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On normal Moishezon surfaces admitting non-isomorphic surjective endomorphisms

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ABSTRACT. Normal Moishezon surfaces admitting non-isomorphic surjective endomorphisms are classified in some cases by using the original notion: "characteristic completely invariant divisor." A surface in our list has a finite Galois cover étale in codimension 1 from one of the following surfaces: a toric surface, an abelian surface, a $\mathbb{P}^1$-bundle over an elliptic curve, a projective cone over an elliptic curve, and the direct product of a non-singular projective curve of genus ≥ 2 with a rational or elliptic curve. As a corollary of our classification, any normal Moishezon surface admitting non-isomorphic surjective endomorphisms is shown to be projective.

CONTENTS

INTRODUCTION

We are interested in classifying compact complex analytic varieties admitting non-isomorphic surjective endomorphisms. In the case of non-singular projective curves, such a curve is rational or elliptic. In the case of non-singular projective surfaces, we have the following complete classification theorem by [15, §3] and [40]:

Theorem. *A non-singular complex projective surface admits a non-isomorphic surjective endomorphism if and only if it is one of the surfaces listed below*:

- *Toric surfaces.*
- $\mathbb{P}^1$*-bundles over elliptic curves.*

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- *$\mathbb{P}^1$-bundles over curves of genus ≥ 2 which are trivialized after finite étale base changes.*
- *Abelian surfaces.*
- *Hyperelliptic surfaces.*
- *Surfaces with Kodaira dimension $= 1$ and euler number $= 0$.*

We have complete classification results also in the cases of non-singular compact complex surfaces (cf. [16]) and non-singular complex projective 3-folds with non-negative Kodaira dimension (cf. [15] and [17]).

This article deals with *normal Moishezon surfaces* admitting non-isomorphic surjective endomorphisms. A Moishezon surface is by definition a compact complex analytic surface bimeromorphic to a projective surface (cf. [36]): This is associated with a 2-dimensional integral algebraic space proper over $\mathbb{C}$ by [2]. The main purpose of this article is to prove the following:

Theorem A. *Let X be a normal Moishezon surface with a reduced divisor S. Suppose that $K_X + S$ is pseudo-effective and that S is completely invariant under a non-isomorphic surjective endomorphism $f\colon X \to X$. Then $K_X + S$ is semi-ample and there exists a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 satisfying one of conditions (1)–(6) below with a non-isomorphic surjective endomorphism $f_V\colon V \to V$ such that $\nu \circ f_V = f^l \circ \nu$ for some positive integer l. Here, one can take $l = 1$ in cases (3)–(6):*

1. *$V = \mathbb{P}^1 \times T$ and $\nu^*S = \mathrm{pr}_1^*(P_1 + P_2) + \mathrm{pr}_2^* D$ for a non-singular projective curve T, two points P_1 $P_2 \in \mathbb{P}^1$, and a reduced divisor $D \subset T$ such that $\deg(K_T + D) > 0$, where $\mathrm{pr}_1\colon V \to \mathbb{P}^1$ and $\mathrm{pr}_2\colon V \to T$ are projections.*
2. *$V = C \times T$ and $\nu^*S = \mathrm{pr}_2^* D$ for an elliptic curve C, a non-singular projective curve T, and a reduced divisor $D \subset T$ such that $\deg(K_T + D) > 0$, where $\mathrm{pr}_2\colon V \to T$ is the second projection;*
3. *V is an abelian surface and $S = 0$;*
4. *V is a $\mathbb{P}^1$-bundle over an elliptic curve and ν^*S is a disjoint union of two sections;*
5. *V is a* projective cone *over an elliptic curve and ν^*S is a* cross section *(cf. Definition 1.16 below);*
6. *V is a toric surface with ν^*S as the boundary divisor.*

Remark. In the statement, K_X denotes the canonical divisor, and a reduced divisor S is said to be *completely invariant under f*, or *f-completely invariant*, if $f^{-1}S = S$; we allow 0 as a completely invariant divisor (cf. Definition 2.12 below). For definitions of *pseudo-effective* and *semi-ample*, see Remark 1.3 and Section 1.2 below. A finite surjective morphism $\nu\colon V \to X$ is said to be *étale in codimension* 1 if $\nu|_{V \setminus Z}\colon V \setminus Z \to X$ is étale for a Zariski-closed subset Z of codimension ≥ 2.

Remark. We have $K_X + S \not\sim_{\mathbb{Q}} 0$ (resp. $\sim_{\mathbb{Q}} 0$) in (1) and (2) (resp. (3)–(6)). Furthermore, $S = 0$ (resp. $\neq 0$) in (3) (resp. (4)–(6)).

Corollary B. *A normal Moishezon surface admitting a non-isomorphic surjective endomorphism is always projective.*

Proof. For the normal Moishezon surface X, if K_X is not pseudo-effective, then X is projective by Brenton's criterion [4, Prop. 7]. If K_X is pseudo-effective, then we can apply Theorem A to the case where $S = 0$. In this case, we have a finite Galois cover $V \to X$ from one of projective surfaces V listed in Theorem A. Thus, X is always projective. □

The proof of Theorem A is given in Section 5. We shall explain briefly the strategy of the proof, where we need:

(A.1) Theorem E below on log-canonicity of (X, S).
(A.2) Theorem 2.24 below concerning the semi-ampleness of $K_X + S$.
(A.3) Some properties on the Galois closure of the k-th power $f^k = f \circ f \circ \cdots \circ f \colon X \to X$ for $k \gg 0$ (cf. Section 2.2).
(A.4) Theorem 4.9 below on the structure of (X, S, f) in which X admits a fibration $X \to T$ to a non-singular projective curve T and f descends to an automorphism of T.
(A.5) Theorem 4.16 below on the structure of (X, S, f) in which X is irrational and ruled.

We see that (X, S) is log-canonical and $K_X + S$ is semi-ample with $(K_X + S)^2 = 0$ by (A.1) and (A.2). In the case where $K_X + S \not\sim_{\mathbb{Q}} 0$, we have a fibration $\pi \colon X \to T$ to a non-singular projective curve T such that some multiple of $K_X + S$ is linearly equivalent to the pullback of an ample divisor on T. Here, the endomorphism $f \colon X \to X$ induces an automorphism $h \colon T \to T$ satisfying $\pi \circ f = h \circ \pi$ (cf. Section 5.1). Theorem A in this case is deduced from (A.4).

For the case where $K_X + S \sim_{\mathbb{Q}} 0$, the proof of Theorem A is given in Section 5.2 (resp. 5.3) when $S = 0$ (resp. $\neq 0$). For the subcase: $S = 0$, Theorem A is deduced from (A.3) with calculation of euler numbers. For the other subcase: $S \neq 0$, Theorem A is deduced from (A.3), (A.5), the theory of toric surfaces (cf. [30], [49], [20]), and so on.

Before stating other results, we shall explain some important notions. The intersection theory of (Weil) divisors is essential to our study of normal Moishezon surfaces. This is defined by Mumford in [38, II, (b)] and is applied to the study of normal surfaces by Sakai in a series of papers [52], [53], [54], [56]. See also [44, §2] and [45, §1.3] for details. By the intersection theory, one can consider the numerical equivalence $\approx$ for $\mathbb{R}$-divisors on a normal Moishezon surface X. We set $\mathsf{N}(X)$ to be the real vector space generated by $\mathbb{R}$-divisors modulo $\approx$, and define the *Weil–Picard number* as $\hat{\boldsymbol{\rho}}(X) := \dim \mathsf{N}(X)$. The usual Picard number $\boldsymbol{\rho}(X) = \operatorname{rank} \mathrm{NS}(X)$, where $\mathrm{NS}(X)$ denotes the Néron–Severi group, is not greater than $\hat{\boldsymbol{\rho}}(X)$, since $\mathrm{NS}(X) \otimes \mathbb{R} \subset \mathsf{N}(X)$. However, we can show:

Proposition C. *If a normal projective surface X has a non-isomorphic surjective endomorphism, then $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$.*

We can extend the cone and contraction theorems of the minimal model theory to pairs (X, B) of a normal Moishezon surface X and a *pseudo-effective* $\mathbb{R}$-divisor B as in Theorems 1.9 and 1.10 below. These theorems concern the cone $\overline{\mathrm{NE}}(X)$ in

$\mathsf{N}(X)$ consisting of the numerical classes of pseudo-effective $\mathbb{R}$-divisors and concern extremal rays $\mathsf{R} \subset \overline{\mathrm{NE}}(X)$ satisfying $(K_X + B)\mathsf{R} < 0$. The theorems seem to be well known, but we give their complete proofs.

For a surjective morphism $f \colon Y \to X$ of normal Moishezon surfaces, we have defined the *numerical pullback* f^*D of a divisor D on X in [44, Def. 2.4(3)] (cf. [45, §1.3]), where f^*D is a $\mathbb{Q}$-divisor. This is a generalization of Mumford's pullback [38, II, (b)] defined when f is a bimeromorphic morphism and Y is non-singular. Since the numerical pullback of a Cartier divisor is just the usual pullback, we consider the numerical pullback as the "standard" pullback of a divisor. For a divisor E on Y, the pushforward f_*E is defined as usual, and we have the equality $(f^*D)E = D(f_*E)$ of intersection numbers. The pullback $D \mapsto f^*D$ and the pushforward $E \mapsto f_*E$ give rise to linear maps $f^* \colon \mathsf{N}(X) \to \mathsf{N}(Y)$ and $f_* \colon \mathsf{N}(Y) \to \mathsf{N}(X)$, respectively (cf. [44, Rem. 2.9]), where $f_* \circ f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ is just the multiplication map by $\deg f$, the *degree* of f, which is the cardinality of a general fiber of f.

For a non-isomorphic surjective endomorphism $f \colon X \to X$ of a normal Moishezon surface X, we define the *first dynamical degree* λ_f as the spectral radius of $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ (cf. Definition 3.1 below). Then $\lambda_f^2 \geq \deg f$, and λ_f equals the spectral radius of $f_* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ (cf. Proposition 3.3 below). We can prove the following by applying Theorem A:

Theorem D. *For a non-isomorphic surjective endomorphism f of a normal projective surface X and for the first dynamical degree λ_f, one of the following holds*:

(1) *The pullback homomorphism $(f^k)^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ is a scalar map for some power $f^k \colon X \to X$. In particular, $(\lambda_f)^2 = \deg f$.*
(2) *There is a fibration $X \to T$ to a non-singular projective curve T such that the support of any fiber is isomorphic to $\mathbb{P}^1$, λ_f is an integer dividing $\deg f$, and $(\lambda_f)^2 > \deg f$. In this case, X has only quotient singularities and has no negative curve, and $\boldsymbol{\rho}(X) = 2$.*
(3) *There is a finite Galois cover $C \times T \to X$ étale in codimension 1 for an elliptic curve C and a non-singular projective curve T of genus ≥ 2. In this case, $\lambda_f = \deg f$.*
(4) *There is a finite Galois cover $A \to X$ étale in codimension 1 from an abelian surface A with an endomorphism $f_A \colon A \to A$ as a lift of f. Here, $\lambda_f = \lambda_{f_A}$ and $\deg f = \deg f_A$.*

Remark. The canonical divisor K_X is not pseudo-effective in (2). Moreover, K_X is *nef* (cf. Remark 1.3) but not numerically trivial in (3), and $K_X \sim_{\mathbb{Q}} 0$ in (4). By a *negative curve*, we mean a prime divisor with negative self-intersection number.

Remark. There is a well-known definition of dynamical degrees of a meromorphic endomorphism of a compact Kähler manifold in the study of complex dynamical systems (cf. [50, p. 917, Def.], [9, p. 960], [25, Def. 1.1]). In Corollary A.10 in Appendix A below, our λ_f is shown to be equal to the first dynamical degree of the meromorphic map $\nu^{-1} \circ f \circ \nu \colon Z \dashrightarrow Z$ for any birational map $\nu \colon Z \dashrightarrow X$ from a non-singular projective surface Z. When X is non-singular, this result is known by [25, Prop. 1.2]. Note that the second dynamical degree of $\nu^{-1} \circ f \circ \nu$ is nothing but $\deg f$.

The following theorem on *completely invariant divisors* (cf. Definition 2.12) is a consequence of [45, Cor. 3.6], since f is a finite morphism by Remark 2.10 below:

Theorem E. *Let X be a normal Moishezon surface with a non-isomorphic surjective endomorphism f. Then (X, S) is log-canonical for any f-completely invariant divisor S.*

In particular, if K_X+S is nef, then K_X+S is semi-ample by the 2-dimensional abundance theorem for log-canonical pairs: The theorem for normal Moishezon surfaces is prepared in Theorem 1.12 below. Theorem E is applied to prove Theorem 2.24 (cf. (A.2)). Moreover, by the classification of 2-dimensional log-canonical pairs (cf. [28, Thm. 9.6], [33, Ch. 3]), the following hold for (X, S) (cf. [44, Thm. 3.22], [45, Fact. 2.5]):

- X has only quotient singularities along S;
- $X \setminus S \subset X$ is a toroidal embedding at any point of the singular locus of S;
- $S|_{X_{\mathrm{reg}}}$ is a normal crossing divisor on the non-singular locus X_{reg} of X.

The *characteristic completely invariant divisor* is a key notion in our study: This is a reduced divisor S_f defined by the following property: A prime divisor Γ on X is contained in S_f if and only if $(f^k)^*\Gamma = b\Gamma$ for some $k \geq 1$ and $b \geq 2$. We have the following in Section 2.4:

- the number of prime components of S_f is finite;
- S_f is f-completely invariant and $S_{f^k} = S_f$ for any $k > 0$;
- any negative curve is contained in S_f.

In particular, X has only a finitely many negative curves, which generalizes [40, Prop. 11]. The *refined ramification divisor* Δ_f defined in Definition 2.16 equals $R_f - f^*S_f + S_f$ for the ramification divisor R_f of f, where S_f and Δ_f have no common prime component. In particular, $K_X + S_f = f^*(K_X + S_f) + \Delta_f$, and every prime component of Δ_f is nef (cf. Proposition 2.20(4)). We can prove that $\Delta_f = 0$ if and only if $f|_{X \setminus S} \colon X \setminus S \to X \setminus S$ is étale in codimension 1 for an f-completely invariant divisor S (cf. Proposition 2.21). Under the assumption of Theorem A, we have $S \geq S_f$ and $\Delta_f = 0$ by Theorem 2.24. Moreover, $\nu^*S_f = \mathrm{pr}_1^*(P_1 + P_2)$ in Theorem A(1), and $S_f = 0$ in Theorem A(2), by Lemma 5.1 below.

Organization of this article. In Section 1, we discuss properties on normal Moishezon surfaces. Some basic properties on divisors are prepared in Section 1.1. Section 1.2 concerns semi-ampleness criteria for nef divisors. In Section 1.3, we shall prove cone and contraction theorems for certain pairs (X, B) generalizing the same theorems in the well-known minimal model theory for projective surfaces. There is added Theorem 1.12 as a version of abundance theorem for normal Moishezon surfaces. Section 1.4 concerns $\mathbb{P}^1$-bundles over a non-singular projective curve. Here, we introduce the notion of a *projective cone* over a curve (cf. Definition 1.16).

General properties of non-isomorphic surjective endomorphisms of normal varieties are explained in Section 2. Elementary properties on endomorphisms of sets, cyclic covers, Galois closures, and endomorphisms of curves are discussed in Sections 2.1–2.3. In Section 2.4, we introduce many key notions such as the characteristic completely invariant divisor S_f and the refined ramification divisor Δ_f for a non-isomorphic finite surjective endomorphism f of a compact normal variety. Some results on Δ_f and completely invariant divisors are given in Section 2.5, which include Theorem 2.24 on the semi-ampleness of $K_X + S$.

From Section 3, we concentrate on the study of normal Moishezon surfaces with non-isomorphic surjective endomorphisms. The first dynamical degree λ_f is introduced and studied in Section 3.1. The singularity of the pair (X, S) for a completely invariant divisor S is studied in detail in Section 3.2. Section 3.3 deals with endomorphisms preserving fibrations or bimeromorphic morphisms. An application of the minimal model program to the study of endomorphisms is given in Section 3.4, where we obtain further properties on the first dynamical degree.

In Section 4, proceeding works in Section 3.3, we study non-isomorphic surjective endomorphisms $f \colon X \to X$ preserving a fibration $\pi \colon X \to T$ to a non-singular projective curve T such that $\pi \circ f = h \circ \pi$ for an endomorphism h of T. In Section 4.1, we study the effect of base change by a surjective morphism $\tau \colon T' \to T$ from another non-singular projective curve T' with an endomorphism $h' \colon T' \to T'$ such that $\tau \circ h' = h \circ \tau$. In Lemma 4.2 and Proposition 4.3 below, we give a sufficient condition for having a good morphism τ so that the normalization X' of $X \times_T T'$ is étale in codimension 1 over X. Some fundamental results are proved in Section 4.2 in the case where the endomorphism h is étale; for example, it is proved in Corollary 4.7 that *if h is étale with $\deg h > 1$, then π is smooth.* Section 4.3 is devoted to proving Theorem 4.9, which determines the structure of X when $\deg h = 1$. In Section 4.4, applying results in Sections 4.2 and 4.3, we prove Theorem 4.16 classifying the irrational ruled surfaces admitting non-isomorphic surjective endomorphisms.

Section 5 is devoted to proving Theorem A. Section 5.1 treats the case where $K_X + S \not\sim_{\mathbb{Q}} 0$. Sections 5.2 (resp. 5.3) treats the case where $K_X + S \sim_{\mathbb{Q}} 0$ and $S = 0$ (resp. $\neq 0$). Some applications of Theorem A are given in Section 6, where we shall prove Proposition C and Theorem D.

We have an appendix, where we compare our definition of the first dynamical degree (cf. Definition 3.1) with the definition of the same degree defined in the study of complex dynamical systems. After discussing elementary properties of spectral radii of endomorphisms of finite-dimensional real vector spaces in Section A.1, we shall prove Theorem A.9 and Corollary A.10 on the comparison in Section A.2.

Background. During the joint work [47] with D.-Q. Zhang on polarized endomorphisms of normal complex varieties, the author recognized gradually the importance of studying them in the 2-dimensional case. Independently of [47], the author began the study of normal Moishezon surfaces admitting non-isomorphic surjective endomorphisms, and was preparing a paper in several versions. One version [43] written in 2008 is referred to the published version of [47]. This incomplete version [43] is non-public and was sent to limited persons; however, it has been distributed so widely than what the author expected. Some ideas and results there have already been applied or generalized by many other people in several papers in these 10 years and more. On the other hand, further modified (but incomplete) versions of [43] seem to have never been taken into account by the people. The author continued the modification work up to 2010, but after that, it was interrupted many times by his new study of subjects in different areas of algebraic geometry. The author gave talks on results in [43] and modified versions several times at symposiums in 2008–2014. The current article is thought of as the core part of a revised version of [43]. Some contents of [43] have already been included in [44] and [45]. The other contents of [43] with further progress will appear in the forthcoming paper [46].

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Notation and conventions. We use standard notation and conventions of the birational (resp. bimeromorphic) geometry of complex algebraic (resp. analytic) varieties as in [29], [41], [44], and [45]. However, some of them are different from those used in [33], [57], and [34].

In this article, we deal with complex analytic spaces rather than schemes over $\mathbb{C}$, and a complex analytic space is always assumed to be Hausdorff and to have a countable open base. A *complex analytic variety* is by definition an irreducible and reduced complex analytic space, which is simply called a *variety*. A variety of dimension 1 (resp. 2) is called a *curve* (resp. *surface*). A *compact* variety is said to be *Moishezon* if its transcendence degree of the function field is equal to the dimension (cf. [36]). In other words, a Moishezon variety is a compact variety bimeromorphic to a projective variety. Sometimes, we call a dominant meromorphic map of Moishezon varieties simply *a rational map*, since it is determined by a $\mathbb{C}$-algebra homomorphism of function fields. We write arrows $\cdots\to$ with dotted tail for meromorphic maps. A list of notations used frequently in this article is in Table 1.

1. Some basic results on normal Moishezon surfaces

We recall some basic properties on divisors on normal surfaces in Section 1.1. In Section 1.2, we prepare results on semi-ample $\mathbb{Q}$-divisors, and in Section 1.3, we prove some versions of cone and contraction theorems for normal Moishezon surfaces with an application to the case when $-K_X$ is big. Some elementary properties of $\mathbb{P}^1$-bundles over non-singular projective curves are explained in Section 1.4 with properties of *projective cones* (cf. Definition 1.16).

1.1. **Basics on divisors on normal surfaces.** We recall some notation and conventions for divisors explained in [44, §§2.1 and 2.2], and [45, §1.2].

Let X be a normal variety. A divisor on X always means a *Weil* divisor. The *prime decomposition* of a divisor is an expression as a formal linear combination of prime divisors. The *multiplicity* $\mathrm{mult}_\Gamma D$ of D along a prime divisor Γ is the

Table 1. List of specific notations

Z_{reg}	the non-singular locus of Z.
$\operatorname{Sing} Z$	the singular locus of Z.
Z_{red}	the reduced structure of Z, i.e., the closed reduced subspace with the same underlying set.
D_{red}	the reduced divisor associated to an $\mathbb{R}$-divisor D with $\operatorname{Supp} D_{\mathrm{red}} = \operatorname{Supp} D$.
$\sim_{\mathbb{Q}}$	the $\mathbb{Q}$-linear equivalence relation for $\mathbb{R}$-divisors.
$\approx$	the numerical equivalence relation for $\mathbb{R}$-divisors.
$\mathsf{N}(X)$	the real vector space of $\mathbb{R}$-divisors on X modulo $\approx$.
$\boldsymbol{\rho}(V)$	the Picard number of V, the rank of the Néron–Severi group $\mathrm{NS}(V)$.
$\hat{\boldsymbol{\rho}}(X)$	the Weil–Picard number of X ($= \dim \mathsf{N}(X)$).
K_V	the canonical divisor of V.
R_f	the ramification divisor of a non-degenerate morphism f.
$\llcorner D \lrcorner$	the round-down of an $\mathbb{R}$-divisor D.
$\ulcorner D \urcorner$	the round-up of an $\mathbb{R}$-divisor D.
$f^{-1}D$	$= (f^*D)_{\mathrm{red}}$ for a reduced divisor D and for a certain morphism f.
D_1D_2	the intersection number of $\mathbb{R}$-divisors D_1 and D_2 on a normal surface.
$(D_1 \cdot D_2)$	$= D_1D_2$.
$\mathrm{cl}(D)$	the numerical class in $\mathsf{N}(X)$ of an $\mathbb{R}$-divisor D.
$\langle\ ,\ \rangle$	the intersection pairing on $\mathsf{N}(X)$ such that $\langle \mathrm{cl}(D_1), \mathrm{cl}(D_2)\rangle = D_1D_2$.
$\overline{\mathrm{NE}}(X)$	the pseudo-effective cone of X.
$\mathrm{Nef}(X)$	the nef cone of X.
$\boldsymbol{e}(Z)$	the euler number $= \sum_{i \geq 0} (-1)^i \dim H_i(Z, \mathbb{C})$.
$\boldsymbol{g}(C)$	the genus of a non-singular projective curve C.
$\kappa(D, X)$	the D-dimension for an $\mathbb{R}$-divisor D on X.
f^m	the m-th power $f \circ f \circ \cdots \circ f \colon V \to V$ of an endomorphism $f \colon V \to V$.
S_f	the characteristic completely invariant divisor of a finite endomorphism $f \colon V \to V$.
Δ_f	the refined ramification divisor of a finite endomorphism $f \colon V \to V$.
λ_f	the first dynamical degree of a surjective endomorphism $f \colon X \to X$.
δ_f	the positive square root $(\deg f)^{1/2}$ of the degree $\deg f$ of a finite endomorphism $f \colon X \to X$.

(Here, Z is a complex analytic space, V is a compact normal variety, and X is a normal Moishezon surface.)

coefficient of Γ in the prime decomposition of D. If $\mathrm{mult}_\Gamma D \in \{0, 1\}$ for any prime divisor Γ, then D is said to be *reduced*. In particular, we allow 0 as a reduced divisor. The group of Weil (resp. Cartier) divisors on X is denoted by $\mathrm{Div}(X)$ (resp. $\mathrm{CDiv}(X)$). Similarly, we have the group $\mathrm{Div}(X, \mathbb{Q})$ (resp. $\mathrm{Div}(X, \mathbb{R})$) of $\mathbb{Q}$ (resp. $\mathbb{R}$)-divisors on X, which is isomorphic to $\mathrm{Div}(X) \otimes \mathbb{Q}$ (resp. $\mathrm{Div}(X) \otimes \mathbb{R}$) when X is compact. A $\mathbb{Q}$-divisor D on X is said to be $\mathbb{Q}$-*Cartier* if mD is Cartier for some $m > 0$ locally on X. The associated reduced divisor, the round-down, and the round-up, respectively, of an $\mathbb{R}$-divisor D are divisors defined by

$$D_{\mathrm{red}} := \sum\nolimits_{r_i \neq 0} \Gamma_i, \quad \llcorner D \lrcorner := \sum \llcorner r_i \lrcorner \Gamma_i \quad \text{and} \quad \ulcorner D \urcorner := \sum \ulcorner r_i \urcorner \Gamma_i,$$

where $D = \sum r_i \Gamma_i$ is the prime decomposition with $r_i \in \mathbb{R}$ and where

$$\llcorner r \lrcorner := \max\{m \in \mathbb{Z} \mid m \leq r\} \quad \text{and} \quad \ulcorner r \urcorner := \min\{m \in \mathbb{Z} \mid m \geq r\}.$$

Let $f\colon Y \to X$ be a morphism of normal varieties. Assume that f is of *maximal rank* (cf. [45, Def. 1.1]) and that $\operatorname{codim}(f^{-1} \operatorname{Sing} X, Y) \geq 2$. Here, f is of maximal rank if and only if f is smooth on a non-empty Zariski open subset of Y (cf. [45, Lem. 1.3]). Then one can consider the pullback f^*D of a divisor D on X as a divisor on Y (cf. [45, Lem. 1.19]): This is defined by the composite homomorphism

$$f^*\colon \operatorname{Div}(X) = \operatorname{CDiv}(X_{\mathrm{reg}}) \xrightarrow{f'^*} \operatorname{CDiv}(Y') = \operatorname{Div}(Y)$$

for $Y' = Y_{\mathrm{reg}} \cap f^{-1}X_{\mathrm{reg}}$, $f' = f|_{Y'}\colon Y' \to X_{\mathrm{reg}}$, and the pullback homomorphism f'^* of Cartier divisors.

Convention 1.1. For a reduced divisor D on X, we write $f^{-1}D$ for $(f^*D)_{\mathrm{red}}$ by abuse of notation, where $f^{-1} \operatorname{Supp} D = \operatorname{Supp} f^*D$ as a set. In particular, $f^{-1}0 = 0$.

In the situation above, assume in addition that $\dim X = \dim Y$, i.e., $f\colon Y \to X$ is *non-degenerate* (cf. [45, Def. 1.1]). Then the *ramification divisor* R_f is defined as the closure of the ramification divisor $R_{f'}$ of $f'\colon Y' \to X_{\mathrm{reg}}$, and we have the *ramification formula*: $K_Y = f^*K_X + R_f$ (cf. [45, §1.5]). In this case, $R_f = 0$ if and only if f is étale in codimension 1, i.e., f is étale on $Y \setminus Z$ for a closed subset Z of codimension ≥ 2.

To a non-degenerate morphism $f\colon Y \to X$ of normal *surfaces*, without assuming $\operatorname{codim}(f^{-1} \operatorname{Sing} X, Y) \geq 2$, we can associate the pullback homomorphism $f^*\colon \operatorname{Div}(X, \mathbb{Q}) \to \operatorname{Div}(Y, \mathbb{Q})$ of $\mathbb{Q}$-divisors by the numerical pullback (cf. [44, §2.1], [45, §1.3]). Here, f^*D is a $\mathbb{Q}$-divisor for any divisor D on X. The homomorphism f^* extends the pullback homomorphism $f^*\colon \operatorname{CDiv}(X) \to \operatorname{CDiv}(Y)$ of Cartier divisors, and extends the pullback homomorphism $f^*\colon \operatorname{Div}(X) \to \operatorname{Div}(Y)$ above in the case where $\operatorname{codim}(f^{-1} \operatorname{Sing} X, Y) \geq 2$ (cf. [45, Lem.-Def. 1.23]). For a reduced divisor D on X, we also write $f^{-1}D$ for $(f^*D)_{\mathrm{red}}$ as in Convention 1.1, where $f^{-1} \operatorname{Supp} D = \operatorname{Supp} f^*D$ as a set.

Remark 1.2. For an $\mathbb{R}$-divisor D on a compact normal variety X, the D-dimension $\kappa(D, X)$ is defined in [41, II, Def. 3.2] by generalizing Iitaka's definition for Cartier divisors [27]. When $\dim X = 2$, for any resolution $\mu\colon \widetilde{X} \to X$ of singularities, we have $\kappa(D, X) = \kappa(\mu^*D, \widetilde{X})$ for the numerical pullback μ^*D, by [52, Thm. (2.1)] (cf. [45, Lem. 1.28]).

The intersection numbers of $\mathbb{R}$-divisors on a normal surface is defined by the numerical pullback. Let X be a normal surface and let D_1 and D_2 be $\mathbb{R}$-divisors on X. If $\operatorname{Supp} D_1 \cap \operatorname{Supp} D_2$ is compact, then the intersection number $D_1 D_2 = (D_1 \cdot D_2)$ is defined as $(\mu^*D_1 \cdot \mu^*D_2)$ for the minimal resolution $\mu\colon \widetilde{X} \to X$ of singularities. The *numerical factorial index* of X, denoted by $\mathbf{nf}(X)$, is defined as the smallest positive integer n such that $\mu^*(nD)$ is Cartier for any divisor D on

X (cf. [45, Def. 1.26]). A *negative curve* on X is a prime divisor Γ with negative self-intersection number $\Gamma^2 = (\Gamma \cdot \Gamma)$. For a compact reduced divisor $E = \sum_{i=1}^{n} E_i$ on X, if the intersection matrix $(E_i E_j)_{1 \le i,j \le n}$ is negative definite, then we have the *contraction morphism of* E as a bimeromorphic morphism $\pi \colon X \to \overline{X}$ to another normal surface $\overline{X}$ such that the π-exceptional locus is just E. This is unique up to isomorphism and its existence is shown in [52, Thm. (1.2)] as a generalization of Grauert's contraction criterion [21, (e), pp. 366–367] (cf. [44, Thm. 2.6]).

The intersection numbers define a numerical equivalence $\approx$ for $\mathbb{R}$-divisors on a normal Moishezon surface X; two $\mathbb{R}$-divisors D_1 and D_2 are numerically equivalent, i.e., $D_1 \approx D_2$, if and only if $D_1\Gamma = D_2\Gamma$ for any prime divisor Γ. We define $\mathsf{N}(X)$ to be the real vector space $\mathrm{Div}(X, \mathbb{R})/\approx$. Then $\mathsf{N}(X) \supset \mathrm{NS}(X) \otimes \mathbb{R}$ for the Néron–Severi group $\mathrm{NS}(X)$. The dimension $\hat{\boldsymbol{\rho}}(X)$ of $\mathsf{N}(X)$ is finite and is called the *Weil–Picard number* of X. Note that the Picard number $\boldsymbol{\rho}(X)$ is the rank of $\mathrm{NS}(X)$. The numerical equivalence class $\mathrm{cl}(D)$ of an $\mathbb{R}$-divisor D is called the *numerical class.* We write $\langle\ ,\ \rangle \colon \mathsf{N}(X) \times \mathsf{N}(X) \to \mathbb{R}$ for the bilinear map induced by the intersection pairing, i.e., $\langle \mathrm{cl}(D_1), \mathrm{cl}(D_2)\rangle = D_1 D_2$. The *pseudo-effective* (resp. *nef*) *cone* of X is defined as the set of numerical classes of pseudo-effective (resp. nef) $\mathbb{R}$-divisors on X, which is denoted by $\overline{\mathrm{NE}}(X)$ (resp. $\mathrm{Nef}(X)$). Two sets $\overline{\mathrm{NE}}(X)$ and $\mathrm{Nef}(X)$ are both strictly convex closed cones of $\mathsf{N}(X)$, and these are dual to each other. Note that an $\mathbb{R}$-divisor D on X is big (resp. numerically ample) if and only if $\mathrm{cl}(D)$ lies in the interior of $\overline{\mathrm{NE}}(X)$ (resp. $\mathrm{Nef}(X)$).

Remark 1.3. An $\mathbb{R}$-divisor D on a normal Moishezon surface X is said to be:

- *numerically trivial* if $D \approx 0$;
- *nef* if $DC \ge 0$ for any prime divisor C;
- *pseudo-effective* if $DB \ge 0$ for any nef $\mathbb{R}$-divisor B;
- *numerically ample* if $D^2 > 0$ and if $DC > 0$ for any prime divisor C;
- *big* if $D - A$ is pseudo-effective for a numerically ample $\mathbb{R}$-divisor A

(cf. [44, Def. 2.11], [53, p. 629]). These are numerical properties, i.e., depending on the numerical class $\mathrm{cl}(D)$ in $\mathsf{N}(X)$.

For a surjective morphism $f \colon Y \to X$ of normal Moishezon surfaces, the pushforward f_*E is defined as usual for any $\mathbb{R}$-divisor E on Y. As the projection formula, we know that

$$(D \cdot f_*E) = (f^*D \cdot E) \quad \text{and} \quad f_*(f^*D) = (\deg f)D$$

for any $D \in \mathrm{Div}(X, \mathbb{R})$ and $E \in \mathrm{Div}(Y, \mathbb{R})$, where $\deg f$, the *degree* of f, is the cardinality of a general fiber of f. Then we have linear maps

$$f^* \colon \mathsf{N}(X) \to \mathsf{N}(Y) \quad \text{and} \quad f_* \colon \mathsf{N}(Y) \to \mathsf{N}(X)$$

satisfying $f^*\mathrm{cl}(D) = \mathrm{cl}(f^*D)$ and $f_*\mathrm{cl}(E) = \mathrm{cl}(f_*E)$ for any D and E (cf. [44, Rem. 2.9]). Here, the composite $f_* \circ f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ is the multiplication map by $\deg f$. Moreover, we have $f^*\mathrm{Nef}(X) \subset \mathrm{Nef}(Y)$, $f^*\overline{\mathrm{NE}}(X) \subset \overline{\mathrm{NE}}(Y)$, $f_*\mathrm{Nef}(Y) = \mathrm{Nef}(X)$, and $f_*\overline{\mathrm{NE}}(Y) = \overline{\mathrm{NE}}(X)$.

1.2. **Semi-ampleness criteria.** We shall prove some results on *semi-ampleness* which are well known for non-singular projective surfaces. Recall that a $\mathbb{Q}$-divisor D on a normal Moishezon surface X is said to be *semi-ample* if there is a positive integer m such that mD is Cartier and the linear system $|mD|$ is base point free.

Lemma 1.4. *Let D be a nef $\mathbb{Q}$-divisor on a normal Moishezon surface X such that $D^2 = 0$. If either $\kappa(D, X) \geq 1$ or $DK_X < 0$, then D is semi-ample and $\kappa(D, X) = 1$.*

Proof. We can reduce to the case where X is a non-singular projective surface, as follows: Let $\mu\colon M \to X$ be a birational morphism from a non-singular projective surface M. Then μ^*D is a nef $\mathbb{Q}$-divisor satisfying $(\mu^*D)^2 = D^2 = 0$ and $(\mu^*D)K_M = D(\mu_*K_M) = DK_X$. Moreover, $\kappa(\mu^*D, M) = \kappa(D, X)$ (cf. Remark 1.2). Suppose that μ^*D is semi-ample. Then there exist a positive integer m and a morphism $\Phi\colon M \to \mathbb{P}^N$ to a projective space such that $m\mu^*D \sim \Phi^*H$ for a hyperplane H of $\mathbb{P}^N$. Here, $\dim \Phi(M) \leq 1$, since $D^2 = 0$. If $\dim \Phi(M) = 0$, then $\mu^*D \sim 0$ contradicting $\kappa(D, X) \geq 1$ or $DK_X < 0$. Hence, $\dim \Phi(M) = 1$ and $\kappa(\mu^*D, M) = 1$. Moreover, Φ factors through X, i.e., there is a morphism $\varphi\colon X \to \mathbb{P}^N$ such that $\Phi = \varphi \circ \mu$, since $(\mu^*D)\Gamma = 0$ for any μ-exceptional prime divisor Γ. Hence, $mD \sim \varphi^*H$. In particular, D is semi-ample and $\kappa(D, X) = 1$.

Therefore, we may assume that X is non-singular and projective. Then the assertion in the case $\kappa(D, X) \geq 1$ is well known (cf. [19, Thm. (4.1)]). If $DK_X < 0$, then $\kappa(D, X) \geq 1$ by the Riemann–Roch formula for $\chi(X, \mathcal{O}_X(mD))$ for $m \in \mathbb{Z}$. Thus, we are done. □

We have the following by Lemma 1.4 and [44, Lem. 2.31]:

Proposition 1.5. *Let X be a normal Moishezon surface. Assume that X is rational and $-K_X$ is big. Then X is a projective surface with only rational singularities. Furthermore, any nef $\mathbb{Q}$-divisor on X is semi-ample.*

Proof. Since $-K_X$ is big, X is projective by Brenton's criterion [4, Prop. 7]. Let $\mu\colon M \to X$ be a resolution of singularities. Then $H^1(M, \mathcal{O}_M) = 0$ as M is rational. Hence, X has only rational singularities by [44, Lem. 2.31(3)]. It remains to prove that any nef $\mathbb{Q}$-divisor D on X is semi-ample. If $D \approx 0$, then $D \sim_{\mathbb{Q}} 0$ by [44, Lem. 2.31(4)], since $H^1(X, \mathcal{O}_X) = H^1(M, \mathcal{O}_M) = 0$; thus, D is semi-ample in this case. If $D \not\approx 0$ with $D^2 = 0$, then $DK_X < 0$. In fact, since $-K_X - A$ is pseudo-effective for an ample $\mathbb{Q}$-divisor A, we have $-DK_X \geq DA > 0$. Hence, in this case, D is semi-ample by Lemma 1.4. Finally, we consider the case where $D^2 > 0$. Then the set $\{\Gamma_1, \Gamma_2, \ldots\}$ of prime divisors Γ on X satisfying $D\Gamma = 0$ is finite, and the intersection matrix $(\Gamma_i\Gamma_j)$ is negative definite, by the Hodge index theorem. Let $\varphi\colon X \to Y$ be the contraction morphism of $\sum \Gamma_i$. Then $-K_Y = \varphi_*(-K_X)$ is also big. Hence, Y is also a projective rational surface with only rational singularities by the previous argument. Then the $\mathbb{Q}$-Cartier divisor $D_Y := \varphi_*D$ is ample, and $D = \varphi^*D_Y$ by the construction of φ. Thus, D is semi-ample. □

Remark. The last assertion on the semi-ampleness has been proved in [59, §2] and [7, §3] when X is non-singular. This assertion can be proved by reducing to the non-singular case, since $-K_M$ is big for the minimal resolution M of singularities.

1.3. **Minimal model program.** For a normal Moishezon surface X and a pseudo-effective $\mathbb{R}$-divisor B on X, we shall prove the cone and contraction theorems as Theorems 1.9 and 1.10 below, respectively, which generalize the same theorems in the minimal model theory (e.g. [29, Thms. 4-2-1, 3-2-1]) in the 2-dimensional case. Moreover, the minimal model program works in the case where the round-up $\ulcorner N \urcorner$ is reduced for the negative part N of the *Zariski-decomposition* of B (cf. Corollary 1.11). For the Zariski-decomposition of a pseudo-effective $\mathbb{R}$-divisor, see [62, §7], [18, §1], [52, §7], [54, App.], [41, III], [45, Lem.-Def. 2.16], and so on. Although the statements are quite different from usual cone and contraction theorems, the proofs are essentially known by the study of open surfaces in 1980s. For the study of endomorphisms in this article, we need these theorems only for log-canonical pairs (X, B), but we present here the generalized versions. Even for Moishezon surfaces, we have the abundance theorem for log-canonical pairs (X, B) as Theorem 1.12 below, where B is a $\mathbb{Q}$-divisor. At the end of Section 1.3, as an application of the cone and contraction theorems, we shall prove Theorem 1.13 on *negative curves* for surfaces with big anti-canonical divisor, which is a generalization of [42, Prop. 3.3].

Definition. The cone and contraction theorems concern the pseudo-effective cone $\overline{\mathrm{NE}}(X)$ in $\mathsf{N}(X)$. For an $\mathbb{R}$-divisor D, we set

$$\overline{\mathrm{NE}}(X)_D := \{z \in \overline{\mathrm{NE}}(X) \mid \langle \mathrm{cl}(D), z \rangle \geq 0\}.$$

An *extremal ray* R of $\overline{\mathrm{NE}}(X)$ is a 1-dimensional cone in $\mathsf{N}(X)$ such that

$$\mathsf{R} = \overline{\mathrm{NE}}(X) \cap H^{\perp} := \{z \in \overline{\mathrm{NE}}(X) \mid \langle \mathrm{cl}(H), z \rangle = 0\}$$

for a nef $\mathbb{R}$-divisor H.

Remark. When X is projective, the usual cone theorem concerns the dual vector space $\mathsf{N}_1(X)$ of $\mathsf{N}^1(X) := \mathrm{NS}(X) \otimes \mathbb{R}$. Note that $\mathsf{N}_1(X)$ is a quotient vector space of $\mathsf{N}(X) = \mathrm{Div}(X, \mathbb{R})/\approx$ identified with $\mathrm{Div}(X, \mathbb{R})/\approx^{\dagger}$ for a restricted numerical equivalence relation $\approx^{\dagger}$, where $E \approx^{\dagger} 0$ if and only if $DE = 0$ for any *Cartier* divisor D. Instead of the cone $\overline{\mathrm{NE}}(X) \subset \mathsf{N}(X)$ above, its image in $\mathsf{N}_1(X)$ is treated in the usual cone theorem.

