RIMS-1921

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July 2020

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ON SURJECTIVE HOMOMORPHISMS FROM A CONFIGURATION SPACE GROUP TO A SURFACE GROUP

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Abstract. In the present paper, we classify all surjective homomorphisms from the étale fundamental group of the configuration space of a hyperbolic curve (over an algebraically closed field of characteristic zero) to the étale fundamental group of a hyperbolic curve. We can show that such a surjective homomorphism is necessarily "geometric" in some sense, that is, it factors through one of the homomorphisms which arise from specific morphisms of schemes.

Introduction

Let n be a positive integer, k an algebraically closed field of characteristic zero, and X a hyperbolic curve of type (g, r) over k. Write X_n for the n-th configuration space (cf. Definition 1.2) and Π for the étale fundamental group of X_n or the maximal pro-l quotient of the étale fundamental group of X_n. We obtain some homomorphisms from X_n to a hyperbolic curve over k, so called "projection morphisms (of co-length 1)" and "exceptional morphisms" (cf. Definitions 1.3, 1.4). These morphisms induce surjective homomorphisms between fundamental groups. In [S], we show that, under some conditions, any surjective homomorphism from Π to a surface group (cf. Definition 1.7) factors through one of the above homomorphisms:

Theorem A (cf. [S] Theorem 7.12)**.** *Let H be a surface group and $\varphi : \Pi \twoheadrightarrow H$ a surjective homomorphism of profinite groups. Suppose that at least one of the following holds:*

1. *$g \neq 1$ or $r \leq 1$.*
2. *H is not isomorphic to the maximal pro-Σ completion of the free group of rank 2 (where Σ is a nonempty set of prime numbers).*

Then there exists a surjective homomorphism $\varphi' : \Pi \twoheadrightarrow H'$ induced by a projection morphism of co-length 1 or an exceptional morphism such that φ factors through φ'.

The main theorem of the present paper is a generalization of Theorem A:

Theorem B (cf. Corollary 3.3)**.** *Let H be a surface group and $\varphi : \Pi \twoheadrightarrow H$ a surjective homomorphism of profinite groups. Then there exists a surjective homomorphism $\varphi' : \Pi \twoheadrightarrow H'$ induced by a projection morphism of co-length 1 or an exceptional morphism such that φ factors through φ'.*

2020 *Mathematics Subject Classification.* 14H30, 20F36.

Key words and phrases. hyperbolic curve, configuration space, fundamental group, surjective homomorphism.

1. Configuration spaces of curves

In the present §1, we review generalities on the configuration spaces of curves and their fundamental groups. Let l be a prime number and n a positive integer.

Definition 1.1. Let S be a scheme and X a scheme over S.

(i) We shall say that X is a *smooth curve* (*of type* (g,r)) over S if there exist a pair of nonnegative integers (g,r), a scheme X^{cpt} over S, and a (possibly empty) closed subscheme $D \subset X^{\mathrm{cpt}}$ of X^{cpt} such that
- X^{cpt} is smooth, proper, and of relative dimension one over S;
- any geometric fiber of $X^{\mathrm{cpt}} \to S$ is connected (hence a smooth proper curve) of genus g;
- the composite $D \hookrightarrow X^{\mathrm{cpt}} \to S$ is finite étale of degree r;
- X is isomorphic to $X^{\mathrm{cpt}} \setminus D$ over S.

(ii) We shall say that X is a *hyperbolic curve* (*of type* (g,r)) over S if X is a smooth curve of type (g,r) over S such that $2g-2+r>0$.

Definition 1.2 (cf. [MT] Definition 2.1)**.**

(i) Let S be a scheme and X be a smooth curve over S. Then we shall write $P_n := \overbrace{X \times_S \cdots \times_S X}^{n}$.

(ii) Let (n,S,X) be as in (i). For (i,j) a pair of integers such that $1 \le i < j \le n$, write $\pi_{i,j} : P_n \twoheadrightarrow P_2 = X \times_S X$ for the projection to the i-th and j-th factors. Moreover, we shall write $X_n := P_n \setminus (\bigcup_{i,j} \pi_{i,j}^{-1}(\Delta))$, where $\Delta \subset \Pi_2$ is the diagonal of P_2. We shall refer to X_n as the *n-th configuration space* of X over S. (For convenience, we set $X_0 := S$.)

