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# Singularity of Normal Complex Analytic Surfaces Admitting Non-Isomorphic Finite Surjective Endomorphisms

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# SINGULARITY OF NORMAL COMPLEX ANALYTIC SURFACES ADMITTING NON-ISOMORPHIC FINITE SURJECTIVE ENDOMORPHISMS

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Abstract. For a non-isomorphic finite endomorphism of the germ of a complex analytic normal surface at a point, the pair of the surface and a completely invariant reduced divisor is shown to be log-canonical. In many situations, the endomorphism or its square lifts to an endomorphism of another surface by an essential blowing up.

## 0. Introduction

We study the singularity of a complex analytic normal surface admitting a non-isomorphic finite surjective endomorphism. More precisely, we consider an endomorphism $\mathfrak{f}$ of the germ $\mathfrak{X} = (X, x)$ of a normal surface $X$ at a point $x$ in which $\mathfrak{f}$ is finite of degree $> 1$. The singularity of $\mathfrak{X}$ has been shown to be log-canonical by Wahl [58]: In the proof, an invariant $-P \cdot P$ concerning the relative Zariski-decomposition plays an essential role. In [6, Thm. B], Favre proves the log-canonicity by another method applying the theory of valuation spaces, where he proves furthermore that $\mathfrak{X}$ is a quotient singularity when $\mathfrak{f}$ ramifies on $X \setminus \{x\}$. There are also some remarkable results in [6] on the liftability of $\mathfrak{f}$ by bimeromorphic morphisms $Y \to X$ from normal surfaces $Y$. In this article, we classify the singularity of $\mathfrak{X}$ and check the liftability of $\mathfrak{f}$ by standard arguments of algebraic geometry not using valuation spaces.

For the singularity, we consider not only $\mathfrak{X}$ but also the germ at $x$ of the pair $(X, S)$ with a reduced divisor $S$ such that $\mathfrak{f}^{-1}S = S$; such a divisor $S$ is said to be *completely invariant* under $\mathfrak{f}$. As a generalization of [58] and [6, Thm. B], we can prove:

**Theorem 0.1.** *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ be a finite surjective endomorphism of the germ $\mathfrak{X} = (X, x)$ of a normal surface $X$ at a point $x$. Let $\mathfrak{S}$ be the germ $(S, x)$ of a reduced divisor $S \subset X$ at $x$. Here, $\mathfrak{S}$ may not contain $x$. Assume that* $\deg \mathfrak{f} > 1$ *and $\mathfrak{f}^{-1}\mathfrak{S} = \mathfrak{S}$. Then $(X, S)$ is log-canonical at $x$. If $\mathfrak{f}$ is not étale on $\mathfrak{X} \setminus \mathfrak{S}$, then $(X, S)$ is* 1-log-terminal *at $x$ (cf. Definition* 2.1*).*

The 1-log-terminal is called "purely log terminal" in many articles (see Remark 2.3 below). Note that singularities of 2-dimensional log-canonical pairs (with

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reduced boundaries) are classified by [29, Thm. 9.6] (cf. [51, App.], [34, Ch. 3]). Theorem 0.1 is a direct consequence of Theorem 3.5 in Section 3.

For the liftability of $\mathfrak{f}$, we can prove the following as a generalization of [6, Prop. 2.1]:

**Theorem 0.2.** *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ be a non-isomorphic finite endomorphism of the germ $\mathfrak{X} = (X, x)$ of a normal surface $X$ at a point $x$. Let $\varphi\colon Y \to X$ be a bimeromorphic morphism such that $E = \varphi^{-1}(x)$ is a divisor and $\varphi$ is an isomorphism over $X \setminus \{x\}$. Let $\Phi\colon \mathfrak{Y} \to \mathfrak{X}$ be the morphism induced by $\varphi$ for the* germ $\mathfrak{Y} = (Y, E)$ of $Y$ along $E$ *(cf. Notation and conventions, below). Then there is an endomorphism $\mathfrak{g}\colon \mathfrak{Y} \to \mathfrak{Y}$ such that $\Phi \circ \mathfrak{g} = \mathfrak{f}^2 \circ \Phi$ for the power $\mathfrak{f}^2 = \mathfrak{f} \circ \mathfrak{f}$ provided that one of the following conditions is satisfied:*

(I) *The endomorphism $\mathfrak{f}$ is étale outside $\{x\}$, $\varphi$ is an* essential blowing up *(cf. Definition 4.24 below) of the log-canonical singularity $\mathfrak{X}$, and $\mathfrak{X}$ is not a cusp singularity.*

(II) *There is a reduced divisor $S \ni x$ such that*
- *$\mathfrak{f}^*\mathfrak{S} = d\mathfrak{S}$ for an integer $d > 0$ and $\mathfrak{f}$ is étale on $\mathfrak{X} \setminus \mathfrak{S}$ for the germ $\mathfrak{S} = (S, x)$ of $S$ at $x$, and*
- *$\varphi$ is an* essential blowing up *at $x$ with respect to $(X, S)$.*

*Remark.* If $\varphi$ is an essential blowing up with respect to a log-canonical pair $(X, S)$ of a normal surface $X$ and a reduced divisor $S$, then $K_Y + S_Y = \varphi^*(K_X + S)$ for the reduced divisor $S_Y = \varphi^{-1}S$, in which $(Y, S_Y)$ is log-canonical, and moreover, it is 1-log-terminal at any point of the non-singular locus $(S_Y)_{\mathrm{reg}}$ (cf. Definition 4.24); in particular, $Y$ has only quotient singularities. Since $\varphi$ is not an isomorphism, the singularity $\mathfrak{X} = (X, x)$ is not log-terminal in (I), and the pair $(X, S)$ is not 1-log-terminal at $x$ in (II). Hence, by the classification of log-canonical singularities (cf. [29, Thm. 9.6]), in case (I), $\mathfrak{X}$ is a simple elliptic singularity or a rational singularity whose index 1 cover is either a simple elliptic singularity or a cusp singularity. In case (II), one of the cases (1) and (3) in Fact 2.5 below occurs for $(X, S)$ at $x$.

*Remark.* The case (I) is treated in [6, Prop. 2.1] for a certain partial resolution of singularities of $\mathfrak{X}$ and it is stated that not only $\mathfrak{f}^2$ the endomorphism $\mathfrak{f}$ itself lifts to an endomorphism of $\mathfrak{Y}$: The corresponding result is given in Lemmas 5.23 and 5.24 below. Unfortunately, the proof of [6, Prop. 2.1] seems to omit the case where "$F_\bullet$ permutes two branched points of $\Gamma(\mu)$," and the author could not understand why "$F$ (not only $F^2$) lifts to a holomorphic endomorphism of $\overline{X}$" as stated in [6, Prop. 2.1]. This question is solved in Lemma 5.24 below, as a consequence of our key theorem, Theorem 5.10. We need to exclude cusp singularities in (I) by the remarkable example constructed in [6, Prop. 2.2].

Theorem 0.2 is a direct consequence of Theorem 5.3 in Section 5. In Theorems 3.5 and 5.3, instead of an endomorphism of a germ $\mathfrak{X} = (X, x)$ of normal surface $X$ at a point $x$, we consider more generally a morphism $f\colon X^\circ \to X$ from an open neighborhood $X^\circ$ of $x$ such that $f$ has only discrete fibers, $f^{-1}(x) = \{x\}$, and $\deg_x f > 1$ (cf. Definition 1.9): A non-isomorphic finite endomorphism of the germ $\mathfrak{X}$ is induced by such a morphism $f$ (cf. Remark 3.2). Our methods proving these theorems are based on standard arguments on the following topics, which are discussed in Sections 1 and 4:

(1) Some morphisms of complex analytic varieties.
(2) Numerical pullbacks of divisors on normal surfaces by non-generate morphisms.
(3) Logarithmic ramification formula.
(4) Classification of 2-dimensional log-canonical singularities of pairs with reduced boundaries.
(5) The 2-dimensional relative abundance theorem for log-canonical pairs.
(6) Theory of toric surfaces.
(7) Description of cyclic covers.
(8) Essential blowings up.
(9) Dual $\mathbb{R}$-divisors.

The organization of this article is as follows: In Section 1, we shall discuss topics (1), (2), and (3). Concerning (1), we consider the following morphisms in Section 1.1: *morphisms of maximal rank*, *non-degenerate morphisms*, *fully equi-dimensional morphisms*, and *discretely proper morphisms*. Here, the notion of a morphism of maximal rank (resp. a non-degenerate morphism) of complex analytic varieties is a generalization of that of a dominant (resp. generically finite and dominant) morphism of integral algebraic schemes. The basics on divisors on normal complex analytic varieties are explained in Section 1.2, and the topic (2) on divisors on normal surfaces is treated in Section 1.3. Note that the pullback of a Cartier divisor by a morphism of maximal rank is canonically defined, but the pullback of a (Weil) divisor is not defined in general. We have the numerical pullback of a (Weil) divisor by a non-degenerate morphism of normal surfaces: this is known as the Mumford pullback (cf. [35, II, §(b)]) in the case of bimeromorphic morphism. In this article, the numerical pullback of divisor is regarded as the standard pullback. Remarks on pullbacks and pushforwards of divisors by meromorphic mappings are mentioned in Section 1.4, which are used in Section 5.3. In Section 1.5 concerning (3), the logarithmic ramification formula due to Iitaka and its generalizations are given with explanations of the canonical divisor and the ramification divisor.

In Section 2, we treat topics (4) and (5). The log-canonical, log-terminal, and 1-log-terminal singularities for pairs of normal surfaces and effective $\mathbb{Q}$-divisors are defined in Section 2.1 in a little different style from the popular one (cf. Definition 2.1). See Remarks 2.3 and 2.8 for a difference from similar definitions in other articles. In Section 2.2, we give comparison results on log-canonicity etc. for some non-degenerate morphisms of normal surfaces by applying formulas in Section 1.5. The relative abundance theorem in (5) is treated in Section 2.3. The theorem is known in the algebraic case, but the proof seems to be omitted and not given in the analytic case. Our proof is based on ideas of Fujita [11] and Kawamata [29] (cf. Theorem 2.19 below). By (5), we define the *log-canonical modification* (see Lemma-Definition 2.22), which is important in the proof of Theorem 3.5 below.

Someone may think that Sections 1 and 2 are superfluous, since most results there are well known at least in the algebraic case. But, we need to confirm some of them in the analytic case, since we can not work in the algebraic category. Not all the results in Sections 1 and 2 are used in the other sections, but it is worthwhile to prove them in a general form, since there seems to be no good references discussing similar topics in the analytic case.

The purpose of Section 3 is to prove Theorem 3.5, from which Theorem 0.1 is deduced directly. In Section 3.1, we give the statement and corollaries, and we prove its 1-dimensional analogue (cf. Proposition 3.4 below). Theorem 3.5 is proved in Section 3.2 gradually by applying results in Sections 1.5, 2.1, and 2.3.

In Section 4, we shall discuss topics (6)–(9). For (6), basics on affine toric surfaces are explained briefly in Section 4.1 with some properties of morphisms of toric surfaces. For (7), we review the construction of cyclic covers by Esnault and Viehweg in Section 4.2 in a slightly different way from the original, and give a criterion on the liftability of an endomorphism to the cyclic cover. The essential blowing up in (8) is defined in Section 4.3 for log-canonical pairs $(X, B)$ of normal surfaces with reduced divisors, where we discuss the comparison of two essential blowings up. The name comes from the "essential divisor" on the resolution of a normal surface singularity (cf. [26, Def. 3.3]). The dual $\mathbb{R}$-divisor in (9) is defined and discussed in Section 4.4; it is defined for a divisor on a normal surface with respect to a compact connected divisor having negative definite intersection matrices. The notion of dual $\mathbb{R}$-divisors comes from arguments in [6, §1.2], where the duals are considered as projective limits of Weil divisors on resolutions (cf. [6, Def. 1.3]).

Section 5 is devoted to proving Theorem 5.3, from which Theorem 0.2 is deduced directly. In Section 5.1, we give the statement explaining our setting on the lifting property. The proof of Theorem 5.3 in the case (II) is given in Section 5.2 by applying results in Sections 4.1, 4.2, and 4.3. For Theorem 5.3 in the case (I), we prove a key theorem (Theorem 5.10) in Section 5.3, and we complete the proof in Section 5.4.

We shall explain a background of this article. This is a revised version of a part of an unpublished preprint [39] of the author written in 2008, which deals with the classification of normal Moishezon surfaces $X$ admitting non-isomorphic surjective endomorphisms. Revised versions of classification parts of [39] are now in preparation. Note that [40] includes a preliminary part of [39]. In [39], the author proves that the singularity of $(X, S)$ is log-canonical for any completely invariant divisor $S$. The log-canonicity of $(X, S)$ at $x \in S$ is shown by using the log-canonical modification (see Lemma-Definition 2.22 below). The log-canonicity of $(X, S)$ at $x \notin S$ is a consequence of results of Wahl [58] or Favre [6]. The author was informed by Favre of their results when preparing [39], and gave a modified proof in [39]. Theorem 3.5 below gives a further modification. The liftability problem of $\mathfrak{f}$ is not treated in [39] but in some modified versions of [39] around 2010.

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**Notation and conventions.** In this article, any complex analytic space is assumed to be Hausdorff and to have a countable open base.

• A *variety* means a complex analytic variety, i.e., an irreducible and reduced complex analytic space. Note that an open subset of a variety is not necessarily irreducible, but a *Zariski-open subset*, the complement of an analytic subset, is a variety (cf. [15, IX, §1.2]).

• For a variety $X$, the non-singular (resp. singular) locus is denoted by $X_{\mathrm{reg}}$ (resp. $\operatorname{Sing} X$). Note that the dimension of $X$ is defined as that of the complex manifold $X_{\mathrm{reg}}$.

• A local isomorphism of complex analytic spaces is called an *étale* morphism.

• A morphism $f\colon X \to Y$ of normal complex analytic spaces is said to be *étale in codimension* 1 if $f|_{X \setminus Z} \colon X \setminus Z \to Y$ is étale for an analytic subset $Z$ of codimension $\geq 2$.

• For the local ring $\mathcal{O}_{X,x}$ of a point $x$ of a complex analytic space $X$, the maximal ideal is denoted by $\mathfrak{m}_x$ and the residue field by $\mathbb{C}(x)$. The *local dimension* of $X$ at $x$ denoted by $\dim_x X$ is defined as the Krull-dimension of $\mathcal{O}_{X,x}$ (cf. [7, §3.1]).

• The *germ* $\mathfrak{X} = (X, S)$ of a complex analytic space $X$ *along* a subset $S$ is a *pro-object* (cf. [19, §8.10], [27, Def. 6.1.1]) of the category (An) of complex analytic spaces defined as

$$\text{“}\varprojlim\text{”}_{X' \in \mathsf{U}(S)} X',$$

where $\mathsf{U}(S)$ is the category of open neighborhoods of $S$ whose morphisms are open immersions and where “$\varprojlim$” is the projective limit in the category of presheaves on (An) (cf. [19, (8.5.3.2)], [27, Not. 2.6.2]). For the germ $\mathfrak{Y} = (Y, T)$ of another complex analytic space $Y$ along a subset $T$, a morphism $\mathfrak{X} = (X, S) \to \mathfrak{Y} = (Y, T)$ of germs is defined as a morphism of pro-objects. Since $Y$ is Hausdorff and since

$$\mathrm{Hom}_{\mathrm{Pro(An)}}(\mathfrak{X}, \mathfrak{Y}) = \varprojlim_{Y' \in \mathsf{U}(T)} \varinjlim_{X' \in \mathsf{U}(S)} \mathrm{Hom}_{\mathrm{(An)}}(X', Y')$$

for the category Pro(An) of pro-objects of (An) (cf. [19, (8.2.5.1), (8.10.5)], [27, (2.6.3), (2.6.4)]), a morphism $\mathfrak{X} \to \mathfrak{Y}$ of germs is represented by a morphism $f\colon X' \to Y'$ in (An) for some $X' \in \mathsf{U}(S)$ and $Y' \in \mathsf{U}(T)$ such that $f(S) \subset T$.

## 1. Preliminaries on complex analytic varieties

We shall discuss some morphisms of complex analytic spaces (Section 1.1), basics on divisors (Section 1.2), numerical pullbacks of divisors on normal surfaces (Section 1.3), pullbacks and pushforwards of divisors by meromorphic maps (Section 1.4), canonical divisors, and the ramification formula (Section 1.5).

1.1. **Some morphisms of complex analytic spaces.** We shall explain basic properties of some morphisms of complex analytic spaces, which include: morphisms of maximal rank, non-degenerate morphisms, fully equi-dimensional morphisms, and discretely proper morphisms. A base change property by a fully equi-dimensional morphism is also given (cf. Lemma 1.13). We refer to [7] for some basics on complex analytic spaces.

**Definition 1.1.** Let $f\colon X \to Y$ be a morphism of *varieties*.

(1) If $f$ is smooth at a point of $X_{\mathrm{reg}} \cap f^{-1}(Y_{\mathrm{reg}}) \neq \emptyset$, then $f$ is said to be *of maximal rank*.
(2) If $f$ is of maximal rank and $\dim X = \dim Y$, then $f$ is said to be *non-degenerate*.
(3) If $\dim_x f^{-1}(f(x)) = \dim X - \dim Y$ for any $x \in X$, then $f$ is said to be *fully equi-dimensional*.

*Remark* 1.2. For a point $x \in X_{\mathrm{reg}} \cap f^{-1}(Y_{\mathrm{reg}})$, the smoothness of $f$ at $x$ is equivalent to each of the following conditions:

- The tangent map $T_xX \to T_{f(x)}Y$ is surjective, where $T_xX$ denotes the tangent space of $X$ at $x$.
- The canonical pullback homomorphism $f^*\Omega^1_Y \to \Omega^1_X$ of holomorphic 1-forms is injective at $x$ and its cokernel $\Omega^1_{X/Y}$ is free at $x$, where $\Omega^1_{X/Y}$ denotes the sheaf of relative 1-forms, and $\Omega^1_X := \Omega^1_{X/\operatorname{Spec}\mathbb{C}}$.
- The morphism $f$ is flat at $x$ and the scheme-theoretic fiber $f^{-1}(f(x))$ over $f(x)$ is non-singular at $x$.
- The morphism $f$ is a *submersion* at $x$ (cf. [7, §2.18]) in the sense that an open neighborhood $\mathcal{U}$ of $x$ is isomorphic to the product $F \times \mathcal{V}$ of an open neighborhood $\mathcal{V}$ of $f(x)$ in $Y$ and a non-singular variety $F$ such that $f|_{\mathcal{U}}$ is isomorphic to the composite of the projection $F \times \mathcal{V} \to \mathcal{V}$ and the immersion $\mathcal{V} \hookrightarrow Y$.

*Remark.* Let $\mathsf{f}\colon \mathsf{X} \to \mathsf{Y}$ be a morphism of integral separated algebraic schemes over $\mathbb{C}$ and assume that $f$ is the associated morphism $\mathsf{f}^{\mathrm{an}}\colon \mathsf{X}^{\mathrm{an}} \to \mathsf{Y}^{\mathrm{an}}$ of complex analytic varieties. Then $f$ is of maximal rank if and only if $\mathsf{f}$ is dominant. Moreover, $f$ is fully equi-dimensional if and only if $\mathsf{f}$ is dominant and equi-dimensional in the sense of [16, Déf. (13.2.2)].

**Lemma 1.3.** *For a morphism $f\colon X \to Y$ of varieties, the following conditions are equivalent*:

(i) *The morphism $f$ is of maximal rank.*
(ii) *The image $f(X)$ contains a non-empty open subset of $Y$.*
(iii) *The equality $\min_{x\in X} \dim_x f^{-1}(f(x)) = \dim X - \dim Y$ holds.*
(iv) *There is a dense Zariski-open subset $X'$ of $X$ such that $f|_{X'}\colon X' \to Y$ is smooth.*
(v) *There is a dense Zariski-open subset $X''$ of $X$ such that $f|_{X''}\colon X'' \to Y$ is fully equi-dimensional.*

*Proof.* Obviously, (iv) is the strongest among conditions except (iii), but we can prove (iv) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (v) by the upper semi-continuity of the function $x \mapsto \dim_x f^{-1}(f(x))$ (cf. [7, §3.4]). Implications (i) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (ii) follow from [7, §2.17, Lem.] and [7, §3.7, Cor.], respectively.

It remains to prove (ii) $\Rightarrow$ (i). We use an argument in the proof of [8, Lem. (IV, 13)]. Replacing $Y$ with $Y_{\mathrm{reg}}$, we may assume that $Y$ is non-singular. The rank of the tangent map $T_x X \to T_{f(x)} Y$ of $f$ is lower-semi-continuous on $x \in X_{\mathrm{reg}}$, and we have a unique maximal Zariski-open subset $X_o$ of $X_{\mathrm{reg}}$ on which the rank is constant and attains the maximum. Since $X$ is assumed to have a countable open basis, $X \setminus X_o$ is a locally finite countable union of subvarieties $\overline{X}_i$ of dimension less than $\dim X$. Similarly to the above, for each $i$, we can find a unique maximal Zariski-open subset $X_i$ of $(\overline{X}_i)_{\mathrm{reg}}$ such that the rank of the tangent map $T_x \overline{X}_i \to T_{f(x)} Y$ of the induced morphism $\overline{X}_i \to Y$ is constant on $x \in X_i$ attaining the maximum. Then the complement of $X_o \cup \bigcup X_i$ in $X$ is also a locally finite countable union of subvarieties of dimension less than $\dim X - 1$. By continuing the process, we have a locally finite countable disjoint union $X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$ of locally closed non-singular analytic subspaces $X_\lambda$ of $X$ such that the tangent map $T_x X_\lambda \to T_{f(x)} Y$ of $f|_{X_\lambda}$ has constant rank for $x \in X_\lambda$. By [7, §2.19, Cor. 2], locally on $X_\lambda$, the morphism $X_\lambda \to Y$ is isomorphic to a submersion to a locally closed submanifold of $Y$. Since $f(X)$ contains an open subset, $f(X_\lambda)$ is open for some $\lambda \in \Lambda$. We fix such an index $\lambda$. Then, for any $x \in X_\lambda$, the composite

$$\Omega^1_Y \otimes \mathbb{C}(f(x)) \to \Omega^1_X \otimes \mathbb{C}(x) \to \Omega^1_{X_\lambda} \otimes \mathbb{C}(x)$$

of canonical linear maps is injective. It implies that the canonical homomorphism $f^*\Omega^1_Y \to \Omega^1_X$ is injective on an open subset $U$ of $X$ containing $X_\lambda$. The cokernel $\Omega^1_{X/Y}$ is locally free on a non-empty Zariski-open subset $U'$ of $U$, since $U$ is reduced (cf. [7, §2.13, Cor.]). Therefore, $f^*\Omega^1_Y$ is a subbundle of $\Omega^1_X$ on $U'$, and $f|_{U'} \colon U' \to Y$ is smooth by Remark 1.2. This shows (ii) $\Rightarrow$ (i), and we are done. □

*Remark.* If $X$ and $Y$ are non-singular, then (ii) $\Rightarrow$ (i) is a consequence of Sard's theorem on critical values.

**Corollary.** *A fully equi-dimensional morphism of varieties is of maximal rank. A surjective morphism of varieties is of maximal rank.*

**Corollary 1.4.** *For a morphism $f \colon X \to Y$ of varieties of the same dimension, the following conditions are equivalent*:

(i) *The morphism $f$ is non-degenerate.*
(ii) *The image $f(X)$ contains a non-empty open subset.*
(iii) *There is a point $x \in X$ such that $x$ is isolated in the fiber $f^{-1}(f(x))$.*
(iv) *There is a dense Zariski-open subset $X'$ of $X$ such that $f|_{X'}$ is étale.*

**Corollary 1.5.** *Let $f \colon X \to Y$ be a morphism of varieties. Let $U$ be a non-empty open subset of $X$ and let $Z$ be an irreducible component of $U$.*

(1) *If $f$ is of maximal rank, then so is $f|_Z \colon Z \to Y$.*
(2) *If $f$ is a fully equi-dimensional morphism, then so is $f|_Z$.*

*Proof.* Assume that $f$ is of maximal rank. Then $Z \cap X' \neq \emptyset$ for the open subset $X'$ of Lemma 1.3(iv) as $X \setminus X'$ is nowhere dense. Since $f|_Z$ is smooth on $Z \cap X'$, it is of maximal rank. Next, assume that $f$ is fully equi-dimensional. Then

$$\dim X = \dim Z \leq \dim_x(f^{-1}(f(x)) \cap Z) + \dim Y \leq \dim_x f^{-1}(f(x)) + \dim Y = \dim X$$

for any $x \in Z$ (cf. [7, §3.9, Prop.]). Thus, $f|_Z$ is fully equi-dimensional. $\square$

**Definition** ($\deg f$)**.** Let $f\colon X \to Y$ be a proper non-degenerate morphism of varieties. The *degree* of $f$, denoted by $\deg f$, is defined as the rank of the coherent $\mathcal{O}_Y$-module $f_*\mathcal{O}_X$. Hence,

$$\deg f = \dim_{\mathbb{C}(y)} f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathbb{C}(y) = \dim_{\mathbb{C}} H^0(\mathcal{O}_{f^{-1}(y)})$$

for a general point $y \in Y$. By Corollary 1.4, we see that $\deg f$ equals the cardinality of $f^{-1}(y)$ for a general point $y \in Y$.

**Definition 1.6.** A morphism of complex analytic spaces is said to be *discretely proper* if the connected components of the fibers are compact.

Proper morphisms and morphisms with only discrete fibers are discretely proper. Moreover, we know the following as a strong version of the Stein factorization (cf. [53], [2, Thm. 3]):

*Fact.* A morphism $f\colon X \to Y$ of complex analytic space is discretely proper if and only if $f = g \circ \pi$ for a proper morphism $\pi\colon X \to Y'$ with an isomorphism $\mathcal{O}_{Y'} \simeq \pi_*\mathcal{O}_X$ and for a morphism $g\colon Y' \to Y$ with only discrete fibers.

By [7, §1.10, Lem. 1 and §3.2, Lem.], we have:

**Lemma 1.7.** *Let $f\colon X \to Y$ be a morphism of complex analytic spaces. For a point $x \in X$ and a connected component $\Gamma$ of $f^{-1}(f(x))$, if $\Gamma$ is compact, then there exist an open neighborhood $V$ of $f(x)$ in $Y$ and an open neighborhood $U$ of $\Gamma$ in $f^{-1}V$ such that $U \cap f^{-1}(f(x)) = \Gamma$ and $f|_U\colon U \to V$ is proper. If $\Gamma = \{x\}$, then one can choose $U$ and $V$ so that $f|_U$ is a finite morphism.*

**Corollary 1.8.** *Let $f\colon X \to Y$ be a morphism of varieties of the same dimension. If $x \in X$ is isolated in $f^{-1}(f(x))$ and if $Y$ is locally irreducible at $f(x)$, then there is an open neighborhood $\mathcal{U}$ of $x$ such that $\mathcal{U} \cap f^{-1}(f(x)) = \{x\}$, $f(\mathcal{U})$ is open, and $f|_{\mathcal{U}}\colon \mathcal{U} \to f(\mathcal{U})$ is a finite morphism. In particular, if $f$ has only discrete fibers and $Y$ is locally irreducible, then $f(X)$ is open.*

*Proof.* By Lemma 1.7, we have an open neighborhood $\mathcal{V}$ of $f(x)$ in $Y$ and an open neighborhood $\mathcal{U}$ of $x$ in $f^{-1}\mathcal{V}$ such that $\mathcal{U} \cap f^{-1}(f(x)) = \{x\}$ and $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{V}$ is finite. It suffices to prove that $f(\mathcal{U}) = \mathcal{V}$ as a set. Here, we may assume that $\mathcal{V}$ is irreducible, i.e., $\mathcal{V}$ is a variety. An irreducible component $\mathcal{U}'$ of $\mathcal{U}$ containing $x$ is a variety and the induced morphism $f' = f|_{\mathcal{U}'}\colon \mathcal{U}' \to \mathcal{V}$ of varieties is non-degenerate. Thus, $f'(\mathcal{U}')$ contains a non-empty open subset of $\mathcal{V}$ by Corollary 1.4. Therefore, $f'(\mathcal{U}') = f(\mathcal{U}) = \mathcal{V}$ by the local irreducibility of $\mathcal{V}$, since $f'(\mathcal{U}')$ is a closed analytic subset of $\mathcal{V}$. $\square$

**Definition 1.9.** In the situation of Corollary 1.8, we define the *local degree* of $f$ at $x$ as the degree of the finite morphism $f|_{\mathcal{U}} \colon \mathcal{U} \to f(\mathcal{U})$: This is independent of the choice of such $\mathcal{U}$ and is denoted by $\deg_x f$. Note that $\deg_x f = 1$ if and only if $f$ is an isomorphism at $x$.

**Lemma 1.10.** *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be morphisms of complex analytic spaces.*

(1) *If $f$ is proper and if $g$ is discretely proper, then $g \circ f$ is discretely proper.*
(2) *If $g \circ f$ is discretely proper, then $f$ is discretely proper.*
(3) *Assume that $f \colon X \to Y$ is a morphism of varieties of maximal rank. If $Y$ is locally irreducible, $g$ has only connected fibers, and $g \circ f$ is surjective and discretely proper, then $f$ is surjective.*

*Proof.* (1) and (2): For a point $x \in X$ and $y = f(x)$, let $\Gamma_x$ (resp. $\Theta_x$) be the connected component of $f^{-1}g^{-1}(g(y))$ (resp. $f^{-1}(y)$) containing $x$. Then $\Theta_x$ is a connected component of a fiber of $\Gamma_x \to g^{-1}(g(y))$. In case (1), $f(\Gamma_x)$ is compact, since it is a closed subset of a connected component of $g^{-1}(g(y))$; thus, $\Gamma_x$ is also compact as a closed subset of $f^{-1}f(\Gamma_x)$. This shows (1). In case (2), $\Gamma_x$ is compact, and hence, $\Gamma_x \to f(\Gamma_x)$ is proper and $\Theta_x$ is compact. This shows (2).

(3): For a point $x \in X$ and the connected component $\Gamma_x$ of $f^{-1}g^{-1}(g(f(x)))$ containing $x$, by Lemma 1.7, we have an open neighborhood $\mathcal{U}_x$ of $\Gamma_x$ in $X$ and an open neighborhood $\mathcal{W}_x$ of $g(f(x))$ in $Z$ such that $g \circ f$ induces a proper morphism $\mathcal{U}_x \to \mathcal{W}_x$. We may assume that $\mathcal{W}_x$ is connected. Then $g^{-1}\mathcal{W}_x$ is a connected open subset of $Y$, which is irreducible as $Y$ is locally irreducible. Now, $f$ induces a proper morphism $\mathcal{U}_x \to g^{-1}\mathcal{W}_x$. For an irreducible component $\mathcal{U}'$ of $\mathcal{U}_x$, the induced morphism $f|_{\mathcal{U}'} \colon \mathcal{U}' \to g^{-1}\mathcal{W}_x$ is of maximal rank, and hence, $f(\mathcal{U}')$ contains a non-empty open subset by Lemma 1.3. Thus, $f(\mathcal{U}_x) = f(\mathcal{U}') = g^{-1}\mathcal{W}_x$. Therefore, $f(X) = \bigcup f(\mathcal{U}_x) = \bigcup g^{-1}\mathcal{W}_x = Y$, since $g \circ f$ is surjective. $\square$

**Corollary 1.11.** *For a surjective morphism $f \colon X \to Y$ of normal varieties and for a proper surjective morphism $\tau \colon Y' \to Y$ of normal varieties with only connected fibers, let*

$$\begin{array}{ccc} X' & \xrightarrow{\tau'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\tau} & Y \end{array}$$

*be a commutative diagram of varieties such that the induced morphism $X' \to X \times_Y Y'$ is an isomorphism over a non-empty open subset of $Y'$. If $\tau'$ is proper surjective and $f$ is discretely proper, then $f'$ is surjective and discretely proper.*

*Proof.* The composite $f \circ \tau'$ is surjective and is discretely proper by Lemma 1.10(1). Hence, $f'$ is discretely proper by Lemma 1.10(2) applied to $X' \to Y' \to Y$. The morphism $f'$ is of maximal rank by Lemma 1.3, since $f'(X')$ contains the open subset of $Y'$ over which $X' \to X \times_Y Y'$ is an isomorphism. Thus, $f'$ is surjective by Lemma 1.10(3) applied to $X' \to Y' \to Y$. $\square$

The openness property in Corollary 1.8 is generalized to:

**Lemma 1.12.** *Let $f\colon X \to Y$ be a fully equi-dimensional morphism of varieties and assume that $Y$ is locally irreducible. Then $f$ is* universally open *in the sense that the base change $f'\colon X \times_Y Y' \to Y'$ is an open holomorphic map for any morphism $\tau\colon Y' \to Y$ from a complex analytic space $Y'$. If $Y'$ is a variety, then $f'|_V\colon V \to Y'$ is fully equi-dimensional for any irreducible component $V$ of $X \times_Y Y'$.*

*Proof.* The morphism $f$ is open by [7, §3.10, Thm.]. For any point $y' \in Y'$, we have an open neighborhood $\mathcal{Y}'$ with a closed immersion $\iota\colon \mathcal{Y}' \hookrightarrow \mathcal{U}$ into a connected open subset $\mathcal{U}$ of an affine space $\mathbb{C}^n$. Then the induced morphism $(\iota, \tau|_{\mathcal{Y}'})\colon \mathcal{Y}' \hookrightarrow \mathcal{U} \times Y$ is a closed immersion and $\tau|_{\mathcal{Y}'}\colon \mathcal{Y}' \to Y$ is the composite of $(\iota, \tau|_{\mathcal{Y}'})$ and the second projection $\mathcal{U} \times \mathcal{Y}' \to \mathcal{Y}'$. In order to prove the openness of $f'$, we may replace $Y'$ with $\mathcal{Y}'$. If $\tau$ is the second projection $Y' = \mathcal{U} \times Y \to Y$, then $Y'$ is locally irreducible and $f'$ is open by [7, §3.10, Thm.]. Thus, we are reduced to the case where $\tau$ is a closed immersion, but in this case, the openness of $f'$ is obvious. This proves the first assertion.

For the second assertion, we set $X' := X \times_Y Y'$. Then the function $x \mapsto \dim_x f'^{-1}(f'(x))$ on $X'$ is constant with value $\dim X - \dim Y$, since $f$ is fully equi-dimensional. The openness of $f'$ implies that

$$\dim_x f'^{-1}(f'(x)) = \dim_x X' - \dim_{f'(x)} Y' = \dim_x X' - \dim Y'$$

for any $x \in X'$ by [7, §3.10, Thm.]. In particular, $x \mapsto \dim_x X'$ is constant. For the morphism $g = f'|_V\colon V \to Y'$ of varieties, we have

$$\dim_v X' - \dim Y' \geq \dim_v g^{-1}g(v) \geq \dim_v V - \dim_{g(v)} Y' = \dim V - \dim Y'$$

for any $v \in V$ by [7, §3.9, Prop.], since $f'^{-1}(f'(v)) \supset g^{-1}(g(v))$. For the open dense subset $V^\circ = V \cap (X'_{\mathrm{red}})_{\mathrm{reg}}$ of $V$, if $v \in V^\circ$, then $\dim V = \dim_v V = \dim_v X'$. Hence, the upper semi-continuous function $v \mapsto \dim_v g^{-1}(g(v))$ on $V$ attains the maximum at any point $V^\circ$. Thus, the function is constant with value $\dim V - \dim Y' = \dim X - \dim Y$. As a consequence, $g$ is fully equi-dimensional. $\square$

*Remark.* For morphisms locally of finite type of schemes, we have a result similar to Lemma 1.12 by [16, Prop. (14.3.2) and Cor. (14.4.4)].

*Remark.* Lemma 1.12 is not true in general if we drop the assumption on the local irreducibility of $Y$. For example, if $Y$ is a nodal cubic plane curve and if $f\colon X \to Y$ and $\tau\colon Y' \to Y$ are the normalization of $Y$, then $X \times_Y Y'$ contains two isolated points.

**Lemma 1.13.** *Let $\tau\colon Y' \to Y$ be a proper surjective morphism of normal varieties with connected fibers and let $f\colon X \to Y$ be a fully equi-dimensional morphism of varieties. Then $X \times_Y Y'$ is irreducible and is* generically reduced*, i.e., a dense open subset is reduced.*

*Proof.* We set $X' = X \times_Y Y'$ and consider the Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\tau'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\tau} & Y. \end{array}$$

By assumption, there exist Zariski-open dense subsets $X^\circ \subset X$ and $Y'^\circ \subset Y'$, and an open dense subset $Y^\circ \subset Y$ such that

- $f$ is smooth on $X^\circ$, $\tau$ is smooth on $Y'^\circ$, and
- $Y^\circ$ is non-singular containing $f(X^\circ) \cup \tau(Y'^\circ)$.

In particular, $X^\circ$ and $Y'^\circ$ are also non-singular. We set $U_1 := \tau'^{-1}(X^\circ) = X^\circ \times_Y Y'$, $U_2 := f'^{-1}(Y'^\circ) = X \times_Y Y'^\circ$, and $U_3 := U_1 \cap U_2 = X^\circ \times_{Y^\circ} Y'^\circ$. Then $U_1$ is normal, $U_2$ is reduced, and $U_3$ is non-singular, since $U_1 \to Y'$ and $U_2 \to X$ are smooth. Here, $U_3$ is Zariski-open and dense in $U_1$ and also in $U_2$. Since $\tau'|_{U_1} : U_1 \to X^\circ$ is a proper surjective morphism with connected fibers and since $X^\circ$ is a non-singular variety, we see that $U_1$ is also normal variety by considering the Stein factorization. Thus, $U_3$ and $U_2$ are also irreducible. For any irreducible component $Z$ of $X'$, the morphism $f'|_Z \colon Z \to Y'$ is fully equi-dimensional by Lemma 1.12. Thus, $Z \cap U_2 \neq \emptyset$, and this non-empty subset is a closed analytic subset of the variety $U_2$ of the same dimension. Hence $Z \supset U_2$, and moreover, $Z$ is the closure of $U_2$ in $X \times_Y Y'$. Therefore, $X \times_Y Y'$ is irreducible. It is generically reduced, since $U_3$ is non-singular. $\square$

**Corollary 1.14.** *Let $\pi_1 \colon X_1 \to Y_1$ and $\pi_2 \colon X_2 \to Y_2$ be proper surjective morphisms of normal varieties with connected fibers. If $f \colon X_1 \to X_2$ and $g \colon Y_1 \to Y_2$ are finite surjective morphisms such that $\pi_2 \circ f = g \circ \pi_1$, then $\deg g \mid \deg f$.*

*Proof.* By Lemma 1.13, $X_2 \times_{Y_2} Y_1$ is irreducible and generically reduced. For the normalization $X_1'$ of $X_2 \times_{Y_2} Y_1$, we can consider the commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "\tau"] \arrow[rr, bend left, "f"] \arrow[dr, "\pi_1"'] & X_1' \arrow[r, "p_1"] \arrow[d, "p_2"] & X_2 \arrow[d, "\pi_2"] \\ & Y_1 \arrow[r, "g"] & Y_2. \end{tikzcd}$$

Here, $p_1$ and $\tau$ are finite surjective morphisms, and $\deg p_1 = \deg g$. Therefore, $\deg f / \deg g = \deg f / \deg p_1 = \deg \tau \in \mathbb{Z}$. $\square$

1.2. **Glossaries on divisors.** We recall basic properties of divisors on normal complex analytic spaces fixing some notation used in this article. Especially, pullbacks of divisors by morphisms of maximal rank are explained in detail.

**Convention** (Divisor)**.** Let $X$ be a *normal* complex analytic space. A *divisor* on $X$ always means a *Weil divisor*, i.e., a locally finite $\mathbb{Z}$-linear combination of closed subvarieties of codimension 1. A *prime divisor* means a closed subvariety of codimension 1. The *divisor group* of $X$, i.e., the group of divisors on $X$, is denoted by $\mathrm{Div}(X)$. We use the following conventions for a divisor $D$ on $X$:

- The *prime decomposition* of $D$ is the expression $D = \sum_{i \in I} m_i \Gamma_i$ as a locally finite $\mathbb{Z}$-linear combination, where $m_i \in \mathbb{Z}$ and $\Gamma_i$ are prime divisors and where the set $I_x = \{i \in I \mid m_i \neq 0 \text{ and } x \in \Gamma_i\}$ is finite for any $x \in X$, by the local finiteness. The integer $m_i$ is called the *multiplicity* of $D$ along $\Gamma_i$ and denoted by $\mathrm{mult}_{\Gamma_i} D$. If $m_i \neq 0$, then $\Gamma_i$ is called a *prime component* of $D$.

- We say that $D$ is *effective* (resp. *reduced*) if $\mathrm{mult}_\Gamma D \geq 0$ (resp. $\mathrm{mult}_\Gamma D \in \{0, 1\}$) for any prime divisor $\Gamma$ on $X$. For another divisor $D'$, we write $D \geq D'$ or $D' \leq D$ if $D - D'$ is effective.
- The *support* of $D$, $\mathrm{Supp}\, D$, is the union of prime components of $D$: This is identified with the reduced divisor $D_{\mathrm{red}} := \sum_{m_i \neq 0} \Gamma_i$ for the prime decomposition of $D$ above. For a closed subset $T$, $\mathrm{Div}_T(X)$ denotes the group of divisors on $X$ whose supports are contained in $T$.
- For an open subset $U$ of $X$, the restriction $D|_U$ is defined as follows: Let $\Theta$ be a prime divisor on $U$ such that $\Theta \subset \mathrm{Supp}\, D$. Then $\Theta \subset \Gamma$ for a unique prime component $\Gamma$ of $D$. We set $m_\Theta := \mathrm{mult}_\Gamma D$. Then the divisor $D|_U$ on $U$ is defined by $\mathrm{mult}_\Theta(D|_U) = m_\Theta$ for any prime divisor $\Theta$ on $U$.

*Remark.* The restriction $D \mapsto D|_U$ gives rise to a group homomorphism $\mathrm{Div}(X) \to \mathrm{Div}(U)$ for any open subset $U$. The correspondence $U \mapsto \mathrm{Div}(U)$ gives rise to a sheaf $\mathcal{D}iv_X$ of abelian groups. In particular, $\mathrm{Div}(X) = H^0(X, \mathcal{D}iv_X)$. If $Z \subset X$ is a closed analytic subset of codimension $\geq 2$, then $\mathrm{Div}(X) \to \mathrm{Div}(X \setminus Z)$ is bijective. Thus, $\mathcal{D}iv_X \simeq j_* \mathcal{D}iv_{X \setminus Z}$ for the open immersion $j \colon X \setminus Z$, and we have $\mathrm{Div}(X) \simeq \mathrm{Div}(X_{\mathrm{reg}})$ for the non-singular locus $X_{\mathrm{reg}}$.

**Definition 1.15.** For a divisor $D$, there exist effective divisors $D_+$ and $D_-$ uniquely such that $D_+$ and $D_-$ have no common prime component and $D_+ - D_- = D$. In fact, $D_+ = \sum_{i \in I_+} m_i \Gamma_i$ and $D_- = \sum_{i \in I_-} (-m_i)\Gamma_i$ for the prime decomposition $D = \sum_{i \in I} m_i \Gamma_i$ and for $I_\pm = \{i \in I \mid \pm m_i > 0\}$. We call $D_+$ (resp. $D_-$) the *positive* (resp. *negative*) *part* of the *prime decomposition* of $D$.

**Convention** (Cartier divisor)**.** A *Cartier divisor* on a complex analytic space $Y$ is defined as a divisor on the ringed space $(Y, \mathcal{O}_Y)$ in the sense of [16, §21.1]. This is an element of $H^0(Y, \mathfrak{M}_Y^\star/\mathcal{O}_Y^\star)$ for the sheaf $\mathfrak{M}_Y^\star$ (resp. $\mathcal{O}_Y^\star$) of invertible meromorphic (resp. holomorphic) functions on $X$. We set $\mathcal{CD}iv_Y := \mathfrak{M}_Y^\star/\mathcal{O}_Y^\star$ and set $\mathrm{CDiv}(Y) := H^0(Y, \mathcal{CD}iv_Y)$ as the Cartier divisor group. A *principal* divisor is a Cartier divisor belonging to the image of $H^0(Y, \mathfrak{M}_Y^\star) \to \mathrm{CDiv}(Y)$. For an invertible meromorphic function $\varphi$, we consider the $\mathcal{O}_Y$-module $\mathcal{O}_Y\varphi^{-1}$ generated by $\varphi^{-1}$ in the sheaf $\mathfrak{M}_Y$ of meromorphic functions on $Y$. Then $\mathcal{O}_Y\varphi^{-1} \simeq \mathcal{O}_Y$. The correspondence $\varphi \mapsto \mathcal{O}_X\varphi^{-1}$ for "local" invertible meromorphic functions $\varphi$ defines a bijection from $\mathrm{CDiv}(Y)$ to the set of invertible sheaves contained in $\mathfrak{M}_Y$ as $\mathcal{O}_Y$-submodules. For a Cartier divisor $D$, the associated invertible sheaf is denoted by $\mathcal{O}_Y(D)$ (cf. [16, (21.2.8)]).

*Remark.* The correspondence $D \mapsto \mathcal{O}_Y(D)$ defines a homomorphism $\mathrm{CDiv}(Y) \to \mathrm{Pic}(Y)$, which is isomorphic to a connecting homomorphism of the exact sequence $0 = \{1\} \to \mathcal{O}_Y^\star \to \mathfrak{M}_Y^\star \to \mathcal{CD}iv_Y \to 0$. In particular, we have isomorphisms

$$\mathcal{O}_Y(-D) \simeq \mathcal{O}_Y(D)^{\otimes(-1)} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y(D), \mathcal{O}_Y),$$
$$\mathcal{O}_Y(D_1 + D_2) \simeq \mathcal{O}_Y(D_1) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(D_2)$$

for any $D$, $D_1$, $D_2 \in \mathrm{CDiv}(Y)$. A Cartier divisor $D$ is principal if and only if $\mathcal{O}_Y(D) \simeq \mathcal{O}_Y$, by the exactness of $H^0(Y, \mathfrak{M}_Y^\star) \to \mathrm{CDiv}(Y) \to \mathrm{Pic}(Y)$.

**Convention 1.16.** Let $\mathcal{L}$ be an invertible sheaf on $Y$. A holomorphic section $\sigma$ of $\mathcal{L}$ is said to be *nowhere vanishing* if $\sigma$ induces an isomorphism $\mathcal{O}_Y \to \mathcal{L}$, or equivalently,

$$\sigma(y) := \sigma_y \bmod \mathfrak{m}_y \in \mathcal{L}_y \otimes \mathbb{C}(y)$$

is not zero for any $y \in Y$. A *meromorphic section* $\varphi$ of $\mathcal{L}$ is by definition a global section of $\mathcal{L} \otimes_{\mathcal{O}_Y} \mathfrak{M}_Y$. We say that $\varphi$ is *regular* if $\varphi$ induces an isomorphism $\mathfrak{M}_Y \simeq \mathcal{L} \otimes_{\mathcal{O}_Y} \mathfrak{M}_Y$ (cf. [16, (20.1.8)]). We note the following on the regularity of $\varphi$:

- When $\mathcal{L} \simeq \mathcal{O}_Y$, $\varphi$ is regular if and only if $\varphi$ is an invertible meromorphic function.
- When $Y$ is irreducible and locally irreducible, $\varphi$ is regular if and only if $\varphi \neq 0$.
- Even if $\varphi$ is regular, it is not necessarily holomorphic.

*Remark.* A Cartier divisor $D$ on $Y$ is in one-to-one correspondence with a pair $(\mathcal{L}, \varphi)$ of an invertible sheaf $\mathcal{L}$ and a regular meromorphic section $\varphi$ of $\mathcal{L}$. In fact, the inclusion $\mathcal{O}_Y(D) \hookrightarrow \mathfrak{M}_Y$ defines an isomorphism $\mathcal{O}_Y(D) \otimes \mathfrak{M}_Y \simeq \mathfrak{M}_Y$, and we have $\varphi$ for $\mathcal{L} = \mathcal{O}_Y(D)$ as the inverse of the isomorphism. Conversely, $\varphi^{-1}$ induces an injection $\mathcal{L} \hookrightarrow \mathfrak{M}_Y$.

**Lemma 1.17.** *Let $f \colon X \to Y$ be a morphism of varieties* of maximal rank (*cf. Definition* 1.1). *Then there exist a canonical morphism*

$$\begin{array}{ccccc} H^0(Y, \mathfrak{M}_Y^\star) & \longrightarrow & \mathrm{CDiv}(Y) & \longrightarrow & \mathrm{Pic}(Y) \\ {\scriptstyle f^*}\big\downarrow & & {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ H^0(X, \mathfrak{M}_X^\star) & \longrightarrow & \mathrm{CDiv}(X) & \longrightarrow & \mathrm{Pic}(X) \end{array}$$

*of exact sequences of abelian groups, where $f^*$ are pullback homomorphisms of meromorphic functions, Cartier divisors, and invertible sheaves, respectively. In particular, $f^*\mathcal{O}_Y(D) \simeq \mathcal{O}_X(f^*D)$ for any Cartier divisor $D$ on $Y$.*

*Proof.* For a non-zero meromorphic function $\varphi$ on $Y$, its restriction $\varphi|_U$ to any non-empty open subset $U \subset Y$ is not zero, and the inverse image $f^*\varphi = \varphi \circ f$ is also not zero as $f(X)$ contains a non-empty open subset (cf. Lemma 1.3). Thus, we have a group homomorphism $\mathfrak{M}_Y^\star \to f_*\mathfrak{M}_X^\star$ extending $\mathcal{O}_Y^\star \to f_*\mathcal{O}_X^\star$, and it defines a morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^{-1}\mathcal{O}_Y^\star & \longrightarrow & f^{-1}\mathfrak{M}_Y^\star & \longrightarrow & f^{-1}\mathcal{CD}iv_Y & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X^\star & \longrightarrow & \mathfrak{M}_X^\star & \longrightarrow & \mathcal{CD}iv_X & \longrightarrow & 0 \end{array}$$

of exact sequences of sheaves on $X$. By taking cohomologies, we are done. $\square$

**Convention** ($\mathrm{div}(\varphi)$)**.** Let $X$ be a *normal* complex analytic space. Let $\varphi$ be a *regular* meromorphic section of an invertible sheaf $\mathcal{L}$ on $X$ (cf. Convention 1.16). The divisor $\mathrm{div}(\varphi) = \mathrm{div}_{\mathcal{L}}(\varphi)$ on $X$ associated with $(\mathcal{L}, \varphi)$ is defined by the property that $\mathrm{mult}_\Gamma \mathrm{div}(\varphi)$ equals the order of zeros or the minus of the order of poles of $\varphi$ along $\Gamma$ for any prime divisor $\Gamma$ on $X$. If $\mathcal{L} = \mathcal{O}_X$, then $\mathrm{div}(\varphi)$ is just the *principal divisor* associated with an invertible meromorphic function $\varphi$.

*Remark.* For a Cartier divisor $D$ on $X$, if a holomorphic section $\sigma$ of $\mathcal{O}_X(D)$ is not zero on each connected component of $X$, then $\sigma$ is regular as a meromorphic section, and $\mathrm{div}(\varphi) + D = \mathrm{div}(\sigma) \geq 0$ for the meromorphic function $\varphi$ defined as as the image of $\sigma$ by the inclusion $\mathcal{O}_X(D) \subset \mathfrak{M}_X$.