We note the following on extremal rays:

Lemma 1.6. *Let* R *be a* 1*-dimensional cone in* $\overline{\mathrm{NE}}(X)$*.*

(1) *If* $\overline{\mathrm{NE}}(X) = \mathcal{C} + \mathsf{R}$ *for a closed convex cone* $\mathcal{C}$ *not containing* R*, then* R *is an extremal ray.*
(2) *If* R *is an extremal ray and if* $v_1 + v_2 \in \mathsf{R}$ *for two vectors* v_1*,* $v_2 \in \overline{\mathrm{NE}}(X)$*, then* v_1 *and* $v_2 \in \mathsf{R}$*.*
(3) *Assume that* R *is an extremal ray contained in* $\mathrm{Nef}(X)$*. If* D *is an effective* $\mathbb{R}$*-divisor such that* $\langle \mathrm{cl}(D), v \rangle = 0$ *for any* $v \in \mathsf{R}$*, then* $\mathrm{cl}(\Theta) \in \mathsf{R}$ *for any prime component* Θ *of* D*.*

Proof. (1): We can find a functional $\beta\colon \mathsf{N}(X) \to \mathbb{R}$ such that $\beta > 0$ on $\mathcal{C} \setminus \{0\}$ and $\beta(v) < 0$ for $v \in \mathsf{R} \setminus \{0\}$. Since the pairing $\langle\ ,\ \rangle$ is non-degenerate, we have an $\mathbb{R}$-divisor B such that $\beta(z) = \langle z, \mathrm{cl}(B)\rangle$ for any $z \in \mathsf{N}(X)$. For a numerically ample divisor A and for $c := -\langle v, \mathrm{cl}(B)\rangle/\langle v, \mathrm{cl}(A)\rangle$, we see that the $\mathbb{R}$-divisor $H = B + cA$ is nef and $\overline{\mathrm{NE}}(X) \cap H^{\perp} = \mathsf{R}$. Thus, R is an extremal ray.

(2): Let H be a nef $\mathbb{R}$-divisor such that $\mathsf{R} = \overline{\mathrm{NE}}(X) \cap H^{\perp}$. Then $0 = \langle \mathrm{cl}(H), v_1 + v_2\rangle = \langle \mathrm{cl}(H), v_1\rangle + \langle \mathrm{cl}(H), v_2\rangle \geq 0$, and we have v_1, $v_2 \in \mathsf{R}$.

(3): For the $\mathbb{R}$-divisor H above, we have $H^2 = 0$ by $\mathsf{R} \subset \mathrm{Nef}(X)$ and by the Hodge index theorem. In particular, $\mathrm{cl}(H) \in \mathsf{R}$. Since $HD = 0$, we have $\mathrm{cl}(D) \in \mathsf{R}$ and also $\mathrm{cl}(\Theta) \in \mathsf{R}$ by (2). □

The following is a consequence of Mori's cone theorem [37, Thm. (1.4)]:

Lemma 1.7. *A normal Moishezon surface X contains a rational curve if K_X is not nef.*

Proof. Let $\mu\colon M \to X$ be the minimal resolution of singularities. Then K_M is not nef, since $K_X = \mu_* K_M$ (cf. [44, Rem. 2.13]). Hence, by [37, Thm. (1.4)], there is a rational curve Γ on M such that $K_M \Gamma < 0$. Now, K_M is *μ-nef*, i.e., $K_M E \geq 0$ for any μ-exceptional prime divisor E. Thus, Γ is not μ-exceptional and $\mu(\Gamma)$ is a rational curve on X. □

The following is shown by the idea in the proof of [60, Prop. 2.5], and it is essential in the proof of Theorem 1.9 below.

Lemma 1.8. *For a normal Moishezon surface X, let $\mathcal{C}$ be a closed convex cone in $\overline{\mathrm{NE}}(X)$ and let $\Gamma_1, \ldots, \Gamma_n$ be finitely many prime divisors such that*

$$\overline{\mathrm{NE}}(X) = \mathcal{C} + \sum\nolimits_{i=1}^{n} \mathbb{R}_{\geq 0}\,\mathrm{cl}(\Gamma_j).$$

Then, for any finitely many prime divisors $C_1, C_2, \ldots, C_m$, one has

$$\overline{\mathrm{NE}}(X) = \mathcal{C}' + \sum\nolimits_{i=1}^{m} \mathbb{R}_{\geq 0}\,\mathrm{cl}(C_i) + \sum\nolimits_{i=1}^{n} \mathbb{R}_{\geq 0}\,\mathrm{cl}(\Gamma_j)$$

for the cone

$$\mathcal{C}' := \{z \in \mathcal{C} \mid \langle z, \mathrm{cl}(C_i)\rangle \geq 0 \quad \textit{for any} \quad 1 \leq i \leq m\}.$$

Proof. Let A be a numerically ample divisor on X. Then $\mathrm{cl}(A) > 0$ on $\overline{\mathrm{NE}}(X)$ by the intersection paring $\langle\ ,\ \rangle$. For a pseudo-effective $\mathbb{R}$-divisor D on X, let $\mathcal{S}(D)$ be the set of collections $\zeta = (x_i, y_j)_{1 \leq i \leq m, 1 \leq j \leq n}$ of non-negative real numbers x_i, y_j such that $\mathrm{cl}(D(\zeta)) \in \mathcal{C}$ for

$$D(\zeta) := D - \sum\nolimits_{i=1}^{m} x_i C_i - \sum\nolimits_{j=1}^{n} y_j \Gamma_j.$$

Then $\mathcal{S}(D)$ is a compact subset of $\mathbb{R}^{m+n}_{\geq 0}$, since it is closed and

$$\alpha(\zeta) := \left(\sum\nolimits_{i=1}^{m} x_i C_i + \sum\nolimits_{j=1}^{n} y_j \Gamma_j\right) A = \sum\nolimits_{i=1}^{m} x_i C_i A + \sum\nolimits_{j=1}^{n} y_j \Gamma_j A \leq DA$$

for any $\zeta \in \mathcal{S}(D)$. Thus, we can find an element $\zeta^{\circ} \in \mathcal{S}(D)$ such that $\alpha(\zeta^{\circ})$ is maximal. It suffices to prove that $D(\zeta^{\circ}) \in \mathcal{C}'$. Assume the contrary. Then

$D(\zeta^\circ)C_i < 0$ for some i. Thus, C_i is contained in the negative part of the Zariski-decomposition of $D(\zeta^\circ)$, and $D(\zeta^\circ) - bC_i$ is pseudo-effective for some $b > 0$. Hence,

$$\mathrm{cl}\left(D(\zeta^\circ) - bC_i - \sum\nolimits_{j=1}^{n} y'_j \Gamma_j\right) \in \mathcal{C}$$

for some $y'_j \geq 0$. Note that $D(\zeta^\dagger) = D(\zeta^\circ) - bC_i - \sum y'_j\Gamma_j$ for a collection $\zeta^\dagger = (x_i^\dagger, y_j^\dagger)$. Then $\zeta^\dagger \in \mathcal{S}(D)$ and $\alpha(\zeta^\dagger) \geq \alpha(\zeta^\circ) + bC_iA > \alpha(\zeta^\circ)$. This contradicts the maximality of $\alpha(\zeta^\circ)$. Thus, we are done. □

The following is our version of the cone theorem for normal Moishezon surfaces (cf. [37, Thm. (1.4)], [60, Props. 2.5, 2.9], [53, Prop. 4.8], [29, Thm. 4-2-1], [56, Thm. 2], [1, Thm. 10.2]):

Theorem 1.9. *Let X be a normal Moishezon surface with a big $\mathbb{R}$-divisor B. If $K_X + B$ is not nef, then there exist finitely many extremal rays R_i such that*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+B} + \sum \mathsf{R}_i$$

and that each R_i is generated by the numerical class of a prime divisor C_i satisfying $(K_X + B)C_i < 0$.

Proof. We can take a numerically ample $\mathbb{Q}$-divisor A and an effective $\mathbb{R}$-divisor B_X such that $B \approx A + B_X$, since B is big. Let $\mu\colon M \to X$ be the minimal resolution of singularities and set B_M to be the proper transform of B_X in M. Since K_M and B_M are μ-nef, there is a μ-exceptional effective $\mathbb{R}$-divisor E' such that $K_M + B_M = \mu^*(K_X + B_X) - E'$ (cf. [44, Rem. 2.15]). Moreover, there is a μ-exceptional effective $\mathbb{Q}$-divisor E'' such that $H := \mu^*A - E''$ is an ample $\mathbb{Q}$-divisor. Then $\mu^*(K_X + B) \approx K_M + H + B_M + E' + E''$. By the cone theorem [37, Thm. (1.4)],

$$\overline{\mathrm{NE}}(M) = \overline{\mathrm{NE}}(M)_{K_M+H} + \sum \mathbb{R}_{\geq 0}\,\mathrm{cl}(\Gamma_j)$$

for finitely many rational curves Γ_j. Let $G_1, G_2, \ldots, G_m$ be the prime components of $B_M + E' + E''$. Then, by Lemma 1.8,

$$\overline{\mathrm{NE}}(M) = \mathcal{C}_M + \sum\nolimits_{k=1}^{m} \mathbb{R}_{\geq 0}\,\mathrm{cl}(G_k) + \sum \mathbb{R}_{\geq 0}\,\mathrm{cl}(\Gamma_j)$$

for the cone

$$\mathcal{C}_M := \{z \in \overline{\mathrm{NE}}(M)_{K_M+H} \mid \langle \mathrm{cl}(G_k), z\rangle \geq 0 \quad \text{for any} \quad 1 \leq k \leq m\}.$$

For an $\mathbb{R}$-divisor D on M, if $\mathrm{cl}(D) \in \mathcal{C}_M$, then

$$\begin{aligned}(K_X + B \cdot \mu_*D) &= (\mu^*(K_X + B) \cdot D) = (K_M + H + B_M + E' + E'')D \\ &= (K_M + H)D + (B_M + E' + E'')D \geq 0.\end{aligned}$$

Thus, $\mu_*\mathcal{C}_M \subset \overline{\mathrm{NE}}(X)_{K_X+B}$ for $\mu_*\colon \mathsf{N}(M) \to \mathsf{N}(X)$. Since $\mu_*\overline{\mathrm{NE}}(M) = \overline{\mathrm{NE}}(X)$, we have

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+B} + \sum \mathbb{R}_{\geq 0}\,\mathrm{cl}(C_i)$$

for finitely many prime divisors C_i, where C_i is expressed as μ_*G_k or $\mu_*\Gamma_j$. We set $\mathsf{R}_i = \mathbb{R}_{\geq 0}\,\mathrm{cl}(C_i)$. By Lemma 1.6(1), removing redundant R_i, we may assume that R_i are all extremal and $\mathsf{R}_i \not\subset \overline{\mathrm{NE}}(X)_{K_X+B}$, i.e., $(K_X + B)C_i < 0$. Thus, we are done. □

Remark. The proof of [1, Thm. 10.2] is sketchy but essentially the same as above.

The following is our version of the contraction theorem (cf. [37, Thm. (2.1)], [60, Props. 2.10, 2.12, 2.13], [53, Thm. 4.9], [29, Thm. 3-2-1], [56, Thm. 3], [1, Thm. 10.3]).

Theorem 1.10. *Let X be a normal Moishezon surface with a pseudo-effective $\mathbb{R}$-divisor B. Let R be an extremal ray of $\overline{\mathrm{NE}}(X)$ such that $(K_X + B)\mathsf{R} < 0$, i.e., $\langle \mathrm{cl}(K_X + B), v\rangle < 0$ for any $0 \neq v \in \mathsf{R}$. Then there exists a unique fibration $\pi\colon X \to Y$ to a normal Moishezon variety Y, called the* contraction morphism *of R, such that, for a prime divisor $C \subset X$, the image $\pi(C)$ is a point if and only if $\mathrm{cl}(C) \in \mathsf{R}$. Moreover, $\hat{\boldsymbol{\rho}}(X) = \hat{\boldsymbol{\rho}}(Y) + 1$ for the Weil–Picard number $\hat{\boldsymbol{\rho}}$, and the following hold for a non-zero vector $v \in \mathsf{R}$:*

1. *If $\langle v, v\rangle > 0$, then $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X) = 1$, $\overline{\mathrm{NE}}(X) = \mathsf{R}$, X is a projective surface containing a rational curve, and π is the structure morphism $X \to \operatorname{Spec}\mathbb{C}$.*
2. *If $\langle v, v\rangle = 0$, then X is a projective surface with only rational singularities, $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X) = 2$, $\dim Y = 1$, and $F_{\mathrm{red}} \simeq \mathbb{P}^1$ and $\mathrm{cl}(F) \in \mathsf{R}$ for any fiber F of π.*
3. *If $\langle v, v\rangle < 0$, then $\mathsf{R} = \mathbb{R}_{\geq 0}\,\mathrm{cl}(C)$ for a negative curve C and π is the contraction morphism of C. Assume that $(B - C)C \geq 0$. Then C is a rational curve and $\boldsymbol{\rho}(X) = \boldsymbol{\rho}(Y) + 1$. Moreover, in this case, if X is projective, then so is Y.*

Proof. There is a numerically ample $\mathbb{R}$-divisor A such that $(K_X + B + A)\mathsf{R} < 0$. By Lemma 1.6(2), R is one of R_i in Theorem 1.9 applied to the big $\mathbb{R}$-divisor $B + A$. Thus, we may assume that $v = \mathrm{cl}(C)$ for a prime divisor C. Then $\langle v, v\rangle = C^2$. When $C^2 < 0$, i.e., C is a negative curve, we can take π as the contraction morphism of C. Here, $\hat{\boldsymbol{\rho}}(X) = \hat{\boldsymbol{\rho}}(Y) + 1$ by [44, Lem. 2.10]. Assume that $(B - C)C \geq 0$, or equivalently, $r \leq 1$ for the real number r defined by $BC = rC^2$. Then $(K_X + B)C = (K_X + rC)C < 0$. Hence, there is a real number $0 \leq t < 1$ such that $(K_X + tC)C < 0$, i.e., $-(K_X + tC)$ is π-ample. Then $R^1\pi_*\mathcal{O}_X = 0$ by a version of Kawamata–Viehweg's vanishing theorem [52, Thm. (6.3)] (cf. [44, Thm. 2.17], [45, Prop. 2.15]), since $\llcorner tC \lrcorner = 0$. In particular, the exceptional curve C is rational. Moreover, $R^1\pi_*\mathcal{O}_X^\star \simeq R^2\pi_*\mathbb{Z}_X$ is a skyscraper sheaf at $\pi(C)$ of the abelian group $H^2(C, \mathbb{Z}) \simeq \mathbb{Z}$. Hence, $\mathrm{Pic}(Y)$ is isomorphic to the kernel of the homomorphism $\mathrm{Pic}(X) \to \mathbb{Z}$ given by $\mathcal{L} \mapsto \deg \mathcal{L}|_C$. This proves $\boldsymbol{\rho}(X) = \boldsymbol{\rho}(Y) + 1$. The last assertion of (3) is shown by the same argument as in [44, Rem. 2.22] from $R^1\pi_*\mathcal{O}_X = 0$. Thus, we are done in the case where $C^2 < 0$.

Note that if $C^2 \geq 0$, then C is nef and K_X is not pseudo-effective by $K_X C \leq (K_X + B)C < 0$. Thus, in this case, X is projective by Brenton's criterion [4, Prop. 7] and X contains a rational curve by Lemma 1.7. Assume that $C^2 > 0$. Then $v = \mathrm{cl}(C)$ is in the interior of $\overline{\mathrm{NE}}(X)$, and hence, $\overline{\mathrm{NE}}(X) = \mathsf{R}$ and $\hat{\boldsymbol{\rho}}(X) = \dim \mathsf{N}(X) = 1 \geq \boldsymbol{\rho}(X) > 0$. This shows (1).

Assume that $C^2 = 0$. Then C is semi-ample by Lemma 1.4 and by $K_X C < 0$, and we have a fibration $\pi\colon X \to Y$ to a non-singular projective curve Y such that $mC \sim \pi^*H$ for some $m > 0$ and an ample divisor H on Y. Since $\mathsf{R} = \mathbb{R}_{\geq 0}\,\mathrm{cl}(C)$

is an extremal ray, the numerical classes of prime components of fibers of π are all belonging to R. It implies that every fiber F of π is irreducible. Moreover, π is a $\mathbb{P}^1$-fibration by $K_X F < 0$. Then $F_{\mathrm{red}} \simeq \mathbb{P}^1$ for any F, $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X) = 2$, and X has only rational singularities by [44, Prop. 2.33]. This shows (2), and we are done. □

Corollary 1.11. *Let X be a normal Moishezon surface.*

(1) *If K_X is not pseudo-effective, then X is a projective surface containing a rational curve, and $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$, i.e., $\mathsf{N}(X) = \mathrm{NS}(X) \otimes \mathbb{R}$.*

Let B be a pseudo-effective $\mathbb{R}$-divisor on X such that $\ulcorner N \urcorner$ is reduced for the negative part N of the Zariski-decomposition of B. Then:

(2) *For any numerically ample $\mathbb{R}$-divisor A, there exist at most finitely many rational curves C_i such that*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+B+A} + \sum \mathbb{R}_{\geq 0} \, \mathrm{cl}(C_i),$$

where $(K_X + B + A)C_i < 0$ and $\mathbb{R}_{\geq 0}\,\mathrm{cl}(C_i)$ is an extremal ray for any i.

(3) *Assume that $K_X + B$ is pseudo-effective. Let $\phi\colon X \to X'$ be the contraction morphism of the negative part E of the Zariski-decomposition of $K_X + B$ and set $B' := \phi_* B$. Then $\hat{\boldsymbol{\rho}}(X) - \boldsymbol{\rho}(X) = \hat{\boldsymbol{\rho}}(X') - \boldsymbol{\rho}(X')$, $K_{X'} + B'$ is nef, and $K_X + B = \phi^*(K_{X'} + B') + E$. Moreover, if X is projective, then so is X'.*

Proof. We set $B = 0$ for the proof of (1). We may assume that $K_X + B$ is not nef for assertions (1)–(3). Thus, there is an extremal ray $\mathsf{R} \subset \overline{\mathrm{NE}}(X)$ such that $(K_X + B)\mathsf{R} < 0$ by Theorem 1.9, and we have the contraction morphism $\pi\colon X \to Y$ of R by Theorem 1.10. If π is not birational, then $K_X + B$ is not pseudo-effective, $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$, and $\mathsf{R} = \mathbb{R}_{\geq 0}\,\mathrm{cl}(C)$ for a rational curve C, by (1) and (2) of Theorem 1.10.

Assume that π is birational. Then π is the contraction morphism of a negative curve C and $\mathsf{R} = \mathbb{R}_{\geq 0}\,\mathrm{cl}(C)$ by Theorem 1.10(3). Here, $(B - C)C = (B - N)C + (N - C)C \geq 0$, since $B - N$ is nef and $\ulcorner N \urcorner$ is reduced. Then C is rational, and $\boldsymbol{\rho}(Y) = \boldsymbol{\rho}(X) - 1$ and $\hat{\boldsymbol{\rho}}(Y) = \hat{\boldsymbol{\rho}}(X) - 1$ by Theorem 1.10(3). Moreover, Y is projective when X is so. We set $B_Y := \pi_* B$ and let N_Y be the negative part of the Zariski-decomposition of B_Y. Then $B_Y - \pi_* N = \pi_*(B - N)$ is nef (cf. [44, Rem. 2.13]), and $N_Y \leq \pi_* N$. Thus, $\ulcorner N_Y \urcorner$ is also reduced. Hence, in this case, we can consider the same statements for (Y, B_Y) instead of (X, B). Moreover, we have

$$K_X + B = \pi^*(K_Y + B_Y) + \alpha C \tag{I-1}$$

for a rational number $\alpha > 0$, since $(K_X + B)C < 0$.

Assertion (2) follows from Theorem 1.9 applied to the big $\mathbb{R}$-divisor $B + A$ and from the rationality of C_i shown in Theorem 1.10 with the observation above.

For (1), it is enough to prove: $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$ by [4, Prop. 7] and Lemma 1.7. By the observation above for birational and non-birational contraction morphisms of extremal rays, we have $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$ by induction on $\hat{\boldsymbol{\rho}}(X)$.

In the situation of (3), the contraction morphism $\pi\colon X \to Y$ is always birational, and $K_Y + B_Y$ is also pseudo-effective. By (I-1), $E = \pi^* E_Y + \alpha C$ for the negative part E_Y of the Zariski decomposition of $K_Y + E_Y$. In particular, $\phi\colon X \to X'$ factors through the contraction morphism $\phi_Y : Y \to X'$ of E_Y, where $B' = \phi_{Y*} B_Y$. Therefore, by induction on $\hat{\boldsymbol{\rho}}(X)$, we see that ϕ is expressed as the composite of birational contraction morphisms of extremal rays. Hence, $\hat{\boldsymbol{\rho}}(X) - \boldsymbol{\rho}(X) = \hat{\boldsymbol{\rho}}(X') - \boldsymbol{\rho}(X')$, and X' is projective when X is so. Thus, we are done. □

Remark. If B is an effective $\mathbb{R}$-divisor such that $\ulcorner B \urcorner$ is reduced, then assertions (2) and (3) hold for B, since $N \leq B$ for the negative part N of the Zariski-decomposition of B. In particular, these assertions hold for any log-canonical pair (X, B) for a normal Moishezon surface X; in this case, if $K_X + B$ is pseudo-effective, then (X', B') in (3) is also log-canonical by $K_X + B = \phi^*(K_{X'} + B') + E$.

We have the following *abundance theorem* for normal Moishezon surfaces:

Theorem 1.12. *Let X be a normal Moishezon surface and B an effective $\mathbb{Q}$-divisor on X such that (X, B) is log-canonical, or more generally,* MR log-canonical *in the sense of* [1, Def. 1.7]. *If $K_X + B$ is pseudo-effective, then the positive part of the Zariski-decomposition of $K_X + B$ is semi-ample.*

Proof. Let $\mu\colon M \to X$ be the minimal resolution of singularities. Then M is projective, and there exist effective $\mathbb{Q}$-divisors B_μ and T_μ on M such that B_μ and T_μ have no common prime component, T_μ is μ-exceptional, $\ulcorner B_\mu \urcorner$ is reduced, and $K_M + B_\mu = \mu^*(K_X + B) + T_\mu$. Hence, for the positive part P of the Zariski-decomposition of $K_X + B$, the pullback $\mu^* P$ is the positive part of the Zariski-decomposition of $K_M + B_\mu$. It is known by [19, Main Thm. (1.4)] that $\mu^* P$ is semi-ample. Thus, P is so by an argument in the proof of Lemma 1.4. In fact, we have a morphism $\Phi\colon M \to \mathbb{P}^N$ to a projective space such that $m\mu^* P \sim \Phi^* H$ for a hyperplane H of $\mathbb{P}^N$ and a positive integer m. Then $\Phi = \varphi \circ \mu$ for a morphism $\varphi\colon X \to \mathbb{P}^N$, and we have $mP \sim \varphi^* H$. Hence, P is semi-ample. □

The following is a generalization of [42, Prop. 3.3] (cf. [51, Prop. 4.4]):

Theorem 1.13. *Let X be a normal Moishezon surface such that $-K_X$ is big. Then X has only finitely many negative curves. If $\hat{\boldsymbol{\rho}}(X) \geq 3$ in addition, then $\overline{\mathrm{NE}}(X)$ is generated by the numerical classes of negative curves.*

Proof. Note that X is projective and $\boldsymbol{\rho}(X) = \hat{\boldsymbol{\rho}}(X)$ by Corollary 1.11(1). Let us fix an ample divisor A and take a rational number $\alpha > 0$ such that $-(K_X + \alpha A)$ is big. Then prime divisors Γ satisfying $(K_X + \alpha A)\Gamma > 0$ are prime components of the negative part N_α of the Zariski-decomposition of $-(K_X + \alpha A)$. We fix a rational number t such that $0 < t < \alpha$. By Theorem 1.9, there exist finitely many extremal rays R_j such that $(K_X + tA)\mathsf{R}_j < 0$ and

(I-2) $$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X + tA} + \sum \mathsf{R}_j.$$

For a negative curve Γ on X, if $\mathrm{cl}(\Gamma) \notin \mathsf{R}_j$ for any j, then $(K_X + tA)\Gamma \geq 0$, and hence, Γ is a prime component of N_α. Therefore, X has only finitely many negative curves.

Assume that $\hat{\boldsymbol{\rho}}(X) \geq 3$. Then $\mathsf{R}_j = \mathbb{R}_{\geq 0}\operatorname{cl}(\Gamma_j)$ for a negative curve Γ_j, by Theorem 1.10. Let $\operatorname{Neg}(X) \subset \overline{\mathrm{NE}}(X)$ be the polyhedral cone generated by the numerical classes of negative curves. For a pseudo-effective $\mathbb{R}$-divisor D, let $\mathfrak{S}(D)$ be the set of elements $z \in \operatorname{Neg}(X)$ such that $\operatorname{cl}(D) - z \in \overline{\mathrm{NE}}(X)_{K_X+tA}$. Then $\mathfrak{S}(D)$ is compact, since it is closed and since

$$\langle \operatorname{cl}(A), z\rangle = AD - \langle \operatorname{cl}(A), \operatorname{cl}(D) - z\rangle \leq AD$$

for any $z \in \mathfrak{S}(D)$. We can find an element $z^\circ \in \mathfrak{S}(D)$ such that

$$\langle \operatorname{cl}(A), z^\circ\rangle = \max\{\langle \operatorname{cl}(A), z\rangle \mid z \in \mathfrak{S}(D)\}$$

and we have an effective $\mathbb{R}$-divisor B such that $\operatorname{cl}(B) = z^\circ$ and that every prime component of B is a negative curve. Then $D - B$ is pseudo-effective.

We consider the Zariski-decomposition $D - B = P + N$, where P (resp. N) is the positive (resp. negative) part. If $P \approx 0$, then $D \approx B + N$ and $\operatorname{cl}(D) \in \operatorname{Neg}(X)$. Thus, it suffices to derive a contradiction assuming $P \not\approx 0$. Then $-(K_X + tA)P > 0$ by the Hodge index theorem as $-(K_X + tA)$ is big. By (I-2), there exist real numbers $r_j \geq 0$ such that one of r_j is positive and that

$$\operatorname{cl}(P - \sum r_j\Gamma_j) \in \overline{\mathrm{NE}}(X)_{K_X+tA}.$$

Then $\operatorname{cl}(B + N + \sum r_j\Gamma_j) \in \mathfrak{S}(D)$ and we have $A(B + N + \sum r_j\Gamma_j) > AB = \langle \operatorname{cl}(A), z^\circ\rangle$ contradicting the choice of z°. Thus, we are done. □

1.4. **$\mathbb{P}^1$-bundles and projective cones over curves.** Here, we note some elementary properties of $\mathbb{P}^1$-bundles and projective cones (see Definition 1.16 below) over a non-singular projective curve.

Let $\pi\colon X \to T$ be a $\mathbb{P}^1$-bundle over a non-singular projective curve T. Then $X \simeq \mathbb{P}_T(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ of rank 2 on T. By [35, Thm. 3.1], the following conditions are equivalent:

- $\mathcal{E}$ is semi-stable;
- X contains no negative curves, i.e., $\operatorname{Nef}(X) = \overline{\mathrm{NE}}(X)$;
- $-K_{X/T} = -K_X + \pi^*K_T$ is nef;
- $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\geq 0}\operatorname{cl}(F) + \mathbb{R}_{\geq 0}\operatorname{cl}(-K_{X/T})$ for a fiber F of π.

In particular, if X contains a negative curve, then $\mathcal{E}$ is not semi-stable; hence, the maximal destabilizing subsheaf of $\mathcal{E}$ produces a *negative section* of π, i.e., a section with negative self-intersection number. The negative section is a unique negative curve by the following:

Lemma 1.14. *Assume that π has a negative section Θ. If C is a prime divisor on X such that $\pi(C) = T$ and $C \neq \Theta$, then $C^2 > 0$.*

Proof. There is a divisor L on T such that $C \sim d\Theta + \pi^*L$ for $d := \deg(C/T) > 0$. Then $C\Theta = d\Theta^2 + \deg L \geq 0$ by $\Theta \neq C$, and

$$C^2 = d^2\Theta^2 + 2d\deg L = 2d(d\Theta^2 + \deg L) - d\Theta^2 \geq -d\Theta^2.$$

Thus, $C^2 > 0$ by $\Theta^2 < 0$. □

The following lemma is used in the proof of Proposition 4.11 below.

Lemma 1.15. *Assume that π has no negative section. Let D be a non-zero effective divisor on X such that $\mathrm{cl}(D) \in \mathbb{R}_{\geq 0}\,\mathrm{cl}(-K_{X/T})$. Then $\mathrm{cl}(\Theta) \in \mathbb{R}_{\geq 0}\,\mathrm{cl}(-K_{X/T})$ for any prime component Θ of D. Suppose that D is reduced. Then D is non-singular and $\pi|_D \colon D \to T$ is an étale morphism. In particular, there is a finite étale cover $\tau \colon T' \to T$ from a non-singular projective curve T' such that $D \times_T T'$ is a disjoint union of copies of sections of the induced $\mathbb{P}^1$-bundle $X \times_T T' \to T'$. Here, if $\deg \pi|_D \geq 3$ in addition, then $X \times_T T' \simeq \mathbb{P}^1 \times T'$ over T'.*

Proof. By assumption, $\mathbb{R}_{\geq 0}\,\mathrm{cl}(-K_{X/T})$ is an extremal ray of $\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)$, and the first assertion follows from Lemma 1.6(2). As a consequence, every Θ dominates T. Suppose that D is reduced. Let $\nu \colon \widetilde{D} \to D$ be the normalization and consider the composite $\alpha := \pi|_D \circ \nu \colon \widetilde{D} \to T$. Then $K_{\widetilde{D}} = \nu^* K_D - \boldsymbol{c}$ for the conductor $\boldsymbol{c}$, and the ramification formula $K_{\widetilde{D}} = \alpha^* K_T + R_\alpha$ implies that

$$\deg K_D = (\deg \alpha) \deg K_T + \deg R_\alpha + \deg \boldsymbol{c}.$$

On the other hand, $\deg K_D = (\deg \alpha) \deg K_T$ by $(K_{X/T} + D)D = 0$. Thus, $R_\alpha = \boldsymbol{c} = 0$. This means that D is non-singular, and $\pi|_D$ is étale. There is a finite étale Galois cover $\tau \colon T' \to T$ which factors through the étale cover $\pi|_\Theta \colon \Theta \to T$ for any prime component Θ of D. Then $D \times_T T'$ is a disjoint union of copies of T', since $T' \times_T T' \simeq G \times T'$ for the Galois group G of τ. Thus, τ is a finite étale cover satisfying the required condition. If $\deg \pi|_D \geq 3$, then $X \times_T T' \to T'$ has at least three mutually disjoint sections of self-intersection number zero. Thus, $X \times_T T' \simeq \mathbb{P}^1 \times T'$ over T' by [40, Lem. 7]. □

Definition 1.16 (cf. [23, §8.3])**.** A normal projective surface X is called a *projective cone* over a non-singular projective curve C if there is an ample invertible sheaf $\mathcal{A}$ on C such that $X \simeq \mathrm{Projan}\,(R(C, \mathcal{A})[\mathtt{x}])$ for the graded $\mathbb{C}$-algebra

$$R(C, \mathcal{A}) = \bigoplus\nolimits_{m=0}^{\infty} R(C, \mathcal{A})_m = \bigoplus\nolimits_{m=0}^{\infty} H^0(C, \mathcal{A}^{\otimes m})$$

and a variable $\mathtt{x}$, where the grading of $R(C, \mathcal{A})[\mathtt{x}] = R(C, \mathcal{A}) \otimes_{\mathbb{C}} \mathbb{C}[\mathtt{x}]$ is given by

$$(R(C, \mathcal{A})[\mathtt{x}])_m = \bigoplus\nolimits_{j=0}^{m} H^0(C, \mathcal{A}^{\otimes j})\mathtt{x}^{m-j}.$$

For the maximal ideal $R(C, \mathcal{A})_+ = \bigoplus_{m>0} R(C, \mathcal{A})_m$ of $R(C, \mathcal{A})$, the point of X determined by the ideal $R(C, \mathcal{A})_+[\mathtt{x}]$ is called the *vertex*. The analytic subspace defined by the ideal $(\mathtt{x} - a)R(C, \mathcal{A})[\mathtt{x}]$ for $a \in R(C, \mathcal{A})_1$ is called a *cross section*.

Remark. If $\mathcal{A}$ is very ample in Definition 1.16, then the definition of projective cone coincides with the usual geometric definition of projective cone over C embedded by the complete linear system $|\mathcal{A}|$. More precisely, we have the following: Let $C \hookrightarrow \mathbb{P}^n$ be the closed embedding defined by $|\mathcal{A}|$. Fixing a hyperplane H of $\mathbb{P}^{n+1}$, we embed C into $\mathbb{P}^{n+1}$ by an isomorphism $\mathbb{P}^n \simeq H$. For a point P in $\mathbb{P}^{n+1} \setminus H$, let $V(C, \mathcal{A})$ be the union of lines of $\mathbb{P}^{n+1}$ passing through P and intersecting C. Then $V(C, \mathcal{A})$ is isomorphic to the projective cone $\mathrm{Projan}\,(R(C, \mathcal{A})[\mathtt{x}])$. Here, P is the vertex and $C = H \cap V(C, \mathcal{A})$ is a cross section.

Remark 1.17. For C and $\mathcal{A}$ in Definition 1.16, let M be the $\mathbb{P}^1$-bundle $\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{A})$ over C. Then there is a birational morphism $\mu \colon M \to X$ such that $\mu^* \mathcal{O}_X(1) \simeq$

$\mathcal{O}_M(1)$ for the tautological invertible sheaf $\mathcal{O}_X(1)$ with respect to the graded algebra $R(C, A)[\mathtt{x}]$ and for the tautological invertible sheaf $\mathcal{O}_M(1)$ with respect to $\mathcal{O}_C \oplus \mathcal{A}$. The inverse image $\mu^{-1}(P)$ of the vertex P equals the μ-exceptional locus, and it is a section of the $\mathbb{P}^1$-bundle $M \to C$ corresponding to the projection $\mathcal{O}_C \oplus \mathcal{A} \to \mathcal{O}_C$. If $C \not\simeq \mathbb{P}^1$ or $\deg \mathcal{A} > 1$, then the vertex is a singular point of X and μ is the minimal resolution of singularity.

Lemma 1.18. *Let X be a projective cone over a non-singular projective curve C. Then $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X) = 1$ and $H^1(X, \mathcal{O}_X) = 0$. If X is irrational and if there is an elliptic curve B on X not containing the vertex, then C is an elliptic curve, and B is a cross section.*

Proof. Let $\mu\colon M = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{A}) \to X$ be the birational morphism in Remark 1.17. Then $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(M) - 1 = 1$ by [44, Lem. 2.10]. Since X is projective, we have $\boldsymbol{\rho}(X) = \hat{\boldsymbol{\rho}}(X) = 1$. We set $E := \mu^{-1}(P)$. Let D be a section of the $\mathbb{P}^1$-bundle $\pi\colon M \to C$ corresponding to a surjection $\mathcal{O}_C \oplus \mathcal{A} \to \mathcal{A}$. Then $D \cap E = \emptyset$ and $K_M + D + E \sim \pi^* K_C$. The image $\mu(D)$ is a cross section of X as $\mathcal{O}_M(1) \simeq \mathcal{O}_M(D)$. By Leray's spectral sequence, we have an exact sequence

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(M, \mathcal{O}_M) \xrightarrow{r} H^0(X, R^1\mu_*\mathcal{O}_M).$$

For the isomorphism $\pi^*\colon H^1(C, \mathcal{O}_C) \to H^1(M, \mathcal{O}_M)$ and for the canonical homomorphism $R^1\mu_*\mathcal{O}_M \to R^1\mu_*\mathcal{O}_E$, the composite

$$H^1(C, \mathcal{O}_C) \xrightarrow{\pi^*} H^1(M, \mathcal{O}_M) \xrightarrow{r} H^0(X, R^1\mu_*\mathcal{O}_M) \to H^0(X, R^1\mu_*\mathcal{O}_E) \simeq H^1(E, \mathcal{O}_E)$$

is an isomorphism induced by $\pi|_E\colon E \to C$. Hence, r is injective, and we have $H^1(X, \mathcal{O}_X) = 0$.

Assume that X is irrational, i.e., $\boldsymbol{g}(C) \geq 1$. Let B be an elliptic curve on X_{reg}. Then $\mathcal{O}_X(K_X + B)|_B \simeq \mathcal{O}_B$, and we have an exact sequence

$$H^0(X, \mathcal{O}_X(K_X + B)) \to H^0(B, \mathcal{O}_B) \to H^1(X, \mathcal{O}_X(K_X)) \simeq H^1(X, \mathcal{O}_X)^\vee = 0.$$

Thus, there is an effective divisor $B^\dagger$ on X such that $K_X + B \sim B^\dagger$ and $B \cap B^\dagger = \emptyset$. Since $\boldsymbol{\rho}(X) = 1$, B is ample and $B^\dagger = 0$. In particular, X is Gorenstein by $K_X + B \sim 0$, and $K_M = \mu^* K_X + mE$ for an integer m. Here, we have $m = -1$ by $0 \leq 2\boldsymbol{g}(C) - 2 = (K_M + E)E = (m + 1)E^2$ and

$$0 < B\mu_*F = (\mu^*B)F = (-\mu^*K_X)F = (mE - K_M)F = m + 2$$

for a fiber F of π. Therefore, $\boldsymbol{g}(C) = 1$, i.e., C is an elliptic curve, and $\mu^*B \sim D$. As a consequence, B is a cross section. □

2. Endomorphisms of normal varieties

We discuss some general properties of endomorphisms of normal varieties. After giving elementary properties of endomorphisms of sets in Section 2.1, we discuss Galois closures of powers of an endomorphism and endomorphisms of curves in Sections 2.2 and 2.3, respectively. In Sections 2.4 and 2.5, we define and study the *characteristic completely invariant divisor* and the *refined ramification divisor*: These are key notions of our study of non-isomorphic surjective endomorphisms.

2.1. **Endomorphisms of sets.** We present two elementary lemmas on endomorphisms of sets, which are useful in our study of endomorphisms. The proofs are left to the reader.

Lemma 2.1 (cf. [40, Prop. 11], [16, Lem. 3.4])**.** *Let $f\colon \mathcal{X} \to \mathcal{X}$ be an injection for a set $\mathcal{X}$. Assume that*

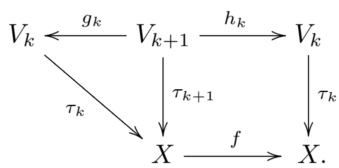
$$\mathcal{X} = \bigcup\nolimits_{m=1}^{\infty} (f^m)^{-1}\mathcal{S}$$

for a finite subset $\mathcal{S}$, where $f^m = f \circ \cdots \circ f\colon \mathcal{X} \to \mathcal{X}$ denotes the m-th power (*iteration*) *of f. Then $\mathcal{X}$ is a finite set. In particular, $f^n\colon \mathcal{X} \to \mathcal{X}$ is the identity map for some $n > 0$.*

Lemma 2.2. *Let $f\colon \mathcal{X} \to \mathcal{X}$ be a surjective map for a set $\mathcal{X}$. For a finite subset $\mathcal{S} \subset \mathcal{X}$, suppose that $f^{-1}\mathcal{S} \subset \mathcal{S}$. Then $f^{-1}\mathcal{S} = \mathcal{S}$ and $f|_{\mathcal{S}}\colon \mathcal{S} \to \mathcal{S}$ is a bijection. In particular, there is a positive integer k such that $(f^k)^{-1}(s) = \{s\}$ for any $s \in \mathcal{S}$.*

2.2. **Galois closure of an endomorphism.** Let f be a non-isomorphic *finite* surjective endomorphism of a *compact* normal variety X. We discuss Galois closures of iterations $f^k\colon X \to X$ in Lemmas 2.3 and 2.4 below. The following is originally in [43] and is written as [47, Lem. 2.5]:

Lemma 2.3. *Let $\theta_k\colon V_k \to X$ be the Galois closure of $f^k\colon X \to X$ for an integer $k \geq 1$ and let $\tau_k\colon V_k \to X$ be the induced finite Galois cover such that $\theta_k = f^k \circ \tau_k$. Then there exist finite Galois morphisms $g_k, h_k\colon V_{k+1} \to V_k$ such that $\tau_k \circ g_k = \tau_{k+1}$ and $\tau_k \circ h_k = f \circ \tau_{k+1}$, i.e., the diagram below is commutative*:



Proof. The composite $f^k \circ \tau_{k+1}\colon V_{k+1} \to X \to X$ is Galois, since so is $f^{k+1} \circ \tau_{k+1} = \theta_{k+1}$. Hence, $f^k \circ \tau_{k+1}$ factors through the Galois closure θ_k of f^k. Thus, $\tau_{k+1} = \tau_k \circ g_k$ for a morphism $g_k\colon V_{k+1} \to V_k$. Let H_i be the Galois group of $f^i \circ \tau_{k+1}\colon V_{k+1} \to X$ for $0 \leq i \leq k+1$. Then $\theta_k\colon V_k \to X$ is regarded as the Galois closure of $H_1 \backslash V_{k+1} \to H_{k+1} \backslash V_{k+1}$; thus $V_k \simeq H \backslash V_{k+1}$ for the maximal normal subgroup H of H_{k+1} contained in H_1. Hence, we have a morphism $h_k\colon V_{k+1} \to V_k$ satisfying $\tau_k \circ h_k = f \circ \tau_{k+1}$. $\square$

Lemma 2.4. *Let $U \subset X$ be a Zariski-open dense subset such that $f^{-1}U = U$, $f|_U\colon U \to U$ is étale in codimension* 1, *and that U has only isolated quotient singularities. Then, for the morphism τ_k in Lemma* 2.3, *$\tau_k^{-1}U$ is non-singular and the euler number $\mathbf{e}(\tau_k^{-1}U)$ is zero for $k \gg 0$.*

Proof. Note that $\operatorname{Sing} U$ is a finite set. By assumption, $f|_U$ is étale over U_{reg} and $f(\operatorname{Sing} U) \subset \operatorname{Sing} U$. For an integer $k > 0$, let $\mathcal{T}_k$ be the set of points $Q \in \operatorname{Sing} U$ such that $(f^k)^{-1}(Q) \subset \operatorname{Sing} U$. Then $\operatorname{Sing} U \supset \mathcal{T}_1 \supset \mathcal{T}_2 \supset \cdots$ and $f^{-1}\mathcal{T}_k \subset \mathcal{T}_{k-1}$ for any $k > 1$. Thus, $f^{-1}\mathcal{T}_\infty = \mathcal{T}_\infty$ and $f\colon \mathcal{T}_\infty \to \mathcal{T}_\infty$ is bijective for the intersection $\mathcal{T}_\infty := \bigcap_{k>0} \mathcal{T}_k$ by Lemma 2.2. In particular, there is a positive integer l such that $(f^l)^{-1}(Q) = Q$ for any $Q \in \mathcal{T}_\infty$. Since the local fundamental group of U at $Q \in \mathcal{T}_\infty$ is finite, f^l gives an isomorphism between two open neighborhoods of Q; this contradicts $\deg f > 1$. Therefore, $\mathcal{T}_\infty = \emptyset$, and $\mathcal{T}_k = \emptyset$ for $k \gg 0$, i.e., $(f^k)^{-1}(Q) \not\subset \operatorname{Sing} U$ for any $Q \in \operatorname{Sing} U$. For the integer k, we have $\operatorname{Sing} \theta_k^{-1} U = \operatorname{Sing} \tau_k^{-1} U = \emptyset$, since $\theta_k \colon V_k \to X$ is Galois and étale in codimension 1 over U. Then there exist two étale morphisms $g_k, h_k \colon \tau_{k+1}^{-1} U \to \tau_k^{-1} U$ satisfying $\deg g_k = (\deg f)(\deg h_k)$ by Lemma 2.3. Hence, $\boldsymbol{e}(\tau_k^{-1} U) = 0$ by $\boldsymbol{e}(\tau_{k+1}^{-1} U) = (\deg g_k)\, \boldsymbol{e}(\tau_k^{-1} U) = (\deg h_k)\, \boldsymbol{e}(\tau_k^{-1} U)$. □

An argument similar to the above is used in the proof of [47, Thm. 3.3]. The following is useful for analyzing the singularity of the Galois closure of f^k:

Lemma 2.5. *Let $\phi \colon V \to W$ be a finite surjective morphism of normal varieties. Let $\theta \colon U \to W$ be the Galois closure of ϕ and let τ be the induced morphism $U \to V$ such that $\theta = \phi \circ \tau$. For a point $P \in W$, suppose that*

- (i) *the morphism $\phi \colon (V, P') \to (W, P)$ of germs of normal varieties is Galois for any $P' \in \phi^{-1}(P)$,*
- (ii) *for any two points P', $P'' \in \phi^{-1}(P)$, there is an isomorphism $\varphi \colon (V, P') \to (V, P'')$ of germs over (W, P), i.e., $\phi \circ \varphi = \phi$.*

Then τ is étale along $\theta^{-1}(P)$.