Definition 1.3 (cf. [S] Definition 1.6)**.** Let K be a field and X a hyperbolic curve of type (g,r) over K. Write $\varepsilon := r$ if $X \cong \mathbb{P}^1_K \setminus \{0,1,\infty\}$ (hence $(g,r)=(0,3)$) or $(g,r)=(1,1)$, and write $\varepsilon := 0$ if otherwise. Let $I \subset \{1,\dots,n+\varepsilon\}$ be such that $0 \le \sharp I \le n$. We shall define $p_I : X_n \to X_{n-\sharp I}$ as follows:

(a) If $X \cong \mathbb{P}^1_K \setminus \{0,1,\infty\}$ (e.g., the case where $(g,r)=(0,3)$ and K is algebraically closed), then there is a natural K-isomorphism $X_n \xrightarrow{\sim} (\mathcal{M}_{0,n+3})_K$, $(x_1,\dots,x_n) \mapsto [x_1,\dots,x_n,0,1,\infty]$, where $\mathcal{M}_{0,n+3}$ is the moduli space of ordered $(n+3)$-pointed curves of genus zero. We shall define p_I as

$$p_I : X_n \xrightarrow{\sim} (\mathcal{M}_{0,n+3})_K \to (\mathcal{M}_{0,n-\sharp I+3})_K \xrightarrow{\sim} X_{n-\sharp I},$$

where $(\mathcal{M}_{0,n+3})_K \to (\mathcal{M}_{0,n-\sharp I+3})_K$ is the morphism obtained by forgetting the marked points corresponding to the elements of I.

(b) If $(g,r)=(1,1)$, then write $E := X^{\mathrm{cpt}}$, and write O for the unique point of $E \setminus X$. Since E is an elliptic curve over K, E has an addition whose identity element is O. In this case, there is a natural K-isomorphism $X_n \xrightarrow{\sim} E_{n+1}/E$, $(x_1,\dots,x_n) \mapsto [x_1,\dots,x_n,O]$, where the action of E on E_{n+1} is the diagonal translation determined by the addition of E. We shall define p_I as

$$p_I : X_n \xrightarrow{\sim} E_{n+1}/E \to E_{n-\sharp I+1}/E \xrightarrow{\sim} X_{n-\sharp I},$$

where $E_{n+1}/E \to E_{n-\sharp I+1}/E$ is the morphism obtained by forgetting the factors corresponding to the elements of I.

(c) If $X \not\cong \mathbb{P}^1_K \setminus \{0, 1, \infty\}$ and $(g, r) \neq (1, 1)$, then we shall define p_I as the projection obtained by forgetting the factors corresponding to the elements of I.

(In particular, $p_\emptyset = \mathrm{id}_{X_n}$. Moreover, if $\sharp I = n$, then p_I is the structure morphism $X_n \to X_0 = \mathrm{Spec}\, K$.) We shall refer to p_I as a *generalized projection morphism*. If p_I coincides with a projection from X_n to $X_{n-\sharp I}$ obtained by forgetting some $\sharp I$ factors (i.e., $I \subset \{1, \ldots, n\}$ or $\sharp I = n$), then we shall also refer to p_I as a *projection morphism*. We shall refer to $n - \sharp I$ as the *co-length* of p_I.

Definition 1.4. Let K be an algebraically closed field of characteristic zero and X a hyperbolic curve of type (g, r) over K. Suppose that $g = 0$ (resp. $g = 1$). Then, since X is hyperbolic and K is algebraically closed, there is a hyperbolic curve Y of type $(0, 3)$ (resp. $(1, 1)$) such that X is an open subscheme of Y. We shall say that a morphism $p : X_n \to Y$ is an *exceptional morphism* if p is a composite of an immersion $X_n \hookrightarrow Y_n$ determined by the open immersion $X \hookrightarrow Y$ and a generalized projection morphism $Y_n \to Y$ of co-length 1 which is not a projection morphism.

Remark 1.4.1. (cf. [S] Remark 4.5.1) If $g = 1$, then any exceptional morphism factors as $X_n \hookrightarrow (X^{\mathrm{cpt}})_n \twoheadrightarrow Y$. This implies that "the set of exceptional morphisms to Y" does not depend on the choose of Y (up to isomorphism between Y's).

