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Remarks on the periodic Zakharov system

By

Nobu Kishimoto

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

REMARKS ON THE PERIODIC ZAKHAROV SYSTEM

NOBU KISHIMOTO

Abstract. We consider the Cauchy problem associated with the Zakharov system on the torus:

$$i\partial_t u + \Delta u = nu, \qquad \alpha^{-2}\partial_t^2 n - \Delta n = \Delta(|u|^2), \qquad (t,x) \in \mathbb{R}\times\mathbb{T}^d;$$
$$(u,n,\partial_t n)\big|_{t=0} = (u_0,n_0,n_1) \ \in H^s\times H^l\times H^{l-1}.$$

Here, u and n are $\mathbb{C}$- and $\mathbb{R}$-valued unknown functions, respectively, α is a positive constant, and H^s denotes Sobolev space on the torus. We obtain unconditional uniqueness result in a range of (s,l), which includes the energy space $(s,l)=(1,0)$ in one and two dimensions, and also prove convergence of solutions in the energy space to the solution of a cubic nonlinear Schrödinger equation as α tends to ∞ for dimensions one and two. Our proof of unconditional uniqueness is based on the method of infinite iteration of the Poincaré-Dulac normal form reduction; actually, we simply show a certain set of multilinear estimates, which was presented as a criterion for unconditional uniqueness in [Kishimoto, 2019 (preprint)]. The convergence result is obtained by a similar argument to the non-periodic case [Masmoudi and Nakanishi, 2008], which exploits conservation laws and unconditional uniqueness for the limit equation.

1. Introduction

We consider the Cauchy problem associated with the Zakharov system under the periodic boundary condition:

$$\begin{cases} i\partial_t u + \Delta u = nu, \quad \dfrac{1}{\alpha^2}\partial_t^2 n - \Delta n = \Delta(|u|^2); \qquad t\in\mathbb{R}, \quad x\in\mathbb{T}^d_\lambda, \\ (u,n,\partial_t n)\big|_{t=0} = (u_0,n_0,n_1) \ \in \mathcal{H}^{s,l}(\mathbb{T}^d_\lambda), \end{cases} \tag{1.1}$$

where $\alpha>0$ is a constant, $\lambda\in(0,\infty)^d$, and $\mathbb{T}^d_\lambda := \mathbb{R}^d/(2\pi\lambda_1\mathbb{Z})\times\cdots\times(2\pi\lambda_d\mathbb{Z})$ is the torus with period $2\pi\lambda=(2\pi\lambda_1,\dots,2\pi\lambda_d)$. We treat the torus of arbitrary period and (by rescaling) normalize the coefficient of the Laplace operator; $\Delta := \partial_{x_1}^2+\cdots+\partial_{x_d}^2$. Write $\mathbb{Z}^d_\lambda$ to denote the lattice $\frac{1}{\lambda_1}\mathbb{Z}\times\cdots\times\frac{1}{\lambda_d}\mathbb{Z}$ corresponding to $\mathbb{T}^d_\lambda$. The unknown functions u,n are $\mathbb{C}$- and $\mathbb{R}$-valued, respectively, and $\mathcal{H}^{s,l}(\mathbb{T}^d_\lambda) := H^s(\mathbb{T}^d_\lambda;\mathbb{C})\times H^l(\mathbb{T}^d_\lambda;\mathbb{R})\times H^{l-1}(\mathbb{T}^d_\lambda;\mathbb{R})$ for $s,l\in\mathbb{R}$. For an interval $I\subset\mathbb{R}$, we denote by $C(I;\mathcal{H}^{s,l}(\mathbb{T}^d_\lambda))$ the space of all functions (u,n) such that

$$u\in C(I;H^s(\mathbb{T}^d_\lambda;\mathbb{C})), \quad n\in C(I;H^l(\mathbb{T}^d_\lambda;\mathbb{R}))\cap C^1(I;H^{l-1}(\mathbb{T}^d_\lambda;\mathbb{R})).$$

If $I=[0,T]$, we further abbreviate as $C_T\mathcal{H}^{s,l}(\mathbb{T}^d_\lambda)$.

The (vector-valued) Zakharov system was derived as a model for propagation of Langmuir waves in a plasma; see, e.g., [17] for more details. There is a wealth of literature on local and global well-posedness, as well as asymptotic behavior of global solutions, of the Cauchy problem (1.1) on $\mathbb{R}^d$ and on $\mathbb{T}^d$; we refer to the recent article [5] and references therein. The aim of this note is to give two results on the property of the solutions to the periodic Cauchy problem (1.1); unconditional uniqueness and convergence to a cubic nonlinear Schrödinger equation as $\alpha\to\infty$ (subsonic limit). These properties have also been studied in the non-periodic case, while there seems no result in the periodic setting.

Let us recall existing results on local well-posedness of the periodic Cauchy problem (1.1) in Sobolev spaces, which were given by Takaoka [18] for $d = 1$ and the author [9] for $d \geq 2$ (see also an earlier work of Bourgain [3]):

Theorem 1.1 ([18, 9]). *The Cauchy problem (1.1) is locally well-posed in $\mathcal{H}^{s,l}(\mathbb{T}^d_\lambda)$ in the following cases:*

- $d = 1$, $\alpha\lambda \notin \mathbb{Z}$, $-\frac{1}{2} \leq l \leq 2s - \frac{1}{2}$, $0 \leq s - l \leq 1$;
- $d = 1$, $\alpha\lambda \in \mathbb{Z}$, $0 \leq l \leq 2s - 1$, $0 \leq s - l \leq 1$;
- $d = 2$, α, λ *are arbitrary,* $0 \leq l \leq 2s - 1$, $0 \leq s - l \leq 1$;
- $d \geq 3$, α, λ *are arbitrary,* $\frac{d-2}{2} < l \leq 2s - \frac{d}{2}$, $0 \leq s - l \leq 1$.

These results were obtained by the iteration argument using the Fourier restriction norm (Bourgain norm), and thus uniqueness is ensured only for those solutions with such an auxiliary norm being finite. In very low regularities (e.g., the case $d = 1$, $\alpha\lambda \notin \mathbb{Z}$, and $(s, l) = (0, -\frac{1}{2})$ in the theorem), one has to impose some additional requirement on solutions (not only to be in $C_T\mathcal{H}^{s,l}$) to ensure that both of the nonlinear terms nu, $\Delta(|u|^2)$ are well-defined in a certain sense. However, at least when $s + l \geq 0$ and $s \geq 0$, these nonlinear terms make sense in the framework of distribution for any $(u, n) \in \mathcal{H}^{s,l}$, so that one can ask uniqueness within the class of all (distributional) solutions in $C_T\mathcal{H}^{s,l}$, which we refer to as *unconditional uniqueness.*

Our result on unconditional uniqueness reads as follows:

Theorem 1.2. *Let $T > 0$. For any $(u_0, n_0, n_1) \in \mathcal{H}^{s,l}(\mathbb{T}^d_\lambda)$, there is at most one solution (in the sense of distribution) to the Cauchy problem (1.1) in $C_T\mathcal{H}^{s,l}(\mathbb{T}^d_\lambda)$ in the following cases:*

- $d = 1$, $\alpha\lambda \notin \mathbb{Z}$, $s > \frac{1}{6}$, $l > -\frac{1}{2}$ *and* $s + l \geq 0$;
- ([10, Theorem 6.1]) $d = 1$, $\alpha\lambda \in \mathbb{Z}$, $s \geq \frac{1}{2}$ *and* $l \geq 0$;
- $d = 2$, α, λ *are arbitrary,* $s \geq \frac{1}{2}$ *and* $l \geq 0$;
- $d \geq 3$, α, λ *are arbitrary,* $s > \frac{d-1}{2}$ *and* $l > \frac{d-2}{2}$.

A result on unconditional uniqueness for the non-periodic problem was obtained in [14] by means of various estimates in Strichartz- and Bourgain-type norms. We prove the theorem by a different approach; infinite iteration of the Poincaré-Dulac normal form reduction. In [10], the author developed this methodology for unconditional uniqueness, which had been introduced in the work of Guo, Kwon, and Oh [6] for the cubic nonlinear Schrödinger equation on $\mathbb{T}$, in an abstract setting and proved that the overall argument can be reduced to a certain set of multilinear estimates associated with the nonlinearity of the equation. In this note, we employ the abstract theory and simply show these multilinear estimates. The case $d = 1$, $\alpha = \lambda = 1$ of Theorem 1.2 was treated in [10] as a demonstration of the method, and the same proof works in the case $\alpha\lambda \in \mathbb{Z}$. Note that, in the above theorem, we only consider (s, l) satisfying $s \geq 0$ and $s + l \geq 0$, so that the nonlinear terms make sense in the framework of distribution.

Combining it with Theorem 1.1, we obtain unconditional well-posedness of (1.1). In particular, when $d = 1, 2$, the energy space $(s, l) = (1, 0)$ is included for arbitrary α, λ.

Corollary 1.3. *The Cauchy problem (1.1) is unconditionally locally well-posed in $\mathcal{H}^{s,l}(\mathbb{T}^d_\lambda)$ if:*

- $d = 1$, $\alpha\lambda \notin \mathbb{Z}$, $-s \leq l \leq 2s - \frac{1}{2}$, $0 \leq s - l \leq 1$ *and* $(s, l) \neq (\frac{1}{6}, -\frac{1}{6}), (\frac{1}{2}, -\frac{1}{2})$;
- $d = 1$, $\alpha\lambda \in \mathbb{Z}$, $0 \leq l \leq 2s - 1$, $0 \leq s - l \leq 1$;
- $d = 2$, α, λ *are arbitrary,* $0 \leq l \leq 2s - 1$, $0 \leq s - l \leq 1$;
- $d \geq 3$, α, λ *are arbitrary,* $\frac{d-2}{2} < l \leq 2s - \frac{d}{2}$, $0 \leq s - l \leq 1$.

Next, we study convergence of the solutions (u^α, n^α) of the periodic Zakharov system

$$\begin{cases} i\partial_t u^\alpha + \Delta u^\alpha = n^\alpha u^\alpha, \quad \frac{1}{\alpha^2}\partial_t^2 n^\alpha - \Delta n^\alpha = \Delta(|u^\alpha|^2), \qquad t \in \mathbb{R}, \quad x \in \mathbb{T}^d_\lambda, \\ (u^\alpha, n^\alpha, \partial_t n^\alpha)\big|_{t=0} = (u^\alpha_0, n^\alpha_0, n^\alpha_1) \end{cases} \tag{1.2}$$

as $\alpha \to \infty$. This problem has also been well studied in the $\mathbb{R}^d$ case; in principle, the Schrödinger part u^α of the solution converges to the unique solution u of the focusing cubic nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = -|u|^2 u, \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}^d \tag{1.3}$$

with initial condition $u(0) = \lim_{\alpha\to\infty} u^\alpha_0$, and the wave part n^α converges to $-|u|^2$. In the non-compatible case $n^\alpha_0 + |u^\alpha_0|^2 \not\to 0$, the strong convergence of the wave part is verified after correction by a fast oscillating linear wave solution; this is called the initial layer. The strong convergence in Sobolev spaces was first proved in [16] for the compatible data, and then the initial layer phenomenon and the rate of convergence were investigated in subsequent works [1, 15, 8]. While a certain amount of regularity (H^5, for instance) had been assumed in the above results, Masmoudi and Nakanishi [13] proved the strong convergence in the energy class $H^1 \times L^2 \times \dot{H}^{-1}(\mathbb{R}^d)$. Their proof is substantially simpler than the previous ones, only using local well-posedness (conservation laws) of (1.2), (1.3) and unconditional uniqueness for the limit equation (1.3) in the energy class, though the rate of convergence is difficult to obtain by it.