Proof. There exists a connected open neighborhood $\mathcal{W}$ of P such that $\theta^{-1}(\mathcal{W})$ is a disjoint union $\bigsqcup \mathcal{U}_Q$ of connected open neighborhoods $\mathcal{U}_Q$ of all $Q \in \theta^{-1}(P)$. We may assume that every connected component of $\phi^{-1}(\mathcal{W})$ is Galois over $\mathcal{W}$ by (i). Let G be the Galois group of θ and let $G_Q \subset G$ be the stabilizer at $Q \in \theta^{-1}(P)$ for the action of G on U. Then, for any $Q \in \theta^{-1}(P)$, the inverse image $\theta^{-1}(P)$ is identified with the factor set G/G_Q, and $\mathcal{U}_Q \to \mathcal{W}$ is a Galois cover with the Galois group G_Q. For $g \in G$, the action $g \colon \theta^{-1}(\mathcal{W}) \to \theta^{-1}(\mathcal{W})$ induces an isomorphism $\mathcal{U}_Q \to \mathcal{U}_{g(Q)}$ over $\mathcal{W}$ and $G_{g(Q)} = g G_Q g^{-1}$.

Let H be the Galois group of τ. Then $\bigcap_{g \in G} g H g^{-1} = \{1\}$, since θ is the Galois closure of ϕ. The stabilizer $H_Q \subset H$ at $Q \in \theta^{-1}(P)$ is just $G_Q \cap H$. The connected component $\mathcal{V}_{\tau(Q)}$ of $\phi^{-1}(\mathcal{W})$ containing $\tau(Q)$ is just isomorphic to the quotient space $H_Q \backslash \mathcal{U}_Q$. By (i) and by our choice of $\mathcal{W}$, we see that H_Q is a normal subgroup of G_Q. We have an isomorphism

$$\mathcal{V}_{\tau(g(Q))} \simeq (g^{-1} H_{g(Q)} g) \backslash \mathcal{U}_Q = (G_Q \cap g^{-1} H g) \backslash \mathcal{U}_Q$$

over $\mathcal{W}$ for $g \in G$ by the action $g \colon \theta^{-1}(\mathcal{W}) \to \theta^{-1}(\mathcal{W})$.

The normalization of the fiber product $\mathcal{U}_Q \times_{\mathcal{W}} \mathcal{U}_Q$ is isomorphic to $G_Q \times \mathcal{U}_Q$ by $G_Q \times \mathcal{U}_Q \ni (g, x) \mapsto (gx, x) \in \mathcal{U}_Q \times \mathcal{U}_Q$. Thus, for two subgroups H_1, $H_2 \subset G_Q$, a connected component of the normalization of $(H_1 \backslash \mathcal{U}_Q) \times_{\mathcal{W}} (H_2 \backslash \mathcal{U}_Q)$ is expressed as $(H_1 \cap k H_2 k^{-1}) \backslash \mathcal{U}_Q$ for some $k \in G_Q$. We apply this to the subgroups $H_1 = H_Q$ and $H_2 = G_Q \cap g^{-1} H g$ for $g \in G$. Then we may assume that $H_1 \backslash \mathcal{U}_Q \simeq H_2 \backslash \mathcal{U}_Q$ over $\mathcal{W}$ by (ii) and by our choice of $\mathcal{W}$. Since $H_1 \backslash \mathcal{U}_Q$ is Galois over $\mathcal{W}$, any connected component of the normalization of $(H_1 \backslash \mathcal{U}_Q) \times_{\mathcal{W}} (H_2 \backslash \mathcal{U}_Q)$ is isomorphic to $H_1 \backslash \mathcal{U}_Q$. Thus, $\sharp(H_1 \cap k H_2 k^{-1}) = \sharp H_1$ for some $k \in G_Q$. Hence, $H_1 = H_2$, since $H_1 = H_Q$

is a normal subgroup of G_Q. Therefore, $G_Q \cap g^{-1}Hg = H_Q$ for any $g \in G$ and

$$\{1\} = G_Q \cap \bigcap\nolimits_{g\in G} gHg^{-1} = \bigcap\nolimits_{g\in G} G_Q \cap gHg^{-1} = H_Q.$$

It implies that $\mathcal{U}_Q \to \mathcal{V}_{\tau(Q)}$ is an isomorphism. As a consequence, $\tau\colon U \to V$ is étale along $\theta^{-1}(P)$. □

2.3. **Endomorphisms of curves.** We present here some basic results on endomorphisms of non-singular projective curves.

Lemma 2.6. *Let f be a finite surjective endomorphism of a non-singular projective curve X. Let Σ be a finite subset of X such that $f^{-1}\Sigma = \Sigma$. If $2\boldsymbol{g}(X) - 2 + \sharp\Sigma > 0$, then f is an automorphism of finite order.*

Proof. We consider Σ as a reduced divisor on X. Then there is an effective divisor Δ on X such that $K_X + \Sigma = f^*(K_X + \Sigma) + \Delta$ (cf. [45, Lem. 1.39]). Thus,

$$\deg(K_X + \Sigma) = (\deg f)\deg(K_X + \Sigma) + \deg\Delta \geq (\deg f)\deg(K_X + \Sigma).$$

Since $\deg(K_X + \Sigma) = 2\boldsymbol{g}(X) - 2 + \sharp\Sigma > 0$, we have $\deg f = 1$, i.e., f is an automorphism. If $\boldsymbol{g}(X) \geq 2$, then it is well known that f is of finite order. If $\boldsymbol{g}(X) = 1$ and $\Sigma \neq \emptyset$, then some power f^k fixes points of Σ; thus, f is of finite order. If $\boldsymbol{g}(X) = 0$ and $\sharp\Sigma \geq 3$, then it is also well known that the order of f is finite. □

Lemma 2.7. *Let $\tau\colon Y \to X$ be a finite surjective morphism of non-singular projective curves and let Σ be a finite subset of X with a collection $\{m_P\}_{P\in\Sigma}$ of integers $m_P \geq 2$ such that*

- *τ is étale over $X \setminus \Sigma$ and*
- *$\tau^*P = m_P\tau^{-1}(P)$ for any $P \in \Sigma$.*

If an automorphism f of X preserves Σ, i.e., $f(\Sigma) = \Sigma$, then there exist an automorphism g of Y and a positive integer k such that $\tau \circ g = f^k \circ \tau$.

Proof. If τ is an isomorphism, then we can take $\tau^{-1} \circ f \circ \tau$ as g. Thus, we may assume that $\deg\tau > 1$. If the order of f is finite, then we can take the identity morphism of Y as g. Hence, we may assume that $2\boldsymbol{g}(X) - 2 + \sharp\Sigma \leq 0$ by Lemma 2.6.

Assume that $\boldsymbol{g}(X) \geq 1$. Then $\boldsymbol{g}(X) = 1$ and $\Sigma = \emptyset$. Hence, X and Y are elliptic curves. For certain complex Lie group structures on X and Y, we may assume that

- some power f^k is the translation morphism $\mathrm{tr}(a)\colon z \mapsto z + a$ by an element $a \in X$, and
- τ is a group homomorphism.

Then, for a point $b \in \tau^{-1}(a)$, the transition morphism $g = \mathrm{tr}(b)\colon Y \to Y$ satisfies $\tau \circ g = f^k \circ \tau$.

Assume next that $\boldsymbol{g}(X) = 0$. Then $\sharp\Sigma \leq 2$. Since τ is étale over $X \setminus \Sigma$ and $\deg\tau > 1$, we have $\sharp\Sigma = 2$, and τ is a cyclic cover of degree $m > 1$ branched at two points P_1 and P_2, where $\Sigma = \{P_1, P_2\}$ and $m_{P_1} = m_{P_2} = m$. By certain isomorphisms $X \simeq \mathbb{P}^1$ and $Y \simeq \mathbb{P}^1$, we may assume that $P_1 = (1:0)$, $P_2 = (0:1)$ and that τ is an endomorphism $\mathbb{P}^1 \to \mathbb{P}^1$ given by $(\mathtt{x}:\mathtt{y}) \mapsto (\mathtt{x}^m:\mathtt{y}^m)$ for a homogeneous coordinate

$(\mathtt{x}:\mathtt{y})$. Since f preserves Σ, it is given by $(\mathtt{x}:\mathtt{y}) \mapsto (c\mathtt{x}:\mathtt{y})$ or $(\mathtt{x}:\mathtt{y}) \mapsto (c\mathtt{y}:\mathtt{x})$ for some $c \in \mathbb{C} \setminus \{0\}$. Hence, we can find an automorphism g of Y satisfying $\tau \circ g = f \circ \tau$. □

Proposition 2.8. *Let X be a non-singular projective curve with a non-isomorphic surjective endomorphism f. Let D be a non-zero effective $\mathbb{Q}$-divisor such that the ramification divisor R_f equals $f^*D - D$, i.e., $K_X + D = f^*(K_X + D)$. Then $X \simeq \mathbb{P}^1$ and $\deg D = 2$. Let $D = \sum a_i P_i$ be the prime decomposition such that $a_1 \geq a_2 \geq \cdots > 0$. Then $\Lambda = (a_1, a_2, \ldots)$ is one of the following:*

$$(1,1), \quad (1,1/2,1/2), \quad (5/6,3/4,1/2),$$
$$(3/4,3/4,1/2), \quad (2/3,2/3,2/3), \quad (1/2,1/2,1/2,1/2).$$

Moreover, the endomorphism f is determined by Λ as follows, where $F_d\colon \mathbb{P}^1 \to \mathbb{P}^1$ is an endomorphism defined by $(\mathtt{x} : \mathtt{y}) \mapsto (\mathtt{x}^d : \mathtt{y}^d)$ for $d := \deg f$, and $\iota\colon \mathbb{P}^1 \to \mathbb{P}^1$ is an involution defined by $(\mathtt{x} : \mathtt{y}) \mapsto (\mathtt{y} : \mathtt{x})$:

1. *If $\Lambda = (1,1)$, then f is isomorphic to F_d or $\iota \circ F_d$.*
2. *If $\Lambda = (1,1/2,1/2)$, then there exists a double-cover $\tau\colon \mathbb{P}^1 \to X$ such that*
   (a) *τ is the quotient morphism by the involution ι, and*
   (b) *$\tau \circ F_d = f \circ \tau$ or $\tau \circ (\iota \circ F_d) = f \circ \tau$.*
3. *If Λ is not $(1,1)$ nor $(1,1/2,1/2)$, then there exist a finite cyclic cover $\tau\colon V \to X$ from an elliptic curve V and an étale endomorphism $f_V\colon V \to V$ such that $K_V = \tau^*(K_X + D)$ and $\tau \circ f_V = f \circ \tau$.*

Proof. We have $\deg(K_X + D) = 0$ by $K_X + D = f^*(K_X + D)$, and it implies that $X \simeq \mathbb{P}^1$ and $\deg D = 2$. For a point $P \in X$, we set $d_P := \operatorname{mult}_P f^*(f(P))$. Then

$$d_P - 1 = \operatorname{mult}_P R_f = d_P \operatorname{mult}_{f(P)} D - \operatorname{mult}_P D$$

by $R_f = f^*D - D$. Hence,

$$d_P(1 - \operatorname{mult}_{f(P)} D) = 1 - \operatorname{mult}_P D \tag{II-1}$$

(cf. the proof of [45, Prop. 3.4]). Let $\mathcal{T}$ be the set of points $Q \in \operatorname{Supp} D$ such that

$$\operatorname{mult}_Q D \notin \{1 - m^{-1} \mid m \in \mathbb{Z}, m \geq 1\}.$$

Then $f^{-1}\mathcal{T} \subset \mathcal{T}$ by (II-1). Hence, $f^{-1}\mathcal{T} = \mathcal{T}$ and f induces a permutation of $\mathcal{T}$ by Lemma 2.2. Thus, $f^{-1}(f(Q)) = \{Q\}$, $d_Q = \deg f > 1$, and $\operatorname{mult}_Q D = 1$ for any $Q \in \mathcal{T}$ by (II-1). If $\sharp\mathcal{T} \geq 2$, then $\sharp\mathcal{T} = 2$ and $\Lambda = (1,1)$. If $\sharp\mathcal{T} = 1$, then $a_1 = 1$ and $a_i = 1 - m_i^{-1}$ for $i \geq 2$ with $m_i \geq 2$; hence $\Lambda = (1,1/2,1/2)$. If $\mathcal{T} = \emptyset$, then $a_i = 1 - m_i^{-1}$ for $m_i \geq 2$ with $m_1 \geq m_2 \geq \cdots$, and hence, $(m_1, m_2, \ldots)$ is one of

$$(6,3,2), \quad (4,4,2), \quad (3,3,3), \quad (2,2,2,2).$$

Therefore, we have the list of Λ. The remaining assertions are shown as follows:

Assume that $\Lambda = (1,1)$. Then $D = P_1 + P_2$ and $f^{-1}D = D$ by $f^{-1}\mathcal{T} = \mathcal{T}$. Replacing f with $\iota \circ f$ if necessary, we may assume that $f^*(P_i) = dP_i$ for $i = 1$, 2. By an isomorphism $X \simeq \mathbb{P}^1$ with a homogeneous coordinate $(\mathtt{x}:\mathtt{y})$ of $\mathbb{P}^1$, we may assume that $P_1 = (1:0)$ and $P_2 = (0:1)$. Then f is written as $(\mathtt{x}:\mathtt{y}) \mapsto (c\mathtt{x}^d:\mathtt{y}^d)$

for a non-zero constant c. Here, we can make $c = 1$ by changing the homogeneous coordinate. Thus, $f = F_d$, and (1) has been proved.

Assume next that $\Lambda \neq (1, 1)$. Let m be the smallest positive integer such that mD is integral. Then $m = 2$ in case $\Lambda = (1, 1/2, 1/2)$, and $a_1 = 1 - m^{-1}$ in case $a_1 \neq 1$ by the list of Λ. For the $\mathbb{Q}$-divisor $L = K_X + D$, its *torsion index* (cf. [45, Def. 4.18(1)]) equals m, and we can consider the *index* 1 *cover* $\tau\colon V \to X$ with respect to L (cf. [45, Def. 4.18(2)]). Then there is an endomorphism $f_V\colon V \to V$ such that $\tau \circ f_V = f \circ \tau$ by [45, Lem. 4.21(1)]. If $a_1 \neq 1$, then V is an elliptic curve by the ramification formula

$$K_V = \tau^*(K_X + \sum (1 - m_i^{-1})P_i)$$

(cf. [45, Cor. 4.15]). Thus, (3) holds. If $\Lambda = (1, 1/2, 1/2)$, then $D = P_1 + (1/2)P_2 + (1/2)P_3$ and $f^{-1}(P_1) = P_1$ by $f^{-1}\mathcal{T} = \mathcal{T}$; hence, $V \simeq \mathbb{P}^1$, the branch locus of τ is $\{P_2, P_3\}$, and $f_V^{-1}(\tau^{-1}(P_1)) = \tau^{-1}(P_1)$. Thus f_V is isomorphic to F_d or $\iota \circ F_d$ by the argument in the case of $\Lambda = (1, 1)$. Hence, (2) also holds, and we are done. □

Proposition 2.9. *Let X be a non-singular projective curve with a non-isomorphic surjective endomorphism f. We define S_f to be the set of points $P \in X$ such that $(f^k)^{-1}(P) = \{P\}$ for some $k > 0$. Let $m\colon X \ni P \mapsto m_P \in \mathbb{Z}_{\geq 1}$ be a function such that $\Sigma := \{P \in X \mid m_P > 1\}$ is a finite set and that*

(II-2) $$d_P m_P = m_{f(P)}$$

for any $P \in X \setminus S_f$, where d_P stands for the ramification index of f at P, i.e., $d_P = \operatorname{mult}_P f^*(f(P)) = 1 + \operatorname{mult}_P R_f$. Then the following hold:

1. *If $P \in S_f$, then $f(P) \in S_f$ and $f^*(f(P)) = (\deg f)P$. Consequently, $S_f \subset \operatorname{Supp} R_f$ and $f^*S_f = (\deg f)S_f$.*
2. *If $\Sigma \neq \emptyset$, then $X \simeq \mathbb{P}^1$ and one of the following holds*:
   - (a) *$\deg S_f = 2$ and $\Sigma = S_f$;*
   - (b) *$\deg S_f = 1$, $\sharp(\Sigma \setminus S_f) = 2$, and $m_{P_1} = m_{P_2} = 2$ for $\{P_1, P_2\} = \Sigma \setminus S_f$;*
   - (c) *$S_f = 0$ and $3 \leq \sharp\Sigma \leq 4$.*
3. *Assume either that $\sharp\Sigma \neq 2$ or that $\Sigma = \{P_1, P_2\}$ with $m_{P_1} = m_{P_2}$. Then there exist a finite Galois cover $\tau\colon Y \to X$ from a non-singular projective curve Y such that*
   - (i) *τ is étale over $X \setminus \Sigma$,*
   - (ii) *$\tau^*(P) = m_P \tau^{-1}(P)$ for any $P \in \Sigma$,*
   - (iii) *there is an endomorphism $g\colon Y \to Y$ satisfying $\tau \circ g = f \circ \tau$.*
4. *In the situation of (3), assume that $\Sigma \neq \emptyset$. If (2a) holds, then $Y \simeq \mathbb{P}^1$ and τ is taken as a cyclic cover. If (2b) holds, then $Y \simeq \mathbb{P}^1$ and τ is taken as a dihedral cover (resp. double-cover) provided that $S_f \cap \Sigma \neq \emptyset$ (resp. $= \emptyset$). If (2c) holds, then Y is an elliptic curve and τ is taken as a cyclic cover.*

Proof. (1): For a point $P \in S_f$, we have $f^{-1}(f(P)) = \{P\}$ by $f(P) \in (f^{k-1})^{-1}(P)$ and $f^{-1}(f(P)) \subset (f^k)^{-1}(P) = \{P\}$ for some $k > 0$. Hence, $f(P) \in S_f$ and $f^*(f(P)) = (\deg f)P$ for any $P \in S_f$.

(2): Let us consider an effective $\mathbb{Q}$-divisor

$$D := \sum\nolimits_{P\in\Sigma\setminus S_f}(1-1/m_P)P.$$

Then $K_X+S_f+D=f^*(K_X+S_f+D)$. In fact, we have

$$K_X+S_f+D-f^*(K_X+S_f+D)=R_f-(\deg f-1)S_f-(f^*D-D),$$

where the right hand side is 0 by the following argument: If $P\in X\setminus S_f$, then

$$\begin{aligned}&\mathrm{mult}_P(R_f-(\deg f-1)S_f-(f^*D-D))\\&=d_P-1-(1-1/m_{f(P)})d_P+(1-1/m_P)=d_P/m_{f(P)}-1/m_P=0\end{aligned}$$

by (II-2), and if $P\in S_f$, then

$$\mathrm{mult}_P(R_f-(\deg f-1)S_f-(f^*D-D))=(\deg f-1)-(\deg f-1)=0$$

by (1). Note that $S_f+D\neq 0$ by $\mathrm{Supp}(S_f+D)\supset\Sigma\neq\emptyset$. By Proposition 2.8 applied to $K_X+S_f+D=f^*(K_X+S_f+D)$, we have $X\simeq\mathbb{P}^1$, and one of the following holds:

(A) $\deg S_f=2$, $D=0$, and $\Sigma=S_f$;
(B) $\deg S_f=\deg D=1$, $\sharp\,\mathrm{Supp}\,D=2$, and $m_{P_1}=m_{P_2}=2$ for $\{P_1,P_2\}=\mathrm{Supp}\,D$;
(C) $S_f=0$, $\deg D=2$, and $3\le\sharp\Sigma\le 4$.

Thus, (2) holds, where (A), (B), and (C), correspond to (2a), (2b), and (2c), respectively.

(3) and (4): We may assume that $\Sigma\neq\emptyset$. In fact, if $\Sigma=\emptyset$, then one can take the identity morphism of X as τ and take f as g. If we have a finite surjective morphism $\tau\colon Y\to X$ satisfying conditions (i) and (ii) of (3), then

$$K_Y=\tau^*(K_X+D+\sum\nolimits_{P\in\Sigma\cap S_f}(1-1/m_P)P).$$

Thus, $Y\simeq\mathbb{P}^1$ if (A) or (B) is satisfied, and Y is an elliptic curve if (C) is satisfied. If (C) is satisfied, then, by Proposition 2.8(3), we have a cyclic cover $\tau\colon Y\to X$ with an endomorphism $g\colon Y\to Y$ satisfying the required conditions in (3) and (4) in the case (2c). Hence, it is enough to consider conditions (A) and (B).

Assume that (A) holds. Then $m_{P_1}=m_{P_2}$ for $\{P_1,P_2\}=\Sigma$ by the assumption on Σ in (3). By Proposition 2.8(1), we may assume that $X=\mathbb{P}^1$ with $P_1=(1:0)$, $P_2=(0:1)$ and that $f=F_d$ or $\iota\circ F_d$ for the endomorphism $F_d\colon(\mathtt{x}:\mathtt{y})\mapsto(\mathtt{x}^d:\mathtt{y}^d)$ of $\mathbb{P}^1$ and the involution $\iota\colon(\mathtt{x}:\mathtt{y})\mapsto(\mathtt{y}:\mathtt{x})$, where $d=\deg f$. Let $\tau\colon Y=\mathbb{P}^1\to X$ be the cyclic cover of degree $m:=m_{P_1}=m_{P_2}$ branched at P_1 and P_2 which is identified with the endomorphism F_m. Then f lifts to an endomorphism g of Y, since $F_d\circ F_m=F_m\circ F_d$ and $\iota\circ F_m=F_m\circ\iota$. Thus, τ and g satisfy the required conditions in (3) and (4) in the case (2a).

Assume that (B) holds. Let P_0 be the unique point of S_f and set $m:=m_{P_0}$. Note that $m\ge 2$ if and only if $P_0\in\Sigma$. We may assume that $X=\mathbb{P}^1$ with $P_0=(0:1)$, $P_1=(1:1)$, and $P_2=(1:-1)$. Let $\tau'\colon X'=\mathbb{P}^1\to X$ be a double-cover given by $(\mathtt{x}:\mathtt{y})\mapsto(2\mathtt{xy}:\mathtt{x}^2+\mathtt{y}^2)$. Then τ' is branched at P_1 and P_2, and $\tau'^{-1}(P_0)=\{(1:0),(0:1)\}$. By Proposition 2.8(2), we have an endomorphism $f'\colon X'\to X'$

such that $\tau' \circ f' = f \circ \tau'$, and moreover, f' is equal to F_d or $\iota \circ F_d$ for $d = \deg f$. Let $\theta\colon Y = \mathbb{P}^1 \to X'$ be the cyclic cover of degree m branched at $\tau'^{-1}(P_0) = \{(1:0), (0:1)\}$ which is identified with the endomorphism F_m. When $m = 1$, we consider θ as the identity morphism of X'. Then the composite $\tau := \tau' \circ \theta\colon Y \to X$ is a Galois cover given by $(\mathtt{x} : \mathtt{y}) \mapsto (2\mathtt{x}^m\mathtt{y}^m : \mathtt{x}^{2m} + \mathtt{y}^{2m})$, and the following hold:

- τ is étale over $X \setminus \{P_0, P_1, P_2\}$;
- $\tau^*(P_i) = m_{P_i}\tau^{-1}(P_i)$ for any $0 \le i \le 2$;
- if $m = 1$, then the Galois group of τ is $\mathbb{Z}/2\mathbb{Z}$;
- if $m > 1$, then the Galois group of τ is the dihedral group D_m of order $2m$.

Moreover, we have an endomorphism $g\colon Y \to Y$ such that $\theta \circ g = f' \circ \theta$. In fact, we can take F_d or $\iota \circ F_d$ as g. Hence, $\tau \circ g = f \circ \tau$. Thus, τ and g satisfy the required conditions in (3) and (4) in the case (2b), and we are done. □

Remark. The set S_f is just the characteristic completely invariant divisor of f defined in Definition 2.16 below.

2.4. **Characteristic completely invariant divisor.** For a *finite* surjective endomorphism f of a *compact* normal variety X, we introduce suitable sets of prime divisors *completely invariant* under powers f^k of f (cf. Definition 2.12 below) and define the *characteristic completely invariant divisor* S_f and the *refined ramification divisor* Δ_f in Definition 2.16 below.

Remark 2.10. If X is a normal projective variety or a normal Moishezon surface, then every surjective endomorphism $f\colon X \to X$ is finite. This is shows as follows: If X is projective, then $f^*\colon \mathrm{NS}(X) \otimes \mathbb{Q} \to \mathrm{NS}(X) \otimes \mathbb{Q}$ is bijective for the Néron–Severi group $\mathrm{NS}(X)$, which implies the finiteness of f, since some ample divisor on X is expressed as f^*A for an ample divisor A. If X is a normal Moishezon surface, then we have a bijection $f_*\colon \mathsf{N}(X) \to \mathsf{N}(X)$, since $f_* \circ f^*$ is the multiplication map by $\deg f$ for the pullback homomorphism $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$. It implies the finiteness of f, since f has no exceptional prime divisor. As an application, we have Theorem E by [45, Cor. 3.6].

Definition (Morphisms flat in codimension 1)**.** A morphism $g\colon Y \to X$ of normal varieties is said to be *flat in codimension* 1 if $g|_{Y \setminus Z}\colon Y \setminus Z \to X$ is flat for an analytic subset Z of codimension ≥ 2.

If g is flat in codimension 1, then g is of *maximal rank* (cf. [45, Def. 1,1. Lem. 1.3]), since every flat morphism is open (cf. [13, §3.19, Prop.]). Morphisms flat in codimension 1 are characterized as follows:

Lemma 2.11. *Let $g\colon Y \to X$ be a morphism of maximal rank of normal varieties. Then g is flat in codimension* 1 *if and only if* $\operatorname{codim}(\Xi, Y) \ge 2$ *for*

$$\Xi := \{y \in Y \mid \dim_y g^{-1}(g(y)) > \dim Y - \dim X\}.$$

Proof. Note that Ξ is an analytic subset (cf. [13, §3.6, Thm.]). The "only if" part is shown by a dimension formula for a flat morphism (cf. [13, §3.19, Lem.]). In fact, if g is flat, then $\Xi = \emptyset$. For the proof of "if" part, we may assume that $\Xi = \emptyset$. Then

$\operatorname{codim}(g^{-1}W, Y) \geq 2$ for any analytic subset W of X of codimension ≥ 2. In fact, for the induced morphism $W \times_X Y \to W$ of complex analytic spaces, we have

$$\begin{aligned}\dim_y W \times_X Y &\leq \dim_{g(y)} W + \dim_y g^{-1}(g(y)) \\ &\leq \dim X - 2 + \dim_y g^{-1}(g(y)) = \dim Y - 2\end{aligned}$$

for any $y \in g^{-1}W = (W \times_X Y)_{\mathrm{red}}$ (cf. [13, §3.9, Prop.]). In particular, $g^{-1} \operatorname{Sing} X$ has codimension ≥ 2, and we may assume in addition that X and Y are both non-singular. In this case, g is known to be flat (cf. [13, §3.20, Cor.]). □

Remark. A similar assertion holds for a dominant morphism $\mathsf{g} \colon \mathsf{Y} \to \mathsf{X}$ of finite type of normal integral Noetherian schemes: *It is flat in codimension* 1 *if and only if* $\operatorname{codim}(\mathsf{g}^{-1}\Xi, \mathsf{Y}) \geq 2$ *for the closed subset*

$$\Xi = \{\mathsf{y} \in \mathsf{Y} \mid \dim_{\mathsf{y}} \mathsf{g}^{-1}(\mathsf{g}(\mathsf{y})) > \dim \mathsf{Y} - \dim \mathsf{X}\}.$$

Remark. For a *non-degenerate* morphism $g \colon Y \to X$ of normal varieties (cf. [45, Def. 1.1]), if g has no exceptional divisor, i.e., $\Gamma \neq \{y \in \Gamma \mid \dim_y g^{-1}(g(y)) > 0\}$ for any prime divisor Γ on Y (cf. [45, §1.2]), then g is flat in codimension 1 by Lemma 2.11. In particular, any finite surjective morphism of normal varieties is flat in codimension 1.

Remark (Pullbacks of divisors). Let $g \colon Y \to X$ be a morphism of normal varieties flat in codimension 1. By Lemma 2.11, we can consider the pullback g^*D of a divisor D on X, and we have the pullback homomorphism $g^* \colon \operatorname{Div}(X) \to \operatorname{Div}(Y)$ extending the pullback homomorphism for Cartier divisors (cf. [45, Lem. 1.19]). This also extends to $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors. If $\dim X = \dim Y = 2$ in addition, then g^*D equals the numerical pullback of D by [45, Lem.-Def. 1.23]. Thus, the use of the same symbol g^* for several pullbacks may not cause any confusion. We may write $g^{-1}D = (g^*D)_{\mathrm{red}}$ for any reduced divisor D as in Convention 1.1. When g is non-degenerate, we have the ramification divisor R_g with the ramification formula $K_Y = g^*K_X + R_g$.

Definition 2.12. For an endomorphism $\mathfrak{f} \colon \mathcal{X} \to \mathcal{X}$ of a set $\mathcal{X}$, a subset $\mathcal{S}$ of $\mathcal{X}$ is said to be *completely invariant* under $\mathfrak{f}$, or $\mathfrak{f}$*-completely invariant*, if $\mathfrak{f}(\mathcal{S}) \subset \mathcal{S}$ and $\mathfrak{f}^{-1}\mathcal{S} \subset \mathcal{S}$. Let $f \colon X \to X$ be a finite surjective endomorphism of a normal variety X. An f*-completely invariant divisor* S is a reduced divisor on X which is completely invariant under f, i.e., $(f^*S)_{\mathrm{red}} = f^{-1}S = S$. In particular, the zero divisor 0 is always completely invariant.

Definition 2.13. For a normal *compact* variety X and a finite surjective endomorphism $f \colon X \to X$, we define several sets of prime divisors on X by Table 2, where R_f denotes the ramification divisor of f.

Remark 2.14. By definition, $\mathcal{S}(X, f; a) \subset \mathcal{S}(X, f^k; a^k)$ for any $k \geq 1$, and

(II-3) $$\mathcal{S}(X, f) = \bigcup\nolimits_{k \geq 1, b > 1} \mathcal{S}(X, f^k; b).$$

Table 2. Some sets of prime divisors

$\mathcal{S}(X,f)^\sharp$	the set of prime divisors Γ on X such that $(f^k)^{-1}(\Gamma)$ is irreducible for any $k \geq 1$.
$\mathcal{S}(X,f)^\sharp_0$	the set of prime divisors $\Gamma \in \mathcal{S}(X,f)^\sharp$ such that $(f^k)^{-1}(\Gamma) \subset \operatorname{Supp} R_f$ for infinitely many $k \geq 1$.
$\mathcal{S}(X,f)^\flat$	the set of prime divisors Γ on X such that $(f^k)^{-1}(f^k(\Gamma)) = \Gamma$ for any $k \geq 1$.
$\mathcal{S}(X,f)^\flat_0$	the set of prime divisors $\Gamma \in \mathcal{S}(X,f)^\flat$ such that $f^k(\Gamma) \subset \operatorname{Supp} R_f$ for infinitely many $k \geq 1$.
$\mathcal{S}(X,f)$	the set of prime divisors Γ on X such that $(f^k)^*(\Gamma) = b\Gamma$ for some $k \geq 1$ and $b > 1$.
$\mathcal{S}(X,f;a)$	the set of prime divisors Γ on X such that $f^*\Gamma = a\Gamma$, where a is a positive integer.

Moreover, $\mathcal{S}(X,f^l) = \mathcal{S}(X,f)$ for any $l \geq 1$. For a positive integer a, if $\mathcal{S}(X,f;a) \neq \emptyset$, then $a \mid \deg f$. In fact, if $f^*\Gamma = a\Gamma$ for a prime divisor Γ, then the degree of $f|_\Gamma \colon \Gamma \to \Gamma$ is $(\deg f)/a$, since $f_*(f^*\Gamma) = (\deg f)\Gamma$.

Proposition 2.15 (cf. [40, Prop. 11])**.** *In Definition* 2.13*, $\mathcal{S}(X,f)$ is a finite set and $\mathcal{S}(X,f)^\sharp_0 = \mathcal{S}(X,f)^\flat_0 = \mathcal{S}(X,f)$. Furthermore, $\Gamma \mapsto f(\Gamma)$ induces a permutation of $\mathcal{S}(X,f)$. In particular, $\mathcal{S}(X,f;a)$ is a finite set for any $a > 1$.*

Proof. The last assertion is a consequence of the first one, since $\mathcal{S}(X,f;a) \subset \mathcal{S}(X,f)$. We have injections $\phi\colon \mathcal{S}(X,f)^\sharp \to \mathcal{S}(X,f)^\sharp$ and $\psi\colon \mathcal{S}(X,f)^\flat \to \mathcal{S}(X,f)^\flat$ defined by $\phi(\Gamma) := f^{-1}(\Gamma)$ and $\psi(\Gamma) := f(\Gamma)$ for prime divisors Γ. Here, one has $\phi(\mathcal{S}(X,f)^\sharp_0) \subset \mathcal{S}(X,f)^\sharp_0$ and $\psi(\mathcal{S}(X,f)^\flat_0) \subset \mathcal{S}(X,f)^\flat_0$. For $\dagger \in \{\sharp, \flat\}$, we define

$$\mathcal{S}(X,f)^\dagger_{00} := \{\Gamma \in \mathcal{S}(X,f)^\dagger_0 \mid \Gamma \subset \operatorname{Supp} R_f\}$$

as a finite subset of $\mathcal{S}(X,f)^\dagger_0$. Then

$$\mathcal{S}(X,f)^\sharp_0 = \bigcup\nolimits_{k\geq 1} (\phi^k)^{-1}\mathcal{S}(X,f)^\sharp_{00} \quad \text{and} \quad \mathcal{S}(X,f)^\flat_0 = \bigcup\nolimits_{k\geq 1} (\psi^k)^{-1}\mathcal{S}(X,f)^\flat_{00}.$$

Hence, $\mathcal{S}(X,f)^\sharp_0$ and $\mathcal{S}(X,f)^\flat_0$ are finite by Lemma 2.1, and there is a positive integer k such that ϕ^k and ψ^k induce identity maps on $\mathcal{S}(X,f)^\sharp_0$ and $\mathcal{S}(X,f)^\flat_0$, respectively. In particular, $(f^k)^{-1}(\Gamma) = \Gamma$ for any $\Gamma \in \mathcal{S}(X,f)^\sharp_0 \cup \mathcal{S}(X,f)^\flat_0$. Therefore, $\mathcal{S}(X,f)^\sharp_0 \cup \mathcal{S}(X,f)^\flat_0 \subset \mathcal{S}(X,f)$. For the rest, by (II-3) in Remark 2.14, it suffices to prove that

$$\mathcal{S}(X,f^k;b) \subset \mathcal{S}(X,f)^\sharp_0 \cap \mathcal{S}(X,f)^\flat_0 \tag{II-4}$$

for any $k \geq 1$ and $b > 1$. Let Γ be a prime divisor in $\mathcal{S}(X,f^k;b)$. Then $\Gamma \in \mathcal{S}(X,f)^\sharp \cap \mathcal{S}(X,f)^\flat$ by $(f^k)^{-1}(\Gamma) = \Gamma$. Since $b > 1$, there is an index $1 \leq i \leq k$ such that $f^i(\Gamma) \subset \operatorname{Supp} R_f$. For, otherwise, $f^*(f^{i+1}(\Gamma)) = f^i(\Gamma)$ for any $1 \leq i \leq k$, and we have a contradiction: $(f^k)^*\Gamma = (f^k)^*(f^k(\Gamma)) = \Gamma$. Therefore, $\Gamma \in \mathcal{S}(X,f)^\sharp_0 \cap \mathcal{S}(X,f)^\flat_0$, since $f^{j+k}(\Gamma) = f^j(\Gamma)$ and $(f^{j+k})^{-1}(\Gamma) = (f^j)^{-1}(\Gamma)$ for any $j \geq 0$. Thus, (II-4) holds, and we are done. ∎

Definition 2.16. For a finite surjective endomorphism $f\colon X \to X$ of a compact normal variety X, we define

$$S_f := \sum\nolimits_{\Gamma \in \mathcal{S}(X,f)} \Gamma \quad \text{and} \quad \Delta_f := \sum\nolimits_{\Gamma \notin \mathcal{S}(X,f)} (\operatorname{mult}_\Gamma R_f)\Gamma,$$

and call them the *characteristic completely invariant divisor* of f and the *refined ramification divisor* of f, respectively. The endomorphism f is said to be *sufficiently iterated* if any prime component of S_f is completely invariant under f (cf. Lemma 2.17(1) below).

Remark. If $\deg f = 1$, i.e., f is an automorphism, then $\mathcal{S}(X,f) = \emptyset$, $S_f = 0$, and $\Delta_f = R_f = 0$. If f is sufficiently iterated, then

$$\mathcal{S}(X,f) = \bigcup\nolimits_{1<a \,|\, \deg f} \mathcal{S}(X,f;a)$$

by Remark 2.14.

Lemma 2.17. *The following hold for S_f and Δ_f:*

(1) *The reduced divisor S_f is f-completely invariant, i.e., $f^{-1}S_f = S_f$. There is a positive integer m such that the power f^m is sufficiently iterated.*

(2) *One has $R_f = f^*S_f - S_f + \Delta_f$, or equivalently,*

$$K_X + S_f = f^*(K_X + S_f) + \Delta_f. \tag{II-5}$$

(3) *For any $k \geq 2$, $S_{f^k} = S_f$ and*

$$\Delta_{f^k} = (f^{k-1})^*\Delta_f + \cdots + f^*\Delta_f + \Delta_f. \tag{II-6}$$

(4) *One has inclusions*

$$\operatorname{Supp}\Delta_f \subset \operatorname{Supp}R_f \subset \operatorname{Supp}R_{f^m} = S_f \cup \operatorname{Supp}\Delta_f,$$

where m is any positive integer such that f^m is sufficiently iterated. In particular, $R_f = 0$ if and only if $S_f = \Delta_f = 0$.

Proof. Assertion (1) follows from Proposition 2.15. We have (2) by (1) and [45, Lem. 1.39]. For (3), we have $S_{f^k} = S_f$ by $\mathcal{S}(X,f^k) = \mathcal{S}(X,f)$ (cf. Remark 2.14), and we have (II-6) by iterating (II-5):

$$K_X + S_f = (f^k)^*(K_X + S_f) + (f^{k-1})^*\Delta_f + \cdots + f^*\Delta_f + \Delta_f.$$

The inclusions in (4) are derived from $R_f = f^*S_f - S_f + \Delta_f$ in (2), where $f^*S_f \geq f^{-1}S_f = S_f$ and $\operatorname{Supp}(f^*S_f - S_f) \subset S_f$. Here, $\operatorname{Supp}((f^m)^*S_f - S_f) = S_f$, and it implies: $\operatorname{Supp}R_{f^m} = S_f \cup \operatorname{Supp}\Delta_f$. By iterating the ramification formula $K_X = f^*K_X + R_f$, we have

$$R_{f^k} = (f^{k-1})^*R_f + \cdots + f^*R_f + R_f$$

for any $k \geq 2$. Hence, $R_f = 0$ if and only if $R_{f^m} = 0$, and this is also equivalent to $S_f = \Delta_f = 0$ by $\operatorname{Supp}R_{f^m} = S_f \cup \operatorname{Supp}\Delta_f$. □

Lemma 2.18. *Let f be a non-isomorphic finite surjective endomorphism of a compact normal variety X and let S be an f-completely invariant divisor. Then $K_X + S = f^*(K_X + S) + \Delta$ for an effective divisor Δ having no common prime component with S. Here, $\Delta \geq \Delta_f$. If $S \geq S_f$, then $\Delta = \Delta_f$.*

Proof. The existence of Δ is shown in [45, Lem. 1.39]. We set $\overline{S} = S \cup S_f = (S + S_f)_{\mathrm{red}}$. Then $f^*(\overline{S} - S_f) = \overline{S} - S_f$ by the definition of S_f, and we have $K_X + \overline{S} = f^*(K_X + \overline{S}) + \Delta_f$ by Lemma 2.17(2). Moreover, we have $\Delta \geq \Delta_f$ by

$$\overline{S} - S = f^*(\overline{S} - S) + \Delta_f - \Delta,$$

since $\overline{S} - S$ is also f-completely invariant. If $S \geq S_f$, then $\overline{S} = S$ and $\Delta_f = \Delta$. □

Lemma 2.19. *Let $\pi\colon X \to Y$ be a surjective morphism of normal compact varieties and let $f\colon X \to X$ and $g\colon Y \to Y$ be non-isomorphic finite surjective endomorphisms such that $\pi \circ f = g \circ \pi$.*

1. *If π is a* fibration*, i.e., $\mathcal{O}_Y \simeq \pi_*\mathcal{O}_X$, then* $\deg g \mid \deg f$.
2. *Assume that π is flat in codimension* 1*. Then $\pi^{-1}S_g \leq S_f$, and $\pi^{-1}S_g$ consists of the prime components of S_f not dominating Y.*
3. *If π is a finite surjective morphism, then $S_f = \pi^{-1}S_g$.*

Proof. We have (1) by [45, Cor. 1.14]. Assertion (3) is a consequence of (2), but we shall prove (3) in the course of the proof of (2). For the proof of (2), we may assume that f and g are sufficiently iterated, since $S_f = S_{f^k}$ and $S_g = S_{g^k}$ for any $k > 1$ (cf. Lemma 2.17(3)). We note the following on prime divisors Γ and Θ on X and Y, respectively, such that $\Theta = \pi(\Gamma)$: *If $f(\Gamma) = \Gamma$, then $g(\Theta) = \Theta$ and*

$$\text{(II-7)} \qquad \operatorname{mult}_\Gamma f^*\Gamma = \frac{\operatorname{mult}_\Gamma f^*(\pi^*\Theta)}{\operatorname{mult}_\Gamma \pi^*\Theta} = \frac{\operatorname{mult}_\Gamma \pi^*(g^*\Theta)}{\operatorname{mult}_\Gamma \pi^*\Theta} = \operatorname{mult}_\Theta g^*\Theta.$$

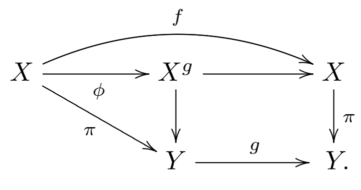
If Θ is a prime component of S_g, then $g^*\Theta = b\Theta$ for some $b > 1$, and we have $\pi^{-1}\Theta \subset S_f$ by $f^*(\pi^*\Theta) = \pi^*(g^*\Theta) = b\pi^*\Theta$, where $\pi^{-1}\Theta = (\pi^*\Theta)_{\mathrm{red}}$ (cf. Convention 1.1). This shows that $\pi^{-1}S_g \leq S_f$.