In the case $g = 0$, by the direct calculation of coordinates, we can show the following: let $X_n \hookrightarrow Y_n \to Y$ be an exceptional morphism. We write $I \subset \{1, \ldots, n+3\}$ for the set corresponds to the generalized projection morphism $Y_n \to Y$ in the notation of Definition 1.3, and write $m := \sharp I \cap \{n+1, n+2, n+3\}$ (since $Y_n \to Y$ is not a projection morphism, it holds that $m > 0$). Then there exists a smooth curve Z of type $(0, 3-m)$ contains Y such that $X_n \to Y$ factors as $X_n \hookrightarrow Z_n \to Y$. (In particular, any exceptional morphism from X_n factors through an n-th configuration space of a smooth curve of type $(0, 2)$.)

Definition 1.5. Let G be a group and Σ a nonempty set of prime numbers. Then we shall write

$$G^\Sigma$$

for the pro-Σ completion of G. If G is a topologically finitely generated profinite group, then G^Σ coincides with the maximal pro-Σ quotient of G.

We often write simply

$$G^l$$

instead of $G^{\{l\}}$. Moreover, we often write simply

$$G^\wedge$$

instead of the profinite completion of G.

Definition 1.6. Let X be a connected noetherian scheme and Σ a nonempty set of prime numbers.

(i) We shall write

$$\pi_1(X) = \pi_1^{\mathrm{prof}}(X)$$

for the étale fundamental group of X (for some choice of base point).

(ii) We shall write

$$\pi_1^{\text{pro-}\Sigma}(X) := (\pi_1(X))^\Sigma.$$

We often write

$$\pi_1^{\text{pro-}l}(X)$$

instead of $\pi_1^{\text{pro-}\{l\}}(X)$.

(iii) If X is a $\mathbb{C}$-scheme of finite type, then we shall write

$$\pi_1^{\text{top}}(X)$$

for the topological fundamental group of the complex analytic space X^{an} associated to X (for some choice of $\mathbb{C}$-rational base point).

(iv) We shall refer to $\pi_1^{\mathcal{C}}(X)$ (where $\mathcal{C} = \text{pro-}\Sigma, \text{prof}, \text{pro-}l, \text{top}$) as a *$\mathcal{C}$-fundamental group* of X.

Remark 1.6.1. Hereinafter, whenever we consider $\pi_1^{\text{top}}(X)$, we always assume that X is a $\mathbb{C}$-scheme of finite type. Note that, for a variety over an algebraically closed field K of characteristic zero, by taking a subfield K' of K such that K' is an algebraic closure of a finitely generated field over $\mathbb{Q}$ and that X has a model X' over K', and fixing an inclusion $K' \hookrightarrow \mathbb{C}$, $\pi_1^{\text{pro-}\Sigma}(X)$ is isomorphic to $(\pi_1^{\text{top}}(X' \times_{K'} \mathbb{C}))^{\Sigma}$.

Definition 1.7 (cf. [HMM] §0)**.** Let K be an algebraically closed field of characteristic zero, X a hyperbolic curve of type (g, r) over K, and $\mathcal{C} \in \{\text{pro-}\Sigma, \text{pro-}l, \text{prof}, \text{top}\}$. Then we shall write

$$\Pi_n^{\mathcal{C}} = \Pi_{n,g,r}^{\mathcal{C}} = \Pi_n^{\mathcal{C}}(X) := \pi_1^{\mathcal{C}}(X_n)$$

If $\mathcal{C} \in \{\text{pro-}\Sigma, \text{pro-}l, \text{prof}\}$, then we shall refer to (a profinite group isomorphic to) $\Pi_n^{\mathcal{C}}(X)$ (resp. $\Pi_1^{\mathcal{C}}(X)$) as a ($\mathcal{C}$-)*configuration space group* (resp. ($\mathcal{C}$-)*surface group*).

Remark 1.7.1. It is well-known that $\pi_1^{\text{top}}(X)$ has a presentation

$$\langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_r \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_r = 1 \rangle.$$

Remark 1.7.2. For the most part of the present paper, we assume that X is hyperbolic. For the case X is not necessarily hyperbolic, see Remark 3.3.2.