*Remark.* The correspondence $\varphi \mapsto \mathrm{div}(\varphi)$ defines an injection $\mathcal{CD}iv_X \hookrightarrow \mathcal{D}iv_X$, which is an isomorphism on $X_{\mathrm{reg}}$. Hence, $\mathrm{CDiv}(X)$ is regarded as a subgroup of $\mathrm{Div}(X)$, and we have $\mathrm{Div}(X) \simeq \mathrm{Div}(X_{\mathrm{reg}}) \simeq \mathrm{CDiv}(X_{\mathrm{reg}})$.

**Definition** ($\mathcal{O}_X(D)$)**.** Let $X$ be a *normal* complex analytic space. For a divisor $D$ on $X$, we set $\mathcal{O}_X(D) := j_*\mathcal{O}_{X_{\mathrm{reg}}}(D|_{X_{\mathrm{reg}}})$ for the open immersion $j\colon X_{\mathrm{reg}} \hookrightarrow X$. The sheaf $\mathcal{O}_X(D)$ is regarded as an $\mathcal{O}_X$-submodule of $\mathfrak{M}_X$ and it is a coherent reflexive sheaf of rank 1 (cf. [45, App. to §1]). Here, a coherent sheaf $\mathcal{F}$ on $X$ is said to be *reflexive* if it is isomorphic to the double-dual $\mathcal{F}^{\vee\vee} = (\mathcal{F}^{\vee})^{\vee}$, where $\mathcal{F}^{\vee} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

*Remark* 1.18. An effective divisor $D$ is identified with a closed analytic subspace of $X$ defined by the ideal sheaf $\mathcal{O}_X(-D)$; the structure sheaf $\mathcal{O}_D$ is the cokernel of the canonical injection $\mathcal{O}_X(-D) \to \mathcal{O}_X$. Hence, $\mathrm{Supp}\, D$ is the underlying set of $D_{\mathrm{red}}$ for any divisor $D$. As a property of a divisor $D$, we consider a property of the analytic space $D$ when $D$ is effective. For example, a divisor $D$ is said to be non-singular if $D$ is effective and the analytic space $D$ is non-singular. Thus, a non-singular divisor is reduced, and the zero divisor is non-singular by considering it as the empty set.

**Convention** ($\mathbb{Q}$-divisors and $\mathbb{R}$-divisors)**.** A $\mathbb{Q}$-*divisor* (resp. $\mathbb{R}$-*divisor*) on a normal complex analytic space $X$ is a locally finite $\mathbb{Q}$ (resp. $\mathbb{R}$)-linear combination of prime divisors. For an $\mathbb{R}$-divisor $D$, the prime decomposition $D = \sum_{i \in I} r_i \Gamma_i$ and the multiplicity $\mathrm{mult}_\Gamma D$ along a prime divisor $\Gamma$ are defined similarly to the case of divisor. Hence, we can speak of effective $\mathbb{R}$-divisors, the support of an $\mathbb{R}$-divisor, prime components of an $\mathbb{R}$-divisor, and the positive and negative parts of the prime decomposition of an $\mathbb{R}$-divisor (cf. Definition 1.15). The group of $\mathbb{Q}$ (resp. $\mathbb{R}$)-divisors on $X$ is denoted by $\mathrm{Div}(X, \mathbb{Q})$ (resp. $\mathrm{Div}(X, \mathbb{R})$), and the group of $\mathbb{Q}$ (resp. $\mathbb{R}$)-divisors on $X$ whose supports are contained in a closed subset $T$ is denoted by $\mathrm{Div}_T(X, \mathbb{Q})$ (resp. $\mathrm{Div}_T(X, \mathbb{R})$) (cf. [38, II, §2.d]); these are $\mathbb{Q}$ (resp. $\mathbb{R}$)-vector spaces. For the prime decomposition of $D$ above, the *round-up* $\ulcorner D \urcorner$, the *round-down* $\llcorner D \lrcorner$, and the *fractional part* $\langle D \rangle$ are defined by

$$\ulcorner D \urcorner := \sum\nolimits_{i \in I} \ulcorner r_i \urcorner \Gamma_i, \quad \llcorner D \lrcorner := \sum\nolimits_{i \in I} \llcorner r_i \lrcorner \Gamma_i, \quad \text{and} \quad \langle D \rangle := D - \llcorner D \lrcorner,$$

where $\llcorner r \lrcorner = \max\{i \in \mathbb{Z} \mid i \leq r\}$ and $\ulcorner r \urcorner = \min\{i \in \mathbb{Z} \mid i \geq r\} = -\llcorner -r \lrcorner$ for $r \in \mathbb{R}$.

*Remark.* For $\mathfrak{K} = \mathbb{Q}$ or $\mathbb{R}$, we have $\mathrm{Div}(X, \mathfrak{K}) = H^0(X, \mathcal{D}iv_X \otimes \mathfrak{K})$, but $\mathrm{Div}(X, \mathfrak{K})$ is not necessarily isomorphic to $\mathrm{Div}(X) \otimes \mathfrak{K}$.

**Convention** (Linear equivalence)**.** Let $X$ be a normal variety. For two $\mathbb{R}$-divisors $D$ and $D'$ on $X$, if $D - D'$ is a principal divisor, i.e., $D - D' = \mathrm{div}(\varphi)$ for a non-zero meromorphic function $\varphi$ on $X$, then $D$ is said to be *linearly equivalent* to $D'$, and we write $D \sim D'$ for the linear equivalence. If $m(D - D') \sim 0$ for a positive integer $m$, then $D$ is said to be $\mathbb{Q}$-*linearly equivalent* to $D'$, and we write $D \sim_{\mathbb{Q}} D'$ for the $\mathbb{Q}$-linear equivalence.

**Definition** ($\mathbb{Q}$-Cartier, $\mathbb{R}$-Cartier)**.** Let $X$ be a normal complex analytic space. A $\mathbb{Q}$-divisor $D$ on $X$ is said to be $\mathbb{Q}$-*Cartier* if there is a positive integer $m$ locally on $X$ such that $mD$ is a Cartier divisor. The group of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on $X$ is denoted by $\mathrm{CDiv}(X, \mathbb{Q})$. Then we have $\mathrm{CDiv}(X, \mathbb{Q}) = H^0(X, \mathcal{CD}iv_X \otimes \mathbb{Q})$. An $\mathbb{R}$-divisor $E$ on $X$ is said to be $\mathbb{R}$-*Cartier* if it is locally expressed as a finite $\mathbb{R}$-linear combination of Cartier divisors. The group of $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on $X$ is denoted by $\mathrm{CDiv}(X, \mathbb{R})$. Then we have $\mathrm{CDiv}(X, \mathbb{R}) = H^0(X, \mathcal{CD}iv_X \otimes \mathbb{R})$.

**Lemma 1.19.** *Let $f\colon X \to Y$ be a morphism of maximal rank of normal varieties. Then the pullback homomorphism $f^*\colon \mathrm{CDiv}(Y) \to \mathrm{CDiv}(X)$ in Lemma 1.17 extends to homomorphisms $f^*\colon \mathrm{CDiv}(Y, \mathbb{Q}) \to \mathrm{CDiv}(X, \mathbb{Q})$ and $f^*\colon \mathrm{CDiv}(Y, \mathbb{R}) \to \mathrm{CDiv}(X, \mathbb{R})$. Assume next that*

$$\mathrm{codim}(X \setminus f^{-1}(Y_{\mathrm{reg}}), X) \geq 2.$$

*Then pullback homomorphisms $f^*$ above extend to homomorphisms $f^*\colon \mathrm{Div}(Y) \to \mathrm{Div}(X)$, $f^*\colon \mathrm{Div}(Y, \mathbb{Q}) \to \mathrm{Div}(X, \mathbb{Q})$, and $f^*\colon \mathrm{Div}(Y, \mathbb{R}) \to \mathrm{Div}(X, \mathbb{R})$. Moreover, there is a functorial isomorphism $(f^*\mathcal{O}_Y(D))^{\vee\vee} \simeq \mathcal{O}_X(f^*D)$ for $D \in \mathrm{Div}(Y)$.*

*Proof.* Let $\mathfrak{K}$ denote $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. By the proof of Lemma 1.17, we have a homomorphism $f^{-1}(\mathcal{CD}iv_Y \otimes \mathfrak{K}) \to \mathcal{CD}iv_X \otimes \mathfrak{K}$, and a homomorphism $\mathcal{CD}iv_Y \otimes \mathfrak{K} \to f_*(\mathcal{CD}iv_X \otimes \mathfrak{K})$ by adjunction. This defines the pullback homomorphism $f^*\colon \mathrm{CDiv}(Y, \mathfrak{K}) \to \mathrm{CDiv}(X, \mathfrak{K})$. We set $X' = f^{-1}(Y_{\mathrm{reg}})$ and assume that $\mathrm{codim}(X \setminus X', X) \geq 2$. Then

$$\begin{aligned} &\mathcal{D}iv_Y \otimes \mathfrak{K} \simeq i_*(\mathcal{D}iv_{Y_{\mathrm{reg}}} \otimes \mathfrak{K}) \simeq i_*(\mathcal{CD}iv_{Y_{\mathrm{reg}}} \otimes \mathfrak{K}) \quad \text{and} \\ &\mathcal{D}iv_X \otimes \mathfrak{K} \simeq j_*(\mathcal{D}iv_{X'} \otimes \mathfrak{K}) \supset j_*(\mathcal{CD}iv_{X'} \otimes \mathfrak{K}), \end{aligned}$$

where $i\colon Y_{\mathrm{reg}} \hookrightarrow Y$ and $j\colon X' \hookrightarrow X$ stand for open immersions. Hence, for the restriction $f' := f|_{X'}\colon X' \to Y_{\mathrm{reg}}$ of $f$, the homomorphism $(f')^{-1}\mathcal{CD}iv_{Y_{\mathrm{reg}}} \to \mathcal{CD}iv_{X'}$ in the proof of Lemma 1.17 defines a homomorphism

$$\mathcal{D}iv_Y \otimes \mathfrak{K} \to f_*(\mathcal{D}iv_X \otimes \mathfrak{K}).$$

For $\mathfrak{K} = \mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, it induces pullback homomorphisms $\mathrm{Div}(Y) \to \mathrm{Div}(X)$, $\mathrm{Div}(Y, \mathbb{Q}) \to \mathrm{Div}(X, \mathbb{Q})$, and $\mathrm{Div}(Y, \mathbb{R}) \to \mathrm{Div}(X, \mathbb{R})$, respectively. For $D \in \mathrm{Div}(Y)$, we have $(f^*\mathcal{O}_Y(D))^{\vee\vee} \simeq \mathcal{O}_X(f^*D)$ by applying $j_*$ to

$$(f^*\mathcal{O}_Y(D))|_{X'} \simeq f'^*\mathcal{O}_{Y_{\mathrm{reg}}}(D|_{Y_{\mathrm{reg}}}) \simeq \mathcal{O}_{X'}(f'^*(D|_{Y_{\mathrm{reg}}})) \simeq \mathcal{O}_X(f^*D)|_{X'}. \qquad \square$$

*Remark.* When $\mathrm{codim}(X \setminus f^{-1}(Y_{\mathrm{reg}}), X) \geq 2$, the pullback $f^*D$ is regarded as the closure of $f'^*(D|_{Y_{\mathrm{reg}}})$.

**Definition** (Pushforward)**.** Let $f\colon X \to Y$ be a *non-degenerate morphism* (cf. Definition 1.1) of normal varieties. Let $B$ be an $\mathbb{R}$-divisor on $X$ such that $f|_\Gamma\colon \Gamma \to$

$Y$ is proper for any prime component $\Gamma$ of $B$. Then the *pushforward* $f_*B$ is defined as an $\mathbb{R}$-divisor on $Y$ such that

$$\mathrm{mult}_\Theta f_*B = \sum\nolimits_{\Gamma \in \mathcal{C}(B;\Theta)} d_{\Gamma/\Theta} \,\mathrm{mult}_\Gamma B$$

for any prime divisor $\Theta$ on $Y$, where $\mathcal{C}(B;\Theta)$ is the set of prime components $\Gamma$ of $B$ such that $f(\Gamma) = \Theta$ and where $d_{\Gamma/\Theta}$ is the degree of the finite morphism $f|_\Gamma \colon \Gamma \to \Theta$. Note that if $B$ is a divisor (resp. $\mathbb{Q}$-divisor), then $f_*B$ is so.

*Remark.* Assume that $f$ is proper. Then $f_*$ gives rise to homomorphisms $\mathrm{Div}(X) \to \mathrm{Div}(Y)$, $\mathrm{Div}(X,\mathbb{Q}) \to \mathrm{Div}(Y,\mathbb{Q})$, and $\mathrm{Div}(X,\mathbb{R}) \to \mathrm{Div}(Y,\mathbb{R})$. If $B \in \mathrm{Div}(X)$, then $\mathcal{O}_Y(f_*B)$ is isomorphic to the double-dual of

$$\bigwedge^{\deg f} (f_*\mathcal{O}_X(B)) \otimes_{\mathcal{O}_Y} \Big(\bigwedge^{\deg f} f_*\mathcal{O}_X\Big)^\vee$$

(cf. [38, II, §2.e]). Moreover, $f_*(f^*D) = (\deg f)D$ for any $D \in \mathrm{CDiv}(Y,\mathbb{Q})$.

**Definition** (Exceptional divisor)**.** Let $f \colon X \to Y$ be a non-degenerate morphism of normal varieties. A prime divisor $\Gamma$ on $X$ is said to be $f$-*exceptional*, or *exceptional for* $f$, if $\dim_x \Gamma \cap f^{-1}(f(x)) > 0$ for any $x \in \Gamma$. An $\mathbb{R}$-divisor on $X$ is called $f$-*exceptional* if its prime components are all $f$-exceptional.

*Remark.* When $f$ is proper, an $\mathbb{R}$-divisor $D$ on $X$ is $f$-exceptional if and only if $f_*D = 0$.

*Remark* 1.20. If a prime divisor $\Gamma$ is not $f$-exceptional, then $\Gamma \cap X' \neq \emptyset$ for $X' := f^{-1}(Y_{\mathrm{reg}})$. Moreover $\Gamma|_{X'}$ is also a prime divisor of $X'$, since $X'$ is a Zariski-open subset of $X$ (cf. [15, IX, §1.2]). Hence, we can consider the multiplicity of $f'^*(D|_{Y_{\mathrm{reg}}})$ along $\Gamma|_{X'}$ for the morphism $f' = f|_{X'} \colon X' \to Y_{\mathrm{reg}}$. If $f$ has no exceptional divisor, then $\mathrm{codim}(X \setminus X', X) \geq 2$.

*Remark* 1.21. Let $f \colon X \to Y$ be a non-degenerate morphism of normal surfaces without exceptional divisors. Then $f$ has only discrete fibers. Conversely, any morphism $f \colon X \to Y$ of normal surfaces with only discrete fibers is non-degenerate by Corollary 1.4. In this case, $f$ is open and is locally a finite morphism by Corollary 1.8, i.e., for any $x \in X$, there exists an open neighborhood $\mathcal{U}$ of $x$ in $X$ such that $\mathcal{U} \cap f^{-1}(f(x)) = \{x\}$, $f(\mathcal{U})$ is open in $Y$, $f|_\mathcal{U} \colon \mathcal{U} \to f(\mathcal{U})$ is finite.

**Definition 1.22** (Strict pullback)**.** Let $f \colon X \to Y$ be a non-degenerate morphism of normal varieties. For an $\mathbb{R}$-divisor $D$ on $Y$, let $\mathcal{S}_f(D)$ be the set of non-$f$-exceptional prime divisors on $X$ contained in $f^{-1}(\mathrm{Supp}\, D)$. The *strict pullback* $f^{[*]}D$ of $D$ is a $\mathbb{Q}$-divisor on $X$ defined by

$$\mathrm{mult}_\Gamma f^{[*]}D = \begin{cases} \mathrm{mult}_{\Gamma|_{X'}} f'^*(D|_{Y_{\mathrm{reg}}}), & \text{if } \Gamma \in \mathcal{S}_f(D), \\ 0, & \text{if } \Gamma \notin \mathcal{S}_f(D), \end{cases}$$

for prime divisors $\Gamma$ on $X$, where $X' = f^{-1}(Y_{\mathrm{reg}})$ and $f' = f|_{X'} \colon X' \to Y_{\mathrm{reg}}$ (cf. Remark 1.20, [38, II, §2.e]). If $f$ is a *bimeromorphic* morphism, i.e., a proper surjective morphism such that $f^{-1}U \to U$ is an isomorphism for a non-empty open subset $U \subset Y$, then $f^{[*]}D$ is called the *proper transform* of $D$ in $X$. In this case, $f_*(f^{[*]}D) = D$.

1.3. **Numerical pullbacks of a divisor on a normal surface.** For a bimeromorphic morphism $f\colon X \to Y$ of normal surfaces and a divisor $D$ on $Y$, we have the *numerical pullback* $f^*D$ as a $\mathbb{Q}$-divisor on $X$, which is introduced by Mumford [35, II, §(b)]. These pullbacks define intersection numbers of two divisors on normal surfaces which are not necessarily Cartier. We can extend the definition of numerical pullback for $\mathbb{R}$-divisors and for non-generate morphisms of normal surfaces. We shall explain some elementary properties of numerical pullbacks. The following is proved by the same method as in [35], [48, §1], or [40, §2.1].

**Lemma-Definition 1.23** (Numerical pullback)**.** *For a non-degenerate morphism $f\colon X \to Y$ of normal surfaces, there is a functorial linear map $f^*\colon \mathrm{Div}(Y, \mathbb{Q}) \to \mathrm{Div}(X, \mathbb{Q})$ of $\mathbb{Q}$-vector spaces satisfying the following conditions*:

(1) *For non-degenerate morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ of normal surfaces, one has $f^* \circ g^* = (g \circ f)^*$.*
(2) *If $f$ is an open immersion, then $f^*$ is the restriction map $D \mapsto D|_X$.*
(3) *The homomorphism $f^*$ extends the pullback homomorphism* $\mathrm{CDiv}(Y) \to \mathrm{CDiv}(X)$ *of groups of Cartier divisors.*
(4) *In case $X$ is non-singular and $f$ is proper, for a $\mathbb{Q}$-divisor $D$ on $Y$, the intersection number $(f^*D)E$ is zero for any $f$-exceptional $\mathbb{Q}$-divisor $E$.*

*The $\mathbb{Q}$-divisor $f^*D$ is called the* numerical pullback *of a $\mathbb{Q}$-divisor $D$ on $Y$.*

*Remark.* When $X$ is non-singular and $f$ is a bimeromorphic morphism, the numerical pullback $f^*D$ is expressed as the sum $f^{[*]}D + E$ of the proper transform $f^{[*]}D$ and an $f$-exceptional $\mathbb{Q}$-divisor $E$ such that $(f^{[*]}D + E)\Gamma = 0$ for any $f$-exceptional prime divisor $\Gamma$. Here, $E$ is uniquely determined, since the intersection matrix $(\Gamma_i\Gamma_j)$ of $f$-exceptional prime divisors $\Gamma_i$ contracted to a fixed point of $Y$ is negative definite (cf. [35, p. 6]).

*Remark.* By resolution of singularities and indeterminacy of meromorphic maps, for the morphism $f$, we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\mu} & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ N & \xrightarrow{\nu} & Y \end{array}$$

of normal surfaces such that $M$ and $N$ are non-singular and that $\mu$ and $\nu$ are bimeromorphic morphisms. Then the numerical pullback is given by $f^*D = \mu_*(g^*(\nu^*D))$ for a divisor $D$, where $g^*$ indicates the pullback of a Cartier divisor, and $\mu_*$ indicates the pushforward of a divisor by the proper morphism $\mu$.

**Definition** (Intersection number)**.** Let $D$ and $E$ be $\mathbb{Q}$-divisors on a normal surface $X$ such that $\mathrm{Supp}\, D \cap \mathrm{Supp}\, E$ is compact. Let $\mu\colon M \to X$ be a bimeromorphic morphism from a non-singular surface $M$. Here, $\mathrm{Supp}\, \mu^*D \cap \mathrm{Supp}\, \mu^*E$ is also compact, and one can consider the intersection number $DE := (\mu^*D)\mu^*E$. Then $DE$ is independent of the choice of $\mu$, and it is called the *intersection number* of $D$ and $E$.

*Remark* 1.24. The numerical pullback $f^*$ in Lemma-Definition 1.23 and the intersection numbers above are defined also for $\mathbb{R}$-divisors by linearity. The following properties are known or shown easily for $f\colon X \to Y$ and an $\mathbb{R}$-divisor $D$ on $Y$:

(1) If $D$ is effective, then $f^*D$ is so and $\operatorname{Supp} f^*D = f^{-1}(\operatorname{Supp} D)$.
(2) For an $\mathbb{R}$-divisor $E$ on $X$, if $f^{-1}(\operatorname{Supp} D) \cap \operatorname{Supp} E$ is compact, then the projection formula: $(f^*D)E = D(f_*E)$ holds.
(3) If $f$ is proper, then $(\deg f)D = f_*(f^*D)$.
(4) If an $\mathbb{R}$-divisor $D'$ on $Y$ has no common prime component with $D$ and if $DD' = 0$, then $\operatorname{Supp} D \cap \operatorname{Supp} D' = \emptyset$.
(5) Assume that $\operatorname{codim}(X \setminus X', X) \geq 2$ for $X' = f^{-1}(Y_{\mathrm{reg}})$. Then the pullback $f^*D$ given in Lemma 1.19 coincides with the numerical pullback. In fact, if $D$ is a divisor, then $f^*D|_{X'}$ coincides with the pullback of the Cartier divisor $D|_{Y_{\mathrm{reg}}}$.

*Remark* 1.25. Let $S$ be a non-zero reduced compact divisor on a normal surface $X$ such that the intersection matrix $(\Gamma_i\Gamma_j)$ of prime components $\Gamma_i$ of $S$ is negative definite. Let $D$ is an $\mathbb{R}$-divisor on $X$ such that $\operatorname{Supp} D \subset S$ and that $D$ is *nef on* $S$ (cf. [40, Def. 2.14(ii)]), i.e., $D\Gamma \geq 0$ for any prime component $\Gamma$ of $S$. Then $-D$ is effective by [59, Lem. 7.1]. If $S$ is connected in addition, then either $D = 0$ or $\operatorname{Supp} D = S$. In fact, if $\Gamma_i \not\subset \operatorname{Supp} D$ for a prime component $\Gamma_i$ of $S$, then $D\Gamma_i = 0$, and hence, $\Gamma_i \cap D = \emptyset$ and $\Gamma_j \cap D = \emptyset$ for any other prime component $\Gamma_j$ such that $\Gamma_i \cap \Gamma_j \neq \emptyset$; this implies that $D = 0$.

**Definition 1.26.** Let $X$ be a normal surface and let $\mu\colon M \to X$ be the minimal resolution of singularity. A divisor $D$ on $X$ is said to be *numerically Cartier* if the numerical pullback $\mu^*D$ is Cartier (cf. "numerically $\mathbb{Q}$-Cartier" in [38, II, §2.e]). We say that $D$ is numerically Cartier at a point $P \in X$ if $D$ is numerically Cartier on an open neighborhood of $P$. The *numerical factorial index* $\operatorname{nf}(X, P)$ at $P \in X$ is defined as the smallest positive integer $r$ such that $rD$ is numerically Cartier at $P$ for any divisor $D$ defined on any open neighborhood of $P$. The *numerical factorial index* $\operatorname{nf}(X)$ of $X$ is defined as $\operatorname{lcm}_{P \in X} \operatorname{nf}(X, P)$.

The numerical factorial index $\operatorname{nf}(X, P)$ is calculated by the intersection matrix as follows:

**Lemma 1.27.** *Let $X$ be a normal surface and let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular surface $Y$. Let $P$ be a point on $X$ such that $f^{-1}(P)$ is a divisor, and let $\Gamma_1, \ldots, \Gamma_k$ be the prime components of $f^{-1}(P)$. Then* $\operatorname{nf}(X, P)$ *equals the smallest positive integer $r$ such that $r\mathsf{M}^{-1}$ is integral for the intersection matrix $\mathsf{M} = (\Gamma_i\Gamma_j)_{1 \leq i,j \leq k}$.*

*Proof.* We can find an open neighborhood $\mathcal{U}$ of $P$ and prime divisors $B_1, B_2, \ldots, B_k$ on $f^{-1}\mathcal{U}$ such that $B_i\Gamma_j = \delta_{i,j}$ for any $1 \leq i, j \leq k$. We set $D_i := f_*B_i$ as a prime divisor on $\mathcal{U}$. Then $f^*D_i = B_i + \sum_{j=1}^{k} a_{i,j}\Gamma_j$ for non-negative rational numbers $a_{i,j}$ such that $(a_{i,j})_{1 \leq i,j \leq k} = -\mathsf{M}^{-1}$. For a positive integer $m$, if $f^*(mD_i)$ is Cartier along $f^{-1}(P)$ for any $i$, then $m(a_{i,j}) = -m\mathsf{M}^{-1}$ is integral. Thus, $r \mid \operatorname{nf}(X, P)$. For a divisor $D$ on an open neighborhood of $P$, we write $f^*D = f^{[*]}D + \sum_{i=1}^k c_i\Gamma_i$ for rational numbers $c_i$. Since $f^{[*]}D$ is Cartier, we have $d_j := (f^{[*]}D)\Gamma_j \in \mathbb{Z}$ and

$$(f^{[*]}D - \sum\nolimits_{i=1}^k d_iB_i)\Gamma_j = 0$$

for any $1 \leq j \leq k$. This implies that $(c_1, c_2, \ldots, c_k) = -(d_1, d_2, \ldots, d_k)\mathsf{M}^{-1}$. Then $rc_i \in \mathbb{Z}$ for any $1 \leq i \leq n$, and $f^*(rD)$ is Cartier. Therefore, $\mathrm{nf}(X, P) = r$. $\square$

The following is a generalization of [48, Thm. (2.1)] and is shown by properties of relative Zariski-decomposition (cf. [38, III, Lem. 5.10(2)]); here, we shall give a direct proof.

**Lemma 1.28.** *Let $f\colon Y \to X$ be a bimeromorphic morphism from a non-singular surface $Y$ to a normal surface $X$. Let $D$ be a divisor on $X$ and let $B$ be a $\mathbb{Q}$-divisor on $Y$ such that $f_*B = D$. Then the canonical injection*

$$\lambda_m\colon f_*\mathcal{O}_Y(\llcorner mB \lrcorner) \to (f_*\mathcal{O}_Y(\llcorner mB \lrcorner))^{\vee\vee} \simeq \mathcal{O}_X(mD)$$

*is an isomorphism for any integer $m > 0$ if and only if $B \geq f^*D$, where $^{\vee\vee}$ stands for the double-dual.*

*Proof.* Since the assertion is local on $X$, we may assume that $X$ is Stein and that $\operatorname{Sing} X$ consists of at most one point. For any $m > 0$, we have an $f$-exceptional $\mathbb{Q}$-divisor $F_m$ on $Y$ such that $mf^*D - F_m$ is Cartier and

$$(f^*\mathcal{O}_X(mD))^{\vee\vee} \simeq \mathcal{O}_Y(mf^*D - F_m).$$

Since the cokernel of $f^*\mathcal{O}_X(mD) \to (f^*\mathcal{O}_X(mD))^{\vee\vee}$ is supported on discrete points, the intersection number $(mf^*D - F_m)\Gamma = -F_m\Gamma$ is non-negative for any $f$-exceptional prime divisor $\Gamma$. Then $F_m$ is effective by Remark 1.25, since the intersection matrix of $f$-exceptional prime divisors is negative definite.

Assume that $B \geq f^*D$. Then $mB \geq \llcorner mB \lrcorner \geq mf^*D - F_m$ for any $m > 0$. Hence, we have an injection $\mathcal{O}_X(mD) \to f_*\mathcal{O}_Y(\llcorner mB \lrcorner)$ giving the inverse of $\lambda_m$. This shows the "if" part. The "only if" part is shown as follows: Suppose that $\lambda_m$ is an isomorphism for any $m > 0$. Then $f^*f_*\mathcal{O}_Y(\llcorner mB \lrcorner) \to \mathcal{O}_Y(\llcorner mB \lrcorner)$ induces an injection $\mathcal{O}_Y(mf^*D - F_m) \to \mathcal{O}_Y(\llcorner mB \lrcorner)$, which corresponds to an inequality $f^*D - (1/m)F_m \leq B$ of $\mathbb{Q}$-divisors. Hence, we are reduced to proving that $F_\infty := \lim_{m\to\infty}(1/m)F_m = 0$. Note that the $\mathbb{R}$-divisor $F_\infty$ exists, since $F_m + F_n \geq F_{m+n}$ for any positive integers $m$ and $n$ (cf. [38, III, Lem. 1.3]).

Let $\Gamma_1, \ldots, \Gamma_l$ be the $f$-exceptional prime divisors. Then there exist positive integers $a_1, \ldots, a_l$ such that $A\Gamma_i > 0$ for any $1 \leq i \leq l$ for the divisor $A = -\sum a_i\Gamma_i$. This implies that $f$ is a projective morphism over an open neighborhood of $\operatorname{Sing} X$ and that $A$ is $f$-ample (cf. [36, Prop. 1.4]). Hence, $mf^*D + A$ is also $f$-ample for any $m > 0$. We choose an integer $b > 0$ such that $bD$ is Cartier. Then there is an integer $k = k_b > 0$ such that

$$f^*f_*\mathcal{O}_Y(k(bf^*D + A)) \to \mathcal{O}_Y(k(bf^*D + A))$$

is surjective. Hence, $k(bf^*D + A) \leq kbf^*D - F_{kb}$; equivalently, $\mathrm{mult}_{\Gamma_i} F_{kb} \leq ka_i$ for any $1 \leq i \leq l$. By taking $b \to \infty$, we have

$$\mathrm{mult}_{\Gamma_i} F_\infty = \lim_{b\to\infty}(1/k_b b)\,\mathrm{mult}_{\Gamma_i} F_{k_b b} \leq \lim_{b\to\infty} a_i/b = 0.$$

Therefore, $F_\infty = 0$ and we are done. □

1.4. **Pullback and pushforward by meromorphic maps.** We shall define pullbacks and pushforwards of $\mathbb{R}$-divisors by "non-degenerate meromorphic maps" under certain conditions.

**Definition 1.29.** Let $f\colon X \dashrightarrow Y$ be a meromorphic map of normal varieties, and let $V$ be the normalization of the graph of $f$. Then $f = \pi \circ \mu^{-1}$ for the bimeromorphic morphism $\mu = \mu_f\colon V \to X$ and the morphism $\pi = \pi_f\colon V \to Y$ defined by projections (cf. [46, §6, Def. 15], [56, I, §2, Def. 2.2]). We say that $f$ is proper (resp. of maximal rank, resp. non-degenerate) when $\pi$ is so.

**Definition 1.30.** In the situation of Definition 1.29 above, assume that $f$ is non-degenerate. We set $n := \dim X = \dim Y$. Let $B$ and $D$ be $\mathbb{R}$-divisors on $X$ and $Y$, respectively.

(1) The *strict pullback* $f^{[*]}D$ of $D$ by $f$ is defined as the $\mathbb{R}$-divisor $\mu_*(\pi^{[*]}D)$ on $X$, where $\pi^{[*]}D$ is defined in Definition 1.22.
(2) When $D$ is $\mathbb{R}$-Cartier or when $n = 2$, the (*total*) *pullback* $f^*D$ of $D$ by $f$ is defined as the $\mathbb{R}$-divisor $\mu_*(\pi^*D)$ on $X$.
(3) When $\mathrm{Supp}\, B$ is compact or when $f$ is proper, the *strict pushforward* $f_{[*]}B$ of $B$ by $f$ is defined as $\pi_*(\mu^{[*]}B)$.
(4) Assume that $B$ is $\mathbb{R}$-Cartier or $n = 2$. When $\mathrm{Supp}\, B$ is compact or when $f$ is proper, the (*total*) *pushforward* $f_*B$ of $B$ by $f$ is defined as $\pi_*(\mu^*B)$.

*Remark.* (1) When $B$ and $D$ are $\mathbb{R}$-Cartier, we have pullbacks $\mu^*B$ and $\pi^*D$ by Lemma 1.19. When $n = 2$, we have $\mu^*B$ and $\pi^*D$ as the numerical pullbacks (cf. Lemma-Definition 1.23).
(2) If $f$ is holomorphic, then $f^{[*]}D$, $f^*D$, and $f_*B$ above, respectively, are equal to the same ones defined for the morphism $f$, since $\mu_f$ is an isomorphism. Moreover, in this case, we have $f_{[*]}B = f_*B$.
(3) When $f$ is a bimeromorphic morphism, the strict transform $f^{[*]}D$ is called also the *proper transform* of $D$. This is expressed as $f_*^{-1}D$ in some articles (e.g. [34]), but this is not equal to the total pushforward $(f^{-1})_*D$ of $D$ by $f^{-1}\colon Y \dashrightarrow X$.

**Lemma 1.31.** *Let $f\colon X \dashrightarrow Y$ be a non-degenerate meromorphic map of varieties of dimension $n$ and let $\nu\colon W \to X$ be a bimeromorphic morphism from a normal variety $W$ such that $\varpi = f \circ \nu\colon W \to Y$ is holomorphic. Let $B$ and $D$ be $\mathbb{R}$-divisors on $X$ and $Y$, respectively.*

(1) *The strict pullback $f^{[*]}D$ equals $\nu_*(\varpi^{[*]}D)$.*
(2) *If $D$ is $\mathbb{R}$-Cartier or $n = 2$, then $f^*D = \nu_*(\varpi^*D)$.*
(3) *If $\mathrm{Supp}\, B$ is compact or if $f$ is proper, then $f_{[*]}B = \varpi_*(\nu^{[*]}B)$.*

(4) *Assume that $B$ is $\mathbb{R}$-Cartier or $n = 2$. If* $\operatorname{Supp} B$ *is compact or $f$ is proper, then $f_* B = \varpi_*(\nu^* B)$.*

*Proof.* For the normalization $V$ of the graph of $f$, there is a bimeromorphic morphism $\sigma\colon W \to V$ such that $\nu = \mu \circ \sigma$ and $\varpi = \pi \circ \sigma$ for morphisms $\mu = \mu_f$ and $\pi = \pi_f$ in Definition 1.29. Then $\varpi^{[*]} D = \sigma^{[*]}(\pi^{[*]} D)$ and $\nu^{[*]} B = \sigma^{[*]}(\mu^{[*]} B)$. Hence, we have (1) and (3) by using $\sigma_*(\varpi^{[*]} D) = \pi^{[*]} D$ and $\sigma_*(\nu^{[*]} B) = \mu^{[*]} B$. Similarly, we can prove (2) and (4), respectively, by $\varpi^* D = \sigma^*(\pi^* D)$ and $\sigma_*(\varpi^* D) = \pi^* D$ and by $\nu^* B = \sigma^*(\mu^* B)$ and $\sigma_*(\nu^* B) = \mu^* B$. $\square$

**Lemma 1.32.** *Let $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$ be non-degenerate meromorphic maps of normal varieties of dimension $n$. Then we have a commutative diagram*

$$\begin{tikzcd}
 & & U \arrow[dl, "\mu_h"'] \arrow[dr, "\pi_h"] & & \\
 & V \arrow[rr, dashed, "h"] \arrow[dl, "\mu_f"'] \arrow[dr, "\pi_f"] & & W \arrow[dl, "\mu_g"'] \arrow[dr, "\pi_g"] & \\
X \arrow[rr, dashed, "f"] & & Y \arrow[rr, dashed, "g"] & & Z
\end{tikzcd}$$

*of meromorphic maps of normal varieties, where $V$ (resp. $W$) is the normalization of the graph of $f$ (resp. $g$), morphisms $\mu_f$ (resp. $\mu_g$) and $\pi_f$ (resp. $\pi_g$) are as in Definition* 1.29*, $U$ is the normalization of the graph of the meromorphic map $h := \mu_g^{-1} \circ \pi_f\colon V \dashrightarrow W$, and morphisms $\mu_h$ and $\pi_h$ are as in Definition* 1.29*. We consider two conditions*:

(a) *every $\pi_f$-exceptional divisor is $\mu_f$-exceptional*;
(b) *every $\mu_g$-exceptional divisor is $\pi_g$-exceptional.*

*Then $\mathbb{R}$-divisors $B$ and $D$ on $X$ and $Z$, respectively, have the following properties*:

(1) *If* (a) *or* (b) *holds, then $(g \circ f)^{[*]} D = f^{[*]}(g^{[*]} D)$.*
(2) *Assume either that* $\operatorname{Supp} B$ *is compact or that $f$ and $g$ are proper. If* (b) *or* (a) *holds, then $(g \circ f)_{[*]} B = g_{[*]}(f_{[*]} B)$.*
(3) *Assume either that $n = 2$ or that $D$ and $g^* D$ are $\mathbb{R}$-Cartier. If* (a) *holds, then $(g \circ f)^* D = f^*(g^* D)$.*
(4) *Assume either that* $\operatorname{Supp} B$ *is compact or that $f$ and $g$ are proper. Moreover, assume either that $n = 2$ or that $B$ and $g_* B$ are $\mathbb{R}$-Cartier. If* (b) *holds, then $(g \circ f)_* B = g_*(f_* B)$.*

*Proof.* We consider $\mathbb{R}$-divisors

$$E = \pi_g^{[*]} D - \mu_g^{[*]}(\mu_{g*}(\pi_g^{[*]} D)) \quad \text{and} \quad \widetilde{E} = \pi_g^* D - \mu_g^*(\mu_{g*}(\pi_g^* D))$$

on $W$ in the cases (1) and (3), respectively, and $\mathbb{R}$-divisors

$$C = \pi_{h*}(\mu_h^{[*]} \mu_f^{[*]} B) - \mu_g^{[*]}(\pi_{f*}(\mu_f^{[*]} B)) \quad \text{and} \quad \widetilde{C} = \pi_{h*}(\mu_h^* \mu_f^* B) - \mu_g^*(\pi_{f*}(\mu_f^* B))$$

on $W$ in the cases (2) and (4), respectively. Here, we have

$$h^{[*]} E = \mu_{h*}(\pi_h^{[*]} \pi_g^{[*]} D) - \pi_f^{[*]}(\mu_{g*}(\pi_g^{[*]} D)), \quad h^* \widetilde{E} = \mu_{h*}(\pi_h^* \pi_g^* D) - \pi_f^*(\mu_{g*}(\pi_g^* D))$$

by $\mu_h^{[*]} \circ \pi_f^{[*]} = \pi_h^{[*]} \circ \mu_g^{[*]}$, $\mu_h^* \circ \pi_f^* = \pi_h^* \circ \mu_g^*$, and $\mu_{f*} \circ \mu_f^{[*]} = \mu_{f*} \circ \mu_f^* = \mathrm{id}$. These $\mathbb{R}$-divisors have the following properties:

(i) $E$ and $\widetilde{E}$ are $\mu_g$-exceptional;
(ii) if every prime component of $\pi_g^{[*]}D$ is not $\mu_g$-exceptional, then $E = 0$;
(iii) $h^{[*]}E$ and $h^*\widetilde{E}$ are $\pi_f$-exceptional;
(iv) $C$ and $\widetilde{C}$ are $\mu_g$-exceptional;
(v) if every prime component of $\mu_f^{[*]}B$ is not $\pi_f$-exceptional, then $C = 0$.

In fact, by linearity, we may assume that $D$ and $B$ are prime divisors for proving (i)–(v), and we have

$$\mu_{g*}E = \mu_{g*}\widetilde{E} = \mu_{g*}C = \mu_{g*}\widetilde{C} = 0$$

by $\mu_{g*} \circ \mu_g^{[*]} = \mu_{g*} \circ \mu_g^* = \mathrm{id}$, $\mu_{g*} \circ \pi_{h*} = \pi_{f*} \circ \mu_{h*}$, and $\mu_{h*} \circ \mu_h^{[*]} = \mu_{h*} \circ \mu_h^* = \mathrm{id}$. This shows (i) and (iv), and we have (iii) as a consequence of (i). Moreover, in case (ii), $E$ has no $\mu_g$-exceptional prime component but $\mu_{g*}E = 0$; this implies that $E = 0$. Thus, (ii) holds. In case (v), $\pi_{f*}(\mu_f^{[*]}B) = m\Theta$ for a prime divisor $\Theta$ on $Y$ and a positive integer $m$, and $\pi_{h*}(\mu_h^{[*]}\mu_f^{[*]}B) = m\mu_g^{[*]}\Theta$, since $\mu_h$ and $\mu_g$ are bimeromorphic morphisms; thus, $C = 0$, and we have proved (v).

By Lemma 1.31, we have four equalities

$$(g \circ f)^{[*]}D - f^{[*]}(g^{[*]}D) = \mu_{f*}(h^{[*]}E), \qquad (g \circ f)^*D - f^*(g^*D) = \mu_{f*}(h^*\widetilde{E}),$$
$$(g \circ f)_{[*]}B - g_{[*]}(f_{[*]}B) = \pi_{g*}C, \qquad (g \circ f)_*B - g_*(f_*B) = \pi_{g*}\widetilde{C}.$$

For example, we have

$$(g \circ f)^{[*]}D = (\mu_f \circ \mu_g)_*((\pi_g \circ \pi_h)^{[*]}D) = \mu_{f*}(\mu_{g*}(\pi_h^{[*]}(\pi_g^{[*]}D)))$$

by Lemma 1.31(1), and this implies the first equality. Hence, for the proof of (1)–(4), it suffices to verify:

(I) $h^{[*]}E$ and $h^*\widetilde{E}$ are $\mu_f$-exceptional, and
(II) $h_{[*]}C$ and $h_*\widetilde{C}$ are $\pi_g$-exceptional.

If (a) holds, then we have (I) and $C = 0$ by (iii) and (v). It implies (1) in the case (a), (2) in the case (a), and (3). If (b) holds, then we have (II) and $E = 0$ by (ii) and (iv). It implies (1) in the case (b), (2) in the case (b), and (4). Thus, we are done. □

**Corollary 1.33.** *In the situation of Lemma* 1.32*, assume that* $n = 2$ *and that* $\pi_g^*D$ *is* $\mu_g$-nef (*cf. Convention* 2.14(1) *below*), *i.e.,* $(\pi_g^*D)\Gamma \geq 0$ *for any* $\mu_g$*-exceptional prime divisor* $\Gamma$*. Then* $(g \circ f)^*D \leq f^*(g^*D)$.

*Proof.* The $\mathbb{R}$-divisor $\widetilde{E}$ in the proof of Lemma 1.32 is $\mu_g$-exceptional and $\mu_g$-nef. Then $-\widetilde{E}$ is effective by Remark 1.25, since the intersection matrix of prime components of any connected non-zero $\mu_g$-exceptional divisor is negative definite (cf. [35, p. 6]). Hence,

$$(g \circ f)^*D - f^*(g^*D) = \mu_{f*}(h^*\widetilde{E}) \leq 0. \qquad \square$$

*Remark.* An inequality of currents similar to the above is noticed in the study of dynamical systems (cf. [4, Prop. 1.13] and (†) in the proof of [20, Prop. 1.2]).

1.5. **Canonical divisors and ramification formulas for normal varieties.** In the first half of Section 1.5, we shall explain the *canonical divisor* $K_Y$ of a normal variety $Y$ and the *ramification formula* $K_X = f^*K_Y + R_f$ for a non-degenerate morphism $f\colon X \to Y$ of normal varieties in some special cases (cf. Situation 1.36), which include the case where $\dim X = \dim Y = 2$. Especially, we want to emphasize that $K_Y$ is unique up to linear equivalence but the ramification formula is regarded as an equality not only as a linear equivalence. In the last half, we shall give some variants of the ramification formula including the logarithmic ramification formula due to Iitaka (cf. (I-2) in Proposition 1.40 below).

**Convention** (Canonical divisor)**.** The *canonical divisor* $K_Y$ of a normal variety $Y$ is regarded as the following object: We set $n = \dim Y$. In case $Y$ is non-singular, the *canonical sheaf* $\omega_Y$ is defined as the sheaf $\Omega^n_Y = \Omega^n_{Y/\operatorname{Spec}\mathbb{C}}$ of germs of holomorphic $n$-forms on $Y$. In general, the *canonical sheaf* $\omega_Y$ is a coherent reflexive sheaf of rank 1 on $Y$ defined as $j_*\omega_{Y_{\mathrm{reg}}}$ for the open immersion $j\colon Y_{\mathrm{reg}} \hookrightarrow Y$ (cf. [45, App. of §1, Cor. (8)]); this is isomorphic to the $(-n)$-th cohomology sheaf $\mathcal{H}^{-n}(\omega^\bullet_Y)$ of the dualizing complex $\omega^\bullet_Y$ (cf. [21], [44]). If $\omega_Y$ has a non-zero meromorphic section $\eta$, then $\eta|_{Y_{\mathrm{reg}}}$ is a meromorphic $n$-form on $Y_{\mathrm{reg}}$, and there is a unique divisor $\operatorname{div}(\eta)$ on $Y$ satisfying $\operatorname{div}(\eta)|_{Y_{\mathrm{reg}}} = \operatorname{div}(\eta|_{Y_{\mathrm{reg}}})$, since $\operatorname{codim}(Y \setminus Y_{\mathrm{reg}}) \geq 2$. The divisor $\operatorname{div}(\eta)$ is called the *canonical divisor* and is denoted by $K_Y$, even though it depends on the choice of $\eta$. Hence, $\mathcal{O}_Y(K_Y) \simeq \omega_Y$, and $K_Y$ is unique up to linear equivalence. Even if $\omega_Y$ has no non-zero meromorphic section, the symbol $K_Y$ is used virtually, which means just the canonical sheaf $\omega_Y$.

*Remark.* If $Y$ is Stein, or more generally, if $Y$ is *weakly 1-complete* with a positive line bundle, then every non-zero reflexive sheaf on $Y$ admits a non-zero meromorphic section (cf. [9, Lem. 3]), thus, we can consider $K_Y$ as a divisor.

*Remark.* Even when $Y$ is a reducible normal complex analytic space, one can consider $K_Y$ as a divisor on $Y$ whose restriction to each connected component (= irreducible component) is the canonical divisor.

**Definition 1.34** ($f^\dagger\eta$)**.** Let $f\colon X \to Y$ be a non-degenerate morphism of non-singular varieties of dimension $n \geq 1$. For a holomorphic $n$-form $\eta$ on $Y$, we write $f^\dagger\eta$ for the pullback of $\eta$ by $f$ as a holomorphic $n$-form on $X$. This is given by the canonical homomorphism $\phi\colon f^*\omega_Y = f^*\Omega^n_Y \to \omega_X = \Omega^n_X$. Even for a meromorphic $n$-form $\eta$ on $Y$, we have the pullback $f^\dagger\eta$ as a meromorphic $n$-form on $X$ by

$$f^*(\mathfrak{M}_Y \otimes \omega_Y) \simeq f^*\mathfrak{M}_Y \otimes f^*\omega_Y \xrightarrow{\psi\otimes\mathrm{id}} \mathfrak{M}_X \otimes f^*\omega_Y \xrightarrow{\mathrm{id}\otimes\phi} \mathfrak{M}_X \otimes \omega_X,$$

where $\psi\colon f^*\mathfrak{M}_Y \to \mathfrak{M}_X$ is the pullback homomorphism of meromorphic functions, which exists as $f$ is non-degenerate (cf. the proof of Lemma 1.17).

*Remark.* The pullback $f^\dagger\eta$ is usually denoted by $f^*\eta$, but we use $f^\dagger$ for avoiding confusions with other $f^*$.

**Lemma-Definition 1.35.** *Let $f\colon X \to Y$ be a non-degenerate morphism of normal varieties of dimension $n \geq 1$ and let $\eta$ be a non-zero meromorphic section of $\omega_Y$. For the open subset $X_\diamond = X_{\mathrm{reg}} \cap f^{-1}(Y_{\mathrm{reg}})$ and for the induced morphism $f_\diamond =$*

*$f|_{X_\diamond} \colon X_\diamond \to Y_{\mathrm{reg}}$, the pullback $f_\diamond^\dagger(\eta|_{Y_{\mathrm{reg}}})$ as a meromorphic $n$-form on $X_\diamond$ extends to a meromorphic section of $\omega_X$. This section is denoted by $f^\dagger\eta$.*

*Proof.* The uniqueness of $f^\dagger\eta$ is obvious. Thus, we can replace $Y$ with any open subset. By the local theory of analytic spaces, we may assume that there is a finite surjective morphism $\tau \colon Y \to \Omega$ to a domain $\Omega$ of the affine space $\mathbb{C}^n$ (cf. [7, §3.1, Thm. 1]). Let $\zeta$ be the standard holomorphic $n$-form on $\Omega$, i.e., $\zeta = d\mathtt{z}_1 \wedge d\mathtt{z}_2 \wedge \cdots \wedge d\mathtt{z}_n$ for a coordinate $(\mathtt{z}_1, \mathtt{z}_2, \ldots, \mathtt{z}_n)$ of $\mathbb{C}^n$. For the induced morphism $\tau_{\mathrm{reg}} \colon Y_{\mathrm{reg}} \to \Omega$ of non-singular varieties, we have a meromorphic function $\varphi$ on $Y$ such that

$$\tau_{\mathrm{reg}}^\dagger \zeta = \varphi\eta|_{Y_{\mathrm{reg}}}.$$

Let $\xi$ be a meromorphic section of $\omega_X$ such that the restriction $\xi|_{X_{\mathrm{reg}}}$ equals the pullback $(\tau \circ f_{\mathrm{reg}})^\dagger\zeta$ as a holomorphic $n$-form on $X_{\mathrm{reg}}$ for the induced morphism $f_{\mathrm{reg}} := f|_{X_{\mathrm{reg}}} \colon X_{\mathrm{reg}} \to Y$. Then

$$\xi|_{X_\diamond} = (f^*\varphi) f_\diamond^\dagger(\eta|_{Y_{\mathrm{reg}}}).$$

Thus, it is enough to set $f^\dagger\eta := (f^*\varphi)^{-1}\xi$. □

*Remark.* If $\operatorname{codim}(X \setminus f^{-1}(Y_{\mathrm{reg}}), X) \geq 2$, then $\operatorname{codim}(X \setminus X_\diamond, X) \geq 2$. In this case, for any holomorphic section $\eta$ of $\omega_Y$, the pullback $f^\dagger\eta$ is also a holomorphic section of $\omega_X$. In fact, the section $f^\dagger\eta$ is holomorphic if and only if the restriction $f^\dagger\eta|_{X_\diamond}$ is so by $\operatorname{codim}(X \setminus X_\diamond, X) \geq 2$, and now $f_\diamond^\dagger(\eta|_{Y_{\mathrm{reg}}})$ is holomorphic.

**Situation 1.36.** Let $f \colon X \to Y$ be a non-degenerate morphism of normal varieties. As a pullback homomorphism $f^*$ for certain $\mathbb{R}$-divisors, we consider one of the following:

(I) The homomorphism $f^* \colon \operatorname{CDiv}(Y, \mathbb{R}) \to \operatorname{CDiv}(X, \mathbb{R})$ in Lemma 1.19.

(II) The homomorphism $f^* \colon \operatorname{Div}(Y, \mathbb{R}) \to \operatorname{Div}(X, \mathbb{R})$ in Lemma 1.19, which is defined only when $\operatorname{codim}(X \setminus f^{-1}(Y_{\mathrm{reg}}), X) \geq 2$.

(III) The numerical pullback homomorphism $f^* \colon \operatorname{Div}(Y, \mathbb{R}) \to \operatorname{Div}(X, \mathbb{R})$ in Lemma-Definition 1.23, which is defined only when $\dim X = \dim Y = 2$. This $f^*$ extends the homomorphisms $f^*$ in (I) and (II), but does not induce $\operatorname{Div}(Y) \to \operatorname{Div}(X)$ in general.