We aim here to give an analogous result of [13] in the periodic setting. We focus on one and two dimensions, because local well-posedness for (1.2) in the energy class has been shown only in one and two dimensions. In the limit $\alpha \to \infty$, we formally obtain $\Delta(n^\alpha + |u^\alpha|^2) \sim 0$, namely, $P_{\neq c}(n^\alpha + |u^\alpha|^2) \sim 0$, where P_c and $P_{\neq c}$ denote the orthogonal projections onto zero and non-zero frequency modes, respectively. In contrast to the non-periodic (spatially decaying) case, one cannot determine the asymptotic behavior of the zero mode (spatial mean) of n^α from the relation $\Delta(n^\alpha + |u^\alpha|^2) \sim 0$. In the periodic case, however, the zero mode of the wave part of the system (1.2) can be decoupled and explicitly solved as

$$\begin{cases} \partial_t^2 P_c n^\alpha = 0, \\ (P_c n^\alpha, \partial_t P_c n^\alpha)\big|_{t=0} = (P_c n^\alpha_0, P_c n^\alpha_1) \end{cases} \quad \Longrightarrow \quad P_c n^\alpha(t) = P_c n^\alpha_0 + t P_c n^\alpha_1 \quad (t \in \mathbb{R}).$$

This suggests that

$$n^\alpha(t,x) = P_{\neq c} n^\alpha(t,x) + P_c n^\alpha(t) \sim -P_{\neq c}(|u^\alpha|^2)(t,x) + P_c n^\alpha_0 + t P_c n^\alpha_1$$

as $\alpha \to \infty$, and that the Schrödinger part u^α converges to the solution of a "shifted" cubic NLS:

$$i\partial_t u + \Delta u = -\Big(|u|^2 - P_c(|u|^2) - \lim_{\alpha\to\infty}\big[P_c n^\alpha_0 + t P_c n^\alpha_1\big]\Big)u.$$

Note that, even in the case of mean-zero wave initial data $P_c n^\alpha_0 = P_c n^\alpha_1 \equiv 0$, the expected limit equation in the periodic setting differs by $P_c(|u|^2)$ from the usual focusing cubic NLS (1.3).[1] We also remark that, if the initial data (u^α_0, n^α_0) do not satisfy the condition $P_{\neq c}(n^\alpha_0 + |u^\alpha_0|^2) = 0$ in the limit $\alpha \to \infty$ (i.e., non-compatible), the initial layer should appear as $\alpha \to \infty$.

We denote by $P_{\le R}$, $P_{>R}$ the projection in spatial frequency onto $\{|k| \le R\}$ and $\{|k| > R\}$, respectively. Here is our theorem on convergence:

[1]This is also different from the renormalized (or Wick-ordered) cubic NLS, where $2P_c(|u|^2)$ is subtracted.

Theorem 1.4. *Let $d=1,2$ and $\lambda\in(0,\infty)^d$ be arbitrary. Let $\{u_0^\alpha,n_0^\alpha,n_1^\alpha\}_\alpha\subset\mathcal{H}^{1,0}(\mathbb{T}^d_\lambda)$ be a family of initial data satisfying*

$$\exists\, u_0^\infty:=\lim_{\alpha\to\infty}u_0^\alpha\qquad \textit{in } H^1(\mathbb{T}^d_\lambda),$$
$$\sup_\alpha\big\|(P_{\neq c}n_0^\alpha,|\alpha\nabla|^{-1}P_{\neq c}n_1^\alpha)\big\|_{L^2\times L^2}<\infty,$$
$$\lim_{R\to\infty}\limsup_{\alpha\to\infty}\big\|(P_{>R}n_0^\alpha,|\alpha\nabla|^{-1}P_{>R}n_1^\alpha)\big\|_{L^2\times L^2}=0,$$

and

$$\exists\,(\nu_0,\nu_1):=\lim_{\alpha\to\infty}\big(P_cn_0^\alpha,P_cn_1^\alpha\big).\tag{1.4}$$

Let $(u^\alpha,n^\alpha)\in C([0,T^\alpha);\mathcal{H}^{1,0}(\mathbb{T}^d_\lambda))$ be the (forward-in-time) maximal-lifespan solution of (1.2),[2] *and let $u\in C([0,T^\infty);H^1)$ be the (forward-in-time) maximal-lifespan solution*[3] *of*

$$\begin{cases} i\partial_tu+\Delta u=-\Big(|u|^2-P_c(|u|^2)-\nu_0-\nu_1t\Big)u, & (t,x)\in(0,T^\infty)\times\mathbb{T}^d_\lambda,\\ u(0,x)=u_0^\infty(x), & x\in\mathbb{T}^d_\lambda.\end{cases}\tag{1.5}$$

Then, we have $T^\infty\le\liminf_{\alpha\to\infty}T^\alpha$, and for any $T\in(0,T^\infty)$,

$$\begin{aligned} u^\alpha &\to u && \textit{in } C([0,T];H^1),\\ P_{\neq c}n^\alpha-n_{il}^\alpha &\to -P_{\neq c}(|u|^2) && \textit{in } C([0,T];L^2),\\ |\alpha\nabla|^{-1}\partial_t(P_{\neq c}n^\alpha-n_{il}^\alpha) &\to 0 && \textit{in } C([0,T];L^2),\\ P_cn^\alpha &\to \nu_0+\nu_1t && \textit{in } C^1([0,T])\end{aligned}$$

as $\alpha\to\infty$, where the initial layer n_{il}^α is given by

$$n_{il}^\alpha(t):=\cos\big(t|\alpha\nabla|\big)P_{\neq c}(n_0^\alpha+|u_0^\alpha|^2)+\frac{\sin\big(t|\alpha\nabla|\big)}{|\alpha\nabla|}P_{\neq c}n_1^\alpha.$$

Remark 1.5. The assumptions trivially hold if the initial data are independent of α; $(u_0^\alpha,n_0^\alpha,n_1^\alpha)\equiv(u_0,n_0,n_1)\in\mathcal{H}^{1,0}$. In this case, one can simply take $n_{il}^\alpha=\cos\big(t|\alpha\nabla|\big)P_{\neq c}(n_0^\alpha+|u_0^\alpha|^2)$ as the initial layer, since the remaining part is of $O(\alpha^{-1})$. On the other hand, (non-zero modes of) the initial data $n_1^\alpha\in H^{-1}$ is allowed to diverge with growth order at most $O(\alpha)$ as $\alpha\to\infty$. For instance, the data $n_1^\alpha=\alpha P_{\neq c}n_1+P_cn_1$ for a fixed $n_1\in H^{-1}$ also satisfy the assumptions. In this case, one needs to modify the initial layer depending on n_1^α as in the theorem.

Remark 1.6. The first three assumptions on initial data in the theorem are the same as those in the $\mathbb{R}^d$ case [13]. The last one (1.4), which was not assumed in [13], is necessary for the convergence of u^α in the periodic case. To see this, we first note that, in the periodic case, for any solution (u^α,n^α) of (1.2) in the energy class, the transformation

$$(u^\alpha,n^\alpha)\ \mapsto\ (u^\alpha e^{i(c_0t+\frac12c_1t^2)},n^\alpha-c_0-c_1t),\qquad c_0,c_1\in\mathbb{R}$$

gives another energy-class solution of (1.2). Then, consider three families of solutions

$$(u^\alpha,n^\alpha),\qquad (u^\alpha e^{it\sin\alpha},n^\alpha-\sin\alpha),\qquad (u^\alpha e^{it^2\sin\alpha},n^\alpha-2t\sin\alpha).$$

We observe that the first three assumptions are equivalent for all of them. However, the claimed convergence cannot hold for any two of them at the same time, unless $u\equiv0$.

[2] The maximal-lifespan solution is uniquely defined in the energy class $C_t\mathcal{H}^{1,0}_x$ by the existence result given in [18, 9] and the uniqueness result established in Theorem 1.2.

[3] This is also uniquely defined in $C_tH^1_x$. See Remark 3.4 (i) below.

The rest of this note is devoted to the proofs of Theorems 1.2 and 1.4, which will be given in Sections 2 and 3, respectively. Throughout this note, we often use the notation

$$X \sim Y, \qquad X \lesssim Y, \qquad X \gg Y$$

as abbreviations for

$$C^{-1}Y \le X \le CY, \qquad X \le CY, \qquad X > CY$$

with a suitably large positive constant C.

2. Proof of unconditional uniqueness

2.1. **Reduction to the fundamental bilinear estimates.** For $p \in [1,\infty]$ and $s \in \mathbb{R}$, let $\ell^p_s = \ell^p_s(\mathbb{Z}^d_\lambda)$ be the weighted ℓ^p space on $\mathbb{Z}^d_\lambda$ with the norm $\|f_k\|_{\ell^p_s} := \|\langle k\rangle^s f_k\|_{\ell^p}$, where $\langle k\rangle := 1+|k|$.

We employ the infinite normal form reduction machinery. As discussed in [10, Sections 1 and 6], unconditional uniqueness of solutions to (1.1) in $\mathcal{H}^{s,l}(\mathbb{T}^d_\lambda)$ is established once we have the following bilinear estimates with some $\varepsilon > 0$:

$$\Big\| \sum_{k_1=k_0+k_2} \frac{f_{k_0} h_{k_2}}{\langle \mu_\pm\rangle^{1/2}} \Big\|_{\ell^2_s((\mathbb{Z}^d_\lambda)_{k_1})} \lesssim \|f\|_{\ell^2_l} \|h\|_{\ell^2_s},$$

$$\Big\| |k_0| \sum_{k_0=k_1-k_2} \frac{g_{k_1} h_{k_2}}{\langle \mu_\pm\rangle^{1/2}} \Big\|_{\ell^2_l((\mathbb{Z}^d_\lambda)_{k_0})} \lesssim \|g\|_{\ell^2_s} \|h\|_{\ell^2_s},$$

$$\Big\| \sum_{k_1=k_0+k_2} \frac{\langle k_0\rangle + \langle k_2\rangle}{\langle k_1\rangle} \frac{f_{k_0} h_{k_2}}{\langle \mu_\pm\rangle^{1-\varepsilon}} \Big\|_{\ell^2_s((\mathbb{Z}^d_\lambda)_{k_1})} \lesssim \|f\|_{\ell^2_l} \|h\|_{\ell^2_s},$$

$$\Big\| |k_0| \sum_{k_0=k_1-k_2} \frac{\langle k_1\rangle + \langle k_2\rangle}{\langle k_0\rangle} \frac{g_{k_1} h_{k_2}}{\langle \mu_\pm\rangle^{1-\varepsilon}} \Big\|_{\ell^2_l((\mathbb{Z}^d_\lambda)_{k_0})} \lesssim \|g\|_{\ell^2_s} \|h\|_{\ell^2_s},$$

$$\|f * h\|_{\ell^2_{s-1}} \lesssim \|f\|_{\ell^2_l} \|h\|_{\ell^2_s},$$

$$\|g * h\|_{\ell^2_l} \lesssim \|g\|_{\ell^2_s} \|h\|_{\ell^2_s}$$

for any non-negative sequences $f \in \ell^2_l(\mathbb{Z}^d_\lambda)$, $g, h \in \ell^2_s(\mathbb{Z}^d_\lambda)$, where[4]

$$\mu_\pm := |k_1|^2 - |k_2|^2 \pm \alpha |k_0|$$

and $*$ denotes the convolution.

We see that the first four estimates are equivalent by duality to the trilinear estimates

$$\sum_{\substack{k_0,k_1,k_2 \in \mathbb{Z}^d_\lambda \\ k_0 = k_1 - k_2}} W_j(k_0,k_1,k_2) f_{k_0} g_{k_1} h_{k_2} \lesssim \|f\|_{\ell^2} \|g\|_{\ell^2} \|h\|_{\ell^2}, \qquad j = 1,\dots,4 \tag{2.1}$$

for non-negative sequences $f, g, h \in \ell^2(\mathbb{Z}^d_\lambda)$, where

$$W_1 = \frac{\langle k_1\rangle^s}{\langle \mu_\pm\rangle^{1/2} \langle k_0\rangle^l \langle k_2\rangle^s}, \qquad W_2 = \frac{\langle k_0\rangle^l |k_0|}{\langle \mu_\pm\rangle^{1/2} \langle k_1\rangle^s \langle k_2\rangle^s},$$

$$W_3 = \frac{\langle k_1\rangle^{s-1} (\langle k_0\rangle + \langle k_2\rangle)}{\langle \mu_\pm\rangle^{1-\varepsilon} \langle k_0\rangle^l \langle k_2\rangle^s}, \qquad W_4 = \frac{\langle k_0\rangle^{l-1} |k_0| (\langle k_1\rangle + \langle k_2\rangle)}{\langle \mu_\pm\rangle^{1-\varepsilon} \langle k_1\rangle^s \langle k_2\rangle^s}.$$

The next proposition is the main ingredient of the proof of Theorem 1.2:

[4] In [10], $\widetilde{\mu}_\pm = |k_1|^2 - |k_2|^2 \pm \langle \alpha k_0\rangle$ was used instead of $\mu_\pm$ (and α was taken to be 1). Since $\langle \mu_\pm\rangle \sim \langle \widetilde{\mu}_\pm\rangle$, there is no difference in the above estimates.