Definition 1.8. Let K be an algebraically closed field of characteristic zero, X a hyperbolic curve of type (g, r) over K, $\mathcal{C} \in \{\text{pro-}l, \text{prof}, \text{top}\}$, and I as in Definition 1.3. Then we shall write

$$\phi_I : \Pi_n^{\mathcal{C}}(X) \twoheadrightarrow \Pi_{n-\sharp I}^{\mathcal{C}}(X)$$

for the natural (outer) surjection induced by p_I. For $i \in \{1, \ldots, n\}$, we often write simply

$$\phi_i$$

instead of $\phi_{\{i\}}$.

Proposition 1.9 (cf. [H] Proposition 2.4(i))**.** *In the notation of Definition 1.8, let $\overline{x} \to X_{n-\sharp I}$ be a geometric point. Then* $\ker \phi_I$ *is isomorphic to the $\mathcal{C}$-fundamental group of the geometric fiber $X_n \times_{X_{n-\sharp I}} \overline{x}$ (which is the $\sharp I$-th configuration space of a hyperbolic curve $X_{n-\sharp I+1} \times_{X_{n-\sharp I}} \overline{x}$ of type $(g, r+n-\sharp I)$ over $\overline{x}$). In particular, a configuration space group is topologically finitely generated.*

The following Lemma 1.10 is used in the next section.

Lemma 1.10. *Let K be an algebraically closed field of characteristic zero, X a hyperbolic curve of type (g, r) over K, and $\mathcal{C} \in \{\text{pro-}l, \text{prof}, \text{disc}\}$. Suppose that $g > 0$. Then the natural open immersion $X_n \hookrightarrow P_n$ determines an isomorphism $\Pi_n^{\mathcal{C},\text{ab}} \xrightarrow{\sim} (\Pi_1^{\mathcal{C},\text{ab}})^n$.*

Proof. The case $\mathcal{C}$ = prof follows immediately from the case $\mathcal{C}$ = pro-l, which is proved in [HMM] Proposition 2.2(ii). If $\mathcal{C}$ = disc, then $\Pi_n^{\mathrm{disc,ab}} \twoheadrightarrow (\Pi_1^{\mathrm{disc,ab}})^n$ is a surjective homomorphism between finitely generated abelian groups such that the homomorphism obtained by taking profinite completions is isomorphic (cf. Remark 1.6.1). Thus, $\Pi_n^{\mathrm{disc,ab}} \twoheadrightarrow (\Pi_1^{\mathrm{disc,ab}})^n$ itself is isomorphic. This completes the proof of Lemma 1.10. □

2. Configuration space group via pure braids

In the present §2, we treat a configuration space group as the group of isotopy classes of pure braids, and prove the key theorem by using a topological argument (cf. Theorem 2.3 below). Let l be a prime number, n a positive integer, K an algebraically closed field of characteristic zero, and X a hyperbolic curve of type (g, r) over K.

By definition, (if $K = \mathbb{C}$, then) $\Pi_n^{\mathrm{top}}(X) = \pi_1(X^{\mathrm{an}}, (x_1, \ldots, x_n))$ is identified with the group of isotopy classes of pure braids of $M := X^{\mathrm{an}}$ on n strands whose endpoints are $x_1, \ldots, x_n$. Note that M^{cpt} is a compact Riemann surface of genus g and $\sharp(M^{\mathrm{cpt}} \setminus M) = r$ (denote $M^{\mathrm{cpt}} \setminus M = \{y_1, \ldots, y_r\}$). Under this identification, $\phi_i : \Pi_n^{\mathrm{top}}(X) \to \Pi_{n-1}^{\mathrm{top}}(X)$ is the morphism obtained by forgetting the i-th braid (i.e., the braid starting from x_i). Moreover, if we write $M_i := M \setminus \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\} \subset M$, then, via the natural inclusion $\iota_i : \pi_1^{\mathrm{top}}(M_i, x_i) \hookrightarrow \Pi_n^{\mathrm{top}}(X)$ obtained by attaching trivial strands, $\ker \phi_i$ is identified with $\pi_1^{\mathrm{top}}(M_i, x_i)$. Here, $\pi_1^{\mathrm{top}}(M_i, x_i)$ has the presentation

$$\pi_1^{\mathrm{top}}(M_i, x_i) = \langle \alpha_1^{(i)}, \ldots, \alpha_g^{(i)}, \beta_1^{(i)}, \ldots, \beta_g^{(i)}, \gamma_1^{(i)}, \ldots, \gamma_r^{(i)}, \delta_1^{(i)}, \ldots, \delta_n^{(i)} \mid \delta_i^{(i)} = 1,$$
$$\prod_{s=1}^{g} [\alpha_s^{(i)}, \beta_s^{(i)}] \prod_{t=1}^{r} \gamma_t^{(i)} \prod_{u=1} \delta_u^{(i)} = 1 \rangle,$$