**Lemma 1.37.** *Let $D$ be an $\mathbb{R}$-divisor on $Y$ such that the pullback $f^*(K_Y + D)$ is defined in one of cases in Situation* 1.36*. Then $K_X - f^*(K_Y + D)$ is uniquely determined as an $\mathbb{R}$-divisor on $X$ when $\omega_Y$ has a non-zero meromorphic section $\eta$, by setting $K_X = \operatorname{div}(f^\dagger\eta)$ and $K_Y = \operatorname{div}(\eta)$.*

*Proof.* For non-zero meromorphic sections $\eta_1$ and $\eta_2$ of $\omega_Y$, there is a non-zero meromorphic function $\varphi$ on $Y$ such that $\eta_1 = \varphi\eta_2$. Then $f^\dagger\eta_1 = (f^*\varphi)f^\dagger\eta_2$, and we have

$$\operatorname{div}(\eta_1) + D = \operatorname{div}(\eta_2) + D + \operatorname{div}(\varphi) \quad \text{and} \quad \operatorname{div}(f^\dagger\eta_1) = \operatorname{div}(f^\dagger\eta_2) + \operatorname{div}(f^*\varphi).$$

Since $f^*\operatorname{div}(\varphi) = \operatorname{div}(f^*\varphi)$ (cf. Lemma 1.17), we have

$$\operatorname{div}(f^\dagger\eta_1) - f^*(\operatorname{div}(\eta_1) + D) = \operatorname{div}(f^\dagger\eta_2) - f^*(\operatorname{div}(\eta_2) + D).$$

Thus, $K_X - f^*(K_Y + D)$ is uniquely determined. □

**Convention.** Let $f\colon X \to Y$ be a non-degenerate morphism of normal varieties and let $B$ and $D$ be $\mathbb{R}$-divisors on $X$ and $Y$, respectively. By an equality $K_X + B = f^*(K_Y + D)$, we mean the following:

(1) Assume that $\omega_Y$ admits a non-zero meromorphic section $\eta$. Then the pullback $f^*(\mathrm{div}(\eta) + D)$ exists in one of cases in Situation 1.36 and $\mathrm{div}(f^\dagger\eta) + B = f^*(\mathrm{div}(\eta) + D)$ as an $\mathbb{R}$-divisors on $X$.

(2) If $Y = \bigcup_\lambda Y_\lambda$ for open subsets $Y_\lambda$ such that each $\omega_{Y_\lambda}$ admits a non-zero meromorphic section on $Y_\lambda$, then

$$K_{X_\lambda} + B|_{X_\lambda} = f_\lambda^*(K_{Y_\lambda} + D|_{Y_\lambda})$$

for any $\lambda$, where $X_\lambda = f^{-1}Y_\lambda$ and $f_\lambda = f|_{X_\lambda}\colon X_\lambda \to Y_\lambda$.

Note that (1) is independent of the choice of $\eta$ by Lemma 1.37.

**Definition** (Ramification divisor (cf. [24, §5.6]))**.** In Situation 1.36, we define the ramification divisor of $f$ as a $\mathbb{Q}$-divisor $R_f$ on $X$ such that $K_X = f^*K_Y + R_f$.

*Remark.* If $X$ and $Y$ are non-singular, then $R_f$ is the usual ramification divisor in the sense that $R_f$ is an effective divisor and that the canonical injection $f^*\omega_Y \to \omega_X$ induces an isomorphism $f^*\omega_Y \simeq \omega_X \otimes \mathcal{O}_X(-R_f)$ (cf. [24, §5.6]). In Situation 1.36(I), $R_f$ exists when $K_Y$ is $\mathbb{Q}$-Cartier, but $R_f$ is not necessarily effective. In fact, when $f$ is a resolution of singularities, $R_f$ is effective if and only if $Y$ has only canonical singularities (cf. [45, Def. (1.1)], [31, Def. 0-2-6]). In Situation 1.36(II), $R_f$ exists always as an effective divisor as the closure of the ramification divisor $R_{f_\diamond}$ of the induced morphism $f_\diamond = f|_{X_\diamond}\colon X_\diamond \to Y_{\mathrm{reg}}$ for $X_\diamond = X_{\mathrm{reg}} \cap f^{-1}Y_{\mathrm{reg}}$. In Situation 1.36(III), $R_f$ exists always, but it is not necessarily effective.

Now, we shall present some variations of ramification formula for non-degenerate morphisms.

**Lemma 1.38.** *Let $f\colon X \to Y$ be a non-degenerate morphism of non-singular varieties of dimension $n \geq 1$ and let $B$ and $D$ be non-singular prime divisors on $X$ and $Y$, respectively, such that $B = f^{-1}D$.*

(1) *If $B$ is not $f$-exceptional, then $\mathrm{mult}_B R_f = m - 1$ for $m = \mathrm{mult}_B f^*D$ and for the ramification divisor $R_f$.*

(2) *If $B$ is $f$-exceptional, then the image of the pullback homomorphism*

$$\phi^n\colon f^*\Omega_Y^n(\log D) \to \Omega_X^n(\log B)$$

*of logarithmic $n$-forms is contained in the subsheaf $\Omega_X^n$.*

*Proof.* We shall give a sheaf-theoretic proof even though (1) is obvious by a local description of $f$. For each $1 \leq p \leq n$, there is a commutative diagram

$$\text{(I-1)}\qquad \begin{CD} 0 @>>> f^*\Omega_Y^p @>>> f^*\Omega_Y^p(\log D) @>>> f^*\Omega_D^{p-1} @>>> 0 \\ @. @V{\psi^p}VV @V{\phi^p}VV @V{\varphi^{p-1}}VV \\ 0 @>>> \Omega_X^p @>>> \Omega_X^p(\log B) @>{r^p}>> \Omega_B^{p-1} @>>> 0 \end{CD}$$

of exact sequences on sheaves of holomorphic and logarithmic $p$-forms, where the pullback homomorphisms $\psi^p = \wedge^p \psi^1$ and $\phi^p = \wedge^p \phi^1$ are injective as $f$ is non-degenerate. Moreover, $r^1$ is induced by the residue isomorphism $\Omega^1_X(\log B) \otimes \mathcal{O}_B \simeq \mathcal{O}_B$, and $\varphi^{p-1}$ is expressed as the composite homomorphism

$$f^*\Omega^{p-1}_D \xrightarrow{\pi^{p-1}} g^*\Omega^{p-1}_D \xrightarrow{\psi^{p-1}_g} \Omega^{p-1}_B$$

for $g := f|_B \colon B \to D$, where $\psi^{p-1}_g$ is the pullback homomorphism of holomorphic $(p-1)$-forms, and $\pi^{p-1}$ is a surjection induced by $f^*\mathcal{O}_D \simeq \mathcal{O}_{mB} \to \mathcal{O}_B$ and by tensor products with the locally free $\mathcal{O}_X$-module $f^*\Omega^p_Y(\log D)$.

Assume that $B$ is not $f$-exceptional. Then $g$ is non-degenerate and $\psi^{n-1}_g$ is injective. Hence, $\varphi^{n-1}$ is generically surjective on $B$, and the kernel of $\varphi^{n-1}$ is isomorphic to

$$\mathcal{O}_{(m-1)B} \otimes \mathcal{O}_X(-B) \otimes f^*\Omega^n_Y(\log Y)$$

if $m > 1$, and is zero if $m = 1$. In particular, $\phi^n$ is surjective on a dense open subset of $B$. By the snake lemma, we have $\mathrm{mult}_B R_f = m - 1$, since the cokernel of $\psi^n$ is isomorphic to $\omega_X \otimes \mathcal{O}_{R_f}$. This shows (1).

Assume next that $B$ is $f$-exceptional. Then $n \geq 2$, and $\psi^{n-1}_g = 0$ as $g$ is degenerate. Hence, the image of $\phi^n$ is contained in $\Omega^n_X$. This shows (2). □

**Lemma 1.39.** *Let $f \colon X \to Y$ be a non-degenerate morphism of normal varieties without exceptional divisors and let $B \subset X$ and $D \subset Y$ be reduced divisors such that $B = f^{-1}D$. Then $K_X + B = f^*(K_Y + D) + \Delta$ for an effective divisor $\Delta$ having no common prime component with $B$. In particular, the induced morphism $X \setminus B \to Y \setminus D$ is étale in codimension 1 if and only if $\Delta = 0$.*

*Proof.* We can consider the pullback homomorphism $f^* \colon \mathrm{Div}(Y) \to \mathrm{Div}(X)$ in Situation 1.36(II), since $\mathrm{codim}(X \setminus f^{-1}(Y_{\mathrm{reg}}), X) \geq 2$. Thus, we may assume that $X$ and $Y$ are non-singular by replacing $Y$ and $X$ with $Y_{\mathrm{reg}}$ and $X_{\mathrm{reg}} \cap f^{-1}(Y_{\mathrm{reg}})$, respectively. For the ramification divisor $R_f = K_X - f^*K_Y$, we have $\Delta = R_f + B - f^*D$. Let $\Gamma$ be a prime divisor on $X$. If $\Gamma \not\subset B = f^{-1}D$, then $\mathrm{mult}_\Gamma \Delta = \mathrm{mult}_\Gamma R_f \geq 0$. If $\Gamma \subset B$, then $\Gamma \subset f^{-1}\Theta$ for a prime component $\Theta$ of $D$. In this case, since $B$ is not $f$-exceptional, we have

$$1 + \mathrm{mult}_\Gamma R_f = \mathrm{mult}_\Gamma f^*\Theta = \mathrm{mult}_\Gamma f^*D$$

by applying Lemma 1.38 to suitable open subsets $U \subset X$ and $V \subset Y$ such that $U \subset f^{-1}V$ and that $\Gamma|_U = B|_U$ and $\Theta|_U = D|_U$ are non-singular prime divisors; hence, $\mathrm{mult}_\Gamma \Delta = \mathrm{mult}_\Gamma(R_f + B - f^*D) = 0$. Thus, $\Delta$ is effective and has no common prime component with $B$. □

The following equality (I-2) is known as the logarithmic ramification formula due to Iitaka (cf. [23, §4, (R)], [24, Thm. 11.5]). The generalization (I-3) is obtained by Suzuki [54] and Iitaka [25, Part 2, Prop. 1]. We shall prove them by a sheaf-theoretic argument.

**Proposition 1.40.** *Let $f \colon X \to Y$ be a non-degenerate morphism of normal varieties and let $B$ and $D$ be reduced divisors on $X$ and $Y$, respectively, such that $Y$ is non-singular, $D$ is normal crossing, and $f^{-1}D \subset B$.*

(1) *There is an effective divisor $\overline{R}$ on $X$ such that*

$$K_X + B = f^*(K_Y + D) + \overline{R} \tag{I-2}$$

*and that any common prime component of $f^{-1}D$ and $\overline{R}$ is $f$-exceptional.*

(2) *Let $C$ be a non-singular divisor on $Y$ and $A$ a reduced divisor on $X$ such that $(f^{[*]}C)_{\mathrm{red}} \leq A$, $A + B$ is reduced, and $C + D$ is reduced and normal crossing. Then there is an effective divisor $R^{\&}$ on $X$ such that*

$$K_X + A + B = f^*(K_Y + C + D) + R^{\&}. \tag{I-3}$$

*Proof.* By replacing $X$ with a Zariski-open subset whose complement has codimension at least 2, we may assume that $X$ and $B$ are non-singular and that $\widetilde{B} = (f^*C + A + B)_{\mathrm{red}}$ is also non-singular in the situation of (2).

(1): The pullback homomorphism

$$\phi^n \colon f^*\Omega^n_Y(\log D) \simeq f^*(\omega_Y \otimes \mathcal{O}_Y(D)) \to \Omega^n_X(\log B) \simeq \omega_X \otimes \mathcal{O}_X(B)$$

of logarithmic $n$-forms is injective as $f$ is non-degenerate, and it implies that $\overline{R} \geq 0$. It is enough to prove that $\Gamma \not\subset \operatorname{Supp} \overline{R}$ for any non-$f$-exceptional prime component $\Gamma$ of $f^{-1}D$. For this, by replacing $X$ and $Y$ with suitable open subsets, we may assume that $\Gamma = B = f^{-1}D$. Then $\Gamma = B \not\subset \operatorname{Supp} \overline{R}$ by Lemma 1.39.

(2): By (1), we have $K_X + \widetilde{B} = f^*(K_Y + C + D) + \widehat{R}$ for an effective divisor $\widehat{R}$. It is enough to prove that $\widehat{R} \geq \widetilde{B} - (A + B)$, or equivalently that $\widehat{R} \geq \Gamma$ for any prime component $\Gamma$ of $\widetilde{B} - (A+B)$. By assumption, $\Gamma$ is $f$-exceptional, $\Gamma \subset f^{-1}C$, and $\Gamma \not\subset B$. By replacing $X$ and $Y$ with open subsets, we may assume that $B = 0$, $\widetilde{B} - (A + B) = (f^*C + A)_{\mathrm{red}} - A = f^{-1}C$, and $\Gamma = f^{-1}C$. Then the image of

$$f^*\Omega^n_Y(\log C) \simeq f^*(\omega_Y \otimes \mathcal{O}_Y(C)) \to \Omega^n_X(\log \Gamma) \simeq \omega_X \otimes \mathcal{O}_X(\Gamma)$$

is contained in $\omega_X$ by Lemma 1.38(2). It implies that $\widehat{R} \geq \Gamma$, and we are done. $\square$

*Remark.* We have a little generalization of [25, Part 2, Prop. 1] in [38, II, Thm. 4.2]. But the assumption $\rho^{[*]}X \leq Y$ in the statement is stronger than what we expect. The correct assumption is $(\rho^{[*]}X)_{\mathrm{red}} \leq Y$. This correct case has been treated in the proof of [25, Part 2, Prop. 1], where $(f^{[*]}C)_{\mathrm{red}}$ is written as $f^{-1}[C]$. The stronger assumption affects [38, II, Lem. 4.4] given as an application of [38, II, Thm. 4.2].

The following lemma is borrowed from [38, II, Lems. 4.3 and 4.4], which are stated for generically finite morphisms.

**Lemma 1.41.** *Let $f\colon X \to Y$ be a non-degenerate morphism of normal varieties and let $D$ be an effective $\mathbb{Q}$-divisor on $Y$. Assume that $Y$ is non-singular and $\ulcorner D \urcorner$ is reduced and normal crossing.*

(1) *There is an effective $\mathbb{Q}$-divisor $\overline{R}_D$ on $X$ such that*

$$K_X + (f^*D)_{\mathrm{red}} = f^*(K_Y + D) + \overline{R}_D.$$

(2) *If $\llcorner D \lrcorner = 0$, then there is a $\mathbb{Q}$-divisor $R_D$ on $X$ such that $\ulcorner R_D \urcorner$ is effective and $K_X = f^*(K_Y + D) + R_D$.*

(3) *If* $C := \llcorner D \lrcorner$ *is non-singular, then there is a* $\mathbb{Q}$*-divisor* $R^{\&}_D$ *on* $X$ *such that* $\ulcorner R^{\&}_D \urcorner$ *is effective and*

$$K_X + (f^{[*]}C)_{\mathrm{red}} = f^*(K_Y + D) + R^{\&}_D.$$

*Proof.* We may assume that $D \neq 0$, since the ramification divisor $R_f = K_X - f^*K_Y$ is effective. Hence $D_{\mathrm{red}} = \operatorname{Supp} D = \ulcorner D \urcorner$. By replacing $X$ with a Zariski-open subset whose complement has codimension at least 2, we may assume that $X$ and $(f^*D)_{\mathrm{red}}$ are non-singular.

(1) and (2): By Proposition 1.40(1), $K_X + (f^*D)_{\mathrm{red}} = f^*(K_Y + D_{\mathrm{red}}) + \widetilde{R}$ for an effective divisor $\widetilde{R}$. Then $\overline{R}_D$ is effective by

$$\widetilde{R} = R_f + (f^*D)_{\mathrm{red}} - f^*(D_{\mathrm{red}}) = \overline{R}_D - f^*(D_{\mathrm{red}} - D).$$

This proves (1). Assume that $\llcorner D \lrcorner = 0$. Then $\overline{R}_D = R_D + (f^*D)_{\mathrm{red}} \geq 0$. For a prime component $\Gamma$ of $f^*D$, we have $\operatorname{mult}_\Gamma f^*(D_{\mathrm{red}} - D) > 0$, and

$$\operatorname{mult}_\Gamma R_D + 1 = \operatorname{mult}_\Gamma \overline{R}_D = \operatorname{mult}_\Gamma \widetilde{R} + \operatorname{mult}_\Gamma f^*(D_{\mathrm{red}} - D) > 0.$$

Hence, $\ulcorner R_D \urcorner$ is effective, and we have proved (2).

(3): We set $\Delta := \langle D \rangle = D - C$. By Proposition 1.40(2), we have

$$K_X + (f^{[*]}C)_{\mathrm{red}} + (f^*\Delta)_{\mathrm{red}} = f^*(K_Y + C + \Delta_{\mathrm{red}}) + R^{\&}$$

for an effective divisor $R^{\&}$ on $X$. Then

$$R^{\&}_D + (f^*\Delta)_{\mathrm{red}} = R^{\&} + f^*(\Delta_{\mathrm{red}} - \Delta)$$

is effective. For a prime component $\Gamma$ of $f^*\Delta$, we have $\operatorname{mult}_\Gamma f^*(\Delta_{\mathrm{red}} - \Delta) > 0$, and

$$1 + \operatorname{mult}_\Gamma R^{\&}_D = \operatorname{mult}_\Gamma(R^{\&}_D + (f^*\Delta)_{\mathrm{red}}) > 0.$$

Therefore, $\ulcorner R^{\&}_D \urcorner$ is effective, and (3) has been proved. □

## 2. Log-canonical singularities for complex analytic surfaces

We explain basic properties of log-canonical singularities and their variants only in the surface case, in Section 2.1, and give results related to ramification formulas in Section 2.2. The relative abundance theorem and the log-canonical modifications for surfaces are given in Section 2.3.

### 2.1. Log-canonical singularities.

**Definition 2.1.** Let $X$ be a normal surface with an effective $\mathbb{Q}$-divisor $B$ and let $\mu\colon M \to X$ be a bimeromorphic morphism from a non-singular surface $M$. We set $\Sigma = \Sigma_\mu(X, B)$ to be the union of $\mu^{-1} \operatorname{Supp} B$ and the $\mu$-exceptional locus. Note that $\Sigma \supset \mu^{-1} \operatorname{Sing} X$. Let $B_\mu = B_\mu(X, B)$ and $T_\mu = T_\mu(X, B)$ be the positive and negative parts, respectively, of the prime decomposition of $\mu^*B - R_\mu$ (cf. Definition 1.15) for the ramification divisor $R_\mu$, i.e., $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$. Note that $B_\mu \geq \mu^{[*]}B$ for the proper transform $\mu^{[*]}B$ in $M$ (cf. Definition 1.22) and that $T_\mu$ is $\mu$-exceptional. If there is a bimeromorphic morphism $\mu$ above such that $\Sigma$ is a normal crossing divisor, then $(X, B)$ is said to be

- *log-canonical* if $\ulcorner B_\mu \urcorner$ is reduced;

- *log-terminal* if $\llcorner B_\mu \lrcorner = 0$;
- 1-*log-terminal* if $\ulcorner B_\mu \urcorner$ is reduced and if $\llcorner B_\mu \lrcorner$is a non-singular divisor identified with the proper transform of $\llcorner B \lrcorner$ in $M$.

Here, the zero divisor is considered as a reduced and non-singular divisor (cf. Remark 1.18). For a point $P \in X$, the pair $(X, B)$ is said to be log-canonical (resp. log-terminal, resp. 1-log-terminal) at $P$ if $(U, B|_U)$ is so for some open neighborhood $U$ of $P$.

*Remark* 2.2. The conditions above are independent of the choice of such bimeromorphic morphisms $\mu\colon M \to X$. This follows from special cases of Lemma 2.10 below.

*Remark.* If $(X, B)$ is log-terminal, then $\mathrm{mult}_\Gamma B_\mu < 1$ for any prime component $\Gamma$ of $\Sigma$. The prefix "1-" of 1-log-terminal comes from a property that we allow $\mathrm{mult}_\Gamma B_\mu = 1$ only for the proper transforms $\Gamma$ of prime components of $B$.

*Remark* 2.3. It is known that $K_X + B$ is $\mathbb{Q}$-Cartier if $(X, B)$ is log-canonical in the sense above (cf. [29, Cor. 9.5], [33, §4.1]). We shall prove it in Corollary 2.21 below by applying the relative abundance theorem, Theorem 2.19. As a consequence, our definitions of log-canonical and log-terminal coincide with those given in [31, Def. 0-2-10]. The log-terminal and 1-log-terminal are called "Kawamata log terminal" (klt) and "purely log terminal" (plt), respectively, in [52] and [34]. As our policy, we do not use the notion of "log terminal" in [52] and [34], since it is not analytically local (cf. Remark 2.8 below). Therefore, the use of "purely log terminal" is not allowed, since it is weaker than our log-terminal. Thus, we use 1-log-terminal instead.

*Remark.* The pair $(X, B)$ is 1-log-terminal if and only if $(X\&B, 0)$ is log-terminal for the bimeromorphic pair $X\&B$ in the sense of [38, II, Def. 4.8].

Bimeromorphic contraction morphisms of extremal rays in the minimal model program preserve log-canonical (resp. log-terminal, resp. 1-log-terminal) pairs by:

**Lemma 2.4.** *Let $\nu\colon X \to X'$ be a bimeromorphic morphism of normal surfaces with a unique exceptional prime divisor $\Gamma$. Let $B$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(K_X + B)\Gamma \le 0$. If $(X, B)$ is log-canonical (resp. log-terminal), then $(X', B')$ is so for $B' := \nu_* B$. If $(K_X + B)\Gamma < 0$ and $(X, B)$ is log-canonical, then $(X', B')$ is 1-log-terminal at $\nu(\Gamma)$.*

*Proof.* By assumption, there is a rational number $\alpha \ge 0$ such that $K_X + B = \nu^*(K_{X'} + B') + \alpha\Gamma$. Here, $\alpha > 0$ if and only if $(K_X + B)\Gamma < 0$. Let $\mu\colon M \to X$, $B_\mu$, and $T_\mu$ be as in Definition 2.1 for $(X, B)$. Here, we may assume that the union of $\mu^{-1}(\Gamma \cup \mathrm{Supp}\, B)$ and the $\mu$-exceptional locus is normal crossing and that the proper transform of $(\llcorner B \lrcorner + \Gamma)_{\mathrm{red}}$ is non-singular. Then

$$K_M + B_\mu = (\nu \circ \mu)^*(K_{X'} + B') + T_\mu + \alpha\mu^*\Gamma.$$

In particular, the first assertion holds when $\alpha = 0$. Thus, we may assume that $\alpha > 0$, i.e., $(K_X + B)\Gamma < 0$. Let $B_{\nu\circ\mu}$ and $T_{\nu\circ\mu}$ be the positive and negative parts, respectively, of the prime decomposition of $B_\mu - (T_\mu + \alpha\mu^*\Gamma)$. Then the following holds for any prime divisor $\Theta$ on $M$:

- If $\Theta \not\subset \mu^{-1}\Gamma$, then $\mathrm{mult}_\Theta B_\mu = \mathrm{mult}_\Theta B_{\nu\circ\mu}$.
- If $\Theta \subset \mu^{-1}\Gamma$ but $\Theta \not\subset \mathrm{Supp}\, B_\mu$, then $\mathrm{mult}_\Theta B_\mu = \mathrm{mult}_\Theta B_{\nu\circ\mu} = 0$.
- If $\Theta \subset \mu^{-1}\Gamma \cap \mathrm{Supp}\, B_\mu$, then $1 \geq \mathrm{mult}_\Theta B_\mu > \mathrm{mult}_\Theta B_{\nu\circ\mu}$.

In particular, if $(X, B)$ is log-terminal, then $(X', B')$ is so, since $\llcorner B_\mu \lrcorner = 0$ implies $\llcorner B_{\nu\circ\mu} \lrcorner = 0$. If $(X, B)$ is log-canonical, then $\ulcorner B_{\nu\circ\mu} \urcorner$ is reduced and $\llcorner B_{\nu\circ\mu} \lrcorner$ is a reduced subdivisor of $\llcorner B_\mu \lrcorner$ having no prime component contracted to $\nu(\Gamma)$ by $\nu\circ\mu$; thus, $(X', B')$ is 1-log-terminal at $\nu(\Gamma)$. Therefore the first assertion for $\alpha > 0$ and the second assertion have been proved, and we are done. $\square$

*Remark.* The first assertion is a special case of Proposition 2.12(1) below.

*Fact* 2.5. The analytic germs of log-canonical pairs $(X, S)$ of a normal surface $X$ and a reduced divisor $S$ at a point $x \in S$ are classified in [29, Thm. 9.6] (cf. [34, Ch. 3]). In particular, one of the following three cases occurs (cf. [40, Thm. 3.22]):

(1) The case where $x \in \mathrm{Sing}\, S$ and $(X, S)$ is *toroidal at* $x$: The latter condition means that $X \setminus S \hookrightarrow X$ is a *toroidal embedding* at $x$ (cf. [32, II, §1]), or equivalently, there exist an affine toric variety $V$ and an open immersion $\theta\colon \mathcal{U} \hookrightarrow V$ of analytic spaces from an open neighborhood of $\mathcal{U}$ of $x$ such that $\theta^{-1}(\mathbb{T}) = \mathcal{U} \setminus S$ for the open torus $\mathbb{T}$ of $V$.

(2) The case where $x \in S_{\mathrm{reg}}$ and $(X, S + S')$ is toroidal embedding at $x$ for a non-singular divisor $S' \not\subset S$ such that $x \in S'$.

(3) The case where $x \in S_{\mathrm{reg}} \cap \mathrm{Sing}\, X$ and there is a double-cover $\tau\colon \widetilde{X} \to X$ such that
   - $\tau$ is étale over $X \setminus \{x\}$,
   - $\tau^{-1}(x) = \{\tilde{x}\}$ for a point $\tilde{x} \in \mathrm{Sing}\, \widetilde{S}$, where $\widetilde{S} := \tau^* S$, and
   - $(\widetilde{X}, \widetilde{S})$ is toroidal at $\tilde{x}$.

Moreover, for the minimal resolution $\mu\colon M \to X$ of singularities, the dual graph of prime components of the union of $\mu^{-1}(S)$ and the $\mu$-exceptional locus is completely described (cf. [29, Thm. 9.6], [40, Thm. 3.22]). In particular, $(X, x)$ is a cyclic quotient singularity in (1) and (2), and is a quotient singularity by an action of a dihedral group in (3). The pair $(X, S)$ is 1-log-terminal at $x$ if and only if (2) occurs. The divisor $K_X + S$ is Cartier at $x$ if and only if either (1) occurs or $x \in X_{\mathrm{reg}} \cap S_{\mathrm{reg}}$.

**Lemma 2.6.** *Let $(X, B)$ be a log-canonical pair of a normal surface $X$ and an effective $\mathbb{Q}$-divisor $B$. If $(X, B)$ is not 1-log-terminal at a point $x \in X$, then $(X, B + C)$ is not log-canonical for any effective $\mathbb{Q}$-divisor $C$ such that $x \in \mathrm{Supp}\, C$; in particular,* $\mathrm{Supp}\langle B \rangle \cap \mathrm{Sing}\, \llcorner B \lrcorner = \emptyset$.

*Proof.* The last assertion follows from the first one, since $(X, S)$ is log-canonical for $S := \llcorner B \lrcorner$ and $(X, S)$ is not 1-log-terminal at any point of $\mathrm{Sing}\, S$.

For the bimeromorphic morphism $\mu\colon M \to X$ in Definition 2.1, we may assume that the union of $\mu^{-1}(\mathrm{Supp}\, B \cup \mathrm{Supp}\, C)$ and the $\mu$-exceptional locus is normal crossing. For the $\mathbb{Q}$-divisors $B_\mu$ and $T_\mu$ above, let $B'_\mu$ and $T'_\mu$ be the positive and negative parts, respectively, of the prime decomposition of $B_\mu + \mu^* C - T_\mu$. Then

$$K_M + B'_\mu = \mu^*(K_X + B + C) + T'_\mu.$$

The first assertion holds if the following condition (*) is satisfied:

(*) There is a prime component $\Gamma$ of $\llcorner B_\mu \lrcorner$ such that $\mu(\Gamma) = \{x\}$.

In fact, if (*) holds, then $\ulcorner B'_\mu \urcorner$ is not reduced by

$$\operatorname{mult}_\Gamma B'_\mu = \operatorname{mult}_\Gamma B_\mu + \operatorname{mult}_\Gamma \mu^*C = 1 + \operatorname{mult}_\Gamma \mu^*C > 1,$$

and $(X, B+C)$ is not log-canonical by the independence of $\mu$ for the log-canonicity (cf. Remark 2.2).

For the rest, we shall check (*). If $x \in S_{\text{reg}}$ for $S = \llcorner B \lrcorner$, then (*) holds, since $(X, B)$ is not 1-log-terminal at $x$. Thus, we may assume that $x \in \operatorname{Sing} S$. Then $(X, S)$ is toroidal at $x$ by Fact 2.5. Let $U$ be an open neighborhood of $x$ in $X$ such that $\operatorname{Sing} U \subset \{x\}$ and $U \cap \operatorname{Sing} S = \{x\}$. When $x \in \operatorname{Sing} X$, let $\eta\colon Y \to U$ be the minimal resolution of singularity. When $x \in X_{\text{reg}}$, let $\eta\colon Y \to U$ be the blowing up at $x$. Then

$$K_Y + S_Y = \eta^*(K_U + S|_U) \tag{II-1}$$

for the reduced divisor $S_Y = \eta^{-1}(S|_U)$. In fact, if $x \in \operatorname{Sing} X$, then $\eta$ is described by Hirzebruch–Jung's method or a toric method (cf. [40, Exam. 3.2]), which induces (II-1). If $x \in X_{\text{reg}}$, then we have (II-1) by a direct calculation. Since $\mu^{-1}(U) \to U$ factors through $\eta$, an $\eta$-exceptional component of $S_Y$ gives a prime component $\Gamma$ of $\llcorner B_\mu \lrcorner$ lying over $x$. Thus (*) is satisfied also in the case where $x \in \operatorname{Sing} S$, and we are done. ☐

**Corollary 2.7.** *For a normal surface $X$ and an effective $\mathbb{Q}$-divisor $B$, the pair $(X, B)$ is* weak log-terminal *in the sense of* [31, Def. 0-2-10] *if and only if*

(a) *$(X, B)$ is 1-log-terminal at any point of $X \setminus \operatorname{Sing} \llcorner B \lrcorner$,*
(b) *$\operatorname{Sing} \llcorner B \lrcorner \subset X_{\text{reg}} \setminus \operatorname{Supp}\langle B \rangle$, and*
(c) *$\llcorner B \lrcorner|_{X_{\text{reg}}}$ is a normal crossing divisor.*

*Proof.* Assume that $(X, B)$ is weak log-terminal. Then we have (a) by (ii) and (iii) of [31, Def. 0-2-10]. By Fact 2.5 and Lemma 2.6, we see that $\operatorname{Sing} \llcorner B \lrcorner \cap \operatorname{Supp}\langle B \rangle = \emptyset$, and $(X, B)$ is toroidal at any point of $\operatorname{Sing} \llcorner B \lrcorner$. Moreover, $X$ is non-singular along $\operatorname{Sing} \llcorner B \lrcorner$ by (iii) of [31, Def. 0-2-10]. This shows (b) and (c).

Conversely, assume (a), (b), and (c). Then we can find a bimeromorphic morphism $\mu\colon M \to X$ from a non-singular surface $M$ such that

- the union of the $\mu$-exceptional locus and $\mu^{-1} \operatorname{Supp} B$ is a normal crossing divisor, and
- $\mu$ is an isomorphism over an open neighborhood of $\operatorname{Sing} \llcorner B \lrcorner$.

For the effective $\mathbb{Q}$-divisors $B_\mu$ and $T_\mu$ in Definition 2.1, $\ulcorner B_\mu \urcorner$ is reduced as $(X, B)$ is log-canonical (cf. Remark 2.2), and moreover, $\llcorner B_\mu \lrcorner$ is the proper transform of $\llcorner B \lrcorner$ in $M$ by (a). Thus, $(X, B)$ is weak log-terminal. ☐

*Remark* 2.8. By the proof above, we see that $(X, B)$ is "log terminal" in the sense of [52] and [34] if and only if (a), (b), and the following stronger version (c′) of (c) are satisfied:

(c′) $\llcorner B \lrcorner|_{X_{\text{reg}}}$ is a *simple* normal crossing divisor.

Note that the condition (c′) is not analytically local. When $B$ is reduced, the "log terminal" condition for $(X, B)$ is equivalent to the condition that $(X, B)$ has only "Kawamata singularities" in the sense of Tsunoda–Miyanishi (cf. [55, 1.1]).

2.2. **Relations with ramification formulas.** We shall show that singularities on $(X, B)$ such as log-canonical, log-terminal, and 1-log-terminal are preserved by a non-degenerate morphism under certain conditions. The results here give refinements of a similar result [40, Lem. 3.19] in the case of schemes.

**Lemma 2.9.** *Let $X$ be a normal surface with an effective $\mathbb{Q}$-divisor $B$ and let $f\colon Y \to X$ be a non-degenerate morphism from another normal surface $Y$. Then there exist bimeromorphic morphisms $\mu\colon M \to X$ and $\nu\colon N \to Y$ from non-singular surfaces $M$ and $N$ with a commutative diagram*

$$\begin{array}{ccc} N & \xrightarrow{\nu} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\mu} & X \end{array} \tag{II-2}$$

*for a non-degenerate morphism $g$ which satisfy the following conditions*:

1. *For the $\mu$-exceptional locus $E_\mu$, the union $E = E_\mu \cup \mu^{-1} \operatorname{Supp} B$ is a normal crossing divisor.*
2. *For the $\nu$-exceptional locus $E_\nu$ and for*
$$\widetilde{\Sigma}_f := f^{-1}(\operatorname{Sing} X \cup \operatorname{Supp} B) \cup \operatorname{Supp} R_f,$$
*the union $F = E_\nu \cup \nu^{-1}\widetilde{\Sigma}_f$ is a normal crossing divisor.*
3. *The equality $F = g^{-1}E \cup \operatorname{Supp} R_g$ holds for the divisors $E$ and $F$ above.*

*Here, $R_f$ and $R_g$ denote the ramification divisors of $f$ and $g$, respectively. Moreover, there is an effective divisor $\overline{R}_g$ in $N$ such that $K_N + F = g^*(K_M + E) + \overline{R}_g$ and that any common prime component of $\overline{R}_g$ and $g^*E$ is $g$-exceptional.*

*Proof.* By Hironaka's resolution of singularity and indeterminacy of meromorphic maps, we have such a commutative diagram satisfying the conditions except (3). The last assertion on $\overline{R}_g$ follows from $g^{-1}E \subset F$ and from Proposition 1.40(1). Thus, it is enough to prove (3): We set $F' = g^{-1}E \cup \operatorname{Supp} R_g$. Then $N \setminus F'$ is the maximum among open subsets of $N \setminus g^{-1}(\mu^{-1} \operatorname{Supp} B)$ étale over $X_{\mathrm{reg}} \setminus \operatorname{Supp} B$. Since $f$ induces an étale morphism $Y \setminus \widetilde{\Sigma}_f \to X_{\mathrm{reg}} \setminus \operatorname{Supp} B$, the complement $N \setminus F$ is étale over $X_{\mathrm{reg}} \setminus \operatorname{Supp} B$. Hence, $F \supset F'$. If a prime divisor $\Gamma$ on $N$ is not contained in $F'$, then $f \circ \nu\colon N \to X$ is étale along a non-empty open subset of $\Gamma$, and hence, $\Gamma$ is not $\nu$-exceptional and $\nu(\Gamma) \not\subset \widetilde{\Sigma}_f$. This shows $F \subset F'$, and (3) has been proved. □

**Lemma 2.10.** *Let $X$ be a normal surface with an effective $\mathbb{Q}$-divisor $B$ and let $f\colon Y \to X$ be a non-degenerate morphism from another normal surface $Y$. Let $B_f$ and $T_f$ be the positive and negative parts, respectively, of the prime decomposition of $f^*B - R_f$ for the ramification divisor $R_f$, i.e., $K_Y + B_f = f^*(K_X + B) + T_f$.*

(1) *If $(X, B)$ is log-canonical (resp. log-terminal), then $\ulcorner B_f \urcorner$ is reduced (resp. $\llcorner B_f \lrcorner = 0$). If $T_f = 0$ in addition, then $(Y, B_f)$ is log-canonical (resp. log-terminal).*

(2) *If $(X, B)$ is 1-log-terminal, then $\llcorner B_f \lrcorner$ has no $f$-exceptional prime component. If $T_f = 0$ in addition, then $(Y, B_f)$ is 1-log-terminal.*

*Proof.* We use the commutative diagram (II-2) in Lemma 2.9. When we consider (2), we may assume that

($*$) the proper transform of $\operatorname{Supp} \llcorner B \lrcorner = (\llcorner B \lrcorner)_{\mathrm{red}}$ in $M$ and the proper transform of $\operatorname{Supp} \llcorner B_f \lrcorner = (\llcorner B_f \lrcorner)_{\mathrm{red}}$ in $N$ are both non-singular,

by taking further blowings up. We consider $\mathbb{Q}$-divisors $B_\mu$ and $T_\mu$ in Definition 2.1 defined for $\mu$, where $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$. First, we shall prove the first half of (1): Assume that $(X, B)$ is log-canonical. Then $\ulcorner B_\mu \urcorner$ is reduced, and

$$K_N + (g^* B_\mu)_{\mathrm{red}} = g^*(K_M + B_\mu) + R'$$

for an effective $\mathbb{Q}$-divisor $R'$ by Lemma 1.41(1). By applying $\nu_*$, we have

$$K_Y + \nu_*((g^* B_\mu)_{\mathrm{red}}) = f^*(K_X + B) + \nu_*(g^* T_\mu + R').$$

Then $B_f \leq \nu_*((g^* B_\mu)_{\mathrm{red}})$, and $\ulcorner B_f \urcorner$ is reduced. Assume next that $(X, B)$ is log-terminal, i.e., $\llcorner B_\mu \lrcorner = 0$. Then

$$K_N = g^*(K_M + B_\mu) + R''$$

for a $\mathbb{Q}$-divisor $R''$ such that $\ulcorner R'' \urcorner$ is effective, by Lemma 1.41(2). Hence,

$$K_Y = f^*(K_X + B) + \nu_*(g^* T_\mu + R'')$$

and we have $\llcorner B_f \lrcorner = 0$ by $T_f - B_f = \nu_*(g^* T_\mu + R'')$. This shows the first half of (1).

Next, we shall prove the first half of (2): Assume that $(X, B)$ is 1-log-terminal and $\llcorner B \lrcorner \neq 0$. We set $C := \llcorner B_\mu \lrcorner$. Then $C$ is just the proper transform of $\llcorner B \lrcorner$ in $M$, and it is reduced and non-singular by ($*$). By Lemma 1.41(3),

$$K_N + g^{[*]} C = g^*(K_M + B_\mu) + R'''$$

for a $\mathbb{Q}$-divisor $R'''$ such that $\ulcorner R''' \urcorner$ is effective. Applying $\nu_*$, we have

$$K_Y + \nu_*(g^{[*]} C) = f^*(K_X + B) + \nu_*(g^* T_\mu + R''') \quad \text{and}$$
$$T_f - B_f = \nu_*(g^* T_\mu + R''') - \nu_*(g^{[*]} C).$$

Hence, $\llcorner B_f \lrcorner \leq \nu_*(g^{[*]} C)$, and every prime component of $\nu_*(g^{[*]} C)$ is not exceptional for $f$. This proves the first half of (2).

Finally, we shall prove the remaining parts of (1) and (2): Assume that $T_f = 0$. Let $B_\nu$ and $T_\nu$, respectively, be the positive and negative parts of the prime decomposition of $\nu^* B_f - R_\nu$. Then

$$K_N + B_\nu = \mu^*(K_Y + B_f) + T_\nu = \mu^*(f^*(K_X + B)) + T_\nu.$$

Moreover, we have $B_f = \nu_* B_\nu$ and $T_f = \nu_* T_\nu = 0$ by applying $\nu_*$.

In the situation of (1), $\ulcorner B_\nu \urcorner$ is reduced (resp. $\llcorner B_\nu \lrcorner = 0$) by the first half of (1) applied to $f \circ \nu \colon N \to X$ and $(X, B)$. Hence, $(Y, B_f)$ is log-canonical (resp. log-terminal).

In the situation of (2), $\llcorner B_\nu \lrcorner$ has no $f \circ \nu$-exceptional prime component by the first half of (2) applied to $f \circ \nu$ and $(X, B)$. Hence, $\llcorner B_\nu \lrcorner$ equals the proper transform of $\llcorner B_f \lrcorner$ in $N$, and it is reduced and non-singular by (1) and $(*)$. Therefore $(Y, B_f)$ is 1-log-terminal by (1). Thus, we are done. $\square$

*Remark.* Some reader may think that Lemma 2.10 can be proved by the same argument as in the proof of [33, Prop. 5.20]. But there is a difficulty in constructing the "fiber product diagram" in the proof, since our $f$ is only a non-degenerate morphism, which is not necessarily proper (cf. Remark of [40, Cor. 3.20]).

**Lemma 2.11.** *Let $X$ be a normal surface with an effective $\mathbb{Q}$-divisor $B$ and let $f \colon Y \to X$ be a surjective and* discretely proper *morphism (cf. Definition* 1.6*) from another normal surface $Y$ with effective $\mathbb{Q}$-divisors $B_Y$ and $\Delta$ such that $R_f = f^*B + \Delta - B_Y$, i.e., $K_Y + B_Y = f^*(K_X + B) + \Delta$. For the diagram* (II-2) *of Lemma* 2.9*, let $B_\nu$, $T_\nu$, $C_\nu$, and $S_\nu$ be effective divisors on $N$ such that*

- *$B_\nu$ and $T_\nu$ are the positive and negative parts, respectively, of the prime decomposition of $\nu^*B_Y - R_\nu$, and*
- *$C_\nu$ and $S_\nu$ are the positive and negative parts, respectively, of the prime decomposition of $B_\nu - \nu^*\Delta$.*

*In particular, one has*

$$K_N + B_\nu = \nu^*(K_Y + B_Y) + T_\nu \quad \text{and} \quad K_N + C_\nu = \nu^*(f^*(K_X + B)) + S_\nu + T_\nu.$$

1. *If $\ulcorner C_\nu \urcorner$ is reduced (resp. $\llcorner C_\nu \lrcorner = 0$), then $(X, B)$ is log-canonical (resp. log-terminal).*
2. *If $\ulcorner C_\nu \urcorner$ is reduced and if $\llcorner C_\nu \lrcorner$ is a non-singular divisor having no $f \circ \nu$-exceptional prime component, then $(X, B)$ is* 1*-log-terminal.*
3. *Suppose that* $\operatorname{Supp} B_Y \subset \widetilde{\Sigma}_f$ *(cf. Lemma* 2.9(2)*). If $\ulcorner B \urcorner$ and $\ulcorner B_Y \urcorner$ are reduced, then there is an effective $\mathbb{Q}$-divisor $\overline{\Delta}$ such that*

$$K_N + \nu^{[*]}B_Y + E_\nu = g^*(K_M + \mu^{[*]}B + E_\mu) + \overline{\Delta} \tag{II-3}$$

*and that any $\nu$-exceptional prime component of $\overline{\Delta}$ is $g$-exceptional.*

*Proof.* Note that $g$ is surjective and discretely proper by Corollary 1.11. Divisors $C_\nu$ and $S_\nu + T_\nu$ have no common prime component, since $C_\nu \leq B_\nu$. Thus, $C_\nu$ and $S_\nu + T_\nu$ are the positive and negative parts, respectively, of the prime decomposition of $(f \circ \nu)^*B - R_{f \circ \nu}$. In particular, $\nu_*C_\nu = B_f$ and $\nu_*(S_\nu + T_\nu) = \nu_*S_\nu = T_f$ for divisors $B_f$ and $T_f$ in Lemma 2.10. Note that $\operatorname{Supp} C_\nu \subset F$ by

$$\operatorname{Supp} C_\nu \subset \nu^{-1}(\operatorname{Supp} B_f) \cup E_\nu \quad \text{and} \quad \operatorname{Supp} B_f \subset \widetilde{\Sigma}_f.$$

Equalities $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$ and $K_N + F = g^*(K_M + E) + \overline{R}_g$ (cf. Lemma 2.9) induce

$$K_N + F = g^*(\mu^*(K_X + B)) + g^*(E + T_\mu - B_\mu) + \overline{R}_g.$$

Comparing with $K_N + C_\nu = \nu^*(f^*(K_X + B)) + S_\nu + T_\nu$, we have

$$g^*(E + T_\mu - B_\mu) + \overline{R}_g = F - C_\nu + S_\nu + T_\nu. \tag{II-4}$$

We shall prove (1) and (2). Assume that $\ulcorner C_\nu \urcorner$ is reduced. Then $F \geq C_\nu$, and we see that $E + T_\mu - B_\mu$ is effective by (II-4), by $\operatorname{Supp} B_\mu \subset E$, and by a property of $\overline{R}_g$ in the last assertion of Lemma 2.9. Moreover, $E \geq B_\mu$, since $B_\mu$ and $T_\mu$ have no common prime component. As a consequence, $\ulcorner B_\mu \urcorner$ is reduced, and hence, $(X, B)$ is log-canonical. Thus, we have proved (1) in the log-canonical case.

For the proof of (1) in the log-terminal case and for that of (2), we consider a prime component $\Gamma$ of $B_\mu$ and set $d := \operatorname{mult}_\Theta g^*\Gamma$. We can take a non-$g$-exceptional prime component $\Theta$ of $f^*\Gamma$, since $g$ is surjective. Then $\Theta \not\subset \operatorname{Supp} \overline{R}_g$ by $\operatorname{Supp} B_\mu \subset E$ and by the last assertion of Lemma 2.9. Moreover,

$$\begin{aligned} d \operatorname{mult}_\Gamma(E - B_\mu) &= \operatorname{mult}_\Theta g^*(E - B_\mu) = \operatorname{mult}_\Theta g^*(E - B_\mu + T_\mu) \\ &= \operatorname{mult}_\Theta(F - C_\nu) + \operatorname{mult}_\Theta(S_\nu + T_\nu) \end{aligned} \tag{II-5}$$

by (II-4).

Assume that $\llcorner C_\nu \lrcorner = 0$. Then $F \geq C_\nu$ and $\operatorname{Supp} F = \operatorname{Supp}(F - C_\nu)$. Thus, $\operatorname{mult}_\Gamma(E - B_\mu) > 0$ for any prime component $\Gamma$ of $B_\mu$ by (II-5). In other words, $E \geq B_\mu$ and $\operatorname{Supp} E = \operatorname{Supp}(E - B_\mu)$. Hence, $\llcorner B_\mu \lrcorner = 0$ and $(X, B)$ is log-terminal. Thus, (1) has been proved.

Next, assume the condition for $C_\nu$ in (2). Then $F \geq C_\nu$ and $E \geq B_\mu$ by the proof above for (1) in the log-canonical case. Assume that $\Gamma$ is a prime component of $\llcorner B_\mu \lrcorner$. Then $\Gamma \not\subset \operatorname{Supp}(E - B_\mu)$, and we have $\Theta \not\subset \operatorname{Supp}(F - C_\nu)$ by (II-5). Thus, $\Theta$ is a prime component of $\llcorner C_\nu \lrcorner$, which is not exceptional for $f \circ \nu \colon N \to X$. Hence, $\Gamma$ is not $\mu$-exceptional. This implies that $(X, B)$ is 1-log-terminal, and we have proved (2).

Finally, we shall prove (3). Note that $E = \operatorname{Supp}(\mu^{[*]}B + E_\mu)$. By the assumption on $B_Y$, we have

$$\operatorname{Supp}(\nu^{[*]}B_Y + E_\nu) \subset \nu^{-1}\widetilde{\Sigma}_f \cup E_\nu = F.$$

Since $\ulcorner B \urcorner$ and $\ulcorner B_Y \urcorner$ are reduced, there exist effective $\mathbb{Q}$-divisors $D_M$ and $D_N$ on $M$ and $N$, respectively, such that

$$E = \mu^{[*]}B + E_\mu + D_M \quad \text{and} \quad F = \nu^{[*]}B_Y + E_\nu + D_N.$$

Then the equality (II-3) holds for

$$\overline{\Delta} := g^*D_M - D_N + \overline{R}_g. \tag{II-6}$$

Here, any prime component of $D_M$ (resp. $D_N$) is not exceptional for $\mu$ (resp. $\nu$), and $\operatorname{mult}_\Theta \overline{\Delta} \geq 0$ for any $\nu$-exceptional prime divisor $\Theta$. On the other hand, we have $\nu_*\overline{\Delta} = \Delta$ by applying $\nu_*$ to (II-3). Thus, $\overline{\Delta}$ is effective. It remains to prove that any $\nu$-exceptional prime component $\Theta$ of $\overline{\Delta}$ is $g$-exceptional. Assume that $\Theta$ is not $g$-exceptional. Then $\Theta \subset g^{-1}\Gamma$ for a prime divisor $\Gamma$ on $M$, and $g|_\Theta \colon \Theta \to \Gamma$ is non-degenerate. Here, $\Gamma$ is $\mu$-exceptional as $\Theta$ is $\nu$-exceptional. Thus, $\Gamma \subset E_\mu$ and $\Gamma \not\subset \operatorname{Supp} D_M$. Hence, $\Theta \subset \operatorname{Supp} \overline{R}_g$ by (II-6). This contradicts the last assertion of Lemma 2.9, since $\Theta$ is a common prime component of $g^*E$ and $\overline{R}_g$. Therefore, $\Theta$ is $g$-exceptional. Thus, we are done. $\square$

**Proposition 2.12.** *Let $X$ be a normal surface with an effective $\mathbb{Q}$-divisor $B$ and let $f\colon Y \to X$ be non-degenerate morphism from another normal surface $Y$ with effective $\mathbb{Q}$-divisors $B_Y$ and $\Delta$ such that $R_f = f^*B + \Delta = B_Y$, i.e., $K_Y + B_Y = f^*(K_X + B) + \Delta$. Then the following hold for any $x \in f(Y)$:*

(1) *If $(Y, B_Y)$ is log-canonical (resp. log-terminal) along a non-empty compact connected component of $f^{-1}(x)$, then $(X, B)$ is log-canonical (resp. log-terminal) at $x$.*

(2) *If $(Y, B_Y)$ is 1-log-terminal along a non-empty compact connected component $C$ of $f^{-1}(x)$ such that $C \cap \operatorname{Supp} \llcorner B_Y \lrcorner$ is finite, then $(X, B)$ is 1-log-terminal at $x$.*

*Proof.* For a compact connected component $C$ of $f^{-1}(x)$, there exist an open neighborhood $U$ of $x$ and an open neighborhood $V$ of $C$ such that $V \subset f^{-1}U$, $V \cap f^{-1}(x) = C$, and $f|_V \colon V \to U$ is proper and surjective, by Lemma 1.7. Hence, by replacing $X$ and $Y$ with $U$ and $V$, respectively, we may assume that $f$ is proper and surjective, $(Y, B_Y)$ is log-canonical (resp. log-terminal) in case (1), and $(Y, B_Y)$ is 1-log-terminal in case (2). Moreover, in case (2), we may assume that

($\natural$) $f|_{\llcorner B_Y \lrcorner} \colon \llcorner B_Y \lrcorner \to X$ is a finite morphism

by Lemma 1.7. We consider the commutative diagram (II-2) in Lemma 2.9 and divisors $B_\nu$ and $C_\nu$ in Lemma 2.11.