Proposition 2.1. *The estimate* (2.1) *holds with some* $\varepsilon > 0$ *in the following cases:*

(i) $d = 1$, $\alpha\lambda \notin \mathbb{N}$, $\frac{1}{6} < s < \frac{1}{2}$, *and* $l = -s$.
(ii) $d = 2$, $(s, l) = (\frac{1}{2}, 0)$.
(iii) $d \geq 3$, $s > \frac{d-1}{2}$, $l = s - \frac{1}{2}$.

We observe that the last two estimates on the convolution, which are equivalent to the Sobolev estimates on the product, hold if and only if

$$A_1 := \min\{1 - s + l, s + l\} \geq 0, \quad B_1 := 1 + l - \frac{d}{2} \geq 0 \quad \text{with } (A_1, B_1) \neq (0, 0)$$
$$\text{and} \quad A_2 := \min\{s - l, 2s\} \geq 0, \quad B_2 := 2s - l - \frac{d}{2} \geq 0 \quad \text{with } (A_2, B_2) \neq (0, 0).$$

These conditions are satisfied in each of the cases (i)–(iii) in Proposition 2.1. Finally, note that uniqueness of solution in $C_T\mathcal{H}^{s,l}$ implies that in $C_T\mathcal{H}^{s',l'}$ for any $s' \geq s$ and $l' \geq l$. Therefore, to establish Theorem 1.2 it suffices to show Proposition 2.1.

2.2. **One dimensional case.** In this subsection, we shall prove Proposition 2.1 (i).

It is easy to check $W_j \lesssim 1$ when $k_0 = 0$, which implies (2.1) in this case. Assume $k_0 \neq 0$, then it holds $\langle \mu_\pm \rangle = \langle k_0(k_0 + 2k_2 \pm \alpha\,\mathrm{sgn}(k_0)) \rangle$ under the relation $k_0 = k_1 - k_2$. If $\alpha\lambda \notin \mathbb{N}$, we have $|k_0 + 2k_2 \pm \alpha\,\mathrm{sgn}(k_0)| \geq \mathrm{dist}(\frac{1}{\lambda}\mathbb{Z}, \alpha) > 0$, and in particular,

$$\langle \mu_\pm \rangle \sim \langle k_0 \rangle \langle k_0 + 2k_2 \pm \alpha\,\mathrm{sgn}(k_0) \rangle \sim \langle k_0 \rangle \langle k_0 + 2k_2 \rangle. \tag{2.2}$$

Let $l = -s$. Using this factorization, for W_1 and W_2, we see that

$$W_1 \sim \frac{\langle k_0 + k_2 \rangle^s}{\langle k_0 \rangle^{1/2-s} \langle k_0 + 2k_2 \rangle^{1/2} \langle k_2 \rangle^s} \lesssim \frac{\mathbf{1}_{|k_0+k_2| \gg |k_2|}}{\langle k_0 \rangle^{1/2-s} \langle k_0 + 2k_2 \rangle^{1/2-s} \langle k_2 \rangle^s} + \frac{\mathbf{1}_{|k_0+k_2| \lesssim |k_2|}}{\langle k_0 \rangle^{1/2-s} \langle k_0 + 2k_2 \rangle^{1/2}},$$
$$W_2 \sim \frac{\langle k_0 \rangle^{1/2-s}}{\langle k_0 + 2k_2 \rangle^{1/2} \langle k_0 + k_2 \rangle^s \langle k_2 \rangle^s} \lesssim \frac{\mathbf{1}_{|k_0+2k_2| \gtrsim |k_0|}}{\langle k_0 + 2k_2 \rangle^s \langle k_0 + k_2 \rangle^s \langle k_2 \rangle^s} + \frac{\mathbf{1}_{|k_0+2k_2| \ll |k_0|}}{\langle k_0 + 2k_2 \rangle^{1/2} \langle k_2 \rangle^{3s-1/2}},$$

where $\mathbf{1}_A$ denotes the characteristic function of the set A or the set of variables satisfying the condition A. For W_3, we take $\varepsilon = \frac{1}{2}$;

$$\begin{aligned} W_3 &\sim \frac{\langle k_0 \rangle + \langle k_2 \rangle}{\langle k_0 \rangle^{1/2-s} \langle k_0 + 2k_2 \rangle^{1/2} \langle k_0 + k_2 \rangle^{1-s} \langle k_2 \rangle^s} \\ &\lesssim \frac{\mathbf{1}_{|k_0+k_2| \gg |k_2|}}{\langle k_0 \rangle^{1-2s} \langle k_2 \rangle^s} + \frac{\mathbf{1}_{|k_0+k_2| \sim |k_2|}}{\langle k_0 \rangle^{1/2-s} \langle k_0 + 2k_2 \rangle^{1/2}} + \frac{\mathbf{1}_{|k_0+k_2| \ll |k_2|}}{\langle k_0 + k_2 \rangle^{1-s}}, \end{aligned}$$

and for W_4 we take $\varepsilon = \frac{1}{3}$, so that

$$\begin{aligned} W_4 &\sim \frac{\langle k_0 + k_2 \rangle + \langle k_2 \rangle}{\langle k_0 \rangle^{2/3+s} \langle k_0 + 2k_2 \rangle^{2/3} \langle k_0 + k_2 \rangle^s \langle k_2 \rangle^s} \\ &\lesssim \frac{\mathbf{1}_{|k_0+2k_2| \gg |k_0|}}{\langle k_0 \rangle^{2/3+s} \langle k_2 \rangle^{2s-1/3}} + \frac{\mathbf{1}_{|k_0+2k_2| \sim |k_0|}}{\langle k_0 \rangle^{1/3+s} \langle k_0 + k_2 \rangle^s \langle k_2 \rangle^s} + \frac{\mathbf{1}_{|k_0+2k_2| \ll |k_0|}}{\langle k_0 + 2k_2 \rangle^{2/3} \langle k_2 \rangle^{3s-1/3}}. \end{aligned}$$

If $\frac{1}{6} < s < \frac{1}{2}$, we deduce from these estimates that

$$W_j \lesssim \frac{1}{\langle k_0 \rangle^{1/2+\delta}} + \frac{1}{\langle k_0 + 2k_2 \rangle^{1/2+\delta}} + \frac{1}{\langle k_0 + k_2 \rangle^{1/2+\delta}} + \frac{1}{\langle k_2 \rangle^{1/2+\delta}}, \qquad j = 1, \dots, 4$$

for some $\delta > 0$. We then apply the Hölder inequality to obtain (2.1).

2.3. **Two and higher dimensional cases.** In this subsection, we shall prove Proposition 2.1 (ii), (iii). The main difficulty comes from the fact that we do not have a factorization like (2.2). We divide the analysis into three cases according to the size of $|\mu_\pm|$. Let $k_{\max}$ and $k_{\min}$ be the largest and the smallest quantities among $|k_0|, |k_1|, |k_2|$, respectively.

2.3.1. *High modulation interactions.* We begin with the case $|\mu_\pm| \gtrsim k_{\max}^2$ and prove (2.1) with $\varepsilon = \frac{1}{2}$. Under the condition $l = s - \frac{1}{2}$, it holds that

$$W_j \lesssim \frac{1}{\langle k_{\max}\rangle^{1/2}\langle k_{\min}\rangle^{s}}, \qquad j = 1, \dots, 4.$$

This and the Sobolev inequality imply (2.1); in fact, the desired estimate

$$\sum_{\substack{k_0,k_1,k_2\in\mathbb{Z}^d_\lambda \\ k_0=k_1-k_2}} \frac{f_{k_0} g_{k_1} h_{k_2}}{\langle k_{\max}\rangle^{1/2}\langle k_{\min}\rangle^{s}} \lesssim \|f\|_{\ell^2}\|g\|_{\ell^2}\|h\|_{\ell^2}$$

is the dual of the product estimate

$$\|uv\|_{L^2(\mathbb{T}^d_\alpha)} \lesssim \|u\|_{H^{1/2}(\mathbb{T}^d_\alpha)}\|v\|_{H^s(\mathbb{T}^d_\alpha)},$$

which holds true if $d \geq 2$ and $s \geq \frac{d-1}{2}$.

2.3.2. *Middle modulation interactions.* Hereafter, we assume $|\mu_\pm| \ll k_{\max}^2$. This in particular implies $|k_0| \lesssim |k_1| \sim |k_2|$. Taking $s = l + \frac{1}{2}$ $(\geq \frac{1}{2})$ and $\varepsilon = \frac{1}{2}$, we see that

$$W_j \lesssim \frac{1}{\langle \mu_\pm\rangle^{1/2}\langle k_0\rangle^{l}}, \qquad j = 1, \dots, 4.$$

If $|k_0| \lesssim 1$, then the left-hand side of (2.1) is bounded by $\|\mathbf{1}_{|n_0|\lesssim 1} f\|_{\ell^1}\|g\|_{\ell^2}\|h\|_{\ell^2}$, which is sufficient. It then suffices to prove

$$\sum_{\substack{k_0=k_1-k_2 \\ 1\ll|k_0|\lesssim|k_1|\sim|k_2| \\ |\mu_\pm|\ll k_{\max}^2}} \frac{f_{k_0} g_{k_1} h_{k_2}}{\langle \mu_\pm\rangle^{1/2}\langle k_0\rangle^{l}} \lesssim \|f\|_{\ell^2}\|g\|_{\ell^2}\|h\|_{\ell^2} \tag{2.3}$$

for $l = 0$ if $d = 2$ and $l > \frac{d-2}{2}$ if $d \geq 3$.

Here, we consider the middle-modulation case $k_{\max} \lesssim |\mu_\pm| \ll k_{\max}^2$, following the idea in [9, Section 3.2] for the corresponding bilinear estimates in Bourgain spaces. First, restrict k_0, k_1, k_2 to $\langle k_j\rangle \sim N_j$ for dyadic numbers N_0, N_1, N_2 with $N_1 \sim N_2 \gtrsim N_0 \gg 1$, and then restrict $\mu_\pm$ to $\langle \mu_\pm\rangle \sim M$ for a dyadic $N_1 \lesssim M \ll N_1^2$, so that

$$\text{L.H.S. of (2.3)} \lesssim \sum_{N_1\sim N_2} \sum_{\substack{1\ll N_0\lesssim N_1 \\ N_1\lesssim M\ll N_1^2}} \frac{1}{M^{1/2}N_0^l} \sum_{\substack{k_0=k_1-k_2 \\ \langle k_j\rangle\sim N_j \\ \langle\mu_\pm\rangle\sim M}} f_{k_0} g_{k_1} h_{k_2}.$$

Since in the last sum we have

$$\big||k_1| - |k_2|\big| = \frac{\big|\mu_\pm \mp \alpha|k_0|\big|}{|k_1| + |k_2|} = O\Big(\frac{M}{N_1}\Big),$$

decomposition into annuli:

$$g_{k_1} h_{k_2} = \sum_{m_1,m_2} (\mathbf{1}_{A_{m_1}} g)_{k_1} (\mathbf{1}_{A_{m_2}} h)_{k_2},$$

$$A_m := \left\{ k \in \mathbb{Z}^d_\lambda \,\middle|\, m\frac{M}{N_1} \leq |k| \leq (m+1)\frac{M}{N_1} \right\}, \qquad m \in \mathbb{Z}, \quad m \sim \frac{N_1^2}{M}$$

exhibits almost orthogonality. If $N_0 \ll N_1$, we make further decomposition into cubes:

$$g_{k_1}h_{k_2} = \sum_{n_1,n_2} (\mathbf{1}_{Q_{n_1}} g)_{k_1} (\mathbf{1}_{Q_{n_2}} h)_{k_2},$$

$$Q_n := \Big\{ k \in \mathbb{Z}^d_\lambda \,\Big|\, |k-n| \in [0,N_0]^d \Big\}, \qquad n \in (N_0\mathbb{Z})^d, \quad |n| \sim N_1$$

and make use of its almost orthogonality. Hence,

$$\text{L.H.S. of (2.3)} \lesssim \sum^*_{N_1,N_2} \sum_{\substack{1 \ll N_0 \lesssim N_1 \\ N_1 \lesssim M \ll N_1^2}} \frac{1}{M^{1/2}N_0^l} \sum^*_{m_1,m_2} \sum^*_{n_1,n_2} \sum_{\substack{k_0 = k_1 - k_2 \\ \langle k_0\rangle \sim N_0,\, \langle \mu_\pm\rangle \sim M \\ k_1 \in A_{m_1} \cap Q_{n_1} \\ k_2 \in A_{m_2} \cap Q_{n_2}}} f_{k_0} g_{k_1} h_{k_2},$$

where $\sum^*$ stand for almost orthogonal sums (i.e., one index determines the other up to $O(1)$ ambiguity). Now, we recall another identity

$$\frac{k_0}{|k_0|} \cdot k_1 = \frac{1}{2|k_0|}\Big(|k_0|^2 \mp \alpha|k_0| + \mu_\pm\Big) = \frac{1}{2|k_0|}\Big(|k_0|^2 \mp \alpha|k_0|\Big) + O\Big(\frac{M}{N_0}\Big),$$

which restricts $\frac{k_0}{|k_0|}$-component of k_1 into an interval of length $O(\frac{M}{N_0})$ for each k_0 fixed. Therefore, for fixed k_0, k_1 is confined to the intersection of a cube, an annulus, and a plate. An elementary computation (see [9, Lemma 2.9 (i)]) evaluates the number of frequencies $k_1 \in \mathbb{Z}^d_\lambda$ in such a region by

$$C \min\Big\{ N_0^d, \frac{M}{N_1}N_0^{d-1}, \frac{M}{N_0}M^{\frac12}N_0^{d-2} \Big\} \lesssim MN_0^{d-2}\Big(\frac{N_0}{N_1}\Big)^{\frac12} \min\Big\{ \frac{N_0^2}{M}, \frac{M^{1/2}}{N_0} \Big\}^{\frac12}.$$

