of the profinite group* Π_2^{prf}.

(iv) *If* $\Sigma_{\mathrm{Gal}} = \{p\}$, *then we suppose further that* k *contains a primitive* p*-th root of unity (cf. Remark 4.3). Write*

$$\Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}] \overset{\mathrm{def}}{=} \begin{cases} \Pi_2^{\mathrm{prf},\Sigma_\Delta} & (\textit{if } \Sigma_\Delta = \Sigma_{\mathrm{Gal}}) \\ \Pi_2^{\mathrm{prf},[p]} & (\textit{if } \Sigma_\Delta \subsetneq \Sigma_{\mathrm{Gal}}) \end{cases}$$

(cf. (i); Notation 4.1),

$$\mathcal{D}_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}] \overset{\mathrm{def}}{=} (\Pi_2^{\mathrm{prf}} \twoheadrightarrow \Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}])(\mathcal{D}_2[\Pi_2^{\mathrm{prf}}]),$$

and $G^{\Sigma_{\mathrm{Gal}}}[\Pi_2^{\mathrm{prf}}]$ *for the maximal pro-*Σ_{Gal} *quotient of* $G[\Pi_2^{\mathrm{prf}}]$. *Then one may construct a PGCS-collection*

$$\mathscr{B}[\Pi_2^{\mathrm{prf}}] \overset{\mathrm{def}}{=} (\Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}], G^{\Sigma_{\mathrm{Gal}}}[\Pi_2^{\mathrm{prf}}], \mathcal{D}_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}])$$

of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ *associated to the intrinsic structure of the profinite group* Π_2^{prf}.

(v) *One may construct a collection* $\mathcal{E}_2[\Pi_2^{\mathrm{prf}}]$ *of ("generalized fiber") subgroups of* Π_2^{prf} *associated to the intrinsic structure of the profinite group* Π_2^{prf} *such that the following property is satisfied: Any isomorphism* $\Pi_2^{\mathrm{prf}} \overset{\sim}{\to} \pi_1(U_{X_2})$ *of profinite groups induces a bijection*

$$\mathcal{E}_2[\Pi_2^{\mathrm{prf}}] \overset{\sim}{\to} \mathcal{E}_{X_2} \overset{\mathrm{def}}{=} \{\textit{generalized fiber subgroups of } \pi_1(U_{X_2})\}$$

(cf. Proposition 4.7, (ii)). For $E \in \mathcal{E}_2[\Pi_2^{\mathrm{prf}}]$, *write* $\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] \overset{\mathrm{def}}{=} \Pi_2^{\mathrm{prf}}/E$.

(vi) *Let $E \in \mathcal{E}_2[\Pi_2^{\mathrm{prf}}]$. Then one may construct a field $F_1[\Pi_2^{\mathrm{prf}}, E]$ equipped with an action by $\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E]$ associated to the intrinsic structure of the profinite group $\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E]$ such that the following property is satisfied: Any isomorphism $\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] \xrightarrow{\sim} \pi_1(U_X)$ of profinite groups induces an isomorphism*

$$F_1[\Pi_2^{\mathrm{prf}}, E] \xrightarrow{\sim} \mathrm{Fnct}(Z),$$

where Z denotes the pro-finite étale covering of U_X corresponding to $\pi_1(U_X)$ (so $\pi_1(U_X) = \mathrm{Gal}(Z/U_X)$); $\mathrm{Fnct}(Z)$ denotes the function field of Z; the isomorphism "$\xrightarrow{\sim}$" is equivariant with respect to the respective actions of the profinite groups $\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E]$, $\pi_1(U_X)$.

(vii) *Let $E \in \mathcal{E}_2[\Pi_2^{\mathrm{prf}}]$. Then one may construct a subfield*

$$\bar{k}[\Pi_2, E] \subseteq F_1[\Pi_2^{\mathrm{prf}}, E]$$

equipped with an action by $\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] \twoheadrightarrow G[\Pi_2^{\mathrm{prf}}]$ associated to the intrinsic structure of the profinite group $\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E]$ such that the following property is satisfied: Any isomorphism $\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] \xrightarrow{\sim} \pi_1(U_X)$ of profinite groups induces a commutative diagram

$$\begin{array}{ccc} F_1[\Pi_2^{\mathrm{prf}}, E] & \xrightarrow{\sim} & \mathrm{Fnct}(Z) \\ \cup & & \cup \\ \bar{k}[\Pi_2^{\mathrm{prf}}, E] & \xrightarrow{\sim} & \bar{k}, \end{array}$$

where the horizontal arrows are isomorphisms; the vertical arrows are the natural inclusions, i.e., $\bar{k} \subseteq \mathrm{Fnct}(Z)$ is the subfield of constant functions; the isomorphisms "$\xrightarrow{\sim}$" are equivariant with respect to the respective actions of the profinite groups $\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] \twoheadrightarrow G[\Pi_2^{\mathrm{prf}}]$, $\pi_1(U_X) \twoheadrightarrow G_k$.

Proof. Assertion (i) follows from [AbsTpI], Theorem 2.6, (ii), (iii), (iv), (v), (vi). Assertion (ii) follows, by applying Proposition 4.7, (ii); [CmbGC], Corollary 2.7, (i); [NodNon], Theorem A; [NodNon], Remark 2.4.2, from [AbsTpIII], Theorem 1.9, (a); [AbsTpIII], Corollary 1.10, (e), applied successively to the morphisms induced on étale fundamental groups by the composite morphism $U_{X_2} \to U_X \to \mathrm{Spec}(k)$ (where the first arrow is a projection morphism as in Proposition 4.7, (ii)), i.e., by the second arrow and the fibers over closed points of the first arrow. Assertion (iii) follows from assertions (i), (ii). Assertion (iv) follows immediately from assertion (iii). Assertion (v) follows from Proposition 4.7, (ii). Assertions (vi), (vii) follow from [AbsTpIII], Theorem 1.9, (e); [AbsTpIII], Corollary 1.10, (h). □

Theorem 5.2. (From PGCS-collections of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ to function fields of tripods) *We maintain the following notation of Definition 4.2: $(g, r^\square, n, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$; $(\Pi_n, G, \mathcal{D}_n)$; $p \in \Sigma_\Delta$; k; $\bar{k}$; $G_k^{\Sigma_{\mathrm{Gal}}}$; $X^{\log}$; K; $\alpha\colon \Pi_n \xrightarrow{\sim} \Pi_{U_{X_n}}$; $\mathcal{D}_{X_n}$. Let Π_2^{prf} be a profinite group which is isomorphic to the étale fundamental group $\Pi_{U_{X_2}}^{\mathrm{prf}} \overset{\mathrm{def}}{=} \pi_1(U_{X_2})$ (relative to a suitable choice of basepoint). Write $\mathscr{B} \overset{\mathrm{def}}{=} (\Pi_n, G, \mathcal{D}_n)$. Suppose that $(g, r^\square, n) \overset{\mathrm{def}}{=} (0, 3^{\mathrm{ord}}, 2)$. Then:*

(i) *Let $E_{\mathscr{B}} \in \mathcal{E}_2[\mathscr{B}]$ (cf. Definition 4.8, (ii)), $\phi\colon H[\mathscr{B}] \xrightarrow{\sim} S_5$ an isomorphism, and $\lambda \in M[\mathscr{B}, E_{\mathscr{B}}]_1[\phi] \subseteq M[\mathscr{B}, E_{\mathscr{B}}]$. Then one may construct from the PGCS-collection $\mathscr{B}$ a collection of isomorphisms between the fields $F[\mathscr{A}[\mathscr{B}, E_{\mathscr{B}}], \phi, \lambda]$*

associated to any two choices of the data $(E_{\mathscr{B}}, \phi, \lambda)$ *that is compatible with composition, i.e., satisfies the "cocycle condition" that arises when one considers three choices of the data* $(E_{\mathscr{B}}, \phi, \lambda)$. *In particular, one may construct*

- *a field* $K[\mathscr{B}] \overset{\text{def}}{=} F[\mathscr{A}[\mathscr{B}, E_{\mathscr{B}}], \phi, \lambda]$ *equipped with a natural action by* G *(cf. Theorem 4.9, (iii)),*
- $k[\mathscr{B}] \overset{\text{def}}{=} K[\mathscr{B}]^G$

associated to the intrinsic structure of the PGCS-collection $\mathscr{B}$, *i.e., which is independent of the choice of data* $(E_{\mathscr{B}}, \phi, \lambda)$.