For the rest, it is enough to prove: $\pi^{-1}S_g = S_f^{\mathrm{ver}}$ for the reduced divisor

$$S_f^{\mathrm{ver}} := \sum\nolimits_{\Gamma \subset S_f, \pi(\Gamma) \neq Y} \Gamma.$$

Let Γ be a prime component of S_f^{ver}. Then $f^*\Gamma = b\Gamma$ for some $b > 1$, and $\Theta = \pi(\Gamma)$ is a prime divisor on Y, since π is flat in codimension 1 and proper surjective. In order to show: $\pi^{-1}S_g = S_f^{\mathrm{ver}}$, it is enough to prove that $g^*\Theta = b\Theta$, and this is equivalent to: $g^{-1}\Theta = \Theta$, since we have $\operatorname{mult}_\Theta g^*\Theta = b$ by (II-7).

We shall show $g^{-1}\Theta = \Theta$ in the case where π is a fibration. For the fiber product $X^g := X \times_{Y,g} Y$ of π and g over Y, there is a commutative diagram



Here, X^g is irreducible by [45, Lem. 1.13], and hence, the induced morphism ϕ is surjective. For a prime component Θ' of $g^{-1}\Theta$ and for the fiber product $\Gamma \times_\Theta \Theta'$, the second projection $\Gamma \times_\Theta \Theta' \to \Theta'$ and the morphism $\phi^{-1}(\Gamma \times_\Theta \Theta') \to \Gamma \times_\Theta \Theta'$ induced by ϕ are both surjective, and $\phi^{-1}(\Gamma \times_\Theta \Theta') \subset f^{-1}\Gamma = \Gamma$. These properties imply that $\Theta' = \pi(\Gamma) = \Theta$, and we have $g^{-1}\Theta = \Theta$ as a consequence.

Next, we shall show $g^{-1}\Theta = \Theta$ in the case where π is finite (cf. (3)). In this case, we have $\deg f = \deg g$ by $\pi \circ f = g \circ \pi$. Since π and f are finite, $\pi_*\Gamma = m\Theta$ and $f_*\Gamma = e\Gamma$ for $m := \deg(\pi|_\Gamma \colon \Gamma \to \Theta)$ and $e := \deg(f|_\Gamma \colon \Gamma \to \Gamma)$. Then $be = \deg f$ by $f_*(f^*\Gamma) = (\deg f)\Gamma$, and $g_*\Theta = e\Theta$ by $g_*(\pi_*(\Gamma)) = \pi_*(f_*\Gamma)$. By (II-7), there is an effective divisor B on Y such that $\Theta \not\subset B$ and that $g^*\Theta = b\Theta + B$. Then

$$(\deg g)\Theta = g_*(g^*\Theta) = be\Theta + g_*B.$$

Hence, $B = 0$ and $g^*\Theta = b\Theta$. This shows $g^{-1}\Theta = \Theta$.

Finally, we shall show $g^{-1}\Theta = \Theta$ in the general case. Let $X \to Y' \to Y$ be the Stein factorization of π. Then the fibration $\pi' \colon X \to Y'$ and the finite morphism $\tau \colon Y' \to Y$ are both flat in codimension 1. By the uniqueness of Stein factorization, there is an endomorphism $g' \colon Y' \to Y'$ such that $g' \circ \pi' = \pi' \circ f$ and $g \circ \tau = \tau \circ g'$. We set $\Theta' := \pi'(\Theta)$. Then $g'^{-1}\Theta' = \Theta'$ by the argument above in the case of fibration, and $g^{-1}\Theta = \Theta$ by the argument above in the case of finite morphism. Thus, we are done. □

When X is a normal Moishezon surface, we have:

Proposition 2.20. *Let f be a non-isomorphic surjective endomorphism of a normal Moishezon surface X.*

(1) *If $C_1C_2 \neq 0$ for some $C_1 \in \mathcal{S}(X, f; a_1)$ and $C_2 \in \mathcal{S}(X, f; a_2)$, then $a_1a_2 = \deg f$.*
(2) *If $C \in \mathcal{S}(X, f)^\flat \setminus \mathcal{S}(X, f)$, then $C^2 = 0$.*
(3) *Every negative curve is contained in S_f. In particular, X has only finitely many negative curves.*
(4) *Every prime component of the refined ramification divisor Δ_f is nef.*
(5) *If $C \in \mathcal{S}(X, f; a)$, then*

$$(1 - (\deg f)/a)(K_X + S)C = \Delta_f C \geq 0$$

for any f-completely invariant divisor S such that $S \geq S_f$.

Proof. (1): This follows from $(f^*C_1)f^*C_2 = (\deg f)C_1C_2$.

(2): For an integer $k \geq 0$, let a_k be the positive integer defined by $f^*(f^{k+1}(C)) = a_k f^k(C)$. Then $a_k = 1$ except for finitely many k, since $\mathcal{S}(X, f) = \mathcal{S}(X, f)^\flat_0$ (cf. Proposition 2.15). Thus, we can consider $a := \prod_{a_k > 1} a_k$. Since $a_k^2 f^k(C)^2 = (\deg f) f^{k+1}(C)^2$ for any $k \geq 1$, we have

$$C^2 = \frac{(\deg f)^l}{a^2} f^l(C)^2 \in \frac{(\deg f)^l}{a^2\, \mathbf{nf}(X)}\mathbb{Z}$$

for $l \gg 0$, where $\mathbf{nf}(X)$ is *numerical factorial index* (cf. [45, Def. 1.26]). Therefore, $C^2 = 0$.

(3): Let C be a negative curve on X. If $f(C) = f(C')$ for a prime divisor C', then $f_*C = \lambda f_*(C')$ for some rational number $\lambda > 0$, which implies that $C - \lambda C'$ is numerically trivial, since $f_* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ is bijective; thus, $C = C'$ by $CC' < 0$. Hence, $f(C)$ is a negative curve and $f^{-1}(f(C)) = C$. Therefore, $C \in \mathcal{S}(X, f)^\flat$, and moreover, $C \in \mathcal{S}(X, f)$ by (2).

(4): This follows from (3) and the definition of Δ_f (cf. Definition 2.16).

(5): We have $\Delta_f C \geq 0$ by (4) and we have $K_X + S = f^*(K_X + S) + \Delta_f$ by Lemma 2.18, since $S \geq S_f$. Then $f^*C = aC$ implies that

$$(\deg f)(K_X + S)C = (f^*(K_X + S) \cdot f^*C) = a(K_X + S - \Delta_f)C.$$

This shows (5), and we are done. □

2.5. **Refined ramification divisor.** Let f be a non-isomorphic surjective endomorphism of a normal compact variety X, and assume that X is either a projective variety or a Moishezon surface. Then f is finite by Remark 2.10. We know that the ramification divisor R_f is zero if and only if f is étale in codimension 1. In Section 2.5, we shall give a criterion for such an endomorphism f to satisfy $\Delta_f = 0$, in Proposition 2.21 below, and prove Theorem 2.24 below on f-completely invariant divisors S such that $K_X + S$ are pseudo-effective.

Proposition 2.21. *Let $f \colon X \to X$ be non-isomorphic surjective endomorphism in which X is either a normal Moishezon surface or a normal projective variety. Then each of the following conditions is equivalent to $\Delta_f = 0$:*

(i) *There is an effective divisor S such that $R_f = f^*S - S$.*
(ii) *For any f-completely invariant divisor S, if $S \geq S_f$, then $R_f = f^*S - S$.*
(iii) *There is an f-completely invariant divisor S such that $f|_{X \setminus S} \colon X \setminus S \to X \setminus S$ is étale in codimension 1.*

Moreover, the following hold for any effective divisor S satisfying (i):

(1) S_{red} *is f-completely invariant*;
(2) $S_{\mathrm{red}} \geq S_f$ *and* $f^*(S - S_{\mathrm{red}}) = S - S_{\mathrm{red}}$.

Proof. If $\Delta_f = 0$, then (ii) holds by Lemma 2.18. If (ii) holds, then $R_f = f^*S_f - S_f$, and (iii) holds for S_f by [45, Lem 1.39]. Moreover, we have (iii) ⇒ (i) by [45, Lem 1.39]. If (1) and (2) hold for a divisor S in (i), then $R_f = f^*S_{\mathrm{red}} - S_{\mathrm{red}}$ and we have $\Delta_f = 0$ by Lemma 2.18. Thus, it is enough to prove (1) and (2).

Let Γ be a prime divisor on X, and B a prime component of $f^{-1}\Gamma$. Then

$$b - 1 = \operatorname{mult}_B R_f = b \operatorname{mult}_\Gamma S - \operatorname{mult}_B S$$

for $b := \operatorname{mult}_B f^*\Gamma$ by $R_f = f^*S - S$, or equivalently:

$$\operatorname{mult}_B S - 1 = b(\operatorname{mult}_\Gamma S - 1). \tag{II-8}$$

In particular, if $\Gamma \subset S$, then $B \subset S$. Let $\mathcal{X}$ be the set of prime divisors on X and let $\mathcal{S}$ be the set of prime components of S. We consider a map $F \colon \mathcal{X} \to \mathcal{X}$ defined by $F(\Gamma) = f(\Gamma)$. Then $F^{-1}(\mathcal{S}) \subset \mathcal{S}$ by the implication $\Gamma \subset S \Rightarrow B \subset S$. Since $\mathcal{S}$ is finite, $F^{-1}(\mathcal{S}) = \mathcal{S}$ and $F|_{\mathcal{S}} \colon \mathcal{S} \to \mathcal{S}$ is bijective by Lemma 2.2. In particular, $f^{-1}(\operatorname{Supp} S) = \operatorname{Supp} S$. Thus, we have (1).

Let k be a positive integer such that f^k is sufficiently iterated and that the power F^k induces the identity map of $\mathcal{S}$. Then $(f^k)^{-1}\Gamma = \Gamma$ for any prime component Γ of $S \cup S_f$. If $\Gamma \subset S_f$, then $(f^k)^*(\Gamma) = r\Gamma$ for some $r > 1$, and we have $\operatorname{mult}_\Gamma S = 1$ by applying (II-8) to f^k. This shows that $S_{\mathrm{red}} \geq S_f$. If $\operatorname{mult}_\Gamma S \geq 2$, then $(f^k)^*\Gamma = \Gamma$ by (II-8) applied to f^k. This implies that $f^*(S - S_{\mathrm{red}}) = S - S_{\mathrm{red}}$. Thus, (2) holds, and we are done. □

The following result is well known in the case where X is a non-singular projective variety and $S = 0$ (cf. [15, Lem. 2.3]).

Lemma 2.22. *Let f be a non-isomorphic surjective endomorphism of a normal compact variety X with a $\mathbb{Q}$-divisor S such that $R_f = f^*S - S + \Delta$ for an effective $\mathbb{Q}$-divisor Δ, i.e., $K_X + S = f^*(K_X + S) + \Delta$. If one of the following two conditions is satisfied, then $\Delta = 0$:*

(i) *X is projective of dimension n and $(K_X + S)H^{n-1} \geq 0$ for any ample divisor H on X;*

(ii) *X is a normal Moishezon surface and $K_X + S$ is pseudo-effective.*

In particular, if $S = 0$ in (i) *and* (ii)*, then $R_f = 0$.*

Proof. By setting $\Delta_k := R_{f^k} + S - (f^k)^*S$ for $k \geq 1$, we have an equality

$$K_X + S = (f^k)^*(K_X + S) + \Delta_{f^k}. \tag{II-9}$$

Iterating the equality $K_X + S = f^*(K_X + S) + \Delta$, we have $\Delta = \Delta_1$ and

$$\Delta_k = (f^{k-1})^*\Delta + \cdots + f^*\Delta + \Delta$$

for any $k \geq 2$ (cf. Lemma 2.17(3)).

Assume (i) and let H be an ample divisor on X. Then $H \approx (f^k)^*A$ for an ample $\mathbb{Q}$-divisor A on X, since $(f^k)^* \colon \mathrm{NS}(X) \otimes \mathbb{Q} \to \mathrm{NS}(X) \otimes \mathbb{Q}$ is bijective (cf. Remark 2.10). Hence,

$$((f^k)^*(K_X + S))H^{n-1} = (\deg f^k)(K_X + S)A^{n-1} \geq 0$$

for any k by (i), and we have $(K_X + S)H^{n-1} \geq \Delta_k H^{n-1}$ by (II-9). If $\Delta \neq 0$, then

$$(K_X + S)H^{n-1} \geq \Delta_k H^{n-1} = ((f^{k-1})^*\Delta + \cdots + \Delta)H^{n-1} \geq \frac{k}{c} \longrightarrow \infty \quad (k \to \infty)$$

for a positive integer c such that $c\Delta$ is a divisor. This is a contradiction. Therefore, $\Delta = 0$, and the assertion holds under (i).

Next, assume (ii) and let H be a numerically ample divisor on X. Then $(K_X + S)H \geq \Delta_k H$ for any $k \geq 1$, since $(f^k)^*(K_X + S)$ is pseudo-effective. If $\Delta \neq 0$, then

$$(K_X + S)H \geq \Delta_k H = ((f^{k-1})^*\Delta + \cdots + \Delta)H \geq \frac{k}{c\,\mathbf{nf}(X)} \longrightarrow \infty \quad (k \to \infty)$$

for a positive integer c such that $c\Delta$ is a divisor. This is a contradiction. Therefore, $\Delta = 0$. □

Lemma 2.23. *Let X be a normal Moishezon surface with an $\mathbb{R}$-divisor D such that $R_f \approx f^*D - D$ for a non-isomorphic surjective endomorphism f of X. Then $(K_X + D)^2 = 0$ and $(K_X + D)C = 0$ for any negative curve C on X.*

Proof. By assumption, $K_X + D \approx f^*(K_X + D)$. We have $(K_X + D)^2 = 0$ by $(f^*(K_X + D))^2 = (\deg f)(K_X + D)^2$, since $\deg f > 1$. For a negative curve C, there exist positive integers k and r such that $(f^k)^*C = rC$ by Lemma 2.17(1) and Proposition 2.20(3). Then $r = (\deg f^k)^{1/2}$ by $((f^k)^*C)^2 = (\deg f^k)C^2$, and

$$(\deg f^k)(K_X + D)C = ((f^k)^*(K_X + D) \cdot (f^k)^*C) = r(K_X + D)C$$

by $K_X + D \approx (f^k)^*(K_X + D)$. Thus, $(K_X + D)C = 0$, since $\deg f^k > r$. □

Theorem 2.24. *Let X be a normal Moishezon surface with a reduced divisor S such that $K_X + S$ is pseudo-effective. If S is completely invariant under a non-isomorphic surjective endomorphism f of X, then $S \geq S_f$, $R_f = f^*S - S$, $\Delta_f = 0$, and $K_X + S$ is semi-ample with $(K_X + S)^2 = 0$.*

Proof. There is an effective divisor Δ such that $K_X + S = f^*(K_X + S) + \Delta$ by [45, Lem. 1.39]. Then $\Delta = 0$ by Lemma 2.22; thus, $R_f = f^*S - S$. Hence, $\Delta_f = 0$, $S \geq S_f$, $(K_X + S)^2 = 0$, and $(K_X + S)C = 0$ for any negative curve C on X by Proposition 2.21 and Lemma 2.23. Since $K_X + S$ is pseudo-effective, it implies that $K_X + S$ is nef. Then $K_X + S$ is semi-ample by Theorem 1.12, since (X, S) is log-canonical (cf. Theorem E). □

Corollary 2.25. *Let f be a non-isomorphic surjective endomorphism of a normal Moishezon surface X. Let S be a reduced divisor such that $R_f = f^*S - S$ and that $K_X + S$ is not semi-ample. Then $-(K_X + S)$ is nef, $(K_X + S)^2 = 0$, and $K_X + S \not\approx 0$.*

Proof. By Lemma 2.23 and by [44, Lem. 2.16(2)], we have $(K_X + S)^2 = 0$, and either $K_X + S$ or $-(K_X + S)$ is nef. If $K_X + S$ is nef, then $K_X + S$ is semi-ample by Theorem 2.24. Thus, $-(K_X + S)$ is nef but not numerically trivial. □

3. Endomorphisms of normal Moishezon surfaces

For a non-isomorphic surjective endomorphism f of a normal Moishezon surface X, we study the first dynamical degree λ_f in Section 3.1 and the singularity of the pair (X, S) for an f-completely invariant divisor S in Section 3.2. An application of the minimal model program to the study of endomorphisms is given in Section 3.4.

3.1. **The first dynamical degree.**

Definition 3.1. Let f be a surjective endomorphism of a normal Moishezon surface X. Then we have an automorphism $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ of the real vector space $\mathsf{N}(X) = \mathrm{Div}(X, \mathbb{R})/\approx$ such that $f^* \operatorname{cl}(D) = \operatorname{cl}(f^*D)$ for any $\mathbb{R}$-divisor D on X (cf. [44, Rem. 2.9]). The *first dynamical degree* λ_f is defined as the *spectral radius* of $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$, i.e., the maximum of the absolute values of eigenvalues of f^*.

Remark. If X is a non-singular projective surface, then λ_f equals the first dynamical degree in the sense of the complex dynamical systems of compact Kähler manifolds by [25, Prop. 1.2(iii)]. Let $\nu\colon Z \to X$ be a birational morphism from a non-singular projective surface Z. Then we have a dominant rational map $\nu^{-1} \circ f \circ \nu\colon Z \dashrightarrow Z$. In Appendix A, following the idea of Guedj [25], we shall show that λ_f equals the first dynamical degree of $\nu^{-1} \circ f \circ \nu$. Note that the *second dynamical degree* of $\nu^{-1} \circ f \circ \nu$ is just the usual degree: $\deg f = \operatorname{rank} f_*\mathcal{O}_X$.

Definition 3.2. For $c \in \mathbb{R}$, we define $\mathsf{N}(X, f; c)$ as the eigenspace of $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ with eigenvalue c. We set $\lambda_f^\dagger := \deg f/\lambda_f$ and set δ_f to be the positive square root $(\deg f)^{1/2}$ of $\deg f$.

Remark. For any $k \geq 1$ and for the power $f^k \colon X \to X$, we have

$$\lambda_{f^k} = (\lambda_f)^k, \quad \lambda^\dagger_{f^k} = (\lambda^\dagger_f)^k, \quad \text{and} \quad \delta_{f^k} = (\delta_f)^k.$$

Proposition 3.3. *The following hold for* $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$*, where* $\langle\ ,\ \rangle \colon \mathsf{N}(X) \times \mathsf{N}(X) \to \mathbb{R}$ *denotes the intersection paring (cf. Section* 1.1*):*

(1) *If* $\langle v, v' \rangle \neq 0$ *for some* $v \in \mathsf{N}(X, f; c)$ *and* $v' \in \mathsf{N}(X, f; c')$*, then* $cc' = \deg f$ *(cf. Proposition* 2.20(1)*).*
(2) *The spectral radius of* $f_* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ *equals* λ_f*, and* $\lambda_f \geq \delta_f$*.*
(3) *There exist non-zero vectors* v_+ *and* v_- *in the nef cone* $\mathrm{Nef}(X)$ *such that*

$$f^* v_+ = \lambda_f\, v_+, \quad f_* v_+ = \lambda^\dagger_f\, v_+, \quad f^* v_- = \lambda^\dagger_f\, v_-, \quad \textit{and} \quad f_* v_- = \lambda_f\, v_-.$$

(4) *A real eigenvalue of* $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ *is one of* δ_f*,* $-\delta_f$*,* λ_f*, and* $\lambda^\dagger_f$*. In particular, if* 1 *or* $\deg f$ *is an eigenvalue of* f^**, then* $\lambda_f = \deg f$ *and* $\lambda^\dagger_f = 1$*.*
(5) *The pairing* $\langle\ ,\ \rangle$ *restricted to* $\mathsf{N}(X, f; c)$ *is negative definite if* c *is a real eigenvalue of* f^* *different from* λ_f *and* $\lambda^\dagger_f$*.*
(6) *If* $\lambda_f > \delta_f$*, then* $\langle v_+, v_- \rangle > 0$*,* $\langle v_+, v_+ \rangle = \langle v_-, v_- \rangle = 0$*,* $\mathsf{N}(X, f; \lambda_f) = \mathbb{R} v_+$*, and* $\mathsf{N}(X, f; \lambda^\dagger_f) = \mathbb{R} v_-$ *for vectors* v_+ *and* v_- *in* (3)*.*
(7) *If* $\lambda_f = \deg f$*, then all the numerical classes of prime components of* Δ_f *belong to* $\mathbb{R}_{\geq 0} v_+$*. In particular,* $S_f \cap \mathrm{Supp}\, \Delta_f = \emptyset$*.*

Proof. First, we shall prove (1)–(3). We have $\langle f^* x, y \rangle = \langle x, f_* y \rangle$ and $\langle f^* x, f^* y \rangle = (\deg f) \langle x, y \rangle$ for any x, $y \in \mathsf{N}(X)$ by the projection formula on intersection numbers of divisors. Thus (1) holds by $(\deg f) \langle v, v' \rangle = \langle f^* v, f^* v' \rangle = cc' \langle v, v' \rangle$, and we have $f_* = (\deg f)(f^*)^{-1}$. We write $\lambda^\vee_f$ for the spectral radius of f_*.

We set $\alpha := \delta_f^{-1} f^*$ as an automorphism of $\mathsf{N}(X)$. Then $\langle \alpha(x), \alpha(y) \rangle = \langle x, y \rangle$ for any x, $y \in \mathsf{N}(X)$, and $\alpha^{-1} = \delta_f^{-1} f_*$. Moreover, the spectral radius of α (resp. α^{-1}) equals $\delta_f^{-1} \lambda_f$ (resp. $\delta_f^{-1} \lambda^\vee_f$). For a basis of $\mathsf{N}(X)$, α is expressed as a matrix $\boldsymbol{A}$ satisfying ${}^t\boldsymbol{A}\boldsymbol{B}\boldsymbol{A} = \boldsymbol{B}$ for the symmetric matrix $\boldsymbol{B}$ representing the pairing $\langle\ ,\ \rangle$. Fixing a norm $\|\ \|$ of $\mathsf{N}(X)$, let $\|T\|$ denote the L^1-norm $\sup\{\|Tx\| \,;\, \|x\| = 1\}$ for any linear transformation T of $\mathsf{N}(X)$. Then $\lim_{m \to +\infty} \|\boldsymbol{A}^m\|^{1/m}$ (resp. $\lim_{m \to -\infty} \|\boldsymbol{A}^m\|^{1/m}$) equals the spectral radius of α (resp. α^{-1}), and we have

$$\|\boldsymbol{B}\| = \|({}^t\boldsymbol{A})^m \boldsymbol{B} \boldsymbol{A}^m\| \leq \|({}^t\boldsymbol{A})^m\| \cdot \|\boldsymbol{B}\| \cdot \|\boldsymbol{A}^m\| = \|\boldsymbol{B}\| \cdot \|\boldsymbol{A}^m\|^2$$

for any $m \in \mathbb{Z}$. Thus $\|\boldsymbol{A}^m\| \geq 1$ for any m, and we have $\lambda_f \geq \delta_f$ and $\lambda^\vee_f \geq \delta_f$.

The automorphisms f^* and f_* of $\mathsf{N}(X)$ preserve the nef cone $\mathrm{Nef}(X)$, i.e., $f^* \mathrm{Nef}(X) \subset \mathrm{Nef}(X)$ and $f_* \mathrm{Nef}(X) \subset \mathrm{Nef}(X)$. By a version of Perron–Frobenius theorem (cf. [3]), we can find non-zero vectors v_+ and v_- in $\mathrm{Nef}(X)$ such that $f^* v_+ = \lambda_f\, v_+$ and $f_* v_- = \lambda^\vee_f\, v_-$. Then $f_* v_+ = (\deg f / \lambda_f) v_+$ and $f^* v_- = (\deg f / \lambda^\vee_f) v_-$ by $f_* = (\deg f)(f^*)^{-1}$.

Assume that $\langle v_+, v_- \rangle = 0$. Then $\langle v_+, v_+ \rangle = \langle v_-, v_- \rangle = 0$ and $\mathbb{R} v_+ = \mathbb{R} v_-$ by the Hodge index theorem (cf. Lemma just before [44, Def. 2.11]). Thus, $\lambda_f \lambda^\vee_f = \delta_f^2$, and we have $\lambda_f = \lambda^\vee_f = \delta_f$ by $\lambda_f \geq \delta_f$ and $\lambda^\vee_f \geq \delta_f$. Next, assume that $\langle v_+, v_- \rangle \neq 0$. Then $\langle v_+, v_- \rangle > 0$ by v_+, $v_- \in \mathrm{Nef}(X)$, and the formula $\langle f^* v_+, v_- \rangle = \langle v_+, f_* v_- \rangle$ implies: $\lambda_f = \lambda^\vee_f$. Therefore, $\lambda_f = \lambda^\vee_f$ in any case and we have proved (2) and (3).

Second, we shall show (4) and (5). Let v be an eigenvector of $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ with real eigenvalue c, i.e., $0 \neq v \in \mathsf{N}(X, f; c)$. Then

$$\text{(III-1)} \qquad \begin{aligned} c\langle v, v_+\rangle &= \langle f^*v, v_+\rangle = \langle v, f_*v_+\rangle = \lambda_f^\dagger \langle v, v_+\rangle \qquad \text{and} \\ c\langle v, v_-\rangle &= \langle f^*v, v_-\rangle = \langle v, f_*v_-\rangle = \lambda_f \langle v, v_-\rangle \end{aligned}$$

for vectors v_+, v_- in (3). Suppose that $c \notin \{\lambda_f, \lambda_f^\dagger\}$. Then $\langle v, v_+\rangle = \langle v, v_-\rangle = 0$ by (III-1). Hence, $\langle v, v\rangle < 0$ or $v \in \mathbb{R}v_+ = \mathbb{R}v_-$ with $\langle v, v\rangle = 0$ by the Hodge index theorem. The latter case does not occur, since the condition implies: $c = \lambda_f = \lambda_f^\dagger$. Thus, $\langle v, v\rangle < 0$ and hence $c = \pm\delta_f$ by (1). This shows the first half of (4) and (5). The latter half of (4) follows directly from the first half by $\lambda_f \geq \delta_f$ (cf. (2)).

Finally, we shall show (6) and (7) separately.

(6): By the proof of (2) and (3) above, we have $\langle v_+, v_-\rangle > 0$, since $\lambda_f > \delta_f$. Moreover, $\langle v_+, v_+\rangle = \langle v_-, v_-\rangle = 0$ by (1). Let v be a vector in $\mathsf{N}(X, f; \lambda_f)$. Then $\langle v, v_+\rangle = 0$ by (III-1), and $\langle v, v\rangle = 0$ by (1). Thus, $v \in \mathbb{R}v_+$ by the Hodge index theorem, and we have $\mathsf{N}(X, f; \lambda_f) = \mathbb{R}v_+$. The other equality $\mathsf{N}(X, f; \lambda_f^\dagger) = \mathbb{R}v_-$ is similarly proved.

(7): We may assume that $\Delta_f \neq 0$. Now, $\lambda_f^\dagger = 1$, and $f_*v_+ = v_+$ by (3). We have $\langle v_+, \mathrm{cl}(\Delta_f)\rangle = 0$ by (II-5) in Lemma 2.17 and by

$$\langle v_+, \mathrm{cl}(K_X + S_f)\rangle = \langle f_*v_+, \mathrm{cl}(K_X + S_f)\rangle = \langle v_+, \mathrm{cl}(f^*(K_X + S_f)\rangle.$$

Then $\langle v_+, \mathrm{cl}(C)\rangle = 0$ for any prime component C of Δ_f, since $v_+ \in \mathrm{Nef}(X)$. Thus $\mathrm{cl}(C) \in \mathbb{R}_{\geq 0}v_+$ by the Hodge index theorem, since C is nef (cf. Proposition 2.20(4)). In particular, $f^*C \approx (\deg f)C$. Hence, $S_fC = 0$ by (1), and $S_f \cap C = \emptyset$ by $C \notin \mathcal{S}(X, f)$. Thus, $S_f \cap \mathrm{Supp}\,\Delta_f = \emptyset$, and we are done. □

Remark. A generalization of (7) is given in Proposition 3.24 below.

Corollary 3.4. *On divisors on X, the following hold*:

1. *If $\mathcal{S}(X, f; a) \neq \emptyset$ (cf. Definition 2.13), then $a \mid \deg f$ and $a \in \{\delta_f, \lambda_f, \lambda_f^\dagger\}$.*
2. *Assume that $f^*D \approx rD$ for an $\mathbb{R}$-divisor D and a real number r.*
   - (a) *If $D^2 \neq 0$ and if D is pseudo-effective, then $r = \delta_f$.*
   - (b) *If D is nef and big, then $r = \delta_f = \lambda_f$.*
   - (c) *If $r = \delta_f$ and D is a $\mathbb{Q}$-divisor, then $\delta_f \in \mathbb{Z}$.*
3. *Assume that $\deg f > 1$ and f is* sufficiently iterated *(cf. Definition 2.16). Then $\mathcal{N}(X) \subset \mathcal{S}(X, f; \delta_f)$ for the set $\mathcal{N}(X)$ of negative curves on X. In this case, if $\mathcal{N}(X) \neq \emptyset$, then $\delta_f \in \mathbb{Z}$.*

Proof. (1): We have $a \mid \deg f$ in Remark 2.14. The latter half of (1) follows from Proposition 3.3(4).

(2): In the situation of (2a) or (2b), $r^2 = \deg f = \delta_f^2$ by $(f^*D)^2 = (\deg f)D^2$. Here, we have $r = \delta_f$, since D is pseudo-effective. Here, if $D^2 > 0$, then $r \in \{\lambda_f, \lambda_f^\dagger\}$ by Proposition 3.3(5). Hence, $r = \lambda_f$ in (2b). In the situation of (2c), the square root δ_f is rational, and hence, $\delta_f \in \mathbb{Z}$.

(3): Let C be a negative curve on X. Then $C \in \mathcal{S}(X, f)$ by Proposition 2.20(3), and we have $f^*C = \delta_f C$ by (2a), since f is sufficiently iterated. This shows the first assertion of (3). The latter assertion follows from (2c). Thus, we are done. □

Corollary 3.5. *Let $\tau\colon X' \to X$ be a surjective morphism of normal Moishezon surfaces and let $f\colon X \to X$ and $f'\colon X' \to X'$ be surjective endomorphisms such that $\tau \circ f' = f \circ \tau$. Then $\deg f = \deg f'$ and $\lambda_f = \lambda_{f'}$.*

Proof. Considering the degree of $\tau \circ f' = f \circ \tau$, we have $\deg f = \deg f'$. Moreover, there exist commutative diagrams

$$\begin{array}{ccc} \mathsf{N}(X') & \xrightarrow{f'^*} & \mathsf{N}(X') \\ {\scriptstyle \tau^*}\uparrow & & \uparrow{\scriptstyle \tau^*} \\ \mathsf{N}(X) & \xrightarrow{f^*} & \mathsf{N}(X) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathsf{N}(X') & \xrightarrow{f'_*} & \mathsf{N}(X') \\ {\scriptstyle \tau_*}\downarrow & & \downarrow{\scriptstyle \tau_*} \\ \mathsf{N}(X) & \xrightarrow{f_*} & \mathsf{N}(X) \end{array}$$

of real vector spaces. Any eigenvalue of f^* is an eigenvalue of f'^*, since τ^* is injective. Hence, $\lambda_{f'} \geq \lambda_f$. On the other hand, there is an eigenvector v' of f'_* with eigenvalue $\lambda_{f'}$ in the nef cone $\mathrm{Nef}(X')$ by Proposition 3.3(3). If $\tau_* v' = 0$, then $\langle \tau_* v', \mathrm{cl}(A)\rangle = \langle v', \mathrm{cl}(\tau^* A)\rangle = 0$ for any numerically ample divisor A in X, and hence, $\langle v', v'\rangle < 0$ by the Hodge index theorem; this contradicts $v' \in \mathrm{Nef}(X')$. Thus, $\tau_* v' \neq 0$ and it is an eigenvector of f_* with eigenvalue $\lambda_{f'}$. Since the spectral radius $\lambda_f^{\vee}$ of f_* equals λ_f, we have $\lambda_{f'} \leq \lambda_f$. Therefore, $\lambda_{f'} = \lambda_f$. □

Example 3.6. Let us consider the following two conditions:

(A) $\lambda_f > \deg f$, or equivalently, $\lambda_f^{\dagger} < 1$;

(B) $\hat{\boldsymbol{\rho}}(X) > 1$ and $(f^l)^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is not a scalar map but has exactly one real eigenvalue for any $l > 0$.

We shall show that each case of (A) and (B) has an example.

Let T be an elliptic curve with a fixed abelian group structure as a complex torus. Let X be the abelian surface $T \times T$. Then $3 \leq \boldsymbol{\rho}(X) \leq 4$, and $\boldsymbol{\rho}(X) = 4$ if and only if T has a complex multiplication. An integral 2×2 matrix $\boldsymbol{M}$ with $\det \boldsymbol{M} \neq 0$ defines a surjective endomorphism $f_{\boldsymbol{M}}\colon X \to X$ by

$$T \times T \ni (t_1, t_2) \mapsto (t_1, t_2)\boldsymbol{M} \in T \times T.$$

Here, $\deg f_{\boldsymbol{M}} = (\det \boldsymbol{M})^2$. We shall consider the first dynamical degree of $f_{\boldsymbol{M}}$. We have an isomorphism $H^{1,1}(X, \mathbb{C}) = H^{1,0}(X,\mathbb{C}) \otimes H^{0,1}(X,\mathbb{C})$, where $H^{p,q}(X,\mathbb{C})$ stands for the Hodge component of $H^*(X,\mathbb{C})$ of type (p,q). The pullback homomorphism $f_{\boldsymbol{M}}^*\colon H^{1,0}(X,\mathbb{C}) \to H^{1,0}(X,\mathbb{C})$ is represented by the matrix $\boldsymbol{M}$. Let $\{\alpha, \beta\}$ be the set of eigenvalues of $\boldsymbol{M}$ and assume that $\alpha \neq \beta$. Then the set of eigenvalues of $f_{\boldsymbol{M}}^*\colon H^{1,1}(X,\mathbb{C}) \to H^{1,1}(X,\mathbb{C})$ is $\{|\alpha|^2, |\beta|^2, \alpha\overline{\beta}, \overline{\alpha}\beta\}$. Since $\mathsf{N}(X) \otimes \mathbb{C}$ is an $f_{\boldsymbol{M}}^*$-invariant subspace of $H^{1,1}(X,\mathbb{C})$ and since $\deg f_{\boldsymbol{M}} = |\alpha\beta|^2$, we have

$$\lambda_{f_{\boldsymbol{M}}} = \max\{|\alpha|^2, |\beta|^2\} \quad \text{and} \quad \lambda_{f_{\boldsymbol{M}}}^{\dagger} = \min\{|\alpha|^2, |\beta|^2\}.$$

We consider the following two matrices:

$$\boldsymbol{M}_1 := \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{M}_2 := \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}.$$

Assume that $\boldsymbol{M} = \boldsymbol{M}_1$. Then $\det \boldsymbol{M} = -2$, and we can take $\alpha = 1 + \sqrt{3}$ and $\beta = 1 - \sqrt{3}$. Thus,

$$\lambda_{f_{\boldsymbol{M}}} = 4 + 2\sqrt{3} > \deg f_{\boldsymbol{M}} = 4 \quad \text{and} \quad \lambda_{f_{\boldsymbol{M}}}^{\dagger} = 4 - 2\sqrt{3} < 1.$$

Hence, this gives an example satisfying (A).

Assume next that $\boldsymbol{M} = \boldsymbol{M}_2$. Then $\det \boldsymbol{M} = 6$, and we can take $\alpha = 1 + \sqrt{-5}$ and $\beta = 1 - \sqrt{-5}$. Thus, $\deg f_{\boldsymbol{M}} = 36$, $\lambda_{f_{\boldsymbol{M}}} = \delta_{f_{\boldsymbol{M}}} = 6$, and the set of eigenvalues of $f_{\boldsymbol{M}}^* \colon H^{1,1}(X, \mathbb{C}) \to H^{1,1}(X, \mathbb{C})$ is $\{6, (1+\sqrt{-5})^2, (1-\sqrt{-5})^2\}$. In particular, the set of eigenvalues of $(f_{\boldsymbol{M}}^l)^* \colon H^{1,1}(X, \mathbb{C}) \to H^{1,1}(X, \mathbb{C})$ is $\{6^l, (1+\sqrt{-5})^{2l}, (1-\sqrt{-5})^{2l}\}$ for any positive integer $l > 0$. Here, $(1 \pm \sqrt{-5})^{2l} \notin \mathbb{R}$ for any $l > 0$. Therefore, $\boldsymbol{M}_2$ gives an example satisfying (B).

Lemma 3.7. *Assume that* $\hat{\boldsymbol{\rho}}(X) = 2$. *Then* $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ *is expressed by a real diagonal matrix. In particular,* f^* *has only real eigenvalues. Moreover, the following hold*:

(1) *Assume that* $\lambda_f = \delta_f$. *Then* $(f^2)^* = (f^*)^2 \colon \mathsf{N}(X) \to \mathsf{N}(X)$ *is the multiplication map by* $\deg f$. *If* f^* *preserves each extremal ray of* $\overline{\mathrm{NE}}(X)$ (*resp.* $\mathrm{Nef}(X)$), *then* f^* *is the multiplication map by* δ_f. *If* f^* *exchanges two extremal rays, then the eigenvalues of* f^* *are* δ_f *and* $-\delta_f$.

(2) *Assume that* $\lambda_f > \delta_f$. *Then* X *contains no negative curves,* $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X) = \mathbb{R}_{\geq 0} v_+ + \mathbb{R}_{\geq 0} v_-$ *for two vectors* v_+, v_- *in Proposition* 3.3, *and* f^* *preserves extremal rays* $\mathbb{R}_{\geq 0} v_+$ *and* $\mathbb{R}_{\geq 0} v_-$.

Proof. Since $\hat{\boldsymbol{\rho}}(X) = 2$, the cone $\overline{\mathrm{NE}}(X)$ is fan-shaped, i.e., $\overline{\mathrm{NE}}(X) = \mathsf{R} + \mathsf{R}'$ for two extremal rays R and R'. Since $f^* \overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)$, $(f^2)^* = (f^*)^2$ preserves R and R', and $(f^2)^*$ is expressed by a diagonal matrix composed of two positive numbers. Hence, f^* is also expressed by a real diagonal matrix. If $\lambda_f = \delta_f$, then an eigenvalue of f^* is δ_f or $-\delta_f$ by Proposition 3.3(4); thus, $(f^2)^*$ is the multiplication map by $\deg f$. This implies (1).

Assume that $\lambda_f > \delta_f$. Then f^* has two eigenvalues $\lambda_f > \lambda_f^\dagger$. Since v_+ and v_- are eigenvectors of f^* contained in $\mathrm{Nef}(X)$, we have $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X) = \mathsf{R} + \mathsf{R}'$ for $\mathsf{R} = \mathbb{R}_{\geq 0} v_+$ and $\mathsf{R}' = \mathbb{R}_{\geq 0} v_-$. This implies (2). □

3.2. **The singularities on the pair** (X, S) **along** S**.** Let f be a non-isomorphic surjective endomorphism of a normal Moishezon surface X and let S be a non-zero f-completely invariant divisor. Then (X, S) is log-canonical by Theorem E. We shall study the singularity on (X, S) along S more in details, e.g., on $\mathrm{Sing}\, S$ and on the subsets $\mathcal{P}(X, S)$ and $\mathcal{D}(X, S)$ defined in [44, Def. 3.27].

Remark 3.8. We have the following by the classification of 2-dimensional log-canonical pairs [28, Thm. 9.6] (cf. [33, Ch. 3], [44, Thm. 3.22], [45, Fact 2.5]):

(1) X has only quotient singularities along S;
(2) $X \setminus S \subset X$ is a toroidal embedding at any point of $\mathrm{Sing}\, S$;
(3) $S \cap X_{\mathrm{reg}}$ is a normal crossing divisor on X_{reg};
(4) $\mathcal{P}(X, S)$ is the set of points P of S_{reg} such that $P \in \mathrm{Sing}\, X$ and that (X, S) is 1-*log-terminal* at P (cf. [45, Def. 2.1]);
(5) $\mathcal{D}(X, S)$ is the set of points P of S_{reg} such that $P \in \mathrm{Sing}\, X$ and (X, S) is not 1-log-terminal at P. Thus, $(S \cap \mathrm{Sing}\, X) \setminus \mathrm{Sing}\, S = \mathcal{P}(X, S) \sqcup \mathcal{D}(X, S)$.