By the Cauchy-Schwarz inequality in k_1, we have (for $d \geq 2$ and $l \geq \frac{d-2}{2}$)

$$\begin{aligned}
&\text{L.H.S. of (2.3)} \\
&\lesssim \sum^*_{N_1,N_2} \sum_{N_0,M} \frac{1}{M^{1/2}N_0^l} \sum^*_{m_1,m_2} \sum^*_{n_1,n_2} \Big[MN_0^{d-2}\Big(\frac{N_0}{N_1}\Big)^{\frac12} \min\Big\{ \frac{N_0^2}{M}, \frac{M^{1/2}}{N_0} \Big\}^{\frac12} \Big]^{\frac12} \\
&\qquad\qquad \times \sum_{k_0} f_{k_0} \Big(\sum_{k_1} (\mathbf{1}_{A_{m_1} \cap Q_{n_1}} g)^2_{k_1} (\mathbf{1}_{A_{m_2} \cap Q_{n_2}} h)^2_{k_1 - k_0} \Big)^{\frac12} \\
&\lesssim \|f\|_{\ell^2} \sum^*_{N_1,N_2} \|\mathbf{1}_{\langle k_1\rangle \sim N_1} g\|_{\ell^2} \|\mathbf{1}_{\langle k_2\rangle \sim N_2} h\|_{\ell^2} \sum_{N_0 \lesssim N_1} \Big(\frac{N_0}{N_1}\Big)^{\frac14} \sum_M \min\Big\{ \frac{N_0^2}{M}, \frac{M^{1/2}}{N_0} \Big\}^{\frac14} \\
&\lesssim \|f\|_{\ell^2} \|g\|_{\ell^2} \|h\|_{\ell^2}.
\end{aligned}$$

2.3.3. *Low modulation interactions.* The remaining case $|\mu_\pm| \ll k_{\max}$ can also be treated by mimicking the proof of the corresponding bilinear estimates in [9, Section 3.3]. Note that we need more delicate analysis including decomposition with respect to the angles between frequencies.

Here, we take a different approach. It was mentioned in [10, Remark 1.2] that some of the multilinear estimates required for the normal form reduction argument have close relationship with the standard multilinear estimates in Bourgain spaces, which are used to prove *conditional* well-posedness. In our setting, the desired estimate (2.3) corresponds to the bilinear estimate

$$\begin{aligned}
&\Big\| \frac{1}{\langle \tau_1 + |k_1|^2\rangle^{b_1}} \int_{\mathbb{R}} \sum_{\substack{k_0 \in \mathbb{Z}^d_\lambda \\ 1 \ll |k_0| \lesssim |k_1| \sim |k_1 - k_0| \\ |\mu_\pm| \ll k_{\max}}} \widetilde{w}(\tau_0,k_0)\widetilde{u}(\tau_1 - \tau_0, k_1 - k_0)\,d\tau_0 \Big\|_{L^2_{\tau_1,k_1}} \\
&\lesssim \big\| \langle k_0\rangle^l \langle \tau_0 \mp \alpha|k_0|\rangle^{b_0} \widetilde{w}(\tau_0,k_0) \big\|_{L^2_{\tau_0,k_0}} \big\| \langle \tau_2 + |k_2|^2\rangle^{b_2} \widetilde{u}(\tau_2,k_2) \big\|_{L^2_{\tau_2,k_2}}
\end{aligned} \tag{2.4}$$

with $b_0 = b_1 = b_2 = \frac{1}{2}$. It is not clear whether the equivalence of these estimates holds in a general setting. Nevertheless, we will see that (2.4) implies (2.3) if $b_0 + b_1 + b_2 < 1$:

Lemma 2.2. *Let $s_1, s_2, l \in \mathbb{R}$, $\gamma \geq 0$, and Ω be a subset of $\{(k_0, k_1, k_2) \in (\mathbb{Z}^d_\lambda)^3 \,|\, k_0 = k_1 - k_2\}$. Assume that there exist $b_0, b_0', b_1, b_2, b_2' \geq 0$ with $b_0 + b_1 + b_2, b_0' + b_1 + b_2' < \frac{1}{2} + \gamma$ such that*

$$\begin{aligned}
&\Big\| \frac{\langle k_1\rangle^{s_1}}{\langle \rho_1\rangle^{b_1}} \int_{\tau_1=\tau_0+\tau_2} \sum_{k_0,k_2\in\mathbb{Z}^d_\lambda} \mathbf{1}_\Omega(k_0,k_1,k_2)\mathbf{1}_{\langle\rho_0\rangle\leq\langle\rho_2\rangle\lesssim\langle\rho_1\rangle\sim\langle\mu_\pm\rangle} \widetilde{w}(\tau_0,k_0)\widetilde{u}(\tau_2,k_2)\,d\tau_0\Big\|_{L^2_{\tau_1,k_1}} \\
&\quad\lesssim \big\|\langle k_0\rangle^l\langle\rho_0\rangle^{b_0}\widetilde{w}(\tau_0,k_0)\big\|_{L^2_{\tau_0,k_0}} \big\|\langle k_2\rangle^{s_2}\langle\rho_2\rangle^{b_2}\widetilde{u}(\tau_2,k_2)\big\|_{L^2_{\tau_2,k_2}}, \\
&\Big\| \frac{\langle k_1\rangle^{s_1}}{\langle \rho_1\rangle^{b_1}} \int_{\tau_1=\tau_0+\tau_2} \sum_{k_0,k_2\in\mathbb{Z}^d_\lambda} \mathbf{1}_\Omega(k_0,k_1,k_2)\mathbf{1}_{\langle\rho_2\rangle\leq\langle\rho_0\rangle\lesssim\langle\rho_1\rangle\sim\langle\mu_\pm\rangle} \widetilde{w}(\tau_0,k_0)\widetilde{u}(\tau_2,k_2)\,d\tau_0\Big\|_{L^2_{\tau_1,k_1}} \\
&\quad\lesssim \big\|\langle k_0\rangle^l\langle\rho_0\rangle^{b_0'}\widetilde{w}(\tau_0,k_0)\big\|_{L^2_{\tau_0,k_0}} \big\|\langle k_2\rangle^{s_2}\langle\rho_2\rangle^{b_2'}\widetilde{u}(\tau_2,k_2)\big\|_{L^2_{\tau_2,k_2}},
\end{aligned} \tag{2.5}$$

where $\rho_0 := \tau_0 \mp \alpha|k_0|$, $\rho_1 := \tau_1 + |k_1|^2$, and $\rho_2 := \tau_2 + |k_2|^2$. Then, we have

$$\Big\| \sum_{k_0,k_2\in\mathbb{Z}^d_\lambda} \frac{\mathbf{1}_\Omega(k_0,k_1,k_2) f_{k_0} h_{k_2}}{\langle\mu_\pm\rangle^\gamma}\Big\|_{(\ell^2_{s_1})_{k_1}} \lesssim \|f\|_{\ell^2_l}\|h\|_{\ell^2_{s_2}}.$$

Proof. Let

$$\begin{aligned}
\mathcal{I} &:= \big\{ (\tfrac{101}{100})^n \,\big|\, n\in\mathbb{Z},\ n\geq 0\big\}, \\
\Omega_{L,\sigma} &:= \big\{ (k_0,k_1,k_2)\in\Omega \,\big|\, 1+|\mu_\pm| \in [L, \tfrac{101}{100}L),\ \sigma\mu_\pm \geq 0\big\} \qquad (L\in\mathcal{I},\ \sigma\in\{\pm1\}).
\end{aligned}$$

Take arbitrary non-negative sequences $f\in\ell^2_l$, $h\in\ell^2_{s_2}$, and define

$$\widetilde{w}_L(\tau,k) := \mathbf{1}_{[-\frac{L}{10},\frac{L}{10}]}(\tau\mp\alpha|k|) f_k, \qquad \widetilde{u}_L(\tau,k) := \mathbf{1}_{[-\frac{L}{10},\frac{L}{10}]}(\tau+|k|^2) h_k \qquad (L\in\mathcal{I}).$$

We observe that, for $(k_0,k_1,k_2)\in\Omega_{L,\sigma}$ and $\tau_1\in\mathbb{R}$,

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbf{1}_{[-\frac{L}{10},\frac{L}{10}]}(\tau_0\mp\alpha|k_0|)\mathbf{1}_{[-\frac{L}{10},\frac{L}{10}]}(\tau_1-\tau_0+|k_2|^2)\,d\tau_0 \\
&\qquad\geq \frac{L}{10}\mathbf{1}_{[-\frac{L}{10},\frac{L}{10}]}(\tau_1+|k_1|^2-\mu_\pm) \geq \frac{L}{10}\mathbf{1}_{[-\frac{L}{20},\frac{L}{20}]}(\tau_1+|k_1|^2-\sigma(L-1)),
\end{aligned}$$

and

$$\int_{\mathbb{R}} \mathbf{1}_{[-\frac{L}{10},\frac{L}{10}]}(\tau_0\mp\alpha|k_0|)\mathbf{1}_{[-\frac{L}{10},\frac{L}{10}]}(\tau_1-\tau_0+|k_2|^2)\,d\tau_0 \neq 0 \qquad\Rightarrow\qquad \langle\rho_1\rangle\sim\langle\mu_\pm\rangle\sim L.$$

Hence, for each $L\in\mathcal{I}$ and $\sigma\in\{\pm1\}$, we have

$$\begin{aligned}
&\Big\| \sum_{k_0,k_2\in\mathbb{Z}^d_\lambda} \frac{\mathbf{1}_{\Omega_{L,\sigma}}(k_0,k_1,k_2) f_{k_0} h_{k_2}}{\langle\mu_\pm\rangle^\gamma}\Big\|_{(\ell^2_{s_1})_{k_1}} \\
&\sim L^{-\frac{1}{2}}\Big\| \langle k_1\rangle^{s_1} \|\mathbf{1}_{[-\frac{L}{20},\frac{L}{20}]}(\rho_1-\sigma(L-1))\|_{L^2_{\tau_1}} \sum_{k_0,k_2\in\mathbb{Z}^d_\lambda} \frac{\mathbf{1}_{\Omega_{L,\sigma}} f_{k_0} h_{k_2}}{\langle\mu_\pm\rangle^\gamma}\Big\|_{(\ell^2)_{k_1}} \\
&\lesssim L^{-\frac{3}{2}}\Big\| \langle k_1\rangle^{s_1} \int_{\mathbb{R}} \sum_{k_0,k_2\in\mathbb{Z}^d_\lambda} \frac{\mathbf{1}_{\Omega_{L,\sigma}}}{\langle\mu_\pm\rangle^\gamma} \widetilde{w}_L(\tau_0,k_0)\widetilde{u}_L(\tau_1-\tau_0,k_2)\,d\tau_0\Big\|_{L^2_{\tau_1,k_1}} \\
&\lesssim L^{-\frac{3}{2}-\gamma+b_1}\Big\| \frac{\langle k_1\rangle^{s_1}}{\langle\rho_1\rangle^{b_1}} \int_{\tau_1=\tau_0+\tau_2} \sum_{k_0,k_2\in\mathbb{Z}^d_\lambda} \mathbf{1}_{\Omega_{L,\sigma}}\mathbf{1}_{\langle\rho_0\rangle,\langle\rho_2\rangle\lesssim\langle\rho_1\rangle\sim\langle\mu_\pm\rangle} \widetilde{w}_L(\tau_0,k_0)\widetilde{u}_L(\tau_2,k_2)\,d\tau_0\Big\|_{L^2_{\tau_1,k_1}},
\end{aligned}$$

and then, using (2.5),

$$\lesssim L^{-\frac{3}{2}-\gamma+b_1}\Big(\big\|\langle k_0\rangle^l\langle\rho_0\rangle^{b_0}\widetilde{w}_L(\tau_0,k_0)\big\|_{L^2_{\tau_0,k_0}}\big\|\langle k_2\rangle^{s_2}\langle\rho_2\rangle^{b_2}\widetilde{u}_L(\tau_2,k_2)\big\|_{L^2_{\tau_2,k_2}}$$
$$+\big\|\langle k_0\rangle^l\langle\rho_0\rangle^{b_0'}\widetilde{w}_L(\tau_0,k_0)\big\|_{L^2_{\tau_0,k_0}}\big\|\langle k_2\rangle^{s_2}\langle\rho_2\rangle^{b_2'}\widetilde{u}_L(\tau_2,k_2)\big\|_{L^2_{\tau_2,k_2}}\Big)$$
$$\lesssim \|f\|_{\ell^2_l}\|h\|_{\ell^2_{s_2}}\big(L^{-\frac{1}{2}-\gamma+b_0+b_1+b_2}+L^{-\frac{1}{2}-\gamma+b_0'+b_1+b_2'}\big).$$