(ii) Suppose that k *is a number field or a* p*-adic local field. Then there exists an isomorphism of PGCS-collections* $\mathscr{B} \overset{\sim}{\to} \mathscr{B}[\Pi_2^{\mathrm{prf}}]$. *In particular, there exists an isomorphism*

$$\Pi_2 \overset{\sim}{\to} \Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}].$$

Let $E \in \mathcal{E}_2[\Pi_2^{\mathrm{prf}}]$ *(cf. Theorem 5.1, (v)) and* $\beta\colon \mathscr{B} \overset{\sim}{\to} \mathscr{B}[\Pi_2^{\mathrm{prf}}]$ *an isomorphism of PGCS-collections. Then the isomorphism* $\beta\colon \mathscr{B} \overset{\sim}{\to} \mathscr{B}[\Pi_2^{\mathrm{prf}}]$ *induces a commutative diagram*

$$\begin{array}{ccccc}
\Pi_2^{\mathrm{prf}} & \twoheadrightarrow & \Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}] & \overset{\sim}{\leftarrow} & \Pi_2 \\
\downarrow & & & & \downarrow \\
\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] & & \twoheadrightarrow & & \Pi_1[\mathscr{B}, E|_{\Pi_2}] \\
\downarrow & & & & \downarrow \\
G[\Pi_2^{\mathrm{prf}}] & & \twoheadrightarrow & & G,
\end{array}$$

where $\Pi_2 \overset{\sim}{\to} \Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}]$ *denotes the isomorphism determined by* β; $\Pi_2^{\mathrm{prf}} \twoheadrightarrow \Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}]$ *denotes the natural surjection (cf. Theorem 5.1, (iv));* $E|_{\Pi_2} \subseteq \Pi_2$ *denotes the generalized fiber subgroup of* Π_2 *given by forming the image of* E *via the composite of arrows* $\Pi_2^{\mathrm{prf}} \twoheadrightarrow \Pi_2^{\twoheadrightarrow}[\Pi_2^{\mathrm{prf}}] \overset{\sim}{\leftarrow} \Pi_2$ *in the upper line of the diagram; the arrows* $\Pi_2^{\mathrm{prf}} \twoheadrightarrow \Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] \twoheadrightarrow G[\Pi_2^{\mathrm{prf}}]$ *denote the natural surjections (cf. Theorem 5.1, (i), (v)); the arrows* $\Pi_2 \twoheadrightarrow \Pi_1[\mathscr{B}, E|_{\Pi_2}] \twoheadrightarrow G$ *denote the natural surjections (cf. Definition 4.8, (i), (ii));* $\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] \twoheadrightarrow \Pi_1[\mathscr{B}, E|_{\Pi_2}]$, $G[\Pi_2^{\mathrm{prf}}] \twoheadrightarrow G$ *denote the unique surjections that render the diagram commutative. In particular, we obtain a field*

$$F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] \overset{\text{def}}{=} F_1[\Pi_2^{\mathrm{prf}}, E]^{\mathrm{Ker}(\Pi_1^{\mathrm{prf}}[\Pi_2^{\mathrm{prf}}, E] \twoheadrightarrow \Pi_1[\mathscr{B}, E|_{\Pi_2}])}$$

equipped with a natural action by $(\Pi_2 \twoheadrightarrow)\ \Pi_1[\mathscr{B}, E|_{\Pi_2}]$ *(cf. Theorem 5.1, (vi)).*

(iii) In the notation of (ii), one may construct a field $F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta]$ *(cf. (ii)) equipped with an action by* Π_2 *associated to the intrinsic structure of the following collection of data:*

- *the PGCS-collection* $\mathscr{B}$;
- *a profinite group* Π_2^{prf} *isomorphic to* $\Pi_{U_{X_2}}^{\mathrm{prf}}$;
- $E \in \mathcal{E}_2[\Pi_2^{\mathrm{prf}}]$;
- *an isomorphism* $\beta\colon \mathscr{B} \overset{\sim}{\to} \mathscr{B}[\Pi_2^{\mathrm{prf}}]$;

such that if

$$\beta_X\colon \mathscr{B} \overset{\sim}{\to} \mathscr{B}[X] \overset{\text{def}}{=} (\Pi_{U_{X_2}}, G_k^{\Sigma_{\mathrm{Gal}}}, \mathcal{D}_{X_2})$$

is an isomorphism of PGCS-collections of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$*, then one may construct a field isomorphism*

$$F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] \overset{\sim}{\to} \mathrm{Fnct}(W)$$

associated to the intrinsic structure of the data $(\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \beta_X)$*, where* W *denotes the pro-finite étale covering of* U_X *corresponding to* Π_{U_X} *(so* $\Pi_{U_X} = \mathrm{Gal}(W/U_X)$*);* $\mathrm{Fnct}(W)$ *denotes the function field of* W*; the isomorphism "*$\overset{\sim}{\to}$*" is equivariant with respect to the respective natural actions of the profinite groups* $(\Pi_2 \twoheadrightarrow)\ \Pi_1[\mathscr{B}, E|_{\Pi_2}]$, Π_{U_X}.

(iv) *In the notation of (i), (ii), (iii), suppose that* $E_{\mathscr{B}} = E|_{\Pi_2}$*. Let* $\phi\colon H[\mathscr{B}] \overset{\sim}{\to} S_5$ *be an isomorphism,* $\lambda \in M[\mathscr{B}, E_{\mathscr{B}}]_1[\phi] \subseteq M[\mathscr{B}, E_{\mathscr{B}}]$*, and*

$$T \in F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta].$$

Then T *induces, by restriction to decomposition groups (cf. also Proposition 4.7, (iv)), a map*

$$T(-)\colon \mathcal{D}_1[\mathscr{B}, E_{\mathscr{B}}] \to K[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] \cup \{\infty\} \overset{\mathrm{def}}{=} \bar{k}[\Pi_2^{\mathrm{prf}}, E]^{\mathrm{Ker}(G[\Pi_2^{\mathrm{prf}}] \twoheadrightarrow G)} \cup \{\infty\}$$

(cf. (ii); Theorem 5.1, (vii)); there exists a unique element $T[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \phi, \lambda] \in F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta]^{\Pi_1[\mathscr{B}, E|_{\Pi_2}]}$ *such that the zero divisor of* $T[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \phi, \lambda]$ *is of degree 1 (cf.* [AbsTpIII]*, Proposition 1.6, (iii)) and supported on* $0[\phi, \lambda]$*,*

$$T[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \phi, \lambda](1[\phi, \lambda]) = 1 \in K[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta],$$

the divisor of poles of $T[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \phi, \lambda]$ *is of degree 1 (cf.* [AbsTpIII]*, Proposition 1.6, (iii)) and supported on* $\infty[\phi, \lambda]$ *(cf. Proposition 2.8). Moreover, the map*

$$T[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \phi, \lambda](-)\colon \mathcal{D}_1[\mathscr{B}, E_{\mathscr{B}}] \to K[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] \cup \{\infty\}$$

induces a field isomorphism

$$K[\mathscr{B}] \overset{\sim}{\to} K[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta],$$

where the isomorphism "$\overset{\sim}{\to}$*" is equivariant with respect to the respective natural actions of* G.

(v) *In the notation of (i), (iii), (iv) (cf. also, Theorem 5.1, (vii)), the isomorphism* $\beta_X : \mathscr{B} \overset{\sim}{\to} \mathscr{B}[X]$ *induces a commutative diagram*

$$\begin{array}{ccccc} & & F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] & \overset{\sim}{\longrightarrow} & \mathrm{Fnct}(W) \\ & & \cup & & \cup \\ K[\mathscr{B}] & \overset{\sim}{\longrightarrow} & K[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] & \overset{\sim}{\longrightarrow} & K \end{array}$$

associated to the intrinsic structure of the data $(\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta, \beta_X)$*, where the horizontal arrows are the isomorphisms discussed so far in (iii), (iv), and Theorem 5.1, (vii); the* $\cup$*'s are the natural inclusions.*

Proof. First, we consider assertion (i). Let ${}^{\dagger}E, {}^{\ddagger}E \in \mathcal{E}_2[\mathscr{B}]$, ${}^{\dagger}\phi, {}^{\ddagger}\phi\colon H[\mathscr{B}] \overset{\sim}{\to} S_5$ isomorphisms, ${}^{\dagger}\lambda \in M[\mathscr{B}, {}^{\dagger}E]_1[{}^{\dagger}\phi] \subseteq M[\mathscr{B}, {}^{\dagger}E]$, ${}^{\ddagger}\lambda \in M[\mathscr{B}, {}^{\ddagger}E]_1[{}^{\ddagger}\phi] \subseteq M[\mathscr{B}, {}^{\ddagger}E]$. Consider the subset of $\mathrm{Out}_G(\mathscr{B})$ (cf. Definition 4.5)

$$\mathrm{Out}_G^*(\mathscr{B}) \overset{\mathrm{def}}{=} \{\sigma \in \mathrm{Out}_G(\mathscr{B}) \mid \ \sigma({}^{\dagger}E) = {}^{\ddagger}E\} \subseteq \mathrm{Out}_G(\mathscr{B}).$$

Let $\sigma \in \mathrm{Out}^*_G(\mathscr{B})$. Then σ induces an isomorphism of CFS-collections $\mathscr{A}[\mathscr{B}, {}^\dagger E] \xrightarrow{\sim} \mathscr{A}[\mathscr{B}, {}^\ddagger E]$ (cf. Definition 3.9; Theorem 4.9, (ii)), hence, in particular, a bijection $\sigma_B \colon B[\mathscr{B}, {}^\dagger E] \xrightarrow{\sim} B[\mathscr{B}, {}^\ddagger E]$ (cf. Definition 4.8, (iii)). By Theorem 3.13, (ii), σ_B induces a field isomorphism $\sigma_B|_F$

$$(B[\mathscr{B}, {}^\dagger E] \supseteq) \quad F[\mathscr{A}[\mathscr{B}, {}^\dagger E], {}^\dagger\phi, {}^\dagger\lambda] \xrightarrow{\sim} F[\mathscr{A}[\mathscr{B}, {}^\ddagger E], \sigma({}^\dagger\phi), \sigma({}^\dagger\lambda)] \quad (\subseteq B[\mathscr{B}, {}^\ddagger E]).$$

By Theorem 3.13, (iii), the data $(\mathscr{A}[\mathscr{B}, {}^\ddagger E], \sigma({}^\dagger\phi), \sigma({}^\dagger\lambda), {}^\ddagger\phi, {}^\ddagger\lambda)$ determines a field isomorphism

$$\sigma_F \colon F[\mathscr{A}[\mathscr{B}, {}^\ddagger E], \sigma({}^\dagger\phi), \sigma({}^\dagger\lambda)] \xrightarrow{\sim} F[\mathscr{A}[\mathscr{B}, {}^\ddagger E], {}^\ddagger\phi, {}^\ddagger\lambda].$$

Thus, the data $(\mathscr{B}, {}^\dagger E, {}^\dagger\phi, {}^\dagger\lambda, {}^\ddagger E, {}^\ddagger\phi, {}^\ddagger\lambda, \sigma)$ determines a field isomorphism

$$\sigma^* \stackrel{\mathrm{def}}{=} \sigma_F \circ \sigma_B|_F \colon F[\mathscr{A}[\mathscr{B}, {}^\dagger E], {}^\dagger\phi, {}^\dagger\lambda] \xrightarrow{\sim} F[\mathscr{A}[\mathscr{B}, {}^\ddagger E], {}^\ddagger\phi, {}^\ddagger\lambda].$$

One verifies easily that this construction is compatible with "composition of σ's" in the evident sense. In particular, (one verifies easily that) by applying these field isomorphisms "σ^*", to complete the proof of assertion (i), it suffices to verify the following assertions:

(a) $\mathrm{Out}_G(\mathscr{B})$ acts transitively on the set of generalized fiber subgroups of Π_2 (i.e., the set $\mathrm{Out}^*_G(\mathscr{B})$ is always nonempty).