We shall show (1). In this case, $\ulcorner B_\nu \urcorner$ is reduced (resp. $\llcorner B_\nu \lrcorner = 0$) as $(Y, B_Y)$ is log-canonical (resp. log-terminal). Hence, $\ulcorner C_\nu \urcorner$ is reduced (resp. $\llcorner C_\nu \lrcorner = 0$), by $C_\nu \leq B_\nu$. Thus, $(X, B)$ is log-canonical (resp. log-terminal) by Lemma 2.11(1).

Finally, we shall show (2). In this case, $\llcorner B_\nu \lrcorner$ is a non-singular divisor having no $\nu$-exceptional component as $(Y, B_Y)$ is 1-log-terminal. Since $\nu_* B_\nu = B_Y$, $\llcorner B_\nu \lrcorner$ has no $f \circ \nu$-exceptional component by ($\natural$). Hence, $\llcorner C_\nu \lrcorner$ is also a non-singular divisor having no $f \circ \nu$-exceptional component by $C_\nu \leq B_\nu$. Thus, $(X, B)$ is 1-log-terminal by Lemma 2.11(2), and we are done. □

2.3. **Relative abundance theorem.** The abundance theorem is one of the main results of the theory of open algebraic surfaces (or logarithmic algebraic surfaces), which is proved in several versions in [28], [47], [55], and [11]. Theorem 2.19 below is a relative version of the abundance theorem, and Lemma 2.18 below is its special case. We shall prove them for the sake of completeness not using the classification of log-canonical singularities but using Fujita's argument in [11] and Kawamata's argument in the proof of [29, Lem. 9.3] with some modifications.

Let us consider a proper surjective morphism $\pi\colon X \to Y$ of normal complex analytic varieties such that $\dim X = 2$, and assume either that $\dim Y > 0$ or that $X$ is a normal Moishezon surface with $\dim Y = 0$. Before Lemma 2.18, we fix the morphism $\pi$. We shall explain relative versions of the Kawamata–Viehweg vanishing theorem (cf. Proposition 2.15) and Zariski-decompositions (cf. Lemma-Definition 2.16) for the morphism $\pi$. The relative abundance theorem (cf. Theorem 2.19) concerns the case where $X$ is non-singular, but it applies to log-canonical pairs by taking resolutions. As an application of the relative abundance theorem, we shall define the log-canonical modification for pairs $(X, B)$ of a normal surface and an effective $\mathbb{Q}$-divisor $B$ on $X$ (cf. Lemma-Definition 2.22), and show a compatibility for certain morphisms with only discrete fibers (cf. Proposition 2.23).

**Lemma 2.13.** *If* $\dim Y > 0$*, then* $\pi$ *is a projective morphism locally over* $Y$*, i.e., for any point* $y \in Y$*, there exist an open neighborhood* $\mathcal{Y} \subset Y$ *and an invertible sheaf on* $\pi^{-1}(\mathcal{Y})$ *which are relatively ample over* $\mathcal{Y}$ (*cf.* [36, Prop. 1.4]).

*Proof.* Since finite morphisms are projective locally over the base varieties, we may assume that every fiber of $\pi$ is connected by considering Stein factorization. If $\dim Y = 2$, then $\pi$ is a bimeromorphic morphism and is projective locally over $Y$ by an argument in the last paragraph of the proof of Lemma 1.28. Thus, we may assume that $\dim Y = 1$. Then $Y$ is a non-singular curve and every fiber is 1-dimensional. We fix a point $y \in Y$ and let $\Gamma$ be an irreducible component of $\pi^{-1}(y)$. For a point $x \in \Gamma_{\mathrm{reg}} \cap X_{\mathrm{reg}}$, there is an open neighborhood $\mathcal{U}$ of $x$ with a coordinate system $(\mathtt{z}_1, \mathtt{z}_2)$ such that $\Gamma|_{\mathcal{U}} = \mathrm{div}(\mathtt{z}_2)$ and $\pi|_{\mathcal{U}} \colon \mathcal{U} \to Y$ is defined by the function $u(\mathtt{z}_1, \mathtt{z}_2)\mathtt{z}_2^m$ on $\mathcal{U}$ for a positive integer $m$ and a unit function $u(\mathtt{z}_1, \mathtt{z}_2)$. Then $\pi^{-1}(y) \cap \Theta = \{x\}$ for the non-singular divisor $\Theta = \mathrm{div}(\mathtt{z}_1)$ on $\mathcal{U}$. Hence, $\pi|_{\Theta} \colon \Theta \to Y$ is a finite morphism over an open neighborhood of $y$ by Corollary 1.8. By considering divisors $\Theta$ for all the irreducible components $\Gamma$ of $\pi^{-1}(y)$, we can find an open neighborhood $\mathcal{Y}$ of $y$ and a non-singular divisor $D$ on $\pi^{-1}(\mathcal{Y})$ such that $D\Gamma > 0$ for any irreducible component $\Gamma$ of $\pi^{-1}(y)$. Then, by [36, Prop. 1.4], $\pi^{-1}(\mathcal{Y}) \to \mathcal{Y}$ is a projective morphism over an open neighborhood of $y$ in which $D$ is relatively ample. $\square$

**Convention 2.14.** For the morphism $\pi \colon X \to Y$ with $\dim Y > 0$, a $\mathbb{Q}$-divisor $D$ on $X$ is said to be:

(1) $\pi$-*nef* (resp. $\pi$-*numerically trivial*), if $DC \geq 0$ (resp. $DC = 0$) for any prime divisor $C \subset X$ such that $\dim \pi(C) = 0$ (cf. [38, II, Def. 5.14], [40, Def. 2.14(i)]);
(2) $\pi$-*semi-ample*, if there is a positive integer $m$ locally over $Y$ such that $mD$ is Cartier and the canonical homomorphism $\pi^*\pi_*\mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$ is surjective (cf. [38, II, Def. 1.9(4)]);
(3) $\pi$-*pseudo-effective*, if $D|_C$ is pseudo-effective for any irreducible component $C$ of a sufficiently general fiber of $\pi$ (cf. [38, II, Cor. 5.17]);
(4) $\pi$-*big*, if $D|_C$ is big for any irreducible component $C$ of a general fiber of $\pi$ (cf. [38, II, Cor. 5.17]).

Note that if $\dim Y = 2$, then any $D$ is $\pi$-big. Similarly, if $\dim Y = 1$, then $D$ is $\pi$-pseudo-effective (resp. $\pi$-big) if and only if $DC \geq 0$ (resp. $DC > 0$) for any irreducible component $C$ of a general fiber of $\pi$. For the morphism $\pi$ with $\dim Y = 0$, i.e., for a normal Moishezon surface $X$, we use the same notions of nef, numerically trivial, semi-ample, pseudo-effective, and big, respectively, as in [40, Def. 2.11] for $\mathbb{Q}$-divisors on $X$. Sometimes we add the prefix "$\pi$-" even when $\dim Y = 0$.

The Kawamata–Viehweg vanishing theorem for non-singular projective surfaces is generalized to the relative situation as follows (cf. [48, Thms. (2.2) and (5.1)]):

**Proposition 2.15.** *For any $\pi$-nef and $\pi$-big $\mathbb{Q}$-divisor $D$ on $X$ and for any $i > 0$, one has $R^i\pi_*\mathcal{O}_X(K_X + \ulcorner D \urcorner) = 0$.*

*Proof.* Our proof is slightly different from Sakai's one in [48, Thm. 5.1]. Since the assertion is local on $Y$, we may assume the existence of a bimeromorphic morphism $\mu\colon M \to X$ from a non-singular surface $M$ such that the union of the $\mu$-exceptional locus and $\mu^{-1}(\operatorname{Supp} D)$ is a normal crossing divisor and that $\pi \circ \mu\colon M \to Y$ is a projective morphism. In fact, if $\dim Y = 0$, then $M$ is projective as $X$ is Moishezon, and if $\dim Y > 0$, then $\pi$ is locally projective by Lemma 2.13. Then

$$R^i(\pi \circ \mu)_*\mathcal{O}_M(K_M + \ulcorner \mu^*D \urcorner) = 0 \quad \text{and} \quad R^i\mu_*\mathcal{O}_M(K_M + \ulcorner \mu^*D \urcorner) = 0$$

for any $i > 0$ as a relative version of Kawamata–Viehweg's vanishing theorem on $M$ (cf. [36, Thm. 3.7]). Let $\mathcal{F}$ be the direct image sheaf $\mu_*\mathcal{O}_M(K_M + \ulcorner \mu^*D \urcorner)$. Then $R^i\pi_*\mathcal{F} = 0$ for any $i > 0$ by a standard argument on Leray's spectral sequence. Since $\mathcal{F}$ is a subsheaf of the double-dual $\mathcal{F}^{\vee\vee} = \mathcal{O}_X(K_X + \ulcorner D \urcorner)$ and since $\mathcal{F}^{\vee\vee}/\mathcal{F}$ is supported on discrete points, we have $R^i\pi_*\mathcal{O}_X(K_X + \ulcorner D \urcorner) \simeq R^i\pi_*\mathcal{F} = 0$ for any $i > 0$. $\square$

We have a relative version of the notion of *Zariski-decomposition* (cf. [59], [10], [48, §7], [50, App.], [38]) as follows:

**Lemma-Definition 2.16.** *Let $D$ be a $\pi$-pseudo-effective $\mathbb{Q}$-divisor on $X$. Then there exists a unique effective $\mathbb{Q}$-divisor $N$ satisfying the following conditions*:

- *Every prime component of $N$ is contained in a fiber of $\pi$.*
- *The difference $P := D - N$ is $\pi$-nef and satisfies $PN = 0$.*
- *For a point $y \in Y$, let $N_y$ be the partial sum of $N$ over the prime components contained in a fiber $\pi^{-1}(y)$. Then either $N_y = 0$ or the intersection matrix $(N_iN_j)_{i,j}$ of prime components $N_i$ of $N_y$ is negative definite.*

*The decomposition $D = P + N$ is called the* relative Zariski-decomposition *of $D$* with respect to $\pi$, *where $P$ and $N$ are called the* positive part *and the* negative part *of the decomposition, respectively.*

*Proof.* First assume that $\dim Y = 0$. For the minimal resolution $\mu\colon M \to X$ of singularities, we have the unique Zariski-decomposition $\mu^*D = P^\sim + N^\sim$ on the non-singular projective surface $M$ by [10], since $\mu^*D$ is pseudo-effective, where $P^\sim$ (resp. $N^\sim$) is the positive (resp. negative) part. Here, $P^\sim$ is $\mu$-numerically trivial. In fact, for a $\mu$-exceptional prime divisor $\Gamma$, if $\Gamma \subset \operatorname{Supp} N^\sim$, then $P^\sim\Gamma = 0$ by $P^\sim N^\sim = 0$, and if $\Gamma \not\subset \operatorname{Supp} N^\sim$, then $P^\sim\Gamma = 0$ by $(\mu^*D)\Gamma = 0$, $P^\sim\Gamma \geq 0$, and $N^\sim\Gamma \geq 0$. Thus, $P^\sim = \mu^*P$ and $N^\sim = \mu^*N$ for $P := \mu_*P^\sim$ and $N := \mu_*N^\sim$, and $D = P + N$ is the Zariski-decomposition of $D$.

Second, assume that $\dim Y > 0$. Our proof in this case is based on Sakai's argument in [48, §7] and [50, App.]. By the uniqueness of the decomposition, we can localize $Y$. Thus, we may assume the finiteness of the number $s(X/Y)$ of prime divisors $\Gamma$ on $X$ such that $\Gamma^2 < 0$ and $\dim \pi(\Gamma) = 0$. Note that $s(X/Y)$ is the number of $\pi$-exceptional prime divisors when $\dim Y = 2$ and that $s(X/Y)$ is the sum of numbers of irreducible components of reducible fibers of $\pi$ when $\dim Y = 1$.

We shall prove the existence and the uniqueness of relative Zariski-decomposition by induction on $s(X/Y)$. We may assume that $D$ is not $\pi$-nef; otherwise, $N = 0$ satisfies the condition and it is unique. Then $D\Gamma < 0$ for an irreducible component $\Gamma$ of a fiber of $\pi$. In particular, $s(X/Y) > 0$. Moreover, $\Gamma^2 < 0$. In fact, if $\dim Y = 2$, then $\Gamma$ is $\pi$-exceptional, and it implies: $\Gamma^2 < 0$. If $\dim Y = 1$ and $\Gamma^2 \geq 0$, then $\Gamma^2 = 0$ and $\Gamma$ is a connected component of a fiber of $\pi$; this implies $D\Gamma \geq 0$, a contradiction. Let $\nu\colon X \to X'$ be the contraction morphism of $\Gamma$, i.e., a bimeromorphic morphism to a normal surface $X'$ such that $\nu(\Gamma)$ is a point $x'$, $\nu^{-1}(x') = \Gamma$, and $\nu$ is an isomorphism outside $x'$. The existence of $\nu$ follows from a generalization [49, Thm. 1.2] of the Grauert contraction criterion [13, (e), pp. 366–367] (cf. [40, Thm. 2.6]). Let $\pi'\colon X' \to Y$ be the induced morphism such that $\pi' \circ \nu = \pi$. Then $s(X'/Y) = s(X/Y) - 1$. We have $D = \nu^*(\nu_* D) + \alpha\Gamma$ for the positive rational number $\alpha = D\Gamma/\Gamma^2$. By induction, the $\pi'$-pseudo-effective $\mathbb{Q}$-divisor $\nu_* D$ admits a relative Zariski-decomposition over $Y$. For the negative part $N'$ of $\nu_* D$, the $\mathbb{Q}$-divisor $N := \nu^* N' + \alpha\Gamma$ satisfies the condition of the negative part of the relative Zariski-decomposition of $D$ over $Y$. In order to prove the uniqueness, assume that another effective $\mathbb{Q}$-divisor $\widetilde{N}$ satisfies the condition of negative part. Then $D\Gamma < 0$ implies that $\widetilde{N}\Gamma < 0$ and $(D - \widetilde{N})\Gamma = 0$. Thus, $\widetilde{N} = \nu^*(\nu_*\widetilde{N}) + \alpha\Gamma$, and $\nu_*\widetilde{N}$ equals the negative part $N'$ of the relative Zariski-decomposition of $\nu_* D$. Hence, $\widetilde{N} = N$. Therefore, $D$ admits a unique relative Zariski-decomposition. $\square$

The relative Zariski-decomposition also has the following well-known properties as in the absolute case:

**Lemma 2.17.** *In the situation of Lemma-Definition* 2.16*, let $E$ be an effective $\mathbb{Q}$-divisor on $X$ such that $D - E$ is $\pi$-nef. Then $E \geq N$. In particular, for any rational number $t \geq 0$,*

$$\pi_*\mathcal{O}_X(\llcorner tP \lrcorner) \simeq \pi_*\mathcal{O}_X(\llcorner tD \lrcorner).$$

*Proof.* For the first assertion, we may assume that $N \neq 0$. Let $B_1$ and $B_2$ be the positive and negative parts, respectively, of the prime decomposition of $E - N$. Then $\operatorname{Supp} B_2 \subset \operatorname{Supp} N$, and

$$(B_1 - B_2)B_2 = (E - N)B_2 \leq (D - N)B_2 = PB_2 = 0.$$

Hence, $B_2^2 \geq B_1 B_2 \geq 0$, and we have $B_2 = 0$, since the intersection matrix $(N_i N_j)$ is negative-definite for prime components $N_i$ of $N$ contained in a fiber of $\pi$. Thus, $E \geq N$. For the last assertion, let us consider the image $\mathcal{F}$ of the canonical homomorphism

$$\pi^*\pi_*\mathcal{O}_X(\llcorner tD \lrcorner) \to \mathcal{O}_X(\llcorner tD \lrcorner).$$

The double-dual $\mathcal{F}^{\vee\vee}$ is expressed as $\mathcal{O}_X(\llcorner tD \lrcorner - F)$ for an effective divisor $F$. The divisor $\llcorner tD \lrcorner - F$ is $\pi$-nef, since the support of $\mathcal{F}^{\vee\vee}/\mathcal{F}$ is 0-dimensional. Hence, by applying the first assertion to $E = (1/t)(\langle tD \rangle + F)$, where $\langle tD \rangle = tD - \llcorner tD \lrcorner$, we have $\langle tD \rangle + F \geq tN$, since $D - E = (1/t)(\llcorner tD \lrcorner - F)$ is $\pi$-nef. Hence, $\llcorner tP \lrcorner \geq \llcorner tD \lrcorner - F$, and as a consequence, $\pi_*\mathcal{O}_X(\llcorner tD \lrcorner - F) = \pi_*\mathcal{O}_X(\llcorner tP \lrcorner) = \pi_*\mathcal{O}_X(\llcorner tD \lrcorner)$. $\square$

The following is a special case of the relative abundance theorem.

**Lemma 2.18.** *For a normal surface $X$, let $\mu\colon M \to X$ be the minimal resolution of singularities. Let $B$ be an effective $\mathbb{Q}$-divisor on $M$ and $m$ a positive integer such that $\ulcorner B \urcorner$ is reduced and that $mB$ is Cartier. If $K_M + B$ is $\mu$-numerically trivial, then $m(K_X + \mu_* B)$ is Cartier and $m(K_M + B) \sim \mu^*(m(K_X + \mu_* B))$.*

*Proof.* Since the assertion is local on $X$, we may assume that $X$ is Stein and $\operatorname{Sing} X$ consists of one point $x$. Then $\Sigma := \mu^{-1}(x)$ is the $\mu$-exceptional locus. We consider $\Sigma$ as a compact connected reduced divisor on $M$. First, we consider the case where $(X, x)$ is a rational singularity, i.e., $(R^1\mu_*\mathcal{O}_M)_x = 0$. Then the element of the Picard group $\operatorname{Pic}(M) = H^1(M, \mathcal{O}_M^\star)$ corresponding to the invertible sheaf $\mathcal{O}_X(m(K_M + B))$ is sent to zero by the canonical homomorphism

$$\begin{aligned}\operatorname{Pic}(M) \to H^0(X, R^1\mu_*\mathcal{O}_M^\star) &\simeq (R^1\mu_*\mathcal{O}_M^\star)_x \simeq (R^2\mu_*\mathbb{Z}_M)_x \\ &\simeq H^2(\Sigma, \mathbb{Z}) \simeq \bigoplus\nolimits_{\Gamma\subset\Sigma} H^2(\Gamma, \mathbb{Z}) \simeq \bigoplus\nolimits_{\Gamma\subset\Sigma} \mathbb{Z},\end{aligned}$$

since $(K_M + B)\Gamma = 0$ for any prime component $\Gamma$ of $\Sigma$. Thus, $m(K_M + B) \sim \mu^* L$ for a Cartier divisor $L$ on $X$, and we have $L \sim \mu_*(m(K_M + B)) = m(K_X + \mu_* B)$. This proves the assertion for rational singularities $(X, x)$.

Next, we consider the case where $(X, x)$ is not a rational singularity. We set

$$B^\dagger := \sum\nolimits_{\Gamma\subset\Sigma} (\operatorname{mult}_\Gamma B)\Gamma \quad \text{and} \quad D := \llcorner B^\dagger \lrcorner.$$

Then $B - B^\dagger$ is $\mu$-nef, and $-B^\dagger - K_M = (B - B^\dagger) - (K_M + B)$ is also $\mu$-nef. Hence, $R^1\mu_*\mathcal{O}_M(-D) = 0$ by Proposition 2.15, since $\ulcorner -B^\dagger \urcorner = -D$. Thus, $D \neq 0$ and $H^1(D, \mathcal{O}_D) \neq 0$ by isomorphisms

$$0 \neq (R^1\mu_*\mathcal{O}_M)_x \simeq (R^1\mu_*\mathcal{O}_D)_x \simeq H^1(D, \mathcal{O}_D),$$

and $D$ is connected by the surjection $\mathcal{O}_X \simeq \mu_*\mathcal{O}_M \to \mu_*\mathcal{O}_D$, since $\mu_*\mathcal{O}_D$ is the skyscraper sheaf of the residue field $\mathbb{C}(x)$ at $x$. In particular, $(K_M + D)D = \deg \omega_D = -2\chi(D, \mathcal{O}_D) \geq 0$ by Riemann–Roch. On the other hand, $(K_M + D)D \leq (K_M + B^\dagger)D \leq 0$, since $-(K_M + B^\dagger)$ is $\mu$-nef. Hence, $(K_M + D)D = 0$ and $H^1(D, \mathcal{O}_D) \simeq H^0(D, \omega_D)^\vee \simeq \mathbb{C}$, which imply $\mathcal{O}_M(K_M + D)|_D \simeq \omega_D \simeq \mathcal{O}_D$. Moreover, $D \cap \operatorname{Supp}(B - D) = \emptyset$ by $0 = (K_M + B)D - (K_M + D)D = (B - D)D$. If $\Sigma \neq D$, then $\Gamma \cap D \neq \emptyset$ for an prime component $\Gamma$ of $\Sigma - D$, since $\Sigma$ is connected. In this case, $\Gamma \not\subset \operatorname{Supp} B$ by $D \cap \operatorname{Supp}(B - D) = \emptyset$, but $K_M\Gamma \geq 0$, $B\Gamma \geq 0$, and $(K_M + B)\Gamma = 0$ imply that $\Gamma \cap \operatorname{Supp} B = \emptyset$; this is a contradiction. Therefore, $\Sigma = D$. Since $m(K_M + B) - B^\dagger - K_M$ is $\mu$-nef, we have $R^1\mu_*\mathcal{O}_M(m(K_M + B) - \Sigma) = 0$ by Proposition 2.15, and have a surjection

$$\mu_*\mathcal{O}_M(m(K_M + B)) \to \mu_*\mathcal{O}_\Sigma(m(K_M + B)|_\Sigma) \simeq \mu_*\mathcal{O}_\Sigma.$$

Hence, a section of $\mathcal{O}_M(m(K_M + B))$ on an open neighborhood of $\Sigma$ is nowhere vanishing. This means that $m(K_M + B) \sim \mu^* L$ for a Cartier divisor $L$ on $X$, and $L \sim \mu_*(m(K_M + B)) = m(K_X + \mu_* B)$. Thus, we are done. □

**Theorem 2.19** (Relative Abundance Theorem)**.** *Let $M$ be a non-singular surface with an effective $\mathbb{Q}$-divisor $B$ such that $\ulcorner B \urcorner$ is reduced. Let $\pi\colon M \to Y$ be a proper surjective morphism to a normal variety $Y$ such that either $\dim Y > 0$ or $M$ is* *projective with* $\dim Y = 0$*. Assume that* $K_M + B$ *is* $\pi$*-pseudo-effective. Then the positive part* $P$ *of the relative Zariski-decomposition* $K_M + B = P + N$ *with respect to* $\pi$ *is* $\pi$*-semi-ample.*

*Proof.* We may assume that $\dim Y > 0$, since the assertion has been proved by [11, Main Thm. (1.4)] in case $\dim Y = 0$. We may assume that $\pi$ is a fibration by taking Stein factorization. Since the assertion is local on $Y$, we may assume further that $Y$ is Stein and $\pi$ is a smooth morphism over $Y \setminus \{y\}$ for a point $y \in Y$.

First, we consider the case where $K_M + B$ is $\pi$-big. By the arguments in (3.2)–(3.5) of [11], we can reduce to the case where $K_M + B$ is $\pi$-nef and $(K_M + B)C > 0$ for any $(-1)$-curve $C$ on $M$ contained in fibers of $\pi$. In particular, $P = K_M + B$ and $N = 0$. Let $\Lambda$ be the set of irreducible components $\Gamma$ of fibers of $\pi$ such that $(K_M + B)\Gamma = 0$. Then $\Lambda$ is finite and the intersection matrix $(\Gamma\Gamma')_{\Gamma,\Gamma' \in \Lambda}$ is negative definite, since $K_M + B$ is $\pi$-nef and $\pi$-big. Let $\mu \colon M \to X$ be the contraction morphism of $\bigcup_{\Gamma \in \Lambda} \Gamma$ and $\bar{\pi} \colon X \to Y$ be the induced morphism such that $\pi = \bar{\pi} \circ \mu$. Note that $\mu$ is the minimal resolution of singularities. If $mB$ is Cartier for a positive integer $m$, then $m(K_M + B) \sim \mu^* L$ for a Cartier divisor $L$ on $X$, by Lemma 2.18. Here, $LG > 0$ for any prime divisor $G$ contained in fibers of $\bar{\pi}$ by the choice of $\Lambda$. Thus, $L$ is relatively ample over $Y$ (cf. the proof of Lemma 2.13), and hence, $K_M + B = P$ is $\pi$-semi-ample.

Second, we consider the case where $K_M + B$ is not $\pi$-big. Then $\dim Y = 1$ and $(K_M + B)F = 0$ for any smooth fiber $F$ of $\pi$. If $BF > 0$, then $F \simeq \mathbb{P}^1$ and $BF = 2$. If $BF = 0$, then $F$ is an elliptic curve and $\operatorname{Supp} B$ is contained in fibers of $\pi$. In both cases, $\mathcal{O}_F(m(K_M + B)|_F) \simeq \mathcal{O}_F$ for a positive integer $m$ such that $mB$ is Cartier. In particular, $\pi_* \mathcal{O}_M(m(K_X + B)) \neq 0$. Then there is an effective divisor $E$ on $M$ such that $\operatorname{Supp} E \subset \pi^{-1}(y)$ and

$$\mathcal{O}_X(m(K_M + B)) \simeq \mathcal{O}_X(E) \otimes \pi^* \pi_* \mathcal{O}_X(m(K_M + B)).$$

The negative part of the relative Zariski-decomposition of $E$ with respect to $\pi$ equals $mN$. Thus, it is enough to show that the positive part $P_E$ of the relative Zariski-decomposition of $E$ is $\pi$-semi-ample. Now $\operatorname{Supp} P_E \subset \pi^{-1}(y)$. As is well known, the intersection matrix of components of $\pi^{-1}(y)$ is negative semi-definite with signature $(0, r-1)$ for the number $r$ of irreducible components of $\pi^{-1}(y)$. Hence, $P_E = q\pi^*(y)$ for a rational number $q \geq 0$, since $P_E$ is $\pi$-nef and since $P_E F = 0$. Therefore, $P_E$ is $\pi$-semi-ample. Thus, we are done. □

By Lemma 2.17 and Theorem 2.19, we have:

**Corollary 2.20.** *In the situation of Theorem* 2.19*, the graded* $\mathcal{O}_Y$*-algebra*

$$\bigoplus\nolimits_{m \geq 0} \pi_* \mathcal{O}_M(\llcorner m(K_M + B) \lrcorner)$$

*is finitely generated locally on* $Y$*.*

**Corollary 2.21.** *Let* $X$ *be a normal surface with an effective* $\mathbb{Q}$*-divisor* $B$*. If* $(X, B)$ *is log-canonical at a point* $x \in X$ *(in the sense of Definition* 2.1*), then* $K_X + B$ *is* $\mathbb{Q}$*-Cartier at* $x$*.*

*Proof.* By localizing $X$, we may assume that $X$ is Stein, $\operatorname{Sing} X = \{x\}$, and $(X, B)$ is log-canonical. Let $\mu\colon M \to X$, $B_\mu$, and $T_\mu$ be as in Definition 2.1. Then $\ulcorner B_\mu \urcorner$ is reduced, and $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$. Hence, $\mu^*(K_X + B)$ is the positive part of the relative Zariski-decomposition of $K_M + B_\mu$ over $X$ and $\mu^*(K_X + B)$ is $\pi$-semi-ample by Theorem 2.19. Therefore, there is a positive integer $m$ such that $mB$ is a divisor and that $m\mu^*(K_X + B) \sim 0$. It implies that $m(K_X + B)$ is Cartier. □

**Lemma-Definition 2.22.** *Let $X$ be a normal surface and $B$ an effective $\mathbb{Q}$-divisor on $X$ such that $\ulcorner B \urcorner$ is reduced. Then there exist a bimeromorphic morphism $\rho\colon Y \to X$ from a normal surface $Y$ and an effective $\mathbb{Q}$-divisor $B_Y$ such that*

- *$(Y, B_Y)$ is log-canonical,*
- *$K_Y + B_Y$ is $\rho$-ample, and*
- *$B_Y = \rho^{[*]}B + E_\rho$ for the $\rho$-exceptional locus $E_\rho$.*

*The pair $(Y, B_Y)$ is unique up to isomorphism over $X$, and it is called the* log-canonical modification *of $(X, B)$; we also say that $\rho\colon (Y, B_Y) \to (X, B)$ is the log-canonical modification. Here, one has $\operatorname{Sing} Y \cup \operatorname{Supp} B_Y = \rho^{-1}(\operatorname{Sing} X \cup \operatorname{Supp} B)$.*

*Proof.* First, we shall show the existence of $(Y, B_Y)$. Let $\mu\colon M \to X$ be a bimeromorphic morphism from a non-singular surface $M$ such that the union of $\mu^{-1}B$ and the $\mu$-exceptional locus $E_\mu$ is a normal crossing divisor. We set $B_M := \mu^{[*]}B + E_\mu$. Then $\ulcorner B_M \urcorner$ is reduced, $\operatorname{Supp} B_M$ is normal crossing, and $\mu_* B_M = B$. Let $P$ be the positive part of the relative Zariski-decomposition of $K_M + B_M$ with respect to $\mu\colon M \to X$. Then $P$ is $\mu$-semi-ample by Theorem 2.19. Therefore, there exist bimeromorphic morphisms $\phi\colon M \to Y$, $\rho\colon Y \to X$, and a $\rho$-ample $\mathbb{Q}$-divisor $A$ such that $Y$ is a normal surface, $\mu = \rho \circ \phi$, and $P \sim_{\mathbb{Q}} \phi^* A$. In particular, $Y \simeq \mathbf{Projan}_X \mathcal{R}$ over $X$ for the graded $\mathcal{O}_X$-algebra

$$\mathcal{R} = \bigoplus\nolimits_{m \geq 0} \mu_* \mathcal{O}_M(\llcorner m(K_M + B_M) \lrcorner) \simeq \bigoplus\nolimits_{m \geq 0} \mu_* \mathcal{O}_M(\llcorner mP \lrcorner),$$

which is finitely generated locally over $X$ (cf. Lemma 2.17 and Corollary 2.20). The negative part $N$ of the relative Zariski-decomposition of $K_M + B_M$ is $\phi$-exceptional, since $PN = (\phi^* A)N = 0$. Hence,

$$\phi_* P = \phi_*(K_M + B_M) = K_Y + B_Y \sim_{\mathbb{Q}} A$$

for the $\mathbb{Q}$-divisor $B_Y := \phi_* B_M$. Hence, $(Y, B_Y)$ is log-canonical, $K_Y + B_Y$ is $\rho$-ample, and $\rho_* B_Y = B$. Moreover, $B_Y = \rho^{[*]}B + E_\rho$ for the $\rho$-exceptional locus $E_\rho$, since $B_M = \mu^{[*]}B + E_\mu$. Therefore, $(Y, B_Y)$ is a log-canonical modification of $(X, B)$.

Second, we shall show the uniqueness of $(Y, B_Y)$. Let $\rho'\colon (Y', B_{Y'}) \to (X, B)$ be another log-canonical modification of $(X, B)$. Then we have bimeromorphic morphisms $\phi'\colon M' \to Y'$ and $\theta\colon M' \to M$ from a non-singular surface $M'$ such that $\mu \circ \theta = \rho' \circ \phi'$ and that the union of $\theta^{-1}(\mu^{-1}B)$ and the $\mu \circ \theta$-exceptional locus $E_{\mu\circ\theta}$ is a normal crossing divisor. We set $B_{M'} = (\mu \circ \theta)^{[*]}B + E_{\mu\circ\theta}$ as above. Then $K_{M'} + B_{M'} = \phi'^*(K_{Y'} + B_{Y'}) + R'$ for a $\phi'$-exceptional effective $\mathbb{Q}$-divisor $R'$, since $(Y', B_{Y'})$ is a log-canonical modification. Thus, $\phi'^*(K_{Y'} + B_{Y'})$ is the positive part of the relative Zariski-decomposition of $K_{M'} + B_{M'}$ over $X$. On the other hand, we have $K_{M'} + B_{M'} = \theta^*(K_M + B_M) + R''$ for a $\theta$-exceptional effective $\mathbb{Q}$-divisor $R''$, since $(M, B_M)$ is log-canonical. Hence, $\nu^* P = \nu^*(K_Y + B_Y)$ is equal to $\phi'^*(K_{Y'} + B_{Y'})$ as the positive part of the relative Zariski-decomposition of $K_{M'} + B_{M'}$ over $X$. Therefore, $Y \simeq Y'$ over $X$.

Finally, we shall show the last assertion. We have $\operatorname{Supp} B_Y = E_\rho \cup \rho^{-1} \operatorname{Supp} B$ by $B_Y = \rho^{[*]} B + E_\rho$, and we have $\operatorname{Sing} Y \cup E_\rho = (\rho^{-1} \operatorname{Sing} X) \cup E_\rho$ by the isomorphism $Y \setminus E_\rho \simeq X \setminus \rho(E_\rho)$. Therefore,

$$\operatorname{Sing} Y \cup \operatorname{Supp} B_Y = \operatorname{Sing} Y \cup E_\rho \cup \rho^{-1} \operatorname{Supp} B = \rho^{-1}(\operatorname{Sing} X \cup \operatorname{Supp} B)$$

by $E_\rho = \rho^{-1}(\rho(E_\rho))$. Thus, we are done. □

A certain morphism of normal surfaces with only discrete fibers lifts to log-canonical modifications as follows:

**Proposition 2.23.** *Let $f \colon Y \to X$ be a morphism of normal surfaces with only discrete fibers and let $B_X$ and $B_Y$ be effective $\mathbb{Q}$-divisors on $X$ and $Y$, respectively, such that $\ulcorner B_X \urcorner$ and $\ulcorner B_Y \urcorner$ are reduced and $K_Y + B_Y = f^*(K_X + B_X)$. Let $\sigma \colon (V, B_V) \to (X, B_X)$ and $\tau \colon (W, B_W) \to (Y, B_Y)$ be the log-canonical modifications. Then there is a morphism $h \colon W \to V$ with only discrete fibers such that $f \circ \tau = \sigma \circ h$ and $K_W + B_W = h^*(K_V + B_V)$.*

*Proof.* We set $B = B_X$ and apply results in Section 2.2. For the commutative diagram (II-2) of Lemma 2.9 for $(X, B) = (X, B_X)$, we can find bimeromorphic morphisms $\phi \colon M \to V$, $\sigma \colon V \to X$, $\psi \colon N \to W$, and $\tau \colon W \to Y$ such that $\mu = \sigma \circ \phi$ and $\nu = \tau \circ \psi$ and that $\phi^*(K_V + B_V)$ (resp. $\psi^*(K_W + B_W)$) is the positive part of the relative Zariski-decomposition of $K_M + \mu^{[*]} B_X + E_\mu$ (resp. $K_N + \nu^{[*]} B_Y + E_\nu$) over $X$ (resp. $Y$), $E_\mu$ (resp. $E_\nu$) is the exceptional locus for $\mu$ (resp. $\nu$). In particular, we have a commutative diagram

$$\begin{array}{ccccc} N & \xrightarrow{\psi} & W & \xrightarrow{\tau} & Y \\ {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\phi} & V & \xrightarrow{\sigma} & X. \end{array}$$

By assumption, $B_f = B_Y$ and $T_f = \Delta = 0$ for $\mathbb{Q}$-divisors $B_f$, $T_f$, and $\Delta$ in Lemmas 2.10 and 2.11. Hence, $\operatorname{Supp} B_Y \subset \widetilde{\Sigma}_f$, and

$$K_N + \nu^{[*]} B_Y + E_\nu = g^*(K_M + \mu^{[*]} B_X + E_\mu) + \overline{\Delta}$$

for an effective $\mathbb{Q}$-divisor $\overline{\Delta}$ which is exceptional for both $\nu$ and $g$ by Lemma 2.11(3) as $\nu_* \overline{\Delta} = \Delta = 0$. Therefore,

$$\text{(II-7)} \qquad K_N + \nu^{[*]} B_Y + E_\nu = g^*(\phi^*(K_V + B_V)) + G$$

for an effective $\mathbb{Q}$-divisor $G$ exceptional for $\phi \circ g$. The fiber product $V \times_X Y$ is irreducible and generically reduced by Lemma 1.13. For the normalization $V'$ of

$V \times_X Y$, we have a commutative diagram

$$\begin{array}{ccccc} N & \xrightarrow{\phi'} & V' & \xrightarrow{\sigma'} & Y \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ M & \xrightarrow{\phi} & V & \xrightarrow{\sigma} & X, \end{array}$$

in which $\phi'$ and $\sigma'$ are bimeromorphic morphisms and $p$ is induced by the first projection $V \times_X Y \to V$. Note that $p$ also has only discrete fibers. Then $G$ is exceptional for $\phi'$, $g^*(\phi^*(K_V + B_V)) = \phi'^* p^*(K_V + B_V)$, and $p^*(K_V + B_V)$ is $\sigma'$-ample. Hence, by (II-7), we have an equality

$$\psi^*(K_W + B_W) = g^*(\phi^*(K_V + B_V))$$

as the positive part of the relative Zariski-decomposition of $K_N + \nu^{[*]}B_Y + E_\nu$. Consequently, there is an isomorphism $\lambda \colon W \to V'$ such that $\lambda \circ \psi = \phi'$, $\tau = \sigma' \circ \lambda$, and $K_W + B_W = \lambda^*(p^*(K_V + B_V))$. Then the morphism $h = p \circ \lambda$ satisfies the required conditions. $\square$

## 3. Singularities of pairs for endomorphisms of surfaces

As a generalization of an endomorphism of a normal surface $X$, we shall consider a morphism $X^\circ \to X$ with only discrete fibers from an open subset $X^\circ$ of $X$. The main result in Section 3 is Theorem 3.5 below on the log-canonicity of singularities of pairs $(X, B)$ in which $X$ admits a morphism $X^\circ \to X$ as above and $B$ satisfies a special condition. Theorem 0.1 in the introduction is a direct consequence of Theorem 3.5. As a corollary of Theorem 3.5, we can prove results of Wahl [58] and Favre [6] on the log-canonicity of a normal surface singularity which admits a non-isomorphic finite surjective endomorphism (cf. Corollary 3.7). In Section 3.1, we explain the situation, the statement, and corollaries of Theorem 3.5, as well as a 1-dimensional analogue, Proposition 3.4. The proof of Theorem 3.5 is given in Section 3.2.

### 3.1. **Setting and statements.**

**Definition 3.1.** For a normal variety $X$, let $f \colon X^\circ \to X$ be a morphism from an open subset $X^\circ$ of $X$. We define inductively open subsets $X^{(k)} = X_f^{(k)}$ of $X$ for $k \geq 0$ by

$$X^{(0)} := X, \quad X^{(1)} = X^\circ, \quad \text{and} \quad X^{(k+1)} = f^{-1}(X^{(k)}).$$

Composing $f$ and its restrictions to $X^{(i)}$, we have a morphism

$$f^{(k)} \colon X^{(k)} \xrightarrow{f} X^{(k-1)} \xrightarrow{f} \cdots \xrightarrow{f} X^{(0)} = X$$

for any $k \geq 0$, where $f^{(0)} = \mathrm{id}_X$ and $f^{(1)} = f$. Note that $f^{(k)}$ has a meaning when $X^{(k)} \neq \emptyset$. We define $X_{(k)} = X_{f,(k)}$ to be the image $f^{(k)}(X^{(k)})$ for any $k \geq 0$. Note that $X_{(k)}$ is an open subset of $X$ when $f$ has only discrete fibers (cf. Corollary 1.8). The intersection $\bigcap_{k \geq 1} X_{(k)}$ is called the limit set of $f$ and is denoted by $X_{(\infty)} = X_{f,(\infty)}$.

*Remark* 3.2. For the germ $\mathfrak{X} = (X, x)$ of a normal variety $X$ at a point $x$, an endomorphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ is assumed to be induced by a morphism $f\colon X^\circ \to X$ from an open neighborhood $X^\circ$ of $x$ such that $f(x) = x$. Then the $k$-th power $\mathfrak{f}^k = \mathfrak{f} \circ \cdots \circ \mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ is induced by $f^{(k)}\colon X^{(k)} \to X$. The endomorphism $\mathfrak{f}$ also corresponds to an endomorphism $\mathfrak{f}^*\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ as a local ring homomorphism. When $\mathfrak{f}^*$ is finite, $\mathfrak{f}$ is said to be finite. In this case, $x$ is an isolated point of $f^{-1}(x)$, and we may assume that $f^{-1}(x) = \{x\}$ and $f$ has only discrete fibers by replacing $X^\circ$ with an open neighborhood of $x$ (cf. Corollaries 1.4 and 1.8).

*Remark.* For the germ $\mathfrak{X} = (X, x)$ above, assume that $x$ is an isolated singular point. Then we may take $X$ as the analytic space $\mathsf{X}^{\mathrm{an}}$ associated with an algebraic scheme $\mathsf{X}$ over $\operatorname{Spec} \mathbb{C}$ by [1, Thm. 3.8]. Hence, an endomorphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ is represented by a morphism $\mathsf{f}\colon \mathsf{U} \to \mathsf{X}$ of algebraic schemes from an étale neighborhood $\mathsf{U}$ of $x$. It is not clear that one can choose $\mathsf{U}$ as a Zariski-open neighborhood of $x$.

We use the following notation for $\mathbb{Q}$-divisors in Section 3.

**Notation 3.3.** Let $X$ be a normal variety and $B$ a $\mathbb{Q}$-divisor on $X$. Let $B = \sum b_i \Gamma_i$ be the prime decomposition, where $b_i \in \mathbb{Q}$, and $\Gamma_i$ are prime divisors. For a rational number $c$, we define

$$B^{\geq c} := \sum\nolimits_{b_i \geq c} b_i \Gamma_i, \quad B^{\leq c} := \sum\nolimits_{b_i \leq c} b_i \Gamma_i, \quad \text{and} \quad B_{=c} := \sum\nolimits_{b_i = c} \Gamma_i.$$

The following deals with the 1-dimensional case, which improves a part of [39, Lem. 3.5.1].

**Proposition 3.4.** *Let $X$ be a non-singular curve and $B$ an effective $\mathbb{Q}$-divisor on $X$ such that* $\operatorname{Supp} B^{\geq 1}$ *is a finite set. Let $f\colon X^\circ \to X$ be a non-degenerate morphism from an open subset $X^\circ$ of $X$ such that*

$$K_{X^\circ} + B|_{X^\circ} = f^*(K_X + B) + \Delta$$

*for an effective $\mathbb{Q}$-divisor $\Delta$ on $X^\circ$. Then the following hold for any point $P \in X_{f,(\infty)} = X_{(\infty)}$:*

1. *If* $\operatorname{mult}_P B \geq 1$, *then* $(f^{(k)})^{-1}(P) \cap X_{(\infty)} = \{P\}$ *for some* $k > 0$.
2. *If* $\operatorname{mult}_P B > 1$, *then $f$ is a local isomorphism at $P$ and* $\operatorname{mult}_{f(P)} B = \operatorname{mult}_P B$.
3. *If* $\operatorname{mult}_P B = 1$, *then* $P \notin \operatorname{Supp} \Delta$ *and* $\operatorname{mult}_{f(P)} B = 1$.
4. *If $f(P) = P$, then*

$$(d-1)(\operatorname{mult}_P B - 1) = -\operatorname{mult}_P \Delta$$

*for $d := \operatorname{mult}_P f^*P$. In particular, when $f$ is not an isomorphism at $P$,* $\operatorname{mult}_P B < 1$ *if and only if* $\operatorname{mult}_P \Delta > 0$.

*Proof.* For a point $Q \in X^\circ$, we set $d_Q := \operatorname{mult}_Q f^*(f(Q))$. Note that $f$ is a local isomorphism at $Q$ if and only if $d_Q = 1$. We have equalities

$$d_Q - 1 = \operatorname{mult}_Q R_f = d_Q \operatorname{mult}_{f(Q)} B - \operatorname{mult}_Q B + \operatorname{mult}_Q \Delta$$

for the ramification divisor $R_f = K_{X^\circ} - f^*K_X = f^*B - B|_{X^\circ} + \Delta$ of $f$. Hence,

$$\text{(III-1)} \qquad \operatorname{mult}_Q B - 1 = d_Q(\operatorname{mult}_{f(Q)} B - 1) + \operatorname{mult}_Q \Delta \geq d_Q(\operatorname{mult}_{f(Q)} B - 1).$$

Then we have (4) by the first equality of (III-1) for $P = Q$. Moreover, (III-1) implies that

$$f^{-1}(\operatorname{Supp} B^{\geq 1}) \subset \operatorname{Supp} B^{\geq 1}.$$

We set $\mathcal{S} := X_{(\infty)} \cap \operatorname{Supp} B^{\geq 1}$. We may assume that $\mathcal{S} \neq \emptyset$ for assertions (1)–(3). Then $\emptyset \neq f^{-1}(P) \cap X_{(\infty)} \subset \mathcal{S}$ for any $P \in \mathcal{S}$ as $\operatorname{Supp} B^{\geq 1}$ is finite. By choosing $Q \in f^{-1}(P) \cap X_{(\infty)}$ for each $P \in \mathcal{S}$, we have a map $\psi\colon \mathcal{S} \to \mathcal{S}$ by $P \mapsto Q$. This map is injective as $f(Q) = P$, and moreover, it is bijective as $\mathcal{S} \subset \operatorname{Supp} B$ is finite. We write $\mathcal{S} = \{P_1, P_2, \ldots, P_n\}$. Then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that $\psi(P_i) = P_{\sigma^{-1}(i)}$ for any $i$. Hence, $f^{-1}(P_{\sigma(i)}) \cap X_{(\infty)} = \{P_i\}$ and $f(P_i) = P_{\sigma(i)}$ for any $i$. Let $k$ be the order of $\sigma$. Then $(f^{(k)})^{-1}(P) \cap X_{(\infty)} = \{P\}$ for any $P \in \mathcal{S}$; this shows (1). We set

$$d_i = d_{P_i} = \operatorname{mult}_{P_i} f^*(f(P_i)), \quad \beta_i = \operatorname{mult}_{P_i} B, \quad \text{and} \quad \delta_i = \operatorname{mult}_{P_i} \Delta$$

for each $1 \leq i \leq n$. Then

$$\text{(III-2)} \qquad \beta_i - 1 = d_i(\beta_{\sigma(i)} - 1) + \delta_i \geq d_i(\beta_{\sigma(i)} - 1)$$

by (III-1) for $Q = P_i$, and hence,

$$\text{(III-3)} \qquad \beta_i - 1 \geq d_i d_{\sigma(i)} \cdots d_{\sigma^{k-1}(i)}(\beta_i - 1)$$

for any $1 \leq i \leq n$. If $\beta_i > 1$, then $d_i = 1$, $\beta_i = \beta_{\sigma(i)}$, and $\delta_i = 0$ by (III-2) and (III-3); this proves (2). If $\beta_i = 1$, then $\beta_{\sigma(i)} = 1$ and $\delta_i = 0$ by (III-2); this proves (3). Thus, we are done. $\square$

*Remark.* The idea of the proof above is originally in the proof of [39, Lem. 3.5.1]. It is used in Lemma 5.3 of the preprint version of [41], preprint RIMS-**1613**, Kyoto Univ. 2007, and in [22, Prop. 2.4].

The following is the main result of Section 3, and it is regarded as a 2-dimensional analogue of Proposition 3.4:

**Theorem 3.5.** *Let $X$ be a normal complex analytic surface and $B$ an effective $\mathbb{Q}$-divisor on $X$ such that* $\operatorname{Sing} X \cup \operatorname{Sing} B_{\mathrm{red}}$ *is a finite set. Let $f\colon X^\circ \to X$ be a morphism with only discrete fibers from an open subset $X^\circ$ of $X$ such that*

$$K_{X^\circ} + B|_{X^\circ} = f^*(K_X + B) + \Delta$$

*for an effective $\mathbb{Q}$-divisor $\Delta$ on $X^\circ$. Then the following hold for the $\mathbb{Q}$-divisor $\widetilde{B} := B^{\leq 1} + \sum_{c>1} B_{=c}$ (cf. Notation* 3.3*) and for any point $x$ of the limit set $X_{(\infty)} = X_{f,(\infty)}$:*

(1) *If $x \in \operatorname{Supp} \Delta$, then $(X, \widetilde{B})$ is* 1*-log-terminal at $x$ (cf. Definition* 2.1*).*
(2) *If $(X, \widetilde{B})$ is not log-canonical at $x$, then $f$ is a local isomorphism at $x$, and $(f^{(k)})^{-1}(x) \cap X_{(\infty)} = \{x\}$ for some $k \geq 1$.*

By Remark 3.2, we have Theorem 0.1 directly from Theorem 3.5. We have two corollaries of Theorem 3.5. The first corollary below is a generalization of [39, Thm. 4.3.1], where $X$ is assumed to be a normal Moishezon surface:

**Corollary 3.6.** *Let $f\colon X \to X$ be a non-isomorphic finite surjective endomorphism of normal complex analytic surface $X$ and let $S$ be a reduced divisor on $X$ such that* $\operatorname{Sing} X \cup \operatorname{Sing} S$ *is a finite set and that $f^{-1}(S) = S$. Then $(X, S)$ is log-canonical.*

*Proof.* There is an effective $\mathbb{Q}$-divisor $\Delta$ such that $K_X + S = f^*(K_X + S) + \Delta$ by Lemma 1.39. Thus, we can apply Theorem 3.5 to the situation where $X^\circ = X$ and $B = S$. Here, $X_{f,(\infty)} = X$, since $f$ is surjective. Assume that $(X, S)$ is not log-canonical at a point $x$. Then $f$ is a local isomorphism at $x$ and $(f^k)^{-1}(x) = \{x\}$ for some $k$ by Theorem 3.5(2). This contradicts: $\deg f > 1$. Thus, $(X, S)$ is log-canonical. □

The second corollary below is well known: The first assertion has been proved by Wahl in [58] by using an invariant $-P \cdot P$, and the second assertion has been proved by Favre in [6, Thm. B(3)] by using the theory of valuation spaces of normal surface singularities.

**Corollary 3.7** (Wahl, Favre)**.** *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{X}$ be a non-isomorphic finite surjective endomorphism of a germ $\mathfrak{X} = (X, x)$ of a normal surface $X$ at a point $x$. Then $\mathfrak{X}$ is log-canonical. If $x$ is contained in the support of the ramification divisor $R_{\mathfrak{f}}$, then $\mathfrak{X}$ is log-terminal.*

*Proof.* By Remark 3.2, we may assume that $\mathfrak{f}$ is induced from a morphism $f\colon X^\circ \to X$ with only discrete fibers from an open neighborhood $X^\circ$ of $x$, in which $f(x) = x$, and $f$ is not a local isomorphism at $x$. Then $x \in X_{f,(\infty)}$. Moreover, $x \in \operatorname{Supp} R_f$ when $x \in \operatorname{Supp} R_{\mathfrak{f}}$. Obviously, we may assume that $\operatorname{Sing} X$ is finite. Hence, assertions are derived from Theorem 3.5 applied to the case where $B = 0$. □

3.2. **Proof of Theorem 3.5.** We shall prove Theorem 3.5 after proving preliminary results Lemma 3.8, Proposition 3.9, and Lemma 3.10, in which the latter two are special cases of Theorem 3.5.