where $\alpha_s^{(i)}$ (resp. $\beta_s^{(i)}$) is determined by a loop going once around the s-th hole (resp. going once through the s-th hole), and $\gamma_t^{(i)}$ (resp. $\delta_u^{(i)}$ $(u \neq i)$) is determined by a loop going once around y_t (resp. x_u) (by choosing suitable orientations of loops).

Lemma 2.1. *Let $s \in \{1, \ldots, g\}$, $t \in \{1, \ldots, r\}$, $i, j \in \{1, \ldots, n\}$ be such that $i \neq j$. Then, in the notation above, $\iota_i(\alpha_s^{(i)})$ (resp. $\iota_i(\beta_s^{(i)})$) commutes with some conjugate of $\iota_j(\gamma_t^{(j)})$. Moreover, if $t' \in \{1, \ldots, r\} \setminus \{t\}$ (hence $r \geq 2$), then $\iota_i(\gamma_{t'}^{(i)})$ commutes with some conjugate of $\iota_j(\gamma_t^{(j)})$.*

Proof. Fix a loop f representing the element $\alpha_s^{(i)}$ (resp. $\beta_s^{(i)}, \gamma_{t'}^{(i)}$). Then we can easily take a loop f' in M_j based at x_j such that f' is a loop going once around y_t, and that the image of f and the image of f' are disjoint in M. (Figure 1 is the case $g = 1$ and f represents $\alpha_1^{(i)}$. Since a compact Riemann surface of genus $g > 0$ is obtained by gluing the sides of a regular $4g$-sided polygon, we can find a loop f' in the general case similarly as in Figure 1.)

Then the element $\gamma \in \pi_1^{\mathrm{top}}(M_j, x_j)$ determined by f' is a conjugate of $\gamma_t^{(j)}$ or $(\gamma_t^{(j)})^{-1}$. Moreover, since $\mathrm{Im}\, f \cap \mathrm{Im}\, f' = \emptyset$, $\iota_i(\alpha_s^{(i)})$ (resp. $\iota_i(\beta_s^{(i)})$, $\iota_i(\gamma_{t'}^{(i)})$) commutes with $\iota_j(\gamma)$. This completes the proof of Lemma 2.1. □

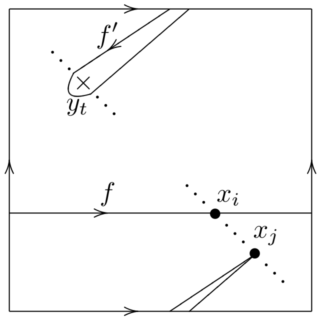

FIGURE 1.

Hereinafter, we regard as $\alpha_s^{(i)}, \beta_s^{(i)}, \gamma_t^{(i)} \in \Pi_n^{\mathrm{prof}}(X)$ via the composite of ι_i, the natural injection $\Pi_{n,g,r}^{\mathrm{top}} \hookrightarrow (\Pi_{n,g,r}^{\mathrm{top}})^\wedge$ (cf. [MT] Proposition 7.1), and the isomorphism $(\Pi_{n,g,r}^{\mathrm{top}})^\wedge \xrightarrow{\sim} \Pi_n^{\mathrm{prof}}(X)$ (cf. Remark 1.6.1). It is immediate that $\alpha_s^{(i)}, \beta_s^{(i)}, \gamma_t^{(i)} \in \ker \phi_i \subset \Pi_n^{\mathrm{prof}}(X)$.