From the assumption on b_0,b_0',b_1,b_2,b_2', we have

$$\Big\|\sum_{k_0,k_2\in\mathbb{Z}^d_\lambda}\frac{\mathbf{1}_\Omega f_{k_0}h_{k_2}}{\langle\mu_\pm\rangle^\gamma}\Big\|_{(\ell^2_{s_1})_{k_1}}\le\sum_{L\in\mathcal{I},\,\sigma\in\{\pm1\}}\Big\|\sum_{k_0,k_2\in\mathbb{Z}^d_\lambda}\frac{\mathbf{1}_{\Omega_{L,\sigma}}f_{k_0}h_{k_2}}{\langle\mu_\pm\rangle^\gamma}\Big\|_{(\ell^2_{s_1})_{k_1}}\lesssim\|f\|_{\ell^2_l}\|h\|_{\ell^2_{s_2}},$$

as desired. □

From [9, Propositions 3.9, 3.6], we can easily deduce the bilinear estimates (2.5) for $s_1=s_2=0$, $l=0$ if $d=2$ and $l>\frac{d-2}{2}$ if $d\ge3$, and $\Omega=\{(k_0,k_1,k_2)\,|\,k_0=k_1-k_2,|\mu_\pm|\ll k_{\max},|k_0|\gg1\}$, under the condition that $b_1=b_2=b_0'>\frac{3}{8}$, $b_0=b_2'>0$. In view of Lemma 2.2, the desired estimate (2.3) is obtained.

This completes the proof of Proposition 2.1.

3. Proof of convergence as $\alpha\to\infty$

3.1. **Preliminaries.** Before the proof, we first reduce the problem to the case of mean-zero wave part. As mentioned in Section 1, any solution $(u^\alpha,n^\alpha)\in C_T\mathcal{H}^{1,0}$ to (1.2) (in the sense of distribution) is also a solution to

$$\begin{cases} i\partial_tu^\alpha+\Delta u^\alpha=\big(P_{\neq c}n^\alpha+P_cn_0^\alpha+tP_cn_1^\alpha\big)u^\alpha,\\ \dfrac{1}{\alpha^2}\partial_t^2P_{\neq c}n^\alpha-\Delta P_{\neq c}n^\alpha=\Delta(|u^\alpha|^2),\qquad (t,x)\in(0,T)\times\mathbb{T}^d_\lambda,\\ \big(u^\alpha,P_{\neq c}n^\alpha,\partial_tP_{\neq c}n^\alpha\big)\big|_{t=0}=(u_0^\alpha,P_{\neq c}n_0^\alpha,P_{\neq c}n_1^\alpha).\end{cases}$$

We introduce

$$(\widetilde{u}^\alpha,\widetilde{n}^\alpha)(t):=\Big(u^\alpha(t)e^{i(tP_cn_0^\alpha+\frac{t^2}{2}P_cn_1^\alpha)},P_{\neq c}n^\alpha(t)\Big),$$

which solves

$$\begin{cases} i\partial_t\widetilde{u}^\alpha+\Delta\widetilde{u}^\alpha=\widetilde{n}^\alpha\widetilde{u}^\alpha,\quad \dfrac{1}{\alpha^2}\partial_t^2\widetilde{n}^\alpha-\Delta\widetilde{n}^\alpha=\Delta(|\widetilde{u}^\alpha|^2),\qquad (t,x)\in(0,T)\times\mathbb{T}^d_\lambda,\\ \big(\widetilde{u}^\alpha,\widetilde{n}^\alpha,\partial_t\widetilde{n}^\alpha\big)\big|_{t=0}=(u_0^\alpha,\widetilde{n}_0^\alpha,\widetilde{n}_1^\alpha):=(u_0^\alpha,P_{\neq c}n_0^\alpha,P_{\neq c}n_1^\alpha)\ \in\mathcal{H}_0^{1,0}(\mathbb{T}^d_\lambda),\end{cases}\tag{3.1}$$

where

$$H_0^l(\mathbb{T}^d_\lambda):=P_{\neq c}H^l(\mathbb{T}^d_\lambda),\qquad \mathcal{H}_0^{1,0}(\mathbb{T}^d_\lambda):=H^1(\mathbb{T}^d_\lambda;\mathbb{C})\times L_0^2(\mathbb{T}^d_\lambda;\mathbb{R})\times H_0^{-1}(\mathbb{T}^d_\lambda;\mathbb{R}).$$

Conversely, for any $(u_0^\alpha,P_{\neq c}n_0^\alpha,P_{\neq c}n_1^\alpha)\in\mathcal{H}_0^{1,0}$ the maximal-lifespan solution of (3.1) exists uniquely in $C([0,T^\alpha);\mathcal{H}_0^{1,0})$, and (with $P_cn_0^\alpha,P_cn_1^\alpha\in\mathbb{R}$ given) the maximal-lifespan solution of the original equation (1.2) (with the same maximal existence time) is given by

$$(u^\alpha,n^\alpha)(t)=\Big(\widetilde{u}^\alpha(t)e^{-i(tP_cn_0^\alpha+\frac{t^2}{2}P_cn_1^\alpha)},\widetilde{n}^\alpha(t,x)+P_cn_0^\alpha+tP_cn_1^\alpha\Big).$$

Clearly, Theorem 1.4 follows once we prove the following:

Proposition 3.1. *Let* $\{u_0^\alpha, \widetilde{n}_0^\alpha, \widetilde{n}_1^\alpha\}_\alpha \subset \mathcal{H}_0^{1,0}(\mathbb{T}_\lambda^d)$ *be a family of initial data satisfying*

$$\exists\, u_0^\infty := \lim_{\alpha\to\infty} u_0^\alpha \qquad \text{in } H^1(\mathbb{T}_\lambda^d), \tag{3.2}$$

$$\sup_\alpha \big\| (\widetilde{n}_0^\alpha, |\alpha\nabla|^{-1}\widetilde{n}_1^\alpha) \big\|_{L^2\times L^2} < \infty, \tag{3.3}$$

$$\lim_{R\to\infty} \limsup_{\alpha\to\infty} \big\| (P_{>R}\widetilde{n}_0^\alpha, |\alpha\nabla|^{-1}P_{>R}\widetilde{n}_1^\alpha) \big\|_{L^2\times L^2} = 0. \tag{3.4}$$

Let $(\widetilde{u}^\alpha, \widetilde{n}^\alpha) \in C([0,T^\alpha); \mathcal{H}_0^{1,0}(\mathbb{T}_\lambda^d))$ *be the (unique) maximal-lifespan solution of* (3.1)*, and let* $\widetilde{u} \in C([0,T^\infty); H^1)$ *be the (unique) maximal-lifespan solution of the Cauchy problem*

$$\begin{cases} i\partial_t \widetilde{u} + \Delta \widetilde{u} = -P_{\neq c}(|\widetilde{u}|^2)\widetilde{u}, & (t,x) \in (0,T^\infty)\times \mathbb{T}_\lambda^d, \\ \widetilde{u}(0,x) = u_0^\infty(x), & x \in \mathbb{T}_\lambda^d. \end{cases} \tag{3.5}$$

Then, we have

$$T^\infty \le \liminf_{\alpha\to\infty} T^\alpha, \tag{3.6}$$

and for any $T \in (0, T^\infty)$*,*

$$\big(\widetilde{u}^\alpha,\, \widetilde{n}^\alpha - \widetilde{n}_{il}^\alpha,\, |\alpha\nabla|^{-1}\partial_t(\widetilde{n}^\alpha - \widetilde{n}_{il}^\alpha)\big) \;\to\; \big(\widetilde{u},\, -P_{\neq c}(|\widetilde{u}|^2),\, 0\big) \quad \text{in } C([0,T]; H^1\times L_0^2 \times L_0^2) \tag{3.7}$$

as $\alpha\to\infty$*, where* $\widetilde{n}_{il}^\alpha \in C(\mathbb{R}; L_0^2(\mathbb{T}_\lambda^d;\mathbb{R})) \cap C^1(\mathbb{R}; H_0^{-1}(\mathbb{T}_\lambda^d;\mathbb{R}))$ *is the solution of the following linear wave equation:*

$$\begin{cases} \dfrac{1}{\alpha^2}\partial_t^2 \widetilde{n}_{il}^\alpha - \Delta \widetilde{n}_{il}^\alpha = 0, & (t,x)\in\mathbb{R}\times\mathbb{T}_\lambda^d, \\ \big(\widetilde{n}_{il}^\alpha,\, \partial_t \widetilde{n}_{il}^\alpha\big)\big|_{t=0} = \big(\widetilde{n}_0^\alpha + P_{\neq c}(|u_0^\alpha|^2),\, \widetilde{n}_1^\alpha\big). \end{cases}$$

For the solution of (3.1) in $C_T\mathcal{H}_0^{1,0}$, with the property $P_c\partial_t\widetilde{n}^\alpha(t) \equiv 0$, the mass and the energy

$$M(\widetilde{u}^\alpha(t)) := \big\|\widetilde{u}^\alpha(t)\big\|_{L^2}^2,$$

$$\mathcal{E}^\alpha(\widetilde{u}^\alpha(t), \widetilde{n}^\alpha(t)) := \big\|\nabla\widetilde{u}^\alpha(t)\big\|_{L^2}^2 + \frac12\big\|\widetilde{n}^\alpha(t)\big\|_{L^2}^2 + \frac12\big\||\alpha\nabla|^{-1}\partial_t\widetilde{n}^\alpha(t)\big\|_{L^2}^2 + \int_{\mathbb{T}_\lambda^d} \widetilde{n}^\alpha(t)|\widetilde{u}^\alpha(t)|^2$$

are well-defined and formally conserved. The solution of (3.5) (as well as that of the standard NLS (1.3)) in the energy class $\widetilde{u}\in H^1$ also (formally) conserves the mass $M(\widetilde{u}(t))$ and the energy

$$\mathcal{E}(\widetilde{u}(t)) := \big\|\nabla\widetilde{u}(t)\big\|_{L^2}^2 - \frac12\big\|\widetilde{u}(t)\big\|_{L^4}^4.$$

It is worth noticing that the energy functionals for (3.1) and (3.5) have the following relation:

$$\mathcal{E}^\alpha(\widetilde{u}^\alpha, \widetilde{n}^\alpha) = \mathcal{E}(\widetilde{u}^\alpha) + \frac12\big\|\widetilde{n}^\alpha + |\widetilde{u}^\alpha|^2 - i|\alpha\nabla|^{-1}\partial_t\widetilde{n}^\alpha\big\|_{L^2}^2.$$

We recall the result on local well-posedness of these Cauchy problems in the energy space including (rigorous) conservation laws, which is a crucial tool to prove Proposition 3.1.