(b) The field isomorphism "σ^*" is completely determined by the data

$$(\mathscr{B}, {}^\dagger E, {}^\dagger\phi, {}^\dagger\lambda, {}^\ddagger E, {}^\ddagger\phi, {}^\ddagger\lambda).$$

Now suppose that ${}^\dagger E = {}^\ddagger E$, ${}^\dagger\phi = {}^\ddagger\phi$, and ${}^\dagger\lambda = {}^\ddagger\lambda$. By Propositions 2.2, (i), (ii); 4.7, (ii), (vii), there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{Out}_G(\mathscr{B}) & \xrightarrow{\sim} & S_5 \\ \cup & & \cup \\ \mathrm{Out}^*_G(\mathscr{B}) & \xrightarrow{\sim} & S_4, \end{array}$$

where the horizontal arrows are isomorphism of groups; the vertical arrows are the natural inclusions. In particular, we conclude that assertion (a) holds. Since $\sigma \in \mathrm{Out}^*_G(\mathscr{B}) \xrightarrow{\sim} S_4$, it holds that

$$\sigma_B \in H[\mathscr{B}]_{B[\mathscr{B}, {}^\dagger E]} \quad (\xrightarrow{\sim} S_3)$$

(cf. Definitions 3.6; 4.8, (v)). Thus, (cf. Theorem 3.13, (iii)) the field isomorphism

$$\sigma^* \colon F[\mathscr{A}[\mathscr{B}, {}^\dagger E], {}^\dagger\phi, {}^\dagger\lambda] \xrightarrow{\sim} F[\mathscr{A}[\mathscr{B}, {}^\ddagger E], {}^\ddagger\phi, {}^\ddagger\lambda]$$

constructed above arises from an element $\in H[\mathscr{B}]_{B[\mathscr{B}, {}^\dagger E]}$ $(\xrightarrow{\sim} S_3)$. This implies that $\sigma^* = \mathrm{id}$. Assertion (b) then follows formally. This completes the proof of assertion (i). Assertion (ii) follows immediately from our construction.

Next, we consider assertion (iii). First, observe that the PGCS-collection $\mathscr{B}[X] = (\Pi_{U_{X_2}}, G_k^{\Sigma_{\mathrm{Gal}}}, \mathcal{D}_{X_2})$ may be naturally identified with the PGCS-collection $\mathscr{B}[\Pi^{\mathrm{prf}}_{U_{X_2}}]$ of Theorem 5.1, (iv). Let $\sigma \in \mathrm{Aut}(\mathscr{B}[\Pi^{\mathrm{prf}}_{U_{X_2}}])$ (cf. Definition 4.5). By assertion (i), σ induces an field isomorphism

$$(K \xleftarrow{\sim}) \quad K[\mathscr{B}[\Pi^{\mathrm{prf}}_{U_{X_2}}]] \xrightarrow{\sim} K[\mathscr{B}[\Pi^{\mathrm{prf}}_{U_{X_2}}]] \quad (\xrightarrow{\sim} K)$$

that is equivariant, relative to the isomorphism

$$(\mathrm{Gal}(K/k) =) \quad G_k^{\Sigma_{\mathrm{Gal}}} \xrightarrow{\sim} G_k^{\Sigma_{\mathrm{Gal}}} \quad (= \mathrm{Gal}(K/k))$$

induced by σ, with respect to the respective natural actions of the profinite groups. In particular, the isomorphism $G_k^{\Sigma_{\mathrm{Gal}}} \xrightarrow{\sim} G_k^{\Sigma_{\mathrm{Gal}}}$ induced by σ arises from an isomorphism of fields $K \xrightarrow{\sim} K$. Since the natural morphism

$$\mathrm{Aut}_k(U_{X_2}) \to \mathrm{Out}_{G_k^{\Sigma_{\mathrm{Gal}}}}(\mathscr{B}[\Pi^{\mathrm{prf}}_{U_{X_2}}])$$

is bijective (cf. Definition 4.5; Proposition 4.7, (vii)), we thus conclude that the natural morphism

$$\mathrm{Aut}(U_{X_2}) \to \mathrm{Out}(\mathscr{B}[\Pi^{\mathrm{prf}}_{U_{X_2}}])$$

is bijective. In particular, it follows that the composite isomorphism of PGCS-collections

$$\beta_X \circ \beta^{-1} \colon \mathscr{B}[\Pi_2^{\mathrm{prf}}] \xleftarrow{\sim} \mathscr{B} \xrightarrow{\sim} \mathscr{B}[X] = \mathscr{B}[\Pi^{\mathrm{prf}}_{U_{X_2}}]$$

is induced by some isomorphism of profinite groups

$$\Pi_2^{\mathrm{prf}} \xrightarrow{\sim} \Pi^{\mathrm{prf}}_{U_{X_2}}$$

which is unique up to composition with an inner automorphism arising from an element of $\mathrm{Ker}(\Pi^{\mathrm{prf}}_{U_{X_2}} \twoheadrightarrow \Pi_{U_{X_2}})$ (where we recall that $\Pi_{U_{X_2}}$ is *center-free* — cf. Remark 4.3; Lemma 5.3 below; [MzTa], Proposition 2.2, (ii)). Thus, by assertion (ii); Theorem 5.1, (iv), (vi), the isomorphism of PGCS-collections

$$\beta_X \colon \mathscr{B} \xrightarrow{\sim} \mathscr{B}[X]$$

determines an isomorphism of profinite groups

$$\Pi_1[\mathscr{B}, E|_{\Pi_2}] \xrightarrow{\sim} \Pi_{U_X}$$

and a compatible isomorphism of fields

$$F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E, \beta] \ \xrightarrow{\sim}\ \mathrm{Fnct}(W).$$

This complete the proof of assertion (iii). Assertions (iv), (v) follow from assertions (i), (iii); Theorem 5.1, (vii). ☐

Lemma 5.3. *Let $p \in \mathfrak{Primes}$, Σ_{Gal} a set of prime numbers such that $\Sigma_{\mathrm{Gal}} = \{p\}$ or equal to $\mathfrak{Primes}$, k a generalized sub-p-adic local field, and σ an automorphism of the field k. Suppose that σ induces the identity outer automorphism of $G_k^{\Sigma_{\mathrm{Gal}}}$. If $\Sigma_{\mathrm{Gal}} \neq \mathfrak{Primes}$, then suppose that k contains a primitive p-th root of unity (cf. Remark 4.3). Then σ is the identity automorphism of k. In particular, $G_k^{\Sigma_{\mathrm{Gal}}}$ is center-free.*

Proof. Suppose that σ is not the identity automorphism of k. Thus, there exists an element $\alpha \in k$ such that $\sigma(\alpha) \neq \alpha$. Let X be the complement of the points $0, 1, \infty, \alpha$ in the projective line $\mathbb{P}^1_k$. Thus, X is a hyperbolic curve over k. Let Y be the result of base-changing X by $\sigma \colon k \to k$. Thus, it follows that X and Y are isomorphic as schemes over $\mathbb{Q}$. Moreover, conjugating by σ defines an outer isomorphism $\pi_1(X)^{\Sigma_{\mathrm{Gal}}} \simeq \pi_1(Y)^{\Sigma_{\mathrm{Gal}}}$ (cf. Remark 4.3) which lies over the identity outer automorphism of $G_k^{\Sigma_{\mathrm{Gal}}}$ (cf. our assumption on σ). Thus, we obtain that this outer isomorphism arises from an k-isomorphism of X with Y (cf. [Topics], Theorem 4.12). But since conjugation by σ preserves the points $0, 1, \infty$ of $\mathbb{P}^1_k$, this implies that $\sigma(\alpha) = \alpha \in k$, a contradiction. The fact that $G_k^{\Sigma_{\mathrm{Gal}}}$ is center-free now follows by considering automorphisms of finite Galois extensions of k arising from open subgroups of $G_k^{\Sigma_{\mathrm{Gal}}}$ that arise from elements of the center of $G_k^{\Sigma_{\mathrm{Gal}}}$ (cf. the proof of [LocAn], Lemma 15.8). ☐

6. Construction of a function field associated to a PGCS-collection

In the present §6, we apply the theory developed thus far in the present paper to prove a semi-absolute bi-anabelian result (cf. Theorem 6.4 below) and also to construct a certain function field associated to the intrinsic structure of a PGCS-collection of type $(g, r^{\square}, n, \Sigma_{\Delta}, \Sigma_{\mathrm{Gal}})$ (cf. Theorem 6.6 below).