Lemma 2.2. *Suppose that $g > 0$. Write π for the quotient map $\Pi_n^{\mathrm{prof}}(X) \twoheadrightarrow \Pi_n^{\mathrm{prof}}(X)^{\mathrm{ab}}$ and J for the closed subgroup of $\ker \phi_i$ generated by $\alpha_1^{(i)}, \ldots, \alpha_g^{(i)}$, $\beta_1^{(i)}, \ldots, \beta_g^{(i)}$, $\gamma_1^{(i)}, \ldots, \gamma_{r-1}^{(i)}$. Then it holds that $\pi(J) = \pi(\ker \phi_i)$.*

Proof. By Lemma 1.10, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \ker \phi_i & \longrightarrow & \Pi_n^{\mathrm{prof}}(X) & \longrightarrow & \Pi_{n-1}^{\mathrm{prof}}(X) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \pi & & \downarrow & & \\
 & & (\ker \phi_i)^{\mathrm{ab}} & \longrightarrow & \Pi_n^{\mathrm{prof}}(X)^{\mathrm{ab}} & \longrightarrow & \Pi_{n-1}^{\mathrm{prof}}(X)^{\mathrm{ab}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \wr & & \downarrow \wr & & \\
1 & \longrightarrow & \Pi_1^{\mathrm{prof}}(X)^{\mathrm{ab}} & \longrightarrow & (\Pi_1^{\mathrm{prof}}(X)^{\mathrm{ab}})^n & \longrightarrow & (\Pi_1^{\mathrm{prof}}(X)^{\mathrm{ab}})^{n-1} & \longrightarrow & 1,
\end{array}$$

where the horizontal sequences are exact. Now it follows from the explicit descriptions of $\ker \phi_i$ and $\Pi_1^{\mathrm{prof}}(X)$ that the composite $J \hookrightarrow \ker \phi_i \twoheadrightarrow (\ker \phi_i)^{\mathrm{ab}} \to \Pi_1^{\mathrm{prof}}(X)^{\mathrm{ab}}$ is surjective. This implies that $\mathrm{Im}(J \to \Pi_1^{\mathrm{prof}}(X)^{\mathrm{ab}}) = \mathrm{Im}(\ker \phi_i \to \Pi_1^{\mathrm{prof}}(X)^{\mathrm{ab}})$. Thus, it follows from the above diagram that

$$\pi(J) = \mathrm{Im}(J \to \Pi_n^{\mathrm{prof}}(X)^{\mathrm{ab}}) = \mathrm{Im}(\ker \phi_i \to \Pi_n^{\mathrm{prof}}(X)^{\mathrm{ab}}) = \pi(\ker \phi_i).$$

This completes the proof of Lemma 2.2. □

Theorem 2.3. *Suppose that $g, r > 0$. Let H be a pro-l group, $\varphi : \Pi_n^{\text{pro-}l}(X) \twoheadrightarrow H$ a surjective morphism, $x \in X^{\mathrm{cpt}} \setminus X$, and $j \in \{1, \ldots, n\}$. Write $I \subset \ker(p_j)(\subset \Pi_n^{\text{pro-}l}(X))$ for the inertia subgroup corresponding to x (which is well-defined up to conjugation). Suppose that the following conditions are satisfied:*

1. *$\varphi(I) \neq \{1\}$.*
2. *Any abelian subgroup of H is (topologically) generated by single element.*
3. *$\dim_{\mathbb{Q}_l}(H^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \geq 2$.*
4. *Any topologically fintely generated normal closed subgroup of H is trivial or open in H.*

Then φ factors through the morphism $\phi_{\{1,\ldots,n\}\setminus\{j\}} : \Pi_n^{\text{pro-}l}(X) \twoheadrightarrow \Pi_1^{\text{pro-}l}(X)$.

Proof. For $a \in \Pi_n^{\text{prof}}(X)$, write $\overline{a}$ for the image of a by the quotient map $\Pi_n^{\text{prof}}(X) \twoheadrightarrow \Pi_n^{\text{pro-}l}(X)$.

In light of [MT] Proposition 2.4(vi), it suffices to show that $\ker\phi_i \subset \ker\varphi$ for all $i \in \{1,\ldots,n\}\setminus\{j\}$. We may assume that the inertia subgroup I is corresponding to y_r, which implies that $\varphi(\overline{\gamma}_r^{(j)}) \neq 1$.

Write $A := \{\overline{\alpha}_1^{(i)},\ldots,\overline{\alpha}_g^{(i)},\overline{\beta}_1^{(i)},\ldots,\overline{\beta}_g^{(i)},\overline{\gamma}_1^{(i)},\ldots,\overline{\gamma}_{r-1}^{(i)}\} \subset \Pi_n^{\text{pro-}l}(X)$, and φ^{ab} for the composite of φ and the quotient map $H \twoheadrightarrow H^{\text{ab}}$. Then it follows from Lemmas 2.1, 2.2 that

- for any $a \in A$, $\varphi(a)$ commutes with $b_a\varphi(\overline{\gamma}_r^{(j)})b_a^{-1}$ for some $b_a \in H$;
- $\varphi^{\text{ab}}(\ker\phi_i)$ is (topologically) generated by $\varphi^{\text{ab}}(A)$.