Lemma 3.2 (Local well-posedness; [18, 9, 2, 4])**.** *Let* $d = 1,2$ *for* (3.1) *and* $d = 1,2,3$ *for* (3.5)*,* $\lambda \in (0,\infty)^d$ *be arbitrary. Then, the initial value problems for* (3.1) *(with any* $\alpha > 0$*) and* (3.5) *on* $\mathbb{T}_\lambda^d$ *are locally well-posed in the energy space* $\mathcal{H} = \mathcal{H}_0^{1,0}(\mathbb{T}_\lambda^d)$ *and* $H^1(\mathbb{T}_\lambda^d)$*, respectively. In particular, for any initial data in* $\mathcal{H}$*, there exists a local-in-time solution in* $C([0,T];\mathcal{H})$*, with existence time* $T > 0$ *depending only on the size of the initial data in* $\mathcal{H}$ *(and also on* α *in the case of* (3.1)*), which depends continuously on the initial data. Moreover, the mass and the energy are conserved for these solutions.*[5]

[5]This can be deduced from the local well-posedness result in the energy space by a standard approximation argument based on persistence of regularity and continuous dependence of solutions upon initial data.

Another important ingredient of the proof is the following:

Lemma 3.3 (Unconditional uniqueness; [7]). *Let $d = 1, 2, 3$, $\lambda \in (0,\infty)^d$ be arbitrary, and $T > 0$. For any $u_0 \in H^1(\mathbb{T}^d_\lambda)$, there are at most one solution (in the sense of distribution) of (3.5) in $L^\infty(0,T;H^1(\mathbb{T}^d_\lambda))$ satisfying $u(0) = u_0$.*[6]

Remark 3.4. (i) The known results [2, 4, 7] on local well-posedness and unconditional uniqueness in the energy space for the cubic NLS (1.3) on $\mathbb{T}^d_\lambda$ are transformed into the same results for shifted NLS (3.5) and (1.5) by the changes of unknown function

$$u(t,x) \quad \mapsto \quad u(t,x)\exp\Big\{ i\int_0^t \frac{1}{|\mathbb{T}^d_\lambda|}\|u(t')\|^2_{L^2(\mathbb{T}^d_\lambda)}\,dt'\Big\} \qquad \text{for (3.5)},$$

$$u(t,x) \quad \mapsto \quad u(t,x)\exp\Big\{ i\int_0^t \Big(\frac{1}{|\mathbb{T}^d_\lambda|}\|u(t')\|^2_{L^2(\mathbb{T}^d_\lambda)} + \nu_0 + \nu_1 t'\Big)\,dt'\Big\} \qquad \text{for (1.5)}.$$

As easily seen, these maps are homeomorphisms on $L^\infty(0,T;H^1(\mathbb{T}^d_\lambda))$ or $C([0,T];H^1(\mathbb{T}^d_\lambda))$ for any $T > 0$ and transform a solution (in the sense of distribution) of (3.5) and (1.5), respectively, to a solution of (1.3).

(ii) In [7], uniqueness of solutions to (1.3) on $\mathbb{T}^d_\lambda$, $d = 2,3$, was shown in the class of mild H^s-solutions (see [7, Definition 1.1]) for some $s < 1$. First, we see that any distributional solution in $C([0,T];H^s)$ turns out to be a mild H^s-solution if $d = 2,3$ and s is close to 1; see [11, Remark 1.3] for details. Then, any distributional solution in $L^\infty(0,T;H^1)$ belongs to $W^{1,\infty}(0,T;H^{-1}) \subset C([0,T];H^{-1})$ by the equation and hence to $C([0,T];H^s)$ for any $s < 1$ by interpolation. Consequently, we can deduce uniqueness in $L^\infty(0,T;H^1)$ from the result in [7]. In the one-dimensional case, uniqueness holds in $C([0,T];H^s)$ for $s > \frac{1}{2}$ by the Sobolev inequality, which implies uniqueness in $L^\infty(0,T;H^1)$ as above.

(iii) To prove Proposition 3.1 we need uniqueness of the solution to (3.5) in $L^\infty(0,T;H^1)$; in fact, uniqueness in $C([0,T];H^1)$ is not sufficient. For the Zakharov system (1.2), we have proved uniqueness in $C([0,T];\mathcal{H}^{s,l})$ as "unconditional uniqueness" in Theorem 1.2. Concerning the energy-space regularity, uniqueness in a wider class $L^\infty(0,T;\mathcal{H}^{1,0})$ follows from Theorem 1.2 in the case $d = 1$ and $\alpha\lambda \notin \mathbb{Z}$ by the same argument as above, whereas it does not follow if $\alpha\lambda \in \mathbb{Z}$ or in the two-dimensional case, since we do not have uniqueness in $C([0,T],\mathcal{H}^{s,l})$ with $l < 0$. Note, however, that uniqueness in $L^\infty(0,T;\mathcal{H}^{1,0})$ for (1.2) will not be required in our proof of Proposition 3.1.

3.2. **Proof.** Now, we present a proof of Proposition 3.1. We follow closely the argument for the non-periodic case given in [13, Section 6].

Proof of Proposition 3.1. We focus on the two-dimensional case; the one-dimensional case can be treated by the same argument with some modifications on exponents related to the Sobolev embedding. We proceed in several steps.

Step 1: We shall show uniform-in-α a priori bound on the energy norm of $(\widetilde{u}^\alpha, \widetilde{n}^\alpha)$: There exists $T_0 > 0$ and $C > 0$ independent of α such that

$$X_{\alpha,T_0} := \max_{0\le t\le T_0}\Big(\|\widetilde{u}^\alpha(t)\|^2_{H^1} + \frac{1}{2}\|\widetilde{n}^\alpha(t)\|^2_{L^2} + \frac{1}{2}\big\||\alpha\nabla|^{-1}\partial_t\widetilde{n}^\alpha(t)\big\|^2_{L^2}\Big) \le C. \tag{3.8}$$

[6]Any distributional solution $u(t)$ in $L^\infty(0,T;H^1)$ belongs to $W^{1,\infty}(0,T;H^{-1})$ by the equation, and thus has limits in H^{-1} at endpoints $t \to 0, T$ and is extended to a function in $C([0,T];H^{-1})$. The initial condition then makes sense in H^{-1}.

In particular, by Lemma 3.2, it holds that $T^\alpha > T_0$ for any α.

By the conservation laws and (3.2), (3.3), together with the Hölder inequality and the Sobolev embedding, the conserved quantities $M(\widetilde{u}^\alpha(t))$ and $\mathcal{E}^\alpha(\widetilde{u}^\alpha(t),\widetilde{n}^\alpha(t))$ are bounded uniformly in α as long as the solution exists. Since

$$X_{\alpha,T} = \max_{0\le t\le T}\Big(M(\widetilde{u}^\alpha(t)) + \mathcal{E}^\alpha(\widetilde{u}^\alpha(t),\widetilde{n}^\alpha(t)) - \int \widetilde{n}^\alpha(t)|\widetilde{u}^\alpha(t)|^2\Big),$$

it suffices to control the cubic term $\int \widetilde{n}^\alpha|\widetilde{u}^\alpha|^2$. By the Hölder inequality, the Sobolev embedding, interpolation and the Duhamel formula, we see that, for $t\in[0,T]$,

$$\begin{aligned}\Big|\int \widetilde{n}^\alpha(t)|\widetilde{u}^\alpha(t)|^2\Big| &\lesssim \|\widetilde{n}^\alpha(t)\|_{L^2}\Big(\|e^{it\Delta}u_0^\alpha\|_{H^{1/2}}^2 + \|\widetilde{u}^\alpha(t)-e^{it\Delta}u_0^\alpha\|_{H^{1/2}}^2\Big)\\ &\lesssim X_{\alpha,T}^{\frac12}\Big(\|u_0^\alpha\|_{H^{1/2}}^2 + \|\widetilde{u}^\alpha(t)-e^{it\Delta}u_0^\alpha\|_{H^1}^{\frac43}\|\widetilde{n}^\alpha\widetilde{u}^\alpha\|_{L^1(0,T;H^{-1/2})}^{\frac23}\Big),\end{aligned}$$

which is, by Sobolev and interpolation again as well as the mass conservation law, bounded by

$$\begin{aligned}&\lesssim X_{\alpha,T}^{\frac12}\Big(\|u_0^\alpha\|_{H^{1/2}}^2 + (X_{\alpha,T}^{\frac23}+\|u_0^\alpha\|_{H^1}^{\frac43})T^{\frac23}\|\widetilde{n}^\alpha\|_{L^\infty(0,T;L^2)}^{\frac23}\|\widetilde{u}^\alpha\|_{L^\infty(0,T;H^{1/2})}^{\frac23}\Big)\\ &\lesssim X_{\alpha,T}^{\frac12}\Big(\|u_0^\alpha\|_{H^{1/2}}^2 + (X_{\alpha,T}^{\frac23}+\|u_0^\alpha\|_{H^1}^{\frac43})T^{\frac23}X_{\alpha,T}^{\frac13}X_{\alpha,T}^{\frac16}\|u_0^\alpha\|_{L^2}^{\frac13}\Big)\\ &\lesssim \|u_0^\alpha\|_{H^{1/2}}^2X_{\alpha,T}^{\frac12} + T^{\frac23}\|u_0^\alpha\|_{H^1}^{\frac53}X_{\alpha,T} + T^{\frac23}\|u_0^\alpha\|_{L^2}^{\frac13}X_{\alpha,T}^{\frac53}.\end{aligned}$$

Using (3.2) again, we have

$$X_{\alpha,T} \le C_0(1+T^{\frac23}) + C_1T^{\frac23}X_{\alpha,T}^{\frac53}$$

for some constants $C_0, C_1>0$ independent of α. Since $X_{\alpha,T}$ is continuous in T, a bootstrap argument shows $X_{\alpha,T}\le 2C_0$ if T is sufficiently small depending on C_0, C_1, which yields (3.8).

Step 2: Let T_0 be as in Step 1. We shall show that for any sequence $\alpha_k\to\infty$ there exist a subsequence α_{k_l} and $\widetilde{u}^\infty\in L^\infty(0,T_0;H^1)\cap C([0,T_0];H^{1/2})$ such that

$$\begin{aligned}\widetilde{u}^{\alpha_{k_l}} &\to \widetilde{u}^\infty \qquad \text{in } C([0,T_0];\text{w-}H^1\cap H^{1/2}),\\ \widetilde{n}^{\alpha_{k_l}} + |\widetilde{u}^{\alpha_{k_l}}|^2 &\rightharpoonup P_c(|\widetilde{u}^\infty|^2) \qquad \text{weakly in } L^2((0,T_0)\times\mathbb{T}^2_\lambda).\end{aligned}$$

Here, convergence in $C([0,T_0];\text{w-}H^1)$ means that

$$\sup_{0\le t\le T_0}\big|\big\langle \widetilde{u}^{\alpha_{k_l}}(t)-\widetilde{u}^\infty(t),\,\psi(t)\big\rangle_{H^1}\big|\to 0,\qquad \psi\in C([0,T_0];H^1).$$

In particular, by the Sobolev embedding, $\widetilde{u}^{\alpha_{k_l}}\to\widetilde{u}^\infty$ strongly in $C([0,T_0];L^4)$.

Let us first establish the convergence of $\widetilde{u}^\alpha$. By Step 1, $\{(\widetilde{u}^\alpha,\widetilde{n}^\alpha)\}_\alpha$ is bounded in $C([0,T_0];H^1\times L^2)$, so that $\{\partial_t\widetilde{u}^\alpha = i(\Delta\widetilde{u}^\alpha - \widetilde{n}^\alpha\widetilde{u}^\alpha)\}_\alpha$ is bounded in $C([0,T_0];H^{-1})$. This implies that $\{\widetilde{u}^\alpha\}_\alpha$ is equicontinuous in H^{-1} at any $t\in[0,T_0]$, and thus in H^s for any $s<1$ by interpolation. Since $\{\widetilde{u}^\alpha(t)\}_\alpha$ is relatively compact in H^s for $s<1$ by the compact embedding $H^1\hookrightarrow H^s$, Ascoli's theorem (cf. [12, Chapter III, Theorem 3.1]) shows that $\{\widetilde{u}^\alpha\}_\alpha$ is relatively compact in $C([0,T_0];H^s)$ for $s<1$. The case of $s=1/2$ implies, for any $\{\alpha_k\}_k$, existence of a subsequence $\{\widetilde{u}^{\alpha_{k_l}}\}_l$ converging to some $\widetilde{u}^\infty$ strongly in $C([0,T_0];H^{1/2})$. Moreover, since for each $t\in[0,T_0]$ (any subsequence of) the bounded sequence $\{\widetilde{u}^{\alpha_{k_l}}(t)\}_l\subset H^1$ has a weakly convergent subsequence, we see the sequence itself converges to $\widetilde{u}^\infty(t)$ weakly in H^1. The weak lower semicontinuity of the norm and the bound from Step 1 then show that $\widetilde{u}^\infty\in L^\infty(0,T_0;H^1)$. Finally, for any $\psi \in C([0,T_0];H^1)$, we use strong convergence in $C([0,T_0];H^{1/2})$ and boundedness of $\widetilde{u}^\infty(t)$ in H^1 obtained so far and notice $\lim_{R\to\infty}\|P_{>R}\psi\|_{L^\infty(0,T_0;H^1)}=0$ to have

$$\begin{aligned}
&\limsup_{l\to\infty}\sup_{0\le t\le T_0}\big|\big\langle \widetilde{u}^{\alpha_{k_l}}(t)-\widetilde{u}^\infty(t),\,\psi(t)\big\rangle_{H^1}\big| \\
&\quad\le \lim_{l\to\infty}\|\widetilde{u}^{\alpha_{k_l}}-\widetilde{u}^\infty\|_{L^\infty(0,T_0;H^{1/2})}\|P_{\le R}\psi\|_{L^\infty(0,T_0;H^{3/2})} \\
&\qquad\qquad +\Big(\sup_l\|\widetilde{u}^{\alpha_{k_l}}\|_{L^\infty(0,T_0;H^1)}+\|\widetilde{u}^\infty\|_{L^\infty(0,T_0;H^1)}\Big)\|P_{>R}\psi\|_{L^\infty(0,T_0;H^1)} \\
&\quad\to 0 \qquad (R\to\infty),
\end{aligned}$$

which shows convergence in $C([0,T_0];\text{w-}H^1)$.