Note that our "1-log-terminal" is identical to "purely log terminal (plt)" in [57] and [33]. See [45, Rem. 2.3] for our policy.

Definition 3.9. Let k be a positive integer such that $(f^k)^{-1}(\Gamma) = \Gamma$ for any prime component Γ of S. For Γ, we define an integer $b_\Gamma > 0$ by $(f^k)^*(\Gamma) = b_\Gamma \Gamma$. We set

$$S^{(\star)} := \sum\nolimits_{b_\Gamma = 1} \Gamma, \quad S^{(\dagger)} := \sum\nolimits_{b_\Gamma < \deg f^k} \Gamma, \quad \text{and} \quad S^{(\ddagger)} := \sum\nolimits_{b_\Gamma = \deg f^k} \Gamma.$$

Remark. These divisors are independent of the choice of such an integer k. By definition, we have $S = S^{(\dagger)} + S^{(\ddagger)}$, $S^{(\star)} \le S^{(\dagger)}$, and $S^{(\ddagger)} \le S - S^{(\star)} \le S_f$, and there is no common prime component of $S^{(\star)}$ and S_f.

Lemma 3.10. *The following hold for $S^{(\star)}$, $S^{(\dagger)}$, and $S^{(\ddagger)}$:*

(1) *Connected components of $S^{(\star)}$ and $S^{(\ddagger)}$ are all irreducible and*

$$S^{(\star)} \cap (S^{(\dagger)} - S^{(\star)}) = S^{(\ddagger)} \cap (S_f - S^{(\ddagger)}) = \emptyset.$$

(2) *One has: $f^*(S^{(\star)}) = S^{(\star)}$, $f^*(S^{(\ddagger)}) = (\deg f)S^{(\ddagger)}$, and $f^{-1}(S^{(\dagger)}) = S^{(\dagger)}$.*

(3) *If $S^{(\star)} \cup S^{(\ddagger)} \ne \emptyset$, then $\lambda_f = \deg f$ and $S^{(\ddagger)} \subset S_f \subset X \setminus \operatorname{Supp} \Delta_f$.*

(4) *If Γ is a prime component of $S^{(\dagger)}$, then $(K_X + S)\Gamma \le 0$. Moreover, if $(K_X + S)\Gamma = 0$ in addition, then $\Gamma \cap \operatorname{Supp} \Delta_f = \emptyset$.*

Proof. (1): This follows from Proposition 2.20(1), since $S_f = S_{f^k}$ for any $k \ge 1$.

(2): Divisors $S^{(\star)}$, $S^{(\dagger)}$, and $S^{(\ddagger)}$ are f-completely invariant by Definition 3.9. Here, we have $f^* S^{(\star)} = S^{(\star)}$, since any prime component of $S^{(\star)}$ is not contained in $\operatorname{Supp} R_{f^k} \supset \operatorname{Supp} R_f$ for the integer k in Definition 3.9. For a prime component Γ of $S^{(\ddagger)}$, the restriction $f^k|_\Gamma \colon \Gamma \to \Gamma$ is birational. Hence, for the prime divisor $\Gamma' = f^{-1}\Gamma$, the morphism $f|_{\Gamma'} \colon \Gamma' \to \Gamma$ is also birational. This implies that $f^*\Gamma = (\deg f)\Gamma'$. Therefore, $f^* S^{(\ddagger)} = (\deg f) S^{(\ddagger)}$.

(3): We have $\lambda_f = \deg f$ by Proposition 3.3(4), and $S_f \subset X \setminus \operatorname{Supp} \Delta_f$ by Proposition 3.3(7).

(4): We set $\overline{S} := S \cup S_f$. Then $(\overline{S} - S)\Gamma \ge 0$, and $(K_X + \overline{S})\Gamma \ge (K_X + S)\Gamma$. On the other hand,

$$(1 - (\deg f^k)/b_\Gamma)(K_X + \overline{S})\Gamma = \Delta_{f^k}\Gamma \ge 0$$

by Proposition 2.20(5) applied to $\Gamma \in \mathcal{S}(X, f^k, b_\Gamma)$. Hence, $(K_X + S)\Gamma \le 0$ by $b_\Gamma < \deg f^k$. Assume that $(K_X + S)\Gamma = 0$. Then $\Delta_{f^k}\Gamma = 0$ by the inequality above, and we have $\Delta_f \Gamma = 0$ by Lemma 2.17(3), since Δ_f is nef. It is enough to prove: $\Gamma \not\subset \operatorname{Supp} \Delta_f$. If $b_\Gamma = 1$, this holds, since $\Gamma \not\subset \operatorname{Supp} R_{f^k} \supset \operatorname{Supp} \Delta_{f^k}$. If $b_\Gamma > 1$, then $\Gamma \subset S_f$ and $\Gamma \not\subset \operatorname{Supp} \Delta_f$, since S_f and Δ_f have no common prime component (cf. Definition 2.16). Thus, we are done. □

Lemma 3.11. *Let D be a non-zero reduced divisor D on X which is either*

- *a connected component of $S^{(\ddagger)}$ satisfying $(K_X + S)D \le 0$, or*
- *a connected component of $S^{(\dagger)}$.*

Then D is an elliptic curve, a cyclic chain of rational curves*, or a* linear chain of rational curves ([44, Defs. 4.1 and 4.3]). *Moreover, the following hold:*

(1) *If D is an elliptic curve or a cyclic chain of rational curves, then $K_X + S$ is Cartier along D, $\mathcal{O}_X(K_X + S)|_D \simeq \mathcal{O}_D$, and $D \cap \operatorname{Sing} X \subset \operatorname{Sing} D$.*

(2) *If $(K_X + S)\Gamma = 0$ for any prime component Γ of D, then $D \cap \operatorname{Supp} \Delta_f = \emptyset$.*

(3) *For any prime component* Γ *of* D,

$$\sharp(\Gamma \cap (\mathcal{D}(X,S) \cup \operatorname{Sing} S)) \le 2.$$

If this is an equality, then $\Gamma \simeq \mathbb{P}^1$, $\Gamma \cap \mathcal{P}(X,S) = \emptyset$, $(K_X + S)\Gamma = 0$, *and* $\Gamma \cap \operatorname{Supp} \Delta_f = \emptyset$.

Proof. We have $(K_X + D)\Gamma \le (K_X + S)\Gamma \le 0$ for any prime component Γ of D by (1) (resp. (4)) of Lemma 3.10 in case $D \subset S^{(\ddagger)}$ (resp. $D \subset S^{(\dagger)}$). Since (X, S) is log-canonical, the first assertion and (1) follow from [44, Lem. 4.5]. In fact, in (1), D satisfies $\mathcal{O}_X(K_X + D)|_D \simeq \mathcal{O}_D$, and it implies that $D \cap (S - D) = \emptyset$ and $(K_X+S)|_D = (K_X+D)|_D$. Assertion (2) follows from (3) (resp. (4)) of Lemma 3.10 in case $D \subset S^{(\ddagger)}$ (resp. $D \subset S^{(\dagger)}$).

For the last assertion (3), we apply [44, Prop. 3.29], which gives a detailed information on (X, S, Γ). We set $m_\Gamma := \sharp\Gamma \cap (\mathcal{D}(X,S) \cup \operatorname{Sing} S)$. Then

$$m_\Gamma = \sharp\Gamma \cap \mathcal{D}(X,S) + \sharp\Gamma \cap \operatorname{Sing} S = \sharp\Gamma \cap \mathcal{D}(X,S) + \sharp\Gamma \cap (S - \Gamma) + \sharp \operatorname{Sing} \Gamma,$$

since $\mathcal{D}(X,S) \cap \operatorname{Sing} S = \emptyset$. If Γ is an elliptic curve, then $m_\Gamma = 0$, and if Γ a nodal rational curve with one node, then $m_\Gamma = 1$: These are checked by cases (A) and (B) of [44, Prop. 3.29]. Thus, we may assume $\Gamma \simeq \mathbb{P}^1$ by the other cases in [44, Prop. 3.29]. By [44, Lem. 3.28] and by the inequality (III-8) in the proof of [44, Prop. 3.29], we have

$$\begin{aligned}(K_X + S)\Gamma &\ge -2 + \sharp\Gamma \cap (S - \Gamma) + \sharp\Gamma \cap \mathcal{D}(X,S) + \sum\nolimits_{r>1} \frac{r-1}{r} \sharp\Gamma \cap \mathcal{P}_r(X,S) \\ &= -2 + m_\Gamma + \sum\nolimits_{r>1} \frac{r-1}{r} \sharp\Gamma \cap \mathcal{P}_r(X,S),\end{aligned}$$

where $\mathcal{P}_r(X,S)$ is a subset of $\mathcal{P}(X,S)$ defined in [44, Def. 3.27], and $\mathcal{P}(X,S) = \coprod_{r>1} \mathcal{P}_r(X,S)$. Therefore, $m_\Gamma \le 2$. If $m_\Gamma = 2$, then $(K_X + S)\Gamma = 0$ and $\Gamma \cap \mathcal{P}(X,S) = \emptyset$; in particular, $\Gamma \cap \operatorname{Supp} \Delta_f = \emptyset$ by (2). Thus, we are done. □

Lemma 3.12. *The subsets* $\operatorname{Sing} S$ *and* $\mathcal{D}(X,S)$ *are* f-completely invariant (*cf. Definition* 2.12). *In particular, there is a positive integer* n *such that* $(f^n)^{-1}(P) = \{P\}$ *for any* $P \in \mathcal{D}(X,S) \cup \operatorname{Sing} S$. *Moreover, for any* $m \ge 1$,

$$(\mathcal{D}(X,S) \cup \operatorname{Sing} S) \cap \operatorname{Supp} \Delta_{f^m} = \emptyset. \quad \text{(III-2)}$$

Proof. First, we shall show: $f^{-1} \operatorname{Sing} S = \operatorname{Sing} S$. Since (X,S) is toroidal at any point of $\operatorname{Sing} S$ (cf. Remark 3.8(2)), we have $f^{-1} \operatorname{Sing} S \subset \operatorname{Sing} S$. In fact, for $P \in \operatorname{Sing} S$ and $Q \in f^{-1}(P)$, there exist open neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of P and Q, respectively, such that $S|_\mathcal{U}$ is reducible and that $f|_\mathcal{V} \colon \mathcal{V} \to \mathcal{U}$ is finite and surjective. Then $f^{-1}S|_\mathcal{V} = S|_\mathcal{V}$ is also reducible and its prime components all contain Q; hence, $Q \in \operatorname{Sing} S$. This shows: $f^{-1} \operatorname{Sing} S \subset \operatorname{Sing} S$. Then $f^{-1} \operatorname{Sing} S = \operatorname{Sing} S$ by Lemma 2.2 applied to the finite set $\operatorname{Sing} S$.

Second, we shall show: $f^{-1}(\mathcal{D}(X,S) \cup \operatorname{Sing} S) = \mathcal{D}(X,S) \cup \operatorname{Sing} S$. Note that if it holds, then $f^{-1}\mathcal{D}(X,S) = \mathcal{D}(X,S)$ by $f^{-1} \operatorname{Sing} S = \operatorname{Sing} S$, and the second assertion holds by Lemma 2.2. We know that $S \setminus (\mathcal{D}(X,S) \cup \operatorname{Sing} S)$ is the set of points $P \in S$ at which (X,S) is 1-log-terminal (cf. Remark 3.8(5)). Thus, we have $f^{-1}(\mathcal{D}(X,S) \cup \operatorname{Sing} S) \subset \mathcal{D}(X,S) \cup \operatorname{Sing} S$ by [45, Prop. 2.12(2)]. Then $f^{-1}(\mathcal{D}(X,S) \cup \operatorname{Sing} S) = \mathcal{D}(X,S) \cup \operatorname{Sing} S$ by Lemma 2.2.

It remains to prove (III-2). By Lemma 2.17(3), it suffices to prove it when m is the integer k in Definition 3.9. Since $f^{-1}S = S$, by [45, Lem. 1.39], there is an effective divisor Δ such that $K_X + S = f^*(K_X + S) + \Delta$ and that S and Δ have no common prime component. Then $\Delta \geq \Delta_f$ by Lemma 2.18, and

$$(f^{k-1})^*\Delta + \cdots + f^*\Delta + \Delta \geq \Delta_{f^k}$$

by Lemma 2.17(3). Thus, it is enough to prove: $(\mathcal{D}(X,S) \cup \operatorname{Sing} X) \cap \operatorname{Supp} \Delta = \emptyset$. But, this follows from [45, Thm. 3.5(1)]: In fact, if $P \in \operatorname{Supp} \Delta$, then (X,S) is 1-log-terminal at P. Thus, we are done. □

Proposition 3.13. *Let C be a singular prime component of S. Then there is a positive integer n such that $(f^n)^*C = \delta_f^n C$. Moreover,*

- *C is isomorphic to a nodal cubic rational curve,*
- *C is a connected component of S,*
- *$C_{\mathrm{reg}} \subset X_{\mathrm{reg}}$, $C \cap \operatorname{Supp} \Delta_f = \emptyset$, and*
- *$K_X + C$ is Cartier along C with $\mathcal{O}_X(K_X + C)|_C \simeq \mathcal{O}_C$.*

As a consequence, $S^{(\star)}$ and $S^{(\ddagger)}$ are non-singular.

Proof. Let P be a singular point of C. By iteration, we may assume that $f^{-1}C = C$ and $f^{-1}(P) = \{P\}$ by Lemma 3.12. Note that $P \notin \operatorname{Supp} \Delta_f$ by Lemma 3.12. Then $f^*C = \delta_f C$ by [45, Cor. 5.7] applied to the morphism $f^\circ := f|_{X^\circ} \colon X^\circ \to X$ from an open neighborhood X° of P, where f° is étale over $X^\circ \setminus S$ by $P \notin \operatorname{Supp} \Delta_f$. This shows the first assertion. In particular, $C \subset S^{(\dagger)}$, and the required properties of C are verified by Lemma 3.11. Then we have the last assertion by Lemma 3.10(1). □

3.3. **Endomorphisms preserving a fibration.** We shall discuss some elementary properties of endomorphisms preserving fibrations. For more properties, see Section 4 below.

Lemma 3.14. *Let f be a non-isomorphic surjective endomorphism of a normal Moishezon surface X and let $\pi \colon X \to Y$ be a non-isomorphic bimeromorphic morphism to a normal Moishezon surface Y. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be the π-exceptional prime divisors. Then*

1. *there is an integer $k > 0$ such that $(f^k)^*\Gamma_i = \delta_f^k \Gamma_i$ for any $1 \leq i \leq n$, and*
2. *each Γ_i is $\mathbb{Q}$-Cartier.*

Moreover, if the π-exceptional divisor $\sum_{i=1}^n \Gamma_i$ is f-completely invariant, then there is a non-isomorphic surjective endomorphism $g \colon Y \to Y$ such that $\pi \circ f = g \circ \pi$, $\deg f = \deg g$, $\lambda_f = \lambda_g$, and the following holds:

3. *The pullback homomorphism $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ is isomorphic to the direct sum of $g^* \colon \mathsf{N}(Y) \to \mathsf{N}(Y)$ and a linear transformation $\theta \colon \mathbb{R}^{\oplus n} \to \mathbb{R}^{\oplus n}$ such that θ^k is the multiplication map by δ_f^k for the integer k in (1).*

Proof. The negative curves Γ_i are contained in S_f by Proposition 2.20(3). Thus, we have (1) by Corollary 3.4(3). Since X has only quotient singularities along S_f

(cf. Remark 3.8(1)), every prime component of S_f is $\mathbb{Q}$-Cartier. In particular, we have (2).

Assume that $\sum_{i=1}^{n} \Gamma_i$ is f-completely invariant. Then the exceptional locus of $\pi \circ f$ equals the exceptional locus of π, and there is an endomorphism $g\colon Y \to Y$ such that $\pi \circ f = g \circ \pi$. Here, $\deg f = \deg g$ and $\lambda_f = \lambda_g$ by Corollary 3.5. Let L be the vector subspace of $\mathsf{N}(X)$ generated by $\mathrm{cl}(\Gamma_i)$ for $1 \le i \le n$. Then $\dim L = n$, and $\mathsf{N}(X) = \pi^*\mathsf{N}(Y) \oplus L$ by [44, Lem. 2.10], and we have $f^*L = L$, since $\sum_{i=1}^{n} \Gamma_i$ is f-completely invariant. Hence, the automorphism $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is just the direct sum of $\pi^*(g^*)\colon \pi^*\mathsf{N}(Y) \to \pi^*\mathsf{N}(Y)$ and $\theta := f^*|_L\colon L \to L$, where θ^k is the multiplication map by δ_f^k by (1). This shows (3) and we are done. □

Lemma 3.15. *In the situation of Lemma* 3.14, *assume that the* π*-exceptional locus is* f*-completely invariant. Then the following hold on the endomorphism* g *of* Y:

1. *If a reduced divisor* S *on* X *is* f*-completely invariant, then* π_*S *is* g*-completely invariant.*
2. *If a reduced divisor* $\overline{S}$ *on* Y *is* g*-completely invariant, then the proper transform in* X *and the inverse image* $\pi^{-1}\overline{S}$ *are both* f*-completely invariant.*
3. *The characteristic completely invariant divisors* S_f *of* f *is the union of* $\pi^{-1}S_g$ *and the* π*-exceptional locus. In particular,* $S_g = \pi_*S_f$.

Proof. Let B_1 and B_2 be non-π-exceptional prime divisors on X, and we set $\Theta_i = \pi_*B_i$ for $i = 1, 2$. Then $\pi^*\Theta_i = B_i + E_i$ for a π-exceptional effective $\mathbb{Q}$-divisor E_i. If $f^*B_1 = aB_2$ for an integer $a > 0$, then

$$\pi^*(g^*\Theta_1 - a\Theta_2) = f^*(B_1 + E_1) - a(B_2 + E_2) = f^*E_1 - aE_2$$

and we have $g^*\Theta_1 = a\Theta_2$ by applying π_*. If $g^*\Theta_1 = a\Theta_2$ for an integer $a > 0$, then

$$f^*B_1 - aB_2 = \pi^*(g^*\Theta_1 - a\Theta_2) - (f^*E_1 - aE_2) = -(f^*E_1 - aE_2)$$

and we have $f^*B_1 = aB_2$ and $f^*E_1 = aE_2$, since $f^*B_1 - aB_2$ has no π-exceptional prime component. Assertions (1)–(3) are shown by this argument. □

Lemma 3.16. *Let* $\pi\colon X \to T$ *be a fibration from a normal Moishezon surface* X *to a non-singular projective curve* T *and let* $f\colon X \to X$ *be a finite surjective endomorphism. Let* $\mathsf{R} \subset \mathsf{N}(X)$ *be the* 1*-dimensional cone generated by the numerical class of a general fiber of* π*. If* $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ *preserves* R*, i.e.,* $f^*\mathsf{R} = \mathsf{R}$*, then there is an endomorphism* $h\colon T \to T$ *such that* $\pi \circ f = h \circ \pi$.

Proof. Let F be a general fiber of π. Then $f^*F \approx bF$ for some $b > 0$. For any $t \in T$, $f^{-1}(\pi^{-1}(t))$ is contained in fibers of π, since $\pi^*(t) \approx F$ and since $F(f^*F) = bF^2 = 0$. Thus, $f^{-1}(\pi^{-1}(t))$ is mapped to finitely many points by the morphism $(\pi \circ f, \pi)\colon X \to T \times T$. Hence, $\dim T' = 1$ for the Stein factorization $X \to T' \to T \times T$ of $(\pi \circ f, \pi)$. Let $\theta\colon T' \to T \times T$ be the finite morphism and let p_i be the i-th projection $T \times T \to T$ for $i = 1, 2$. Since $\pi\colon X \to T$ is a fibration, the composite $u := p_2 \circ \theta\colon T' \to T$ is an isomorphism. Then $\pi \circ f = h \circ \pi$ for $h = p_1 \circ \theta \circ u^{-1}$. □

Proposition 3.17. *Let X be a normal Moishezon surface with a non-isomorphic surjective endomorphism f, and let $\pi\colon X \to T$ be a fibration to a non-singular projective curve T with an endomorphism h satisfying $\pi \circ f = h \circ \pi$. Then:*

(1) *The fiber product $X^h := X \times_T T$ of π and h is irreducible and f induces a finite surjective morphism $X \to X^h$ of degree $\deg f / \deg h$. In particular, $\deg h \mid \deg f$, and $f|_{\pi^{-1}(t)}\colon \pi^{-1}(t) \to \pi^{-1}(h(t))$ is surjective for any $t \in T$.*

(2) *The numerical class of a general fiber of π is an eigenvector of $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ with eigenvalue $\deg h$.*

(3) *The surface X is projective and has at most quotient singularities.*

Let S be an f-completely invariant divisor and assume either that $S = 0$ or that every prime component of S dominates T. Then

(4) *S is non-singular,*

(5) *(X, S) is 1-log-terminal, and*

(6) *$(X, S + \pi^{-1}(t))$ is log-canonical for any $t \in T$.*

Proof. Assertion (1) is a consequence of [45, Cor. 1.14]. Since $h^*\colon \mathsf{N}(T) \simeq \mathbb{R} \to \mathsf{N}(T)$ is the multiplication map by $\deg h$, the subspace $\pi^*\mathsf{N}(T)$ of $\mathsf{N}(X)$ is an eigenspace of f^* with eigenvalue $\deg h$. This implies (2). Assertion (3) follows from [44, Lem. 2.31(2)] and from (5) in the case where $S = 0$. We have implications (6) $\Rightarrow$ (5) and (5) $\Rightarrow$ (4) by properties on 1-log-terminal pairs (cf. [45, Fact 2.5]). Thus, it is enough to prove (6).

(6): We set $G_t := \pi^{-1}(t)$ for $t \in T$. Let $\mathcal{S}$ be the set of points $t \in T$ such that $(X, S + G_t)$ is not log-canonical along G_t. It suffices to show that $\mathcal{S} = \emptyset$. If π is smooth along G_t and $\pi|_S\colon S \to T$ is étale along $S \cap G_t$, then $t \notin \mathcal{S}$. In particular, $\mathcal{S}$ is a finite set.

We shall show that $h^{-1}\mathcal{S} \subset \mathcal{S}$, or equivalently that if $t \notin \mathcal{S}$, then $h(t) \notin \mathcal{S}$. We have a connected open neighborhood $\mathcal{U}$ of t such that $\mathcal{V} := h(\mathcal{U})$ is open, $h|_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{V}$ is finite, and $h^{-1}(h(t)) \cap \mathcal{U} = \{t\}$, by [45, Cor. 1.8]. Then $\phi := f|_{\pi^{-1}\mathcal{U}}\colon \pi^{-1}\mathcal{U} \to \pi^{-1}\mathcal{V}$ is finite and $\phi^{-1}G_{h(t)} = G_t$. By [45, Lem. 1.39],

$$K_{\pi^{-1}\mathcal{U}} + S|_{\pi^{-1}\mathcal{U}} + G_t = \phi^*(K_{\pi^{-1}\mathcal{V}} + S|_{\pi^{-1}\mathcal{V}} + G_{h(t)}) + \Delta_t$$

for an effective divisor Δ_t on $\pi^{-1}\mathcal{U}$ having no common prime component with $S|_{\pi^{-1}\mathcal{U}} + G_t$. Now $(X, S + G_t)|_{\pi^{-1}\mathcal{U}} = (\pi^{-1}\mathcal{U}, S|_{\pi^{-1}\mathcal{U}} + G_t)$ is log-canonical along G_t by $t \notin \mathcal{S}$. Thus, $(X, S+G_{h(t)})|_{\pi^{-1}\mathcal{V}} = (\pi^{-1}\mathcal{V}, S|_{\pi^{-1}\mathcal{V}}+G_{h(t)})$ is also log-canonical along $G_{h(t)}$ by [45, Prop. 2.12(1)]. Hence, $h(t) \notin \mathcal{S}$.

Therefore, $h^{-1}\mathcal{S} \subset \mathcal{S}$, and there is a positive integer n such that $(h^n)^{-1}(t) = \{t\}$ for any $t \in \mathcal{S}$ by Lemma 2.2. Then $(f^n)^{-1}G_t = G_t$ for any $t \in \mathcal{S}$, and it implies that $(X, S+G_t)$ is log-canonical by Theorem E, since $S+G_t$ is f^n-completely invariant. This is a contradiction. Therefore $\mathcal{S} = \emptyset$, and we are done. □

Finally in Section 3.3, we shall show:

Lemma 3.18. *For a non-singular projective curve T, let $f\colon \mathbb{P}^1 \times T \to \mathbb{P}^1 \times T$ be a surjective endomorphism such that $p_2 \circ f = h \circ p_2$ for an endomorphism h of T and for the second projection $p_2\colon \mathbb{P}^1 \times T \to T$. Then $f = g \times h$ for an endomorphism $g\colon \mathbb{P}^1 \to \mathbb{P}^1$.*

Proof. This is a consequence of Horst's theorem [26, Thm. 3.1] for $\mathbb{P}^1$. We shall give another proof. The endomorphism f is induced by h and a morphism $\Phi\colon \mathbb{P}^1 \times T \to \mathbb{P}^1$, i.e., $f = (\Phi, h \circ p_2)$. Since $\boldsymbol{\rho}(\mathbb{P}^1 \times T) = 2$, Φ is factored by either the first projection $p_1 : \mathbb{P}^1 \times T \to \mathbb{P}^1$ or the second projection $p_2 : \mathbb{P}^1 \times T \to T$. But the latter case does not occur, since f is surjective. Thus, $\Phi = g \circ p_1$ for an endomorphism $g\colon \mathbb{P}^1 \to \mathbb{P}^1$ and we have $f = g \times h$. □

3.4. **Application of the minimal model theory.** We can consider a kind of "equivariant" minimal model program for normal Moishezon surfaces with non-isomorphic surjective endomorphisms.

Proposition 3.19. *Let X be a normal Moishezon surface with a non-isomorphic surjective endomorphism f and let S be an f-completely invariant divisor. Assume that $K_X + S$ is not pseudo-effective. Then X is projective with $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$ and there exists a birational morphism $\phi\colon X \to \overline{X}$ to a normal projective surface $\overline{X}$ with an endomorphism $\bar{f}$ such that*

1. *$\bar{f} \circ \phi = \phi \circ f^k$ for a positive integer k,*
2. *$\overline{S} := \phi_* S$ is completely invariant under $\bar{f}$,*
3. *the characteristic completely invariant divisor $S_{\bar{f}}$ of $\bar{f}$ equals $\phi_* S_f$, and*
4. *one of the following two conditions is satisfied:*
   - (a) *$-(K_{\overline{X}} + \overline{S})$ is ample and $\hat{\boldsymbol{\rho}}(\overline{X}) = \boldsymbol{\rho}(\overline{X}) = 1$;*
   - (b) *$\overline{X}$ has only quotient singularities, $\hat{\boldsymbol{\rho}}(\overline{X}) = \boldsymbol{\rho}(\overline{X}) = 2$, and there is a fibration $\pi\colon \overline{X} \to T$ to a non-singular projective curve T with a surjective endomorphism $h_{(2)} : T \to T$ such that*
     - *$-(K_{\overline{X}} + \overline{S})$ is π-ample,*
     - *$F_{\mathrm{red}} \simeq \mathbb{P}^1$ for any fiber F of π,*
     - *$\pi \circ (\bar{f})^2 = h_{(2)} \circ \pi$,*
     - *$\deg h_{(2)} \mid (\deg \bar{f})^2$.*

In (4b)*, if* $\mathrm{cl}(F)$ *is an eigenvector of* $(\bar{f})^* : \mathsf{N}(\overline{X}) \to \mathsf{N}(\overline{X})$*, then there is an endomorphism $h\colon T \to T$ such that $\pi \circ \bar{f} = h \circ \pi$, $h_{(2)} = h^2$, and $\deg h \mid \deg \bar{f}$.*

Proof. The pair (X, S) is log-canonical by Theorem E, and X is projective with $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$ by Corollary 1.11(1). By applying Theorems 1.9 and 1.10 successively to the non-nef $\mathbb{Q}$-divisor $K_X + S$, we have a birational morphism $\phi\colon X \to \overline{X}$ to a normal projective surface $\overline{X}$ as the composite of contraction morphisms of extremal rays, in which one of the following holds, where $\overline{S} = \phi_* S$:

- (i) $\hat{\boldsymbol{\rho}}(\overline{X}) = \boldsymbol{\rho}(\overline{X}) = 1$, and $-(K_{\overline{X}} + \overline{S})$ is ample;
- (ii) $\hat{\boldsymbol{\rho}}(\overline{X}) = \boldsymbol{\rho}(\overline{X}) = 2$, and there is a fibration $\pi\colon \overline{X} \to T$ to a non-singular curve T such that $-(K_{\overline{X}} + \overline{S})$ is π-ample and that $F_{\mathrm{red}} \simeq \mathbb{P}^1$ for any fiber F of π.

By applying Lemmas 3.14 and 3.15 to the birational morphism ϕ, we have an endomorphism $\bar{f}$ of $\overline{X}$ satisfying (1)–(3) above.

Since (i) is identical to (4a), it remains to check properties in (4b) assuming (ii). By Lemma 3.7, $(\bar{f}^2)^* : \mathsf{N}(\overline{X}) \to \mathsf{N}(\overline{X})$ preserves the extremal ray $\mathsf{R} = \mathbb{R}_{\geq 0}\, \mathrm{cl}(F)$ of $\overline{\mathrm{NE}}(\overline{X})$ for a general fiber F of π. Hence, we have an endomorphism $h\colon T \to T$

satisfying $h_{(2)} \circ \pi = \pi \circ \bar{f}^2$ by Lemma 3.16. Here, $\deg h_{(2)} \mid \deg \bar{f}^2 = (\deg \bar{f})^2$, and $\overline{X}$ has only quotient singularities by Proposition 3.17. If $\mathrm{cl}(F)$ is an eigenvector of $\bar{f}^*$, then $\bar{f}^*$ preserves R and we have $h\colon T \to T$ satisfying $h \circ \pi = \pi \circ \bar{f}$ and $h_{(2)} = h^2$ by Lemma 3.16. Here, $\deg h \mid \deg \bar{f}$ by Proposition 3.17. Thus, we are done. □

We shall give an application of Proposition 3.19 to *polarized endomorphisms*.

Definition 3.20 ([11], [64], [47])**.** An endomorphism $\psi\colon Z \to Z$ of a normal projective variety Z is said to be *polarized* if there is an ample divisor A such that $\psi^* A \sim qA$ for a positive number $q > 1$.

Remark. By [47, Lem. 2.2], we see that the endomorphism $\psi\colon Z \to Z$ is polarized if $\psi^* A \approx qA$ for an ample divisor A and a positive number $q > 1$. Furthermore q is an integer by [47, Lem. 2.1] (cf. Corollary 3.4(2) in the 2-dimensional case).

Remark. For a surjective endomorphism f of a normal projective surface, if a power f^k is polarized, then $\lambda_f = \delta_f$. In fact, this is derived from Corollary 3.4(2b), since $\lambda_{f^k} = (\lambda_f)^k$ and $\delta_{f^k} = (\delta_f)^k$.

The following result is due to Zhang (cf. the proof of [63, Thm. 2.7]):

Lemma 3.21. *In Definition* 3.20*, if a power ψ^k is polarized for some $k > 0$, then ψ is polarized provided that $\deg \psi = q^{\dim Z}$ for an integer q.*

Proof. Let A be an ample divisor such that $(\psi^k)^* A \sim bA$ for some $b > 1$. Then $b = q^k$ by $\deg \psi^k = (\deg \psi)^k = q^{nk} = b^n$ for $n = \dim Z$. Thus, for the ample divisor

$$\widehat{A} = \sum\nolimits_{i=0}^{k-1} q^{k-i} (\psi^i)^* A,$$

we have $\psi^* \widehat{A} \sim q\widehat{A}$. □

Theorem 3.22. *Let X be a normal Moishezon surface admitting a non-isomorphic surjective endomorphism f. If $\boldsymbol{\rho}(X) \neq 2$ and if K_X is not pseudo-effective, then $(f^k)^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is a scalar map for some $k > 0$. In particular, $\lambda_f = \delta_f$. Moreover, in the situation, X is projective and the square f^2 is polarized.*

Proof. By Proposition 3.19 applied to the case where $S = 0$, we see that X is projective and $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$. If $(f^k)^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is a scalar map, then it is the multiplication map by δ_{f^k}, by Proposition 3.3(1). Moreover, in this case, f^k is a polarized endomorphism, and hence, f^2 is also a polarized endomorphism by Lemma 3.21, since $\delta_{f^2} = \deg f \in \mathbb{Z}$. Thus, it is enough to prove that $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is a scalar map assuming that $\boldsymbol{\rho}(X) \geq 3$, K_X is not pseudo-effective, and f is sufficiently iterated.

Let $\phi\colon X \to \overline{X}$ be the birational morphism in Proposition 3.19 in the case where $S = 0$. Since f is sufficiently iterated, any ϕ-exceptional prime divisor is f-completely invariant and $\phi \circ f = \bar{f} \circ \phi$ by Lemma 3.14. It suffices to prove that $\bar{f}^*\colon \mathsf{N}(\overline{X}) \to \mathsf{N}(\overline{X})$ is a scalar map, by Lemma 3.14(3). Thus, we may assume that $\boldsymbol{\rho}(\overline{X}) = 2$.

Since $\boldsymbol{\rho}(X) \geq 3$, there is a negative curve Γ on X contracted to a point P of $\overline{X}$. For the $\mathbb{P}^1$-fibration $\pi\colon \overline{X} \to T$ in Proposition 3.19(4b), let G be the set-theoretic fiber over $\pi(P)$, i.e., $G = \pi^{-1}(\pi(P))$. Here, G is a prime divisor and its proper transform G' in X is a negative curve, since $P \in G$. Hence, $f^*G' = \delta_f G'$ by Corollary 3.4(3), and we have $\bar{f}^*G = \delta_f G$ by applying ϕ_* (cf. the proof of Lemma 3.15). It implies that $\bar{f}^*\colon \mathsf{N}(\overline{X}) \to \mathsf{N}(\overline{X})$ preserves the extremal ray $\mathbb{R}_{\geq 0}\operatorname{cl}(G)$, and hence, $\bar{f}^*$ is a scalar map by Lemma 3.7. Thus, we are done. □

As a corollary of Theorem 3.22, we have the following on endomorphisms f with $\lambda_f = \delta_f$: Essentially the same result is obtained by Zhang in [63, Thm. 2.7] under an assumption similar to that f is a polarized endomorphism:

Corollary 3.23. *Let f be a non-isomorphic surjective endomorphism of a normal Moishezon surface X such that $\lambda_f = \delta_f$. Then $(f^k)^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ is a scalar map for some $k > 0$ unless $K_X \approx 0$ and $\boldsymbol{\rho}(X) \geq 3$.*

Proof. By Lemma 3.7 and Theorem 3.22, we may assume that K_X is pseudo-effective and $K_X \not\approx 0$. Then $K_X = f^*K_X$ by Lemma 2.22. Thus, $\operatorname{cl}(K_X)$ is an eigenvector of f^* with eigenvalue 1, and $\lambda_f = \deg f \neq \delta_f$ by Proposition 3.3(4). □

Proposition 3.24. *Let f be a non-isomorphic surjective endomorphism of a normal Moishezon surface X such that $\lambda_f > \delta_f$. Then $\mathcal{S}(X, f; \delta_f) = \emptyset$. In particular, X contains no negative curve. If $\lambda_f = \deg f$, then $\operatorname{cl}(\Theta) \in \mathbb{R}_{\geq 0}v_+$ for any prime component Θ of R_f for the vector v_+ in Proposition* 3.3; *in particular,* $\operatorname{Supp} R_f$ *is empty or a disjoint union of prime divisors.*

Proof. Now, $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X) = 2$ or K_X is pseudo-effective by Proposition 3.19 and Theorem 3.22. If $\hat{\boldsymbol{\rho}}(X) = 2$, then δ_f is not an eigenvalue of $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ by Lemma 3.7, and hence, $\mathcal{S}(X, f; \delta_f) = \emptyset$. If K_X is pseudo-effective, then $R_f = 0$ by Lemma 2.22, in particular, $S_f = 0$ and $\mathcal{S}(X, f; \delta_f) = \emptyset$. Hence, in any case, $\mathcal{S}(X, f^k, \delta_{f^k}) = \emptyset$ for any $k \geq 1$, and X has no negative curve by Corollary 3.4(3).

Assume that $\lambda_f = \deg f$ and $R_f \neq 0$. Then K_X is not pseudo-effective, $\boldsymbol{\rho}(X) = \hat{\boldsymbol{\rho}}(X) = 2$, and $\overline{\mathrm{NE}}(X) = \operatorname{Nef}(X)$ is generated by v_+ and v_- by Lemma 3.7(2), where $f^*v_+ = (\deg f)v_+$, $f_*v_+ = v_+$, and $\langle v_+, v_+\rangle = 0$ (cf. Proposition 3.3). We have $\langle v_+, \operatorname{cl}(R_f)\rangle = 0$ by the ramification formula $K_X = f^*K_X + R_f$ and by

$$\langle v_+, \operatorname{cl}(K_X)\rangle = \langle f_*v_+, \operatorname{cl}(K_X)\rangle = \langle v_+, \operatorname{cl}(f^*K_X)\rangle.$$

Since $\mathbb{R}_{\geq 0}v_+$ is an extremal ray of $\operatorname{Nef}(X) = \overline{\mathrm{NE}}(X)$, it contains $\operatorname{cl}(\Theta)$ for any prime component Θ of R_f, by Lemma 1.6(3). In particular, $\Theta\Theta' = 0$ for any other prime component Θ' of R_f. Hence, $\operatorname{Supp} R_f$ is a disjoint union of prime divisors. □

Proposition 3.25. *Let X be a normal Moishezon surface with a reduced divisor S such that $K_X + S$ is not pseudo-effective. Suppose that S is completely invariant under a non-isomorphic surjective endomorphism f of X satisfying $\lambda_f > \delta_f$. Then X is a projective surface with only quotient singularities, X contains no negative curve, $\boldsymbol{\rho}(X) = 2$, and there exists a fibration $\pi\colon X \to T$ to a non-singular projective curve T with an endomorphism $h\colon T \to T$ satisfying the following conditions*:

(1) *$\pi \circ f = h \circ \pi$, $\deg h \mid \deg f$, and $\lambda_f = \max\{\deg h, \deg f/\deg h\}$*;

(2) $(K_X + S)F < 0$ *and* $F_{\mathrm{red}} \simeq \mathbb{P}^1$ *for any fiber* F *of* π.

Proof. Theorem 3.22 and Propositions 3.19 and 3.24 prove the assertion except the existence of h. Now, $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ preserves each extremal ray of $\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)$ by Lemma 3.7. Thus, we have the expected endomorphism h by the last assertion of Proposition 3.19. □

4. Fibrations preserved by endomorphisms

We shall study the structure of a fibration $\pi \colon X \to T$ from a normal Moishezon surface X to a non-singular projective curve T in which X admits a non-isomorphic surjective endomorphism f and T admits an endomorphism h satisfying $\pi \circ f = h \circ \pi$. In Section 3.3, we have shown some elementary properties of X, f, and h. Especially, X is a projective surface with only quotient singularities and $\deg h \mid \deg f$ (cf. Proposition 3.17). In Section 4.1, we shall study the base change of π by a finite surjective morphism $T' \to T$ from another non-singular projective curve T' admitting an endomorphism as a lift of h (or a power h^k). In Section 4.2, we shall show some fundamental properties in the case where h is étale. In Section 4.3, we shall prove a structure theorem on $X \to T$ as Theorem 4.9 below in the case where h is an automorphism. Applying results in Sections 4.2 and 4.3, we shall determine the structure of irrational ruled surfaces admitting non-isomorphic surjective endomorphisms in Section 4.4.

4.1. **Base changes of endomorphisms preserving fibrations.**

Lemma 4.1. *Let* $\pi \colon X \to T$ *be a fibration from a normal surface* X *to a non-singular curve* T *and let* $f \colon X \to X$ *and* $h \colon T \to T$ *be finite surjective endomorphisms such that* $\pi \circ f = h \circ \pi$. *Let* $\tau \colon T' \to T$ *be a finite surjective morphism from a non-singular curve* T' *with an endomorphism* $h' \colon T' \to T'$ *such that* $\tau \circ h' = h \circ \tau$. *Then the fiber product* $X \times_T T'$ *is irreducible, and the normalization* X' *of* $X \times_T T'$ *admits a finite surjective endomorphism* $f' \colon X' \to X'$ *such that* $\nu \circ f' = f \circ \nu$ *and* $\pi' \circ f' = h' \circ \pi'$ *for the induced morphisms* $\nu \colon X' \to X$ *and* $\pi' \colon X' \to T'$:

$$\begin{array}{ccc} X' & \xrightarrow{\nu} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ T' & \xrightarrow{\tau} & T. \end{array}$$

Proof. The fiber product $X \times_T T'$ is irreducible by the flatness of π and the connectedness of a general fiber of π (cf. [45, Lem. 1.13]). The endomorphism $f \times h' \colon X \times T' \to X \times T'$ induces the expected endomorphism f' of X'. □

Lemma 4.2. *Let* $\pi \colon X \to T$ *be a fibration from a normal surface* X *to a non-singular curve* T *with a finite subset* Σ *such that*

- *the fiber* $\pi^*(t)$ *is reduced for any* $t \in T \setminus \Sigma$, *and*
- *if* $t \in \Sigma$, *then* $\pi^*(t) = m_t \pi^{-1}(t)$ *for an integer* $m_t > 1$.

Let $\tau \colon T' \to T$ *be a finite surjective morphism from a non-singular curve* T' *and let* X' *be the normalization of* $X \times_T T'$. *Then the following three conditions are equivalent*:

(i) *The induced finite morphism* $\nu\colon X' \to X$ *is étale in codimension* 1 *and the fibration* $\pi'\colon X' \to T'$ *has only reduced fibers.*

(ii) *The morphism* τ *is étale over* $T \setminus \Sigma$ *and* $\tau^*(t) = m_t\tau^{-1}(t)$ *for any* $t \in \Sigma$.