Now, since $\varphi(\overline{\gamma}_r^{(j)}) \neq 0$, it follows from assumption (2) that there exists $c_a \in \mathbb{Q}_l$ such that $\varphi(a) = (b_a\varphi(\overline{\gamma}_r^{(j)})b_a^{-1})^{c_a}$, which implies that $\varphi^{\text{ab}}(a) = c_a\varphi^{\text{ab}}(\overline{\gamma}_r^{(j)}) \in \varphi^{\text{ab}}(\overline{\gamma}_r^{(j)})\mathbb{Q}_l (\subset H^{\text{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$. Thus, it holds that $\varphi^{\text{ab}}(\ker\phi_i) \subset \varphi^{\text{ab}}(\overline{\gamma}_r^{(j)})\mathbb{Q}_l$. In light of assumption (3), it follows that $\varphi^{\text{ab}}(\ker\phi_i) \subset H^{\text{ab}}$, hence also $\varphi(\ker\phi_i) \subset H$, is not open. On the other hand, since φ is surjective, $\varphi(\ker\phi_i) \subset H$ is a normal closed subgroup of H. Thus, it follows from Proposition 1.9, together with assumption (4), that $\varphi(\ker\phi_i)$ is trivial, i.e., $\ker\phi_i \subset \ker\varphi$. This completes the proof of Theorem 2.3. □

3. Classification of surjective homomorphisms

In the present §3, we classify all surjective homomorphisms from a configuration space group to a surface group. Let l be a prime number.

Definition 3.1 (cf. [S] Definition 5.5)**.** Let G be a pro-l group. Then we shall write $\gamma_m(G)$ for the m-th term of the lower central series of G, i.e., $\gamma_1(G) = G$ and $\gamma_{m+1}(G) := \overline{[G,\gamma_m(G)]}$ $(m \in \mathbb{Z}_{>0})$. Moreover, we shall write $\text{Gr}^{\text{lcs}}(G) := \bigoplus_{m\geq 1}\gamma_m(G)/\gamma_{m+1}(G)$. Note that $\text{Gr}^{\text{lcs}}(G)$ can be regarded as a $\mathbb{Z}_{>0}$-graded Lie algebra over $\mathbb{Z}_l$.

Theorem 3.2. *Let H be a profinite group, n a positive integer, (g,r) a pair of nonnegative integers such that $2g-2+r > 0$, and $\varphi : \Pi_{n,g,r}^{\mathcal{C}} \twoheadrightarrow H$ a surjective homomorphism. Suppose that the following conditions are satisfied:*

1. *$\text{Gr}^{\text{lcs}}(H^l)$ is a free $\mathbb{Z}_l$-module of rank ≥ 2.*
2. *For $a,b \in \text{Gr}^{\text{lcs}}(H^l)$, if $[a,b]=0$, then a and b are linearly dependent over $\mathbb{Z}_l$.*
3. *Any topologically finitely generated normal closed subgroup of H (resp. H^l) is trivial or open in H (resp. H^l).*
4. *If $g > 0$ and $r \geq 2$, then any abelian subgroup of H^l is (topologically) generated by single element.*

Then there exists a (surjective) homomorphism $\varphi' : \Pi_{n,g,r}^{\mathcal{C}} \twoheadrightarrow H'$ induced by a projection morphism of co-length 1 or an exceptional morphism such that φ factors through φ'.

Proof. As in the proof of [S] Theorem 5.11, we can reduce to the case $\mathcal{C} =$ pro-l. If $g = 0$ or $r \leq 1$, then Theorem 3.2 follows from (the proof of) [S] Theorem 5.11

(note that, since $\mathrm{Gr}^{\mathrm{lcs}}(H^l)$ is generated by $\gamma_1(H^l)/\gamma_2(H^l) \cong H^{l,\mathrm{ab}}$, the assumptions (1),(2) imply that $\mathrm{rank}_{\mathbb{Z}_l}(H^{l,\mathrm{ab}}) \geq 2$ and that $\mathrm{rank}_{\mathbb{Z}_l}(\mathrm{Gr}^{\mathrm{lcs}}(H^l)) \geq 3$).