Next, we derive weak convergence of $\widetilde{n}^\alpha+|\widetilde{u}^\alpha|^2$. We see $\Delta(\widetilde{n}^{\alpha_{k_l}}+|\widetilde{u}^{\alpha_{k_l}}|^2)=\alpha_{k_l}^{-2}\partial_t^2\widetilde{n}^{\alpha_{k_l}}\to 0$ in $\mathcal{D}'((0,T_0)\times\mathbb{T}^2_\lambda)$ by the uniform bound on $\widetilde{n}^\alpha$ from Step 1. This particularly implies that $\widetilde{n}^{\alpha_{k_l}}+P_{\neq c}(|\widetilde{u}^{\alpha_{k_l}}|^2)\to 0$ in $\mathcal{D}'((0,T_0)\times\mathbb{T}^2_\lambda)$. Moreover, strong convergence of $\{\widetilde{u}^{\alpha_{k_l}}\}_l$ obtained above shows $P_c(|\widetilde{u}^{\alpha_{k_l}}|^2)\to P_c(|\widetilde{u}^\infty|^2)$ in $C([0,T_0])$. Consequently, we have $\widetilde{n}^{\alpha_{k_l}}+|\widetilde{u}^{\alpha_{k_l}}|^2\to P_c(|\widetilde{u}^\infty|^2)$ in $\mathcal{D}'((0,T_0)\times\mathbb{T}^2_\lambda)$. On the other hand, (any subsequence of) $\{\widetilde{n}^{\alpha_{k_l}}+|\widetilde{u}^{\alpha_{k_l}}|^2\}_l$ is bounded in $L^2((0,T_0)\times\mathbb{T}^2_\lambda)$ and therefore has a weakly convergent subsequence. Hence, the sequence $\{\widetilde{n}^{\alpha_{k_l}}+|\widetilde{u}^{\alpha_{k_l}}|^2\}_l$ itself converges to $P_c(|\widetilde{u}^\infty|^2)$ weakly in $L^2((0,T_0)\times\mathbb{T}^2_\lambda)$.[7]

Step 3: We shall show that $T^\infty>T_0$ and $\widetilde{u}^\alpha\to\widetilde{u}$ in $C([0,T_0];\text{w-}H^1\cap H^{1/2})$ as $\alpha\to\infty$.

We first prove that $\widetilde{u}^\infty$ given in Step 2 is a solution of (3.5) on $(0,T_0)\times\mathbb{T}^2_\lambda$ in the sense of distribution. The initial condition is easily verified from strong convergence in Step 2 and (3.2), so it suffices to show that

$$\widetilde{n}^{\alpha_{k_l}}\widetilde{u}^{\alpha_{k_l}}\ \to\ -P_{\neq c}(|\widetilde{u}^\infty|^2)\widetilde{u}^\infty\quad\text{in }\mathcal{D}'((0,T_0)\times\mathbb{T}^2_\lambda)\qquad(l\to\infty).$$

For any $\psi\in C_0^\infty((0,T_0)\times\mathbb{T}^2_\lambda)$, we see that

$$\begin{aligned}
&\Big|\int_0^{T_0}\!\!\int_{\mathbb{T}^2_\lambda}\big(\widetilde{n}^{\alpha_{k_l}}\widetilde{u}^{\alpha_{k_l}}+P_{\neq c}(|\widetilde{u}^\infty|^2)\widetilde{u}^\infty\big)\psi\,dx\,dt\Big| \\
&\quad\le\Big|\int_0^{T_0}\!\!\int_{\mathbb{T}^2_\lambda}\widetilde{n}^{\alpha_{k_l}}\big(\widetilde{u}^{\alpha_{k_l}}-\widetilde{u}^\infty\big)\psi\,dx\,dt\Big|+\Big|\int_0^{T_0}\!\!\int_{\mathbb{T}^2_\lambda}\big(\widetilde{n}^{\alpha_{k_l}}+|\widetilde{u}^{\alpha_{k_l}}|^2-P_c(|\widetilde{u}^\infty|^2)\big)\widetilde{u}^\infty\psi\,dx\,dt\Big| \\
&\qquad+\Big|\int_0^{T_0}\!\!\int_{\mathbb{T}^2_\lambda}\big(|\widetilde{u}^\infty|^2-|\widetilde{u}^{\alpha_{k_l}}|^2\big)\widetilde{u}^\infty\psi\,dx\,dt\Big| \\
&\quad\le\|\widetilde{n}^{\alpha_{k_l}}\|_{L^\infty(0,T_0;L^2)}\|\widetilde{u}^{\alpha_{k_l}}-\widetilde{u}^\infty\|_{L^\infty(0,T_0;L^2)}\|\psi\|_{L^1(0,T_0;L^\infty)} \\
&\qquad+\Big|\big\langle\widetilde{n}^{\alpha_{k_l}}+|\widetilde{u}^{\alpha_{k_l}}|^2-P_c(|\widetilde{u}^\infty|^2),\,\overline{\widetilde{u}^\infty\psi}\big\rangle_{L^2((0,T_0)\times\mathbb{T}^2_\lambda)}\Big| \\
&\qquad+\|\widetilde{u}^\infty-\widetilde{u}^{\alpha_{k_l}}\|_{L^\infty(0,T_0;L^4)}\Big(\|\widetilde{u}^\infty\|_{L^\infty(0,T_0;L^4)}+\|\widetilde{u}^{\alpha_{k_l}}\|_{L^\infty(0,T_0;L^4)}\Big) \\
&\qquad\qquad\qquad\qquad\times\|\widetilde{u}^\infty\|_{L^\infty(0,T_0;L^2)}\|\psi\|_{L^1(0,T_0;L^\infty)}.
\end{aligned}$$

By the uniform bound given in Step 1 and the convergence results proved in Step 2, the right-hand side vanishes as $l\to\infty$. Hence, $\widetilde{u}^\infty$ satisfies (3.5).

Now, we invoke Lemma 3.3 to conclude that $\widetilde{u}^\infty=\widetilde{u}\in C([0,T_0];H^1)$. In particular, $\widetilde{u}^{\alpha_{k_l}}\to\widetilde{u}$ in $C([0,T_0];\text{w-}H^1\cap H^{1/2})$ as $l\to\infty$. This is true for any sequence $\alpha_k\to\infty$, so that $\{\widetilde{u}^\alpha\}_\alpha$ itself converges to $\widetilde{u}$ as $\alpha\to\infty$.

[7]In the non-periodic case [13], $\Delta(n^{\alpha_{k_l}}+|u^{\alpha_{k_l}}|^2)\to 0$ in $\mathcal{D}'((0,T_0)\times\mathbb{R}^d)$ and weak convergence of a subsequence in $L^2((0,T_0)\times\mathbb{R}^d)$ imply that $n^{\alpha_{k_l}}+|u^{\alpha_{k_l}}|^2\rightharpoonup 0$ weakly in $L^2((0,T_0)\times\mathbb{R}^d)$. That is why $u^{\alpha_{k_l}}$ converges to a solution of the standard NLS (1.3).

Step 4: We shall show (3.7) with $T = T_0$.

Let $N^\alpha := \widetilde{n}^\alpha - i|\alpha\nabla|^{-1}\partial_t\widetilde{n}^\alpha$ and $N^\alpha_{il} := \widetilde{n}^\alpha_{il} - i|\alpha\nabla|^{-1}\partial_t\widetilde{n}^\alpha_{il}$. Note that $P_c N^\alpha(t) = P_c N^\alpha_{il}(t) \equiv 0$. N^α and N^α_{il} solve the following inhomogeneous and homogeneous linear Cauchy problems:

$$\begin{cases} \partial_t N^\alpha = i|\alpha\nabla|N^\alpha + i|\alpha\nabla|(|\widetilde{u}^\alpha|^2), \\ N^\alpha(0) = \widetilde{n}^\alpha_0 - i|\alpha\nabla|^{-1}\widetilde{n}^\alpha_1, \end{cases} \qquad \begin{cases} \partial_t N^\alpha_{il} = i|\alpha\nabla|N^\alpha_{il}, \\ N^\alpha_{il}(0) = \widetilde{n}^\alpha_0 - i|\alpha\nabla|^{-1}\widetilde{n}^\alpha_1 + P_{\neq c}(|u^\alpha_0|^2). \end{cases}$$

In particular, we have $\|N^\alpha_{il}(t)\|_{L^2} \equiv \|N^\alpha_{il}(0)\|_{L^2}$. To prove the claim, it suffices to show that

$$\sup_{0\le t\le T_0}\Big(\big\|\nabla\big(\widetilde{u}^\alpha(t)-\widetilde{u}(t)\big)\big\|^2_{L^2} + \frac{1}{2}\big\|N^\alpha(t) - N^\alpha_{il}(t) + P_{\neq c}\big(|\widetilde{u}(t)|^2\big)\big\|^2_{L^2}\Big) \to 0 \qquad (\alpha\to\infty).$$

By a direct calculation, we have

$$\begin{aligned} &\big\|\nabla(\widetilde{u}^\alpha - \widetilde{u})\big\|^2_{L^2} + \frac{1}{2}\big\|N^\alpha - N^\alpha_{il} + P_{\neq c}(|\widetilde{u}|^2)\big\|^2_{L^2} \\ &\quad = \mathcal{E}^\alpha(\widetilde{u}^\alpha, \widetilde{n}^\alpha) - \mathcal{E}(\widetilde{u}) - \frac{1}{2}\big\|N^\alpha_{il}\big\|^2_{L^2} - \frac{1}{2}\big\|P_c(|\widetilde{u}|^2)\big\|^2_{L^2} \qquad (3.9)\\ &\qquad + \mathrm{Re}\big\langle N^\alpha,\, |\widetilde{u}|^2 - |\widetilde{u}^\alpha|^2\big\rangle_{L^2} + 2\mathrm{Re}\big\langle \nabla(\widetilde{u} - \widetilde{u}^\alpha),\, \nabla\widetilde{u}\big\rangle_{L^2} \qquad (3.10)\\ &\qquad - \mathrm{Re}\big\langle N^\alpha - N^\alpha_{il} + P_{\neq c}(|\widetilde{u}|^2),\, N^\alpha_{il}\big\rangle_{L^2}. \qquad (3.11) \end{aligned}$$

The first line (3.9) consists of conserved quantities, and hence for any t,

$$\begin{aligned} (3.9) &= \mathcal{E}(u^\alpha_0) + \frac{1}{2}\big\|N^\alpha(0) + |u^\alpha_0|^2\big\|^2_{L^2} - \mathcal{E}(u^\infty_0) - \frac{1}{2}\big\|N^\alpha_{il}(0)\big\|^2_{L^2} - \frac{1}{2}\big\|P_c(|u^\infty_0|^2)\big\|^2_{L^2} \\ &= \Big(\mathcal{E}(u^\alpha_0) - \mathcal{E}(u^\infty_0)\Big) + \frac{1}{2}\Big(\big\|P_c(|u^\alpha_0|^2)\big\|^2_{L^2} - \big\|P_c(|u^\infty_0|^2)\big\|^2_{L^2}\Big), \end{aligned}$$

which vanishes as $\alpha\to\infty$ by (3.2). The second line (3.10) vanishes uniformly in t by the uniform-in-α bound from Step 1 and the convergence result from Step 3. Therefore, we only have to show that the last line (3.11) vanishes uniformly in t.