Definition 6.1. Let $n \in \mathbb{Z}_{>1}$; $\Sigma_{\Delta}, \Sigma_{\mathrm{Gal}}$ be sets of prime numbers such that $\Sigma_{\Delta} \subseteq \Sigma_{\mathrm{Gal}}$, and $\Sigma_{\Delta}, \Sigma_{\mathrm{Gal}}$ are of cardinality 1 or equal to $\mathfrak{Primes}$; $p \in \Sigma_{\Delta}$; $\square \in \{\mathrm{arb}, \mathrm{ord}\}$; k a generalized sub-p-adic local field; $\bar{k}$ an algebraic closure of k; $X^{\log}$ a smooth log curve over k of type $(g, r^{\square})$. Write K for the maximal pro-Σ_{Gal} subextension of $\bar{k}/k$. Then we shall say that $(X^{\log}, n)$ is *tripodally ample* (resp. *tripodally very ample*) if one of the following conditions (i), (ii), (iii) holds:

(i) $\sharp(X(K) \setminus U_X(K)) = 3$ (resp. $\sharp(X(k) \setminus U_X(k)) = 3$) and $(g, r^{\square}, n) = (0, 3^{\square}, 2)$;
(ii) $X(K) \setminus U_X(K) \neq \emptyset$ (resp. $X(k) \setminus U_X(k) \neq \emptyset$), $n \in \mathbb{Z}_{>2}$, and $r \neq 0$;
(iii) $U_X(K) \neq \emptyset$ (resp. $U_X(k) \neq \emptyset$) and $n \in \mathbb{Z}_{>3}$.

Remark 6.2. We maintain the following notation of Definition 6.1: $(g, r^{\square}, n, \Sigma_{\Delta}, \Sigma_{\mathrm{Gal}})$; $p \in \Sigma_{\Delta}$; k; $\bar{k}$; $X^{\log}$; K. Let $Y^{\log}$ be a smooth log curve over k of type $(0, 3^{\mathrm{ord}})$. Write $Y_2^{\log}$ for the second log configuration space associated to $Y^{\log} \to \mathrm{Spec}(k)$. Suppose that $(X^{\log}, n)$ is *tripodally very ample*. Then:

(i) If $n = 2$, then there exists an isomorphism of k-log schemes $X_2^{\log} \xrightarrow{\sim} Y_2^{\log}$.
(ii) If $n > 2$, $r \neq 0$, and $X(k) \setminus U_X(k) \neq \emptyset$, then there exist projections $X_n^{\log} \to X_3^{\log} \to X_2^{\log} \to X^{\log}$ given by forgetting the respective final factors. Denote the last two of these arrows by $p^X_{3/2} \colon X_3^{\log} \to X_2^{\log}$, $p^X_{2/1} \colon X_2^{\log} \to X^{\log}$. Write $V_{\mathrm{diag}} \subseteq X_2$ for the diagonal divisor, i.e., the strict transform of the diagonal divisor in $X \times X$, relative to the morphism $\iota \colon X_2 \to X \times X$ determined by the projections to the first and second factors. Let $c \in X(k) \setminus U_X(k)$. Then one verifies easily that
$$(p^X_{2/1})^{-1}(c) = V_Y \cup V_X,$$
where V_Y, V_X are log divisors of $X_2^{\log}$, V_Y is a (g, r)-divisor (cf. [Hgsh], Definition 3.1, (iii)) such that $V_Y \cap V_{\mathrm{diag}} \neq \emptyset$, and V_X is a tripodal divisor (cf. [Hgsh], Definition 3.1, (ii)) such that $V_X \cap V_{\mathrm{diag}} = \emptyset$. In particular, $V_Y \cap X_2^{\log \leq 1}$ (cf. Notation 1.2, (v)) is naturally isomorphic to U_Y, and $V_X \cap X_2^{\log \leq 1}$ is naturally isomorphic to U_X. Moreover, one verifies easily that
$$(p^X_{3/2})^{-1}(V_Y) = W_{YY} \cup W_{XY},$$
where W_{YY}, W_{XY} are log divisors of $X_3^{\log}$, and W_{YY} is a (g, r)-divisor. Here, we have natural isomorphisms as follows (cf. [Hgsh], Lemma 6.1, (ii), (iii)):
$$W_{YY} \cap X_3^{\log \leq 1} \xrightarrow{\sim} U_{Y_2}, \quad W_{YY} \xrightarrow{\sim} Y_2,$$
$$W_{XY} \cap X_3^{\log \leq 1} \xrightarrow{\sim} U_X \times U_Y, \quad W_{XY} \xrightarrow{\sim} X \times Y.$$
Finally, we observe that
$$W_{YY} = X_3 \times_{X \times X \times X} (c, c, c),$$
where $X_3 \to X \times X \times X$ denotes the morphism determined by the projections to the first, second, and third factors.

(iii) If $n > 3$, and $U_X(k) \neq \emptyset$, then there exist projections $X_n^{\log} \to X_4^{\log} \to X_3^{\log} \to X_2^{\log} \to X^{\log}$ given by forgetting the respective final factors. Denote the last three of these arrows by $p_{4/3}^X\colon X_4^{\log} \to X_3^{\log}$, $p_{3/2}^X\colon X_3^{\log} \to X_2^{\log}$, $p_{2/1}^X\colon X_2^{\log} \to X^{\log}$. Write $V_{\mathrm{diag}} \subseteq X_2$ for the diagonal divisor, i.e., the strict transform of the diagonal divisor in $X \times X$, relative to the morphism $\iota\colon X_2 \to X \times X$ determined by the projections to the first and second factors. Let $c \in U_X(k)$. Write $(c, c) \in V_{\mathrm{diag}} \subseteq X_2(k)$ for the unique element such that $\iota(c, c) = (c, c)$; $X_c^{\log} \stackrel{\mathrm{def}}{=} X_2^{\log} \times_X c$, where the morphism $X_2^{\log} \to X$ is the morphism determined by $p_{2/1}^X$. Then one verifies easily that

$$X_{c,n-1}^{\log} = X_n^{\log} \times_X c,$$

$(X_c^{\log}, n-1)$ is tripodally very ample, and

$$(p_{3/2}^X)^{-1}(c, c) = V_Y \cup V_{X_c}.$$

Here, the morphism $X_n^{\log} \to X$ is the morphism determined by the composite of the projections considered above; $X_{c,n-1}^{\log}$ denotes the $(n-1)$-th log configuration space associated to the smooth log curve $X_c^{\log} \to \mathrm{Spec}(k)$ of type $(g, (r+1)^{\square})$; V_Y, V_{X_c} are irreducible components of $(p_{3/2}^X)^{-1}(c, c)$; $V_Y \cap X_3^{\log\leq 1}$ is naturally isomorphic to U_Y; $V_{X_c} \cap X_3^{\log\leq 1}$ is naturally isomorphic to U_{X_c}. Moreover, one verifies easily that

$$(p_{4/3}^X)^{-1}(V_Y) = W_{YY} \cup W_{X_cY},$$

where W_{YY}, W_{X_cY} are irreducible components of $(p_{4/3}^X)^{-1}(V_Y)$, and we have natural isomorphisms as follows (cf. [Hgsh], Lemma 6.1, (ii), (iii)):

$$W_{YY} \cap X_4^{\log\leq 1} \xrightarrow{\sim} U_{Y_2}, \quad W_{YY} \xrightarrow{\sim} Y_2$$

$$W_{X_cY} \cap X_4^{\log\leq 1} \xrightarrow{\sim} U_{X_c} \times U_Y, \quad W_{X_cY} \xrightarrow{\sim} X_c \times Y.$$

Finally, we observe that

$$W_{YY} = X_4 \times_{X\times X\times X\times X} (c, c, c, c),$$

where $X_4 \to X \times X \times X \times X$ denotes the morphism determined by the projections to the first, second, third, and fourth factors.

Remark 6.3. We maintain the following notation of Definition 4.2: $(g, r^{\square}, n, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$; $(\Pi_n, G, \mathcal{D}_n)$; $p \in \Sigma_\Delta$; k; $\bar{k}$; $X^{\log}$; K; $\alpha\colon \Pi_n \xrightarrow{\sim} \Pi_{U_{X_n}}$. Let $Y^{\log}$ be a smooth log curve over k of type $(0, 3^{\mathrm{ord}})$. Write $Y_2^{\log}$ for the second log configuration space associated to $Y^{\log} \to \mathrm{Spec}(k)$. Suppose that $(X^{\log}, n)$ is *tripodally very ample*. Then:

(i) Suppose that $n = 2$. Then $\Pi_2^{\mathrm{tpd}} \stackrel{\mathrm{def}}{=} \Pi_2$, $\mathcal{D}_2^{\mathrm{tpd}} \stackrel{\mathrm{def}}{=} \mathcal{D}_2$.

(ii) Suppose that $n > 2$, $r \neq 0$, and $X(k) \setminus U_X(k) \neq \emptyset$.

(a) By [HMM], Theorem 2.5, (v), it makes sense to speak of the *generalized fiber subgroups of length one* associated to the profinite group Π_n. Fix such a subgroup E_{Π_n} of Π_n. Write $\Pi_{n-1} \stackrel{\mathrm{def}}{=} \Pi_n/E_{\Pi_n}$. Similarly, as i ranges from $n-1$ to 2, by applying [HMM], Theorem 2.5, (v), to Π_i, it makes sense to (speak of and hence, in particular, to) fix a *generalized fiber subgroup of length one* E_{Π_i} of Π_i and write $\Pi_{i-1} \stackrel{\mathrm{def}}{=} \Pi_i/E_{\Pi_i}$. For $i \in$

$\{1, \ldots, n\}$, write $\Pi_0 \overset{\text{def}}{=} G$; $p^{\Pi}_{i/i-1} \colon \Pi_i \to \Pi_{i-1}$ for the natural surjection; $\Delta_{i/i-1} \overset{\text{def}}{=} \mathrm{Ker}(p_{i/i-1})^{\Pi}$; $\Delta_i \overset{\text{def}}{=} \mathrm{Ker}(\Pi_i \to G)$. (Note that $\Delta_1 = \Delta_{1/0}$.) Thus, we obtain projections

$$\Pi_n \to \Pi_3 \to \Pi_2 \to \Pi_1,$$

which induce surjections $\mathcal{D}_n \to \mathcal{D}_j \overset{\text{def}}{=} \{C_{\Pi_j}((\Pi_n \to \Pi_j)(D)) \mid D \in \mathcal{D}_n\}$, where $j \in \{1, 2, \ldots, n-1\}$. Let ${}^{\dagger}E_{\Pi_2} \subseteq \Pi_2$, be a generalized fiber subgroup of length one such that ${}^{\dagger}E_{\Pi_2} \neq E_{\Pi_2}$. Write $p^{\Pi}_{1\backslash 2} \colon \Pi_2 \to \Pi_2/{}^{\dagger}E_{\Pi_2}$ for the natural surjection and $\Delta_{1\backslash 2} \overset{\text{def}}{=} \mathrm{Ker}(p^{\Pi}_{1\backslash 2})$.