**Lemma 3.8.** *In the situation of Theorem* 3.5*, there is an inclusion*

$$f^{-1}(\operatorname{Supp} B^{\geq 1}) \subset \operatorname{Supp} B^{\geq 1}|_{X^\circ}, \tag{III-4}$$

*and there is an effective $\mathbb{Q}$-divisor $\widetilde{\Delta}$ on $X^\circ$ such that*

$$K_{X^\circ} + \widetilde{B}|_{X^\circ} = f^*(K_X + \widetilde{B}) + \widetilde{\Delta}. \tag{III-5}$$

*Assume the following three conditions*:

(i) *The $\mathbb{Q}$-divisor $B^{\geq 1}$ has only finitely many prime components.*
(ii) *For any prime component $\Gamma$ of $B^{\geq 1}$, $\Gamma|_{X^\circ}$ is a prime divisor.*
(iii) *For any prime component $\Gamma$ of $B^{\geq 1}$, $f^{-1}\Gamma$ is not empty.*

*Then $f^*(B_{=c}) = B_{=c}|_{X^\circ}$ for any $c > 1$, $f^{-1}(B_{=1}) = B_{=1}|_{X^\circ}$, and $B^{\geq 1}|_{X^\circ}$ has no common prime component with $\Delta$. In particular, $\widetilde{\Delta} = \Delta$ and $K_{X^\circ} + B^{\leq 1}|_{X^\circ} = f^*(K_X + B^{\leq 1}) + \Delta$.*

*Proof.* Let $\widehat{\mathcal{S}}$ be the set of prime divisors on $X$ and let $\mathcal{T}_f$ be the set of prime divisors $\Gamma^\circ$ on $X^\circ$ such that $\Gamma^\circ$ is a prime component of $f^{-1}D$ for an effective divisor $D$ on $X$. Then, for each $\Gamma^\circ \in \mathcal{T}_f$, there is a unique prime divisor $\Gamma$ on $X$ such that

$\Gamma^\circ$ is a prime component of $f^{-1}\Gamma$, and we have a map $\psi\colon \mathcal{T}_f \to \widehat{\mathcal{S}}$ by $\Gamma^\circ \mapsto \Gamma$. For $\Gamma^\circ \in \mathcal{T}_f$ and $\Gamma = \psi(\Gamma^\circ)$, the integer $a := \operatorname{mult}_{\Gamma^\circ} f^*\Gamma$ is the ramification index of $f$ along $\Gamma^\circ$. Hence,

$$a - 1 = \operatorname{mult}_{\Gamma^\circ} R_f = a \operatorname{mult}_\Gamma B - \operatorname{mult}_{\Gamma^\circ} B|_{X^\circ} + \operatorname{mult}_{\Gamma^\circ} \Delta$$

for the ramification divisor $R_f = K_{X^\circ} - f^*K_X = f^*B - B|_{X^\circ} + \Delta$ of $f$. Therefore,

$$\operatorname{mult}_{\Gamma^\circ} B|_{X^\circ} - 1 = a(\operatorname{mult}_\Gamma B - 1) + \operatorname{mult}_{\Gamma^\circ} \Delta \geq a(\operatorname{mult}_\Gamma B - 1). \tag{III-6}$$

If $\Gamma \subset \operatorname{Supp} B^{\geq 1}$, i.e., $\operatorname{mult}_\Gamma B \geq 1$, then $\Gamma^\circ \subset \operatorname{Supp} B^{\geq 1}|_{X^\circ}$ by (III-6). This shows (III-4). Next, we shall prove that the $\mathbb{Q}$-divisor $\widetilde{\Delta}$ defined by (III-5) is effective. The $\mathbb{Q}$-divisor is written as

$$\widetilde{\Delta} = R_f + \widetilde{B}|_{X^\circ} - f^*\widetilde{B} = \Delta - (B - \widetilde{B})|_{X^\circ} + f^*(B - \widetilde{B}),$$

where $B - \widetilde{B} = \sum_{c>1}(c-1)B_{=c}$. It is enough to show that $\operatorname{mult}_{\Gamma^\circ} \widetilde{\Delta} \geq 0$ for any prime divisor $\Gamma^\circ$ such that $\Gamma^\circ \subset \operatorname{Supp}(B - \widetilde{B})|_{X^\circ} \cap f^{-1} \operatorname{Supp} B$. Here, $\Gamma^\circ \in \mathcal{T}_f$ and $\Gamma := \psi(\Gamma^\circ) \subset \operatorname{Supp} B$. Hence, $\operatorname{mult}_{\Gamma^\circ} \widetilde{B}|_{X^\circ} = 1$ and $\operatorname{mult}_\Gamma \widetilde{B} \leq 1$, and we have

$$\operatorname{mult}_{\Gamma^\circ} \widetilde{\Delta} = a - 1 + \operatorname{mult}_{\Gamma^\circ} \widetilde{B}|_{X^\circ} - a \operatorname{mult}_\Gamma \widetilde{B} \geq 0$$

for the ramification index $a$ of $f$ along $\Gamma^\circ$. Therefore, $\widetilde{\Delta}$ is effective.

For the rest of the proof, we assume three conditions (i)–(iii). Let $\mathcal{S}$ be the set of prime components of $B^{\geq 1}$. Then $\mathcal{S}$ is finite by (i), and $\psi\colon \psi^{-1}(\mathcal{S}) \to \mathcal{S}$ is surjective by (iii) and (III-4). By (ii) and by the inclusion (III-4), we have an injection $i\colon \psi^{-1}(\mathcal{S}) \to \mathcal{S}$ such that $\Gamma^\circ = i(\Gamma)|_{X^\circ}$ for any $\Gamma^\circ \in \psi^{-1}(\mathcal{S})$. Then $i\colon \psi^{-1}(\mathcal{S}) \to \mathcal{S}$ and $\psi\colon \psi^{-1}(\mathcal{S}) \to \mathcal{S}$ are both bijective. Let $\Gamma_1, \ldots, \Gamma_n$ be the elements of $\mathcal{S}$. Then, by maps $\psi$ and $i$, there is a permutation $\sigma$ of the set $\{1, \ldots, n\}$ such that

$$f^{-1}(\Gamma_{\sigma(i)}) = \Gamma_i|_{X^\circ}$$

for any $1 \leq i \leq n$. We set

$$a_i = \operatorname{mult}_{\Gamma_i|_{X^\circ}} f^*\Gamma_{\sigma(i)}, \quad \beta_i := \operatorname{mult}_{\Gamma_i} B, \quad \text{and} \quad \delta_i = \operatorname{mult}_{\Gamma_i|_{X^\circ}} \Delta.$$

Here, $a_i \in \mathbb{Z}_{\geq 1}$, $\beta_i \in \mathbb{Q}_{\geq 1}$, and $\delta_i \in \mathbb{Q}_{\geq 0}$. By (III-6) for $\Gamma_i|_{X^\circ}$, we have

$$\beta_i - 1 = a_i(\beta_{\sigma(i)} - 1) + \delta_i \geq a_i(\beta_{\sigma(i)} - 1). \tag{III-7}$$

Let $k$ be the order of the permutation $\sigma$. Then

$$\beta_i - 1 \geq a_i a_{\sigma(i)} \cdots a_{\sigma^{k-1}(i)}(\beta_i - 1) \tag{III-8}$$

for any $1 \leq i \leq n$ by (III-7). If $\beta_i > 1$, then $a_i = 1$, $\beta_{\sigma(i)} = \beta_i$, and $\delta_i = 0$ by (III-7) and (III-8). Therefore, for any $c > 1$, the equality $f^*(B_{=c}) = B_{=c}|_{X^\circ}$ holds, and $B_{=c}|_{X^\circ}$ has no common prime component with $\Delta$. Subtracting $f^*(B_{=c}) = B_{=c}|_{X^\circ}$ from $K_{X^\circ} + B|_{X^\circ} = f^*(K_X + B) + \Delta$, we have

$$K_{X^\circ} + B^{\leq 1}|_{X^\circ} = f^*(K_X + B^{\leq 1}) + \Delta \quad \text{and} \quad \Delta = \widetilde{\Delta}.$$

If $\beta_i = 1$, then $\beta_{\sigma(i)} = 1$ and $\delta_i = 0$ by (III-7). Therefore, $f^{-1}(B_{=1}) = B_{=1}|_{X^\circ}$, and $B_{=1}|_{X^\circ}$ has no common prime component with $\Delta$. Thus, we are done. $\square$

We shall prove the following special case of Theorem 3.5(1).

**Proposition 3.9.** *In the situation of Theorem* 3.5*, assume that* $\ulcorner B \urcorner$ *is reduced, i.e.,* $B = B^{\leq 1}$*. Let* $x$ *be a point in* $X^{\circ}$ *such that* $f(x) = x$ *and* $x \in \operatorname{Supp} \Delta$*. Then* $(X, B)$ *is* 1*-log-terminal at* $x$*.*

*Proof.* For an integer $k \geq 1$, we set $Y = X^{(k)}$, $B_Y := B|_{X^{(k)}}$, $g := f^{(k)} \colon Y \to X$, and $\Sigma_Y := (g^{-1} \operatorname{Supp} B) \cup \operatorname{Supp} R_g$, where $R_g$ is the ramification divisor of $g$. Then $K_Y + B_Y = g^*(K_X + B) + \Delta_Y$ for an effective divisor

$$\Delta_Y = \Delta|_Y + \sum\nolimits_{i=1}^{k-1} g_i^*(\Delta|_{X^{(i)}}),$$

where $g_i$ is the composite $Y = X^{(k)} \to X^{(k-1)} \to \cdots \to X^{(i)}$ of morphisms induced by $f$. Let $\mu \colon M \to X$ be a bimeromorphic morphism from a non-singular surface $M$ such that the union $\Sigma_\mu := \Sigma_\nu(X, B)$ of $\mu^{-1} \operatorname{Supp} B$ and the $\mu$-exceptional locus is a normal crossing divisor and that the proper transform of $\llcorner B \lrcorner$ in $M$ is non-singular. We set $N := \mu^{-1} Y$ and $\nu := \mu|_N \colon N \to Y$. Let $T_\nu$ and $B_\nu$ be the positive and negative parts, respectively, of the prime decomposition of $K_N - \nu^*(K_Y + B_Y)$; hence, $K_N + B_\nu = \nu^*(K_Y + B_Y) + T_\nu$. Here, $\nu_* B_\nu = B_Y$ as $T_\nu$ is $\nu$-exceptional (cf. Definition 2.1), and $\operatorname{Supp} B_\nu$ is normal crossing by $\operatorname{Supp} B_\nu \subset \Sigma_\mu \cap Y$. Let $C_\nu$ and $S_\nu$ be the positive and negative parts, respectively, of the prime decomposition of $B_\nu - \nu^* \Delta_Y$. By Lemma 2.11(2), it suffices to prove that $\ulcorner C_\nu \urcorner$ is reduced and $\llcorner C_\nu \lrcorner$ has no $\nu$-exceptional prime component over an open neighborhood of $x$ in $Y = X^{(k)}$.

Let $\mathcal{U}$ be an open neighborhood of $x$ in $X^{\circ}$ such that $B|_{\mathcal{U}}$ has only finitely many prime components. Let $m$ be a positive integer such that $mB|_{\mathcal{U}}$ is a divisor. Then $m\Delta|_{\mathcal{U}}$ is a divisor by $\Delta = R_f - f^*B + B|_{X^{\circ}}$, since $R_f$ is a divisor by Remark 1.24(5). Thus, $m\Delta$ and $mg_i^*(\Delta|_{X^{(i)}})$ are all divisors on an open neighborhood $\mathcal{V}$ of $x$ in $Y$. Here, we may assume that $\nu$ is an isomorphism over $\mathcal{V} \setminus \{x\}$. Then $mr\nu^*(\Delta|_Y)$ and $mr\nu^* g_i^*(\Delta|_{X^{(i)}})$ are divisors on $\nu^{-1}\mathcal{V}$ for the numerical factorial index $r := \operatorname{nf}(X, x)$ (cf. Definition 1.26). Since $x \in X^{(i)}$ for all $i$ and since $x \in \operatorname{Supp} \Delta$, we have

$$\operatorname{mult}_\Gamma \nu^* \Delta_Y = \operatorname{mult}_\Gamma \nu^*(\Delta|_Y) + \sum\nolimits_{i=1}^{k-1} \operatorname{mult}_\Gamma \nu^*(g_i^*(\Delta|_{X^{(i)}})) \geq k/mr$$

for any $\nu$-exceptional prime divisor $\Gamma$ contained in $\nu^{-1}(x)$. On the other hand, $\operatorname{mult}_\Gamma B_\nu$ does not depend on $k$. Thus, if we take $k$ large enough, then the positive part $C_\nu$ of the prime decomposition of $B_\nu - \mu^* \Delta_Y$ does not contain such prime divisors $\Gamma$. Hence, $C_\nu$ is contained in the proper transform of $B_Y$ on $\nu^{-1}\mathcal{V}$. Since $\ulcorner B_Y \urcorner = \ulcorner B \urcorner|_Y$ is reduced, we see that $\ulcorner C_\nu \urcorner$ is reduced and that $\llcorner C_\nu \lrcorner$ has no $\nu$-exceptional prime component over $\mathcal{V}$. Thus, we are done. $\square$

*Remark.* The iteration $f^{(k)}$ is also considered in the proof of [6, Thm. B(3)].

We shall prove the following special case of Theorem 3.5(2) by applying the *log-canonical modification* (cf. Lemma-Definition 2.22) and Proposition 2.23.

**Lemma 3.10.** *In the situation of Theorem* 3.5*, assume that* $\ulcorner B \urcorner$ *is reduced, i.e.,* $B = B^{\leq 1}$*. Let* $x$ *be a point in* $X^{\circ}$ *such that* $f(x) = x$ *and* $x \notin \operatorname{Supp} \Delta$*. If* $f$ *is not a local isomorphism at* $x$*, then* $(X, B)$ *is log-canonical at* $x$*.*

*Proof.* We shall derive a contradiction by assuming that $(X, B)$ is not log-canonical at $x$. By replacing $X^\circ$ with an open neighborhood of $x$, we may assume that $\Delta = 0$. Let $\rho\colon (Y, B_Y) \to X$ be the log-canonical modification of $(X, B)$. Then $\rho^{-1}(x)$ is a non-zero compact divisor as $(X, B)$ is not log-canonical at $x$. We set $Y^\circ = \rho^{-1}(X^\circ)$, $B_{Y^\circ} = B_Y|_{Y^\circ}$, and $\rho^\circ := \rho|_{Y^\circ}\colon Y^\circ \to X^\circ$. By Proposition 2.23, there is a morphism $f_Y\colon Y^\circ \to Y$ with only discrete fibers such that $\rho \circ f_Y = f \circ \rho^\circ$ and $K_{Y^\circ} + B_{Y^\circ} = f_Y^*(K_Y + B_Y)$. By Remark 1.21, we can find open neighborhoods $V_1$ and $V_2$ of $x$ in $X^\circ$ and $X$, respectively, such that $f(V_1) = V_2$, $f^{-1}(x) \cap V_1 = \{x\}$, and the induced morphism $\tau := f|_{V_1}\colon V_1 \to V_2$ is finite. Here, $\deg \tau > 1$, since $f$ is not a local isomorphism at $x$. We set $Y_i := \rho^{-1}V_i$ for $i = 1, 2$. Then $\tau$ lifts to a finite surjective morphism $\theta := f_Y|_{Y_1}\colon Y_1 \to Y_2$ such that $\deg \theta = \deg \tau$. In particular, $\theta|_{\rho^{-1}(x)}\colon \rho^{-1}(x) \to \rho^{-1}(x)$ is also finite and surjective. Let $\mathcal{S}$ be the set of prime components of $\rho^{-1}(x)$. Then the map $\Gamma \mapsto f_Y(\Gamma) = \theta(\Gamma)$ induces a bijection $\mathcal{S} \to \mathcal{S}$. By replacing $f\colon X^\circ \to X$ with the $k$-th power $f^{(k)}\colon X^{(k)} \to X$ for some $k > 1$, we may assume that $\Gamma = f_Y(\Gamma) = \theta(\Gamma)$ for any $\Gamma \in \mathcal{S}$. Then $\theta^*\Gamma = d_\Gamma \Gamma$ for a positive integer $d_\Gamma$. Since $\Gamma^2 < 0$, we have $d_\Gamma^2 = \deg \theta$ by

$$d_\Gamma^2 \Gamma^2 = (\theta^*\Gamma)^2 = \Gamma \theta_*(\theta^*\Gamma) = (\deg \theta)\Gamma^2$$

(cf. Remark 1.24). Therefore, $\deg \tau = \deg \theta = d^2$ for an integer $d > 1$ and $\theta^*\Gamma = d\Gamma$ for any $\Gamma \in \mathcal{S}$. Then we have $(K_Y + B_Y)\Gamma = 0$ by

$$\begin{aligned} d(K_Y + B_Y)\Gamma &= (K_Y + B_Y)\theta^*\Gamma = (K_{Y^\circ} + B_{Y^\circ})\theta^*\Gamma = (f_Y^*(K_Y + B_Y))\theta^*\Gamma \\ &= (K_Y + B_Y) f_{Y*}(\theta^*\Gamma) = (\deg \theta)(K_Y + B_Y)\Gamma = d^2(K_Y + B_Y)\Gamma. \end{aligned}$$

This contradicts the $\rho$-ampleness of $K_Y + B_Y$. Thus, we are done. □

Now, we are ready to prove Theorem 3.5:

*Proof of Theorem* 3.5. Let $\Sigma \subset X$ be the set of points $x$ such that $(X, \widetilde{B})$ is not 1-log-terminal at $x$. Then $f^{-1}\Sigma \subset \Sigma$ by Proposition 2.12(2) applied to the equality (III-5) in Lemma 3.8. Note that $\Sigma$ is finite by $\Sigma \subset \operatorname{Sing} X \cup \operatorname{Sing} B_{\mathrm{red}}$. We set $\Sigma_{(\infty)} := \Sigma \cap X_{(\infty)}$. For the proof of Theorem 3.5, we may assume that $\Sigma_{(\infty)} \neq \emptyset$. For any point $x \in \Sigma_{(\infty)}$, we have $f^{-1}(x) \cap \Sigma_\infty = f^{-1}(x) \cap X_{(\infty)} \neq \emptyset$. In fact, $f^{-1}(x)$ is finite by $f^{-1}(x) \subset \Sigma$, and if $f^{-1}(x) \cap X_{(\infty)} = \emptyset$, then $f^{-1}(x) \cap X_{(k)} = \emptyset$ for $k \gg 1$, but it contradicts $x \in X_{(\infty)} \subset f(X_{(k)})$. By choosing an element of $f^{-1}(x) \cap \Sigma_{(\infty)}$ for each $x \in X_{(\infty)}$, we have an injection $\psi\colon \Sigma_{(\infty)} \to \Sigma_{(\infty)}$ such that $\psi(x) \in f^{-1}(x)$. This is a bijection as $\Sigma_{(\infty)}$ is finite. Hence, $\Sigma_{(\infty)} \subset X^\circ$ and $f$ induces the inverse map $\Sigma_{(\infty)} \to \Sigma_{(\infty)}$ of $\psi$, i.e., $f(\psi(x)) = x$. There is a positive integer $k$ such that $\psi^k(x) = x$, i.e., $(f^{(k)})^{-1}(x) \cap X_{(\infty)} = \{x\}$, for any $x \in \Sigma_{(\infty)}$. By replacing $f$ with $f^{(k)}$, we may assume that $f^{-1}(x) \cap X_{(\infty)} = \{x\}$ for any $x \in \Sigma_{(\infty)}$.

Let us fix a point $x \in \Sigma_{(\infty)}$. We can choose an open neighborhood $U$ of $x$ in $X$ satisfying the following conditions:

- If $x \notin \operatorname{Supp} B^{\geq 1}$, then $B^{\geq 1}|_U = 0$.
- If $x \in \operatorname{Supp} B^{\geq 1}$, then any prime component $\Gamma$ of $B^{\geq 1}|_U$ contains $x$ and is locally irreducible.

We set $U^\circ := (f^{-1}U) \cap U$. Note that if $x \in \operatorname{Supp} B^{\geq 1}$, then $\Gamma|_{U^\circ}$ is a prime divisor for any prime component $\Gamma$ of $B^{\geq 1}|_U$. Then we can apply Lemma 3.8 to the restriction $U^\circ \to U$ of $f$ and to $B^{\geq 1}|_U$. As a consequence, we have

$$\widetilde{\Delta}|_{U^\circ} = \Delta|_{U^\circ}.$$

Then $x \notin \operatorname{Supp} \Delta$ by Proposition 3.9 applied to $U^\circ \to U$ and to the equality (III-5). This proves Theorem 3.5(1). Moreover, if $(X, \widetilde{B})$ is not log-canonical at $x$, then $f$ is a local isomorphism by Lemma 3.10 applied to the equality (III-5), since $x \notin \operatorname{Supp} \widetilde{\Delta}$. This proves Theorem 3.5(2), and we are done. □

## 4. Some technical notions for the study of endomorphisms

We prepare some technical results on toric surfaces (Section 4.1) and cyclic covers (Section 4.2), and introduce two notions: *essential blowings up* (Section 4.4) and *dual $\mathbb{R}$-divisors* (Section 4.4) with their properties. These results and properties are applied to discussions in Section 5 on the lift of endomorphisms of germs of normal surfaces.

4.1. **Endomorphisms of certain affine toric surfaces.** We shall explain some basic properties of toric surfaces, toric morphisms, and toric endomorphisms, by using the theory of toric varieties (cf. [32], [42], [12], etc.) with some related arguments in [37, §3.1] and [40, §3.1] in addition. An affine toric surface, which is considered as a complex analytic surface, is expressed as

$$\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) = (\operatorname{Spec} \mathbb{C}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}])^{\mathrm{an}},$$

for a free abelian group $\mathsf{N}$ of rank 2, a closed strictly convex rational polyhedral cone $\boldsymbol{\sigma}$ in $\mathsf{N} \otimes \mathbb{R}$, the dual abelian group $\mathsf{M} := \operatorname{Hom}_{\mathbb{Z}}(\mathsf{N}, \mathbb{Z})$, and the dual cone

$$\boldsymbol{\sigma}^\vee = \{m \in \mathsf{M} \otimes \mathbb{R} \mid m(x) \geq 0 \text{ for any } x \in \boldsymbol{\sigma}\}.$$

Here, ${}^{\mathrm{an}}$ stands for the analytic space associated to an algebraic scheme over $\mathbb{C}$ (cf. [18, XII, §1]), the strict convexity means that $\boldsymbol{\sigma} \cap (-\boldsymbol{\sigma}) = \{0\}$, and $\mathbb{C}[\boldsymbol{\sigma}^\vee \cap \mathsf{M}]$ denotes the semi-group ring over $\mathbb{C}$. We write $\mathbb{T}_{\mathsf{N}} = \mathbb{T}_{\mathsf{N}}(\{0\})$, which is canonically isomorphic to the algebraic torus $\mathsf{N} \otimes_{\mathbb{Z}} \mathbb{C}^\star$, where $\mathbb{C}^\star := \mathbb{C} \setminus \{0\}$. The toric surface admits an action of $\mathbb{T}_{\mathsf{N}}$ and an equivariant open immersion $\mathbb{T}_{\mathsf{N}}(\{0\}) \hookrightarrow \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$.

*Remark.* If the cone $\boldsymbol{\sigma}$ is 1-dimensional, then it is a ray $\mathbb{R}_{\geq 0} e$ generated by a primitive element $e$ of $\mathsf{N}$ and we have an isomorphism $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \simeq \mathbb{C} \times \mathbb{C}^\star$ extending $\mathbb{T}_{\mathsf{N}}(\{0\}) \simeq \mathbb{C}^\star \times \mathbb{C}^\star$.

*Fact* 4.1. Assume that the cone $\boldsymbol{\sigma}$ is 2-dimensional. Then $\mathsf{N}$ has two primitive elements $e_1$, $e_2$ such that $(e_1, e_2)$ is a basis of $\mathsf{N} \otimes \mathbb{R}$ and $\boldsymbol{\sigma} = \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2$. Let $\mathcal{E}$ be the set of elements $e \in \boldsymbol{\sigma} \cap \mathsf{N}$ such that $\mathsf{N} = \mathbb{Z}e + \mathbb{Z}e_2$, and let $u \in \mathcal{E}$ be the element attaining the minimum of $e_1^\vee(e)$ for $e \in \mathcal{E}$, where $(e_1^\vee, e_2^\vee)$ is the dual basis of $(e_1, e_2)$ in $\mathsf{M} \otimes \mathbb{R}$. Then there exist integers $n > q \geq 0$ such that $\gcd(n, q) = 1$ and $u = (1/n)(e_1 + q e_2)$. The integer $n$ is uniquely determined by $(\mathsf{N}, \boldsymbol{\sigma})$. But $q$ can be replaced with an integer $0 \leq q^\dagger < n$ by interchanging $e_1$ and $e_2$, where $q^\dagger = 0$ if $q = 0$, and $q q^\dagger \equiv 1 \bmod n$ if $q > 0$.

**Definition 4.2.** When $\dim \boldsymbol{\sigma} = 2$, the number $n$ above is called the *order* of $(\mathsf{N}, \boldsymbol{\sigma})$, and the pair $(n, q)$ is called the *type* of $(\mathsf{N}, \boldsymbol{\sigma})$.

*Remark.* Assume that $\dim \boldsymbol{\sigma} = 2$. Then $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ has a unique fixed point $*$ on the action of $\mathbb{T}_{\mathsf{N}}$: For $e_1$ and $e_2$ in Fact 4.1, the complement of $\mathbb{T}_{\mathsf{N}}(\mathbb{R}_{\geq 0}e_1) \cup \mathbb{T}_{\mathsf{N}}(\mathbb{R}_{\geq 0}e_2)$ in $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ is just $\{*\}$. If $q = 0$, then $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ is isomorphic to the affine plane $\mathbb{C}^2$. If $q > 0$, then $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ is singular at $*$, and it is a cyclic quotient singularity of type $(n, q)$, or $1/n(1, q)$ in some literature; in this case, the exceptional locus of the minimal resolution forms a linear chain of rational curves whose self-intersection numbers are calculated by a certain continued fraction of $n/q$ (cf. [40, Exam. 3.2]).

In general, a toric surface is expressed as

$$\mathbb{T}_{\mathsf{N}}(\triangle) = \bigcup\nolimits_{\boldsymbol{\sigma} \in \triangle} \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$$

for a free abelian group $\mathsf{N}$ of rank 2 and for a *fan* $\triangle$ of $\mathsf{N}$: A finite collection $\triangle$ of closed strictly convex rational polyhedral cones of $\mathsf{N} \otimes \mathbb{R}$ is called a fan if each face of a cone in $\triangle$ belongs to $\triangle$ and the intersection of two cones in $\triangle$ is a face of both cones. The open immersion $\mathbb{T}_{\mathsf{N}}(\{0\}) \subset \mathbb{T}_{\mathsf{N}}(\triangle)$ is also $\mathbb{T}_{\mathsf{N}}$-equivariant. The open orbit $\mathbb{T}_{\mathsf{N}}(\{0\})$ or $\mathbb{T}_{\mathsf{N}}$ is called the *open torus* and the complement $\mathbb{T}_{\mathsf{N}}(\triangle) \setminus \mathbb{T}_{\mathsf{N}}(\{0\})$ is called the *boundary divisor*. We have the following analogy of [40, Exam. 3.4].

*Example* 4.3. Assume that the union $|\triangle| = \bigcup_{\boldsymbol{\sigma} \in \triangle} \boldsymbol{\sigma}$ is a strictly convex cone of dimension 2. Then there exist primitive elements $v_i$ of $\mathsf{N}$ for $i = 1, 2, \ldots, l$ such that $\triangle$ consists of

- 2-dimensional cones $\boldsymbol{\sigma}_i = \mathbb{R}_{\geq 0}v_i + \mathbb{R}_{\geq 0}v_{i+1}$ for $0 \leq i \leq l - 1$,
- 1-dimensional cones $\mathbf{R}_i := \mathbb{R}_{\geq 0}v_i$ for $0 \leq i \leq l$, and
- the 0-dimensional cone $\{0\}$,

where $|\triangle| = \mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_l$. The toric surface $\mathbb{T}_{\mathsf{N}}(\triangle)$ is obtained by gluing $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_i)$ for $0 \leq i \leq l - 1$ by open immersions $\mathbb{T}_{\mathsf{N}}(\mathbf{R}_{i+1}) \subset \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_i)$ and $\mathbb{T}_{\mathsf{N}}(\mathbf{R}_{i+1}) \subset \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_{i+1})$. The boundary $\mathbb{T}_{\mathsf{N}}(\triangle) \setminus \mathbb{T}_{\mathsf{N}}(\{0\})$ consists of prime divisors $\Gamma(v_i)$ for $0 \leq i \leq l$ which are determined by the property that $\Gamma(v_i) \cap \mathbb{T}_{\mathsf{N}}(\mathbf{R}_i) = \mathbb{T}_{\mathsf{N}}(\mathbf{R}_i) \setminus \mathbb{T}_{\mathsf{N}}(\{0\})$.

*Remark* 4.4. For $m \in \mathsf{M}$, let $\boldsymbol{e}(m)$ denote the nowhere vanishing function on $\mathbb{T}_{\mathsf{N}} = (\operatorname{Spec} \mathbb{C}[\mathsf{M}])^{\mathrm{an}}$ corresponding to the invertible element $m$ of $\mathbb{C}[\mathsf{M}]$. We regard $\boldsymbol{e}(m)$ as a meromorphic function on a toric surface $\mathbb{T}_{\mathsf{N}}(\triangle)$ for the fan $\triangle$ in Example 4.3. Then the principal divisor $\operatorname{div}(\boldsymbol{e}(m))$ is written as $\sum_{i=0}^{l} m(v_i)\Gamma(v_i)$ for any $m \in \mathsf{M}$.

*Remark.* If $\triangle$ consists of the faces of the cone $\boldsymbol{\sigma} = \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2$ in Fact 4.1, then $\mathbb{T}_{\mathsf{N}}(\triangle)$ is just the affine toric surface $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$, and $l = 1$ in Example 4.3.

**Definition 4.5.** For toric varieties $\mathbb{T}_{\mathsf{N}}(\triangle)$ and $\mathbb{T}_{\mathsf{N}'}(\triangle')$, a morphism $f\colon \mathbb{T}_{\mathsf{N}'}(\triangle') \to \mathbb{T}_{\mathsf{N}}(\triangle)$ of varieties is called a *toric morphism* if there is a homomorphism $\phi\colon \mathsf{N}' \to \mathsf{N}$ such that $f$ is equivariant under actions of $\mathbb{T}_{\mathsf{N}'}$ and $\mathbb{T}_{\mathsf{N}}$ along the complex Lie group homomorphism $\phi \otimes \mathbb{C}^\star\colon \mathbb{T}_{\mathsf{N}'} = \mathsf{N}' \otimes \mathbb{C}^\star \to \mathbb{T}_{\mathsf{N}} = \mathsf{N} \otimes \mathbb{C}^\star$.

A homomorphism $\phi\colon \mathsf{N}' \to \mathsf{N}$ is said to be *compatible* with $\triangle'$ and $\triangle$ if, for any $\boldsymbol{\sigma}' \in \triangle'$, there is a cone $\boldsymbol{\sigma} \in \triangle$ such that $\phi_{\mathbb{R}}(\boldsymbol{\sigma}') \subset \boldsymbol{\sigma}$, where $\phi_{\mathbb{R}}$ denotes the induced linear map $\phi \otimes \mathbb{R}\colon \mathsf{N}' \otimes \mathbb{R} \to \mathsf{N} \otimes \mathbb{R}$. In this case, the dual homomorphism

$\phi^\vee\colon \mathsf{M}' = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{N}', \mathbb{Z}) \to \mathsf{M} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{N}, \mathbb{Z})$ induces homomorphisms $\boldsymbol{\sigma}'^\vee \cap \mathsf{M}' \to \boldsymbol{\sigma}^\vee \cap \mathsf{M}$ of semi-groups, and toric morphisms $\mathbb{T}_{\mathsf{N}'}(\boldsymbol{\sigma}') \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$. These are glued to a toric morphism $\mathbb{T}_{\mathsf{N}'}(\triangle') \to \mathbb{T}_{\mathsf{N}}(\triangle)$, which is denoted by $\mathbb{T}(\phi)$. By [42, Thm. 1.13], we know that every toric morphism $\mathbb{T}_{\mathsf{N}'}(\triangle') \to \mathbb{T}_{\mathsf{N}}(\triangle)$ is expressed as $\mathbb{T}(\phi)$ for a homomorphism $\phi\colon \mathsf{N}' \to \mathsf{N}$ compatible with $\triangle'$ and $\triangle$.

*Remark* 4.6. The toric morphism $f$ in Definition 4.5 is proper if, for any $\boldsymbol{\sigma} \in \triangle$, the inverse image $\phi_{\mathbb{R}}^{-1}\boldsymbol{\sigma}$ is the union of some cones $\boldsymbol{\sigma}'$ in $\triangle'$ (cf. [42, Thm. 1.15]). In particular, the fan $\triangle$ in Example 4.3 gives a toric bimeromorphic morphism $\mu\colon \mathbb{T}_{\mathsf{N}}(\triangle) \to \mathbb{T}_{\mathsf{N}}(|\triangle|)$, where $\Gamma(v_i)$ is $\mu$-exceptional for $1 \le i \le l-1$. If $\mu$ is an isomorphism, then $l = 1$, i.e., $\triangle$ consists of the faces of the cone $|\triangle|$.

*Remark* 4.7. The toric morphism $\mu\colon \mathbb{T}_{\mathsf{N}}(\triangle) \to \mathbb{T}_{\mathsf{N}}(|\triangle|)$ above is expressed as the blowing up along an ideal as follows: Let $\Gamma_1$ and $\Gamma_2$ be the boundary prime divisors of $\mathbb{T}_{\mathsf{N}}(|\triangle|)$ defined by $\mathbb{R}_{\ge 0}v_0$ and $\mathbb{R}_{\ge 0}v_l$, respectively. We have positive rational numbers $a_i$ and $b_i$ for $1 \le i \le l-1$ such that $v_i = a_i v_0 + b_i v_l$. Then $a_1/b_1 > a_2/b_2 > \cdots > a_{l-1}/b_{l-1}$. Let $p_i$ for $1 \le i \le l-1$ be positive integers such that $-\sum p_i\Gamma(v_i)$ is $\mu$-very ample. Then $\mu$ is regarded as the blowing up of $\mathbb{T}_{\mathsf{N}}(|\triangle|)$ along the ideal sheaf

$$\mathcal{J} := \mathcal{O}_{\mathbb{T}_{\mathsf{N}}(\triangle)}(-\sum\nolimits_{i=1}^{l-1} p_i\Gamma(v_i)).$$

For an element $m \in |\triangle|^\vee \cap M$, the holomorphic function $\boldsymbol{e}(m)$ on $\mathbb{T}_{\mathsf{N}}(|\triangle|)$ belongs to $\mathcal{J}$ if and only if

$$\mathrm{div}(\boldsymbol{e}(m)) - \sum\nolimits_{i=1}^{l-1} p_i\Gamma(v_i) \ge 0$$

as a divisor on $\mathbb{T}_{\mathsf{N}}(\triangle)$, i.e., $m(v_i) = a_i m(v_0) + b_i m(v_l) \ge p_i$ for any $1 \le i \le l-1$. Since $\mathcal{J}$ is preserved by the action of $\mathbb{T}_{\mathsf{N}}$, $\mathcal{J}$ is generated by such $\boldsymbol{e}(m)$. Hence,

$$\mathcal{J} = \bigcap\nolimits_{i=1}^{l-1} \sum\nolimits_{a_i c + b_i d \ge p_i} \mathcal{O}_{\mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau})}(-c\Gamma_1 - d\Gamma_2),$$

where $c$ and $d$ are non-negative integers.

**Lemma 4.8.** *Let $\triangle$ and $\triangle'$ be fans of a free abelian group $\mathsf{N}$ of rank 2 such that $\boldsymbol{\tau} = |\triangle|$ and $\boldsymbol{\tau}' = |\triangle'|$ are strictly convex cones of dimension 2 and $\boldsymbol{\tau}' \subset \boldsymbol{\tau}$. Let*

$$\vartheta\colon \mathbb{T}_{\mathsf{N}'}(\triangle') \xrightarrow{\mu'} \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}') \xrightarrow{t} \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}) \overset{\mu^{-1}}{\cdots\to} \mathbb{T}_{\mathsf{N}}(\triangle)$$

*be the meromorphic map, where $\mu$ and $\mu'$ are canonical bimeromorphic toric morphisms defined by $\boldsymbol{\tau} = |\triangle|$ and $\boldsymbol{\tau}' = |\triangle'|$, and $t$ is the toric morphism defined by $\boldsymbol{\tau}' \subset \boldsymbol{\tau}$. Then $\vartheta$ is holomorphic if and only if any $\boldsymbol{\sigma}' \in \triangle'$ is contained in some cone $\boldsymbol{\sigma} \in \triangle$. In particular, when $\boldsymbol{\tau} = \boldsymbol{\tau}'$, $\vartheta$ is holomorphic if and only if $\triangle = \triangle'$, and in this case, $\vartheta$ is the identity morphism of $\mathbb{T}_{\mathsf{N}}(\triangle)$.*

*Proof.* The second assertion follows from the first one, since fans $\triangle$ and $\triangle'$ give polyhedral decompositions of the same cone $\boldsymbol{\tau} = \boldsymbol{\tau}'$, and both fans have finitely many 2-dimensional cones. For the first assertion, it suffices to prove the "only if" part, and we may assume that $\triangle'$ consists of the faces of a single 2-dimensional cone. Thus, from the beginning we may assume that $\mathbb{T}_{\mathsf{N}}(\triangle') = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}')$ and $\mu'$ is the identity morphism. The normalization of the fiber product of $\mu$ and $t$ over $\mathbb{T}_{\mathsf{N}}(\tau)$

is a toric variety expressed as $\mathbb{T}_{\mathsf{N}}(\triangle'')$ for the fan $\triangle'' = \{\boldsymbol{\tau}' \cap \boldsymbol{\sigma} \mid \boldsymbol{\sigma} \in \triangle\}$. If $\vartheta$ is holomorphic, then $\mathbb{T}_{\mathsf{N}}(\triangle'') \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}')$ is an isomorphism, and it implies that $\triangle''$ consists of the faces of $\boldsymbol{\tau}'$ by Remark 4.6. Hence, $\boldsymbol{\tau}' \subset \boldsymbol{\sigma}$ for some $\boldsymbol{\sigma} \in \triangle$. □

**Lemma 4.9.** *For $(\mathsf{N}, \boldsymbol{\sigma})$ in Fact 4.1, let $\phi \colon \mathsf{N}' \to \mathsf{N}$ be an injective homomorphism of free abelian groups of rank 2, and let $\boldsymbol{\sigma}'$ be a 2-dimensional closed strictly convex rational polyhedral cone of $\mathsf{N}' \otimes \mathbb{R}$ such that $\phi_{\mathbb{R}}(\boldsymbol{\sigma}') \subset \boldsymbol{\sigma}$ for the isomorphism $\phi_{\mathbb{R}} = \phi \otimes \mathbb{R} \colon \mathsf{N}' \otimes \mathbb{R} \to \mathsf{N} \otimes \mathbb{R}$. As in Fact 4.1, we write $\boldsymbol{\sigma}' = \mathbb{R}_{\geq 0} e'_1 + \mathbb{R}_{\geq 0} e'_2$ for two primitive elements $e'_1$ and $e'_2$ of $\mathsf{N}'$ which form a basis of $\mathsf{N}' \otimes \mathbb{R}$. Let $\pi \colon \mathbb{T}_{\mathsf{N}'}(\boldsymbol{\sigma}') \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ be the toric morphism $\mathbb{T}(\phi)$. Then*

$$\pi^*\Gamma(e_1) = a_{11}\Gamma(e'_1) + a_{12}\Gamma(e'_2) \quad \textit{and} \quad \pi^*\Gamma(e_2) = a_{21}\Gamma(e'_1) + a_{22}\Gamma(e'_2)$$

*for non-negative integers $a_{ij}$ defined by*

$$(\phi(e'_1), \phi(e'_2)) = (e_1, e_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

*Moreover, $\sharp \mathsf{N}/\phi(\mathsf{N}') = (n/n')|a_{11}a_{22} - a_{12}a_{21}|$ for the order $n'$ of $(\mathsf{N}', \boldsymbol{\sigma}')$.*

*Proof.* Let $(e_1^\vee, e_2^\vee)$ be the dual basis of $(e_1, e_2)$ in $\mathsf{M} \otimes \mathbb{R}$ and let $(e_1'^\vee, e_2'^\vee)$ be the dual basis of $(e'_1, e'_2)$ in $\mathsf{M}' \otimes \mathbb{R}$, where $\mathsf{M}' = \operatorname{Hom}_{\mathbb{Z}}(\mathsf{M}, \mathbb{Z})$. Let $\phi^\vee \colon \mathsf{M} \to \mathsf{M}'$ be the dual homomorphism of $\phi$. Then $\phi^\vee_{\mathbb{R}} = \phi^\vee \otimes \mathbb{R}$ is given by

$$(\phi^\vee_{\mathbb{R}}(e_1^\vee), \phi^\vee_{\mathbb{R}}(e_2^\vee)) = (e_1'^\vee, e_2'^\vee)\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}.$$

For $i = 1$, 2, let $k_i$ be a positive integer such that $k_i e_i^\vee \in \mathsf{M}$. Then

$$\pi^*\boldsymbol{e}(k_i e_i^\vee) = \boldsymbol{e}(\phi^\vee(k_i e_i^\vee)) = \boldsymbol{e}(k_i a_{i1} e_1'^\vee)\boldsymbol{e}(k_i a_{i2} e'^\vee).$$

By Remark 4.4, we have $\operatorname{div}(\boldsymbol{e}(k_i e_i^\vee)) = k_i\Gamma(e_i)$, and hence,

$$k_i\pi^*\Gamma(e_i) = \operatorname{div}(\pi^*\boldsymbol{e}(k_i e_i^\vee)) = k_i a_{i1}\Gamma(e'_1) + k_i a_{i2}\Gamma(e'_2)$$

for $i = 1$, 2. For the second assertion, we choose an element of $\mathsf{N}'$ of the form $u' = (1/n')(e'_1 + q'e'_2)$ such that $\mathsf{N}' = \mathbb{Z}u' + \mathbb{Z}e'_2$. Then

$$\begin{aligned}(\phi(u'), \phi(e'_2)) &= (\phi(e'_1), \phi(e'_2))\begin{pmatrix} 1/n' & 0 \\ q'/n' & 1 \end{pmatrix} = (e_1, e_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1/n' & 0 \\ q'/n' & 1 \end{pmatrix} \\ &= (u, e_2)\begin{pmatrix} 1/n & 0 \\ q/n & 1 \end{pmatrix}^{-1}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1/n' & 0 \\ q'/n' & 1 \end{pmatrix}.\end{aligned}$$

By taking determinants of matrices above, we have the equality for $\sharp \mathsf{N}/\phi(\mathsf{N}')$. □

**Lemma 4.10.** *For $(\mathsf{N}, \boldsymbol{\sigma})$ in Fact 4.1, let $f \colon \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ be the finite surjective toric morphism $\mathbb{T}(\phi)$ associated with an injective homomorphism $\phi \colon \mathsf{N} \to \mathsf{N}$ such that $\phi_{\mathbb{R}}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}$. Then, for the composite $g = f \circ f$, there exist positive integers $d_1$ and $d_2$ such that*

$$\deg g = d_1 d_2, \quad g^*\Gamma_1 = d_1\Gamma_1, \quad g^*\Gamma_2 = d_2\Gamma_2, \quad \textit{and} \quad d_1 \equiv d_2 \bmod n,$$

*where $\Gamma_1 = \Gamma(e_1)$ and $\Gamma_2 = \Gamma(e_2)$ are prime components of the boundary divisor of $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$, and $n$ is the order of $(\mathsf{N}, \boldsymbol{\sigma})$. Here, if $d_1 = d_2$, then $\phi^2 = \phi \circ \phi$ is the multiplication map by $d_1$.*

*Proof.* By Lemma 4.9, there exist positive integers $d_1$ and $d_2$ such that $\deg g = d_1 d_2$, $g^*\Gamma_i = d_i\Gamma_i$, and $\phi^2(e_i) = d_i e_i$ for $i = 1, 2$. Moreover, for the primitive element $u$ in Fact 4.1, we have

$$\phi^2(u) = (1/n)(d_1e_1 + qd_2e_2) = d_1u + (q/n)(d_2 - d_1)e_2 \in \mathsf{N}.$$

Thus, $d_1 \equiv d_2 \bmod n$. If $d_1 = d_2$, then $\phi^2(u) = d_1 u$, i.e., $\phi^2$ is the multiplication map by $d_1$. □

4.2. **Lifting endomorphisms to certain cyclic covers.** There is a well-known construction of cyclic covers of normal varieties due to Esnault [5, §1] and Viehweg [57, §1]. A similar construction can be found in [43, §5] and [3]. We shall present another construction of cyclic covers from a $\mathbb{Q}$-divisor whose multiple is principal: This is called an *index* 1 *cover* (cf. Definition 4.18(2) below), which is a generalization of the same cover considered in [30]. As a byproduct, we shall give a sufficient condition for an endomorphism of a variety to lift to a cyclic cover (cf. Lemma 4.21). In Section 4.2, varieties are not necessarily 2-dimensional.

**Definition 4.11.** For a normal complex analytic variety $X$ and a $\mathbb{Q}$-divisor $L$ on $X$, assume that $mL$ is a principal divisor for a positive integer $m$; hence, we have an isomorphism $s\colon \mathcal{O}_X(mL) \xrightarrow{\simeq} \mathcal{O}_X$. We consider the $\mathcal{O}_X$-module

$$\mathcal{R}(L, m, s) := \bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_X(\llcorner iL \lrcorner)$$

and endow it an $\mathcal{O}_X$-algebra structure by homomorphisms

$$\tilde{\mu}_{i,j}\colon \mathcal{O}_X(\llcorner iL \lrcorner) \otimes \mathcal{O}_X(\llcorner jL \lrcorner) \to \mathcal{O}_X(\llcorner m\langle \tfrac{i+j}{m}\rangle L \lrcorner)$$

defined as follows for integers $0 \le i, j < m$: If $i + j < m$, then $\tilde{\mu}_{i,j}$ is just the composite

$$\mu_{i,j}\colon \mathcal{O}_X(\llcorner iL \lrcorner) \otimes \mathcal{O}_X(\llcorner jL \lrcorner) \to \mathcal{O}_X(\llcorner iL \lrcorner + \llcorner jL \lrcorner) \to \mathcal{O}_X(\llcorner (i+j)L \lrcorner),$$

where the first homomorphism is given by taking the double-dual and the second one is induced by the inequality $\llcorner iL \lrcorner + \llcorner jL \lrcorner \le \llcorner (i+j)L \lrcorner$ of divisors. If $i + j \ge m$, then $\tilde{\mu}_{i,j}$ is the composite

$$\mathcal{O}_X(\llcorner iL \lrcorner) \otimes \mathcal{O}_X(\llcorner jL \lrcorner) \xrightarrow{\mu_{i,j}} \mathcal{O}_X(\llcorner (i+j)L \lrcorner) \xrightarrow{\otimes s} \mathcal{O}_X(\llcorner (i+j-m)L \lrcorner).$$

The associated finite morphism $\pi\colon \mathbb{V}(L, m, s) := \mathbf{Specan}_X\, \mathcal{R}(L, m, s) \to X$ is called the *cyclic cover* with respect to $(L, m, s)$. For **Specan**, see [7, §1.14]. Note that $\mathcal{R}(L, m, s) = \mathcal{O}_X$ and $\mathbb{V}(L, m, s) = X$ when $m = 1$.

*Remark.* For the variety $X$ above, let $H$ be a Cartier divisor on $X$ with a non-zero global section $\sigma$ of $\mathcal{O}_X(mH)$ for an integer $m > 1$. Then the effective divisor $D = \operatorname{div}(\sigma)$, the *divisor of zeros* of $\sigma$, is linearly equivalent to $mH$, and $\sigma$ induces an isomorphism $\mathcal{O}_X(D) \simeq \mathcal{O}_X(mH)$. We set $L := (1/m)D - H$ as a $\mathbb{Q}$-divisor, and set $s\colon \mathcal{O}_X(mL) = \mathcal{O}_X(D - mH) \to \mathcal{O}_X$ to be the isomorphism induced by $\sigma$. Then $\mathbb{V}(L, m, s)$ is just the usual cyclic cover associated with $(H, m, \sigma)$ in the sense of Esnault [5, §1] and Viehweg [57, (1.1)]. Conversely, for $(L, m, s)$ in Definition 4.11, we set $H := -\llcorner L \lrcorner$ and $D := m\langle L \rangle$. Then $\mathbb{V}(L, m, s)$ coincides with the cyclic cover in the sense of Esnault and Viehweg defined by a section $\sigma$ of $\mathcal{O}_X(mH)$ such that $\operatorname{div}(\sigma) = D$.

*Remark* 4.12. The $\mathcal{O}_X$-algebra $\mathcal{R}(L, m, s)$ is graded by $\mathbb{Z}/m\mathbb{Z}$. Hence, $\mathbb{V}(L, m, s)$ admits an action of the group $\boldsymbol{\mu}_m$ of $m$-th roots of unity over $X$. The action of $\zeta \in \boldsymbol{\mu}_m$ is defined by multiplication maps $\mathcal{O}_X(\llcorner iL \lrcorner) \to \mathcal{O}_X(\llcorner iL \lrcorner)$ by $\zeta^i$. For an open subset $U$ such that $L|_U$ is Cartier, we know that $\mathbb{V}(L|_U, m, s) \to U$ is a $\boldsymbol{\mu}_m$-torsor by [17, Prop. 4.1]. For another isomorphism $s' \colon \mathcal{O}_X(mL) \xrightarrow{\simeq} \mathcal{O}_X$, there is a $\boldsymbol{\mu}_m$-equivariant isomorphism $\mathbb{V}(L, m, s') \simeq \mathbb{V}(L, m, s)$ over $X$ if and only if $s' = \varepsilon^m s$ for a nowhere vanishing function $\varepsilon$ on $X$.