By the condition (3.4) and the Sobolev inequality;

$$\big\|P_{>R}(|u^\alpha_0|^2)\big\|_{L^2} \lesssim R^{-1/2}\big\||u^\alpha_0|^2\big\|_{H^{1/2}} \lesssim R^{-1/2}\big\|u^\alpha_0\big\|^2_{H^1},$$

we see $\limsup_{\alpha\to\infty}\|P_{>R}N^\alpha_{il}(t)\|_{L^2} = \limsup_{\alpha\to\infty}\|P_{>R}N^\alpha_{il}(0)\|_{L^2} \to 0$ as $R\to\infty$. Hence, the uniform-in-α bound from Step 1 implies that for any ε there exist $R>0$ and $\alpha_0>0$ such that for any $\alpha\ge\alpha_0$

$$\sup_{0\le t\le T_0}\Big|\big\langle N^\alpha(t) - N^\alpha_{il}(t) + P_{\neq c}(|\widetilde{u}(t)|^2),\, P_{>R}N^\alpha_{il}(t)\big\rangle_{L^2}\Big| < \varepsilon.$$

We fix such an $R>0$ and estimate the low-frequency part. Noticing

$$\begin{aligned} &\sup_{0\le t\le T_0}\Big|\big\langle N^\alpha(t) - N^\alpha_{il}(t) + P_{\neq c}(|\widetilde{u}(t)|^2),\, P_{\le R}N^\alpha_{il}(t)\big\rangle_{L^2}\Big| \\ &\qquad \lesssim \big\|N^\alpha - N^\alpha_{il} + P_{\neq c}(|\widetilde{u}|^2)\big\|_{L^\infty(0,T_0;H^{-5/2})}R^{\frac{5}{2}}\big\|N^\alpha_{il}(0)\big\|_{L^2}, \end{aligned}$$

we shall estimate the $H^{-5/2}$ norm of $N^\alpha - N^\alpha_{il} + P_{\neq c}(|\widetilde{u}|^2)$.

By the Duhamel formula and an integration by parts in t, we have

$$\begin{aligned} &N^\alpha(t) - N^\alpha_{il}(t) + P_{\neq c}(|\widetilde{u}(t)|^2) \\ &\quad = P_{\neq c}(|\widetilde{u}(t)|^2) - e^{it|\alpha\nabla|}P_{\neq c}(|u^\alpha_0|^2) - \int_0^t e^{i(t-s)|\alpha\nabla|}(-i)|\alpha\nabla|(|\widetilde{u}^\alpha(s)|^2)\,ds \\ &\quad = P_{\neq c}(|\widetilde{u}(t)|^2) - e^{it|\alpha\nabla|}P_{\neq c}(|u^\alpha_0|^2) \\ &\qquad\quad - |\widetilde{u}^\alpha(t)|^2 + e^{it|\alpha\nabla|}(|u^\alpha_0|^2) + \int_0^t e^{i(t-s)|\alpha\nabla|}\partial_s(|\widetilde{u}^\alpha(s)|^2)\,ds \end{aligned}$$

$$= P_{\neq c}(|\widetilde{u}(t)|^2) - P_{\neq c}(|\widetilde{u}^\alpha(t)|^2) + \int_0^t e^{i(t-s)|\alpha\nabla|}\partial_s(|\widetilde{u}^\alpha(s)|^2)\,ds,$$

where we have used the L^2 conservation for $\widetilde{u}^\alpha$ at the last equality. The Sobolev embedding gives a bound for the first two terms as

$$\big\||\widetilde{u}|^2 - |\widetilde{u}^\alpha|^2\big\|_{L^\infty(0,T_0;H^{-5/2})} \lesssim \Big(\|\widetilde{u}\|_{L^\infty(0,T_0;L^2)} + \|\widetilde{u}^\alpha\|_{L^\infty(0,T_0;L^2)}\Big)\|\widetilde{u} - \widetilde{u}^\alpha\|_{L^\infty(0,T_0;L^2)}.$$

On the other hand, by the equation for $\widetilde{u}^\alpha$ we have $\partial_t(|\widetilde{u}^\alpha|^2) = 2\nabla\cdot\mathrm{Re}\big(i\overline{\widetilde{u}^\alpha}\nabla\widetilde{u}^\alpha\big)$. We shall apply integration by parts once more to deal with this term.[8] Since (in the two-dimensional case) the high-frequency components will be difficult to control after integration by parts, we first remove them and then perform integration by parts, as follows: For $t \in [0,T_0]$ and $\widetilde{R} > 0$, we use the 2D Sobolev estimate $\|fg\|_{H^{-3/2}} \lesssim \|f\|_{H^{1/2}}\|g\|_{H^{-1/2}}$ to have

$$\begin{aligned}
&\Big\|2\nabla\cdot\int_0^t e^{i(t-s)|\alpha\nabla|}\mathrm{Re}\Big[i\overline{\widetilde{u}^\alpha(s)}\nabla\widetilde{u}^\alpha(s) - i\overline{P_{\le\widetilde{R}}\widetilde{u}^\alpha(s)}\nabla P_{\le\widetilde{R}}\widetilde{u}^\alpha(s)\Big]\,ds\Big\|_{L^\infty(0,T_0;H^{-5/2})}\\
&\le 2T_0\big\|\overline{\widetilde{u}^\alpha}\nabla\widetilde{u}^\alpha - \overline{P_{\le\widetilde{R}}\widetilde{u}^\alpha}\nabla P_{\le\widetilde{R}}\widetilde{u}^\alpha\big\|_{L^\infty(0,T_0;H^{-3/2})}\\
&\lesssim \|\widetilde{u}^\alpha\|_{L^\infty(0,T_0;H^{1/2})}\|P_{>\widetilde{R}}\widetilde{u}^\alpha\|_{L^\infty(0,T_0;H^{1/2})}\\
&\lesssim \widetilde{R}^{-\frac12}\|\widetilde{u}^\alpha\|_{L^\infty(0,T_0;H^1)}^2.
\end{aligned}$$

On the other hand, using the equation for $\widetilde{u}^\alpha$ again we have

$$\begin{aligned}
\partial_t\big(i\overline{P_{\le\widetilde{R}}\widetilde{u}^\alpha}\nabla P_{\le\widetilde{R}}\widetilde{u}^\alpha\big) &= \Delta\overline{P_{\le\widetilde{R}}\widetilde{u}^\alpha}\nabla P_{\le\widetilde{R}}\widetilde{u}^\alpha - \overline{P_{\le\widetilde{R}}\widetilde{u}^\alpha}\nabla\Delta P_{\le\widetilde{R}}\widetilde{u}^\alpha\\
&\quad - P_{\le\widetilde{R}}\big(\widetilde{n}^\alpha\overline{\widetilde{u}^\alpha}\big)\nabla P_{\le\widetilde{R}}\widetilde{u}^\alpha + \overline{P_{\le\widetilde{R}}\widetilde{u}^\alpha}\nabla P_{\le\widetilde{R}}\big(\widetilde{n}^\alpha\widetilde{u}^\alpha\big),
\end{aligned}$$

so the Sobolev inequality yields that

$$\big\|\partial_t\big(i\overline{P_{\le\widetilde{R}}\widetilde{u}^\alpha}\nabla P_{\le\widetilde{R}}\widetilde{u}^\alpha\big)\big\|_{H^{-5/2}} \lesssim \widetilde{R}^3\|\widetilde{u}^\alpha\|_{L^2}^2 + \widetilde{R}\|\widetilde{n}^\alpha\|_{L^2}\|\widetilde{u}^\alpha\|_{H^{1/2}}^2.$$

Then, integration by parts implies that

$$\begin{aligned}
&\Big\|2\nabla\cdot\int_0^t e^{i(t-s)|\alpha\nabla|}\mathrm{Re}\Big[i\overline{P_{\le\widetilde{R}}\widetilde{u}^\alpha(s)}\nabla P_{\le\widetilde{R}}\widetilde{u}^\alpha(s)\Big]\,ds\Big\|_{L^\infty(0,T_0;H^{-5/2})}\\
&\le 2\Big\||\alpha\nabla|^{-1}\nabla\cdot\Big(\mathrm{Re}\Big[i\overline{P_{\le\widetilde{R}}\widetilde{u}^\alpha(t)}\nabla P_{\le\widetilde{R}}\widetilde{u}^\alpha(t)\Big] - e^{it|\alpha\nabla|}\mathrm{Re}\Big[i\overline{P_{\le\widetilde{R}}u_0^\alpha}\nabla P_{\le\widetilde{R}}u_0^\alpha\Big]\Big)\Big\|_{L^\infty(0,T_0;H^{-5/2})}\\
&\quad + 2\Big\||\alpha\nabla|^{-1}\nabla\cdot\int_0^t e^{i(t-s)|\alpha\nabla|}\mathrm{Re}\partial_s\Big[i\overline{P_{\le\widetilde{R}}\widetilde{u}^\alpha(s)}\nabla P_{\le\widetilde{R}}\widetilde{u}^\alpha(s)\Big]\,ds\Big\|_{L^\infty(0,T_0;H^{-5/2})}\\
&\lesssim \alpha^{-1}\Big(\widetilde{R}\|\widetilde{u}^\alpha\|_{L^\infty(0,T_0;L^2)}^2 + T_0\widetilde{R}^3\|\widetilde{u}^\alpha\|_{L^\infty(0,T_0;L^2)}^2 + T_0\widetilde{R}\|\widetilde{n}^\alpha\|_{L^\infty(0,T_0;L^2)}\|\widetilde{u}^\alpha\|_{L^\infty(0,T_0;H^{1/2})}^2\Big).
\end{aligned}$$

Using the above estimates and the uniform-in-α bound from Step 1, we obtain

$$\big\|N^\alpha - N_{il}^\alpha + P_{\neq c}(|\widetilde{u}|^2)\big\|_{L^\infty(0,T_0;H^{-5/2})} \lesssim \|\widetilde{u}^\alpha - \widetilde{u}\|_{L^\infty(0,T_0;L^2)} + \widetilde{R}^{-\frac12} + \widetilde{R}^3\alpha^{-1}$$

for any $\widetilde{R} > 1$, with the implicit constant independent of $\widetilde{R},\alpha$. We set $\widetilde{R}$ largely enough depending on $\varepsilon > 0$ and $R > 0$ fixed above, and recall strong convergence of $\widetilde{u}^\alpha$ shown in Step 3, to verify

$$\sup_{0\le t\le T_0}\Big|\big\langle N^\alpha(t) - N_{il}^\alpha(t) + P_{\neq c}(|\widetilde{u}(t)|^2),\, N_{il}^\alpha(t)\big\rangle_{L^2}\Big| \le 2\varepsilon$$

for all sufficiently large α, as desired.

[8] In the non-periodic case [13], the integral term was dealt with by the Strichartz estimate for the reduced wave equation, which yields some negative power of α. Although the same argument may be valid in the periodic case as well, we take a different approach here.

Step 5: We shall show (3.6) and (3.7) for any $T \in (0, T^\infty)$, concluding the proof.

This follows once we can show the following: Let $T \in [T_0, \min\{T^\infty, \liminf T^\alpha\})$ be such that (3.7) holds on the time interval $[0, T]$. Then, there exists $T_1 = T_1(\|\widetilde{u}(T)\|_{H^1}) > 0$ such that $\min\{T^\infty, \liminf T^\alpha\} > T + T_1$ and (3.7) holds on $[0, T + T_1]$. Note that the hypothesis is true for $T = T_0$ by the previous steps.

If (3.7) holds for some $T \in [T_0, \min\{T^\infty, \liminf T^\alpha\})$, then $T^\alpha > T$ for sufficiently large α and $\widetilde{u}^\alpha(T) \to \widetilde{u}(T)$ in H^1. A similar argument as Step 1 then gives a uniform a priori bound as (3.8) on the time interval $[T, T + T_1]$, where T_1 depends only on $\sup_\alpha \|\widetilde{u}^\alpha(T)\|_{H^1}$, which is bounded by $2\|\widetilde{u}(T)\|_{H^1}$ for sufficiently large α. Hence, we have $\liminf T^\alpha > T + T_1$ and a uniform a priori bound on the interval $[0, T + T_1]$, and then repeat the arguments in Steps 2–4 to show $T^\infty > T + T_1$ and (3.7) on $[0, T + T_1]$. □

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
Email address: nobu@kurims.kyoto-u.ac.jp