(b) Let $i \in \{1, \ldots, n\}$. Write

$$\mathcal{I}_i \overset{\text{def}}{=} \{I \subseteq \Delta_i \mid \exists D \in \mathcal{D}_i \text{ such that } I = D \cap \Delta_i \neq \{1\}\};$$

$$\mathcal{I}_{i/i-1} \overset{\text{def}}{=} \{I \subseteq \Delta_{i/i-1} \mid I \in \mathcal{I}_i\}.$$

Then it follows from [HMM], Proposition 1.3, (i), (iii), (iv) (cf. also the surjectivity of "$\Delta_{x_n} \twoheadrightarrow \Delta_{x_{n-1}}$" in the proof of [HMM], Proposition 1.3); [HMM], Lemma 1.5, that α maps $\mathcal{I}_i$ to the set of *decomposition groups in* $\alpha(\Delta_i)$ *of closed points of* $X_i \setminus U_{X_i}$ and $\mathcal{I}_{i/i-1}$ to the set of *cuspidal inertia groups of* $\alpha(\Delta_{i/i-1})$. Let $I_c \in \mathcal{I}_1$, $D_c \in \mathcal{D}_1$ be such that $I_c \overset{\text{def}}{=} D_c \cap \Delta_1 \neq \{1\}$ (so α maps D_c to a *decomposition group in* $\alpha(\Pi_1)$ *of a closed point* c *of* $X \setminus U_X$).

(c) One verifies easily that α maps $I \in \mathcal{I}_{2/1}$ to a decomposition group of $\alpha(\Delta_{2/1})$ associated to the diagonal divisor V_{diag} (cf. Remark 6.2, (ii)) if and only if $I \subseteq \Delta_{2/1} \cap \Delta_{1\backslash 2}$. Let $I_{\text{diag}} \in \mathcal{I}_{2/1}$ be such an element of $\mathcal{I}_{2/1}$.

(d) Consider the extensions

$$1 \longrightarrow \Delta_{2/1} \longrightarrow \Pi_{I_c} \overset{\text{def}}{=} \Pi_2 \times_{\Pi_1} I_c \longrightarrow I_c \longrightarrow 1,$$

$$1 \longrightarrow \Delta_{2/1} \longrightarrow \Pi_{D_c} \overset{\text{def}}{=} \Pi_2 \times_{\Pi_1} D_c \longrightarrow D_c \longrightarrow 1.$$

By applying (the algorithms of) [NodNon], Theorem A (cf. also [NodNon], Remark 2.4.2) to the data $(\Pi_{I_c} \twoheadrightarrow I_c, \mathcal{I}_{2/1})$, we obtain a group-theoretic construction of a verticial subgroup $I_{V_Y} \subseteq \Pi_{I_c}$ (unique up to $\Delta_{2/1}$-conjugacy) such that $D_{V_Y} \overset{\text{def}}{=} C_{\Pi_{D_c}}(I_{V_Y})$ contains some $\Delta_{2/1}$-conjugate of I_{diag}. Write $\Pi_{V_Y} \overset{\text{def}}{=} D_{V_Y}/I_{V_Y}$. One verifies easily that α maps I_{V_Y} to a decomposition group in $\Pi_{U_{X_2}}$ associated to V_Y (cf. Remark 6.2, (ii)).

(e) Consider the extensions

$$1 \longrightarrow \Delta_{3/2} \longrightarrow \Pi_{I_{V_Y}} \overset{\text{def}}{=} \Pi_3 \times_{\Pi_2} I_{V_Y} \longrightarrow I_{V_Y} \longrightarrow 1,$$

$$1 \longrightarrow \Delta_{3/2} \longrightarrow \Pi_{D_{V_Y}} \overset{\text{def}}{=} \Pi_3 \times_{\Pi_2} D_{V_Y} \longrightarrow D_{V_Y} \longrightarrow 1.$$

By applying (the algorithms of) [NodNon], Theorem A (cf. also [NodNon], Remark 2.4.2) to the data $(\Pi_{I_{V_Y}} \twoheadrightarrow I_{V_Y}, \mathcal{I}_{3/2})$, we obtain a group-theoretic construction of a verticial subgroup $I_{W_{YY}} \subseteq \Pi_{I_{V_Y}}$ (unique up to $\Delta_{3/2}$-conjugacy) such that if we write

$$D_{W_{YY}} \overset{\text{def}}{=} C_{\Pi_{D_{V_Y}}}(I_{W_{YY}}), \quad \Pi_{W_{YY}} \overset{\text{def}}{=} D_{W_{YY}}/I_{W_{YY}},$$

$$\Delta_{W_{YY}} \stackrel{\text{def}}{=} \operatorname{Ker}(\Pi_{W_{YY}} \twoheadrightarrow G),$$

then $\Delta_{W_{YY}}$ is indecomposable (cf. Remark 6.2, (ii); [Hgsh], Definition 6.2; [Hgsh], Remark 6.3). One verifies easily that α maps $I_{W_{YY}}$ to a decomposition group in $\Pi_{U_{X_3}}$ associated to W_{YY} (cf. Remark 6.2, (ii)) and induces a natural isomorphism

$$\Pi_{W_{YY}} \xrightarrow{\sim} \Pi_{U_{Y_2}},$$

where we write

$$\Pi_{U_{Y_2}} \stackrel{\text{def}}{=} \begin{cases} \pi_1(U_{Y_2})^{\Sigma_\Delta} & (\text{if } \Sigma_\Delta = \Sigma_{\text{Gal}}) \\ \pi_1(U_{Y_2})^{[p]} & (\text{if } \Sigma_\Delta \subsetneq \Sigma_{\text{Gal}}). \end{cases}$$

(f) Write $\Pi_2^{\text{tpd}} \stackrel{\text{def}}{=} \Pi_{W_{YY}}$ and

$$\mathcal{D}_2^{\text{tpd}} \stackrel{\text{def}}{=} \{D^{\text{tpd}} \subseteq \Pi_2^{\text{tpd}} \mid \exists D_3 \in \mathcal{D}_3 \text{ such that } I_{W_{YY}} \subseteq D_3 \subseteq D_{W_{YY}}, D^{\text{tpd}} = D_3/I_{W_{YY}}\}.$$

Note that it follows immediately from the equality

$$W_{YY} = X_3 \times_{X \times X \times X} (c, c, c)$$

(cf. Remark 6.2, (ii)), together with Proposition 4.7, (iv), that α induces an isomorphism $\Pi_2^{\text{tpd}} \xrightarrow{\sim} \Pi_{U_{Y_2}}$ that maps $\mathcal{D}_2^{\text{tpd}}$ to the set of decomposition groups of closed points of $W_{YY} \xrightarrow{\sim} Y_2$ (cf. Remark 6.2, (ii)).

(iii) By [HMM], Theorem 2.5, (iv), it makes sense to speak of the *generalized fiber subgroups of co-length one* associated to the profinite group Π_n. Fix such a subgroup E of Π_n. Write $\Pi_1 \stackrel{\text{def}}{=} \Pi_n/E$; $\Delta_1 \stackrel{\text{def}}{=} \operatorname{Ker}(\Pi_1 \to G)$. Thus, we obtain a projection

$$\Pi_n \to \Pi_1,$$

which induces a surjection $\mathcal{D}_n \to \mathcal{D}_1 \stackrel{\text{def}}{=} \{C_{\Pi_1}(\Pi_n \to \Pi_1)(D) \mid D \in \mathcal{D}_n\}$. Let $D_c \in \mathcal{D}_1$ be such that $D_c \cap \Delta_1 = \{1\}$ (so α maps D_c to a *decomposition group in* $\alpha(\Pi_1)$ *of a closed point* c *of* U_X — cf. (ii), (b)). Write $\Pi_{c,n-1} \stackrel{\text{def}}{=} \Pi_n \times_{\Pi_1} D_c$ and

$$\mathcal{D}_{c,n-1} \stackrel{\text{def}}{=} \{D \subseteq \Pi_{c,n-1} \mid D \in \mathcal{D}_n\}.$$

Note that it follows immediately from the equality $X_{c,n-1}^{\log} = X_n^{\log} \times_X c$ (cf. Remark 6.2, (iii)), together with Proposition 4.7, (iv), that α induces an isomorphism

$$\Pi_{c,n-1} \xrightarrow{\sim} \Pi_{U_{X_{c,n-1}}} \stackrel{\text{def}}{=} \begin{cases} \pi_1(U_{X_{c,n-1}})^{\Sigma_\Delta} & (\text{if } \Sigma_\Delta = \Sigma_{\text{Gal}}) \\ \pi_1(U_{X_{c,n-1}})^{[p]} & (\text{if } \Sigma_\Delta \subsetneq \Sigma_{\text{Gal}}). \end{cases}$$

that maps $\mathcal{D}_{c,n-1}$ to the set of decomposition groups of closed points of $X_{c,n-1}$. Thus, we obtain a PGCS-collection of type $(g, (r+1)^{\square}, n-1, \Sigma_\Delta, \Sigma_{\text{Gal}})$

$$(\Pi_{c,n-1}, G, \mathcal{D}_{c,n-1})$$

(well-defined up to Π_n-conjugacy) associated to the intrinsic structure of the PGCS-collection of type $(g, r^{\square}, n, \Sigma_\Delta, \Sigma_{\text{Gal}})$

$$(\Pi_n, G, \mathcal{D}_n)$$

and the choices of the generalized fiber subgroup E and the decomposition group D_c. Finally, by applying the algorithms of (ii) to the data $(\Pi_{c,n-1}, G, \mathcal{D}_{c,n-1})$, we obtain a group-theoretic construction of $\Pi_2^{\mathrm{tpd}}, \mathcal{D}_2^{\mathrm{tpd}}$.