**Lemma 4.13.** *Let $X$ be a non-singular variety with a non-zero holomorphic function $t$ such that the principal divisor $D = \operatorname{div}(t)$ is non-zero and non-singular. For integers $0 < a < m$, we define $L := (a/m)D$ as a $\mathbb{Q}$-divisor on $X$, and consider $t^a$ as a nowhere vanishing section of $\mathcal{O}_X(-mL) = \mathcal{O}_X(-aD) = \mathcal{O}_X t^a$. Then*

$$\mathcal{R}(L, m, t^a) \simeq \mathcal{O}_X[\mathtt{u}, \mathtt{y}]/(\mathtt{u}^d - 1, \mathtt{y}^{m'} - t)\mathcal{O}_X[\mathtt{u}, \mathtt{y}] \tag{IV-1}$$

*as an $\mathcal{O}_X$-algebra for integers $d := \gcd(a, m)$ and $m' := m/d$, where* $\mathtt{u}$ *and* $\mathtt{y}$ *are variables. In particular, $\mathbb{V}(L, m, t^a)$ is non-singular and is a disjoint union of $d$-copies of $\mathbb{V}((1/m')D, m', t)$.*

*Proof.* Let $\mathcal{B}$ be the $\mathcal{O}_X$-algebra in the right hand side of (IV-1), and let us consider an $\mathcal{O}_X$-algebra

$$\mathcal{A} := \mathcal{O}_X[\mathtt{z}]/(\mathtt{z}^m - t^a)\mathcal{O}_X[\mathtt{z}]$$

for a variable $\mathtt{z}$. Then there an $\mathcal{O}_X$-algebra homomorphism $\mathcal{A} \to \mathcal{B}$ given by $\mathtt{z} \mapsto \mathtt{u}\mathtt{y}^{a'}$ for $a' := a/d$. Since $m'(ai/m) = a'i$ for any $i \in \mathbb{Z}$, we have

$$(\mathtt{u}\mathtt{y}^{a'})^i = \mathtt{u}^i \mathtt{t}^{\llcorner ai/m \lrcorner} \mathtt{y}^{m'\langle ai/m \rangle}$$

for any $i$, and the correspondence

$$i \mapsto (i \bmod d,\ m'\langle ai/m \rangle \bmod m')$$

gives rise to a bijection $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/m'\mathbb{Z}$. Hence, $\mathcal{A} \to \mathcal{B}$ is isomorphic to the canonical injection

$$\bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_X \mathtt{z}^i \to \bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_X t^{-\llcorner ia/m \lrcorner} \mathtt{z}^i.$$

As a consequence, we have (IV-1), i.e., $\mathcal{B} \simeq \mathcal{R}(L, m, t^a)$. The last assertion is deduced from the isomorphism

$$\mathbb{V}(L, m, s) = \mathbf{Specan}_X \mathcal{B} \simeq \boldsymbol{\mu}_d \times \mathbb{V}((1/m')D, m', t)$$

with a property that $\mathbb{V}((1/m')D, m', t) \simeq \mathbf{Specan}_X \mathcal{O}_X[\mathtt{y}]/(\mathtt{y}^{m'} - t')\mathcal{O}_X[\mathtt{y}]$ is non-singular. ☐

**Lemma 4.14.** *Let $\pi \colon \mathbb{V} = \mathbb{V}(L, m, s) \to X$ be the cyclic cover in Definition 4.11 in which $m > 1$. Then $\mathbb{V}$ is normal, $\pi^*L$ is a principal divisor on $\mathbb{V}$, and $\mathcal{O}_{\mathbb{V}}(\pi^*L)$ has a $\boldsymbol{\mu}_m$-linearization such that the associated $\mathbb{Z}/m\mathbb{Z}$-graded $\mathcal{R}(L, m, s)$-module $\pi_*\mathcal{O}_{\mathbb{V}}(l\pi^*L)$ is isomorphic to the twist $\mathcal{R}(L, m, s)(l)$ by $l$ for any $l \in \mathbb{Z}$, i.e.,*

$$\pi_*\mathcal{O}_{\mathbb{V}}(l\pi^*L) \simeq \bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_X(\llcorner (l+i)L \lrcorner). \tag{IV-2}$$

*Here, the image $v$ of $1$ by the injection*

$$\mathcal{O}_X = \mathcal{R}(L, m, s)(-1)_1 \subset \mathcal{R}(L, m, s)(-1) \simeq \pi_*\mathcal{O}_{\mathbb{V}}(-\pi^*L)$$

*is a nowhere vanishing section of $\mathcal{O}_{\mathbb{V}}(-\pi^*L)$ satisfying $\pi^*s = v^m$. If $X$ and $\operatorname{Supp}\langle L\rangle$ are non-singular, then $\mathbb{V}$ is also non-singular.*

*Proof.* We set $X^\circ := X \setminus (\operatorname{Sing} X \cup \operatorname{Sing}\operatorname{Supp}\langle L\rangle)$. For a given point $x \in X^\circ \cap \operatorname{Supp}\langle L\rangle$, we have an open neighborhood $U$ of $x$ and a non-zero holomorphic function $t$ on $U$ such that

- $\operatorname{div}(t)$ is non-singular,
- $\langle L\rangle|_U = (a/m)\operatorname{div}(t)$ for an integer $0 < a < m$, and
- $s|_U = \varepsilon^m t^a$ as a section of $\mathcal{O}_X(-mL)|_U$ for a nowhere vanishing section $\varepsilon$ of $\mathcal{O}_X(-m\llcorner L\lrcorner)|_U$.

In particular, $\mathbb{V}|_U \simeq \mathbb{V}((a/m)\operatorname{div}(t), m, t^a)$ by Remark 4.12 and it is non-singular by Lemma 4.13. Hence, $\mathbb{V}^\circ := \pi^{-1}(X^\circ)$ is non-singular, since $\mathbb{V} \to X$ is a $\boldsymbol{\mu}_m$-torsor over $X^\circ \setminus \operatorname{Supp}\langle L\rangle$ (cf. Remark 4.12). This shows the last assertion. For open immersions $j\colon X^\circ \hookrightarrow X$ and $j'\colon \mathbb{V}^\circ \hookrightarrow \mathbb{V}$, we have isomorphisms $\mathcal{R}(L, m, s) \simeq j_*(\mathcal{R}(L, m, s)|_{X^\circ})$ and $\mathcal{O}_{\mathbb{V}} \simeq j'_*\mathcal{O}_{\mathbb{V}^\circ}$, since $\mathcal{R}(L, m, s)$ is a reflexive $\mathcal{O}_X$-module and $\operatorname{codim}(X \setminus X^\circ, X) \geq 2$. Therefore, $\mathbb{V}$ is normal.

For the rest, by Hartogs' lemma, we may assume that $X$ and $\operatorname{Supp}\langle L\rangle$ are non-singular, by replacing $X$ with $X^\circ$. Let

$$\psi\colon \mathcal{O}_X(\llcorner L\lrcorner) \to \mathcal{R}(L, m, s) = \bigoplus\nolimits_{i=0}^{m-1} \mathcal{O}_X(\llcorner iL\lrcorner) = \pi_*\mathcal{O}_{\mathbb{V}}$$

be the canonical injection from the factor of $i = 1$. Let $\delta_m\colon \mathcal{O}_X(m\llcorner L\lrcorner) \hookrightarrow \mathcal{O}_X(mL)$ be the inclusion corresponding to the inequality $m\llcorner L\lrcorner \leq mL$ of divisors and let $p_m\colon (\pi_*\mathcal{O}_{\mathbb{V}})^{\otimes m} \to \pi_*\mathcal{O}_{\mathbb{V}}$ be the homomorphism defined by $m$-times products in the $\mathcal{O}_X$-algebra $\pi_*\mathcal{O}_{\mathbb{V}}$. Then the diagram

$$\text{(IV-3)}\qquad \begin{array}{ccccc} \mathcal{O}_X(m\llcorner L\lrcorner) & \xrightarrow{\delta_m} & \mathcal{O}_X(mL) & \xrightarrow[\simeq]{s} & \mathcal{O}_X \\ {\scriptstyle\simeq}\uparrow & & & & \downarrow \\ \mathcal{O}_X(\llcorner L\lrcorner)^{\otimes m} & \xrightarrow{\psi^{\otimes m}} & (\pi_*\mathcal{O}_{\mathbb{V}})^{\otimes m} & \xrightarrow{p_m} & \pi_*\mathcal{O}_{\mathbb{V}} \end{array}$$

is commutative, where the right vertical arrow indicates the canonical homomorphism of $\mathcal{O}_X$-algebra. Let

$$\varphi\colon \pi^*\mathcal{O}_{\mathbb{V}}(\llcorner L\lrcorner) \to \mathcal{O}_{\mathbb{V}}$$

be an injection corresponding to $\psi$ by adjunction for $(\pi^*, \pi_*)$. Then the image of $\varphi$ is the ideal sheaf $\mathcal{O}_{\mathbb{V}}(-E)$ of an effective Cartier divisor $E$ on $\mathbb{V}$. By (IV-3), the $m$-th power

$$\varphi^{\otimes m}\colon \pi^*\mathcal{O}_{\mathbb{V}}(\llcorner L\lrcorner)^{\otimes m} \to \mathcal{O}_{\mathbb{V}}^{\otimes m} = \mathcal{O}_{\mathbb{V}}$$

equals $(\pi^*s) \circ \pi^*\delta_m$, and hence,

$$mE = \pi^*(\llcorner mL\lrcorner - m\llcorner L\lrcorner) = m\pi^*\langle L\rangle.$$

Therefore, $E = \pi^*\langle L\rangle$, and $\pi^*L = \pi^*(\llcorner L \lrcorner) + E$ is a principal divisor. For an integer $n$, let us consider the diagram

$$
\text{(IV-4)} \qquad
\begin{tikzcd}
\pi_*\mathcal{O}_{\mathbb{V}}(-n\pi^*L) \arrow[r, dotted] \arrow[d, hook] & \bigoplus_{i=0}^{m-1}\mathcal{O}_X(\llcorner(i-n)L\lrcorner) \arrow[d, hook] \\
\pi_*\mathcal{O}_{\mathbb{V}}(-n\pi^*\llcorner L\lrcorner) \arrow[r, "\simeq"] & \bigoplus_{i=0}^{m-1}\mathcal{O}_X(-n\llcorner L\lrcorner) \otimes \mathcal{O}_X(\llcorner iL\lrcorner)
\end{tikzcd}
$$

of $\mathcal{R}(L, m, v)$-modules in which the bottom isomorphism is derived from the projection formula and vertical arrows are injections defined by inequalities $-\llcorner(i - n)L\lrcorner \leq -n\llcorner L\lrcorner + \llcorner iL\lrcorner$ of divisors for $0 \leq i < m$. We shall show that the dotted arrow exists as the isomorphism (IV-2) for $l = -n$ and it makes the diagram (IV-4)commutative. For the purpose, we can localize $X$ and we may assume that $L = (a/m)D$, $D = \mathrm{div}(t)$, and $s = t^a$ as in Lemma 4.13. In this case, $\llcorner L\lrcorner = 0$, $\pi^*L = aE$, $E = \mathrm{div}(\mathtt{z})$ for $\mathtt{z} = \mathtt{u}\mathtt{y}^{a'}$ in the proof of Lemma 4.13, and the diagram (IV-4) is expressed as

$$
\begin{tikzcd}
(\mathtt{u}\mathtt{y}^{a'})^n\mathcal{O}_X[\mathtt{u},\mathtt{y}]/(\mathtt{u}^d - 1, \mathtt{y}^{m'} - 1)\mathcal{O}_X[\mathtt{u},\mathtt{y}] \arrow[r, dotted] \arrow[d, hook] & \bigoplus_{i=0}^{m-1}\mathcal{O}_X t^{-\llcorner(i-n)a/m\lrcorner}\mathtt{z}^i \arrow[d, hook] \\
\mathcal{O}_X[\mathtt{u},\mathtt{y}]/(\mathtt{u}^d - 1, \mathtt{y}^{m'} - 1)\mathcal{O}_X[\mathtt{u},\mathtt{y}] \arrow[r, "\simeq"] & \bigoplus_{i=0}^{m-1}\mathcal{O}_X t^{-\llcorner ia/m\lrcorner}\mathtt{z}^i.
\end{tikzcd}
$$

Thus, we have the dotted arrow as an isomorphism making the diagram commutative. As a consequence, $\pi_*\mathcal{O}_{\mathbb{V}}(l\pi^*L) \simeq \mathcal{R}(L, m, v)(l)$ for any $l \in \mathbb{Z}$.

For the section $v$ of $\mathcal{O}_{\mathbb{V}}(-\pi^*L)$ in the statement, the section $v^m$ of $\mathcal{O}_{\mathbb{V}}(-m\pi^*L)$ corresponds to the section $s$ of $\mathcal{O}_X(-mL)$ by the isomorphism

$$\pi_*\mathcal{O}_{\mathbb{V}}(-m\pi^*L) \simeq \mathcal{R}(L, m, s)(-m) \simeq \mathcal{R}(L, m, s) \otimes \mathcal{O}_X(-mL).$$

Thus, $\pi^*s = v^m$, and we are done. □

**Corollary 4.15.** *The cyclic cover $\mathbb{V} = \mathbb{V}(L, m, s)$ is reducible if and only if there exist a positive integer $k$ and a nowhere vanishing section $w$ of $\mathcal{O}_X(-kL)$ such that $k < m$, $k \mid m$, $kL$ is Cartier, and $s = w^{m/k}$. When $\mathbb{V}$ is irreducible,*

$$\text{(IV-5)} \qquad K_{\mathbb{V}} = \pi^*\left(K_X + \sum\nolimits_i (1 - 1/e_i)\Gamma_i\right)$$

*for the prime components $\Gamma_i$ of $\langle L\rangle$ and for the denominator $e_i$ of the rational number* $\mathrm{mult}_{\Gamma_i} L$.

*Proof.* We may assume that $X$ and $\mathrm{Supp}\langle L\rangle$ are non-singular as in the proof of Lemma 4.14. The second assertion is reduced to the case where $L = (1/m)D$ for $D = \mathrm{div}(t)$ in Lemma 4.13, and we have (IV-5) from the ramification formula for the cyclic cover $\mathbf{Specan}_X\, \mathcal{O}_X[\mathtt{y}]/(\mathtt{y}^m - t)\mathcal{O}_X[\mathtt{y}] \to X$. For the first assertion, it is enough to prove the "only if" part, since the "if" part is shown by the isomorphism

$$\mathbb{V}(L, m, s) \simeq \boldsymbol{\mu}_{m/k} \times \mathbb{V}(L, k, w).$$

Assume that $\mathbb{V}$ is reducible, and let $Y$ be an irreducible component of $\mathbb{V}$. Then $Y \cap \pi^{-1}(X^\star)$ is a connected component of the $\boldsymbol{\mu}_m$-torsor $\pi^{-1}(X^\star)$ over $X^\star :=$

$X \setminus (\operatorname{Sing} X \cup \operatorname{Supp}\langle L\rangle)$ (cf. Remark 4.12). Let $H \subset \boldsymbol{\mu}_m$ be the subgroup consisting elements $\zeta \in \boldsymbol{\mu}_m$ such that $\zeta(Y) \subset Y$. Then $H$ is the Galois group of the Galois cover $\pi_Y = \pi|_Y : Y \to X$, the order $k := \sharp H$ divides $m$ and is less than $m$, and $\mathbb{V}$ is a disjoint union of $m/k$-copies of $Y$. Let $v$ be the nowhere vanishing section of $\mathcal{O}_{\mathbb{V}}(-\pi^* L)$ in Lemma 4.14. Then, for any $\zeta \in \boldsymbol{\mu}_m$, the pullback $\zeta^* v$ by the automorphism $\zeta \colon \mathbb{V} \to \mathbb{V}$ equals $\zeta v$ as a section of $\mathcal{O}_{\mathbb{V}}(-\pi^* L)$, since $v$ is contained in $\mathcal{R}(L, m, s)(-1)_1$. Thus,

$$(-1)^{k-1} \prod\nolimits_{\zeta \in H} \zeta^*(v|_Y) = (-1)^{k-1} (\prod\nolimits_{\zeta \in H} \zeta)(v^k|_Y) = (v|_Y)^k$$

is an $H$-invariant nowhere vanishing section of $\mathcal{O}_{\mathbb{V}}(-k\pi^* L) \otimes \mathcal{O}_Y \simeq \mathcal{O}_Y(-\pi_Y^*(kL))$. Hence, $kL$ is a principal divisor on $X$ with a nowhere vanishing section $w$ of $\mathcal{O}_X(-kL)$ satisfying $\pi_Y^*(w) = (v|_Y)^k$. Here, $w^{m/k} = s$ by $v^m = \pi^* s$. Thus, we are done. $\square$

**Lemma 4.16.** *Let $(X, L, m, s)$ be as in Definition 4.11 in which $m > 1$. Let $f \colon Y \to X$ be a morphism of* maximal rank *(cf. Definition 1.1) from a normal variety $Y$ such that* $\operatorname{codim}(f^{-1} \operatorname{Sing} X, Y) \geq 2$; *in this situation, one can consider the inverse image $f^* D$ for a divisor $D$ as a divisor on $Y$ by Lemma 1.19. Then the cyclic cover $\mathbb{V}(f^* L, m, f^* s) \to Y$ is isomorphic to the normalization of the fiber product $\mathbb{V}(L, m, s) \times_X Y$.*

*Proof.* For any $i \in \mathbb{Z}$, we have a composite homomorphism

$$\gamma_i \colon f^* \mathcal{O}_X(\llcorner iL \lrcorner) \xrightarrow{\alpha} \mathcal{O}_Y(f^* \llcorner iL \lrcorner) \xrightarrow{\beta} \mathcal{O}_Y(\llcorner i f^* L \lrcorner),$$

where $\alpha$ is the canonical homomorphism on the pullback (cf. Lemma 1.19) and $\beta$ corresponds to the inequality $f^*(\llcorner iL \lrcorner) \leq \llcorner i f^* L \lrcorner$ of divisors. Note that $Y' := Y \setminus f^{-1}(\operatorname{Sing} X \cup \operatorname{Supp}\langle L\rangle)$ is a non-empty open subset of $Y$ as $f$ is of maximal rank and that $\gamma_i$ is an isomorphism over $Y'$. The sum of $\gamma_i$ induces an $\mathcal{O}_Y$-algebra homomorphism

$$f^* \mathcal{R}(L, m, s) \to \mathcal{R}(f^* L, m, f^* s)$$

and the associated finite morphism $\mathbb{V}(f^* L, m, f^* s) \to \mathbb{V}(L, m, s) \times_X Y$ over $Y$, which is an isomorphism over $Y'$. Then the assertion holds, since $\mathbb{V}(f^* L, m, f^* s)$ is normal (cf. Lemma 4.14). $\square$

**Proposition 4.17.** *Let $(X, L, m, s)$ be as in Definition 4.11 in which $m > 1$. Let $f \colon X' \to X$ be a morphism of maximal rank from a normal variety $X'$ such that* $\operatorname{codim}(f^{-1} \operatorname{Sing} X, X') \geq 2$. *Let $L'$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X'$ such that $mL' \sim 0$ and $s'$ a nowhere vanishing section of $\mathcal{O}_{X'}(-mL')$. We set $\pi \colon \mathbb{V} := \mathbb{V}(L, m, s) \to X$ and $\pi' \colon \mathbb{V}' := \mathbb{V}(L', m, s') \to X'$ as the associated cyclic covers. For an integer $k$, assume that $f^* L \sim kL'$ and $f^* s = \varepsilon^m s'^k$ for a nowhere vanishing section $\varepsilon$ of $\mathcal{O}_{X'}(kL' - f^* L)$. Then:*

1. *There is a morphism $g \colon \mathbb{V}' \to \mathbb{V}$ such that $\pi \circ g = f \circ \pi'$ which is equivariant under the actions of $\boldsymbol{\mu}_m$ on $\mathbb{V}$ and $\mathbb{V}'$ explained in Remark 4.12, along the $k$-th power map $\boldsymbol{\mu}_m \to \boldsymbol{\mu}_m$ i.e.,*

$$g(\zeta x) = \zeta^k g(x)$$

*for any $x \in \mathbb{V}'$ and $\zeta \in \boldsymbol{\mu}_m$.*

(2) *If $k$ is coprime to $m$, then $\mathbb{V}'$ is isomorphic over $X'$ to the normalization of the fiber product $\mathbb{V} \times_X X'$.*

*Proof.* By Lemma 4.16, it suffices to construct a certain morphism $\mathbb{V}(L', m, s') \to \mathbb{V}(f^*L, m, f^*s)$ over $X'$. Thus, we may assume that $X' = X$ and $f = \mathrm{id}_X$. Moreover, by Remark 4.12, we may assume that $L = kL'$, $\varepsilon = 1$, and $s = s'^k$. By interchanging $L$ and $L'$, we are reduced to construct a morphism $g_k \colon \mathbb{V}(L, m, s) \to \mathbb{V}(kL, m, s^k)$ over $X$ such that

(a) it is equivariant along the $k$-th power map $\boldsymbol{\mu}_m \to \boldsymbol{\mu}_m$, and
(b) it is an isomorphism when $k$ is coprime to $m$.

For each $0 \le i < m$, by the equality $ik = m\llcorner ik/m \lrcorner + m\langle ik/m \rangle$ and by tensor product with $s^{\llcorner ik/m \lrcorner}$, we have an isomorphism

$$\varphi_i \colon \mathcal{O}_X(\llcorner (ik)L \lrcorner) \simeq \mathcal{O}_X(\llcorner m\langle ik/m\rangle L \lrcorner) \otimes \mathcal{O}_X(m\llcorner ik/m \lrcorner L) \to \mathcal{O}_X(\llcorner m\langle ik/m\rangle L \lrcorner).$$

For any $0 \le i, j < m$, the diagram

$$\begin{array}{ccc} \mathcal{O}_X(\llcorner ikL \lrcorner) \otimes \mathcal{O}_X(\llcorner jkL \lrcorner) & \xrightarrow{\varphi_i \otimes \varphi_j} & \mathcal{O}_X(\llcorner m\langle \frac{ik}{m}\rangle L \lrcorner) \otimes \mathcal{O}_X(\llcorner m\langle \frac{jk}{m}\rangle L \lrcorner) \\ {\scriptstyle \tilde{\mu}_{i,j}} \downarrow & & \downarrow {\scriptstyle \tilde{\mu}_{m\langle \frac{ik}{m}\rangle, m\langle \frac{jk}{m}\rangle}} \\ \mathcal{O}_X(\llcorner (m\langle \frac{i+j}{m}\rangle kL \lrcorner) & \xrightarrow{\varphi_{m\langle \frac{i+j}{m}\rangle}} & \mathcal{O}_X(\llcorner m\langle \frac{(i+j)k}{m}\rangle L \lrcorner) \end{array}$$

is commutative , where $\tilde{\mu}_{\cdot,\cdot}$ are homomorphisms defining $\mathcal{O}_X$-algebra structures of $\mathcal{R}(kL, m, s^k)$ and $\mathcal{R}(L, m, s)$ (cf. Definition 4.11) and where we use

$$m\langle (m\langle \tfrac{ik}{m}\rangle + m\langle \tfrac{jk}{m}\rangle)/m\rangle = m\langle \langle \tfrac{ik}{m}\rangle + \langle \tfrac{jk}{m}\rangle\rangle = m\langle \tfrac{(i+j)k}{m}\rangle.$$

Thus, the sum of $\varphi_i$ for all $0 \le i < m$ gives an $\mathcal{O}_X$-algebra homomorphism

$$\Phi_k \colon \mathcal{R}(kL, m, s^k) \to \mathcal{R}(L, m, s),$$

which corresponds to a finite morphism $g_k \colon \mathbb{V}(L, m, s) \to \mathbb{V}(kL, m, s^k)$ over $X$. It is equivariant along the $k$-th power map $\boldsymbol{\mu}_m \to \boldsymbol{\mu}_m$, since each $\varphi_i$ commutes with multiplication by

$$\zeta^{ik} = \zeta^{m\langle ik/m\rangle}$$

for any $\zeta \in \boldsymbol{\mu}_m$. This shows (a). If $k$ is coprime to $m$, then the correspondence $i \mapsto m\langle ik/m\rangle$ gives a permutation of $\{0, 1, \ldots, m-1\}$, which is identified with the $k$-th power map of $\boldsymbol{\mu}_m$, and hence, $\Phi_k$ and $g_k$ are isomorphisms. This shows (b), and we are done. $\square$

**Definition 4.18.** Let $X$ be a normal variety and $L$ a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$.

(1) The *Cartier* (resp. *torsion*) *index* of $L$ is either the smallest positive integer $r$ such that $rL$ is Cartier (resp. $rL \sim 0$), or $\infty$ if such $r$ does not exist. For a point $P \in X$, the *local Cartier index* of $L$ at $P$ is the smallest positive integer $r$ such that $rL$ is Cartier at $P$.
(2) A finite morphism $Y \to X$ is called an *index* 1 *cover* (or a *global index* 1 *cover*) with respect to $L$ if $Y \simeq \mathbb{V}(L, m, s)$ over $X$ for the torsion index $m$

of $L$ and an isomorphism $s\colon \mathcal{O}_X(mL) \xrightarrow{\simeq} \mathcal{O}_X$. Note that the index 1 cover is normal and irreducible by Lemma 4.14 and Corollary 4.15.

(3) For a point $P \in X$, a *local index* 1 *cover* with respect to $L$ and $P$ is an index 1 cover with respect to $L|_U$ for an open neighborhood $U$ of $P$ such that the torsion index of $L|_U$ equals the local Carter index of $L$ at $P$.

(4) For a point $P \in X$, an *index* 1 *cover* of the germ $(X, P)$ with respect to $L$ is a morphism $(\widetilde{X}, \widetilde{P}) \to (X, P)$ of germs (or the germ $(\widetilde{X}, \widetilde{P})$) induced by a local index 1 cover $\widetilde{X}$ with respect to $L$ and $P$ and for the point $\widetilde{P}$ lying over $P$.

*Remark* 4.19. Let $V = \mathbb{V}(L, m, s)$ and $V' = \mathbb{V}(L, m, s')$ be two index 1 covers with respect to $L$. Then $s = \alpha s'$ for a nowhere vanishing function $\alpha$ on $X$. We have a finite étale morphism $\tau\colon \widehat{X} \to X$ from a normal variety $\widehat{X}$ such that $\tau^*\alpha = \beta^m$ for a nowhere vanishing function $\beta$ on $\widehat{X}$. In fact, $\widehat{X}$ is given as a connected component of $\mathbb{V}(0, m, \alpha)$ (cf. Lemma 4.14). Then $V \times_X \widehat{X} \simeq V' \times_X \widehat{X}$ over $\widehat{X}$ by Remark 4.12. If $H^0(X, \mathcal{O}_X) \simeq \mathbb{C}$, then $\alpha$ is constant, $\widehat{X} \to X$ is an isomorphism, and hence, $V \simeq V'$ over $X$. Similarly, every point $P \in X$ has an open neighborhood $U$ such that $V \times_X U \simeq V' \times_X U$ over $U$. In particular, the index 1 cover of the germ $(X, P)$ with respect to $L$ is unique up to isomorphism.

*Remark.* In [30], an index 1 cover is considered only for $K_X + D \sim_{\mathbb{Q}} 0$, where $X$ is a normal surface and $D$ is a reduced divisor.

**Lemma 4.20.** *For $(X, L, m, s)$ in Definition* 4.11 *in which $m > 1$, let $\tau\colon Y \to X$ be a finite surjective morphism from a normal variety $Y$ such that $m = \deg \tau$ and $\tau^*L \sim 0$.*

(1) *If $H^0(X, \mathcal{O}_X) \simeq \mathbb{C}$ and if $m$ is the torsion index of $L$, then $\tau$ is an index* 1 *cover with respect to $L$.*

(2) *If $m$ is the local Cartier index of $L$ at a point $P \in X$, then $\tau^{-1}U \to U$ is a local index* 1 *cover with respect to $L$ and $P$ for an open neighborhood $U$ of $P$.*

*Proof.* We set $\pi\colon \mathbb{V} := \mathbb{V}(L, m, s) \to X$ as the associated cyclic cover over $X$. By assumption, there is a nowhere vanishing section $t$ of $\mathcal{O}_Y(-\tau^*L)$. Then $\tau^*s = \alpha t^m$ in $H^0(Y, \mathcal{O}_Y(-m\tau^*L))$ for a nowhere vanishing function $\alpha$ on $Y$. Suppose that $\alpha = \beta^m$ for a nowhere vanishing function $\beta$ on $Y$. Then $\tau^*s = (\beta t)^m$ and the normalization of $\mathbb{V} \times_X Y$ is isomorphic to

$$\mathbb{V}(\tau^*L, m, (\beta t)^m) \simeq \boldsymbol{\mu}_m \times \mathbb{V}(\tau^*L, 1, \beta t) \simeq \boldsymbol{\mu}_m \times Y$$

by Lemma 4.16 and Remark 4.12. Thus, there is a finite morphism $\theta\colon Y \to \mathbb{V}$ over $X$. If $\mathbb{V}$ is irreducible, then $\theta$ is an isomorphism, since $\mathbb{V}$ is normal (cf. Lemma 4.14) and since $\deg \tau = \deg \pi$. In the situation of (1), $H^0(Y, \mathcal{O}_Y) \simeq \mathbb{C}$, since it is integral over $H^0(X, \mathcal{O}_X) \simeq \mathbb{C}$ (cf. [7, §2.27, Integrity Lemma]); Hence, such $\beta$ exists and (1) holds, since $\mathbb{V}$ is irreducible (cf. Corollary 4.15).

In the situation of (2), by replacing $X$ with an open neighborhood of $P$, we may assume that $mL \sim 0$. Then $\pi\colon \mathbb{V} \to X$ is an index 1 cover with respect to

$L$, and moreover, $\pi^{-1}(U) \to U$ is an index 1 cover with respect to $L|_U$ for any open neighborhood $U$ of $P$. Thus, $\mathbb{V}$ and $\pi^{-1}(U)$ are irreducible. It remains to find an open neighborhood $U$ of $P$ and a function $\beta_U$ on $\tau^{-1}U$ such that $\alpha|_{\tau^{-1}U} = (\beta_U)^m$. This is shown by the finiteness of $\tau$ as follows: Now, $\tau^{-1}(P)$ is a finite set $\{Q_1, Q_2, \dots, Q_k\}$. For each $1 \le i \le k$, we have an open neighborhood $\mathcal{V}_i$ of $Q_i$ and a nowhere vanishing function $\beta_i$ on $\mathcal{V}_i$ such that $\bigcup_{i=1}^k \mathcal{V}_i$ is a disjoint union of $\mathcal{V}_i$ and that $\alpha|_{\mathcal{V}_i} = \beta_i^m$. Then $\tau^{-1}U \subset \bigcup_{i=1}^k \mathcal{V}_i$ for an open neighborhood $U$ of $P$, and functions $\beta_i$ defines a nowhere vanishing function $\beta_U$ on $\tau^{-1}U$ such that $\alpha|_{\tau^{-1}U} = (\beta_U)^m$. Thus, we are done. □

**Lemma 4.21.** *For a normal variety $X$ and its connected open subset $X^\circ$, let $f\colon X^\circ \to X$ be a non-degenerate morphism without exceptional divisor. Let $L$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ such that $rL \sim 0$ for an integer $r > 0$ and that $f^*L \sim kL|_{X^\circ}$ for an integer $k \in \mathbb{Z}$. Then the following holds for an index 1 cover $\pi\colon V \to X$ with respect to $L$.*

(1) *If $H^0(X^\circ, \mathcal{O}_{X^\circ}) \simeq \mathbb{C}$, then there is a morphism $g\colon V^\circ \to V$ such that $\pi \circ g = f \circ \pi^\circ$, where $V^\circ = \pi^{-1}V$ and $\pi^\circ = \pi|_{V^\circ}\colon V^\circ \to X^\circ$.*
(2) *For any point $P \in X^\circ$, there exist an open neighborhood $U$ of $P$ in $X^\circ$ and a morphism $g_U\colon V_U^\circ \to V$ such that $\pi \circ g_U = f \circ \pi_U^\circ$, where $V_U^\circ := \pi^{-1}(U)$ and $\pi_U^\circ := \pi|_{V_U^\circ}\colon V_U^\circ \to U \hookrightarrow X^\circ$.*
(3) *Assume that $k$ is coprime with respect to the torsion index of $L$. Then the morphism $g$ (resp. $g_U$) in (1) (resp. (2)) induces an isomorphism from $V^\circ$ (resp. $V_U^\circ$) to the normalization of $V \times_{X,f} X^\circ$ (resp. $(V \times_{X,f} X^\circ) \times_{X^\circ} U$).*

*Proof.* Let $m$ be the torsion index of $L$ and we write $V = \mathbb{V}(L, m, s)$ for a nowhere vanishing section $s$ of $\mathcal{O}_X(-mL)$. By $mf^*L \sim mkL|_{X^\circ}$, we have a nowhere vanishing section $\alpha$ of $\mathcal{O}_{X^\circ}(m(kL|_{X^\circ} - f^*L))$ such that $f^*s = \alpha s^k|_{X^\circ}$. For an open subset $U$ of $X^\circ$, assume that

($*$) $\alpha|_U = \beta_U^m$ for a nowhere vanishing section $\beta$ of $\mathcal{O}_{X^\circ}(kL|_{X^\circ} - f^*L)|_U$.

Then there is a morphism $g_U\colon V_U^\circ = \pi^{-1}(U) \to V$ such that $\pi \circ g_U = f \circ \pi_U^\circ$ by Proposition 4.17(1), since $j^*(f^*s) = (\beta_U)^m s^k|_U$ for the open immersion $j\colon U \hookrightarrow X$. Moreover, if $k$ is coprime to $m$, then $V_U^\circ$ is isomorphic to the normalization of $V \times_{X, f\circ j} U$ by Proposition 4.17(2). Thus, it is enough to verify ($*$) for $U = X$ in case (1) and for an open neighborhood $U$ of $P$ in case (2). This is trivial in case (2), and this is deduced from $\alpha \in \mathbb{C}$ in case (1). □

*Remark.* In (1), if $X^\circ = X$, then $g\colon V \to V$ is a lift of the endomorphism $f\colon X \to X$. In (2), if the torsion index of $L$ equals the local Cartier index of $L$ at $P$, then $V \to X$ and $V_U^\circ \to U$ are local index 1 covers with respect to $L$ and $P$.

4.3. **Essential blowings up of log-canonical pairs.** We shall introduce the notion of an essential blowing up for a log-canonical pair $(X, S)$ of a normal surface $X$ and a reduced divisor $S$. This generalizes the notion of toroidal blowing up of a toroidal pair (cf. [40, §4.3]). We begin with some preliminary results on $\llcorner B \lrcorner$ for log-canonical pairs $(X, B)$.

**Lemma 4.22.** *Let $X$ be a normal surface with an effective $\mathbb{Q}$-divisor $B$ such that $(X, B)$ is log-canonical. Let $f : Y \to X$ be a bimeromorphic morphism from a normal surface $Y$ and let $B_f$ and $T_f$ be the positive and negative parts, respectively, of the prime decomposition of $f^*B - R_f$, i.e., $K_Y + B_f = f^*(K_X + B) + T_f$. Then $\llcorner B_f \lrcorner = D + D'$ for two reduced divisors $D$ and $D'$, which might be zero, such that*

- *$D \cap D' = \emptyset$, $f(D) = \operatorname{Supp} \llcorner B \lrcorner$, $f(D') \cap \operatorname{Supp} \llcorner B \lrcorner = \emptyset$,*
- *$f(D')$ is at most 0-dimensional, and*
- *$f$ induces an isomorphism $\mathcal{O}_{\llcorner B \lrcorner} \simeq f_*\mathcal{O}_D$ when $\llcorner B \lrcorner \neq 0$.*

*Proof.* Since $T_f - B_f - K_Y = -f^*(K_X + B)$ is $f$-nef, we have

$$R^1 f_* \mathcal{O}_Y(\ulcorner T_f \urcorner - \llcorner B_f \lrcorner) = 0 \tag{IV-6}$$

by Proposition 2.15. We set $T := \ulcorner T_f \urcorner$ and $C := \llcorner B_f \lrcorner$, and let $\mathcal{F}$ be the cokernel of the canonical injection $\mathcal{O}_Y(T - C) \to \mathcal{O}_Y(T)$. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_Y(-C) \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d] & \mathcal{O}_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_Y(T - C) \arrow[r] & \mathcal{O}_Y(T) \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

of exact sequences of sheaves on $Y$, where vertical arrows are all injective, since $\mathcal{O}_Y(T - C) \cap \mathcal{O}_Y = \mathcal{O}_Y(-C)$ as a subsheaf of $\mathcal{O}_Y(T)$. By (IV-6) and by applying $f_*$ to this diagram, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & f_*\mathcal{O}_Y(-C) \arrow[r] \arrow[d] & \mathcal{O}_X \simeq f_*\mathcal{O}_Y \arrow[r] \arrow[d, "\alpha"'] & f_*\mathcal{O}_C \arrow[d, "\beta"] & \\
0 \arrow[r] & f_*\mathcal{O}_Y(T - C) \arrow[r] & f_*\mathcal{O}_Y(T) \arrow[r] & f_*\mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

of exact sequences of sheaves on $X$. Here,$\alpha$ is an isomorphism as $T$ is $f$-exceptional. Hence, $\beta$ is an isomorphism and $\mathcal{O}_X \to f_*\mathcal{O}_C$ is surjective. On the other hand, we have $f_*B_f = B$ as a $\mathbb{Q}$-divisor on $X$ by applying $f_*$ to $K_Y + B_f = f^*(K_X + B) + T_f$. Thus, $f_*C = \llcorner B \lrcorner$. In other words, the ideal sheaf $\mathcal{O}_X(-\llcorner B \lrcorner)$ equals the double dual of $f_*\mathcal{O}_Y(-C)$. Therefore, there is a surjection $f_*\mathcal{O}_C \to \mathcal{O}_{\llcorner B \lrcorner}$ which is an isomorphism outside a discrete set $Z$. Since $C$ is reduced, $\llcorner B \lrcorner \cap Z = \emptyset$. Thus, $C = D + D'$ for reduced divisors $D$ and $D'$ such that $D \cap D' = \emptyset$ and $f(D') \subset Z$ and $f(D) = \llcorner B \lrcorner$ with an isomorphism $f_*\mathcal{O}_D \simeq \mathcal{O}_{\llcorner B \lrcorner}$. □

**Lemma 4.23.** *In the situation of Lemma 4.22, the following hold for any point $x \in \llcorner B \lrcorner$:*

1. *If $(X, B)$ is 1-log-terminal at $x$, then $f|_D : D \to \llcorner B \lrcorner$ is an isomorphism over an open neighborhood of $x$.*
2. *If $x \in \operatorname{Sing} \llcorner B \lrcorner$ and if $f^{-1}(x)$ is a divisor contained in $\llcorner B_f \lrcorner$, then $f$ is a toroidal blowing up with respect to $(X, \llcorner B \lrcorner)$ over an open neighborhood of $x$.*

*Proof.* (1): By shrinking $X$, we may assume that $(X, B)$ is 1-log-terminal and that $D = \llcorner B_f \lrcorner$. Then $D$ is just the proper transform of $\llcorner B \lrcorner$ in $Y$. Hence, $D \to \llcorner B \lrcorner$ is a finite surjective morphism. This is in fact an isomorphism by $\mathcal{O}_{\llcorner B \lrcorner} \simeq f_*\mathcal{O}_D$.

(2): We know that $(X, \llcorner B \lrcorner)$ is log-canonical and that $B = \llcorner B \lrcorner$ on an open neighborhood of $x$ by Lemma 2.6. By shrinking $X$, we may assume that $B$ is reduced. Moreover, we may assume that $D = B_f$ and $\operatorname{Supp} D = (\operatorname{Supp} f^{[*]}B) \cup f^{-1}(x)$, since $f^{-1}(x) \subset \llcorner B_f \lrcorner$. In particular, $K_Y + D = f^*(K_X + B)$. Since $(X, B)$ is toroidal at $x$ by Fact 2.5(1), we may assume also that $K_X + B$ is Cartier. Thus, $K_Y + D$ is also Cartier. If $\Gamma$ is a prime component of $f^{-1}(x)$, then $(K_Y + D)\Gamma = 0$, and one of the cases (A), (B), (C), and (D) of [40, Prop. 3.29] occurs for $\Gamma$ as $C$. Now, $\Gamma \cap (D - \Gamma) \neq \emptyset$, since $f^{-1}(x)$ is connected and $f^{-1}(x) \cap f^{[*]}B \neq \emptyset$. Hence, only the case (C) occurs, and we have $\sharp(D - \Gamma) \cap \Gamma = 2$ and $\Gamma \cap \operatorname{Sing} Y \subset \Gamma \cap (D - \Gamma) \subset \operatorname{Sing} D$. On the other hand, $f^{[*]}B \cap \operatorname{Sing} Y \subset f^{[*]}B \cap f^{-1}(x) \subset \operatorname{Sing} D$ by (1). Thus, $(Y, D)$ is toroidal along $D$, and $f$ is a toroidal blowing up by [40, Prop. 4.21]. □

**Definition 4.24.** Let $(X, B)$ be a log-canonical pair of a normal surface $X$ and a reduced divisor $B$. A bimeromorphic morphism $f\colon Y \to X$ from a normal surface $Y$ is called an *essential blowing up* of $(X, B)$ if $K_Y + B_Y = f^*(K_X + B)$ for a reduced divisor $B_Y$ such that

- $B_Y$ contains the $f$-exceptional locus, and
- $(Y, B_Y)$ is 1-log-terminal outside $\operatorname{Sing} B_Y$, i.e., $(U, B_Y|_U)$ is 1-log-terminal for $U = Y \setminus \operatorname{Sing} B_Y$.

In this case, we say also that $f\colon (Y, B_Y) \to (X, B)$ is an essential blowing up. Furthermore, if $B = 0$, then $X$ has only log-canonical singularities, and we call $f$ an essential blowing up of $X$.

*Remark.* The pairs $(Y, B_Y)$ is log-canonical by Lemma 2.10(1). We have $B_Y \supset f^{-1}B$, since $f_*B_Y = B$ and since $B_Y$ contains the $f$-exceptional locus. If $(X, B = 0)$ is log-terminal, then any essential blowing up of $X$ is an isomorphism.

**Lemma 4.25.** *For a normal surface $X$ with a reduced divisor $B$, suppose that $(X, B)$ is log-canonical and that $(X, B)$ is 1-log-terminal outside $\operatorname{Sing} B$. Let $f\colon Y \to X$ be a bimeromorphic morphism from a normal surface $Y$. Then the following conditions are equivalent*:

(i) *$f$ is an essential blowing up of $(X, B)$;*
(ii) *$f$ is a toroidal blowing up with respect to $(X, B)$;*
(iii) *there is a reduced divisor $B_Y$ on $Y$ such that $K_Y + B_Y = f^*(K_X + B)$ and that $B_Y$ contains the $f$-exceptional locus.*

*Proof.* We have (i) ⇒ (iii) by Definition 4.24.

(iii) ⇒ (ii): In (iii), any $f$-exceptional prime divisor is contained in $B_Y$, and it is contracted to a point of $\operatorname{Sing} B$ by $K_Y + B_Y = f^*(K_X + B)$, since $(X, B)$ is 1-log-terminal outside $\operatorname{Sing} B$. Thus, $f$ is an isomorphism over $X \setminus \operatorname{Sing} B$, and (ii) is satisfied by Lemma 4.23(2).

(ii) ⇒ (i): In (ii), we have $K_Y + B_Y = f^*(K_X + B)$ for $B_Y := f^{-1}B$, where $B_Y$ contains the $f$-exceptional locus. Let $P$ be a point of $X$ over which $f$ is not an isomorphism. Then $(Y, B_Y)$ is toroidal along $f^{-1}(P)$, and $(Y, B_Y)$ is 1-log-terminal along $f^{-1}(P) \setminus \operatorname{Sing} B_Y$. Hence, $(Y, B_Y)$ is 1-log-terminal outside $\operatorname{Sing} B_Y$, since $(X, B)$ is so outside $\operatorname{Sing} B$. Thus, (i) is satisfied. □

**Lemma 4.26.** *For the log-canonical pair $(X, B)$ of a normal surface $X$ and a reduced divisor $B$, let $M$ be a non-singular surface with a bimeromorphic morphism $\mu\colon M \to V$. Let $B_\mu$ and $T_\mu$ be effective $\mathbb{Q}$-divisors on $M$ without common prime components such that $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$. Let $\sigma\colon M \to Y$ be the contraction morphism of all the $\mu$-exceptional prime divisors not contained in $\llcorner B_\mu \lrcorner$. Let $f\colon Y \to X$ be the induced morphism such that $\mu = f \circ \sigma$ and set $B_Y := \sigma_* B_\mu$. Then $f\colon (Y, B_Y) \to (X, B)$ is an essential blowing up.*

*Proof.* The divisor $\ulcorner B_\mu \urcorner$ is reduced by Lemma 2.10(1). We have an equality $K_Y + B_Y = f^*(K_X + B)$ by applying $\sigma_*$ to $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$, since $T_\mu$ is $\sigma$-exceptional. Then $(Y, B_Y)$ is log-canonical by Lemma 2.10(1). We set $D := \llcorner B_\mu \lrcorner$. By construction, $B_Y$ is reduced, and $D = \sigma^{[*]} B_Y$. The induced morphism $\sigma|_D\colon D \to B_Y$ is an isomorphism, since it is finite and since $\mathcal{O}_{B_Y} \simeq \sigma_* \mathcal{O}_D$ by Lemma 4.22. In particular, $\sigma(\operatorname{Sing} D) = \operatorname{Sing} B_Y$. Then $(U, B_Y|_U)$ is 1-log-terminal for $U := Y \setminus \operatorname{Sing} B_Y$. In fact, for the equality

$$K_M + B_\mu = \sigma^*(K_Y + B_Y) + T_\mu,$$

$\ulcorner B_\mu \urcorner$ is reduced and the divisor $D|_{\sigma^{-1}U} = \llcorner B_\mu \lrcorner|_{\sigma^{-1}U}$ on $\sigma^{-1}U$ is non-singular and contains no $\sigma$-exceptional prime component. Moreover, the $f$-exceptional locus is contained in $\sigma(D) = B_Y$, since the image of the $\mu$-exceptional locus by $\sigma$ is contained in the union of $\sigma(D)$ and a finite set. Therefore, $(Y, B_Y)$ is an essential blowing up of $(X, B)$. □

**Definition 4.27.** The essential blowing up $(Y, B_Y) \to (X, B)$ in Lemma 4.26 is called the *standard partial resolution* if $\mu\colon M \to X$ is the minimal resolution of singularities.

We shall give local descriptions of standard partial resolutions in Examples 4.28 and 4.29 below:

*Example* 4.28. Let $(X, B)$ be a log-canonical pair of a normal surface $X$ and a reduced divisor $B$. Assume that $\operatorname{Sing} X = \{x\}$, $\operatorname{Sing} B \subset \{x\}$, and $x \in B$. Let $f\colon (Y, B_Y) \to (X, B)$ be the standard partial resolution. Then $B_Y$ is the union of $f^{-1}(x)$ and the proper transform $B' = f^{[*]}B$ of $B$. If $x \in \operatorname{Sing} B$, then $(X, B)$ is toroidal at $x$ by Fact 2.5(1), and hence:

- $f$ is the minimal resolution of singularities;
- $f^{-1}(x)$ is a *linear chain $C$ of rational curves* (cf. [40, Def. 4.1]);
- if $C$ is irreducible, then it intersects $B'$ transversely at two points;
- if $C$ is reducible, then each end component of $C$ intersects $B'$ transversely at one point;
- any non-end component of $C$ does not intersect $B'$.

If $x \in B_{\mathrm{reg}}$ and $(X, B)$ is 1-log-terminal at $x$, then $f$ is an isomorphism by Lemma 4.25. Assume that $x \in B_{\mathrm{reg}}$ and $(X, B)$ is not 1-log-terminal at $x$. Then the local description of $(X, B)$ at $x$ as in Fact 2.5(3). For the minimal resolution of singularities of $X$, the dual graph of the union of the exceptional locus and the inverse image of $B$ is well known (cf. [29, Thm. 9.6(6)], [34, Ch. 3], [40, Thm 3.22(iii), Fig. 2]). As a consequence, the following hold:

(1) The $f$-exceptional locus $f^{-1}(x)$ is a linear chain $C = \sum_{i=1}^{k} C_i$ of rational curves.
(2) The divisor $B'$ is non-singular, and it intersects an end component $C_1$ of $C$ transversely at a non-singular point of $Y$ but does not intersect $C - C_1$.
(3) The singularity $Y$ consists of two $\mathsf{A}_1$-singular points lying on $C$. If $k > 1$, then these points are both contained in the other end component $C_k$ of $C$ but not on $C - C_k$.

*Example* 4.29. Let $X$ be a normal surface with a point $x \in X$ such that $(X, 0)$ is log-canonical and $\operatorname{Sing} X = \{x\}$. By the classification of 2-dimensional log-canonical singularities (cf. [51, App.], [29, Thm. 9.6], [34, Ch. 3]), the standard partial resolution $f\colon (Y, B_Y) \to (X, 0)$ is described as follows:

(1) If $(X, x)$ is a quotient singularity, then $f$ is an isomorphism.
(2) If $(X, x)$ is a simple elliptic singularity, then $f$ is the minimal resolution of singularities, and $B_Y$ is an elliptic curve.
(3) If $(X, x)$ is a cusp singularity, then $f$ is the minimal resolution of singularities, and $B_Y$ is a *cyclic chain of rational curves* (cf. [40, Def. 4.3]).
(4) If $(X, x)$ is a rational singularity and its *index* 1 *cover* with respect to $K_X$ (cf. Definition 4.18(4)) is a simple elliptic singularity, then $B_Y$ is a non-singular rational curve, and $\operatorname{Sing} Y$ consists of at least three cyclic quotient singular points lying on $B_Y$.
(5) If $(X, x)$ is a rational singularity and its index 1 cover with respect to $K_X$ is a cusp singularity, then $B_Y$ is a reducible linear chain of rational curves, $\mathsf{Sing}\, Y$ consists of four $\mathsf{A}_1$-singular points lying only on end components of $B_Y$, and each end component has exactly two $\mathsf{A}_1$-singular points.

**Definition 4.30.** Let $\Gamma$ be a prime component of a reduced divisor $B$ on a normal surface. We define $v(\Gamma/B) := \sharp\Gamma \cap (B - \Gamma)$.

**Lemma 4.31.** *Let $f\colon (Y, B_Y) \to (X, B)$ be an essential blowing up of a log-canonical pair $(X, B)$ of a normal surface $X$ and a reduced divisor $B$. Let $\sigma\colon Z \to Y$ be a non-isomorphic bimeromorphic morphism from another normal surface $Z$ with a reduced divisor $D$ such that $D$ contains the $f \circ \sigma$-exceptional locus and that $K_Z + D = \sigma^*(K_Y + B_Y)$. Then:*

(1) *The composite $f \circ \sigma\colon (Z, D) \to (X, B)$ is an essential blowing up, and $\sigma\colon (Z, D) \to (Y, B_Y)$ is a toroidal blowing up with respect to $(Y, B_Y)$.*
(2) *If $\Gamma$ is a non-singular prime component of $B_Y$, then $v(\Gamma/B_Y) = v(\sigma^{[*]}\Gamma/D)$.*
(3) *If $\Theta$ is a $\sigma$-exceptional prime divisor, then $v(\Theta/D) = 2$.*

*Proof.* By Lemma 4.25, $\sigma$ is a toroidal blowing up with respect to $(Y, B_Y)$ and is also is an essential blowing up of $(Y, B_Y)$. In particular, $(Z, D)$ is 1-log-terminal outside $\operatorname{Sing} D$. Thus, we have proved (1). Assertions (2) and (3) are deduced from the property that $\sigma$ is a toroidal blowing up with respect to $(Y, B_Y)$. □

**Lemma 4.32.** *Let $(X, B)$ be a log-canonical pair of a normal surface $X$ and a reduced divisor $B$. For two essential blowings up $f_1\colon (Y_1, B_1) \to (X, B)$ and $f_2\colon (Y_2, B_2) \to (X, B)$, there exists an essential blowing up $f_3\colon (Y_3, B_3) \to (X, B)$ such that $f_i^{-1} \circ f_3\colon Y_3 \to Y_i$ is holomorphic and is a toroidal blowing up with respect to $(Y_i, B_i)$ for any $i = 1, 2$.*

*Proof.* We can take a bimeromorphic morphism $\mu\colon M \to X$ from a non-singular surface $M$ such that the union of $\mu^{-1}B$ and the $\mu$-exceptional locus is a normal crossing divisor and that $\nu_i := f_i^{-1} \circ \mu\colon M \to Y_i$ is holomorphic for any $i = 1, 2$. Let $B_\mu$ and $T_\mu$ be effective $\mathbb{Q}$-divisors on $M$ without common prime components such that $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$. Then, for each $i = 1, 2$,

$$K_M + B_\mu = \nu_i^*(K_{Y_i} + B_i) + T_\mu,$$

and $\langle B_\mu \rangle + T_\mu$ is $\nu_i$-exceptional. Let $\nu_3\colon M \to Y_3$ be the contraction morphisms of all the prime divisors which are exceptional for both $\nu_1$ and $\nu_2$. Let $f_3\colon Y_3 \to X$ be the induced morphism such that $\mu = f_3 \circ \nu_3$. Then $\sigma_i := \nu_i \circ \nu_3^{-1} = f_i^{-1} \circ f_3\colon Y_3 \to Y_i$ is holomorphic for any $i$, and $K_{Y_3} + B_3 = f_3^*(K_X + B)$ for the reduced divisor $B_3 := \nu_{3*}B_\mu = \nu_{3*}\llcorner B_\mu \lrcorner$, since $\langle B_\mu \rangle + T_\mu$ is $\nu_3$-exceptional. Hence,

$$K_{Y_3} + B_3 = \sigma_i^*(K_{Y_i} + B_i) \tag{IV-7}$$

for $i = 1, 2$. Here, $\sigma_i(B_3) \subset B_i$, since $Y_i \setminus B_i$ has only log-terminal singularities, and the induced morphism $\sigma_i|_{B_3}\colon B_3 \to B_i$ is an isomorphism over $B_i \setminus \operatorname{Sing} B_i$ by Lemma 4.23(1). In particular, $B_i = \sigma_i(B_3)$ for $i = 1, 2$.