(iii) *One has an equality*

$$K_{T'} = \tau^*(K_T + \sum\nolimits_{t\in\Sigma}(1 - m_t^{-1})t). \tag{IV-1}$$

Moreover, if one of these conditions is satisfied, then the Galois closure $\tau''\colon T'' \to T$ *of* τ *also satisfies the same condition.*

Proof. The equivalence (ii) $\Leftrightarrow$ (iii) is shown directly by the ramification formula for τ. For the Galois closure τ'' and the induced morphism $T'' \to T'$, the normalization X'' of $X \times_T T''$ is also the normalization of $X' \times_{T'} T''$, and the induced morphism $\nu''\colon X'' \to X$ is the Galois closure of $\nu\colon X' \to X$. Thus, if (i) holds, then every fiber of the induced fibration $X'' \to T''$ is reduced and ν'' is étale in codimension 1, i.e., τ'' also satisfies (i). Thus, it is enough to show the equivalence (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (ii): Let U be the maximal open subset of X such that $\pi|_U\colon U \to T$ is smooth. Then U is the complement of $\Xi \cup \pi^{-1}\Sigma$ in X for a discrete set Ξ by the assumption on Σ. Since ν is étale in codimension 1, $\nu^{-1}U$ is étale over U and the composite $\nu^{-1}(U) \to U \to T$ is smooth. It implies that $T' \setminus \tau^{-1}\Sigma$ is étale over $T \setminus \Sigma$. For a point $t \in T$, we set $\Gamma_t := \pi^{-1}(t)$. Then $\nu^*\Gamma_t$ is reduced as ν is étale in codimension 1. Thus, $\nu^*\Gamma_t = \pi'^*(\tau^{-1}(t))$, and $\tau^*(t) = m_t\tau^{-1}(t)$ for any $t \in \Sigma$ by

$$\pi'^*\tau^*(t) = \nu^*\pi^*(t) = m_t\nu^*\Gamma_t = m_t\pi'^*(\tau^{-1}(t)).$$

(ii) $\Rightarrow$ (i): The morphism $\nu^{-1}U \to T' \setminus \tau^{-1}\Sigma$ induced by π' is again smooth, since it is the base change of $U \to T \setminus \Sigma$. Thus, the fiber of π' over any point of $T' \setminus \tau^{-1}\Sigma$ is reduced. For a point $t \in \Sigma$, let us take a point $x \in \Gamma_t = \pi^{-1}(t)$ at which X and Γ_t are non-singular. For any $t' \in \tau^{-1}(t)$, the local ring of $X \times_T T'$ at the point (x, t') is isomorphic to $\mathbb{C}\{\mathtt{x}, \mathtt{y}\}/(\mathtt{x}^{m_t} - \mathtt{y}^{m_t})$ by (ii). Hence, the morphism $\nu\colon X' \to X$ is étale along $\nu^{-1}(x)$, and $\pi'\colon X' \to T'$ is smooth along $\nu^{-1}(x)$. Thus, π' has only reduced fibers and ν is étale in codimension 1. $\square$

An affirmative answer to Fenchel's conjecture (cf. [5], [14], [6], [48]) is applied in the proof of the following:

Proposition 4.3. *In the situation of Lemma* 4.2, *assume that* X *and* T *are projective and that there exist endomorphisms* $f\colon X \to X$ *and* $h\colon T \to T$ *satisfying*:

(i) $\pi \circ f = h \circ \pi$;

(ii) *every prime component of the* refined ramification divisor Δ_f *(cf. Definition* 2.16*) dominates* T.

Then there is a finite Galois cover $\tau\colon T' \to T$ *satisfying Lemma* 4.2(ii) *with an endomorphism* $h'\colon T' \to T'$ *such that* $\tau \circ h' = h^k \circ \tau$ *for some* $k > 0$, *except the following two cases*:

(1) $\deg h = 1$, $T \simeq \mathbb{P}^1$, *and* $\sharp\Sigma = 1$;

(2) $\deg h = 1$, $T \simeq \mathbb{P}^1$, $\sharp\Sigma = 2$, *and* $m_{t_1} \neq m_{t_2}$ *for* $\Sigma = \{t_1, t_2\}$.

If $\deg h > 1$ *and* $\Sigma \neq \emptyset$*, then* $2 \leq \sharp\Sigma \leq 4$ *and one can take* k *as* 1*. If the following condition* (ii′) *stronger than* (ii) *is satisfied in addition, then* $\sharp\Sigma \geq 3$ *and* τ *can be taken as a cyclic cover from an elliptic curve* T':

(ii′) *Every prime component of the ramification divisor* R_f *of* f *dominates* T.

Proof. If $\Sigma = \emptyset$, then we can take $\tau\colon T' \to T$ as the identity morphism of T, where h' is given as h. Hence, we may assume that $\Sigma \neq \emptyset$.

If $\deg h = 1$ and if any of cases (1) and (2) does not occur, then, by an affirmative answer to Fenchel's conjecture, there exists a finite cover $\tau\colon T' \to T$ satisfying Lemma 4.2(ii) (cf. [48, Thm. 1.2.15]). In this case, we can find an expected automorphism $h'\colon T' \to T'$ by Lemma 2.7.

Thus, we may assume that $\deg h > 1$. By (ii) and by Lemma 2.17(3), every prime component of Δ_{f^l} dominates T and $S_{h^l} = S_h$ for any $l > 0$. Thus, $\pi^{-1}S_h$ is the union of prime components of R_{f^l} not dominating T for some $l > 0$ by Lemma 2.19(2), since we can choose l so that $\operatorname{Supp} R_f = S_f \cup \operatorname{Supp} \Delta_{f^l}$ (cf. Lemma 2.17(4)). In particular, if (ii′) holds, then $S_h = 0$. For $t \in T$, let d_t be the ramification index of h at t, i.e., $d_t = \operatorname{mult}_t h^*(h(t)) = \operatorname{mult}_t R_h + 1$, and let m_t be the positive integer defined by $\pi^*(t) = m_t\pi^{-1}(t)$. Then m_t is the same number as in Lemma 4.2 for $t \in \Sigma$, and $m_t = 1$ for any $t \notin \Sigma$. By the description of $\pi^{-1}S_h$ above and by $\pi^*h^*(h(t)) = f^*\pi^*(h(t))$ (cf. (i)), we see that $f^*(\pi^{-1}(t))$ is reduced and $d_t m_t = m_{h(t)}$ for any $t \in T \setminus S_h$. Then $2 \leq \sharp\Sigma \leq 4$, and $\sharp\Sigma \geq 3$ when $S_h = 0$, by Proposition 2.9(2). Moreover, we have an expected finite Galois cover $\tau\colon T' \to T$ with an endomorphism $h'\colon T' \to T'$ by (3) and (4) in Proposition 2.9 except the case where

(2′) $T \simeq \mathbb{P}^1$, $\deg S_h = 2$, $S_h = \Sigma$, and $m_{t_1} \neq m_{t_2}$ for $\{t_1, t_2\} = \Sigma$.

It is enough to derive a contradiction assuming (2′). We assume that $m_{t_1} < m_{t_2}$ and let $\theta\colon \widehat{T} \to T$ be the cyclic cover of degree m_{t_1} branched at Σ. For $i = 1, 2$, let $\hat{t}_i \in \widehat{T}$ be the point lying over t_i. Since h is a cyclic cover branched at Σ, there is an endomorphism $\hat{h}\colon \widehat{T} \to \widehat{T}$ such that $\theta \circ \hat{h} = h \circ \theta$ and $S_{\hat{h}} = \{\hat{t}_1, \hat{t}_2\} = \theta^{-1}\Sigma$. Let $\widehat{X}$ be the normalization of $X \times_T \widehat{T}$ and let $\hat{\pi}\colon \widehat{X} \to \widehat{T}$ and $\mu\colon \widehat{X} \to X$ be induced morphisms:

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\mu} & X \\ {\scriptstyle\hat{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widehat{T} & \xrightarrow{\theta} & T. \end{array}$$

By an argument in the proof of Lemma 4.2 showing (ii) $\Rightarrow$ (i), we see that $\operatorname{Supp} R_\mu \subset \hat{\pi}^{-1}(\hat{t}_2)$, the fiber $\hat{\pi}^*(\hat{t}_1)$ is reduced, but

$$\hat{\pi}^*(\hat{t}_2) = \frac{m_{t_2}}{\gcd(m_{t_1}, m_{t_2})}\hat{\pi}^{-1}(\hat{t}_2).$$

Since $m_{t_1} < m_{t_2}$, the assumption of Lemma 4.2 for π and Σ is also satisfied for $\hat{\pi}$ and $\widehat{\Sigma} := \{\hat{t}_2\}$. By Lemma 4.1, there is an endomorphism $\hat{f}\colon \widehat{X} \to \widehat{X}$ such that $\hat{\pi} \circ \hat{f} = \hat{h} \circ \hat{\pi}$ and $\mu \circ \hat{f} = f \circ \mu$. Then $\hat{\pi}^{-1}S_{\hat{h}} \subset S_{\hat{f}}$ and $\mu^{-1}S_f = S_{\hat{f}}$ by Lemma 2.19(2), and we have an equality

$$\mu^*R_f + R_\mu = \hat{f}^*R_\mu + R_{\hat{f}} \qquad \text{(IV-2)}$$

for ramification divisors R_μ and $R_{\hat{f}}$ of μ and $\hat{f}$, respectively. Here,

$$\operatorname{Supp} R_\mu \subset \hat{\pi}^{-1}(\hat{t}_2) \subset \hat{\pi}^{-1} S_{\hat{h}} \subset S_{\hat{f}}.$$

Let Θ be a prime component of $\Delta_{\hat{f}}$. Then $\Theta \subset R_{\hat{f}}$ and $\Theta \not\subset S_{\hat{f}}$ by Lemma 2.17(4). Hence, $\Theta \subset \mu^{-1} \operatorname{Supp} \Delta_f$ by (IV-2), since $\mu^{-1} S_f = S_{\hat{f}}$ and $\operatorname{Supp} R_f \subset S_f \cup \operatorname{Supp} \Delta_f$ (cf. Lemma 2.17(4)). Therefore, $\mu(\Theta)$ is a prime component of Δ_f, and $\hat{\pi}(\Theta) = \widehat{T}$ by (ii). Thus, $(\hat{f}, \hat{h}, \hat{\pi})$ also satisfies (i) and (ii) instead of (f, h, π). However, we have $\sharp\widehat{\Sigma} \geq 2$ by Proposition 2.9(2). This is a contradiction. Hence, (2′) does not occur, and we are done. □

4.2. **Endomorphisms inducing étale endomorphisms of base curves.** In Section 4.2, we fix

- a normal Moishezon surface X with a non-isomorphic surjective endomorphism f,
- a fibration $\pi\colon X \to T$ to a non-singular projective curve T, and
- an *étale* endomorphism $h\colon T \to T$ such that $\pi \circ f = h \circ \pi$.

Note that X is a projective surface with only quotient singularities and that $\deg h$ is an eigenvalue of $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ satisfying $\deg h \mid \deg f$ (cf. Proposition 3.17).

Lemma 4.4. *In the situation, the following hold, where F is a general fiber of π:*

1. *Every prime component of S_f dominates T, and S_f is non-singular.*
2. *Every fiber of π is irreducible.*
3. *If $\deg h > 1$, then T is an elliptic curve. If $\deg f = \deg h$, then f is étale, and π is smooth: In particular, $\operatorname{Sing} X = \emptyset$ and $R_f = S_f = 0$.*
4. *If $\deg f \neq \deg h$, then $\deg f > \deg h$, and F is a rational or elliptic curve: In more detail, if $R_f F > 0$ (resp. $= 0$), then F is rational (resp. elliptic).*
5. *If $S_f \neq 0$, then $\deg f > \deg h$, $f^* S_f = (\deg f / \deg h) S_f$, $1 \leq S_f F \leq 2$, and F is rational.*
6. *If $\deg h = 1$ and $S_f \neq 0$, then $S_f \cap \operatorname{Supp} \Delta_f = \emptyset$. Moreover, in this case, $\Delta_f = 0$ if and only if $S_f F = 2$.*

Proof. (1): Since $S_h = 0$, this follows from Lemma 2.19(2) and Proposition 3.17(4).

(2): A prime component of a reducible fiber is a negative curve, which is a prime component of S_f by Proposition 2.20(3). Thus, (2) is a consequence of (1).

(3): The first assertion is trivial, since h is étale. By Lemma 2.17(4), it is enough to prove that f is étale and π is smooth in the case where $\deg f = \deg h$. The fiber product $X^h = X \times_{T,h} T$ is a normal variety, since the second projection $X^h \to T$ has only connected fibers and the first projection $p_1\colon X^h \to X$ is a finite étale morphism. There is a morphism $q\colon X \to X^h$ such that $f = p_1 \circ q$. Since $\deg p_1 = \deg h = \deg f$, q is an isomorphism, and hence, $f\colon X \to X$ is étale. The smoothness of π follows if the scheme-theoretic fiber $F_t = \pi^*(t)$ is non-singular for any $t \in T$. Let Λ be the set of points $t \in T$ such that F_t is singular (including the case where F_t is non-reduced). Since f and h are étale, we have $h^{-1}\Lambda \subset \Lambda$. Thus, $h^{-1}\Lambda = \Lambda$ by Lemma 2.2, which implies that $\Lambda = \emptyset$ as $\deg h > 1$. Therefore, π is smooth.

(4): We set $b := \deg f / \deg h$, which is an integer > 1 by Proposition 3.17(1). Then $f^*F \approxeq (\deg h)F$, and we have $f_*F \approxeq bF$ by $f_*(f^*F) = (\deg f)F$. Thus,

$$0 \le R_f F = -(b-1)K_X F$$

by $K_X = f^*K_X + R_f$. In particular, $2\boldsymbol{g}(F) - 2 = K_X F \le 0$. If $\pi(\operatorname{Supp} R_f) = T$ (resp. $\ne T$), or equivalently, if $R_f F > 0$ (resp. $= 0$), then $K_X F = 2\boldsymbol{g}(F) - 2 < 0$ (resp. $= 0$); thus, F is a rational (resp. elliptic) curve.

(5): Assume that $S_f \ne 0$. Then $S_f F \ge 1$ by (1), and $\deg f > \deg h$ and $F \simeq \mathbb{P}^1$ by (3) and (4). For $b = \deg f / \deg h > 1$, we have

$$0 \le \Delta_f F = -(b-1)(K_X + S_f)F \tag{IV-3}$$

as in Proposition 2.20(5). Thus, $S_f F \le 2$. We shall show $f^*S_f = bS_f$. If S_f is irreducible, this holds by (1) and by $(f^*S_f)F = S_f(f_*F) = bS_f F$. If S_f is reducible, then $S_f = C_1 + C_2$ for two sections C_1 and C_2 of π by (1) and by $S_f F \le 2$. If $f^{-1}C_i = C_j$ for some i, $j \in \{1, 2\}$, then $f^*C_i = bC_j$ by $(f^*C_i)F = C_i(f_*F) = bC_i F = b$. Thus, $f^*S_f = bS_f$ even if S_f is reducible.

(6): Since $\deg h = 1$ is an eigenvalue of $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$, we have $\lambda_f = \deg f$ and $S_f \cap \operatorname{Supp} \Delta_f = \emptyset$ by (4) and (7) of Proposition 3.3. Moreover, $\Delta_f F = 0$ if and only if $S_f F = 2$ by (IV-3). If $\Delta_f \ne 0$, then every prime component of Δ_f dominates T by $S_f \cap \operatorname{Supp} \Delta_f = \emptyset$ and by (1) and (2), and it implies that $\Delta_f F > 0$. Thus, we are done. □

Definition 4.5. For a point $t \in T$, we set F_t to be the scheme-theoretic fiber $\pi^*(t)$. Then F_t is irreducible for any t by Lemma 4.4(2). We set $\Gamma_t := (F_t)_{\mathrm{red}} = \pi^{-1}(t)$ and set m_t to be the multiplicity of the fiber F_t, i.e., $F_t = m_t \Gamma_t$. We define $\Sigma = \Sigma(\pi)$ as the set of points $t \in T$ such that $m_t > 1$.

Lemma 4.6. *The following hold for $\Sigma = \Sigma(\pi)$ and the ramification divisor R_f:*

(1) *The inverse image $h^{-1}\Sigma$ equals Σ, and $h|_\Sigma \colon \Sigma \to \Sigma$ is bijective. In particular, if $\Sigma \ne \emptyset$, then h is an automorphism.*
(2) *If $2\boldsymbol{g}(T) - 2 + \sharp\Sigma > 0$, then h is an automorphism of finite order.*
(3) *Every prime component of R_f dominates T. In particular, if f is not étale in codimension 1, then a general fiber of π is a rational curve.*
(4) *If $\deg f > \deg h$, then $f(\operatorname{Supp} R_f)$ is not a section of π.*
(5) *If a general fiber of π is a rational curve, then $\boldsymbol{\rho}(X) = 2$, $\Gamma_t \simeq \mathbb{P}^1$ for any $t \in T$, and the restriction $X \setminus \pi^{-1}\Sigma \to T \setminus \Sigma$ of π is a $\mathbb{P}^1$-bundle.*
(6) *The restriction $\pi|_{S_f} \colon S_f \to T$ of π is étale over $T \setminus \Sigma$.*
(7) *If a general fiber of π is an elliptic curve, then f is étale in codimension 1 and π is smooth over $T \setminus \Sigma$. Moreover, Γ_t is a non-singular rational curve or an elliptic curve for any $t \in \Sigma$.*

Proof. (1): Let $\pi^h \colon X^h = X \times_{T,h} T \to T$ be the base change of π by h. As in the proof of Lemma 4.4(3), we have a finite surjective morphism $q \colon X \to X^h$ such that $\pi = \pi^h \circ q$ and $f = p_1 \circ q$ for the first projection $p_1 \colon X^h \to X$. The scheme-theoretic fiber F_t^h of π^h over a point t is isomorphic to $F_{h(t)}$, and $q^*(F_t^h) = F_t$. Hence, if F_t is reduced, then so is $F_{h(t)}$. Thus, $h^{-1}\Sigma \subset \Sigma$, which implies that $h^{-1}\Sigma = \Sigma$ and that $h|_\Sigma \colon \Sigma \to \Sigma$ is bijective by Lemma 2.2. If $\Sigma \neq \emptyset$, then h is an automorphism, since h is étale.

(2): This follows from (1) and Lemma 2.6.

(3): For a point $t \in T$, let $r_t(f)$ be the ramification index of f along Γ_t, which is equal to the multiplicity of $f^*(\Gamma_{h(t)})$ along Γ_t. Since h is étale, $f^*(F_{h(t)})$ is the disjoint union of $F_{t'}$ for all $t' \in h^{-1}(h(t))$. Hence, $m_t = r_t(f)m_{h(t)}$. Assume that $r_t(f) > 1$ for some $t \in T$. Then $t \in \Sigma$, and hence, $\deg h = 1$ and $h^k(t) = t$ for some $k > 0$ by (1). Here, we have $r_t(f^k) = 1$ by $m_t = r_t(f^k)m_t$, but $r_t(f^k) \geq r_t(f)$ by $R_{f^k} \geq R_f$. This is a contradiction. Therefore, $r_t(f) = 1$ for any $t \in T$. This means that, if $R_f \neq 0$, then every prime component of R_f dominates T. The last assertion follows from (3) and (4) of Lemma 4.4, since $R_f = 0$ if and only if f is étale in codimension 1.

(4): Assume that $f(\operatorname{Supp} R_f)$ is a section of π. Note that f is étale over $X_{\mathrm{reg}} \setminus f(\operatorname{Supp} R_f)$. We can take a general point $t \in T$ so that F_t and $F_{h(t)}$ are both smooth fibers of π. Then $f|_{F_t} \colon F_t \to F_{h(t)}$ is étale over $F_{h(t)} \setminus f(\operatorname{Supp} R_f) \simeq \mathbb{C}$, but $\deg(f|_{F_t}) = \deg f / \deg h > 1$. This is a contradiction.

(5): This follows from Lemma 4.4(2) and from (4) and (6) of [44, Prop. 2.33].

(6): We may assume that $S_f \neq 0$. Then a general fiber of π is rational by Lemma 4.4(5), and $X \setminus \pi^{-1}\Sigma \to T \setminus \Sigma$ is a $\mathbb{P}^1$-bundle by (5). Furthermore, S_f is non-singular and each prime component dominates T by Lemma 4.4(1). Let Ξ be the set of points $t \in T \setminus \Sigma$ such that $S_f \to T$ is not étale over t. If $\Xi \neq \emptyset$, then S_f is irreducible and $\deg(S_f/T) = 2$ by Lemma 4.4(5). Moreover, if $t \in \Xi$, then the smooth fiber F_t intersect S_f tangentially at one point, and hence, $(X, S_f + F_t)$ is not log-canonical contradicting Proposition 3.17(6). Therefore, $\Xi = \emptyset$.

(7): In this case, f is étale in codimension 1 by (3). First, we shall prove the last assertion of (7) assuming that $\Sigma \neq \emptyset$. Then h is an automorphism and $h^{-1}\Sigma = \Sigma$ by (1). Hence, we have $f^*D = D$ for the reduced divisor $D = \sum_{t \in \Sigma} \Gamma_t$. Thus, $D = D^{(\star)}$ in Definition 3.9, and hence, any prime component of D is $\mathbb{P}^1$ or an elliptic curve by Lemma 3.11 and Proposition 3.13. This proves the last assertion.

It remains to prove that π is smooth over $T \setminus \Sigma$, i.e., $\Lambda = \emptyset$ for the set Λ of points $t \in T \setminus \Sigma$ such that F_t is singular. We shall show that $h^{-1}\Lambda \subset \Lambda$. For a point $t \in T \setminus (\Sigma \cup \Lambda)$, let us take a connected open neighborhood $\mathcal{U}$ of t in T such that $\mathcal{U} \cap h^{-1}(h(t)) = \{t\}$. Then $\mathcal{W} := \pi^{-1}\mathcal{U}$ is a connected open neighborhood of the smooth fiber F_t. Here, $F_{h(t)}$ is reduced by $h(t) \notin \Sigma = h^{-1}\Sigma$, and $F_t = f^*F_{h(t)}|_{\mathcal{W}}$ by $R_f = 0$. By [45, Lem. 1.39] applied to the non-degenerate morphism $f|_{\mathcal{W}} \colon \mathcal{W} \to X$, we have an effective divisor $\mathcal{D}$ on $\mathcal{W}$ such that

$$K_{\mathcal{W}} + F_t = (f|_{\mathcal{W}})^*(K_X + F_{h(t)}) + \mathcal{D}.$$

Since (X, F_t) is 1-log-terminal along F_t, $(X, F_{h(t)})$ is 1-log-terminal along $F_{h(t)}$ by [45, Prop. 2.12(2)]. In particular, $F_{h(t)}$ is non-singular (cf. [45, Fact 2.5]), i.e., $h(t) \notin \Lambda$. Hence, $h^{-1}\Lambda \subset \Lambda$ by $h^{-1}\Sigma = \Sigma$. Then $h^{-1}\Lambda = \Lambda$ and $h|_\Lambda \colon \Lambda \to \Lambda$ is bijective by Lemma 2.2. If $\Lambda \neq \emptyset$, then the étale morphism h is an automorphism and the non-zero reduced divisor $S := \sum_{t \in \Lambda} F_t$ satisfies $f^*S = S$, i.e., $S = S^{(\star)}$ in Definition 3.9. Then S is non-singular by Proposition 3.13, and it contradicts the definition of Λ. Therefore, $\Lambda = \emptyset$, and π is smooth over $T \setminus \Sigma$. Thus, we are done. □

Corollary 4.7. *If $\deg h = 1$, then $\pi|_{X\setminus\pi^{-1}\Sigma} \colon X \setminus \pi^{-1}\Sigma \to T \setminus \Sigma$ is a $\mathbb{P}^1$-bundle or a smooth elliptic fibration. If $\deg h > 1$, then T is an elliptic curve and π is a smooth fibration. In both cases, $S_f \cap \operatorname{Supp} \Delta_f = \emptyset$.*

Proof. Let F be a general fiber of π. First, assume that $\deg h = 1$. Then F is rational or elliptic by Lemma 4.4(4). If F is rational (resp. elliptic), then $X \setminus \pi^{-1}\Sigma \to T \setminus \Sigma$ is a $\mathbb{P}^1$-bundle (resp. smooth elliptic fibration) by (5) (resp. (7)) of Lemma 4.6. Moreover, $S_f \cap \operatorname{Supp} \Delta_f = \emptyset$ by Lemma 4.4(6).

Next, assume that $\deg h > 1$. Then T is elliptic and $\Sigma = \emptyset$ by Lemma 4.6(1). If $\deg f = \deg h$, then π is smooth and $S_f = \Delta_f = 0$ by Lemma 4.4(3) (cf. Lemma 2.17(4)). Thus, we may assume that $\deg f \neq \deg h$. Then $\deg f > \deg h$, and F of π is rational or elliptic by Lemma 4.4(4). In both cases, π is smooth by $\Sigma = \emptyset$ and by (5) and (7) of Lemma 4.6. If $S_f \neq 0$, then F is rational and $f^*S_f = (\deg f/\deg h)S_f$ by Lemma 4.4(5): thus, $S_f = S_f^{(\dagger)}$ in Definition 3.9, and we have $S_f \cap \operatorname{Supp} \Delta_f = \emptyset$ by Lemma 3.11(2). □

Corollary 4.8. *If X contains a negative curve C, then $\deg h = \delta_f$, T is an elliptic curve, $\pi \colon X \to T$ is a $\mathbb{P}^1$-bundle, and C is a unique negative section of π.*

Proof. The existence of negative curve implies $\lambda_f = \delta_f$ by Proposition 3.24. Then δ_f is a unique positive eigenvalue of $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ by Proposition 3.3(4). Therefore, $\delta_f = \deg h$ by Proposition 3.17(2). In particular, $\deg h > 1$. Thus, T is an elliptic curve, and π is a $\mathbb{P}^1$-bundle by Lemma 4.4(5) and Corollary 4.7, since $C \leq S_f$. By Lemma 1.14, C is a unique negative section of π. □

4.3. Endomorphisms inducing automorphisms of base curves.

Theorem 4.9. *Let f be a non-isomorphic surjective endomorphism of a normal Moishezon surface X. Let $\pi \colon X \to T$ be a fibration to a non-singular projective curve T with an* automorphism *h satisfying $\pi \circ f = h \circ \pi$. Then there exists a finite Galois cover $\tau \colon T' \to T$ from a non-singular projective curve T' satisfying the following conditions* (1)–(4) *for the normalization X' of $X \times_T T'$ and for induced morphisms $\pi' \colon X' \to T'$ and $\nu \colon X' \to X$ making a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\nu} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ T' & \xrightarrow{\tau} & T : \end{array}$$

1. *There is an isomorphism $X' \simeq C \times T'$ over T' for a non-singular projective curve C. If $R_f \neq 0$* (*resp.* $= 0$), *then C is rational* (*resp. elliptic*).
2. *The induced Galois cover $\nu \colon X' \to X$ is étale in codimension* 1.
3. *There exist an automorphism h' of T', a non-isomorphic surjective endomorphism f' of X', and a positive integer k such that $\nu \circ f' = f^k \circ \nu$, $\tau \circ h' = h^k \circ \tau$, and $\pi' \circ f' = h' \circ \pi'$.*

(4) *Assume that* C *is rational. Then there is an endomorphism* g *of* C *such that the endomorphism* f' *in* (3) *corresponds to* $g \times h' \colon C \times T' \to C \times T'$ *by the isomorphism* $X' \simeq C \times T'$ *in* (1)*. In particular,* $\deg g = \deg f$*. Moreover,* $\nu^* S_f = S_{f'} = p_C^*(S_g)$ *for the first projection* $p_C \colon X' \simeq C \times T' \to C$*, and* $\deg S_g \le 2$.

Remark 4.10. If $R_f \neq 0$ (resp. $= 0$), then a general fiber of π is rational (resp. elliptic) by Lemma 4.6(3) (resp. Lemma 4.4(4)). Thus, the latter half of Theorem 4.9(1) follows from the first half.

After showing preliminary results, we shall prove Theorem 4.9 at the end of Section 4.3. For the discussion below, we apply results in Sections 4.1 and 4.2 and the theory of elliptic surfaces by Kodaira ([31], [32]). We also use the same notation as in Section 4.2 (e.g. Σ, m_t, etc.).

Proposition 4.11. *Assume that* $\Sigma = \emptyset$ *(cf. Definition* 4.5*). Then there exists a finite étale cover* $\tau \colon T' \to T$ *such that* $X \times_T T' \simeq C \times T'$ *over* T' *for a non-singular projective curve* C *which is rational or elliptic. If* C *is elliptic, then* f *is étale. If* C *is rational, then the inverse image of* $\operatorname{Supp} R_f$ *by the projection* $C \times T' \simeq X \times_T T' \to X$ *is a union of fibers of the projection* $C \times T' \to C$.

Proof. By assumption and by Corollary 4.7, $\pi \colon X \to T$ is a $\mathbb{P}^1$-bundle or a smooth elliptic fibration. In the latter case, $R_f = 0$ by Lemma 4.6(3), and the assertion follows from Lemma 4.12 below, which is well known in the theory of elliptic surfaces. Thus, we may assume that π is a $\mathbb{P}^1$-bundle. In the proof below, we use arguments similar to those in the proof of [40, Thm. 15].

Note that X contains no negative curve by Corollary 4.8. By [35, Thm. 3.1], $-K_{X/T}$ is nef with $(-K_{X/T})^2 = 0$ and $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\ge 0}\,\mathrm{cl}(F) + \mathsf{R}$ for a fiber F of π and for the ray $\mathsf{R} := \mathbb{R}_{\ge 0}\,\mathrm{cl}(-K_{X/T})$ (cf. Section 1.4). Since h is an automorphism, we have $f^*F \approx F$, $K_T = h^*K_T$, and $f^*K_{X/T} \approx (\deg f)K_{X/T}$. In particular, $R_f = K_{X/T} - f^*K_{X/T} \approx (1 - \deg f)K_{X/T}$, and $0 \neq \mathrm{cl}(R_f) \in \mathsf{R}$. We set $D := f^*(f_*(R_f))_{\mathrm{red}}$. Since $f^*\mathsf{R} = \mathsf{R}$, by Lemma 1.15, we see that $\mathrm{cl}(D) \in \mathsf{R}$, D is non-singular, and $\pi|_D \colon D \to T$ is étale; moreover, if $\deg D/T = \deg \pi|_D \ge 3$, then $X \times_T T' \simeq \mathbb{P}^1 \times T'$ for a finite étale cover $\tau \colon T' \to T$, where $D \times_T T'$ is a union of fibers of the projection $X \times_T T' \simeq \mathbb{P}^1 \times T' \to \mathbb{P}^1$. On the other hand, we have $\deg D/T \ge \deg f(\operatorname{Supp} R_f)/T \ge 2$ by Lemma 4.6(4). Hence, we may assume that $\deg D/T = 2$.

Then $D = (R_f)_{\mathrm{red}}$ and $f^{-1}D = D$. If D is irreducible, then the endomorphism $f|_D \colon D \to D$ is compatible with h, i.e., $(\pi|_D) \circ f|_D = h \circ \pi|_D$, and hence, $X_D := X \times_T D$ has an endomorphism compatible with f and $f|_D$ for the projections $X_D \to X$ and $X_D \to D$, by Lemma 4.1. Thus, by replacing $X \to T$ with $X_D \to D$, we may assume that D is reducible. Then D is a disjoint union of two sections Θ_1, Θ_2 of π. By replacing f with f^2, we may assume that $f^{-1}\Theta_i = \Theta_i$ for $i = 1$, 2. Then $\Theta_i^2 = 0$ and $f^*\Theta_i = (\deg f)\Theta_i$ for $i = 1$, 2, since $\mathrm{cl}(\Theta_i) \in \mathsf{R}$. Now, $X \simeq \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{O}_T(L))$ for a divisor L on T such that $\pi^*L \sim \Theta_1 - \Theta_2$. Here, $\deg L = 0$ by $(\Theta_1 - \Theta_2)\Theta_1 = 0$. We have $\pi^*(h^*L) = f^*(\pi^*L) \sim (\deg f)\pi^*L$ by $f^*\Theta_i = (\deg f)\Theta_i$. It implies that $h^*L \sim (\deg f)L$.

It suffices to prove that $L \sim_{\mathbb{Q}} 0$. If $\boldsymbol{g}(T) = 0$, then $L \sim 0$ by $\deg L = 0$. If $\boldsymbol{g}(T) \geq 2$, then $h^k = \mathrm{id}_T$ for some $k > 0$, and $L \sim_{\mathbb{Q}} 0$ by $L = (h^k)^* L \sim (\deg f)^k L$. If $\boldsymbol{g}(T) = 1$, then some power h^k is the translation morphism $\mathrm{tr}(a)\colon z \mapsto z + a$ by some $a \in T$ with respect to a group structure of T. In this case, $(h^k)^* L \sim L$ by $\deg L = 0$, and we have $L \sim_{\mathbb{Q}} 0$ by $L \sim (h^k)^* L \sim (\deg f)^k L$. Thus, we are done. □

In the proof of Proposition 4.11, we use Lemma 4.12 below, for which several proofs are known. We shall give a proof based on Kodaira's theory of elliptic surfaces in [31] (see [39, §1] for a sheaf theoretic argument).

Lemma 4.12. *Let $\varphi\colon \mathcal{Y} \to T$ be a smooth elliptic fibration over a non-singular projective curve T. Then there exists a finite étale cover $T' \to T$ such that $\mathcal{Y} \times_T T' \simeq C \times T'$ over T' for an elliptic curve C.*

Proof. For a point $t \in T$, let

$$\rho_{(T,t)}\colon \pi_1(T, t) \to \mathrm{Aut}(H_1(\varphi^{-1}(t), \mathbb{Z}))$$

be the monodromy representation associated with φ. First, we shall show that the monodromy representation $\rho_{(T',t')}$ associated with the base change $\mathcal{Y} \times_T T' \to T'$ is trivial for a finite étale cover $T' \to T$ and a point $t' \in T'$. The J-function associated with the elliptic fibration is constant, since it is a holomorphic map $T \to \mathbb{C}$ from the compact variety T. Hence, any fiber of π is isomorphic to a fixed elliptic curve C. For the period $z \in \mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$ of C, i.e., $C \simeq \mathbb{C}/(\mathbb{Z}z + \mathbb{Z})$, the monodromy group $\rho(\pi_1(T, t))$ is contained in the stabilizer group of z in $\mathrm{SL}(2, \mathbb{Z})$. Thus, the monodromy group is finite. Let $T' \to T$ be the finite étale cover with a point t' lying over t such that $\pi_1(T', t')$ corresponds to the kernel of $\rho_{(T,t)}$. Then $\rho_{(T',t')}$ is trivial. Therefore, by replacing T with T', we may assume that the monodromy representation $\rho_{(T,t)}$ is trivial.

Second, we apply Kodaira's theory of elliptic surfaces in [31]. The smooth elliptic fibration $\mathcal{Y} \to T$ is expressed as the twist $\mathcal{B}^\eta \to T$ of a *basic* smooth elliptic fibration $\mathcal{B} \to T$ by an element η of the cohomology group $H^1(T, \mathcal{O}_T(\mathcal{B}))$. Here, the basic smooth elliptic fibration $\mathcal{B} \to T$ is characterized by properties that it is a smooth elliptic fibration admitting a global section and that it has the same data of the period map and the monodromy representation as those of $\mathcal{Y} \to T$. Furthermore, $\mathcal{B} \to T$ has a structure of relative Lie group, and $\mathcal{O}_T(\mathcal{B})$ is the sheaf of germs of sections of $\mathcal{B} \to T$. Now, the period map is constant and the monodromy representation is trivial. Hence, $\mathcal{B} \simeq C \times T$ over T. In particular, if $\mathcal{Y} \to T$ admits a global section, then $\mathcal{Y} \simeq C \times T$. Thus, we assume that $\mathcal{Y} \to T$ has no global section. Then $\mathcal{Y} \simeq (C \times T)^\eta$ with $\eta \neq 0$. By [31, Thm. 11.5], η is a torsion element, since $\mathcal{Y}$ is projective. Let m be the order of η. Then we have a finite étale morphism

$$\psi\colon \mathcal{Y} \simeq (C \times T)^\eta \to (C \times T)^{m\eta} \simeq C \times T$$

over T by gluing multiplication maps $C \times T_\lambda \ni (\zeta, t) \to (m\zeta, t) \in C \times T_\lambda$ for an open covering $T = \bigcup T_\lambda$. Let $T' \subset \mathcal{Y}$ be a connected component of $\psi^{-1}(\{P\} \times T)$ for a point $P \in C$. Then T' is étale over T, and $\mathcal{Y} \times_T T' \to T'$ admits a global section. Hence, $\mathcal{Y} \times_T T' \simeq C \times T'$ over T', and we are done. □

We have the following reduction for the proof of Theorem 4.9:

Lemma 4.13. *Suppose that there is a finite surjective morphism $T' \to T$ satisfying Theorem* 4.9(2) *and the following condition* (1′) *weaker than the first half of Theorem* 4.9(1):

(1′) *The induced morphism $\pi' \colon X' \to T'$ is smooth.*

Then there exists a finite Galois cover $T'' \to T$ satisfying all the conditions (1)–(4) *in Theorem* 4.9.

Proof. By assumption, $T' \to T$ satisfies Lemma 4.2(i). We may assume that $T' \to T$ is Galois by Lemma 4.2. It also satisfies Theorem 4.9(3) by Lemmas 2.7 and 4.1. We can apply Proposition 4.11 to $X' \to T'$ and f' by (1′) and by Theorem 4.9(3). Then, by replacing T' by a further étale cover, we may assume that $T' \to T$ satisfies the first half of Theorem 4.9(1), which is stronger than (1′). The latter half of Theorem 4.9(1) is satisfied by Remark 4.10.

The first half of Theorem 4.9(4) follows from Lemma 3.18. It remains to show the last half of Theorem 4.9(4). Since ν is étale in codimension 1 (cf. Theorem 4.9(2)), by Lemmas 2.17(3) and 2.19(3), we have $\nu^* S_f = \nu^* S_{f^k} = \nu^{-1} S_{f^k} = S_{f'}$ for the endomorphism f' and the integer k in Theorem 4.9(3). Moreover, $S_{f'} = p_C^*(S_g)$ by Lemma 2.19(2) for the endomorphism $g \colon C \to C$ in the first half of Theorem 4.9(4). Here, $\deg S_g \le -\deg K_C = 2$ by $\deg g = \deg f > 1$ and by

$$K_C + S_g = g^*(K_C + S_g) + \Delta_g$$

(cf. Lemma 2.17(2)). This shows Theorem 4.9(4), and we are done. □

Proposition 4.14. *Assume that T and a general fiber of π are rational and that $\sharp\Sigma \le 2$. Then there exists a surjective morphism $\Phi \colon X \to \mathbb{P}^1$ such that*

- *the induced morphism $\varphi := (\Phi, \pi) \colon X \to \mathbb{P}^1 \times T$ is finite surjective, and*
- *$\{\Gamma_t \mid t \in \Sigma\}$ equals the set of prime components Γ of the ramification divisor R_φ of φ satisfying $\Phi(\Gamma) = \mathbb{P}^1$.*

Moreover, if $\Sigma = \emptyset$, then φ is an isomorphism, and if $\Sigma \neq \emptyset$, then $\sharp\Sigma = 2$ and $m_t = \deg \varphi \ge 2$ for any $t \in \Sigma$.

Proof. By Lemmas 4.4(4) and 4.6(5) and Corollary 4.8, we see that $\pi(\operatorname{Supp} R_f) = T$, $\boldsymbol{\rho}(X) = 2$, and X contains no negative curves. Since $\lambda_f = \deg f$, by Lemma 3.7, there is a nef $\mathbb{Q}$-divisor L such that $f^*L \approx (\deg f)L$, $L^2 = 0$, and $\overline{\mathrm{NE}}(X) = \mathbb{R}_{\ge 0}\,\mathrm{cl}(F) + \mathbb{R}_{\ge 0}\,\mathrm{cl}(L)$ for a general fiber F of π. The numerical equivalence relation $\approx$ coincides with the $\mathbb{Q}$-linear equivalence relation $\sim_{\mathbb{Q}}$ for $\mathbb{Q}$-divisors on X by [44, Lem. 2.31(4)], since X is rational and has only quotient singularities (cf. Proposition 3.17(3)). In particular, $f^*F \sim_{\mathbb{Q}} F$ and we may assume that L is a Cartier divisor satisfying $f^*L \sim_{\mathbb{Q}} (\deg f)L$. Let a and b be rational numbers defined by $-K_X \sim_{\mathbb{Q}} aF + bL$. Then $R_f = K_X - f^*(K_X) \sim_{\mathbb{Q}} (\deg f - 1)bL$, and $\mathrm{cl}(\Theta) \in \mathbb{R}_{\ge 0}\,\mathrm{cl}(L)$ for any prime component Θ of R_f by Lemma 1.6(3). Thus, $\kappa(L, X) \ge 0$, since some positive multiple of Θ is linearly equivalent to a positive multiple of L.

Claim. The divisor L is semi-ample.

Proof. Assume the contrary. Then $\kappa(L, X) = 0$ by Lemma 1.4. Hence, there is a unique prime divisor Θ such that $\operatorname{Supp} R_f = \Theta$ and that any effective divisor D with $\mathrm{cl}(D) \in \mathbb{R}_{\geq 0} \mathrm{cl}(L)$ is a multiple of Θ, since $\approx$ coincides with $\sim_{\mathbb{Q}}$. In particular, $f^*\Theta = (\deg f)\Theta$. Consequently, $S_f = \Theta$ and $\Delta_f = 0$. Then Θ is non-singular and $\pi|_\Theta \colon \Theta \to T$ is étale over $T \setminus \Sigma$ by Lemmas 4.4(1) and 4.6(6). Moreover, $\deg \pi|_\Theta = 2$ by Lemmas 4.4(5) and 4.6(4). Since $\sharp\Sigma \leq 2$, we have $\Theta \simeq \mathbb{P}^1$ and $\sharp\Sigma = 2$. We set $T_1 := \Theta$ and define $\tau_1 \colon T_1 \to T$ as the double-cover $\pi|_\Theta$. Let $h_1 \colon T_1 \to T_1$ be the automorphism corresponding to $f|_\Theta$. Then $\tau_1 \circ h_1 = h \circ \tau_1$. By Lemma 4.1, the normalization X_1 of $X \times_T T_1$ is irreducible and it admits an endomorphism $f_1 \colon X_1 \to X_1$ such that $\pi_1 \circ f_1 = h_1 \circ \pi_1$ for the induced fibration $\pi_1 \colon X_1 \to T_1$. Now, there is a commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\nu_1} & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ T_1 & \xrightarrow{\tau_1} & T \end{array}$$

and $(\pi_1 \colon X_1 \to T_1, f_1, h_1, \tau_1^{-1}\Sigma)$ satisfies the conditions in the assumption required for $(\pi \colon X \to T, f, h, \Sigma)$. Here, $\nu_1^*\Theta$ is reducible as $\Theta \times_T \Theta$ is so. This implies that $\kappa(\nu_1^*L, X_1) \geq 1$, but this contradicts $\kappa(\nu_1^*L, X_1) = \kappa(L, X)$ (cf. [27, Thm. 4], [41, II, Lem. 3.11]). $\square$

Proof of Proposition 4.14 *continued.* By the Claim, there is a fibration $\Phi \colon X \to B \simeq \mathbb{P}^1$ such that $\mathcal{O}_X(mL) \simeq \Phi^*\mathcal{O}(n)$ for some $m > 0$ and $n > 0$. We may replace L with a general fiber of Φ. Then $\mathcal{O}_X(L) \simeq \Phi^*\mathcal{O}(1)$. By an argument before the Claim, every prime component of R_f is a fiber of Φ. Since f^* preserves the ray $\mathbb{R}_{\geq 0} \mathrm{cl}(L)$, by Lemma 3.16, we have an endomorphism $f_B \colon B \to B$ such that $\deg f_B = \deg f$ and $\Phi \circ f = f_B \circ \Phi$.