Thus, in summary, in either of the situations discussed in (i), (ii), (iii), we obtain a *PGCS-collection of type* $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$

$$(\Pi_2^{\mathrm{tpd}}, G, \mathcal{D}_2^{\mathrm{tpd}})$$

(well-defined up to Π_n-conjugacy) associated to the *intrinsic structure of the PGCS-collection of type* $(g, r^\Box, n, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$

$$(\Pi_n, G, \mathcal{D}_n)$$

and the choices of *generalized fiber subgroups* (cf. (ii), (a); (iii)) and the *decomposition group* D_c (cf. (ii), (b); (iii)).

Theorem 6.4. (Semi-absolute bi-anabelian formulation) *Let* $* \in \{\dagger, \ddagger\}$*;* ${}^*n \in \mathbb{Z}_{>1}$*;* $({}^*g, {}^*r)$ *a pair of nonnegative integers such that* $2({}^*g - 1) + {}^*r > 0$*;* ${}^*\Box \in \{\mathrm{arb}, \mathrm{ord}\}$*;* $\Sigma_\Delta, \Sigma_{\mathrm{Gal}}$ *sets of prime numbers such that* $\Sigma_\Delta \subseteq \Sigma_{\mathrm{Gal}}$*, and* $\Sigma_\Delta, \Sigma_{\mathrm{Gal}}$ *are of cardinality* 1 *or equal to* $\mathfrak{Primes}$*;* $p \in \Sigma_\Delta$*;* ${}^*k$ *a generalized sub-p-adic local field;* ${}^*\bar{k}$ *an algebraic closure of* ${}^*k$*;* ${}^*X^{\log}$ *a smooth log curve over* ${}^*k$ *of type* $({}^*g, {}^*r^{{}^*\Box})$*. Write* ${}^*X^{\log}_{{}^*n}$ *for the* ${}^*n$*-th log configuration space associated to* ${}^*X^{\log} \to \mathrm{Spec}({}^*k)$*;* ${}^*K \subseteq {}^*\bar{k}$ *for the maximal pro-*Σ_{Gal} *subextension of* ${}^*\bar{k}/{}^*k$*;*

$$\Pi_{U_{{}^*X_{{}^*n}}} \stackrel{\mathrm{def}}{=} \begin{cases} \pi_1(U_{{}^*X_{{}^*n}})^{\Sigma_\Delta} & (\textit{if } \Sigma_\Delta = \Sigma_{\mathrm{Gal}}) \\ \pi_1(U_{{}^*X_{{}^*n}})^{[p]} & (\textit{if } \Sigma_\Delta \subsetneq \Sigma_{\mathrm{Gal}}); \end{cases}$$

$$\Delta_{U_{{}^*X_{{}^*n}}} \stackrel{\mathrm{def}}{=} \pi_1(U_{{}^*X_{{}^*n}} \times_{{}^*k} {}^*\bar{k})^{\Sigma_\Delta}; \quad G^{\Sigma_{\mathrm{Gal}}}_{{}^*k} \stackrel{\mathrm{def}}{=} \mathrm{Gal}({}^*\bar{k}/{}^*k)^{\Sigma_{\mathrm{Gal}}};$$

$$\mathcal{D}_{{}^*X_{{}^*n}} \stackrel{\mathrm{def}}{=} \{D \subseteq \Pi_{U_{{}^*X_{{}^*n}}} \mid D \textit{ is a decomposition group associated to some } x \in {}^*X_{{}^*n}({}^*K)\}$$

(cf. Notation 4.1). Suppose that the sequence

$$1 \longrightarrow \Delta_{U_{{}^*X_{{}^*n}}} \longrightarrow \Pi_{U_{{}^*X_{{}^*n}}} \longrightarrow G^{\Sigma_{\mathrm{Gal}}}_{{}^*k} \longrightarrow 1$$

is exact (cf. Notation 4.1; Remark 4.3), and that $({}^*X^{\log}, {}^*n)$ *is tripodally ample (cf. Definition 6.1). Thus,*

$$\mathscr{B}[{}^*X^{\log}_{{}^*n}] \stackrel{\mathrm{def}}{=} (\Pi_{U_{{}^*X_{{}^*n}}}, G^{\Sigma_{\mathrm{Gal}}}_{{}^*k}, \mathcal{D}_{{}^*X_{{}^*n}})$$

is a PGCS-collection of type $({}^*g, {}^*r^{{}^*\Box}, {}^*n, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ *(cf. Definition 4.2). Write*

$$\mathrm{Isom}(U_{{}^\dagger X_{{}^\dagger n}}, U_{{}^\ddagger X_{{}^\ddagger n}})$$

for the set of isomorphisms of schemes $U_{{}^\dagger X_{{}^\dagger n}} \xrightarrow{\sim} U_{{}^\ddagger X_{{}^\ddagger n}}$ *and*

$$\mathrm{Isom}^{\mathrm{Out}}(\mathscr{B}[{}^\dagger X^{\log}_{{}^\dagger n}], \mathscr{B}[{}^\ddagger X^{\log}_{{}^\ddagger n}])$$

for the set of equivalence classes of isomorphisms of PGCS-collections $\mathscr{B}[{}^\dagger X^{\log}_{{}^\dagger n}] \xrightarrow{\sim} \mathscr{B}[{}^\ddagger X^{\log}_{{}^\ddagger n}]$ *(cf. Definition 4.4) with respect to the equivalence relation given by composition with an inner automorphism arising from* $\Pi_{U_{{}^*X_{{}^*n}}}$*. Then the natural morphism*

$$\mathrm{Isom}(U_{{}^\dagger X_{{}^\dagger n}}, U_{{}^\ddagger X_{{}^\ddagger n}}) \to \mathrm{Isom}^{\mathrm{Out}}(\mathscr{B}[{}^\dagger X^{\log}_{{}^\dagger n}], \mathscr{B}[{}^\ddagger X^{\log}_{{}^\ddagger n}])$$

is bijective.

Proof. First, observe that we may assume without loss of generality that

$$\mathrm{Isom}^{\mathrm{Out}}(\mathscr{B}[{}^{\dagger}X^{\log}_{{}^{\dagger}n}], \mathscr{B}[{}^{\ddagger}X^{\log}_{{}^{\ddagger}n}]) \neq \emptyset.$$

Then it follows from [HMM], Theorem A, (i), that $({}^{\dagger}g, {}^{\dagger}r^{{}^{\dagger}\square}, {}^{\dagger}n) = ({}^{\ddagger}g, {}^{\ddagger}r^{{}^{\ddagger}\square}, {}^{\ddagger}n)$. Thus, we shall write

$$({}^{*}g, {}^{*}r^{{}^{*}\square}, {}^{*}n) = (g, r^{\square}, n).$$

Next, recall that it follows from Lemma 5.3 (cf. also Remark 4.3; the final conclusion of Remark 6.3); [MzTa], Proposition 2.2, (ii), that $\Pi_{U_{{}^{*}X_n}}$ is center-free. Thus, by applying Notation 1.9 to a suitable open normal subgroup of $\Pi_{U_{{}^{*}X_n}}$ that arises from a open normal subgroup of $G^{\Sigma_{\mathrm{Gal}}}_{{}^{*}k}$, we conclude that, after replacing ${}^{*}k$ by a suitable finite Galois subextension of ${}^{*}k$ in ${}^{*}K$, we may assume that $({}^{*}X^{\log}, n)$ is tripodally very ample.

Next, observe that the injectivity of the natural morphism

$$\mathrm{Isom}(U_{{}^{\dagger}X_{{}^{\dagger}n}}, U_{{}^{\ddagger}X_{{}^{\ddagger}n}}) \to \mathrm{Isom}^{\mathrm{Out}}(\mathscr{B}[{}^{\dagger}X^{\log}_{{}^{\dagger}n}], \mathscr{B}[{}^{\ddagger}X^{\log}_{{}^{\ddagger}n}])$$

follows immediately from Lemma 5.3 (cf. also Remark 4.3; the final conclusion of Remark 6.3); the injectivity portion of [Topics], Theorem 4.12 (applied to the hyperbolic curves that arise as the codomains of the various natural projections $U_{{}^{*}X_n} \to U_{{}^{*}X}$).