Let $\Gamma$ be an $f_3$-exceptional prime divisor on $Y_3$. Then $\sigma_i(\Gamma)$ is a prime divisor for $i = 1$ or 2, and in this case, $\sigma_i(\Gamma)$ is contained in the $f_i$-exceptional locus; thus, $\sigma_i(\Gamma) \subset B_i$. Here, the proper transform $\Gamma$ of $\sigma_i(\Gamma)$ is contained in $B_3$ by $B_i = \sigma_i(B_3)$. Hence, $B_3$ contains the $f_3$-exceptional locus. Therefore, $\sigma_i\colon (Y_3, B_3) \to (Y_i, B_i)$ is a toroidal blowing up for $i = 1, 2$, and $f_3\colon (Y_3, B_3) \to (X, B)$ is an essential blowing up, by Lemma 4.31 applied to (IV-7). □

**Corollary 4.33.** *Let $f\colon (Y, B_Y) \to (X, B)$ be an essential blowing up of a log-canonical pair $(X, B)$ of a normal surface $X$ and a reduced divisor $B$.*

1. *If an $f$-exceptional prime divisor $\Gamma$ is non-singular, then $v(\Gamma/B_Y) \le 2$.*
2. *If $\Gamma$ is a non-singular prime component of $B_Y$ such that $v(\Gamma/B_Y) \ne 2$, then $\Gamma$ is not contracted to a point by the meromorphic map $g^{-1} \circ f\colon Y \dashrightarrow Z$ for any essential blowing up $g\colon (Z, D) \to (X, B)$, i.e., the proper transform of $\Gamma$ in $Z$ is a prime component of $D$.*
3. *If every $f$-exceptional prime divisor $\Gamma$ is non-singular and satisfies $v(\Gamma/B_Y) \le 1$, then, for any essential blowing up $g\colon (Z, D) \to (X, B)$, there is an essential blowing up $\sigma\colon (Z, D) \to (Y, B_Y)$ such that $g = f \circ \sigma$.*

*Here, $v(\Gamma/B_Y) = \sharp\Gamma \cap (B_Y - \Gamma)$ (cf. Definition 4.30).*

*Proof.* Let us fix an essential blowing up $g\colon (Z, D) \to (X, B)$ and let $f_1\colon (Y_1, B_1) \to (X, B)$ be the standard partial resolution. By Lemma 4.32, we have an essential blowing up $f_2\colon (Y_2, B_2) \to (X, B)$ such that $\sigma_1 := f_1^{-1} \circ f_2\colon Y_2 \to Y_1$, $\sigma := f^{-1} \circ f_2\colon Y_2 \to Y$, and $\tau := g^{-1} \circ f_2\colon Y_2 \to Z$ are all holomorphic. Here, $\sigma_1\colon (Y_2, B_2) \to (Y_1, B_1)$, $\sigma\colon (Y_2, B_2) \to (Y, B_Y)$, and $\tau\colon (Y_2, B_2) \to (Z, D)$ are toroidal blowings up by Lemma 4.31:

$$\begin{tikzcd}
 & Y_2 \arrow[dl, "\sigma_1"'] \arrow[d, "\sigma"] \arrow[dr, "\tau"] \arrow[dd, bend right, "f_2"'] & \\
Y_1 \arrow[dr, "f_1"'] & Y \arrow[d, "f"] & Z \arrow[dl, "g"] \\
 & X. &
\end{tikzcd}$$

Let $\Gamma$ be a non-singular prime component of $B_Y$. Then the proper transform $\Gamma'' = \sigma^{[*]}\Gamma$ in $Y_2$ is also non-singular. By Lemma 4.31, we have $v(\Gamma/B_Y) = v(\Gamma''/B_2)$, and if $v(\Gamma''/B_2) \neq 2$, then $\Gamma''$ is not exceptional for $\tau$ and $\sigma_1$. In particular, we have (2).

Assume that $\Gamma$ is $f$-exceptional and that $\Gamma' = \sigma_1(\Gamma)$ is a divisor, which is a prime component of $B_1$. If $\Gamma'$ is non-singular, then $v(\Gamma'/B_1) = v(\Gamma''/B_2)$ by Lemma 4.31(2), and we have $v(\Gamma'/B_1) \leq 2$ by Examples 4.28 and 4.29. If $\Gamma'$ is singular, then $f(\Gamma) = f_1(\Gamma') \notin B$, $X$ has a cusp singularity at $f(\Gamma)$, and $\Gamma'$ is a nodal rational curve being a connected component of $B_1$, by Examples 4.28 and 4.29. Then $v(\Gamma''/B_2) = 2$, since $\sigma_1$ is a toroidal blowing up with respect to $(Y_1, B_1)$ and is not an isomorphism over the node of $\Gamma'$ as $\Gamma''$ is non-singular. Therefore, $v(\Gamma/B_Y) \leq 2$, and we have (1).

The remaining assertion (3) is deduced from (2). In fact, any $f$-exceptional prime divisor is not contracted to a point by the meromorphic map $\tau \circ \sigma^{-1}\colon Y \dashrightarrow Z$ by (2). Hence, every $\tau$-exceptional divisor is $\sigma$-exceptional. It implies that $\sigma \circ \tau^{-1}\colon Z \dashrightarrow Y$ is holomorphic. Thus, we have (3). $\square$

**Lemma 4.34.** *Let $(X, B)$ and $(X', B')$ be log-canonical pairs of normal surfaces $X$ and $X'$ and reduced divisors $B$ and $B'$, respectively. Let $\tau\colon X' \to X$ be a morphism with only discrete fibers such that $B' = \tau^{-1}B$ and that $\tau|_{X' \setminus B'}\colon X' \setminus B' \to X \setminus B$ is étale in codimension* 1. *Then, for any essential blowing up $f\colon (Y, D) \to (X, B)$, there exists a commutative diagram*

$$\begin{tikzcd}
Y' \arrow[r, "f'"] \arrow[d, "\sigma"'] & X' \arrow[d, "\tau"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

*of normal surfaces such that $Y'$ is the normalization of the fiber product $Y \times_X X'$ and that $f'\colon (Y', D') \to (X', B')$ is an essential blowing up for $D' := \sigma^{-1}D$. In particular, $\sigma\colon Y' \to Y$ is a morphism with only discrete fibers and the induced morphism $Y' \setminus D' \to Y \setminus D$ is étale in codimension* 1.

*Proof.* Let $Y'$ be the normalization of $X' \times_X Y$ and let $\sigma\colon Y' \to Y$ and $f'\colon Y' \to X'$ be induced morphisms. Here, $X' \times_X Y$ is irreducible and generically reduced by Lemma 1.13. Then $\sigma$ has only discrete fibers, and it is étale in codimension 1 outside

$D$, since $D$ contains the $f$-exceptional locus and since $\tau$ is étale in codimension 1 outside $B$. The $f'$-exceptional locus is contained in the inverse image by $\sigma$ of the $f$-exceptional locus as $\sigma$ has only discrete fibers. Thus, $D' = \sigma^{-1}D$ contains the $f'$-exceptional locus. We have $K_{X'} + B' = \tau^*(K_X + B)$ and $K_{Y'} + D' = \sigma^*(K_Y + D)$ by Lemma 1.39, and have $K_Y + D = f^*(K_X + B)$ as $f$ is an essential blowing up. Hence, $K_{Y'} + D' = f'^*(K_{X'} + B')$. In particular, $(Y', D')$ is log-canonical by Lemma 2.10(1).

It remains to prove that $(Y', D')$ is 1-log-terminal outside $\operatorname{Sing} D'$. Now, $(Y', D')$ is 1-log-terminal outside $\sigma^{-1}(\operatorname{Sing} D)$ by Lemma 2.10(2). Thus, it is enough to show: $\sigma^{-1}(\operatorname{Sing} D) \subset \operatorname{Sing} D'$. For a point $y' \in \sigma^{-1}(\operatorname{Sing} D)$, by Corollary 1.8, we have an open neighborhood $\mathcal{U}'$ of $y'$ in $Y'$ such that $\mathcal{U} := \sigma(\mathcal{U}')$ is open and $\sigma_{\mathcal{U}} := \sigma|_{\mathcal{U}'} \colon \mathcal{U}' \to \mathcal{U}$ is finite and surjective. By shrinking $\mathcal{U}$, we may assume that $D|_{\mathcal{U}} = \Gamma_1 + \Gamma_2$ for two distinct prime divisors $\Gamma_1$ and $\Gamma_2$ and $\sigma(y') \in \Gamma_1 \cap \Gamma_2$. Then $\sigma^* D|_{\mathcal{U}'} = \sigma_{\mathcal{U}}^*\Gamma_1 + \sigma_{\mathcal{U}}^*\Gamma_2$ and $y' \in \sigma_{\mathcal{U}}^{-1}\Gamma_1 \cap \sigma_{\mathcal{U}}^{-1}\Gamma_2$, where $\sigma_{\mathcal{U}}^*\Gamma_1$ and $\sigma_{\mathcal{U}}^*\Gamma_2$ have no common prime component, since $\sigma_{\mathcal{U}}$ is surjective. Hence, $y' \in \operatorname{Sing} \sigma^{-1} D = \operatorname{Sing} D'$, and we have $\sigma^{-1}(\operatorname{Sing} D) \subset \operatorname{Sing} D'$. As a consequence, $(Y', D')$ is 1-log-terminal outside $\operatorname{Sing} D'$, and $f' \colon (Y', D') \to (X', B')$ is an essential blowing up. □

4.4. **Dual $\mathbb{R}$-divisors.** We fix a normal surface $X$ and a non-zero reduced connected compact divisor $S$ on $X$ such that the intersection matrix of prime components of $S$ is negative definite; in other words, $S$ is the inverse image of a point by a certain bimeromorphic morphism $X \to \overline{X}$ to a normal surface $\overline{X}$, by the contraction criterion (cf. [13, (e), page 366–367] and [48, Thm. (1.2)]). We shall introduce *primitive dual $\mathbb{Q}$-divisors* and *dual $\mathbb{R}$-divisors* for a prime component of $S$ in Lemma-Definition 4.35 below and show their basic properties.

**Lemma-Definition 4.35.** *Let $\Gamma$ be a prime component of $S$.*

(1) *There is a unique $\mathbb{Q}$-divisor $\boldsymbol{D}(\Gamma/S)$ on $X$ supported on $S$ such that*
$$\operatorname{mult}_\Gamma A = \boldsymbol{D}(\Gamma/S)A$$
*for any divisor $A$ supported on $S$. We call $\boldsymbol{D}(\Gamma/S)$ the* primitive dual $\mathbb{Q}$-divisor *of $\Gamma$ with respect to $S$.*

(2) *For an effective $\mathbb{R}$-divisor $H$ on $X$ such that* $\operatorname{Supp} H = S$, *we set*
$$\boldsymbol{\Delta}(\Gamma, H) := -(\operatorname{mult}_\Gamma H)^{-1}\boldsymbol{D}(\Gamma/S)$$
*and call it the* dual $\mathbb{R}$-divisor *of $\Gamma$ with respect to $H$.*

*The following hold for $\boldsymbol{D}(\Gamma/S)$ and $\boldsymbol{\Delta}(\Gamma, H)$:*

(3) *The $\mathbb{Q}$-divisor $-\boldsymbol{D}(\Gamma/S)$ is effective and* $\operatorname{Supp} \boldsymbol{D}(\Gamma/S) = S$.

(4) *If $\Gamma'$ is a prime component of $S - \Gamma$, then $\boldsymbol{D}(\Gamma/S)\Gamma' = 0$. Moreover,*
$$A = \sum\nolimits_{\Gamma \subset S} (A\Gamma)\boldsymbol{D}(\Gamma/S).$$
*for any $\mathbb{R}$-divisor $A$ supported on $S$.*

(5) *For any effective* $\mathbb{R}$*-divisor* $H$ *on* $X$ *such that* $\operatorname{Supp} H = S$*, the* $\mathbb{R}$*-divisor* $\boldsymbol{\Delta}(\Gamma, H)$ *is effective,* $\operatorname{Supp} \boldsymbol{\Delta}(\Gamma, H) = S$*,* $-\boldsymbol{\Delta}(\Gamma, H)$ *is nef on* $S$*, and* $\boldsymbol{\Delta}(\Gamma, H)H = -1$*.*

*Proof.* Since the intersection matrix of $S$ is definite, the $\mathbb{Q}$-divisor $\boldsymbol{D}(\Gamma/S)$ satisfying (1) exists uniquely, and we have (4). Since $\boldsymbol{D}(\Gamma/S)$ is nef on $S$, we have (3) by Remark 1.25. Assertion (5) is deduced from (3) and (4). □

**Lemma 4.36.** *Let* $\pi\colon Y \to X$ *be a bimeromorphic morphism from a normal surface* $Y$*, and set* $S_Y := \pi^{-1}S$*. Let* $H_Y$ *be an* $\mathbb{R}$*-divisor on* $Y$ *such that* $\operatorname{Supp} H_Y = S_Y$*, and set* $H := \pi_* H_Y$*. Then, for any prime component* $\Gamma$ *of* $S$ *and its proper transform* $\pi^{[*]}\Gamma$ *in* $Y$*, one has*

$$\pi^*\boldsymbol{D}(\Gamma/S) = \boldsymbol{D}(\pi^{[*]}\Gamma/S_Y) \quad \text{and} \quad \pi^*\boldsymbol{\Delta}(\Gamma, H) = \boldsymbol{\Delta}(\pi^{[*]}\Gamma, H_Y).$$

*Proof.* Note that $S_Y$ is compact and connected, the intersection matrix of prime components of $S_Y$ is also negative definite, and $\operatorname{Supp} H = S$. For any $\pi$-exceptional prime divisor $E$, we have $\boldsymbol{D}(\pi^{[*]}\Gamma/S_Y)E = 0$ by Lemma-Definition 4.35(4), since either $E \cap S_Y = \emptyset$ or $E \subset S_Y$. Thus, $\boldsymbol{D}(\pi^{[*]}\Gamma/S_Y) = \pi^*D$ for the pushforward $D := \pi_*\boldsymbol{D}(\pi^{[*]}\Gamma/S_Y)$. Then

$$D\Gamma' = (\pi^*D)\pi^{[*]}\Gamma' = \boldsymbol{D}(\pi^{[*]}\Gamma/S_Y)\pi^{[*]}\Gamma' = \begin{cases} 1, & \text{if } \Gamma' = \Gamma, \\ 0, & \text{otherwise,} \end{cases}$$

for any prime component $\Gamma'$ of $S$, and $D = \boldsymbol{D}(\Gamma/S)$ by Lemma-Definition 4.35(1), and we have the first equality. The second equality follows from the first one by Lemma-Definition 4.35(2), since $\operatorname{mult}_{\pi^{[*]}\Gamma} H_Y = \operatorname{mult}_\Gamma H$. □

We have the following generalization of the first equality in Lemma 4.36:

**Lemma 4.37.** *Let* $\pi\colon Y \to X$ *be a non-degenerate morphism from a normal surface* $Y$ *such that* $S_Y := \pi^{-1}S$ *is compact. Let* $\Theta$ *be a prime component of* $S_Y$*. Then*

$$\pi_*\boldsymbol{D}(\Theta/S_Y) = \sum\nolimits_{\pi(\Theta) \subset \Gamma \subset S} (\operatorname{mult}_\Theta \pi^*\Gamma)\boldsymbol{D}(\Gamma/S).$$

*In particular, if* $\pi(\Theta)$ *is a prime divisor* $\Gamma$*, then*

$$\pi_*\boldsymbol{D}(\Theta/S_Y) = (\operatorname{mult}_\Theta \pi^*\Gamma)\boldsymbol{D}(\Gamma/S).$$

*Conversely, for any prime component* $\Gamma$ *of* $S$*, one has*

$$\pi^*\boldsymbol{D}(\Gamma/S) = \sum\nolimits_{\Gamma \subset \pi(\Theta)} (\operatorname{mult}_\Gamma \pi_*\Theta)\boldsymbol{D}(\Theta/S_Y).$$

*Proof.* For any prime component $\Gamma$ of $S$, we have

$$(\pi_*\boldsymbol{D}(\Theta/S_Y))\Gamma = \boldsymbol{D}(\Theta/S_Y)\pi^*\Gamma = \operatorname{mult}_\Theta \pi^*\Gamma$$

by Lemma-Definition 4.35(1). This implies the first equality, since $\operatorname{mult}_\Theta \pi^*\Gamma \neq 0$ if and only if $\pi(\Theta) \subset \Gamma$. The second equality is a special case of the first one. The third equality is deduced from equalities

$$(\pi^*\boldsymbol{D}(\Gamma/S))\Theta = \boldsymbol{D}(\Gamma/S)\pi_*\Theta = \operatorname{mult}_\Gamma \pi_*\Theta$$

and from Lemma-Definition 4.35(4). □

The following result almost corresponds to the last assertion of [6, Prop. 1.4].

**Proposition 4.38.** *Assume that $(X, S)$ is log-canonical and let $H$ be an effective $\mathbb{R}$-divisor on $X$ such that* $\operatorname{Supp} H = S$*. Then there exist positive rational numbers $c_1 < c_2$ depending only on $(X, S, H)$ such that*

$$c_1\pi^*H \le \mathbf{\Delta}(\Theta, \pi^*H) \le c_2\pi^*H \tag{IV-8}$$

*for any non-degenerate morphism $\pi\colon Y \to X$ from a normal surface $Y$ and any prime component $\Theta$ of $S_Y := \pi^{-1}S$ satisfying the following conditions*:

(i) *$\pi(Y)$ is an open neighborhood of $S$, and $\pi\colon Y \to \pi(Y)$ is a bimeromorphic morphism*;

(ii) $\operatorname{mult}_\Theta \Delta_\pi = 0$ *for the $\mathbb{Q}$-divisor $\Delta_\pi$ defined by $K_Y + S_Y = \pi^*(K_X + S) + \Delta_\pi$.*

*Proof.* We divide the proof into three steps.

*Step* 1. *Reduction to the following two cases of* $(\pi, \Theta)$:

(1) *$\pi$ is the identity morphism*;

(2) *$\pi(Y) = X$ and the exceptional locus of $\pi$ equals the prime component $\Theta$.*

Note that in case (2), we have $\Delta_\pi = 0$ by $\operatorname{mult}_\Theta \Delta_\pi = 0$. Let $c_1$ and $c_2$ be positive rational numbers such that (IV-8) holds only in cases (1) and (2). Let $(\pi\colon Y \to X, \Theta)$ be an arbitrary pair satisfying (i) and (ii). First, assume that $\Theta$ is not $\pi$-exceptional. Then $\Theta = \pi^{[*]}\Gamma$ for a prime component $\Gamma$ of $S$, and we have

$$\mathbf{\Delta}(\Theta, \pi^*H) = \pi^*\mathbf{\Delta}(\Gamma, H)$$

by Lemma 4.36 applied to the bimeromorphic morphism $Y \to \pi(Y)$. Hence, (IV-8) for this $(\pi, \Theta)$ is deduced from that for $(\mathrm{id}_X, \Gamma)$. Second, assume that $\Theta$ is $\pi$-exceptional and let $\varphi\colon Y \to \overline{Y}$ be the contraction morphism of the union of $\pi$-exceptional prime divisors except $\Theta$. Then $\pi = \bar{\pi} \circ \varphi$ for a morphism $\bar{\pi}\colon \overline{Y} \to X$ satisfying (i), the $\bar{\pi}$-exceptional locus is $\overline{\Theta} := \varphi(\Theta)$, and

$$\mathbf{\Delta}(\Theta, \pi^*H) = \varphi^*\mathbf{\Delta}(\overline{\Theta}, \bar{\pi}^*H)$$

by Lemma 4.36. We can construct a bimeromorphic morphism $\hat{\pi}\colon \widehat{Y} \to X$ with an isomorphism $\hat{\pi}^{-1}(\pi(Y)) \simeq Y$ over $X$ by gluing $Y \to \pi(Y)$ and the identity morphism of $X \setminus S$. Then $\widehat{\Theta} = \overline{\Theta}$ and $\hat{\pi}^*H = \bar{\pi}^*H$ are regarded as $\mathbb{Q}$-divisors on $\widehat{Y}$, and we have

$$\mathbf{\Delta}(\widehat{\Theta}, \hat{\pi}^*H) = \mathbf{\Delta}(\overline{\Theta}, \bar{\pi}^*H).$$

Thus, (IV-8) for $(\pi, \Theta)$ is deduced from that for $(\hat{\pi}, \widehat{\Theta})$. Therefore, we are reduced to the cases (1) and (2).

*Step* 2. *Reduction to the case where $X$ is non-singular and $S$ is a simple normal crossing divisor*: Since the assertion is on $\mathbb{R}$-divisors lying over $S$, we may replace $X$ with an open neighborhood of $S$ freely. Thus, we may assume that $X \setminus S$ is non-singular. There is a bimeromorphic morphism $\mu\colon M \to X$ from a non-singular surface $M$ such that $S_M := \mu^{-1}S$ is a simple normal crossing divisor and that $\mu$ is an isomorphism over $X \setminus S$. Then the $\mathbb{Q}$-divisor $\Delta_\mu$ defined by $K_M + S_M = \mu^*(K_X + S) + \Delta_\mu$ is effective as $(X, S)$ is log-canonical. Assume that the assertion holds for $(M, S_M, \mu^*H)$ instead of $(X, S, H)$, i.e., the equality corresponding to (IV-8) holds for $(M, S_M, \mu^* H)$ for some $c_1$ and $c_2$. For the proof, by *Step* 1, it is enough to verify (IV-8) for $(\pi, \Theta)$ such that $\pi$ is a bimeromorphic morphism, $\Theta$ is the exceptional locus of $\pi$, and $\Delta_\pi = 0$. Then $(Y, S_Y)$ is log-canonical by $K_Y + S_Y = \pi^*(K_X + S)$ (cf. Lemma 2.10(1)). We can find a bimeromorphic morphism $\nu\colon N \to Y$ from a non-singular surface $N$ and a bimeromorphic morphism $\phi\colon N \to M$ such that $\nu$ is an isomorphism over $Y \setminus S_Y$, $\phi$ is an isomorphism over $M \setminus S_M$, and the diagram

$$\begin{array}{ccc} N & \xrightarrow{\nu} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\mu} & X \end{array}$$

is commutative. Then

$$\mathbf{\Delta}(\nu^{[*]}\Theta, \nu^*(\pi^* H)) = \nu^* \mathbf{\Delta}(\Theta, \pi^* H)$$

by Lemma 4.36. We set $S_N := \phi^{-1} S_M = \nu^{-1} S_Y$, and let $\Delta_\phi$ and $\Delta_\nu$ be $\mathbb{Q}$-divisors defined by

$$K_N + S_N = \phi^*(K_M + S_M) + \Delta_\phi \quad \text{and} \quad K_N + S_N = \nu^*(K_Y + S_Y) + \Delta_\nu.$$

Then $\Delta_\phi$ is $\phi$-exceptional and effective, and $\Delta_\nu$ is $\nu$-exceptional and effective, as $(M, S_M)$ and $(Y, S_Y)$ are log-canonical. Moreover, we have

$$\phi^* \Delta_\mu + \Delta_\phi = \Delta_\nu + \nu^* \Delta_\pi = \Delta_\nu.$$

Thus, $\nu^{[*]}\Theta \not\subset \operatorname{Supp} \Delta_\phi$ and $\phi(\nu^{[*]}\Theta) \not\subset \operatorname{Supp} \Delta_\mu$. As an equality corresponding to (IV-8) for $(M, S_M, \pi^* H)$, we have

$$c_1 \phi^*(\mu^* H) \leq \mathbf{\Delta}(\nu^{[*]}\Theta, \phi^*(\mu^* H)) \leq c_2 \phi^*(\mu^* H).$$

Applying $\nu_*$ to it, we have

$$c_1 \pi^* H \leq \mathbf{\Delta}(\Theta, \pi^* H) \leq c_2 \pi^* H$$

by Lemma 4.36, since $\phi^*(\mu^* H) = \nu^*(\pi^* H)$. Therefore, for the proof, we may replace $(X, S, H)$ with $(M, S_M, \mu^* H)$.

*Step* 3. *The final step*: We may assume that $X$ is non-singular and $S$ is a simple normal crossing divisor by *Step* 2. Since $S$ has only finitely many prime components, we have positive rational numbers $c_1^0 < c_2^0$ satisfying

$$c_1^0 H \leq \mathbf{\Delta}(\Gamma, H) \leq c_2^0 H \tag{IV-9}$$

for any prime component $\Gamma$ of $S$. We shall show that rational numbers $c_1 = c_1^0$ and $c_2 > c_2^0 + (2h^2)^{-1}$ satisfy the inequality (IV-8) for

$$h := \min\{\operatorname{mult}_\Gamma H \mid \Gamma \text{ is a prime component of } S\}.$$

By *Step* 1, it is enough to verify (IV-8) in the case where $\pi\colon Y \to X$ is a bimeromorphic morphism, $\Theta$ is the exceptional locus of $\pi$, and $\Delta_\pi = 0$. Since $K_Y + S_Y = \pi^*(K_X + S)$, the pair $(Y, S_Y)$ is log-canonical and $\pi$ is a toroidal blowing up at the node $x := \pi(\Theta)$ of $S$. Hence, $x \in \Gamma_1 \cap \Gamma_2$ for two prime components $\Gamma_1$, $\Gamma_2$ of $S$, and $\pi^{[*]}\Gamma_1 \cap \pi^{[*]}\Gamma_2 \cap \Theta = \emptyset$. Therefore, $x \notin \pi(\pi^{[*]}\Gamma_1 \cap \pi^{[*]}\Gamma_2)$, and

$$\Gamma_1 \Gamma_2 = (\pi^{[*]}\Gamma_1)\pi^{[*]}\Gamma_2 + 1. \tag{IV-10}$$

For $i = 1, 2$, we set $a_i := \mathrm{mult}_\Theta\, \pi^*\Gamma_i \in \mathbb{Q}$, i.e., $\pi^*\Gamma_i = \pi^{[*]}\Gamma_i + a_i\Theta$. Then

$$(\text{IV-11}) \qquad (\pi^{[*]}\Gamma_1)\Theta = a_2^{-1}, \quad (\pi^{[*]}\Gamma_2)\Theta = a_1^{-1}, \quad \text{and} \quad \Theta^2 = -(a_1a_2)^{-1}.$$

In fact, the first equality of (IV-11) is obtained by calculation

$$\Gamma_1\Gamma_2 = (\pi^*\Gamma_1)\pi^{[*]}\Gamma_2 = (\pi^{[*]}\Gamma_1)\pi^{[*]}\Gamma_2 + a_1\Theta\pi^{[*]}\Gamma_2 = \Gamma_1\Gamma_2 - 1 + a_1\pi^{[*]}\Theta\Gamma_2$$

using (IV-10): We have the second equality by interchanging $(\Gamma_1, a_1)$ and $(\Gamma_2, a_2)$, and the third one by calculation

$$0 = a_2(\pi^*\Gamma_1)\Theta = a_2(\pi^{[*]}\Gamma_1)\Theta + a_1a_2\Theta^2 = 1 + a_1a_2\Theta^2$$

using the first equality. We set $h_i := \mathrm{mult}_{\Gamma_i} H$ for $i = 1, 2$, and $h_3 := \mathrm{mult}_\Theta\, \pi^*H$. Then $h_3 = a_1h_1 + a_2h_2$ and we have

$$\begin{aligned} h_3\pi_*\boldsymbol{\Delta}(\Theta, \pi^*H) &= -\pi_*\boldsymbol{D}(\Theta/S_Y) = -a_1\boldsymbol{D}(\Gamma_1/S) - a_2\boldsymbol{D}(\Gamma_2/S) \\ &= a_1h_1\boldsymbol{\Delta}(\Gamma_1, H) + a_2h_2\boldsymbol{\Delta}(\Gamma_2, H) \end{aligned}$$

by Lemma 4.37 and Lemma-Definition 4.35(2). Therefore,

$$(\text{IV-12}) \qquad c_1^0 H \le \pi_*\boldsymbol{\Delta}(\Theta, \pi^*H) \le c_2^0 H$$

by (IV-9). For the rational number $e$ defined by

$$\boldsymbol{\Delta}(\Theta, \pi^*H) = \pi^*(\pi_*\boldsymbol{\Delta}(\Theta, \pi^*H)) + e\Theta,$$

we have $e = a_1a_2/h_3 > 0$ by calculation

$$-1/h_3 = \boldsymbol{\Delta}(\Theta, \pi^*H)\Theta = e\Theta^2 = -e/(a_1a_2)$$

using Lemma-Definition 4.35(2) and (IV-11). Therefore,

$$c_1^0\pi^*H \le \boldsymbol{\Delta}(\Theta, \pi^*H) \le c_2^0\pi^*H + \frac{a_1a_2}{h_3}\Theta \le (c_2^0 + \frac{a_1a_2}{h_3^2})\pi^*H$$

by (IV-12) and by $h_3^*\Theta \le \pi^*H$. Here, $a_1a_2h_3^{-2} \le (2h^2)^{-1}$ by

$$h_3^2 = (a_1h_1 + a_2h_2)^2 \ge 2a_1a_2h_1h_2 \ge 2a_1a_2h^2.$$

Thus, we have the expected inequality (IV-8) for $c_1 = c_1^0$ and $c_2 > c_2^0 + (2h^2)^{-1}$. □

## 5. Endomorphisms of normal surface singularities

The purpose of this section is to prove Theorem 5.3 below from which Theorem 0.2 is deduced directly. This is stated in Section 5.1 with our setting. The proof of Theorem 5.3 in the case (I) (resp. (II)) is given in Section 5.4 (resp. 5.2). In Section 5.3, we shall prove Theorem 5.10 which is a key to the proof in the case (I).

5.1. **Setting and statement.** Let $\mathfrak{X} = (X, x)$ be a germ of a normal surface $X$ at a point $x$. We consider a non-isomorphic finite surjective endomorphism $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{X}$ of the germ. Then $\mathfrak{X}$ is a log-canonical singularity by Corollary 3.7. Note that $\mathfrak{f}$ is represented by a morphism $f \colon X^\circ \to X$ of normal surfaces from an open neighborhood $X^\circ$ of $x$ such that $f$ has only discrete fibers, $f^{-1}(x) = \{x\}$, and $\deg_x f > 1$ (cf. Definition 1.9). Here, we may assume that $\operatorname{Sing} X \subset \{x\}$.

*Remark* 5.1. By assumption and by Corollary 1.8, there is an open neighborhood $\mathcal{U}$ of $x$ in $X^\circ$ such that $\mathcal{V} = f(\mathcal{U})$ is open and $f|_{\mathcal{U}} \colon \mathcal{U} \to \mathcal{V}$ is a finite morphism of degree $= \deg_x f > 1$.

*Remark* 5.2. If $\mathfrak{X} = (X, x)$ is a 2-dimensional quotient singularity, then any finite endomorphism $\mathfrak{f} \colon \mathfrak{X} \to \mathfrak{X}$ étale outside $x$ is an isomorphism (cf. [6, §2.1]). This is shown as follows: Let $f \colon X^\circ \to X$ be a representative of $\mathfrak{f}$ as above and let $f|_{\mathcal{U}} \colon \mathcal{U} \to \mathcal{V} = f(\mathcal{V})$ be the finite morphism in Remark 5.1. Now, we may assume that $\mathcal{U} \setminus \{x\}$ is étale over $\mathcal{V} \setminus \{x\}$. Since $(X, x)$ is a quotient singularity, by shrinking $\mathcal{U}$ and $\mathcal{V}$, we may assume that the fundamental group $\boldsymbol{\pi}_1(\mathcal{V} \setminus \{x\})$ of $\mathcal{V} \setminus \{x\}$ is finite. Then $\deg \mathfrak{f}$ is just the index of the subgroup $\boldsymbol{\pi}_1(\mathcal{U} \setminus \{x\})$. As a consequence, $\deg \mathfrak{f}$ is bounded. If $\deg \mathfrak{f} > 1$, then $\deg \mathfrak{f}^k = k \deg \mathfrak{f}$ is sufficiently large for $k \gg 0$ for the $k$-th power $\mathfrak{f}^k = \mathfrak{f} \circ \mathfrak{f} \circ \cdots \circ \mathfrak{f}$. Thus, $\deg \mathfrak{f} = 1$ and $\mathfrak{f}$ is an isomorphism.

**Theorem 5.3.** *Let $X$ be a normal surface with a reduced divisor $S$ such that $\operatorname{Sing} X \subset \{x\}$ and $\operatorname{Sing} S \subset \{x\}$ for a point $x$. Let $f \colon X^\circ \to X$ be a morphism from an open neighborhood of $x$ in $X^\circ$ such that $f$ has only discrete fibers, $f^{-1}(x) = \{x\}$, $\deg_x f > 0$, $f^{-1}S = S|_{X^\circ}$, and $f$ is étale over $X \setminus (\{x\} \cup \operatorname{Supp} S)$. Then $(X, S)$ is log-canonical by Theorem* 3.5. *For any essential blowing up $\varphi \colon Y \to X$ of the log-canonical pair $(X, S)$, the meromorphic map $f_Y^{(2)} \colon Y^{(2)} \cdots\!\to Y$ defined in Definition* 5.4 *below is holomorphic and has only discrete fibers in the following two cases*:

(I) *$S = 0$, and $x$ is not a cusp singularity of $X$*;
(II) *$x \in S$, and $f^*S = dS|_{X^\circ}$ for a positive integer $d$.*

**Definition 5.4.** For an integer $k \geq 1$ and for the morphism $f^{(k)} \colon X^{(k)} \to X$ in Definition 3.1, we set $Y^{(k)} := \varphi^{-1}(X^{(k)})$ and define a meromorphic map $f_Y^{(k)}$ as the composite

$$Y^{(k)} \xrightarrow{\varphi|_{Y^{(k)}}} X^{(k)} \xrightarrow{f^{(k)}} X \overset{\varphi^{-1}}{\cdots\!\to} Y.$$

Since $X^{(1)} = X^\circ$ and $f^{(1)} = f$, we write $Y^\circ := Y^{(1)}$ and $f_Y := f_Y^{(1)}$.

*Remark* 5.5. By the assumption of Theorem 5.3 and by Lemma 1.39, we have $K_{X^\circ} + S|_{X^\circ} = f^*(K_X + S)$.

5.2. **Proof of Theorem 5.3 in the case (II).** In Proposition 5.6 and Corollary 5.7 below, we treat the case where $x \in \operatorname{Sing} S$. The case where $x \in S_{\mathrm{reg}}$ and $(X, S)$ is not 1-log-terminal, is treated in Proposition 5.9 below. Theorem 5.3 in the case (II) is just derived from Corollary 5.7 and Proposition 5.9. Proposition 5.8 below concerns the case where $(X, S)$ is 1-log-terminal at $x$; it is not related to

Theorem 5.3 directly, but where we consider a lifting problem of $f$ by another kind of toroidal blowing up.

**Proposition 5.6.** *In the situation of Theorem* 5.3*, assume that* $\{x\} = S_1 \cap S_2$ *for two distinct prime components* $S_1$ *and* $S_2$ *of* $S$ *and that*

$$f^*S_i = d_i S_i|_{X^\circ}$$

*for some positive integer* $d_i$ *for* $i = 1$, $2$*. Then* $\deg_x f = d_1 d_2$*. Moreover, the meromorphic map* $f_Y = f_Y^{(1)} : Y^\circ = Y^{(1)} \dashrightarrow Y$ *in Definition* 5.4 *is holomorphic if and only if* $d_1 = d_2$*, and in this case,* $f_Y$ *has only discrete fibers.*

*Proof.* The pair $(X, S)$ is toroidal at $x$ by Fact 2.5. Hence, $\varphi \colon Y \to X$ is a toroidal blowing up by Lemma 4.25. For the finite morphism $f|_{\mathcal{U}} \colon \mathcal{U} \to \mathcal{V} = f(\mathcal{U})$ in Remark 5.1, by shrinking $\mathcal{V}$, we may assume that there is an open immersion $j \colon \mathcal{V} \hookrightarrow V$ of analytic spaces to an affine toric surface $V = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$, where $S|_{\mathcal{V}} = j^{-1}D$ for the boundary divisor $D$ of $V$. We assume that $(\mathsf{N}, \boldsymbol{\sigma})$ is as in Fact 4.1 with primitive elements $e_1$ and $e_2$ of $\mathsf{N}$ and that $S_i|_{\mathcal{V}} = j^{-1}\Gamma_i$ for $i = 1$ and 2, for the prime components $\Gamma_1 = \Gamma(e_1)$ and $\Gamma_2 = \Gamma(e_2)$ of $D$. Hence, $j(x)$ is the fixed point $*$ of the action of $\mathbb{T}_{\mathsf{N}}$. By shrinking $\mathcal{V}$ furthermore, we may assume that the open immersion $\mathcal{V} \setminus S \hookrightarrow V \setminus D \simeq \mathbb{T}_{\mathsf{N}}$ induces an isomorphism $\boldsymbol{\pi}_1(\mathcal{V} \setminus S) \simeq \boldsymbol{\pi}_1(V \setminus D) \simeq \mathsf{N}$ of fundamental groups (cf. [37, Cor. 3.1.2]). Let $\mathsf{N}^\dagger$ be a finite index subgroup of $\mathsf{N}$ isomorphic to the image of the homomorphism $\boldsymbol{\pi}_1(\mathcal{U} \setminus S) \to \boldsymbol{\pi}_1(\mathcal{V} \setminus S)$ associated with the finite étale morphism $f|_{\mathcal{U} \setminus S} \colon \mathcal{U} \setminus S \to \mathcal{V} \setminus S$, and let

$$\pi \colon V^\dagger := \mathbb{T}_{\mathsf{N}^\dagger}(\boldsymbol{\sigma}) \to V = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$$

be the toric morphism defined by the inclusion $\mathsf{N}^\dagger \subset \mathsf{N}$ and $\boldsymbol{\sigma} \subset \mathsf{N}^\dagger \otimes \mathbb{R} = \mathsf{N} \otimes \mathbb{R}$. This $\pi$ is a finite surjective morphism, and it is étale over $V \setminus D$. Then $\mathcal{U} \setminus S \to \mathcal{V} \setminus S$ is isomorphic to the base change of $\pi$ by the open immersion $\mathcal{V} \setminus S \hookrightarrow V$. Therefore, $\mathcal{U} \simeq V^\dagger \times_V \mathcal{V}$ over $\mathcal{V}$ by a theorem of Grauert–Remmert (cf. [14], [18, XII, Thm. 5.4]), since normal varieties $\mathcal{U}$ and $V^\dagger \times_V \mathcal{V}$ are finite over $\mathcal{V}$ and these are isomorphic to each other over the Zariski-open subset $\mathcal{V} \setminus S$. In particular, the singularity of $V^\dagger$ is the same as that of $\mathcal{U}$, and the *type* $(n, q)$ of $(\mathsf{N}, \boldsymbol{\sigma})$ equals that of $(\mathsf{N}^\dagger, \boldsymbol{\sigma})$ (cf. Definition 4.2). Hence, we may assume that $\mathsf{N}^\dagger = \mathsf{N}$, $V^\dagger = V$, and $\pi$ is the toric endomorphism $\mathbb{T}(\phi) \colon V \to V$ associated with an injective endomorphism $\phi \colon \mathsf{N} \to \mathsf{N}$ such that $\phi_{\mathbb{R}}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}$. The open immersion $j^\dagger \colon \mathcal{U} \hookrightarrow V^\dagger = V$ induced by $j \colon \mathcal{V} \hookrightarrow V$ is also a toroidal embedding such that $j^{\dagger -1}D = S|_{\mathcal{U}}$. Since $\pi^{-1}\Gamma_1$ is either $\Gamma_1$ or $\Gamma_2$, we have $\pi^*\Gamma_i = d_i\Gamma_i$ for $i = 1$, 2 from the equality $f^*S_i = d_i S_i|_{X^\circ}$. Hence, $\deg_x f = \deg \pi = d_1 d_2$ by Lemma 4.10. Note that $j^\dagger$ and $j$ may not induce the same open immersion to $V$ from a common open neighborhood of $x$.

The toroidal blowing up $\varphi \colon Y \to X$ is induced by the bimeromorphic toric morphism $\mu \colon W = \mathbb{T}_{\mathsf{N}}(\triangle) \to V = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ associated with a fan $\triangle$ of $\mathsf{N}$ such that $|\triangle| = \boldsymbol{\sigma}$ (cf. Example 4.3). More precisely, $\varphi$ is obtained by $\mu$ as follows: Let $\theta \colon \mathcal{W} \to \mathcal{V}$ be the base change of $\mu$ by $j \colon \mathcal{V} \hookrightarrow V$. This is expressed as the blowing up of $\mathcal{V}$ along a closed subscheme $Z$ of $\operatorname{Spec} \mathcal{O}_{V,x}/\mathfrak{m}_x^k$ for $k \gg 0$, where the defining ideal $\mathcal{J}$ of $Z$ in $\mathcal{O}_V$ is written as in Remark 4.7. The morphism $\varphi \colon Y \to X$ is defined as the blowing up of $X$ along the closed analytic subspace $Z$. In other words, $Y$ is obtained by gluing $X$ and $\mathcal{W}$ via the isomorphism $\mathcal{W} \setminus \theta^{-1}(x) \simeq \mathcal{V} \setminus \{x\}$. Here, $\triangle$ contains at least three 1-dimensional cones, since $\mu$ is not an isomorphism.

Let $W^\dagger$ be the normalization of the fiber product $V^\dagger \times_V W$ of $\pi\colon V^\dagger \to V$ and $\mu\colon W \to V$. Then $W^\dagger$ is a toric variety expressed as $\mathbb{T}_{\mathsf{N}}(\triangle^\dagger)$ for the fan $\triangle^\dagger$ consisting of cones $\phi_{\mathbb{R}}^{-1}\boldsymbol{\tau}$ for all $\boldsymbol{\tau} \in \triangle$, and the morphism $\mu^\dagger\colon W^\dagger \to V^\dagger$ induced by the first projection is a bimeromorphic toric morphism defined by $|\triangle^\dagger| = \boldsymbol{\sigma}$. Let $Y^\dagger$ be the normalization of the fiber product $X^\circ \times_X Y$ of $f$ and $\varphi$. Then the morphism $\varphi^\dagger\colon Y^\dagger \to X^\circ$ is a toroidal blowing up induced by the bimeromorphic toric morphism $\mu^\dagger$ and by the open immersion $j^\dagger\colon \mathcal{U} \hookrightarrow V^\dagger$. On the other hand, the morphism $\varphi^\circ\colon Y^\circ = \varphi^{-1}(X^\circ) \to X^\circ$ defined by $\varphi$ is also a toroidal blowing up and it is induced by $\mu\colon W \to V$ and $j\colon \mathcal{V} \hookrightarrow V$. Therefore, the holomorphicity of three meromorphic maps

$$f_Y\colon Y^\circ \dashrightarrow Y, \quad (\varphi^\dagger)^{-1} \circ \varphi^\circ\colon Y^\circ \dashrightarrow Y^\dagger, \quad \text{and} \quad (\mu^\dagger)^{-1} \circ \mu\colon W \dashrightarrow W^\dagger$$

are equivalent to one another. By Lemma 4.8, $(\mu^\dagger)^{-1} \circ \mu$ is holomorphic if and only if $\triangle = \triangle^\dagger$. Hence, $f_Y$ is holomorphic if and only if $W \simeq W^\dagger$ over $V$. Since the morphism $W^\dagger \to W$ induced by the second projection is finite and surjective, $f_Y$ has only discrete fibers if it is holomorphic.

Assume that $d_1 = d_2$. Then $\phi\colon \mathsf{N} \to \mathsf{N}$ is the multiplication map by $d_1$, by Lemma 4.10. It implies that $\triangle = \triangle^\dagger$, and hence, $f_Y$ is holomorphic.

Conversely, assume that $f_Y$ is holomorphic. Then $\phi\colon \mathsf{N}^\dagger = \mathsf{N} \to \mathsf{N}$ is *compatible* with $\triangle^\dagger = \triangle$ and $\triangle$ (cf. Definition 4.5). In particular, $\phi_{\mathbb{R}}$ has at least three eigenvectors, since $\triangle$ contains at least three 1-dimensional cones. This implies that $\phi_{\mathbb{R}}$ is a scalar map, and hence, $d_1 = d_2$ by Lemma 4.9. Thus, we are done. □

**Corollary 5.7.** *In the situation of Theorem* 5.3*, assume that* $x \in \operatorname{Sing} S$ *and* $f^*S = dS|_{X^\circ}$ *for a positive integer* $d$*. Then* $\deg_x f = d^2$*, and* $f_Y^{(2)}\colon Y^{(2)} \to Y$ *is holomorphic with only discrete fibers.*

*Proof.* By replacing $X$ with an open neighborhood of $x$, we may assume that $\{x\} = S_1 \cap S_2$ for two distinct prime components $S_1$ and $S_2$ of $S$. Thus, the assertion follows from Proposition 5.6 applied to $f^{(2)}\colon X^{(2)} \to X$ instead of $f\colon X^\circ \to X$. □

**Proposition 5.8.** *In the situation of Theorem* 5.3*, assume that* $x \in S$ *and that* $(X, S)$ *is* 1*-log-terminal at* $x$*. Then* $f|_{S \cap X^\circ}\colon S \cap X^\circ \to S$ *is an isomorphism at* $x$*. Moreover, for any integer* $k > 0$ *and for any non-isomorphic toroidal blowing up* $\varphi\colon Y \to X$ *at* $x$ *in the sense* ($\diamondsuit$) *below, the meromorphic map* $f_Y^{(k)}\colon Y^{(k)} \dashrightarrow Y$ *in Definition* 5.4 *is not holomorphic*:

($\diamondsuit$) *By Fact* 2.5*,* $x$ *has an open neighborhood* $U$ *with a prime divisor* $S'$ *on* $U$ *such that* $x \in S|_U \cap S'$ *and that* $(U, S|_U + S')$ *is toroidal at* $x$*. The bimeromorphic morphism* $\varphi\colon Y \to X$ *is a toroidal blowing up with respect to* $(U, S|_U + S')$ *for such* $U$ *and* $S'$*.*

*Proof.* For the finite morphism $f|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{V} = f(\mathcal{U})$ in Remark 5.1, we may assume the existence of an open immersion $j\colon \mathcal{V} \hookrightarrow V$ to a toric surface $V = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ satisfying the following conditions by Fact 2.5 and by an argument in the proof of Proposition 5.6:

- $j(x)$ is the fixed point $*$ of an action of $\mathbb{T}_{\mathsf{N}}$;
- $j^{-1}\Gamma_2 = S|_{\mathcal{V}}$ for a prime component $\Gamma_2$ of the boundary divisor $D = \Gamma_1 + \Gamma_2$ of $V$;
- $\varphi$ is a toroidal blowing up with respect to $(\mathcal{V}, j^{-1}D)$;
- the homomorphism $\boldsymbol{\pi}_1(\mathcal{V} \setminus j^{-1}D) \to \boldsymbol{\pi}_1(V \setminus D) = \mathsf{N}$ of fundamental groups is an isomorphism.

Let $\mathsf{N}'$ be the subgroup of $\mathsf{N}$ isomorphic to the image of the homomorphism

$$\boldsymbol{\pi}_1(\mathcal{U} \setminus f^{-1}(j^{-1}D)) \to \boldsymbol{\pi}_1(\mathcal{V} \setminus j^{-1}D)$$

associated with the finite étale morphism $\mathcal{U} \setminus f^{-1}(j^{-1}D) \to \mathcal{V} \setminus j^{-1}D$. Let $\pi \colon V' = \mathbb{T}_{\mathsf{N}'}(\boldsymbol{\sigma}) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ be the toric morphism associated with the inclusion $\mathsf{N}' \subset \mathsf{N}$ and $\boldsymbol{\sigma} \subset \mathsf{N}' \otimes \mathbb{R} = \mathsf{N} \otimes \mathbb{R}$. Then $f|_{\mathcal{U}} \colon \mathcal{U} \to \mathcal{V}$ is isomorphic to the base change of $\pi$ by $j$ by the same argument as in the proof of Proposition 5.6. In particular, the type $(n, q)$ of $(\mathsf{N}, \boldsymbol{\sigma})$ equals that of $(\mathsf{N}', \boldsymbol{\sigma})$. Hence, $\pi$ is isomorphic to a toric morphism $\mathbb{T}(\phi) \colon \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ associated with an injective homomorphism $\phi \colon \mathsf{N} \to \mathsf{N}$ such that $\phi_{\mathbb{R}}(\boldsymbol{\sigma}) = \boldsymbol{\sigma}$. Since $f|_{\mathcal{U}}$ is étale over $\mathcal{V} \setminus j^{-1}\Gamma_2$, we have $\pi^*\Gamma_1 = \Gamma_1$ and $\pi^*\Gamma_2 = d\Gamma_2$ for a positive integer $d > 0$. Hence, $\deg_x f = \deg \pi = d > 1$ by Lemma 4.9. In particular, $\pi|_{\Gamma_2} \colon \Gamma_2 \to \Gamma_2$ is an isomorphism, and hence, $f|_{S \cap X^{\circ}} \colon S \cap X^{\circ} \to S$ is an isomorphism at $x$.