The morphism $\varphi = (\Phi, \pi) \colon X \to B \times T$ over T defined by Φ is finite, since $\varphi^* \colon \mathsf{N}(B \times T) \to \mathsf{N}(X)$ is an isomorphism. The fiber $F_t = \pi^*(t)$ over $t \in T$ equals $\varphi^*(B \times \{t\})$. Since $F_t = m_t\Gamma_t$, we have

$$\deg \varphi = LF_t = m_t L\Gamma_t \quad \text{and} \quad \mathrm{mult}_{\Gamma_t} R_\varphi = m_t - 1 \tag{IV-4}$$

for any $t \in T$ and for the ramification divisor R_φ of φ. Comparing the ramification divisors of $\varphi \colon X \to B \times T$, $f \colon X \to X$, and $f_B \times h \colon B \times T \to B \times T$, we have

$$f^*(R_\varphi) - R_\varphi = -R_f + \Phi^*(R_{f_B}). \tag{IV-5}$$

Let $\mathfrak{S}_\varphi$ be the set of prime components of R_φ which dominates B by $\Phi \colon X \to B$. If $\Gamma \in \mathfrak{S}_\varphi$, then every prime component Γ' of $f^{-1}\Gamma$ belongs to $\mathfrak{S}_\varphi$. In fact, $f_B(\Phi(\Gamma')) = \Phi(f(\Gamma')) = \Phi(\Gamma) = B$, and we have

$$0 < (\mathrm{mult}_{\Gamma'} f^*\Gamma) \, \mathrm{mult}_\Gamma R_\varphi = \mathrm{mult}_{\Gamma'} f^*R_\varphi = \mathrm{mult}_{\Gamma'} R_\varphi$$

by (IV-5), since $\operatorname{Supp} R_f$ and $\operatorname{Supp} \Phi^*(R_{f_B})$ are contained in fibers of Φ. By Lemma 2.2 applied to the finite set $\mathfrak{S}_\varphi$ and by iterating f, we may assume that $f^{-1}\Gamma = \Gamma$ for any $\Gamma \in \mathfrak{S}_\varphi$. Then $f^*\Gamma = \Gamma$ for any $\Gamma \in \mathfrak{S}_\varphi$, since the eigenvalues of $f^* \colon \mathsf{N}(X) \to \mathsf{N}(X)$ are 1 and $\deg f$ and since $\mathrm{cl}(\Gamma) \notin \mathbb{R}\,\mathrm{cl}(L)$. Therefore, every member Γ of $\mathfrak{S}_\varphi$ is set-theoretically a fiber of π. This is never a smooth fiber, since any smooth fiber $\pi^*(t) = \varphi^*(B \times \{t\})$ is not contained in the ramification locus of φ. Thus, $\mathfrak{S}_\varphi = \{\Gamma_t \mid t \in \Sigma\}$ by the second equality of (IV-4).

It remains to prove the last assertion. Now, L is a general fiber of Φ. For the point $o := \Phi(L) \in B$, the induced finite morphism $\psi := \varphi|_L : L \to \{o\} \times T \simeq T$ has degree $FL = \deg \varphi$, and its ramification divisor R_ψ equals

$$R_\varphi|_L = \sum\nolimits_{t \in \Sigma} (m_t - 1)\Gamma_t|_L$$

by (IV-4). In particular,

$$\text{(IV-6)} \qquad \begin{aligned} 2\,\boldsymbol{g}(L) - 2 = \deg K_L &= \deg(\psi^* K_T) + \deg R_\psi = -2FL + R_\varphi L \\ &= FL(-2 + \sum\nolimits_{t \in \Sigma}(1 - 1/m_t)). \end{aligned}$$

Note that $\sum_{t \in \Sigma}(1 - 1/m_t) < 2$ by $\sharp\Sigma \le 2$. Thus, $\boldsymbol{g}(L) = 0$ by (IV-6). If $\Sigma = \emptyset$, then $\deg \varphi = FL = 1$ and φ is an isomorphism. If $\Sigma \ne \emptyset$, then $\deg \varphi = FL \ge 2$, and $\sum_{t \in \Sigma}(1 - 1/m_t) \ge 1$ by (IV-6); thus $\sharp\Sigma = 2$. Assume that $\Sigma = \{t_1, t_2\}$ with $t_1 \ne t_2$. Then

$$2 = (\deg \varphi)(1/m_{t_1} + 1/m_{t_2}) = L\Gamma_{t_1} + L\Gamma_{t_2}$$

by (IV-4) and (IV-6). Therefore, $L\Gamma_{t_1} = L\Gamma_{t_2} = 1$ and $\deg \varphi = m_{t_1} = m_{t_2}$. Thus, we are done. □

Lemma 4.15. *Assume that a general fiber of* π *is an elliptic curve. If* $T \simeq \mathbb{P}^1$ *and if* $1 \le \sharp\Sigma \le 2$, *then* $\Sigma = \{t_1, t_2\}$ *with* $t_1 \ne t_2$ *and* $m_{t_1} = m_{t_2}$.

Proof. The elliptic fibration $\pi : X \to T$ is smooth over $T \setminus \Sigma$ by Corollary 4.7. Thus, the period map is constant, since the universal covering space of $T \setminus \Sigma$ is isomorphic to $\mathbb{C}$. In particular, the associated monodromy representation $\rho : \pi_1(T \setminus \Sigma) \to \mathrm{SL}(2, \mathbb{Z})$ has a finite image. We shall prove the assertion by the following 3 steps.

Step 1. We shall show that *if the monodromy representation* ρ *is trivial, then* X *is non-singular and* Γ_t *is an elliptic curve for any* $t \in \Sigma$: Let $\mu : M \to X$ be the minimal resolution of singularities and let $\pi_Y : Y \to T$ be the relative minimal model over T of the elliptic fibration $\pi \circ \mu : M \to T$. Then $\pi_Y \circ \delta = \pi \circ \mu$ for a birational morphism $\delta : M \to Y$. Here, X and Y are isomorphic to each other over $T \setminus \Sigma$. Since ρ is trivial, by Kodaira's theory of elliptic surfaces [31] (cf. [39, §5]), we know that the scheme-theoretic fiber $\pi_Y^*(t)$ is a multiple of an elliptic curve for any $t \in \Sigma$. The proper transform of the elliptic curve in M can not be contracted by μ, since X has only quotient singularities (cf. Proposition 3.17(3)). Therefore, $Y \simeq X$ over T, and this proves the assertion of *Step* 1.

Step 2. We shall show the $\mathbb{Q}$-linear equivalence relation

$$\text{(IV-7)} \qquad K_X + \sum\nolimits_{t \in \Sigma} \Gamma_t \sim_{\mathbb{Q}} \pi^*(K_T + \Sigma),$$

where Σ in the right hand side is regarded as a reduced divisor on T: Let $\hat{\tau} : \widehat{T} \simeq \mathbb{P}^1 \to T$ be a finite surjective morphism such that $\hat{\tau}$ is étale over $T \setminus \Sigma$ and that the subgroup $\hat{\tau}_* \pi_1(\widehat{T} \setminus \hat{\tau}^{-1}\Sigma)$ of $\pi_1(T \setminus \Sigma)$ is the kernel of the monodromy representation ρ. The morphism $\hat{\tau}$ exists and is unique up to isomorphism. If ρ is trivial, then $\hat{\pi}$ is an isomorphism. Otherwise, $\sharp\Sigma = 2$ and $\hat{\pi}$ is a cyclic cover branched at Σ. By an argument in the proof of Lemma 2.7, we have an automorphism $\hat{h} : \widehat{T} \to \widehat{T}$

such that $\hat{\tau} \circ \hat{h} = h \circ \tau$. Let $\widehat{X}$ be the normalization of $X \times_T \widehat{T}$. Then $\widehat{X}$ admits a non-isomorphic surjective endomorphism compatible with f and $\hat{h}$ by Lemma 4.1. Moreover, the monodromy representation of the induced elliptic fibration $\hat{\pi}\colon \widehat{X} \to \widehat{T}$ is trivial. Hence, by *Step* 1, $\widehat{X}$ is non-singular and fibers of $\hat{\pi}$ are multiples of an elliptic curve. For a point $\hat{t} \in \widehat{T}$, we write $\widehat{\Gamma}_{\hat{t}}$ for the set-theoretic fiber $\hat{\pi}^{-1}(\hat{t})$. Since the period map of $\hat{\pi}$ is constant, we have

$$K_{\widehat{X}} + \sum\nolimits_{\hat{t} \in \hat{\tau}^{-1}\Sigma} \hat{\Gamma}_{\hat{t}} \sim_{\mathbb{Q}} \hat{\pi}^*(K_{\widehat{T}} + \hat{\tau}^{-1}\Sigma) \tag{IV-8}$$

by the canonical bundle formula (cf. [32, Thm. 12], [61, App.]). On the other hand,

$$K_{\widehat{T}} + \hat{\tau}^{-1}\Sigma = \hat{\tau}^*(K_T + \Sigma) \quad \text{and} \quad K_{\widehat{X}} + \sum\nolimits_{\hat{t} \in \hat{\tau}^{-1}\Sigma} \hat{\Gamma}_{\hat{t}} = \hat{\nu}^*(K_X + \sum\nolimits_{t \in \Sigma} \Gamma_t)$$

by [45, Lem. 1.39], since $\hat{\tau}$ is étale over $T \setminus \Sigma$ and the induced finite cover $\hat{\nu}\colon \widehat{X} \to X$ is étale over $X \setminus \pi^{-1}\Sigma$. Combining with (IV-8), we have

$$\nu^*(K_X + \sum\nolimits_{t \in \Sigma} \Gamma_t) \sim_{\mathbb{Q}} \nu^*(\pi^*(K_T + \Sigma)),$$

which implies the expected $\mathbb{Q}$-linear equivalence relation (IV-7) by applying ν_*.

Step 3. Final step. By $\sharp\Sigma \leq 2$ and by (IV-7), $K_X + \Gamma_t$ is not nef for any $t \in \Sigma$. Thus, $(K_X + \Gamma_t)\mathsf{R} < 0$ for an extremal ray R of $\overline{\mathrm{NE}}(X)$ (cf. Theorem 1.9). Now, X contains no negative curve by Corollary 4.8. Hence, the contraction morphism of R is a fibration $\psi\colon X \to B$ to a non-singular projective curve B, and $\boldsymbol{\rho}(X) = 2$ (cf. Theorem 1.10). Here, Γ_t is a section of ψ, since $0 > (K_X + \Gamma_t)G = -2 + \Gamma_t G$ for a general fiber G of ψ. In particular, B is an elliptic curve. Hence, $FG = m_t\Gamma_t G = m_t$ for a general fiber F of π. It is enough to show: $\sharp\Sigma \neq 1$. If $\Sigma = \{t\}$, then $K_X + \Gamma_t \sim_{\mathbb{Q}} \pi^*(K_T + t)$ by (IV-7) and it implies that

$$-1 = (K_X + \Gamma_t)G = \pi^*(K_T + t)G = -FG = -m_t.$$

This contradicts: $m_t \geq 2$. Thus, $\sharp\Sigma \neq 1$, and we are done. □

Finally in Section 4.3, we shall prove Theorem 4.9.

Proof of Theorem 4.9. By Lemma 4.13, it is enough to construct a finite surjective morphism $\tau\colon T' \to T$ such that $\pi'\colon X' \to T'$ is smooth and $\nu\colon X' \to X$ is étale in codimension 1 for the normalization X' of $X \times_T T'$. If $\Sigma = \emptyset$, then we can take τ as the identity morphism of T by Corollary 4.7. If $\Sigma \neq \emptyset$, then one of the following is satisfied by Proposition 4.14 and Lemma 4.15:

- $\boldsymbol{g}(T) \geq 1$;
- $\boldsymbol{g}(T) = 0$ and $\sharp\Sigma \geq 3$;
- $\boldsymbol{g}(T) = 0$, $\sharp\Sigma = 2$, and $m_{t_1} = m_{t_2}$ for $\{t_1, t_2\} = \Sigma$.

Hence, we have an expected cover $\tau\colon T' \to T$ by Lemmas 4.2 and 4.6(3) and by Proposition 4.3 in the case where $\deg h = 1$. Thus, we are done. □

4.4. **Structure of irrational ruled surfaces.** Applying Lemma 4.4 and Theorem 4.9, we shall prove the following structure theorem on irrational ruled normal projective surfaces admitting non-isomorphic surjective endomorphisms. Note that, by [4, Prop. 7], a normal Moishezon surface is projective if it is ruled.

Theorem 4.16. *Let X be an irrational ruled normal projective surface admitting a non-isomorphic surjective endomorphism f. Then one of the following holds*:

(1) *There exist a finite cover $\nu\colon \mathbb{P}^1 \times T \to X$ étale in codimension* 1 *for a non-singular projective curve T of genus ≥ 2 and endomorphisms $g\colon \mathbb{P}^1 \to \mathbb{P}^1$ and $h\colon T \to T$ such that $\nu \circ (g \times h) = f^k \circ \nu$ for some $k > 0$.*

(2) *The surface X is a $\mathbb{P}^1$-bundle over an elliptic curve.*

(3) *The surface X is a* projective cone *over an elliptic curve* (*cf. Definition* 1.16).

Proof. Let $\mu\colon M \to X$ be the minimal resolution of singularities. The Albanese morphism of M gives a $\mathbb{P}^1$-fibration $\pi_M\colon M \to T$ to an irrational non-singular projective curve T. Let Y be the normalization of the graph of the rational map $\pi = \pi_M \circ \mu^{-1}\colon X \cdots\to T$, and let $\pi_Y\colon Y \to T$ and $\sigma\colon Y \to X$ be induced morphisms. Then the endomorphism f induces an étale endomorphism $h\colon T \to T$ such that $\pi \circ f = h \circ \pi$. In fact, the rational map $\mu^{-1} \circ f \circ \mu\colon M \cdots\to M$ induces h, since π_M is given by the Albanese morphism of M. Thus, $f \times h$ induces an endomorphism $f_Y\colon Y \to Y$ such that $\pi_Y \circ f_Y = h \circ \pi_Y$ and $\sigma \circ f_Y = f \circ \sigma$.

First, we consider the case where σ is an isomorphism, i.e., $\pi\colon X \to T$ is holomorphic. Then π is a $\mathbb{P}^1$-fibration. If π is smooth and T is an elliptic curve, then (2) holds. If $\boldsymbol{g}(T) \geq 2$, then h is an automorphism, and (1) holds by Theorem 4.9. Hence, we may assume that T is an elliptic curve and π is not smooth, i.e., the set $\Sigma = \Sigma(\pi)$ in Definition 4.5 is not empty (cf. Lemma 4.6(5)). Then h is an automorphism by Lemma 4.6(2), and we have a finite surjective morphism $\tau\colon T' \to T$ from a non-singular projective curve T' satisfying conditions of Theorem 4.9. Here, $\boldsymbol{g}(T') > 1$ by (IV-1) in Lemma 4.2. Thus, (1) holds.

Next, we consider the case where σ is not an isomorphism. Applying the previous argument to Y and the endomorphism f_Y, we see that Y satisfies (1) or (2). There is a negative curve C on Y as a σ-exceptional divisor dominating T. Thus, T is an elliptic curve, π_Y is a $\mathbb{P}^1$-bundle, and C is a unique negative section of π_Y, by Corollary 4.8. Hence, X is a projective cone over T, i.e., (3) holds. Thus, we are done. □

Remark. In Theorem 4.16, any two of (1)–(3) are not satisfied at the same time. In fact, $\boldsymbol{\rho}(X) = 1$ in the case (3), but $\boldsymbol{\rho}(X) = 2$ in cases (1) and (2). Moreover, in the case (2), any finite cover over X étale in codimension 1 is a $\mathbb{P}^1$-bundle over an elliptic curve; thus X does not satisfy (1).

5. Classification in the pseudo-effective case: Proof of Theorem A

Theorem A in the introduction is a structure theorem on pairs (X, S) of a normal Moishezon surfaces X and a reduced divisor S such that $K_X + S$ is pseudo-effective and that S is completely invariant under a non-isomorphic surjective endomorphism $f\colon X \to X$. This section is devoted to proving Theorem A. The pair (X, S) is log-canonical by Theorem E, and moreover, $K_X + S$ is semi-ample with $(K_X + S)^2 = 0$, $S \geq S_f$, and $f|_{X \setminus S}\colon X \setminus S \to X \setminus S$ is étale in codimension 1, by Theorem 2.24. Thus, we have the following three cases:

- $K_X + S \not\sim_{\mathbb{Q}} 0$;
- $K_X + S \sim_{\mathbb{Q}} 0$ and $S = 0$;
- $K_X + S \sim_{\mathbb{Q}} 0$ and $S \neq 0$.

The proof of Theorem A is divided into these three cases, which are treated separately in Sections 5.1, 5.2, and 5.3 below.

5.1. **The case:** $K_X + S \not\sim_{\mathbb{Q}} 0$**.** Let $f\colon X \to X$ and S be as in Theorem A and assume that $K_X + S \not\sim_{\mathbb{Q}} 0$. Our purpose is to construct a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 satisfying (1) or (2) of Theorem A and to construct an endomorphism $f_V\colon V \to V$ satisfying $\nu \circ f_V = f^l \circ \nu$ for some $l > 0$.

Let $\pi\colon X \to T$ be the fibration associated with the semi-ample divisor $K_X + S$, i.e., T is a non-singular projective curve and $m(K_X + S) \sim \pi^*A$ for a positive integer m and for an ample divisor A on T. Note that $\lambda_f = \deg f$ by Proposition 3.3(4), since $f^*(K_X + S) = K_X + S$ (cf. Theorem 2.24). By Lemma 3.16, we have an endomorphism $h\colon T \to T$ such that $h \circ \pi = \pi \circ f$. Here, $\deg h = 1$, since $h^*A \sim A$ by $m(K_X + S) \sim \pi^*A$. Then, by Theorem 4.9, there exist a finite Galois cover $\tau\colon T' \to T$ from a non-singular curve T' with an endomorphism $h'\colon T' \to T$, an endomorphism f' of the normalization X' of $X \times_T T'$ and a positive integer k such that

- $X' \simeq C \times T'$ over T' for a rational or elliptic curve C,
- the induced Galois cover $\nu\colon X' \to X$ is étale in codimension 1,
- $\tau \circ h' = h^k \circ \tau$, $\nu \circ f' = f^k \circ \nu$, and $\mathrm{pr}_2 \circ f' = h' \circ \mathrm{pr}_2$ for the second projection $\mathrm{pr}_2\colon X' \simeq C \times T' \to T'$.

Here, $\nu^* S_f = \nu^* S_{f^k} = S_{f'}$ by Lemmas 2.17(3) and 2.19(3). Hence, by replacing X with X' and replacing S with $\nu^* S$, we may assume that $X = C \times T$. Thus, the proof of Theorem A in the case where $K_X + S \not\sim_{\mathbb{Q}} 0$ is reduced to the following:

Lemma 5.1. *Let X be the direct product $C \times T$ of non-singular projective curves, where C is either rational or elliptic. Let $\mathrm{pr}_1\colon X \to C$ and $\mathrm{pr}_2\colon X \to T$ be the first and second projections, respectively. Let $f\colon X \to X$ and $h\colon T \to T$ be surjective endomorphisms such that $\deg f > 1$, $\deg h = 1$, and $\mathrm{pr}_2 \circ f = h \circ \pi_2$. Let S be an f-completely reduced divisor on X such that $m(K_X + S) \sim \mathrm{pr}_2^* A$ for an integer $m > 0$ and an ample divisor A on T.*

(1) *If C is rational, then $S_f = \mathrm{pr}_1^*(P_1 + P_2)$ and $S = \mathrm{pr}_1^*(P_1 + P_2) + \mathrm{pr}_2^* D$ for two points $P_1 \neq P_2 \in C$ and for a reduced divisor D on T such that $m(K_T + D) \sim A$.*

(2) *If C is elliptic, then $S_f = 0$ and $S = \mathrm{pr}_2^* D$ for a reduced divisor D on T such that $m(K_T + D) \sim A$.*

Proof. (1): In this case, $S \geq S_f$, $\Delta_f = 0$, and $R_f = f^* S_f - S_f \neq 0$ by Theorems 2.24 and 4.9(1). Thus, $S_f F = 2$ for a general fiber F of pr_2 by Lemma 4.4(6), and $S_f = \mathrm{pr}_1^*(P_1 + P_2)$ for two points $P_1 \neq P_2 \in C$ by Theorem 4.9(4). Hence, $(S - S_f)F = -(K_X + S_f)F = 0$, and $\mathrm{Supp}(S - S_f)$ is a union of fibers of π. Thus, $S - S_f = \mathrm{pr}_2^* D$ for a reduced divisor D on T. Therefore,

$$K_X + S \sim \mathrm{pr}_1^*(K_C + P_1 + P_2) + \mathrm{pr}_2^*(K_T + D) \sim \mathrm{pr}_2^*(K_T + D),$$

and we have $m(K_T + D) \sim A$.

(2): In this case, $S_f = R_f = 0$ by Theorem 4.9(1). Hence, $f^*S = S$, and S is a union of fibers of pr_2, i.e., $S = \mathrm{pr}_2^* D$ for a reduced divisor D on T. Therefore,

$$K_X + S \sim \mathrm{pr}_1^* K_C + \mathrm{pr}_2^*(K_T + D) \sim \mathrm{pr}_2^*(K_T + D),$$

and we have $m(K_T + D) \sim A$. □

5.2. **The case:** $S = 0$ **and** $K_X \sim_{\mathbb{Q}} 0$**.** Let $f\colon X \to X$ and S be as in Theorem A and assume that $S = 0$ and $K_X \sim_{\mathbb{Q}} 0$. We shall construct a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 from an abelian surface V and construct an endomorphism $f_V\colon V \to V$ such that $\nu \circ f_V = f \circ \nu$. Note that X has only log-canonical singularities by Theorem E, and f is étale in codimension 1, i.e., $R_f = 0$, by Lemma 2.22. We begin with the case: $K_X \sim 0$.

Proposition 5.2. *Let X be a normal Moishezon surface admitting non-isomorphic surjective endomorphism f. Assume that $K_X \sim 0$. Then X has only rational double points as singularities, and there is a finite surjective morphism $V \to X$ étale in codimension* 1 *from an abelian surface V.*

Proof. Let Ξ be the set of irrational singular points of X. Then $f^{-1}\Xi \subset \Xi$, since any irrational singularity cannot be dominated by a rational singularity by a finite morphism. By Lemma 2.2, we may assume that $f^{-1}(P) = P$ for any $P \in \Xi$ by iterating f when $\Xi \neq \emptyset$. The open subset $X \setminus \Xi$ has only rational double points as singularities, since X is Gorenstein. Let $\theta\colon V \to X$ be the Galois closure of $f^k\colon X \to X$ for $k \gg 0$. Then $V \setminus \theta^{-1}\Xi$ is non-singular and $\boldsymbol{e}(V \setminus \theta^{-1}\Xi) = 0$ by Lemma 2.4. Here, $K_V \sim 0$ and V has only log-canonical singularities by [45, Lem. 2.10(1)], since θ is étale in codimension 1. In particular, if $\Xi = \emptyset$, i.e., X has only rational double points as singularities, then V is non-singular, $K_V \sim 0$, and $\boldsymbol{e}(V) = 0$; consequently, V is an abelian surface.

Thus, it suffices to derive a contradiction assuming: $\Xi \neq \emptyset$. In this case, $\theta^{-1}\Xi = \operatorname{Sing} V$, and $\operatorname{Sing} V$ consists of simple elliptic singularities or cusp singularities, by the classification of Gorenstein log-canonical singularities (cf. [55, App.], [28, Thm. 9.6], [33, Ch. 3]). Let $\delta\colon W \to V$ be the minimal resolution of singularities. Then $K_W = \delta^* K_V - \Theta \sim -\Theta$ for the reduced exceptional divisor $\Theta := \delta^{-1}(\theta^{-1}\Xi)$. In particular, W is ruled. By [44, Lem. 4.5], a connected component of Θ is an elliptic curve or a *cyclic chain of rational curves* (cf. [44, Def. 4.3]). Thus, $\boldsymbol{e}(\Theta)$ equals the number of rational curves contained in Θ (cf. [44, Rem. 4.4]). Therefore, $\boldsymbol{e}(\Theta) \leq \boldsymbol{\rho}(W) - 1$. Since $\boldsymbol{e}(W \setminus \Theta) = \boldsymbol{e}(V \setminus \theta^{-1}\Xi) = 0$, we have $\boldsymbol{e}(W) = \boldsymbol{e}(\Theta) \leq \boldsymbol{\rho}(W) - 1$. On the other hand,

$$\boldsymbol{e}(W) = 2 - 4\,\boldsymbol{q}(W) + \boldsymbol{b}_2(W) \geq 2 - 4\,\boldsymbol{q}(W) + \boldsymbol{\rho}(W)$$

for the irregularity $\boldsymbol{q}(W) = \dim H^1(W, \mathcal{O}_W)$ and the second Betti number $\boldsymbol{b}_2(W)$ of W. Thus $\boldsymbol{q}(W) \geq 1$, and we have a $\mathbb{P}^1$-fibration $\pi\colon W \to T$ to a non-singular projective curve T with $\boldsymbol{g}(T) = \boldsymbol{q}(W)$. For a general fiber F of π, we have $\Theta F = -K_W F = 2$. Thus, a prime component of Θ dominates T, and we have $\boldsymbol{g}(T) = 1$, since every prime component of Θ has genus ≤ 1. If a connected component of

Θ is not an elliptic curve, then it is a cyclic chain of rational curves, and it is contained in a fiber of π, which contradicts $R^1\pi_*\mathcal{O}_W = 0$. Hence, $\boldsymbol{e}(\Theta) = \boldsymbol{e}(W) = 0$. Consequently, $\boldsymbol{b}_2(W) = 2$, and W is a $\mathbb{P}^1$-bundle over the elliptic curve T. In particular, $K_W^2 = 0$, but it contradicts $K_W^2 = \Theta^2 < 0$. Therefore, $\Xi = \emptyset$, and we are done. $\square$

Proof of Theorem A *in the case where* $S = 0$ *and* $K_X \sim_{\mathbb{Q}} 0$. Let $\widetilde{X} \to X$ be the *index* 1 *cover* with respect to $K_X \sim_{\mathbb{Q}} 0$ (cf. [45, Def. 4.18(2)]). Then $\widetilde{X} \to X$ is étale in codimension 1, $K_{\widetilde{X}} \sim 0$, and f lifts to a non-isomorphic surjective endomorphism $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$, by [45, Lem. 4.21(1)]. Applying Proposition 5.2 to $\widetilde{X}$ and $\tilde{f}$, we have a finite surjective morphism $\nu'\colon V' \to X$ étale in codimension 1 from an abelian surface V'. For the set of finite surjective morphisms $V'' \to X$ étale in codimension 1 from abelian surfaces V'', let us choose one member $\nu\colon V \to X$ such that $\deg \nu$ is minimal in the set. Then ν is Galois and unique up to non-canonical isomorphism over X; this is called the *Albanese closure* in [47, Lem. 2.6]. For the proof of Theorem A in this case, it is enough to construct an endomorphism $f_V\colon V \to V$ as a lift of $f\colon X \to X$. Let $V \times_X X$ be the fiber product of $\nu\colon V \to X$ and $f\colon X \to X$ over X, and let $\widehat{V}$ be a connected component of the normalization of $V \times_X X$ which dominates V and X. Then we have a commutative diagram

$$\begin{array}{ccc} \widehat{V} & \xrightarrow{g} & V \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\nu} \\ X & \xrightarrow{f} & X, \end{array}$$

where induced finite covers g and σ are étale in codimension 1. Since V is abelian, g is étale and $\widehat{V}$ is also an abelian surface. Thus, $\deg \sigma \geq \deg \nu$ by the minimality of $\deg \nu$. On the other hand, $\deg \sigma \leq \deg(V \times_X X \to X) = \deg \nu$ by construction. Therefore, $\deg \sigma = \deg \nu$, and we have an isomorphism $\phi\colon V \to \widehat{V}$ over X, i.e., $\nu = \sigma \circ \phi$. Then $f_V := g \circ \phi$ is an endomorphism of V satisfying $\nu \circ f_V = f \circ \nu$. Thus, we are done. $\square$

5.3. **The case:** $S \neq 0$ **and** $K_X + S \sim_{\mathbb{Q}} 0$**.** Let $f\colon X \to X$ and S be as in Theorem A and assume that $S \neq 0$ and $K_X + S \sim_{\mathbb{Q}} 0$. Then X is projective by [4, Prop. 7]. We shall construct a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 satisfying one of (4), (5), and (6) of Theorem A, and to construct an endomorphism $f_V\colon V \to V$ such that $\nu \circ f_V = f \circ \nu$. The strategy is as follows: First, we study the structure of (X, S, f) in the case where $K_X + S \sim 0$ applying Theorem 4.16 on the irrational ruled surfaces, and considering the Galois closure of $f^k\colon X \to X$ for $k \gg 0$, as in the proof of Proposition 5.2 above. Second, we shall reduce to the first case by taking the index 1 cover $\widetilde{X} \to X$ with respect to $K_X + S \sim_{\mathbb{Q}} 0$.

Lemma 5.3. *Assume that* $K_X + S \sim 0$. *Then* $S \cap \operatorname{Sing} X \subset \operatorname{Sing} S$, *and any connected component of* S *is either an elliptic curve or a cyclic chain of rational curves. If* X *is rational, then* S *is connected and* X *has only rational singularities*; *in particular, any singular point of* $X \setminus S$ *is a rational double point.*

Proof. Since (X, S) is log-canonical (cf. Theorem E), the first assertion follows from [44, Cor. 4.6] by $K_X + S \sim 0$. Assume that X is rational. Then X has only rational singularities by [44, Lem. 2.31(3)]. Hence, $X \setminus S$ has only rational double points as singularities by $K_X + S \sim 0$. From the exact sequence $0 \to \mathcal{O}_X(K_X) \simeq \mathcal{O}_X(-S) \to \mathcal{O}_X \to \mathcal{O}_S \to 0$, we have a surjection $\mathbb{C} \simeq H^0(X, \mathcal{O}_X) \to H^0(S, \mathcal{O}_S)$, since $H^1(X, \mathcal{O}_X(K_X)) \simeq H^1(X, \mathcal{O}_X)^\vee = 0$ by the rationality of X. Therefore, S is connected. □

Lemma 5.4. *If $K_X + S \sim 0$ and X is irrational, then one of the following holds*:

(1) *X is a $\mathbb{P}^1$-bundle over an elliptic curve and S is a disjoint union of two sections*;

(2) *X is a projective cone over an elliptic curve and S is a cross section.*

Proof. Since X is irrational and ruled, one of the three cases of Theorem 4.16 occurs. First, we shall prove (2) in the case Theorem 4.16(3), i.e., X is a projective cone over an elliptic curve T. Since (X, S) is log-canonical, X has at most quotient singularities along S (cf. Remark 3.8). Hence, the vertex P of X is not contained in S. For the minimal resolution $\mu\colon M \to X$ of singularity and for the exceptional curve $E = \mu^{-1}(P)$, we have $K_M + E + \mu^*S = \mu^*(K_X + S) \sim 0$ and $E \cap \mu^*S = \emptyset$. Thus, μ^*S is a section of the $\mathbb{P}^1$-bundle obtained as the Albanese morphism of M. Therefore, S is a cross section of the projective cone X, and (2) holds.

Next, we shall prove (1) in the other cases of Theorem 4.16. The Albanese morphism of the minimal resolution of singularities of X induces a $\mathbb{P}^1$-fibration $\pi\colon X \to T$ to an irrational non-singular projective curve T. There is also an étale endomorphism $h\colon T \to T$ satisfying $\pi \circ f = h \circ \pi$ by the proof of Theorem 4.16. There is a prime component C of S such that $\pi(C) = T$, since $SF = -K_X F = 2$ for a general fiber F of π. Then C is an elliptic curve contained in X_{reg} and is a connected component of S by Lemma 5.3. In particular, T is an elliptic curve and $\pi|_C\colon C \to T$ is étale. Thus, any fiber of π is reduced by $C \subset X_{\mathrm{reg}}$, and π is a $\mathbb{P}^1$-bundle by Lemma 4.6(5). Now S contains no fibers of π, since C is a connected component of S. By $SF = 2$, we see that S is either a disjoint union of two sections of π or an étale double-cover over T. However, the latter case does not occur. In fact, in this case, the exact sequence $0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + S) \to \mathcal{O}_S(K_S) \to 0$ of $\mathcal{O}_X$-modules induces an exact sequence

$$0 \to \pi_*\mathcal{O}_X(K_X + S) \to \pi_*\mathcal{O}_S(K_S) \to R^1\pi_*\mathcal{O}_X(K_X) \to 0$$

of $\mathcal{O}_T$-modules with isomorphisms

$$\pi_*\mathcal{O}_X(K_X + S) \simeq \mathcal{O}_T, \quad \pi_*\mathcal{O}_S(K_S) \simeq \pi_*\mathcal{O}_S, \quad R^1\pi_*\mathcal{O}_X(K_X) \simeq \mathcal{O}_T(K_T) \simeq \mathcal{O}_T.$$

In particular, $\det \pi_*\mathcal{O}_S \simeq \mathcal{O}_S$. On the other hand, since $\pi|_S\colon S \to T$ is étale, $\pi_*\mathcal{O}_S \simeq \mathcal{O}_T \oplus \mathcal{N}$ for an invertible sheaf $\mathcal{N}$ on T such that $\mathcal{N} \not\simeq \mathcal{O}_T$ and $\mathcal{N}^{\otimes 2} \simeq \mathcal{O}_T$. Hence, we have a contradiction: $\mathcal{N} \simeq \det \pi_*\mathcal{O}_S \simeq \mathcal{O}_T$. Therefore, S is a disjoint union of two sections of π, and (1) holds. Thus, we are done. □

In order to study the case where $K_X + S \sim 0$ and X is rational, we introduce a special finite Galois cover over X as follows:

Definition 5.5. For an integer $k > 0$, let $\theta_k : V_k \to X$ be the Galois closure of the k-th power $f^k : X \to X$ of f, and let $\tau_k : V_k \to X$ be the induced morphism such that $\theta_k = f^k \circ \tau_k$ (cf. Lemma 2.3). We fix a sufficiently large integer k, and set $V := V_k$, $\nu := \tau_k : V \to X$, and $S_V := \nu^{-1}S$. Moreover, we define $\delta : W \to V$ to be the minimal resolution of singularities and set $S_W := \mu^{-1}S_V$.

Lemma 5.6. *Assume that* $K_X + S \sim 0$ *and* X *is rational. Then the following hold by a suitable choice of* $k \gg 0$:

(1) $\nu : V \to X$ *is étale over* $X_{\mathrm{reg}} \cup S$;
(2) $V \setminus S_V$ *is non-singular with* $\boldsymbol{e}(V \setminus S_V) = 0$;
(3) (V, S_V) *is log-canonical with* $K_V + S_V \sim 0$;
(4) W *is a ruled surface,* $K_W + S_W \sim 0$, *and* S_W *is a normal crossing divisor whose connected component is either an elliptic curve or a cyclic chain of rational curves.*

Proof. Now, $f|_{X \setminus S} : X \setminus S \to X \setminus S$ is étale in codimension 1 by $R_f = f^*S - S$ (cf. Theorem 2.24). Then (2) holds and $\nu = \tau_k$ is étale in codimension 1 on $V \setminus S_V$ by Lemma 2.4, since $X \setminus S$ has only rational double points as singularities (cf. Lemma 5.3) and since we take $k \gg 0$. We have $K_V + S_V = \nu^*(K_X + S) \sim 0$ by [45, Lem. 1.39]. Thus, (3) holds, since (X, S) is log-canonical. Hence, $S_V \cap \operatorname{Sing} V \subset \operatorname{Sing} S_V$ and a connected component of S_V is either an elliptic curve or a cyclic chain of rational curves, by [44, Cor. 4.6]. In particular, δ is an isomorphism outside $\operatorname{Sing} S_V$, and we have $K_W + S_W = \delta^*(K_V + S_V) \sim 0$, since $V \setminus S_V \hookrightarrow V$ is a toroidal embedding at any point of $\operatorname{Sing} S_V$. Thus, we have (4) by [44, Cor. 4.6].

It remains to prove (1). Note that $X_{\mathrm{reg}} \cup S$ is an open subset of X whose complement $\operatorname{Sing} X \setminus S$ is a finite set. Since ν is étale over $X_{\mathrm{reg}} \setminus S$ and since $(f^k)^{-1}S = S$, it suffices to prove that $\nu = \tau_k$ is étale along $\theta_k^{-1}(P)$ for any $P \in S$.

Assume first that $P \in S_{\mathrm{reg}}$. Then $P \in X_{\mathrm{reg}}$ by $S \cap \operatorname{Sing} X \subset \operatorname{Sing} S$ (cf. Lemma 5.3), and $f^{-1}(P) \subset S_{\mathrm{reg}}$ by $f^{-1} \operatorname{Sing} S = \operatorname{Sing} S$ (cf. Lemma 3.12). Thus, for any point $P' \in (f^k)^{-1}(P)$, the morphism $f^k : (X, P') \to (X, P)$ of germs of surfaces is a cyclic cover branched possibly along S. Moreover the degree m of $(X, P') \to (X, P)$ is independent of the choice of $P' \in (f^k)^{-1}(P)$, since $(f^k)^*C = mC'$ for the prime component C of S containing P and for the prime component $C' = (f^k)^{-1}C$ of $(f^k)^{-1}S = S$. Hence, ν is étale along $\theta_k^{-1}(P)$ by Lemma 2.5.

Assume next that $P \in \operatorname{Sing} S$. Then P is a node of S, and $(f^k)^{-1}(P) = \{P'\}$ for a node P' of S by $f^{-1} \operatorname{Sing} S = \operatorname{Sing} S$. Since $X \setminus S \subset X$ is a toroidal embedding at P, the fundamental group $\pi_1(\mathcal{U} \setminus S)$ is abelian and $f^{-1}(\mathcal{U} \setminus S) \to \mathcal{U} \setminus S$ is étale for a sufficiently small open neighborhood $\mathcal{U} \subset X$ of P. Thus, the morphism $f^k : (X, P') \to (X, P)$ of germs of surfaces is a Galois cover. Hence, ν is étale along $\theta_k^{-1}(P)$ by Lemma 2.5. Thus, (1) holds, and we are done. □

Lemma 5.7. *In the situation of Lemma* 5.6, *assume that* V *is rational. Then*:

(1) V *is a toric surface with* S_V *as the boundary divisor*;
(2) $\nu^{-1}U$ *is the universal cover of* U *for the open subset* $U := X_{\mathrm{reg}} \cup S$;
(3) V *admits an endomorphism* f_V *satisfying* $\nu \circ f_V = f \circ \nu$.

As a consequence of (2), *the isomorphism class of* $\nu = \tau_k \colon V = V_k \to X$ *is independent of the choice of* $k \gg 0$.

Proof. By Lemma 5.6(4) and [44, Rem. 4.4], $\boldsymbol{e}(S_W)$ equals the number of rational curves in S_W. Now, $\boldsymbol{e}(W \setminus S_W) = \boldsymbol{e}(V \setminus S_V) = 0$ by Lemma 5.6(2), and we have $\boldsymbol{e}(S_W) = \boldsymbol{e}(W) = \boldsymbol{\rho}(W)+2$, since W is a non-singular rational surface. Then W is a toric surface with S_W as the boundary divisor by Lemma 5.6(4) and by Shokurov's criterion of toric surface [58, Thm. 6.4] (cf. [44, Thm. 1.3]). Hence, we have (1) by [44, Lem. 3.9], since the exceptional locus of $\delta \colon W \to V$ is in S_W (cf. Lemma 5.6(2)). The complement of $\nu^{-1}U$ in V is a finite set $\nu^{-1}(\operatorname{Sing} X \setminus S)$ contained in V_{reg}. Thus, the complement of $\delta^{-1}(\nu^{-1}U)$ in W is also a finite set, and $\delta^{-1}(\nu^{-1}U) \simeq \nu^{-1}U$ is simply connected. Then Lemma 5.6(1) implies (2) and the last assertion. In particular, the morphism $g_k \colon V_{k+1} \to V_k$ in Lemma 2.3 is an isomorphism, and we have an endomorphism f_V in (3) as the composite $h_k \circ g_k^{-1} \colon V_k \to V_k$ for the other morphism $h_k \colon V_{k+1} \to V_k$ in Lemma 2.3. □

Lemma 5.8. *In the situation of Lemma* 5.6, *assume that* V *is irrational. Then*

(1) V *is a* $\mathbb{P}^1$*-bundle over an elliptic curve,*
(2) S *is an elliptic curve contained in* X_{reg},
(3) $-K_X$ *is nef with* $K_X^2 = 0$, *and*
(4) *there exist a* $\mathbb{P}^1$*-fibration* $\pi \colon X \to T \simeq \mathbb{P}^1$ *and an endomorphism* $h \colon T \to T$ *such that* $\pi \circ f = h \circ \pi$ *and* $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X) = \mathbb{R}_{\geq 0}\,\mathrm{cl}(-K_X) + \mathbb{R}_{\geq 0}\,\mathrm{cl}(F)$ *for a fiber* F *of* π.