Let $\sigma \in \mathrm{Isom}(\mathscr{B}[{}^{\dagger}X^{\log}_n], \mathscr{B}[{}^{\ddagger}X^{\log}_n])$. By Definition 4.4, σ induces a commutative diagram of homomorphisms of profinite groups

$$\begin{array}{ccc}
\Pi_{U_{{}^{\dagger}X_n}} & \xrightarrow[\sigma_{\Pi}]{\sim} & \Pi_{U_{{}^{\ddagger}X_n}} \\
\downarrow & \circlearrowright & \downarrow \\
G^{\Sigma_{\mathrm{Gal}}}_{{}^{\dagger}k} & \xrightarrow[\sigma_G]{\sim} & G^{\Sigma_{\mathrm{Gal}}}_{{}^{\ddagger}k},
\end{array}$$

as well as a bijection

$$\mathcal{D}_{{}^{\dagger}X_n} \xrightarrow[\sigma_{\mathcal{D}}]{\sim} \mathcal{D}_{{}^{\ddagger}X_n}.$$

Since $({}^{*}X^{\log}, n)$ is tripodally very ample, by applying Remark 6.3 to suitable choices of generalized fiber subgroups and decomposition groups, we obtain, for each $* \in \{\dagger, \ddagger\}$, a PGCS-collection of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_{\Delta}, \Sigma_{\mathrm{Gal}})$

$${}^{*}\mathscr{B}^{\mathrm{tpd}}_2 \stackrel{\mathrm{def}}{=} ({}^{*}\Pi^{\mathrm{tpd}}_2, G^{\Sigma_{\mathrm{Gal}}}_{{}^{*}k}, {}^{*}\mathcal{D}^{\mathrm{tpd}}_2)$$

associated to the intrinsic structure of the PGCS-collection $\mathscr{B}[{}^{*}X^{\log}_n]$ (together with the suitable choices of generalized fiber subgroups and decomposition groups) such that $\sigma \in \mathrm{Isom}(\mathscr{B}[{}^{\dagger}X^{\log}_n], \mathscr{B}[{}^{\ddagger}X^{\log}_n])$ induces an isomorphism

$$\sigma^{\mathrm{tpd}} \in \mathrm{Isom}({}^{\dagger}\mathscr{B}^{\mathrm{tpd}}_2, {}^{\ddagger}\mathscr{B}^{\mathrm{tpd}}_2).$$

By Theorem 5.2, (i), σ induces a field isomorphism

$$({}^{\dagger}K \stackrel{\sim}{\leftarrow}) \quad K[{}^{\dagger}\mathscr{B}^{\mathrm{tpd}}_2] \stackrel{\sim}{\to} K[{}^{\ddagger}\mathscr{B}^{\mathrm{tpd}}_2] \quad (\stackrel{\sim}{\to} {}^{\ddagger}K)$$

that is equivariant, relative to the isomorphism σ_G

$$(\mathrm{Gal}({}^{\dagger}K/{}^{\dagger}k) =) \quad G^{\Sigma_{\mathrm{Gal}}}_{{}^{\dagger}k} \stackrel{\sim}{\to} G^{\Sigma_{\mathrm{Gal}}}_{{}^{\ddagger}k} \quad (= \mathrm{Gal}({}^{\ddagger}K/{}^{\ddagger}k)),$$

with respect to the respective natural actions of $G^{\Sigma_{\mathrm{Gal}}}_{{}^{\dagger}k}$, $G^{\Sigma_{\mathrm{Gal}}}_{{}^{\ddagger}k}$ on $K[{}^{\dagger}\mathscr{B}^{\mathrm{tpd}}_2]$, $K[{}^{\ddagger}\mathscr{B}^{\mathrm{tpd}}_2]$. In particular, the isomorphism $\sigma_G \colon G^{\Sigma_{\mathrm{Gal}}}_{{}^{\dagger}k} \stackrel{\sim}{\to} G^{\Sigma_{\mathrm{Gal}}}_{{}^{\ddagger}k}$ arises from an isomorphism of fields ${}^{\dagger}K \stackrel{\sim}{\to} {}^{\ddagger}K$ that induces an isomorphism of fields ${}^{\dagger}k \stackrel{\sim}{\to} {}^{\ddagger}k$. In the following, to simplify the notation, we shall identify $^{\dagger}K$ with $^{\ddagger}K$ and $^{\dagger}k$ with $^{\ddagger}k$ via these isomorphisms and denote the resulting fields by K, k.

Write $\mathrm{Isom}_k(U_{{}^{\dagger}X_n}, U_{{}^{\ddagger}X_n})$ for the set of isomorphisms of schemes $U_{{}^{\dagger}X_n} \xrightarrow{\sim} U_{{}^{\ddagger}X_n}$ that lie over the field k. Next, observe that it follows from Definition 4.6; [HMM], Theorem 2.5, (v); [Topics], Theorem 4.12 (applied successively to the various arrows of the composite morphisms $U_{{}^{*}X_n} \to U_{{}^{*}X_{n-1}} \to \cdots \to U_{{}^{*}X} \to \mathrm{Spec}(k)$ arising from the natural projections), that the natural morphism

$$\mathrm{Isom}_k(U_{{}^{\dagger}X_n}, U_{{}^{\ddagger}X_n}) \to \mathrm{Isom}^{\mathrm{Out}}_{G_k^{\Sigma_{\mathrm{Gal}}}}(\mathscr{B}[{}^{\dagger}X_n^{\log}], \mathscr{B}[{}^{\ddagger}X_n^{\log}])$$

is bijective. We thus conclude that the natural morphism

$$\mathrm{Isom}(U_{{}^{\dagger}X_n}, U_{{}^{\ddagger}X_n}) \to \mathrm{Isom}^{\mathrm{Out}}(\mathscr{B}[{}^{\dagger}X_n^{\log}], \mathscr{B}[{}^{\ddagger}X_n^{\log}])$$

is bijective. □

Definition 6.5. We maintain the following notation of Definition 4.2: $(g, r^{\square}, n, \Sigma_{\Delta}, \Sigma_{\mathrm{Gal}})$; $(\Pi_n, G, \mathcal{D}_n)$; $X^{\log}$. Write $\mathscr{B} \overset{\mathrm{def}}{=} (\Pi_n, G, \mathcal{D}_n)$. Suppose that $(g, r^{\square}, n) = (0, 3^{\mathrm{ord}}, 2)$. Let Π_2^{prf} be a profinite group which is isomorphic to the étale fundamental group $\Pi_{U_{X_2}}^{\mathrm{prf}} \overset{\mathrm{def}}{=} \pi_1(U_{X_2})$ (relative to a suitable choice of basepoint) and $\beta\colon \mathscr{B} \xrightarrow{\sim} \mathscr{B}[\Pi_2^{\mathrm{prf}}]$ an isomorphism of PGCS-collections (cf. Theorem 5.2, (ii)). Recall from Definition 4.8, (ii), that there exists a group-theoretic characterization of the set $\mathcal{E}_2[\mathscr{B}] = \{E_1, \ldots, E_5\}$ of generalized fiber subgroups $\subseteq \Pi_2$. Write

$$E_{\cap} \overset{\mathrm{def}}{=} \bigcap_{i=1}^{5} E_i, \quad \Pi_{2\to 1}[\mathscr{B}] \overset{\mathrm{def}}{=} \Pi_2/E_{\cap}.$$

By Theorem 5.2, (i), one may construct a *field* $K[\mathscr{B}]$ equipped with a natural action by G associated to the intrinsic structure of the PGCS-collection $\mathscr{B}$. Let $E_i \in \mathcal{E}_2[\mathscr{B}]$ and $E_i^{\mathrm{prf}} \in \mathcal{E}_2[\Pi_2^{\mathrm{prf}}]$ (cf. Theorem 5.1, (v)) be such that $E_i^{\mathrm{prf}}|_{\Pi_2} = E_i$ (cf. Theorem 5.2, (ii)). By Theorem 5.2, (ii), one may construct a *field* $F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E_i^{\mathrm{prf}}, \beta]$ equipped with a natural action by $(\Pi_2 \twoheadrightarrow)\Pi_1[\mathscr{B}, E_i]$ associated to the intrinsic structure of the data $(\mathscr{B}, \Pi_2^{\mathrm{prf}}, E_i^{\mathrm{prf}}, \beta)$. Let $T \in F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E_i^{\mathrm{prf}}, \beta]$. Then T induces, by restriction to decomposition groups (cf. also Proposition 4.7, (iv)), a map

$$T(-)\colon \mathcal{D}_1[\mathscr{B}, E_i] \to K[\mathscr{B}] \cup \{\infty\}$$

(cf. Theorem 5.2, (iv)). Thus, it follows immediately from the scheme-theoretic interpretation of this situation given in Theorem 5.2, (iii), (v), that we obtain a *natural* $\Pi_1[\mathscr{B}, E_i]$ *($\twoheadrightarrow G$)-equivariant injection*

$$F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E_i^{\mathrm{prf}}, \beta] \hookrightarrow \mathrm{RatMaps}(\mathcal{D}_1[\mathscr{B}, E_i], K[\mathscr{B}])$$

(cf. Notation 1.12; Definition 4.8, (iii); Theorem 5.2, (i)). Here, in the definition of "RatMaps$(-,-)$" (cf. Notation 1.12), we take the collection of quotients to be the single "identity quotient" $\mathcal{D}_1[\mathscr{B}, E_i] \to \mathcal{D}_1[\mathscr{B}, E_i]$. Write

$$p^{\mathcal{D}}_{2/1}[\mathscr{B}, E_i]\colon \mathcal{D}_2 \to \mathcal{D}_1[\mathscr{B}, E_i]$$

for the surjection induced by the quotient homomorphism $p^{\Pi}_{2/1}[\mathscr{B}, E_i]$ (cf. Definition 4.8, (ii), (iii)). Thus, we obtain a *natural* Π_2 *($\twoheadrightarrow \Pi_1[\mathscr{B}, E_i] \twoheadrightarrow G$)-equivariant* *injection*

$$\begin{array}{rcl}\rho_i\colon F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E_i^{\mathrm{prf}}, \beta] & \hookrightarrow & \mathrm{RatMaps}(\mathcal{D}_2[\mathscr{B}], K[\mathscr{B}]) \\ T & \mapsto & T(-) \circ p^{\mathcal{D}}_{2/1}[\mathscr{B}, E_i].\end{array}$$