If $g > 0$ and $r \geq 2$, then let us consider the surjective morphism $\Pi^{\text{pro-}l}_{n,g,r} \twoheadrightarrow \Pi^{\text{pro-}l}_{n,g,1}$ obtained by the open immersion $X_n \hookrightarrow Y_n$ arising from an open immersion $X \hookrightarrow Y$, where X (resp. Y) is a hyperbolic curve of type (g, r) (resp. $(g, 1)$) over an algebraically closed field of characteristic zero. Then, in light of Theorem 2.3, we may assume that the surjective homomorphism φ factors through $\Pi^{\text{pro-}l}_{n,g,r} \twoheadrightarrow \Pi^{\text{pro-}l}_{n,g,1}$. Thus, we can reduce to the case $r = 1$, which has already been verified. This completes the proof of Theorem 3.2. □

Corollary 3.3. *Let H be a surface group, $\mathcal{C} \in \{\text{pro-}l, \text{prof}\}$, Π a $\mathcal{C}$-configuration space group, and $\varphi : \Pi \twoheadrightarrow H$ a surjective homomorphism. Then there exists a (surjective) homomorphism $\varphi' : \Pi \twoheadrightarrow H'$ induced by a projection morphism of co-length 1 or an exceptional morphism such that φ factors through φ'.*

Proof. We may assume that $l \in \Sigma$, where H is a pro-Σ surface group. Then it follows from [S] Lemma 2.5, [MT] Theorem 1.5, together with the well-known fact that any closed subgroup of H^l of infinite index is a free pro-l group (cf. e.g. the proof of [MT] Theorem 1.5), that H satisfies the conditions of Theorem 3.2. □

Remark 3.3.1. We can prove similar results in the case $\mathcal{C} = \text{top}$ by arguments similar to the above arguments (or arguments appearing in [S] §8).

Remark 3.3.2. Let X be a smooth curve of type (g, r) over an algebraically closed field of characteristic zero. If $2g - 2 + r \leq 0$, then, by taking an integer r' such that $2g - 2 + r' > 0$ and a surjective homomorphism $\pi_1^{\mathcal{C}}(X_n) \twoheadrightarrow \Pi^{\mathcal{C}}_{n,g,r'}$ determined by an open immersion $X \hookrightarrow Y$ (where Y is a hyperbolic curve of type (g, r')), we can classify all surjective homomorphisms from $\pi_1^{\mathcal{C}}(X_n)$ to a surface group.

Remark 3.3.3. Historically, Corollary 3.3 in the case

- H is not isomorphic to the pro-Σ completion of the free group of rank 2 for any nonempty set of prime numbers Σ: proved in [HMM] Proposition 2.3.
- $g \geq 2$: essentially proved in [MT] Corollary 4.8 (see [S] Theorem 7.11).
- $g = 0$ or $r \leq 1$: proved in [S] Theorem 5.11.

Moreover, in the case $g \geq 2$ and $\mathcal{C} = \text{top}$, an alternative proof was given in [Ch2] Lemma 2.5 (see also [Ch1] §3).

Remark 3.3.4. By Theorem 3.2, we can replace the condition "$H \supset \Delta^{\Sigma}_{X_n/K}$ and $(g, r, n) \in \{(0, 3, 3), (1, 1, 2)\}$" appearing in [S] Theorem 7.14 with "$H \supset \Delta^{\Sigma}_{X_n/K}$, and, moreover, $(g, r, n) = (0, 3, 3)$ or $g \geq 1$". In particular, if $g \geq 1$, then, for a positive integer n, a generalized sub-l-adic field K (cf. [M] Definition 4.11), a hyperbolic curve X of type (g, r) over K, and a hyperbolic polycurve Z over K (cf. [H] Definition 2.1), it holds that the natural map

$$\mathrm{Isom}_K(X_n, Z) \to \mathrm{Isom}_{G_K}(\pi_1(X_n), \pi_1(Z)) / \mathrm{Inn}(\pi_1(Z \times_K \overline{K}))$$

is bijective.

Acknowledgement

I would like to thank Professor Akio Tamagawa for valuable discussions and advice. This research was supported by JSPS KAKENHI Grant Numbers JP17J11423, JP20J00323.

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