Let Σ *be the set of points* $t \in T$ *such that* $\pi^*(t)$ *is not reduced. Then*

(5) π *is smooth over* $T \setminus \Sigma$,
(6) Σ *coincides with the branch locus of the double-cover* $\pi|_S \colon S \to T$,
(7) $\pi^*(t) = 2\pi^{-1}(t)$ *for any* $t \in \Sigma$.

Proof. Let $\psi_W \colon W \to Z$ be the $\mathbb{P}^1$-fibration to an irrational non-singular projective curve Z induced by the Albanese morphism of W. By Lemma 5.6(4), we see that Z is an elliptic curve and a connected component of S_W is an elliptic curve dominating Z. If another connected component of S_W is not an elliptic curve, then it is a cyclic chain of rational curves (cf. Lemma 5.6(4)), and hence, it is contained in a fiber of ψ_W: this contradicts $R^1\psi_{W*}\mathcal{O}_W = 0$. Therefore, S_W is a union of elliptic curves. Consequently, S_V is also a union of elliptic curves and V is non-singular by [44, Cor. 4.6] and by (2) and (3) of Lemma 5.6. In particular, $\delta \colon W \to V$ is an isomorphism. Now, $\boldsymbol{e}(V) = \boldsymbol{e}(V \setminus S_V) + \boldsymbol{e}(S_V) = 0$ by Lemma 5.6(2). Thus, $\psi := \psi_W \circ \delta^{-1} \colon V \simeq W \to Z$ is a $\mathbb{P}^1$-bundle, and we have proved (1).

We shall show (2) and (3). Now, S is connected and $S \cap \operatorname{Sing} X \subset \operatorname{Sing} S$ by Lemma 5.3. Moreover, $\nu|_{S_V} \colon S_V \to S$ is étale by Lemma 5.6(1) and we have proved that each connected component of S_V is an elliptic curve. Thus, (2) holds. Note that $K_V = \nu^* K_X$ by Lemma 5.6(1) and that $K_V^2 = 0$ by (1). Hence, $K_X^2 = S^2 = 0$ by $K_X + S \sim 0$. Thus, $-K_X$ is nef, and we have (3).

The action of the Galois group of ν on V descends to Z by the $\mathbb{P}^1$-bundle $\psi \colon V \to Z$. By taking quotients, we have a $\mathbb{P}^1$-fibration $\pi \colon X \to T \simeq \mathbb{P}^1$ with a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\nu} & X \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Z & \longrightarrow & T. \end{array}$$

For a general fiber F of π and a fiber $\widetilde{F}$ of ψ lying over F, we have $SF = S_V\widetilde{F} = -K_XF = -K_V\widetilde{F} = 2$ by $K_X + S \sim 0$ and $K_V + S_V \sim 0$. Thus, $\deg(S_V/Z) = \deg(S/T) = 2$. On the other hand, $\nu|_{S_V} : S_V \to S$ is étale with $\deg(S_V/S) = \deg\nu$ by Lemma 5.6(1). Therefore, $\deg(Z/T) = \deg\nu$. As a consequence, V is the normalization of $Z \times_T X$.

We shall show (4). We have $\boldsymbol{\rho}(X) = 2$ by the diagram. Now, F and $-K_X$ are nef with $F^2 = K_X^2 = 0$ and $K_XF < 0$ (cf. (3)). Thus, $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X) = \mathbb{R}_{\geq 0}\,\mathrm{cl}(-K_X) + \mathbb{R}_{\geq 0}\,\mathrm{cl}(F)$. Since S is irreducible (cf. (2)), $f^*S = mS$ for a positive integer m. In particular, $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ preserves the ray $\mathbb{R}_{\geq 0}\,\mathrm{cl}(-K_X) = \mathbb{R}_{\geq 0}\,\mathrm{cl}(S)$, and hence, f^* preserves also the other extremal ray $\mathbb{R}_{\geq 0}\,\mathrm{cl}(F)$ (cf. the proof of Lemma 3.7). Then we have an endomorphism $h\colon T \to T$ satisfying $\pi \circ f = h \circ \pi$ by Lemma 3.16. Thus, (4) has been shown.

Finally, we shall show remaining assertions (5)–(7) on the set Σ. For a point $t \in T$, let F_t denote the fiber $\pi^*(t)$ and set $\Gamma_t := \pi^{-1}(t) = (F_t)_{\mathrm{red}}$. Then $\Gamma_t \simeq \mathbb{P}^1$ for any $t \in T$ (cf. Theorem 1.10(2)). In particular, $F_t = m_t\Gamma_t$ for an integer $m_t > 0$, and we have $\Sigma = \{t \in T \mid m_t > 1\}$ as in Definition 4.5. We have (5) by [44, Prop. 2.33(4)]. Since $S \subset X_{\mathrm{reg}}$ (cf. (2)), S intersects Γ_t transversely for any $t \in T$ by Proposition 3.17(6). Thus, $2 = SF_t = m_tS\Gamma_t$ and $S\Gamma_t = \sharp S \cap F_t = \sharp(\pi|_S)^{-1}(t)$ for any $t \in T$. Therefore, $t \in \Sigma$ if and only if $m_t = 2$. This implies (6) and (7). Thus, we are done. □

Corollary 5.9. *In the situation of Lemma* 5.8*, let $\tau\colon T' \to T$ be the double-cover $\pi|_S\colon S \to T$, and let X' be the normalization of the fiber product $X \times_T T'$ of π and τ. Then the following hold*:

1. *The double-cover $\nu'\colon X' \to X$ induced by the first projection $X \times_T T' \to X$ is étale in codimension* 1.
2. *The morphism $\pi'\colon X' \to T'$ induced by the second projection $X \times_T T' \to T'$ is a $\mathbb{P}^1$-bundle.*
3. *The pullback $\nu'^{-1}(S)$ is a disjoint union of two sections of π'.*
4. *There is an endomorphism $f'\colon X' \to X'$ such that $f' \circ \nu' = \nu' \circ f$.*

Proof. Let $h'\colon T' \to T'$ be the endomorphism $f|_S\colon S \to S$. Then $\tau \circ h' = h \circ \tau$ for the endomorphism h in Lemma 5.8(4). Thus, we have an endomorphism f' of X' satisfying $\nu' \circ f' = f \circ \nu'$ and $\pi' \circ f' = h' \circ \pi'$ by Lemma 4.1. In particular, (4) holds. Here, h' is étale, since $T' = S$ is an elliptic curve (cf. Lemma 5.8(2)). We can apply results in Section 4 to $(f'\colon X' \to X', \pi'\colon X' \to T', h'\colon T' \to T')$. By (5)–(7) of Lemma 5.8 and by Lemma 4.2, we have (1) and every fiber of π' is reduced. Moreover, we have (2) by Lemma 4.6(5). By construction, $S' = \nu'^{-1}(S)$ is reducible. Now, $\nu'|_{S'}\colon S' \to S$ is an étale morphism of degree 2 by (1) and by

$S \subset X_{\mathrm{reg}}$ (cf. Lemma 5.8(2)). Therefore, S' is a disjoint union of two copies of S which are both sections of π'. This shows (3), and we are done. $\square$

Finally in Section 5, we finish the proof of Theorem A by proving it in the case where $K_X + S \sim_{\mathbb{Q}} 0$ and $S \neq 0$.

Proof of Theorem A *in this case.* Let $\theta\colon \widetilde{X} \to X$ be the *index* 1 *cover* with respect to $K_X + S \sim_{\mathbb{Q}} 0$ (cf. [45, Def. 4.18(2)]), and set $\widetilde{S} := \theta^* S$. By [45, Lem. 4.21(1)], θ is étale in codimension 1, $\widetilde{S}$ is reduced, $K_{\widetilde{X}} + \widetilde{S} \sim 0$, and there is an endomorphism $\tilde{f}\colon \widetilde{X} \to \widetilde{X}$ satisfying $\theta \circ \tilde{f} = f \circ \theta$. In particular, $\widetilde{S}$ is completely invariant under $\tilde{f}$. Therefore, we can apply Lemmas 5.4, 5.7, and 5.8 and Corollary 5.9 to $(\widetilde{X}, \tilde{f}, \widetilde{S})$.

First, assume that $\widetilde{X}$ is irrational. Then, by Lemma 5.4, $(\theta\colon \widetilde{X} \to X, \tilde{f})$ satisfies conditions required for $(\nu\colon V \to X, f_V)$ in Theorem A, where either (4) or (5) of Theorem A is satisfied.

Second, assume that $\widetilde{X}$ is rational and the surface V in Lemma 5.6 is also rational. Then, by Lemma 5.7, there exists a finite Galois cover $\nu\colon V \to \widetilde{X}$ étale in codimension 1 from a toric surface V with $\nu^*(\widetilde{S})$ as the boundary divisor and with a lift $f_V\colon V \to V$ of the endomorphism $\tilde{f}$. The open subset $\widetilde{U} := (\widetilde{X})_{\mathrm{reg}} \cup \widetilde{S}$ of $\widetilde{X}$ is preserved by the action of the Galois group of θ on $\widetilde{X}$, and $\nu^{-1}\widetilde{U}$ is the universal cover of $\widetilde{U}$ by Lemma 5.7. Thus, the composite $\theta \circ \nu\colon V \to X$ is Galois. Hence, $(\theta \circ \nu\colon V \to X, f_V)$ satisfies Theorem A(6).

Finally, assume that $\widetilde{X}$ is rational and the surface V in Lemma 5.6 is irrational. Then, by Lemma 5.8, $\widetilde{S}$ is an elliptic curve contained in $\widetilde{X}_{\mathrm{reg}}$ and there exists a $\mathbb{P}^1$-fibration $\tilde{\pi}\colon \widetilde{X} \to \widetilde{T} \simeq \mathbb{P}^1$ as the contraction morphism of an extremal ray. Let $\tau\colon \widehat{T} \to \widetilde{T}$ be the double-cover $\tilde{\pi}|_{\widetilde{S}}\colon \widetilde{S} \to \widetilde{T}$ and let $\widehat{X}$ be the normalization of $\widetilde{X} \times_{\widetilde{T}} \widehat{T}$. Then, by Corollary 5.9, the induced morphism $\hat{\nu}\colon \widehat{X} \to \widetilde{X}$ is étale in codimension 1 and the other induced morphism $\hat{\pi}\colon \widehat{X} \to \widehat{T}$ is a $\mathbb{P}^1$-bundle in which $\widehat{S} := \hat{\nu}^{-1}(\widetilde{S})$ is a disjoint union of two sections of $\hat{\pi}$:

$$\begin{array}{ccccc} \widehat{X} & \xrightarrow{\hat{\nu}} & \widetilde{X} & \xrightarrow{\theta} & X \\ {\scriptstyle \hat{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\pi}} & & \\ \widehat{T} & \longrightarrow & \widetilde{T}. & & \end{array}$$

Moreover, there is an endomorphism $\hat{f}\colon \widehat{X} \to \widehat{X}$ satisfying $\hat{\nu} \circ \hat{f} = \tilde{f} \circ \hat{\nu}$. Thus, $\hat{f}$ is a lift of $f\colon X \to X$. By construction, the Galois group of θ preserves not only the fibration $\tilde{\pi}\colon \widetilde{X} \to \widetilde{T}$ but also the double-cover $\tilde{\pi}|_{\widetilde{S}}\colon \widetilde{S} \to \widetilde{T}$. Hence, the composite $\theta \circ \hat{\nu}$ is Galois and $(\theta \circ \hat{\nu}\colon \widehat{X} \to X, \hat{f})$ satisfies Theorem A(4). This completes the proof of Theorem A. $\square$

6. Applications of Theorem A

A normal Moishezon surface admitting a non-isomorphic surjective endomorphism is always projective. This is the statement of Corollary B, which has been proved in the introduction by applying Theorem A. We shall give some other applications of Theorem A. First, we shall prove Theorem 6.1 below concerning the case where X is irrational or K_X is pseudo-effective. As a corollary of Theorem 6.1, we have Proposition 6.2 below on possible singularities on X. Proposition C in the introduction is a consequence of Proposition 6.2. Finally, we shall prove Theorem D in the introduction on the first dynamical degree by applying Theorem 6.1 and previous results in Sections 3.1, 3.4, and 5.1.

Theorem 6.1. *Let X be a normal projective surface such that X is irrational or K_X is pseudo-effective. If X admits a non-isomorphic surjective endomorphism, then one of the following conditions is satisfied*:

1. *There is a finite Galois cover $C \times T \to X$ étale in codimension 1 for an elliptic curve C and a non-singular projective curve T of genus at least 2.*
2. *There is a finite Galois cover $A \to X$ étale in codimension 1 from an abelian surface A.*
3. *There is a finite Galois cover $\mathbb{P}^1 \times T \to X$ étale in codimension 1 for a non-singular projective curve T of genus at least 2.*
4. *The surface X is a $\mathbb{P}^1$-bundle over an elliptic curve.*
5. *The surface X is a projective cone over an elliptic curve.*

Proof. Assume that K_X is pseudo-effective. By Theorem A applied to the case where $S = 0$, there is a finite Galois cover $\nu\colon V \to X$ étale in codimension 1 satisfying either (2) or (3) of Theorem A. This is equivalent to saying that either (1) or (2) above is satisfied.

Next assume that X is irrational and K_X is not pseudo-effective. Then X is ruled, and one of the conditions (3), (4), and (5) above is satisfied by Theorem 4.16. Thus, we are done. □

Proposition 6.2. *Let X be a normal projective surface admitting a non-isomorphic surjective endomorphism. If X has an irrational singular point, then X is a projective cone over an elliptic curve.*

Proof. Assume that X has an irrational singular point. Then either X is irrational or K_X is pseudo-effective by [44, Lem. 2.31(3)]. Thus, we can apply Theorem 6.1, where only Theorem 6.1(5) remains as the possible case. □

Proof of Proposition C. Let X be a normal projective surface admitting a non-isomorphic surjective endomorphism. If X is a projective cone over an elliptic curve, then $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X) = 1$ by Lemma 1.18. If not, X has only rational singularities by Proposition 6.2; hence, X is $\mathbb{Q}$-factorial by [44, Lem. 2.31], and it implies that $\hat{\boldsymbol{\rho}}(X) = \boldsymbol{\rho}(X)$. □

Finally, we shall prove Theorem D.

Proof of Theorem D. Let f be a non-isomorphic surjective endomorphism of a normal projective surface X. If $K_X \approx 0$, then Theorem A(3) holds, and we have Theorem D(4) by Corollary 3.5. Assume that K_X is pseudo-effective and $K_X \not\approx 0$. Then Theorem 6.1(1) holds, and we have Theorem D(3) except the equality: $\lambda_f = \deg f$, but it has already been shown in Section 5.1. In fact, the semi-ample divisor K_X

defines a fibration $\pi\colon X \to T$ to a non-singular projective curve T and an automorphism $h\colon T \to T$ satisfying $\pi \circ f = h \circ \pi$. Thus, $\deg h = 1$ is an eigenvalue of $f^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$, and $\lambda_f = \deg f$ by Proposition 3.3(4).

Therefore, we may assume that K_X is not pseudo-effective. If $\boldsymbol{\rho}(X) \neq 2$ or if $(\lambda_f)^2 = \deg f$, then we have Theorem D(1) by Theorem 3.22 and Corollary 3.23. If $(\lambda_f)^2 > \deg f$, then we have Theorem D(2) by Propositions 3.24 and 3.25. Thus, we are done. □

Appendix A. On dynamical degrees

In Section 3.1, we have introduced the first dynamical degree for a surjective endomorphism of a normal Moishezon surface. Originally, dynamical degrees were introduced in the study of complex dynamical systems on compact Kähler manifolds (cf. [50], [9], [25]). The purpose of Appendix A is to prove that the first dynamical degree λ_f of a surjective endomorphism f of a normal Moishezon surface X defined in Definition 3.1 coincides with the first dynamical degree in the sense of complex dynamics of the induced meromorphic map $\nu^{-1} \circ f \circ \nu\colon Z \dashrightarrow Z$ for a non-singular projective surface Z with a birational map $\nu\colon Z \dashrightarrow X$ (cf. Corollary A.10 below). Our discussion simplifies and clarifies arguments in the proof of [25, Prop. 1.2(iii)], which deals with non-singular projective varieties of any dimension.

After proving some useful algebraic results on spectral radii of endomorphisms of real vector spaces in Section A.1, we shall prove Theorem A.9 and Corollary A.10 on the comparison of two first dynamical degrees in Section A.2.

A.1. **Spectral radii of endomorphisms of vector spaces.** The *spectral radius* of an endomorphism of a finite-dimensional real vector space is by definition the maximum of the absolute values of eigenvalues. We shall give some results on spectral radii of endomorphisms which preserve a strictly convex closed cone.

To begin with, we recall some basics on convex cones (cf. [49, App.]). Let V be a real vector space of dimension $n < \infty$. A *convex cone* $\mathcal{C}$ of V is by definition a subset such that $\mathbb{R}_{>0}\mathcal{C} \subset \mathcal{C}$ and $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$. For a convex cone $\mathcal{C}$, the minus $-\mathcal{C}$ is also a convex cone, and $\mathcal{C} + (-\mathcal{C})$ is the vector subspace generated by $\mathcal{C}$. If a convex cone $\mathcal{C}$ contains 0, then the intersection $\mathcal{C} \cap (-\mathcal{C})$ is also a vector subspace. A *strictly convex cone* is by definition a convex cone $\mathcal{C}$ such that $\mathcal{C} \cap (-\mathcal{C}) \subset \{0\}$. The *dual cone* $\mathcal{C}^\vee$ of $\mathcal{C}$ is a closed convex cone of the dual vector space $V^\vee = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ defined by

$$\mathcal{C}^\vee = \{\chi \in V^\vee \mid \chi(x) \geq 0 \text{ for any } x \in \mathcal{C}\}.$$

It is known that $(\mathcal{C}^\vee)^\vee = \mathcal{C}$ and $(\mathcal{C}_1 \cap \mathcal{C}_2)^\vee = \mathcal{C}_1^\vee + \mathcal{C}_2^\vee$ for any closed convex cones $\mathcal{C}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ of V. In particular, $\mathcal{C} + (-\mathcal{C}) = V$ if and only if $\mathcal{C}^\vee$ is strictly convex. It is an exercise to prove the following:

Lemma A.1. *Suppose that $\mathcal{C} + (-\mathcal{C}) = V$ for a closed convex cone $\mathcal{C}$ of V. Then the interior of $\mathcal{C}$ is non-empty. For a vector u in the interior of $\mathcal{C}$, there exists a basis $(x_1, \ldots, x_n)$ of V such that $x_i \in \mathcal{C}$ for any i and that $u = \sum_{i=1}^n x_i$.*

Definition A.2. Let $\mathcal{C}$ be a strictly convex closed cone of V and u a vector in the interior of $\mathcal{C}$. We define a norm $\|\cdot\| = \|\cdot\|_{\mathcal{C},u}$ of V by

$$\|v\| := \inf\{r \in \mathbb{R}_{\geq 0} \mid -v + ru \in \mathcal{C} \text{ and } v + ru \in \mathcal{C}\}$$

for any $v \in V$.

Remark. The norm $\|v\|$ is well-defined. In fact, for any $v \in V$,

- $u \pm (1/r)v \in \mathcal{C}$ for $r \gg 0$,
- $\|rv\| = r \cdot \|v\|$ for any $r \geq 0$, and
- $\|v\| = 0$ implies that $v \in \mathcal{C} \cap (-\mathcal{C}) = \{0\}$,

since $\mathcal{C}$ is strictly convex and closed. Moreover, since $\|v\|u \pm v \in \mathcal{C}$, we have

- $\|v_1\| + \|v_2\| \geq \|v_1 + v_2\|$ for any v_1, $v_2 \in V$.

Remark A.3. For the norm $\|\cdot\|_{\mathcal{C},u}$, we have $\|u\|_{\mathcal{C},u} = 1$. If $\mathcal{C}$ is the polyhedral cone $\sum_{i=1}^n \mathbb{R}_{\geq 0} e_i$ for a basis $(e_1, \ldots, e_n)$ of V and if $u = \sum_{i=1}^n e_i$, then, for any $a_i \in \mathbb{R}$,

$$\left\| \sum_{i=1}^n a_i e_i \right\|_{\mathcal{C},u} = \max\nolimits_{1 \leq i \leq n} |a_i|.$$

Lemma A.4. *Let χ be a vector in the dual space $V^\vee = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two closed convex cones of V such that $\mathcal{C}_1 \subset \mathcal{C}_2$, $\mathcal{C}_1 + (-\mathcal{C}_1) = V$ and that $\chi > 0$ on $\mathcal{C}_2 \setminus \{0\}$. Let u be a vector in the interior of $\mathcal{C}_1$ and let $\|\cdot\|_*$ be a norm of $\mathrm{End}(V) = \mathrm{Hom}_{\mathbb{R}}(V, V)$. Then there exist positive real numbers $c_1 < c_2$ such that*

$$c_1 \|\phi\|_* \leq \chi(\phi(u)) \leq c_2 \|\phi\|_*$$

for any $\phi \in \mathrm{End}(V)$ satisfying $\phi(\mathcal{C}_1) \subset \mathcal{C}_2$.

Proof. Since $\chi > 0$ on $\mathcal{C}_2 \setminus \{0\}$, we have $\mathcal{C}_1 \cap (-\mathcal{C}_1) = \mathcal{C}_2 \cap (-\mathcal{C}_2) = \{0\}$, i.e., $\mathcal{C}_1$ and $\mathcal{C}_2$ are strictly convex. For the proof, we may replace $\mathcal{C}_1$ with any closed convex cone $\mathcal{C}'_1 \subset \mathcal{C}_1$ such that u is in the interior of $\mathcal{C}'_1$. Therefore, by Lemma A.1, we may assume that $\mathcal{C}_1 = \sum_{i=1}^n \mathbb{R}_{\geq 0} x_i$ and $u = \sum_{i=1}^n x_i$ for a basis $(x_1, \ldots, x_n)$ of V. Let $\|\cdot\| := \|\cdot\|_{\mathcal{C}_1,u}$ be the norm of V defined in Definition A.2. Since $\chi > 0$ on $\mathcal{C}_2 \setminus \{0\}$ and since $\{y \in \mathcal{C}_2\,;\, \|y\| = 1\}$ is compact, there exist positive real numbers $\varepsilon_1 < \varepsilon_2$ such that

$$\text{(A-1)} \qquad \varepsilon_1 \|y\| \leq \chi(y) \leq \varepsilon_2 \|y\|$$

for any $y \in \mathcal{C}_2$. Let $\|\cdot\|_{\mathrm{op}}$ be the operator norm of $\mathrm{End}(V)$ defined by

$$\|\psi\|_{\mathrm{op}} := \sup\{\|\psi(v)\|\,;\, v \in V \text{ such that } \|v\| = 1\}$$

for $\psi \in \mathrm{End}(V)$. Since any two norms of the finite-dimensional vector space $\mathrm{End}(V)$ are "equivalent" to each other, we may assume that $\|\cdot\|_* = \|\cdot\|_{\mathrm{op}}$. Let ϕ be an arbitrary endomorphism of V such that $\phi(\mathcal{C}_1) \subset \mathcal{C}_2$. Then

$$\chi(\phi(u)) \leq \varepsilon_2 \|\phi(u)\| \leq \varepsilon_2 \|\phi\|_{\mathrm{op}}$$

by the right inequality of (A-1) and by $\|u\| = 1$. If $v = \sum_{i=1}^n a_i x_i \in V$ satisfies $\|v\| = 1$, then $\max_{1 \leq i \leq n} |a_i| = 1$ by Remark A.3, and moreover,

$$\|\phi(v)\| \leq \sum |a_i| \cdot \|\phi(x_i)\| \leq \sum \|\phi(x_i)\| \leq \varepsilon_1^{-1} \sum \chi(\phi(x_i)) = \varepsilon_1^{-1} \chi(\phi(u))$$

by the left inequality of (A-1). In particular, $\|\phi\|_{\mathrm{op}} \le \varepsilon_1^{-1}\chi(\phi(u))$. Therefore, it is enough to set $(c_1, c_2) = (\varepsilon_1, \varepsilon_2)$. □

Convention. For an endomorphism $\phi\colon V \to V$ of a finite-dimensional real vector space V, the spectral radius is denoted by $\rho(\phi)$.

Remark A.5. By considering the Jordan normal form of the matrix representation of ϕ, we see that $\rho(\phi) < 1$ if and only if $\lim_{m\to\infty} \phi^m = 0$ in $\mathrm{End}(V)$. Based on the property, we have:

$$\|\phi\|_{\mathrm{op}} \ge \rho(\phi) = \lim_{m\to\infty} \|\phi^m\|_*^{1/m}$$

for the operator norm $\|\cdot\|_{\mathrm{op}}$ associated with any norm $\|\cdot\|$ of V and for any norm $\|\cdot\|_*$ of $\mathrm{End}(V)$.

By Lemma A.4 and Remark A.5, we have:

Corollary A.6. *Let V, χ, $\mathcal{C}_1$, and $\mathcal{C}_2$ be the same as in Lemma* A.4. *Let ϕ_1, ϕ_2, ϕ_3, … be an infinite sequence of endomorphisms of V such that $\phi_i(\mathcal{C}_1) \subset \mathcal{C}_2$ for any $i \ge 1$. Then*

$$\overline{\lim}_{m\to\infty} \chi(\phi_m(u))^{1/m} = \overline{\lim}_{m\to\infty} \|\phi_m\|_*^{1/m} \ge \overline{\lim}_{m\to\infty} \rho(\phi_m)^{1/m},$$
$$\underline{\lim}_{m\to\infty} \chi(\phi_m(u))^{1/m} = \underline{\lim}_{m\to\infty} \|\phi_m\|_*^{1/m} \ge \underline{\lim}_{m\to\infty} \rho(\phi_m)^{1/m},$$

for any vector u in the interior of $\mathcal{C}_1$ and for any norm $\|\cdot\|_*$ of $\mathrm{End}(V)$.

Proposition A.7. *Let V be a finite-dimensional real vector space, $\chi\colon V \to \mathbb{R}$ a linear function, and $\mathcal{C}$ a closed convex cone of V such that $\mathcal{C} + (-\mathcal{C}) = V$ and $\chi > 0$ on $\mathcal{C} \setminus \{0\}$. Let u be a vector in the interior of $\mathcal{C}$ and let ϕ_1, ϕ_2, ϕ_3, … be an infinite sequence of endomorphisms $V \to V$ such that*

(i) *$\phi_m(\mathcal{C}) \subset \mathcal{C}$ for any $m \ge 1$, and*
(ii) *$\phi_{m_1}(\phi_{m_2}(u)) - \phi_{m_1+m_2}(u) \in \mathcal{C}$ for any m_1, $m_2 \ge 1$.*

Then one has

$$\begin{aligned}\lim_{m\to\infty} \chi(\phi_m(u))^{1/m} = \lim_{m\to\infty} \|\phi_m\|_*^{1/m} &= \lim_{m\to\infty} \rho(\phi_m)^{1/m} \\ &= \inf_{m\ge 1} \rho(\phi_m)^{1/m}\end{aligned}$$

for any norm $\|\cdot\|_*$ of $\mathrm{End}(V)$.

Proof. We apply the argument in the proof of Lemma A.4 to the case where $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$. Let $\|\cdot\|$ be the norm $\|\cdot\|_{\mathcal{C},u}$ defined in Definition A.2. Then we may replace $\|\cdot\|_*$ with the operator norm $\|\cdot\|_{\mathrm{op}}$ with respect to $\|\cdot\|$ as in the proof of Lemma A.4. By (i) and (ii),

$$\phi_{m_1} \circ \phi_{m_2} \circ \cdots \circ \phi_{m_s}(u) - \phi_{m_1+\cdots+m_s}(u) \in \mathcal{C} \tag{A-2}$$

holds for any positive integers m_1, …, m_s. This is shown by induction on s as follows: If $s = 2$, then this is just (ii). Assume that (A-2) holds for integers m_2, …, m_s. Then

$$\phi_{m_1}\bigl(\phi_{m_2} \circ \phi_{m_2} \circ \cdots \circ \phi_{m_s}(u) - \phi_{m_2+\cdots+m_s}(u)\bigr) \in \mathcal{C}$$

by (i), since $\phi_{m_1}(\mathcal{C}) \subset \mathcal{C}$. On the other hand, $\phi_{m_1}(\phi_l(u)) - \phi_{m_1+l}(u) \in \mathcal{C}$ for $l = m_2 + \cdots + m_s$ by (ii). Thus, (A-2) holds true by induction.

Let k be an arbitrary positive integer and fix it. Any positive integer m is expressed as $m = lk + r$ for non-negative integers $l = \llcorner m/k \lrcorner$ and $r < k$. Then

$$\phi_k^l(\phi_r(u)) - \phi_m(u) \in \mathcal{C}$$

by (A-2). Hence, there is a positive real number c_2 such that

$$\chi(\phi_m(u)) \le \chi(\phi_k^l(\phi_r(u))) \le c_2 \|\phi_k^l\|_*$$

for any m and $l = \llcorner m/k \lrcorner$ by Lemma A.4, and we have

$$\overline{\lim}_{m\to\infty} \chi(\phi_m(u))^{1/m} \le \lim_{l\to\infty} \left(\|\phi_k^l\|_*^{1/l} \right)^{1/k} = \rho(\phi_k)^{1/k}$$

by Remark A.5. As a consequence,

$$\overline{\lim}_{m\to\infty} \chi(\phi_m(u))^{1/m} \le \inf_{k\ge 1} \rho(\phi_k)^{1/k} < \infty.$$

By inequalities in Corollary A.6, we have the expected equalities. □

A.2. **The first dynamical degree for a normal Moishezon surface.** First, we recall the definition of dynamical degrees in the sense of complex dynamics (cf. [50, p. 917, Def.], [9, p. 960], [25, Def. 1.1]):

Definition A.8. Let $\varphi\colon Z \dashrightarrow Z$ be a dominant meromorphic map for a compact Kähler manifold Z of dimension n. Let $\mu\colon Y \to Z$ be a bimeromorphic morphism from another compact Kähler manifold Y such that $\psi := \varphi \circ \mu\colon Y \to Z$ is holomorphic. For a Kähler form ω on Z and for an integer $1 \le l \le n$, we set

$$\delta_l(\varphi, \omega) := \int_Z \varphi^*(\omega^l) \wedge \omega^{n-l} = \int_Y \psi^*(\omega^l) \wedge \mu^*(\omega^{n-l}),$$

where ω^i stands for the (i,i)-form $\wedge^i\omega$, and $\varphi^*(\omega^i) := \mu_*(\psi^*\omega^i)$ as a current on Z. The number $\delta_l(\varphi,\omega)$ is independent of the choice of μ. The *l-th dynamical degree* $\lambda_l(\varphi)$ is defined as

$$\lambda_l(\varphi) := \lim_{m\to\infty} \delta_l(\varphi^m, \omega)^{1/m}.$$

Remark. The limit exists by [10, Cor. 7]. By definition, $\lambda_n(\varphi)$ is equal to the mapping degree $\deg\varphi$ $(= \deg\psi)$. The dynamical degree $\lambda_l(\varphi)$ is independent of the choice of the Kähler form ω, and moreover, it is determined by φ up to conjugation by the bimeromorphic maps $Z \dashrightarrow Z$ (cf. [10, Cor. 7]).

Remark. The dynamical degrees are defined and studied in connection with topological entropies in several articles of complex dynamics including [50], [8], [9], [10], and [25].

The purpose of Section A.2 is to prove:

Theorem A.9. *Let $f\colon X \dashrightarrow X$ be a dominant meromorphic map for a normal Moishezon surface X. Let $\nu\colon Z \dashrightarrow X$ be a birational map from a non-singular projective surface Z and let $\varphi\colon Z \dashrightarrow Z$ be the meromorphic map $\nu^{-1} \circ f \circ \nu$. Then*

$$\lambda_1(\varphi) = \lim_{m\to\infty} \rho\left((f^m)^*\right)^{1/m}.$$

Here, $(f^m)^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ *is a linear map in Definition* A.11 *below defined for the* m*-th power* $f^m\colon X \dashrightarrow X$*, and* $\rho((f^m)^*)$ *is the spectral radius.*

Corollary A.10. *In Theorem* A.9*, assume that* f *is holomorphic. Then* $\lambda_1(\varphi)$ *equals* λ_f *defined in Definition* 3.1*.*

Proof. We have $\rho\left((f^m)^*\right) = \rho(f^*)^m = \lambda_f^m$ for any $m \geq 1$, since $(f^m)^* = (f^*)^m$ as an endomorphism of $\mathsf{N}(X)$. Thus, $\lambda_1(\varphi) = \lambda_f$ by Theorem A.9. □

The proof of Theorem A.9 is given at the end. We begin with

Definition A.11. Let $f\colon X \dashrightarrow Y$ be a dominant meromorphic map of normal Moishezon surfaces. For an $\mathbb{R}$-divisor D on Y, we define the *total pullback* f^*D as $\mu_*(h^*D)$ for a birational morphism $\mu\colon X' \to X$ from another normal Moishezon surface X' such that $h := f \circ \mu\colon X' \to Y$ is holomorphic (cf. [45, Def. 1.30]). Here, f^*D does not depend on the choice of $\mu\colon X' \to X$ (cf. [45, Lem. 1.31(2)]). We set $f^*\colon \mathsf{N}(Y) \to \mathsf{N}(X)$ to be the homomorphism defined by $D \mapsto f^*D$. This is just the composite $\mu_* \circ h^*\colon \mathsf{N}(Y) \to \mathsf{N}(X') \to \mathsf{N}(X)$.

Lemma A.12. *In the situation of Definition* A.11*, if* D *is nef* (*resp. effective, resp. big*)*, then so is* f^*D*. In particular,* $f^*\operatorname{Nef}(Y) \subset \operatorname{Nef}(X)$ *and* $f^*\overline{\mathrm{NE}}(Y) \subset \overline{\mathrm{NE}}(X)$ *for the homomorphism* $f^*\colon \mathsf{N}(Y) \to \mathsf{N}(X)$ (*cf.* [44, Rem. 2.13]).

Proof. If D is effective, then h^*D and $\mu_*(h^*D) = f^*D$ are so. Suppose that D is big. Then h^*D is so, since h is generically finite, and $h^*D \geq \varepsilon\mu^*A$ for a numerically ample divisor A on X and a rational number $\varepsilon > 0$. It implies: $f^*D = \mu_*(h^*D) \geq \varepsilon\mu_*(\mu^*A) = \varepsilon A$. Thus, f^*D is big. Suppose next that D is nef. Then, for any effective divisor E on X, we have

$$(f^*D \cdot E) = (\mu_*(h^*D) \cdot E) = (h^*D \cdot \mu^*E) = (D \cdot h_*(\mu^*E)) \geq 0,$$

since μ^*E and $h_*(\mu^*E)$ are effective. Thus, f^*D is also nef. □

Lemma A.13. *Let* $f\colon X \dashrightarrow X$ *be a dominant meromorphic map for a normal Moishezon surface* X*. Then*

$$\lim\nolimits_{m\to\infty}(B \cdot (f^m)^*B)^{1/m} = \lim\nolimits_{m\to\infty} \rho((f^m)^*)^{1/m} \tag{A-3}$$

for any nef and big $\mathbb{R}$*-divisor* B *on* X*, where* $(f^m)^*\colon \mathsf{N}(X) \to \mathsf{N}(X)$ *is the endomorphism defined in Definition* A.11 *for the* m*-th power* $f^m\colon X \dashrightarrow X$*.*

Proof. Let A be a numerically ample $\mathbb{R}$-divisor on X. Then $\mathrm{cl}(A)$ lies in the interior of $\operatorname{Nef}(X)$, and $\chi_A > 0$ on $\overline{\mathrm{NE}}(X) \setminus \{0\}$ for the linear function $\chi_A\colon \mathsf{N}(X) \to \mathbb{R}$ defined by $D \mapsto (D \cdot A)$ for $\mathbb{R}$-divisors D. Here, $(f^m)^*\,\overline{\mathrm{NE}}(X) \subset \overline{\mathrm{NE}}(X)$ for any $m \geq 1$ by Lemma A.12 and

$$(f^{m_1})^*((f^{m_2})^*A) \geq (f^{m_1+m_2})^*A$$

for any m_1, $m_2 \geq 1$ by [45, Cor. 1.33]. Thus, we can apply Proposition A.7 to: $V = \mathsf{N}(X)$, $\mathcal{C} = \overline{\mathrm{NE}}(X)$, $\phi_m = (f^m)^*$, $u = \mathrm{cl}(A)$, and $\chi = \chi_A$. As a consequence, we have (A-3) for the numerically ample divisor A instead of B. Now, there exist positive real numbers α and β such that $B - \alpha A$ is big and $A - \beta B$ is numerically ample. Then $(f^m)^*A$ and $(f^m)^*B$ are nef and $(f^m)^*(A-\beta B)$ and $(f^m)^*(B-\alpha A)$ are big for any $m>1$ by Lemma A.12. Thus, we have inequalities

$$(A\cdot(f^m)^*A)\geq \beta(B\cdot(f^m)^*A)\geq \beta^2(B\cdot(f^m)^*B) \quad \text{and}$$
$$(B\cdot(f^m)^*B)\geq \alpha(A\cdot(f^m)^*B)\geq \alpha^2(A\cdot(f^m)^*A)$$

and hence,

$$\lim_{m\to\infty}(B\cdot(f^m)^*B)^{1/m}=\lim_{m\to\infty}(A\cdot(f^m)^*A)^{1/m}.$$

Therefore, (A-3) holds. ☐

Corollary A.14. *In Lemma* A.13*, suppose that X is a non-singular projective surface. Then*

$$\lambda_1(f)=\lim_{m\to\infty}\rho((f^m)^*)^{1/m}=\lim_{m\to\infty}(B\cdot(f^m)^*B)^{1/m}$$

for any nef and big $\mathbb{R}$-divisor B.

Proof. By Lemma A.13, it is enough to show

$$\lambda_1(f)=\lim_{m\to\infty}(B\cdot(f^m)^*B)^{1/m} \tag{A-4}$$

assuming that B is an ample divisor. Let ω be a Kähler form on X such that $[\omega]=c_1(B)$ in $H^2(X,\mathbb{R})$. Let $\mu\colon X'\to X$ be a birational morphism from a non-singular projective surface X' such that $h:=f\circ\mu\colon X'\to X$ is holomorphic. Then $\delta_1(f,\omega)=(h^*B\cdot\mu^*B)=(B\cdot f^*B)$ by Definition A.8. Hence, $\delta_1(f^m,\omega)=(B\cdot(f^m)^*B)$ for any $m\geq 1$, and we have (A-4). ☐

Lemma A.15. *Assume that the meromorphic map $\nu\colon Z\dashrightarrow X$ in Theorem* A.9 *is holomorphic. Then*

$$(\nu^*A_1\cdot(\varphi^m)^*(\nu^*A_2))=(A_1\cdot(f^m)^*A_2)$$

for any $m\geq 1$ and for any $\mathbb{R}$-divisors A_1 and A_2 on X, where $\varphi\colon Z\dashrightarrow Z$ is the meromorphic map $\nu^{-1}\circ f\circ\nu$ in Theorem A.9.

Proof. Without loss of generality, we may assume that $m=1$, since $\varphi^m=\nu^{-1}\circ f^m\circ\nu$. Let $\mu\colon X'\to X$ and $\tilde{\mu}\colon Z'\to Z$ be birational morphisms from normal projective surfaces X' and Z' such that $h:=f\circ\mu\colon X'\to X$, $\tilde{h}:=\varphi\circ\tilde{\mu}\colon Z'\dashrightarrow Z$, and $\nu':=\mu^{-1}\circ\nu\circ\tilde{\mu}\colon Z'\to X'$ are holomorphic. Then we have a commutative diagram

$$\begin{array}{ccccc}
Z & \xleftarrow{\tilde{\mu}} & Z' & \xrightarrow{\tilde{h}} & Z\\
{\scriptstyle\nu}\downarrow & & {\scriptstyle\nu'}\downarrow & & \downarrow{\scriptstyle\nu}\\
X & \xleftarrow{\mu} & X' & \xrightarrow{h} & X
\end{array}$$

of holomorphic maps, where $\varphi=\tilde{h}\circ\tilde{\mu}^{-1}$ and $f=h\circ\mu^{-1}$ as meromorphic maps. Hence,

$$\begin{aligned}(\nu^*A\cdot\varphi^*(\nu^*B))&=(\nu^*A\cdot\tilde{\mu}_*(\tilde{h}^*(\nu^*B)))=(\tilde{\mu}^*(\nu^*A)\cdot\tilde{h}^*(\nu^*B))\\
&=(\nu'^*(\mu^*A)\cdot\nu'^*(h^*B))=(\mu^*A\cdot h^*B)=(A\cdot\mu_*(h^*B))=(A\cdot f^*B)\end{aligned}$$

by projection formulas for $_*$ and * on intersection numbers. ☐

Finally, we shall prove Theorem A.9.

Proof of Theorem A.9. Let $\sigma\colon Z' \to Z$ be a birational morphism from a non-singular projective surface Z' such that $\nu' = \nu \circ \sigma\colon Z' \to X$ is also holomorphic. Let $\varphi'\colon Z' \cdots\to Z'$ be the induced meromorphic map $\sigma^{-1} \circ \varphi \circ \sigma = \nu'^{-1} \circ f \circ \nu'$. Then the diagram

$$\begin{array}{ccccc} X & \xleftarrow{\nu'} & Z' & \xrightarrow{\sigma} & Z \\ {\scriptstyle f}\Big\downarrow & & {\scriptstyle \varphi'}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi} \\ X & \xleftarrow{\nu'} & Z' & \xrightarrow{\sigma} & Z \end{array}$$

of meromorphic maps is commutative, where $\nu = \nu' \circ \sigma^{-1}$. For a numerically ample divisor A on X and an ample divisor H on Z, we have

$$\begin{aligned}\lambda_1(\varphi') &= \lim_{m\to\infty}(\nu'^*A \cdot (\varphi'^m)^*(\nu'^*A))^{1/m} = \lim_{m\to\infty}(\sigma^*H \cdot (\varphi'^m)^*(\sigma^*H))^{1/m} \\ &= \lim_{m\to\infty}(A \cdot (f^m)^*A)^{1/m} = \lim_{m\to\infty}(H \cdot (\varphi^m)^*H)^{1/m} = \lambda_1(\varphi)\end{aligned}$$

by Corollary A.14 and Lemma A.15, since ν' and σ are holomorphic and since ν'^*A and σ^*H are nef and big. Therefore,

$$\lambda_1(\varphi) = \lim_{m\to\infty}(A \cdot (f^m)^*A)^{1/m} = \lim_{m\to\infty} \rho((f^m)^*)$$

by Lemma A.13. Thus, we are done. □

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