Here, in the definition of "RatMaps$(-,-)$" (cf. Notation 1.12), we take the collection of quotients to be the collection of quotients $\{p^{\mathcal{D}}_{2/1}[\mathscr{B}, E_i]\}_{i=1,2,3,4,5}$. Next, let us observe that the *field structure* of $K[\mathscr{B}]$ induces a natural *ring structure* on $\mathrm{RatMaps}(\mathcal{D}_2[\mathscr{B}], K[\mathscr{B}])$. Moreover, it follows immediately from the scheme-theoretic interpretation of this situation given in Theorem 5.2, (iii), (v), that ρ_i is a ring homomorphism, relative to the ring structure (just described) on $\mathrm{RatMaps}(\mathcal{D}_2[\mathscr{B}], K[\mathscr{B}])$ and the field structure of $F_1[\mathscr{B}, \Pi_2^{\mathrm{prf}}, E_i, \beta]$, and that the image of ρ_i is independent of the choice of the data $(\Pi_2^{\mathrm{prf}}, \beta)$. In particular, it makes sense to write $F_1[\mathscr{B}, E_i]$ for the image of ρ_i. Finally, we observe that it follows immediately from the scheme-theoretic interpretation of this situation given in Theorem 5.2, (iii), (v), that if we write $R[\mathscr{B}]$ for the subring of

$$\mathrm{RatMaps}(\mathcal{D}_2[\mathscr{B}], K[\mathscr{B}])$$

generated by the subrings

$$F_1[\mathscr{B}, E_i]$$

for $i \in \{1,2,3,4,5\}$, then $R[\mathscr{B}]$ is an integral domain on which the subgroup $E_\cap \subseteq \Pi_2$ acts trivially. In particular, it makes sense to speak of the quotient field

$$\mathrm{Frac}(R[\mathscr{B}])$$

of this integral domain $R[\mathscr{B}]$, which is equipped with an action by $\Pi_{2\to 1}[\mathscr{B}]$.

Theorem 6.6. (From PGCS-collections of type $(g, r^{\square}, n, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ to certain function fields arising from tripods) *We maintain the following notation of Definition 4.2: $(g, r^{\square}, n, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$; $(\Pi_n, G, \mathcal{D}_n)$; $p \in \Sigma_\Delta$; k; $\bar{k}$; $G_k^{\Sigma_{\mathrm{Gal}}}$; $X^{\log}$; K; $\alpha\colon \Pi_n \xrightarrow{\sim} \Pi_{U_{X_n}}$; $\mathcal{D}_{X_n}$. Write $\mathscr{B} \overset{\mathrm{def}}{=} (\Pi_n, G, \mathcal{D}_n)$. Suppose that $(X^{\log}, n)$ is tripodally ample, and that k is a number field or a p-adic local field. Then:*

- (i) *For any sufficiently small open normal subgroup H of G, one may construct a family (cf. the discussion of "choices" in the final portion of Remark 6.3) of PGCS-collections $\{\mathscr{B}^{\mathrm{tpd}} = (\Pi_2^{\mathrm{tpd}}, H, \mathcal{D}_2^{\mathrm{tpd}})\}$ of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ associated to the intrinsic structure of the PGCS-collection $\mathscr{B}$.*
- (ii) *Let $\beta_X\colon \mathscr{B} \xrightarrow{\sim} \mathscr{B}[X] \overset{\mathrm{def}}{=} (\Pi_{U_{X_n}}, G_k^{\Sigma_{\mathrm{Gal}}}, \mathcal{D}_{X_n})$ be an isomorphism of PGCS-collections (cf. Definition 4.4) and $\mathscr{B}^{\mathrm{tpd}} = (\Pi_2^{\mathrm{tpd}}, H, \mathcal{D}_2^{\mathrm{tpd}})$ a PGCS-collection of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ associated to $\mathscr{B}$ (cf. (i)). Write $H[X] \overset{\mathrm{def}}{=} \mathrm{Ker}(G_k^{\Sigma_{\mathrm{Gal}}} \to G/H)$, where $G_k^{\Sigma_{\mathrm{Gal}}} \to G/H$ denotes the composite of the natural quotient $G \to G/H$ with the inverse of the isomorphism $(\beta_X)_G\colon G \xrightarrow{\sim} G_k^{\Sigma_{\mathrm{Gal}}}$ determined by β_X (cf. Definition 4.4). Let $Y^{\log}$ be a smooth log curve over k of type $(0, 3^{\mathrm{ord}})$ (cf. Remark 6.2); write*

$$\Pi_{U_{Y_2}} \overset{\mathrm{def}}{=} \begin{cases} \pi_1(U_{Y_2})^{\Sigma_\Delta} & (\text{if } \Sigma_\Delta = \Sigma_{\mathrm{Gal}}) \\ \pi_1(U_{Y_2})^{[p]} & (\text{if } \Sigma_\Delta \subsetneq \Sigma_{\mathrm{Gal}}). \end{cases}$$

  Then, for a suitable choice $\mathscr{B}^{\mathrm{tpd}}[X] = (\Pi_{U_{Y_2}}, H[X], \mathcal{D}_{Y_2})$ of PGCS-collection of type $(0, 3^{\mathrm{ord}}, 2, \Sigma_\Delta, \Sigma_{\mathrm{Gal}})$ associated to $\mathscr{B}[X]$ (cf. (i); Remarks 6.2, 6.3),

β_X induces an isomorphism of PGCS-collections

$$\beta_Y^{\mathrm{tpd}} \colon \mathscr{B}^{\mathrm{tpd}} \xrightarrow{\sim} \mathscr{B}^{\mathrm{tpd}}[X] = (\Pi_{U_{Y_2}}, H[X], \mathcal{D}_{Y_2}).$$

(iii) One may construct a quotient group $\Pi_2^{\mathrm{tpd}} \twoheadrightarrow \Pi_{2\to 1}^{\mathrm{tpd}}[\mathscr{B}^{\mathrm{tpd}}]$ (cf. Definition 6.5) and a field $\mathrm{Frac}(R[\mathscr{B}^{\mathrm{tpd}}])$ (cf. Definition 6.5) equipped with an action by $\Pi_{2\to 1}^{\mathrm{tpd}}[\mathscr{B}^{\mathrm{tpd}}]$ associated to the intrinsic structure of the PGCS-collection $\mathscr{B}^{\mathrm{tpd}}$.

(iv) In the notation of (ii), (iii), write $\mathcal{E}_2[\mathscr{B}^{\mathrm{tpd}}] = \{E_1, \ldots, E_5\}$ for the set of generalized fiber subgroups $\subseteq \Pi_2^{\mathrm{tpd}}$;

$$\Pi_{U_{Y_{2\to 1}}} \stackrel{\mathrm{def}}{=} \Pi_{U_{Y_2}} / \bigcap_{i=1}^{5} (\beta_Y^{\mathrm{tpd}})_\Pi(E_i),$$

where $(\beta_Y^{\mathrm{tpd}})_\Pi \colon \Pi_2^{\mathrm{tpd}} \xrightarrow{\sim} \Pi_{U_{Y_2}}$ denotes the isomorphism determined by β_Y^{tpd} (cf. Definition 4.4). Then the isomorphism $(\beta_Y^{\mathrm{tpd}})_\Pi$ induces a commutative diagram

$$\begin{array}{ccc} \Pi_2^{\mathrm{tpd}} & \xrightarrow[(\beta_Y^{\mathrm{tpd}})_\Pi]{\sim} & \Pi_{U_{Y_2}} \\ \downarrow & & \downarrow \\ \Pi_{2\to 1}^{\mathrm{tpd}}[\mathscr{B}^{\mathrm{tpd}}] & \xrightarrow{\sim} & \Pi_{U_{Y_{2\to 1}}}, \end{array}$$

where the vertical arrows are the natural projections, and $\Pi_{2\to 1}^{\mathrm{tpd}}[\mathscr{B}^{\mathrm{tpd}}] \xrightarrow{\sim} \Pi_{U_{Y_{2\to 1}}}$ denotes a uniquely determined isomorphism of profinite groups.

(v) In the notation of (iv), write $Z \to U_{Y_2}$ for the profinite étale covering corresponding to $(\Pi_{U_{Y_2}} \twoheadrightarrow)\ \Pi_{U_{Y_{2\to 1}}}$ and $\mathrm{Fnct}(Z)$ for the function field of Z. Then one may construct a field isomorphism

$$\mathrm{Frac}(R[\mathscr{B}^{\mathrm{tpd}}]) \xrightarrow{\sim} \mathrm{Fnct}(Z)$$

associated to the intrinsic structure of the data $(\mathscr{B}^{\mathrm{tpd}}, \mathscr{B}^{\mathrm{tpd}}[X], \beta_Y^{\mathrm{tpd}} \colon \mathscr{B}^{\mathrm{tpd}} \xrightarrow{\sim} \mathscr{B}^{\mathrm{tpd}}[X])$, where the field isomorphism "$\xrightarrow{\sim}$" is equivariant with respect to the respective natural actions of the profinite groups $(\Pi_2^{\mathrm{tpd}} \twoheadrightarrow)\ \Pi_{2\to 1}^{\mathrm{tpd}}[\mathscr{B}]$, $(\Pi_{U_{Y_2}} \twoheadrightarrow)\ \Pi_{U_{Y_{2\to 1}}}$ (cf. the display of (iv)).

Proof. Assertion (i) follows from Remark 6.3. Assertion (ii) follows from assertion (i); Remarks 6.2, 6.3. Assertion (iii) follows from Definition 6.5. Assertion (iv) follows from assertion (iii) and Definition 6.5. Assertion (v) follows from assertions (ii), (iii), (iv); Theorem 5.2, (iii). □

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