Let $\mu \colon W = \mathbb{T}_{\mathsf{N}}(\triangle) \to V = \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ be a toric morphism defined by a fan $\triangle$ such that $|\triangle| = \boldsymbol{\sigma}$ and assume that the toroidal blowing up $\varphi \colon Y \to X$ in the sense of $(\diamondsuit)$ is induced by $\mu$ in the same way as in the proof of Proposition 5.6. For an integer $k > 0$, let $W^{(k)}$ be the normalization of the fiber product $V \times_V W$ of $\mu$ and $k$-th power $\pi^k \colon V \to V$. Then $W^{(k)} \simeq \mathbb{T}_{\mathsf{N}}(\triangle^{(k)})$ for the fan $\triangle^{(k)}$ consisting of cones $(\phi^k_{\mathbb{R}})^{-1}\boldsymbol{\tau}$ for all $\boldsymbol{\tau} \in \triangle$, and the morphism $W^{(k)} \to V$ induced by the first projection is a toric morphism defined by $|\triangle^{(k)}| = \boldsymbol{\sigma}$. As in the proof of Proposition 5.6, if $f^{(k)}_Y$ is holomorphic, then $\triangle^{(k)} = \triangle$, and $\phi^k_{\mathbb{R}}$ is a scalar map. However, $\phi^k_{\mathbb{R}}$ has two eigenvalues 1 and $d > 1$; thus, it is not a scalar map. Therefore, $f^{(k)}_Y$ is not holomorphic for any $k > 0$. □

**Proposition 5.9.** *In the situation of Theorem* 5.3*, assume that* $x \in S_{\mathrm{reg}}$ *and* $(X, S)$ *is not* 1*-log-terminal at* $x$*. Then there is a positive integer* $d$ *such that* $f^*S = dS|_{X^{\circ}}$ *and* $\deg_x f = d^2$*. Moreover, the meromorphic map* $f^{(2)}_Y$ *in Definition* 5.4 *is holomorphic and has only discrete fibers for any essential blowing up* $\varphi \colon Y \to X$ *of the log-canonical pair* $(X, S)$*.*

*Proof.* For the proof, we may replace $X$ with an open neighborhood of $x$ freely. Thus, by Fact 2.5(3), we may assume that $S$ is a non-singular prime divisor, $2(K_X + S) \sim 0$, and $K_X + S$ is Cartier on $X \setminus \{x\}$. In particular, $f^*S = dS|_{X^{\circ}}$ for a positive integer $d$. Let $\lambda \colon \widetilde{X} \to X$ be an index 1 cover with respect to $K_X + S$. Then

- $\lambda$ is a double-cover étale over $X \setminus \{x\}$,
- $\lambda^{-1}(x) = \{\tilde{x}\}$ for a point $\{\tilde{x}\}$, and
- $(\widetilde{X}, \widetilde{S})$ is toroidal at $\{\tilde{x}\}$ and $\tilde{x} \in \operatorname{Sing} \widetilde{S}$ for the divisor $\widetilde{S} := \lambda^*S$,

by Fact 2.5(3). Since $K_{X^{\circ}} + S|_{X^{\circ}} = f^*(K_X + S)$ (cf. Remark 5.5), by Lemma 4.21(2), after replacing $X^{\circ}$ with an open neighborhood of $x$, we have a morphism $\tilde{f} \colon \widetilde{X}^{\circ} =$

$\lambda^{-1}(X^\circ) \to \widetilde{X}$ with a commutative diagram

$$\begin{tikzcd} \widetilde{X}^\circ \arrow[r, "\tilde{f}"] \arrow[d, "\lambda|_{\widetilde{X}^\circ}"'] & \widetilde{X} \arrow[d, "\lambda"] \\ X^\circ \arrow[r, "f"] & X. \end{tikzcd}$$

Here, $\tilde{f}$ has only discrete fibers, $\tilde{f}^{-1}(\tilde{x}) = \{\tilde{x}\}$, and $\tilde{f}^*\widetilde{S} = d\widetilde{S}|_{\widetilde{X}^\circ}$. Then $\deg_x f = \deg_{\tilde{x}} \tilde{f} = d^2$ by Corollary 5.7. By iterating $f$, we have a commutative diagram

$$\begin{tikzcd} \widetilde{X}^{(2)} \arrow[r, "\tilde{f}^{(2)}"] \arrow[d, "\lambda|_{\widetilde{X}^{(2)}}"'] & \widetilde{X} \arrow[d, "\lambda"] \\ X^{(2)} \arrow[r, "f^{(2)}"] & X, \end{tikzcd}$$

where $\widetilde{X}^{(2)} := \tilde{f}^{-1}(\widetilde{X}^\circ)$ and $\tilde{f}^{(2)} := \tilde{f}\circ(\tilde{f}|_{\widetilde{X}^{(2)}})$. Note that $\tilde{f}$ and $\tilde{f}^{(2)}$ are equivariant under the action of the Galois group $\boldsymbol{\mu}_2$ of $\lambda$.

We set $T := \varphi^{-1}S$ and apply Lemma 4.34 to the essential blowing up $\varphi\colon (Y, T) \to (X, S)$ and the index 1 cover $\lambda\colon \widetilde{X} \to X$. Then we have a commutative diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\sigma"'] & \widetilde{X} \arrow[d, "\lambda"] \\ Y \arrow[r, "\varphi"] & X \end{tikzcd}$$

such that $\widetilde{Y}$ is the normalization of the fiber product $Y \times_X \widetilde{X}$ and that $\tilde{\varphi}\colon (\widetilde{Y}, \widetilde{T}) \to (\widetilde{X}, \widetilde{S})$ is an essential blowing up for the reduced divisor $\widetilde{T} = \sigma^{-1}T$. Here, $\sigma$ is also an index 1 cover with respect to $K_Y + T = \varphi^*(K_X + S)$, and $\tilde{\varphi}$ is a toroidal blowing up at $\tilde{x}$ by Lemma 4.25. Then the meromorphic map

$$\tilde{f}^{(2)}_{\widetilde{Y}} : \widetilde{Y}^{(2)} := \sigma^{-1}(Y^{(2)}) = \tilde{\varphi}^{-1}(\widetilde{X}^{(2)}) \xrightarrow{\tilde{\varphi}} \widetilde{X}^{(2)} \xrightarrow{\tilde{f}^{(2)}} \widetilde{X} \overset{\tilde{\varphi}^{-1}}{\dashrightarrow} \widetilde{Y}$$

is $\boldsymbol{\mu}_2$-equivariant and $\sigma \circ \tilde{f}^{(2)}_{\widetilde{Y}} = f^{(2)}_Y \circ \sigma|_{\widetilde{Y}^{(2)}}$. By Corollary 5.7, $\tilde{f}^{(2)}_{\widetilde{Y}}$ is a holomorphic map having only discrete fibers. Hence, $f^{(2)}_Y$ is so. $\square$

5.3. **A key theorem.** We shall prove the following theorem, which is a key to the proof of Theorem 5.3 in the case (I). For the proof, we apply results in Sections 1.4 and 4.4.

**Theorem 5.10.** *Let $X$ be a normal surface with a point $x \in X$ and let $f\colon X^\circ \to X$ be a morphism from an open neighborhood $X^\circ$ of $x$ such that $f^{-1}(x) = \{x\}$, $\deg_x f > 1$, and $f$ is étale over $X \setminus \{x\}$. Let $\varphi\colon Y \to X$ be a bimeromorphic morphism from a normal surface $Y$ such that $S = \varphi^{-1}(x)$ is a divisor, $\varphi$ is an isomorphism over $X \setminus \{x\}$, and $K_Y + S = \varphi^*K_X$. We define $g\colon Y^\circ \dashrightarrow Y$ to be the meromorphic map $f_Y$ in Definition* 5.4 *and assume that*

(*) *any prime component of $S$ is not contracted to a point by $g$.*

*Then $g$ is holomorphic and induces an automorphism of the set of prime components of $S$ by $\Gamma \mapsto g(\Gamma)$. Moreover, the following hold for the positive square root $b$ of* $\deg_x f$*:*

(1) *If $g(\Gamma) = \Gamma$ for a prime component $\Gamma$ of $S$, then $b \in \mathbb{Z}$ and $g^*\Gamma = b\Gamma$.*
(2) *There exists an effective $\mathbb{R}$-divisor $H$ on $Y$ such that* $\operatorname{Supp} H = S$*, $g^*H = bH|_{Y^\circ}$, and $H\Gamma < 0$ for any prime component $\Gamma$ of $S$.*

The proof of Theorem 5.10 is given at the end of Section 5.3. We begin with the following lemma on the graph of the meromorphic map:

**Lemma 5.11.** *Let $V$ be the normalization of the fiber product $Y \times_{X,f} X^\circ$ of $\varphi\colon Y \to X$ and $f\colon X^\circ \to X$ over $X$. Let $\phi\colon V \to Y$ and $\varphi_V\colon V \to X^\circ$ be morphisms induced by projections from the fiber product. Then there is a bimeromorphic morphism $\mu\colon V \to Y^\circ$ such that $\phi = g \circ \mu$ and $\varphi_V = \varphi^\circ \circ \mu$ for $\varphi^\circ := \varphi|_{Y^\circ}\colon Y^\circ \to X^\circ$. In particular, $V$ is isomorphic to the normalization of the graph of the meromorphic map $g$.*

*Proof.* Let $W$ be the normalization of the graph of the bimeromorphic map $\varphi_V^{-1} \circ \varphi^\circ\colon Y^\circ \cdots\to V$. Let $\nu\colon W \to Y^\circ$ and $\psi\colon W \to V$ be induced morphisms such that $\varphi^\circ \circ \nu = \varphi_V \circ \psi$. Then we have a commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{\psi} & V & \xrightarrow{\phi} & Y \\ {\scriptstyle\nu}\downarrow & & {\scriptstyle\varphi_V}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Y^\circ & \xrightarrow{\varphi^\circ} & X^\circ & \xrightarrow{f} & X. \end{array}$$

If a prime divisor $\Xi$ on $W$ is $\psi$-exceptional, then $\Xi \subset \psi^{-1}\phi^{-1}S = \nu^{-1}S$, and $\Xi$ is not expressed as $\nu^{[*]}\Gamma$ for any prime component $\Gamma$ of $S$ by (*) in Theorem 5.10; hence, $\Xi$ is $\nu$-exceptional. Therefore, the meromorphic map $\mu := \nu \circ \psi^{-1}\colon V \cdots\to Y^\circ$ is holomorphic, and $\varphi_V = \varphi^\circ \circ \mu$. Hence, $\psi\colon W \to V$ is an isomorphism, since $W$ is the normalization of the graph of $\mu^{-1} = \varphi_V^{-1} \circ \varphi^\circ$, and we have

$$g \circ \mu = \varphi^{-1} \circ f \circ \varphi^\circ \circ \mu = \varphi^{-1} \circ f \circ \varphi_V = \varphi^{-1} \circ \varphi \circ \phi = \phi.$$

Thus, $V$ is also isomorphic to the normalization of the graph of $g$. $\square$

*Remark.* By Lemma 5.11, we have a commutative diagram

(V-1)
$$\begin{tikzcd} & V \arrow[dl, "\mu"'] \arrow[dr, "\phi"] & \\ Y^\circ \arrow[rr, dashed, "g"] \arrow[d, "\varphi^\circ"'] & & Y \arrow[d, "\varphi"] \\ X^\circ \arrow[rr, "f"] & & X \end{tikzcd}$$

of normal surfaces with the following properties:

- $\varphi$, $\varphi^\circ$, and $\mu$ are bimeromorphic morphisms;
- $\phi$ has only discrete fibers, and is étale over $Y \setminus S$;

- the restriction $\mu^{-1}(\varphi^{\circ-1}\mathcal{U}) \to \varphi^{-1}\mathcal{V}$ of $\phi$ is a finite and surjective morphism of degree $\deg_x f$ for some open neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of $x$ (cf. Remark 5.1).

**Definition 5.12.** We define

$$S^\circ := S|_{Y^\circ} \quad \text{and} \quad S_V := \phi^{-1}S = \mu^{-1}S^\circ$$

as reduced divisors on $Y^\circ$ and $V$, respectively. For an $\mathbb{R}$-divisor $D$ on $Y$ such that $\operatorname{Supp} D \subset S$, we write $D^\circ = D|_{Y^\circ}$ as an $\mathbb{R}$-divisor on $Y^\circ$, and set

$$D^V := \mu^*(D^\circ) \quad \text{and} \quad D_{(V)} := \mu^{[*]}(D^\circ)$$

as $\mathbb{R}$-divisors on $V$ (cf. Definition 1.22). However, sometimes, we write $S = S^\circ$ and $D = D^\circ$ for simplicity. Note that $S_V = (S^V)_{\mathrm{red}}$.

*Remark* 5.13. The pullbacks $g^{[*]}D$ and $g^*D$ and the pushforwards $g_{[*]}D^\circ = g_{[*]}D$ and $g_*D^\circ = g_*D$ by the meromorphic map $g$ are defined in Definition 1.30. Here,

$$g_*D = \phi_*D^V \quad \text{and} \quad g_{[*]}D = \phi_*D_{(V)}$$

by definition, but

$$g^{[*]}D = g^*D = \mu_*(\phi^*D),$$

since $\phi$ has no exceptional divisor. If $g$ is holomorphic, then $g_*D = g_{[*]}D$.

**Definition 5.14.** For an integer $k \geq 0$, we define $g^{(k)} \colon Y^{(k)} \dashrightarrow Y$ to be the meromorphic map $f_Y^{(k)}$ in Definition 5.4.

*Remark* 5.15. For an $\mathbb{R}$-divisor $D$ on $Y$ such that $\operatorname{Supp} D \subset S$, we can consider $g_*^{(k)}D$, $g_{[*]}^{(k)}D$, and $g^{(k)*}D$ as in Remark 5.13. Then

$$g_{[*]}^{(k+l)}D = g_{[*]}^{(k)}(g_{[*]}^{(l)}D) \quad \text{and} \quad g^{(k+l)*}D = g^{(l)*}(g^{(k)*}D)$$

for any $k$, $l \geq 0$ by Lemma 1.32, since $\phi$ has no exceptional divisor. However, we can not expect the equality $g_*^{(k+l)}D = g_*^{(k)}(g_*^{(l)}D)$ in general.

**Definition 5.16.** Let $\mathbb{I}$ be the set of prime components of $S$ and let $\mathbb{J}$ be the set of prime components of $S_V$. We define a map $f_{\mathbb{I}} \colon \mathbb{I} \to \mathbb{I}$ by

$$f_{\mathbb{I}}(\Gamma) = \phi(\Gamma_{(V)}) = \operatorname{Supp} g_{[*]}\Gamma.$$

We define a function $a \colon \mathbb{I} \to \mathbb{Z}_+ = \{m \in \mathbb{R} \mid m > 0\}$ by

$$a(\Gamma) := \operatorname{mult}_\Gamma g^*S = \operatorname{mult}_\Gamma g^*(f_{\mathbb{I}}(\Gamma)) = \operatorname{mult}_{\Gamma_{(V)}} \phi^*S = \operatorname{mult}_{\Gamma_{(V)}} \phi^*(f_{\mathbb{I}}(\Gamma)).$$

For $\Gamma \in \mathbb{I}$, we define $\mathbb{J}_\Gamma$ to be the set of prime components $\Theta$ of $S_V$ such that $\phi(\Theta) = \Gamma$. Then $\mathbb{J} = \bigsqcup_{\Gamma \in \mathbb{I}} \mathbb{J}_\Gamma$. For $\Theta \in \mathbb{J}_\Gamma$, we define

$$a_\Theta := \operatorname{mult}_\Theta \phi^*S = \operatorname{mult}_\Theta \phi^*\Gamma \quad \text{and} \quad m_\Theta := \operatorname{mult}_\Gamma \phi_*\Theta = \deg(\phi|_\Theta \colon \Theta \to \Gamma).$$

*Remark* 5.17. We have $a(\Gamma) \in \mathbb{Z}$ for any $\Gamma \in \mathbb{I}$, since $\phi$ has only discrete fiber and since $\phi^*S$ is a divisor (cf. Lemma 1.19 and Remarks 1.20 and 1.24(5)). Moreover,

$$a(\Gamma) = a_{\Gamma_{(V)}} \quad \text{and} \quad g_{[*]}\Gamma = m_{\Gamma_{(V)}} f_{\mathbb{I}}(\Gamma)$$

for the proper transform $\Gamma_{(V)} = \mu^{[*]}\Gamma^{\circ}$ in $V$, where $\Gamma_{(V)} \in \mathbb{J}_{f_{\mathbb{I}}(\Gamma)}$. If $f_{\mathbb{I}}^{-1}(f_{\mathbb{I}}(\Gamma)) = \{\Gamma\}$, then

$$\text{(V-2)} \qquad g^*(f_{\mathbb{I}}(\Gamma)) = \mu_*\phi^*(f_{\mathbb{I}}(\Gamma)) = a(\Gamma)\mu_*\Gamma_{(V)} = a(\Gamma)\Gamma,$$

since $\mu_*\Theta = 0$ for any other member $\Theta$ of $\mathbb{J}_{f_{\mathbb{I}}(\Gamma)}$. For any integer $k \geq 1$,

(1) the $k$-th power $(f_{\mathbb{I}})^k = f_{\mathbb{I}} \circ \cdots \circ f_{\mathbb{I}} \colon \mathbb{I} \to \mathbb{I}$ equals the map $(f^k)_{\mathbb{I}}$ associated with $f^k \colon X^{(k)} \to X$, which maps $\Gamma \in \mathbb{I}$ to $\operatorname{Supp} g^{(k)}_{[*]}\Gamma$, and

(2) the equality
$$\operatorname{mult}_{\Gamma}(g^{(k)})^*S = \prod\nolimits_{i=0}^{k-1} a(f_{\mathbb{I}}^i(\Gamma))$$
holds for any $\Gamma \in \mathbb{I}$.

These are shown by equalities in Remark 5.15.

*Remark* 5.18. For $\Gamma \in \mathbb{I}$ and $\Theta \in \mathbb{J}_{\Gamma}$, we have

$$\phi^*\Gamma = \sum\nolimits_{\Theta \in \mathbb{J}_{\Gamma}} a_{\Theta}\Theta \quad \text{and} \quad \phi_*\Theta = m_{\Theta}\Gamma$$

by Definition 5.16, and moreover,

$$\phi_*\boldsymbol{D}(\Theta/S_V) = a_{\Theta}\boldsymbol{D}(\Gamma/S) \quad \text{and} \quad \phi^*\boldsymbol{D}(\Gamma/S) = \sum\nolimits_{\Theta \in \mathbb{J}_{\Gamma}} m_{\Theta}\boldsymbol{D}(\Theta/S_V),$$

by Lemma 4.37.

**Lemma 5.19.** *Let $D$ be a non-zero effective $\mathbb{R}$-divisor on $Y$ such that* $\operatorname{Supp} D \subset S$. *We set $H := H_D := \sum_{\Gamma \in \mathbb{I}} h_{\Gamma}\boldsymbol{D}(\Gamma/S)$, where*

$$h_{\Gamma} = \begin{cases} 0, & \textit{if } \operatorname{mult}_{\Gamma} D = 0, \\ -(\operatorname{mult}_{\Gamma} D)^{-1}, & \textit{otherwise.} \end{cases}$$

*Then $H$ is effective,* $\operatorname{Supp} H = S$, *and $-H$ is* nef on $S$ (*cf. Remark* 1.25). *If $f_{\mathbb{I}} \colon \mathbb{I} \to \mathbb{I}$ is bijective and if $g^*D = bD$ for a real number $b > 0$, then $g^{(k)}_* H = b^k H$ for any $k \geq 1$.*

*Proof.* By Lemma-Definition 4.35(3), $H$ is effective and $\operatorname{Supp} H = S$. Moreover, $H\Gamma = h_{\Gamma} \leq 0$ for any $\Gamma \in \mathbb{I}$ by Lemma-Definition 4.35(1). Thus, $-H$ is nef on $S$, and we have proved the first assertion. Assume that $g^*D = bD$. Then

$$a(\Gamma)\operatorname{mult}_{f_{\mathbb{I}}(\Gamma)} D = \operatorname{mult}_{\Gamma} g^*D = b \operatorname{mult}_{\Gamma} D$$

for any $\Gamma \in \mathbb{I}$ by the definition of $a(\Gamma)$. In particular, $\Gamma \subset \operatorname{Supp} D$ if and only if $f_{\mathbb{I}}(\Gamma) \subset \operatorname{Supp} D$, and we have

$$a(\Gamma)h_{\Gamma} = bh_{f_{\mathbb{I}}(\Gamma)}$$

for any $\Gamma \subset \operatorname{Supp} D$. On the other hand, for any $\Gamma \in \mathbb{I}$, we have

$$\mu^*\boldsymbol{D}(\Gamma/S) = \boldsymbol{D}(\Gamma_{(V)}/S_V) \quad \text{and} \quad g_*\boldsymbol{D}(\Gamma/S) = \phi_*\boldsymbol{D}(\Gamma_{(V)}/S_V) = a(\Gamma)\boldsymbol{D}(f_{\mathbb{I}}(\Gamma)/S)$$

by Lemma 4.36 and Remark 5.18. Therefore,

$$\begin{aligned} g_*H = \sum\nolimits_{\Gamma \subset \operatorname{Supp} D} h_{\Gamma} g_*\boldsymbol{D}(\Gamma/S) &= \sum\nolimits_{\Gamma \subset \operatorname{Supp} D} h_{\Gamma} a(\Gamma)\boldsymbol{D}(f_{\mathbb{I}}(\Gamma)/S) \\ &= b\sum\nolimits_{\Gamma \subset \operatorname{Supp} D} h_{f_{\mathbb{I}}(\Gamma)}\boldsymbol{D}(f_{\mathbb{I}}(\Gamma)/S), \end{aligned}$$

and we have $g_*H = bH$ when $f_{\mathbb{I}}$ is bijective. For any $k \geq 1$, we have $g^{(k)*}D = b^k D$ by Remark 5.15, and if $f_{\mathbb{I}}$ is bijective, then $(f^k)_{\mathbb{I}} = (f_{\mathbb{I}})^k$ is bijective by Remark 5.17(1). Hence, if $f_{\mathbb{I}}$ is bijective, then $g^{(k)}_* H = b^k H$ by the argument above applied to $f^{(k)}$ instead of $f$. □

**Lemma 5.20.** *Assume that $X \setminus \{x\}$ is non-singular. Then $(Y, S)$ and $(V, S_V)$ are log-canonical, and $K_V + S_V = \mu^*(K_{Y^\circ} + S^\circ)$.*

*Proof.* The pair $(Y, S)$ is log-canonical by $K_Y + S = \varphi^*(K_X)$ and by Lemma 2.10(1). Since $\phi$ is étale over $Y \setminus S$ and since $f$ is étale over $X \setminus \{x\}$, we have

$$K_V + S_V = \phi^*(K_Y + S) = \phi^*(\varphi^* K_X) = \varphi_V^*(f^* K_X) = \varphi_V^*(K_{X^\circ})$$

by Lemma 1.39. Thus, $(V, S_V)$ is also log-canonical by Lemma 2.10(1). Moreover,

$$\mu^*(K_{Y^\circ} + S^\circ) = \mu^*(\varphi^{\circ *} K_{X^\circ}) = \varphi_V^*(K_{X^\circ}) = K_V + S_V$$

by Lemma 5.11. □

**Proposition 5.21.** *Let $H$ be a non-zero $\mathbb{R}$-divisor on $Y$ and let $b$ be a positive real number such that* $\operatorname{Supp} H \subset S$, *$-H$ is nef on $S$, and $g^{(k)}_* H = b^k H$ for any $k \geq 1$. Then $\phi^* H = bH^V$ and* $\deg_x f = b^2$.

*Proof.* By Remark 1.25, $H$ is effective and $\operatorname{Supp} H = S$. Moreover, we can write

$$H = \sum\nolimits_{\Gamma \in \mathbb{I}} \beta_\Gamma \boldsymbol{\Delta}(\Gamma, H) \tag{V-3}$$

for non-negative real numbers $\beta_\Gamma = -(H\Gamma) \operatorname{mult}_\Gamma H$ by (2) and (4) of Lemma-Definition 4.35. Note that $\beta := \sum_{\Gamma \in \mathbb{I}} \beta_\Gamma > 0$ as $H \neq 0$. In order to prove $\phi^* H = bH^V$ and $\deg_x f = b^2$, we may replace $X$ with an open neighborhood of $x$. Thus, we may assume that $X \setminus \{x\}$ is non-singular. Then there exist positive integers $c_1 < c_2$ depending on $(Y, S, H)$ such that

$$c_1 H^V \leq \boldsymbol{\Delta}(\Theta, H^V) \leq c_2 H^V \tag{V-4}$$

for any $\Theta \in \mathbb{J}$, by Lemma 5.20 and by Proposition 4.38 applied to $(Y^\circ, S^\circ, H^\circ)$, $\mu \colon V \to Y^\circ$, and $\Theta$.

For a prime component $\Theta$ of $S_V$, we define

$$t_\Theta := \frac{\operatorname{mult}_\Theta H^V}{\operatorname{mult}_\Gamma H},$$

where $\Gamma = \phi(\Theta)$, i.e., $\Theta \in \mathbb{J}_\Gamma$. Then

$$\phi^* \boldsymbol{\Delta}(\Gamma, H) = \sum\nolimits_{\Theta \in \mathbb{J}_\Gamma} m_\Theta t_\Theta \boldsymbol{\Delta}(\Theta, H^V)$$

by Lemma 4.37 and Lemma-Definition 4.35(2). We have

$$b = \sum\nolimits_{\Theta \in \mathbb{J}_\Gamma} m_\Theta t_\Theta \tag{V-5}$$

by calculations

$$\boldsymbol{\Delta}(\Gamma, H)\phi_* H^V = b\boldsymbol{\Delta}(\Gamma, H)H = -b \qquad \text{and}$$
$$(\phi^* \boldsymbol{\Delta}(\Gamma, H))H^V = \sum\nolimits_{\Theta \in \mathbb{J}_\Gamma} m_\Theta t_\Theta \boldsymbol{\Delta}(\Theta, H^V)H^V = -\sum\nolimits_{\Theta \in \mathbb{J}_\Gamma} m_\Theta t_\Theta$$

using $\phi_* H^V = g_* H = bH$ and Lemma-Definition 4.35(5). Therefore,

$$c_1 b H^V \le \phi^* \mathbf{\Delta}(\Gamma, H) \le c_2 b H^V$$

for any $\Gamma \in \mathbb{I}$ by (V-4) and (V-5). Applying $\phi_*$, we have

$$c_1 b^2 H \le (\deg_x f) \mathbf{\Delta}(\Gamma, H) \le c_2 b^2 H$$

for any $\Gamma \in \mathbb{I}$, since $\phi$ is a finite morphism of degree $\deg_x f$ over an open neighborhood of $S$. Therefore,

$$c_1 \beta b^2 \le \deg_x f \le c_2 \beta b^2$$

for $\beta = \sum_{\Gamma \in \mathbb{I}} \beta_\Gamma > 0$ by (V-3). Since $g_*^{(k)} H = b^k H$ and since $c_1$, $c_2$, and $\beta$ depend only on $(Y, S, H)$, we can apply the argument above for $f^{(k)}$ instead of $f$. Then

$$c_1 \beta b^{2k} \le \deg_x f^{(k)} = (\deg_x f)^k \le c_2 \beta b^{2k}$$

for any $k \ge 1$. Taking limits for $k \to \infty$, we have $\deg_x f = b^2$. Then

$$\begin{aligned}(\phi^* H - bH^V)^2 &= (\phi^* H)^2 - 2b(\phi^* H)H + b^2(\mu^* H^\circ)^2 \\ &= (\deg_x f)H^2 - 2b^2 H^2 + b^2 H^2 = 0,\end{aligned}$$

by $H^V = \mu^* H^\circ$. Hence, $\phi^* H = bH^V$, since the intersection matrix of prime components of $S$ is negative definite. □

*Remark.* The method of the proof above is borrowed from the proof of [6, Prop. 2.1].

**Lemma 5.22.** *Theorem* 5.10 *holds true if* $f_{\mathbb{I}} \colon \mathbb{I} \to \mathbb{I}$ *is bijective.*

*Proof.* We shall divide the proof into three steps:

*Step* 1. Let $D$ and $H = H_D$ be $\mathbb{R}$-divisors in Lemma 5.19, and assume that $g^* D = bD$ for a real number $b > 0$. Then $\phi^* H = bH^V = b\mu^* H$ and $\deg_x f = b^2$ by Lemma 5.19 and Proposition 5.21. Assuming that $\operatorname{Supp} D = S$, we shall show that $g$ is holomorphic and that $H$ satisfies the condition of Theorem 5.10(2). By assumption, $H\Gamma = h_\Gamma < 0$ for any $\Gamma \in \mathbb{I}$, and $H$ satisfies the condition of Theorem 5.10(2) by Lemma 5.19. On the other hand, $\phi^* H = bH^V$ implies that

$$H(\phi_* \Theta) = (\phi^* H)\Theta = b(\mu^* H)\Theta = 0$$

for any $\mu$-exceptional prime divisor $\Theta$. Hence, $\phi_* \Theta = 0$ for any $\mu$-exceptional prime divisor $\Theta$, and consequently, $\mu$ is an isomorphism and $g$ is holomorphic.

*Step* 2. We shall show that $a(\Gamma)^2 = \deg_x f$ for any $\Gamma \in \mathbb{I}$ satisfying $f_{\mathbb{I}}(\Gamma) = \Gamma$. Now, $g^* \Gamma = a(\Gamma)\Gamma$ by (V-2) in Remark 5.17. By applying *Step* 1 to $D = \Gamma$, we have $a(\Gamma)^2 = \deg_x f$. As a consequence, we have $g^* S = bS$ for $b := (\deg_x f)^{1/2} > 0$ provided that $f_{\mathbb{I}}$ is the identity map.

*Step* 3. Final step. By *Step* 1, it is enough to construct an effective $\mathbb{R}$-divisor $D$ on $Y$ such that $\operatorname{Supp} D = S$ and $g^* D = bD$ for $b := (\deg_x f)^{1/2}$. Let $n$ be the order of the bijection $f_{\mathbb{I}} \colon \mathbb{I} \to \mathbb{I}$. Then $(\deg_x f)^n = b^{2n} = \deg_x f^{(n)}$ and $f_{\mathbb{I}}^{(n)} = (f_{\mathbb{I}})^n = \mathrm{id}_{\mathbb{I}}$ by Remark 5.17(1), and $g^{(n)*} S = b^n S$ by *Step* 2 applied to $f^{(n)} \colon X^{(n)} \to X$ instead of $f$. By Remark 5.17(2), we have

$$b^n = \operatorname{mult}_\Gamma g^{(n)*} S = \prod\nolimits_{k=0}^{n-1} a((f_{\mathbb{I}})^k \Gamma) \tag{V-6}$$

for any $\Gamma \in \mathbb{I}$. Let $\mathbb{M}$ be the multiplicative abelian group defined as the set of maps $\mathbb{I} \to \mathbb{R}_+ = \{r \in \mathbb{R} \mid r > 0\}$. The bijection $f_{\mathbb{I}}$ defines an action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{M}$ in which the transform $\gamma^{\mathsf{T}}$ of $\gamma \in \mathbb{M}$ by the action of $1 \in \mathbb{Z}/n\mathbb{Z}$ is given by $\gamma^{\mathsf{T}}(\Gamma) = \gamma(f_{\mathbb{I}}(\Gamma))$. We define $\varepsilon \in \mathbb{M}$ as a map $\mathbb{I} \to \mathbb{R}_+$ given by $\varepsilon(\Gamma) = b^{-1}a(\Gamma)$. Then

$$\prod\nolimits_{k=0}^{n-1} \varepsilon^{\mathsf{T}^k} = 1$$

by (V-6), and hence, $\varepsilon$ defines a 1-cocycle of the $\mathbb{Z}/n\mathbb{Z}$-module $\mathbb{M}$. The group cohomology $H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{M})$ is trivial, since the $n$-th power map is bijective for $\mathbb{R}_+$ and for $\mathbb{M}$. Thus, we have a map $\delta \colon \mathbb{I} \to \mathbb{R}_+$ such that $\varepsilon = \delta \cdot (\delta^{\mathsf{T}})^{-1}$, i.e.,

$$\varepsilon(\Gamma) = \delta(\Gamma)\delta(f_{\mathbb{I}}(\Gamma))^{-1}$$

for any $\Gamma \in \mathbb{I}$. Then $D = \sum_{\Gamma \in \mathbb{I}} \delta(\Gamma)\Gamma$ satisfies $\operatorname{Supp} D = S$ and

$$\begin{aligned} g^*D = \sum\nolimits_{\Gamma \in \mathbb{I}} \delta(f_{\mathbb{I}}(\Gamma)) g^*(f_{\mathbb{I}}\Gamma) &= \sum\nolimits_{\Gamma \in \mathbb{I}} \delta(f_{\mathbb{I}}(\Gamma)) a(\Gamma)\Gamma \\ &= \sum\nolimits_{\Gamma \in \mathbb{I}} \varepsilon(\Gamma)^{-1} a(\Gamma)\delta(\Gamma)\Gamma = bD \end{aligned}$$

by (V-2) in Remark 5.17. Thus, we are done. $\square$

Now, we shall finish the proof of Theorem 5.10:

*Proof of Theorem* 5.10. We set $\mathbb{I}_\infty := \bigcap_{k \geq 1} f_{\mathbb{I}}^k(\mathbb{I})$. Then $\mathbb{I}_\infty = f_{\mathbb{I}}^m(\mathbb{I})$ for some $m > 0$, and $f_{\mathbb{I}}$ induces a bijection $\mathbb{I}_\infty \to \mathbb{I}_\infty$. By Lemma 5.22, it is enough to derive a contradiction assuming that $\mathbb{I}_\infty \neq \mathbb{I}$. Let $\pi \colon Y \to \overline{Y}$ be the contraction morphism of all the prime components of $S$ not contained in $\mathbb{I}_\infty$. Let $\bar{\varphi} \colon \overline{Y} \to X$ be the induced bimeromorphic morphism satisfying $\varphi = \bar{\varphi} \circ \pi$ and set

$$\bar{g} \colon \overline{Y}^\circ := \bar{\varphi}^{-1}(X^\circ) \xrightarrow{\bar{\varphi}^\circ} X^\circ \xrightarrow{f} X \overset{\bar{\varphi}^{-1}}{\dashrightarrow} \overline{Y}$$

to be a meromorphic map defined by $f$, $\bar{\varphi}$, and $\bar{\varphi}^\circ = \bar{\varphi}|_{\overline{Y}^\circ}$. Then we have a commutative diagram

(V-7)
$$\begin{tikzcd} & V \arrow[dl, "\mu"'] \arrow[dr, "\phi"] & \\ Y^\circ \arrow[rr, dashed, "g"] \arrow[d, "\pi^\circ"'] \arrow[dd, bend right=60, "\varphi^\circ"'] & & Y \arrow[d, "\pi"] \arrow[dd, bend left=60, "\varphi"] \\ \overline{Y}^\circ \arrow[rr, dashed, "\bar{g}"] \arrow[d, "\bar{\varphi}^\circ"'] & & \overline{Y} \arrow[d, "\bar{\varphi}"] \\ X^\circ \arrow[rr, "f"] & & X \end{tikzcd}$$

extending (V-1), where $\pi^\circ = \pi|_{Y^\circ}$. The set $\overline{\mathbb{I}}$ of prime components of $\overline{S} = \pi(S) = \bar{\varphi}^{-1}(x)$ is identified with $\mathbb{I}_\infty$, and the map $f_{\overline{\mathbb{I}}} \colon \overline{\mathbb{I}} \to \overline{\mathbb{I}}$ defined by $\overline{\Gamma} \mapsto \bar{g}_{[*]}\overline{\Gamma}$ is identical to the bijection $\mathbb{I}_\infty \to \mathbb{I}_\infty$ induced by $f_{\mathbb{I}}$. Hence, by Lemma 5.22, $\bar{g}$ is holomorphic, and $\bar{g}^*\overline{H} = b\overline{H}$ for an $\mathbb{R}$-divisor $\overline{H}$ on $\overline{Y}$ such that $\overline{H}\overline{\Gamma} < 0$ for any $\overline{\Gamma} \in \overline{\mathbb{I}}$, where $b^2 = \deg_x f$. Then

$$b\mu^*(\pi^{\circ *}\overline{H}) = \mu^*(\pi^{\circ *}(\bar{g}^*\overline{H})) = \phi^*(\pi^*\overline{H})$$

by the equality $\bar{g}\circ\pi^\circ\circ\mu = \pi\circ\phi$ shown in (V-7). For any $\Gamma\in\mathbb{I}$, if $f_{\mathbb{I}}(\Gamma)\in\mathbb{I}_\infty$, then $\Gamma\in\mathbb{I}_\infty$, by

$$\begin{aligned} &b\overline{H}(\pi_*\Gamma) = b(\pi^*\overline{H})\Gamma = b(\pi^{\circ *}\overline{H})\Gamma^\circ = b(\pi^{\circ *}\overline{H})\mu_*\Gamma_{(V)} = b\mu^*(\pi^{\circ *}\overline{H})\Gamma_{(V)} \\ &= \phi^*(\pi^*\overline{H})\Gamma_{(V)} = (\pi^*\overline{H})\phi_*\Gamma_{(V)} = m_{\Gamma_{(V)}}(\pi^*\overline{H})f_{\mathbb{I}}(\Gamma) = m_{\Gamma_{(V)}}\overline{H}\pi_*(f_{\mathbb{I}}(\Gamma)) < 0. \end{aligned}$$

Therefore, $\mathbb{I}=\mathbb{I}_\infty$, a contradiction. Thus, we are done. □

5.4. **Proof of Theorem 5.3 in the case (I).** We shall finish the proof of Theorem 5.3.

**Lemma 5.23.** *In the situation of the case* (I) *of Theorem* 5.3*, assume that the index* 1 *cover of* $(X,x)$ *with respect to* $K_X$ *is a simple elliptic singularity. Then the exceptional locus* $C=\varphi^{-1}(x)$ *is irreducible, and the meromorphic map* $f_Y\colon Y^\circ \dashrightarrow Y$ *is holomorphic and has only discrete fibers. Moreover,* $\deg_x f = b^2$ *for a positive integer* $b$*, and* $f_Y^*C = bC|_{Y^\circ}$*.*

*Proof.* Every essential blowing up $\varphi\colon Y\to X$ is isomorphic to the *standard partial resolution* (cf. Definition 4.27) and $C=\varphi^{-1}(0)$ is irreducible by Example 4.29. Let $V$ be the normalization of the fiber product $Y\times_{X,f}X^\circ$ of $\varphi$ and $f$ over $X$. Then the induced morphism $\varphi_V\colon V\to X^\circ$ is also an essential blowing up by Lemma 4.34. Thus, the bimeromorphic map $\varphi_V^{-1}\circ\varphi\colon Y^\circ\dashrightarrow V$ is an isomorphism by Corollary 4.33(3), and $f_Y$ is holomorphic with only discrete fibers. We have $f_Y^*C=bC$ for a positive integer $b$ by construction, where $b^2=\deg_x f$ by $C^2<0$. □

*Remark.* We can prove Lemma 5.23 by another method as follows. When $(X,x)$ is a simple elliptic singularity, $\varphi$ is the minimal resolution of singularities and $C$ is an elliptic curve (cf. Example 4.29(2)); in this case, it is easy to prove the assertion. Next, we consider the case where $(X,x)$ is a rational singularity. By localizing $X$, we may have an index 1 cover $\lambda\colon\widetilde{X}\to X$ with respect to $K_X$ such that $(\widetilde{X},\tilde{x})$ is a simple elliptic singularity for the point $\tilde{x}$ lying over $x$. Moreover, we may assume that $f\colon X^\circ\to X$ lifts to a morphism $\tilde{f}\colon\widetilde{X}^\circ=\lambda^{-1}(X^\circ)\to\widetilde{X}$ by Lemma 4.21(2). Thus, in this case, we can prove that $f_Y$ is holomorphic and has only discrete fibers, by the same method as in the proof of Proposition 5.9 using Lemma 4.34.

**Lemma 5.24.** *In the situation of the case* (I) *of Theorem* 5.3*, assume that* $(X,x)$ *is a rational singularity whose index* 1 *cover with respect to* $K_X$ *is a cusp singularity. Assume also that the essential blowing up* $\varphi\colon Y\to X$ *is obtained from the standard partial resolution of* $X$ *by contracting all the non-end components of the exceptional divisor, which forms a reducible linear chain of rational curves* (*cf. Example* 4.29(5))*. Then* $f_Y\colon Y^\circ\dashrightarrow Y$ *is holomorphic and has only discrete fibers. Moreover,* $(f_Y^{(2)})^*\Gamma = (\deg_x f)\Gamma|_{Y^{(2)}}$ *for any* $\varphi$*-exceptional prime divisor* $\Gamma$*.*

*Proof.* The exceptional locus $\varphi^{-1}(x)$ is a linear chain $\Gamma_1+\Gamma_2$ consisting of two prime components by construction and by Example 4.29(5). In particular, $\sharp\Gamma_1\cap\Gamma_2=1$. For the normalization $V$ of the fiber product $Y\times_{X,f}X^\circ$ of $\varphi$ and $f$ over $X$, the induced morphism $\varphi_V\colon V\to X^\circ$ is also an essential blowing up by Lemma 4.34. Thus, the bimeromorphic map $\varphi_V^{-1}\circ\varphi\colon Y^\circ\dashrightarrow V$ does not contract $\Gamma_1$ and $\Gamma_2$ to points by Corollary 4.33(2). Hence, $f_Y$ does not contract $\Gamma_1$ and $\Gamma_2$ to points and the image of $\Gamma_1$ by $f_Y$ is either $\Gamma_1$ or $\Gamma_2$, and vice versa. Therefore, the assertion is a consequence of Theorem 5.10. □

Theorem 5.3 has been proved in the case (II) by Corollary 5.7 and Proposition 5.9 in Section 5.2. Finally, we shall prove Theorem 5.3 in the case (I):

*Proof of Theorem* 5.3 *in the case* (I). Now, $(X, x)$ is a log-canonical singularity but is not a quotient singularity nor a cusp singularity. Hence, either

(a) the index 1 cover of $(X, x)$ with respect to $K_X$ is a simple elliptic singularity, or
(b) $(X, x)$ is a rational singularity and its index 1 cover with respect to $K_X$ is a cusp singularity

by the classification of 2-dimensional log-canonical singularities (cf. [29, Thm. 65]). In the case (a), Theorem 5.3 is a consequence of Lemma 5.23. It is enough to consider the case (b). Let $\widehat{\varphi}\colon \widehat{Y} \to X$ be the essential blowing up $\varphi$ in Lemma 5.24. Then any essential blowing up $\varphi\colon Y \to X$ factors through $\widehat{Y}$ by a toroidal blowing up $Y \to \widehat{Y}$, by Lemma 4.32 and Corollary 4.33(3). By Lemma 5.24, the meromorphic map $f^{(2)}_{\widehat{Y}}\colon \widehat{Y}^{(2)} = \hat{\varphi}^{-1}(X^{(2)}) \cdots\to \widehat{Y}$ defined as a lift of $f^{(2)}$ is holomorphic with only discrete fibers, and

$$(f^{(2)}_{\widehat{Y}})^*\Gamma_i = (\deg_x f)\Gamma_i$$

for $i = 1,\ 2$, for the exceptional locus $\hat{\varphi}^{-1}(x) = \Gamma_1 \cup \Gamma_2$. Hence, the lift $f^{(2)}_Y$ of $f^{(2)}_{\widehat{Y}}$ is also holomorphic with only discrete fibers by Proposition 5.6. Thus, we are done. □

## References


[1] M. Artin, Algebraic approximation of structures over complete local rings, Publ. Math. I.H.É.S. **36** (1969), 23–58.

[2] H. Cartan, Quotients of complex analytic spaces, Contributions of function theory (Internat. Colloq. Function Theory, Bombay, 1960), pp. 1–15, Tata Inst. Fund. Res., Bombay, 1960.

[3] M. Demazure, Anneaux gradués normaux, *Introduction a la Théorie des Singularités II* (ed. Lê Dũng Tráng), pp. 35–68, Travaux Cours **37**, Hermann, 1988.

[4] J. Diller and C. Favre, Dynamics of bimeromorphic maps of surfaces, Amer. J. Math. **123** (2001), 1135–1169.

[5] H. Esnault, Fibre de Milnor d'un cône sur une courbe plane singulière, Invent. Math. **68** (1982), 477–496.

[6] C. Favre, Holomorphic self-maps of singular rational surfaces, Publ. Mat. **54** (2010), 389–432.

[7] G. Fischer, *Complex Analytic Geometry*, Lecture Notes in Math. **538**, Springer-Verlag, 1976.

[8] J. Frisch, Points de platitude d'un morphisme d'espaces analytiques complexes, Invent. Math. **4** (1967), 118–138.

[9] A. Fujiki, On the blowing down of analytic spaces, Publ. RIMS, Kyoto Univ. **10** (1975), 471–507.

[10] T. Fujita, On Zariski Problem, Proc. Japan Acad. **55**, Ser. A (1979), 106–110.

[11] T. Fujita, Fractionally logarithmic canonical rings of algebraic surfaces, J. Fac. Sci. Univ. Tokyo Sect. IA **30** (1984), 685–696.

[12] W. Fulton, *Introduction to Toric Varieties*, Ann. of Math. Studies, **131**, Princeton Univ. Press, 1993.

[13] H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Ann. Math. **146** (1962), 331–368.

[14] H. Grauert and R. Remmert, Komplexe Räume, Math. Ann. **136** (1958), 245–318.

[15] H. Grauert and R. Remmert, *Coherent Analytic Sheaves*, Grundlehren der math. Wiss. **265**, Springer-Verlag, 1984.

[16] A. Grothendieck, Éléments de géométrie algébrique (rédigés avec la collaboration de J. Dieudonné): IV, Publ. Math. I.H.É.S. **20** (1964), **24** (1965), **28** (1966), **32** (1967).

[17] A. Grothendieck, Groupes diagonalisables, Exp. VIII of *Groupes de Type Multiplicatif, et Structure des Schémas en Groupes Généraux* (Schéma en groupes, SGA3, Tome II, eds. M. Demazure and A. Grothendieck), pp. 1–36, Lecture Notes in Math. **152**, Springer-Verlag, 1970.

[18] A. Grothendieck and Mme M. Raynaud, *Revêtements Étales et Groupe Fondamental* (SGA1), Lecture Notes in Math. **224**, Springer-Verlag, 1971; A new updated edition: Documents Math. **3**, Soc. Math. France, 2003.

[19] A. Grothendieck and J. L. Verdier, Préfaisceaux, Exp. I of *Théorie des Toposes et Cohomologie Etale des Schémas* (SGA4, Tome 1, eds. M. Artin, A, Grothendieck, and J. L. Verdier), pp. 1–217, Lecture Notes in Math. **269**, Springer-Verlag, 1972.

[20] V. Guedj, Ergodic properties of rational mappings with large topological degree, Ann. of Math. **161** (2005), 1589–1607.

[21] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math. **20**, Springer-Verlag, 1966.

[22] J.-M. Hwang and N. Nakayama, On endomorphisms of Fano manifolds of Picard number one, Pure and Appl. Math. Quarterly, **7** (2011), 1407–1426.

[23] S. Iitaka, On logarithmic Kodaira dimension of algebraic varieties, *Complex Analysis and Algebraic Geometry* (eds. W. L. Baily, Jr. and T. Shioda), pp. 175–189, Iwanami-Shoten Publishers and Cambridge Univ. Press, 1977.

[24] S. Iitaka, *Algebraic Geometry, An Introduction to Birational Geometry of Algebraic Varieties*, Grad. Texts in Math. **76**, Springer-Verlag, 1982.

[25] S. Iitaka, Basic structure of algebraic varieties, *Algebraic Varieties and Analytic Varieties* (ed. S. Iitaka), Adv. Stud. in Pure Math. **1**, pp. 303–316, Kinokuniya and North-Holland, 1983.

[26] S. Ishii, On isolated Gorenstein singularities, Math. Ann. **270** (1985), 541–554.

[27] M. Kashiwara and P. Schapira, *Categories and Sheaves*, Grundlehren der math. Wiss. **332**, Springer-Verlag, 2006.

[28] Y. Kawamata, On the classification of non-complete algebraic surfaces, *Algebraic Geometry* (Copenhagen 1978, ed. K. Lønsted), Lecture Notes in Math. **732**, pp. 215–232, Springer-Verlag, 1979.

[29] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Ann. of Math. **127** (1988), 93–163.

[30] Y. Kawamata, Index 1 covers of log terminal surface singularities, J. Alg. Geom. **8** (1999), 519–527.

[31] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the minimal model problem, *Algebraic geometry, Sendai, 1985* (ed. T. Oda), Adv. Stud. Pure Math. **10**, pp. 283–360, Kinokuniya and North-Holland, 1987.

[32] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal Embeddings, I*, Lecture Notes in Math. **339**, Springer-Verlag, 1973.

[33] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math. **134**, Cambridge Univ. Press, 1998.

[34] J. Kollár et al, *Flips and Abundance for Algebraic Threefolds*, Astérisque **211**, Soc. Math. de France, 1992.

[35] D. Mumford, The topology of normal surface singularities of an algebraic surface and a criterion for simplicity, Publ. Math. I.H.É.S. **9** (1961), 5–22.

[36] N. Nakayama, The lower semi-continuity of the plurigenera of complex varieties, *Algebraic Geometry, Sendai, 1985* (ed. T. Oda), pp. 551–590, Adv. Stud. in Pure Math. **10**, Kinokuniya and North-Holland, 1987.

[37] N. Nakayama, Global structure of an elliptic fibration, Publ. RIMS, Kyoto Univ. **38** (2002), 451–649.

[38] N. Nakayama, *Zariski-decomposition and Abundance*, MSJ Memoirs vol. **14**, Math. Soc. Japan, 2004.

[39] N. Nakayama, On complex normal projective surfaces admitting non-isomorphic surjective endomorphisms, unpublished preprint, 2008.

[40] N. Nakayama, A variant of Shokurov's criterion of toric surface, *Algebraic Varieties and Automorphism Groups* (eds. K. Masuda, et. al.), pp. 287–392, Adv. Stud. in Pure Math. **75**, Math. Soc. Japan, 2017.

[41] N. Nakayama and D.-Q. Zhang, Polarized endomorphisms of complex normal varieties, Math. Ann. **346** (2010), 991–1018.

[42] T. Oda, *Convex Bodies and Algebraic Geometry – An Introduction to the Theory of Toric Varieties*, Ergebnisse der Math. (3) **15**, Springer-Verlag, 1988.

[43] H. Pinkham, Normal surface singularities with $C^*$ action, Math. Ann. **227** (1977), 183–193.

[44] J-P. Ramis and G. Ruget, Complexe dualisant et théorèmes de dualité en géometrie analytique complexe, Publ. Math. I.H.É.S. **38** (1970), 77–91.

[45] M. Reid, Canonical 3-folds, *Journées de Géometrie Algébrique d'Angers, Julliet 1979/Algebraic Geometry, Angers 1979* (ed. A. Beauville), pp. 273–310, Sijthoff and Noordhoff, 1980.

[46] R. Remmert, Holomorphe und meromorphe Abbildungen komplexer Räume, Math. Ann. **133** (1957), 328–370.

[47] F. Sakai, $D$-dimensions of algebraic surfaces and numerically effective divisors, Compos. Math. **48** (1983), 101–118.

[48] F. Sakai, Weil divisors on normal surfaces, Duke Math. J. **51** (1984), 877–888.

[49] F. Sakai, The structure of normal surfaces, Duke Math. J. **52** (1985), 627–648.

[50] F. Sakai, Ample Cartier divisors on normal surfaces, J. reine und angew. Math. **366** (1986), 121–128.

[51] F. Sakai, Classification of normal surfaces, *Algebraic Geometry, Bowdoin, 1985* (ed. S. Bloch), pp. 451–465, Proc. Sympo. in Pure Math. **46**, Amer. Math. Soc., 1987.

[52] V. V. Shokurov, 3-fold log flips, Izv. Russ. Acad. Nauk. Ser. Mat. **56** (1992), 105–203; (English translation) Russian Acad. Sci. Izv. Math. **40** (1993), 95–202.

[53] K. Stein, Analytische Zerlegungen komplexer Räume, Math. Ann. **132** (1956), 63–93.

[54] S. Suzuki, Birational geometry of birational pairs, Comment. Math. Univ. St. Pauli, **32** (1983), 85–106.

[55] S. Tsunoda and M. Miyanishi, The structure of open algebraic surfaces II, *Classification of Algebraic and Analytic Manifolds* (ed. K. Ueno), Progress in Math. **39**, pp. 499–544, Birkhäuser, 1983.

[56] K. Ueno, *Classification Theory of Algebraic Varieties and Compact Complex Spaces*, Lecture Notes in Math. **439**, Springer-Verlag, 1975.

[57] E. Viehweg, Vanishing theorems, J. reine und angew. Math. **335** (1982), 1–8.

[58] J. Wahl, A characteristic number for links of surface singularities, J. Amer. Math. Soc. **3** (1990), 625–637.

[59] O. Zariski, The theorem of Riemann–Roch for high multiples of an effective divisor on an algebraic surface, Ann. Math. **76** (1962), 560–615.

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